

ON THE SYMPLECTIC STRUCTURE DEFORMATIONS RELATED WITH MONGE–AMPÈRE EQUATION ON THE KÄHLER MANIFOLD $P_2(\mathbb{C})$

ALEXANDER A. BALINSKY AND ANATOLIY K. PRYKARPATSKI^{1,2)}, PETRO YA. PUKACH²⁾,
MYROSLAVA I. VOVK²⁾

In memoriam of the outstanding Ukrainian mathematician Anatoliy M. Samoilenko with admiration

ABSTRACT. There is analyzed the cohomology structure of the fundamental two-form deformation related with a modified Monge–Ampère type on the complex Kähler manifold $P_2(\mathbb{C})$. Based on the Levi-Civita connection and the related vector field deformation of the fundamental two-form there is constructed a hierarchy of bilinear symmetric forms on the tangent bundle to the Kähler manifold $P_2(\mathbb{C})$, generating on it Hermitean metrics and suitable solutions to the studied Monge–Ampère type equation. The classical fundamental two-form construction on the complex Kähler manifold $P_2(\mathbb{C})$ is generalized and the related metric deformations are discussed.

1. INTRODUCTION

Let us consider a compact complex n -dimensional manifold $M_{\mathbb{C}}^n$, endowed with the Kähler [1, 3, 27] fundamental symplectic two-form $\omega \in \Lambda^2(M_{\mathbb{C}}^n)$ and define the related Monge–Ampère equation, describing a deformation of this symplectic structure:

$$(1.1) \quad (\omega + i\bar{\partial}\partial\varphi)^n = (\exp f) \omega^n$$

under the normalizing conditions

$$(1.2) \quad \int_{M_{\mathbb{C}}^n} (\exp f) \omega^n = \int_{M_{\mathbb{C}}^n} \omega^n, \quad \int_{M_{\mathbb{C}}^n} \varphi \omega^n = 0,$$

where $\varphi \in C^\infty(M_{\mathbb{C}}^n; \mathbb{R})$ is a real valued function on $M_{\mathbb{C}}^n$ and $\bar{\partial}$ is the complex ∂ -bar differential, corresponding to the standard differential splitting $d = \partial \oplus \bar{\partial} : \Lambda(M_{\mathbb{C}}^n) \rightarrow \Lambda(M_{\mathbb{C}}^n)$ on the complex manifold $M_{\mathbb{C}}^n$. In a general case it was supposed [28] that if the two-form $(\omega + i\bar{\partial}\partial\varphi) \in \Lambda^2(M_{\mathbb{C}}^n)$ is real valued and the first Chern class $c_1(M_{\mathbb{C}}^n) = 0$ of a Kähler manifold $M_{\mathbb{C}}^n$, then there exists a Riemannian metric $g : T(M_{\mathbb{C}}^n) \times T(M_{\mathbb{C}}^n) \rightarrow \mathbb{C}$ of the Calabi-Yau type, whose holonomy group [9, 10, 12, 13] coincides with a subgroup of the Lie group $SU(2)$, generating, in particular, a so called Einsteinian metric. The equation (1.1) is always [28] solvable, yet its holonomy groups, in general, not classified and its unitarity remains to be open.

One can also mention here that if a Kähler manifold $M_{\mathbb{C}}^2$ is compact with the Chern class $c_1(M_{\mathbb{C}}^2) = 0$, it is well known [24] that it is then hyper-Kähler, possessing exactly three Kähler fundamental forms ω_I, ω_J and $\omega_K \in \Lambda^2(M_{\mathbb{C}}^2)$, corresponding to three compatible complex structures I, J and $K : T(M_{\mathbb{C}}^2) \rightarrow T(M_{\mathbb{C}}^2)$. As for the compact projective two-dimensional Kähler manifold $M_{\mathbb{C}}^2 = P_2(\mathbb{C})$ the Chern class $c_1(M_{\mathbb{C}}^2) \neq 0$, it is not hyper-Kähler, and its holomorphic volume two-form $\Omega \in \Lambda_{hol}^2(M_{\mathbb{C}}^2)$ is not composed of the symplectic forms ω_J and $\omega_K \in \Lambda^2(M_{\mathbb{C}}^2)$.

We here also remark that there exists a slightly different modified Monge–Ampère type equation

$$(1.3) \quad (\omega + dJ^*d\varphi)^n = (\exp f) \omega^n,$$

on a real symplectic manifold $M^{2n} \simeq M_{\mathbb{C}}^n$, where $f \in C^\infty(M^{2n}; \mathbb{R})$ and $J : T(M^{2n}) \rightarrow T(M^{2n}), J^2 = -I$, is a suitably chosen nonintegrable quasi-complex structure on the manifold M^{2n} and $J^* : T^*(M^{2n}) \rightarrow T^*(M^{2n})$ denotes its conjugate. It was proved [5] that if the structure $J : T(M^{2n}) \rightarrow T(M^{2n})$ is integrable, then the equation (1.3) reduces to the Monge–Ampère equation (1.1) on the related complex manifold $M_{\mathbb{C}}^n \simeq M^{2n}$ owing to the classical Newlander-Nirenberg [16] criterion. Otherwise, if the equation (1.3) is solvable for its arbitrarily chosen right hand side,

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then the quasi-complex structure $J : T(M^{2n}) \rightarrow T(M^{2n})$ proves to be necessary [5, 15, 19, 17] a complex one, once more reducing the equation (1.3) to the Monge–Ampère equation (1.1).

In our note we are interested in the following "symplectic" deformation

$$(1.4) \quad (\omega + dd^s\psi)^2 = (\exp f)\omega^2$$

of the Monge–Ampère (1.1) on the complex two-dimensional Kähler manifold $M_{\mathbb{C}}^2 = P_2(\mathbb{C})$, whose Chern class $c_1(M_{\mathbb{C}}^2) \neq 0$, by means of a symplectic deformation

$$(1.5) \quad \omega \rightarrow \omega + dd^s\psi,$$

where $\psi \in \Lambda^2(M_{\mathbb{C}}^n)$ is a searched two-form and, by definition, the mapping $d^s := (-1)^{k+1} \star_s d \star_s$ denotes the adjoint [6, 10, 12, 24] symplectic Hodge type differentiation, satisfying the by-linear scalar relationship

$$(1.6) \quad (\alpha^{(k)} | d\beta^{(m)})_s := (d^s \alpha^{(k)} | \beta^{(m)})_s$$

for all differential k -forms $\alpha, \beta \in \Lambda^k(M^{2n})$, $k = \overline{1, 2n}$, as well as the identity $dd^s = -d^s d$. Here we have put, by definition,

$$(1.7) \quad (\alpha^{(k)} | \beta^{(m)})_s := \delta_{km} \int_{M^n} \langle \bar{\alpha}^{(k)} | \beta^{(m)} \rangle_s d\mu := \delta_{km} \int_{M^n} \bar{\alpha}^{(k)} \wedge \star_s \beta^{(m)}$$

for any differential k -form $\alpha \in \Lambda^k(M^{2n})$ and differential m -form $\beta \in \Lambda^m(M^{2n})$, where $d\mu := \omega^n/n!$ is the volume measure on M^{2n} and the symplectic Hodge-star mapping $\star_s : \Lambda(M^{2n}) \rightarrow \Lambda(M^{2n})$ acts on differential k -forms via the identity

$$(1.8) \quad \alpha \wedge \star_s \beta = \langle \alpha | \beta \rangle_s \omega^n / n!$$

subject to the "symplectic" bilinear form $\langle \cdot | \cdot \rangle_s : \Lambda^k(M^{2n}) \times \Lambda^k(M^{2n}) \rightarrow \mathbb{R}$, looking as

$$(1.9) \quad \langle \alpha | \beta \rangle_s := \frac{1}{k!} \sum_{i_l, j_m, l, m = \overline{1, k}}^{2n} \omega^{i_1 j_1} \omega^{i_2 j_2} \dots \omega^{i_k j_k} \alpha_{i_1 i_2 \dots i_k} \beta_{j_1 j_2 \dots j_k},$$

if $\alpha := \frac{1}{k!} \sum_{i_m, m = \overline{1, k}}^{2n} \alpha_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k}$, $\beta := \frac{1}{k!} \sum_{i_m, m = \overline{1, k}}^{2n} \beta_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} \in \Lambda^k(M^{2n})$, $\omega := \frac{1}{2} \sum_{i, j = 1}^{2n} \omega_{ij} dx^i \wedge dx^j \in \Lambda^2(M^{2n})$ and $\sum_{k=1}^{2n} \omega^{ik} \omega_{kj} := \delta_j^i$, $i, j = \overline{1, 2n}$, being further naturally continued on the complex-valued differential forms. It is worth of mentioning here that in the case of Kähler manifolds the following important equalities

$$(1.10) \quad \text{Im } d \cap \ker d^s = \ker d \cap \text{Im } d^s = \text{Im } dd^s$$

hold, being consequences of the expression (1.6) and (1.8). The scalar product (1.7), naturally extends [3, 27] on the complex valued differential forms on the complex manifold $M^n \simeq M^{2n}$ and gives rise to the following symplectic scalar product on $\Lambda(M_{\mathbb{C}}^n)$:

$$(1.11) \quad (\alpha^{(k)} | \beta^{(m)})_s := \delta_{km} \int_{M_{\mathbb{C}}^n} \langle \bar{\alpha}^{(k)} | \bar{\beta}^{(m)} \rangle_s d\mu := \delta_{km} \int_{M_{\mathbb{C}}^n} \alpha^{(k)} \wedge \star_s \bar{\beta}^{(m)},$$

where $\alpha^{(k)} \in \Lambda^k(M_{\mathbb{C}}^n)$, $\beta^{(m)} \in \Lambda^m(M_{\mathbb{C}}^n)$, $k, m = \overline{1, 2n}$, are complex-valued differential forms on $M_{\mathbb{C}}^n$ and the bar "–" denotes the usual complex conjugation. In addition, the following identities

$$(1.12) \quad \begin{aligned} (\omega | \alpha \wedge \beta)_s &= (\alpha | \beta)_s = -\bar{\beta}(X_{\alpha}) = \bar{\alpha}(X_{\beta}), \\ (\omega + dd^s\psi | \alpha \wedge \beta)_s &= -\bar{\beta}(X_{\alpha}) - (i_{\bar{X}_{\alpha}} d\psi | d\bar{\beta})_s + (i_{\bar{X}_{\beta}} d\psi | d\bar{\alpha})_s, \end{aligned}$$

hold for any one-forms $\alpha, \beta \in \Lambda^1(M_{\mathbb{C}}^n)$, where vector fields $X_{\alpha}, X_{\beta} \in \Gamma(T(M_{\mathbb{C}}^n))$ are defined via the relationships $i_{X_{\alpha}} \omega := \alpha$, $i_{X_{\beta}} \omega := \beta$, respectively.

It is worth remarking also that in general case, when the Chern class $c_1(M_{\mathbb{C}}^2) \neq 0$, notwithstanding this fact, based on the equalities (1.10) and the well known [3, 4, 26, 27] relationship

$$(1.13) \quad \star_s \eta = -\eta$$

for an arbitrary "primitive" holomorphic volume two-form $\eta \in \Lambda_{hol}^2(M_{\mathbb{C}}^2)$, satisfying the additional condition $\eta \wedge \omega = 0$, one easily derives the existence of two cohomological "primitive" holomorphic volume two-forms Ω_1 and $\Omega_2 \in \Lambda_{hol}^2(M^2)$, for which $\Omega_1 \wedge \bar{\Omega}_1 = \omega = \Omega_2 \wedge \bar{\Omega}_2$. Moreover, the following interesting relationship

$$(1.14) \quad \Omega_1 - \Omega_2 = dd^s \chi$$

holds for some smooth two-form $\chi \in \Lambda^2(M_{\mathbb{C}}^2)$, solving the problem (1.4) for the case, when the fundamental symplectic structure $\omega \in \Lambda^2(M_{\mathbb{C}}^2)$ is replaced by a holomorphic volume two-form $\Omega \in \Lambda_{hol}^2(M_{\mathbb{C}}^2)$.

In our report, having analyzed in detail the cohomology structure of the deformed two-form expression $(\omega + dd^s\psi) \in \Lambda^2(M_{\mathbb{C}}^2)$ and applied some generalized transformations which were suggested in the classical works by Enneper [7] and Weierstrass [25] about one and half century ago and recently developed in [11], we succeeded in reformulating the "symplectic" modification of the Monge–Ampère (1.4) by means of specially constructed coordinates, allowing to construct its special solutions. It is here important to mention that in general the considered deformed structures are, in general, stable and conserving [2, 15, 20, 21, 22, 23] their properties only locally, and their global determination is one of main problems under regard.

2. CANONICAL METRIC ON $P_2(\mathbb{C})$ AND THE RELATED FUNDAMENTAL SYMPLECTIC FORM

Let $z := (z^0, z^1, z^2)^T \in \mathbb{C}^3$ be a uniform coordinate frame on the Kähler complex manifold $M_{\mathbb{C}}^2 := P_2(\mathbb{C})$ and define a related linear connection mapping

$$(2.1) \quad E^3 \ni u \rightarrow d_f(u) := du + \vartheta_f u \in E^3 \otimes \Lambda^1(M_{\mathbb{C}}^2),$$

where $E^3 := (\mathbb{C}^3, \pi, M_{\mathbb{C}}^2; \text{SO}(3) \times \mathbb{S}^1)$ is a one-dimensional complex vector bundle over $M_{\mathbb{C}}^2 \simeq \mathbb{E}^3 / (\text{SO}(3) \times \mathbb{S}^1)$ with the structure group $\text{SO}(3) \times \mathbb{S}^1$, completely specified by means of the local holomorphic basis frame vector $f(z) := z \in E^3$, and $\vartheta : E^3 \rightarrow E^3 \otimes \Lambda^1(M_{\mathbb{C}}^2)$ denotes the corresponding connection form. As the basis frame vector $f(z) \in E^3$ makes it possible to define on the Kähler manifold $M_{\mathbb{C}}^2$ the canonical Hermitian metric

$$(2.2) \quad g_f(A(z), B(z)) := \bar{f}(z)^T f(z) a(z) \bar{b}(z)$$

for any vectors $A(z) = a(z)f(z) \in E^3$ and $B(z) = b(z)f(z) \in E^3$ at a point $p(z) \in M_{\mathbb{C}}^2$, one can construct easily the holomorphic connection form $\vartheta_f := \vartheta : E^3 \rightarrow E^3 \otimes \Lambda^{1,0}(M_{\mathbb{C}}^2)$, compatible with the metric (2.2) by means of the well known [27] relationship

$$(2.3) \quad \vartheta^{1,0}(z) := [\bar{f}(z)^T f(z)]^{-1} \bar{f}(z)^T \partial f(z),$$

where, by definition, $\vartheta(f(z)) := \vartheta^{1,0}(z)f(z) \in E^3 \otimes \Lambda^1(M_{\mathbb{C}}^2)$. As the iterated mapping $d_f^2 = d_f \circ d_f : E^3 \otimes \Lambda(M_{\mathbb{C}}^2) \rightarrow E^3 \otimes \Lambda(M_{\mathbb{C}}^2)$, being a linear homomorphism on the module $\Lambda(M_{\mathbb{C}}^2)$, determines [3, 27] the closed global curvature two-form

$$(2.4) \quad \begin{aligned} \Omega(z) &= \frac{i}{2} d_f^2 = \frac{i}{2} (d\vartheta^{1,0}(z) + \vartheta^{1,0}(z) \wedge \vartheta^{1,0}(z)) = \\ &= \frac{i}{2} \bar{\partial} \left([\bar{f}(z)^T f(z)]^{-1} \partial \bar{f}(z)^T f(z) \right) = \\ &= \frac{i}{2} (|z|^{-2} \langle dz | \wedge dz \rangle - |z|^{-4} \langle dz | z \rangle \wedge \langle z | dz \rangle) \end{aligned}$$

at arbitrary point $p(z) \in M_{\mathbb{C}}^2$, generating the first nontrivial Chern class $c_1(M_{\mathbb{C}}^2) := [\Omega] \in H^2(M_{\mathbb{C}}^2; \mathbb{Z})$ of the Kähler complex manifold $M_{\mathbb{C}}^2$. The obtained curvature two-form $\Omega \in \Lambda^2(M_{\mathbb{C}}^2)$, being nondegenerate, can be identified with the fundamental symplectic two-form $\omega \in \Lambda^2(M_{\mathbb{C}}^2)$, that is

$$(2.5) \quad \omega = \frac{i}{2} (|z|^{-2} \langle dz | \wedge dz \rangle - |z|^{-4} \langle dz | z \rangle \wedge \langle z | dz \rangle),$$

naturally determining on the Kähler complex manifold $M_{\mathbb{C}}^2$ the compatible positive definite Fubini–Study [3, 27] metric

$$(2.6) \quad g(dz, dz) := |z|^{-2} \langle dz | dz \rangle - |z|^{-4} \langle dz | z \rangle \langle z | dz \rangle$$

at $p(z) \in M_{\mathbb{C}}^2$.

The Fubini–Study metric (2.6) is obtained as that compatible with the canonical symplectic structure (2.5), naturally generated by the curvature two-form (2.4), corresponding to the canonical connection (2.1) on the one-dimensional vector bundle $E^3 := (\mathbb{C}^3, \pi, M_{\mathbb{C}}^2; \text{SO}(3) \times \mathbb{S}^1)$. The latter depends strongly on a coordinate frame $f(z) \in E^3$ at point $p(z) \in M_{\mathbb{C}}^2$, which was chosen to be trivial as $f(z) := z \in E^3$. It is evident that this choice is not unique and any other coordinate frame $f(z) \in E^3$ will provide a suitable connection $d_f : E^3 \rightarrow E^3 \otimes \Lambda^1(M_{\mathbb{C}}^2)$ on the

Kähler complex manifold $M_{\mathbb{C}}^2$, whose a priori closed curvature two-form $\Omega_f \in \Lambda^2(M_{\mathbb{C}}^2)$ if, none-degenerate, can be interpreted as a symplectic two-form $\omega_f \in \Lambda^2(M_{\mathbb{C}}^2)$, based on which one can naturally present the related bilinear and symmetric form $g_f : T(M_{\mathbb{C}}^2) \times T(M_{\mathbb{C}}^2) \rightarrow \mathbb{C}$. If, moreover, a suitably chosen coordinate frame $f(z) \in E^3$ makes this bilinear form positive definite, we will arrive at some symplectic deformation $\omega_f \in \Lambda^2(M_{\mathbb{C}}^2)$ of the canonical symplectic two-form $\omega \in \Lambda^2(M_{\mathbb{C}}^2)$, constructed before.

To realize this scheme analytically, we will make use below of the classical constructions devised still in classical works by Enneper [7] and Weierstrass [25] about one and half century ago and recently developed in the general case [4, 8, 11, 29] of the Kähler manifold $P_n(\mathbb{C})$, $n \in \mathbb{N}$.

Consider the following linear projective type mapping

$$(2.7) \quad Q_V : E^3 \ni f \rightarrow \langle df|V \rangle - f|f|^{-2} \langle df|V \rangle \in E^3$$

for a fixed nontrivial vector field $V \in \Gamma(T(E^3))$ and iterate it starting from some holomorphic coordinate frame function $f = f_0 \in E^3$. A sequence $f_j \in E, j = \overline{0, 2}$, obtained this way, is parameterized by a fixed nontrivial vector $V \in \Gamma(T(E^3))$ and characterized by the following lemma.

Lemma 2.1. *Let vectors $f_j := f_j(z) \in E, j = \overline{0, 2}$, be defined for any $n \in \mathbb{Z}_+$ as*

$$(2.8) \quad \begin{aligned} f_0 &:= f, \quad f_1 := Q_V f_0, \quad f_2 := Q_V f_1 - f_0|f_0|^{-2} \langle df_1(V)|f_0 \rangle, \\ f_3 &:= Q_V f_2 - f_1|f_1|^{-2} \langle df_2(V)|f_1 \rangle - f_0|f_0|^{-2} \langle df_2(V)|f_0 \rangle, \dots, \\ f_{n+1} &:= Q_V f_n - \sum_{j=\overline{0, n-1}} f_j|f_j|^{-2} \langle df_n(V)|f_j \rangle \end{aligned}$$

at each point $p(z) \in M_{\mathbb{C}}^2$. Then $f_j = 0$ for all $j \geq 3$ and three vectors $f_j \in E^3, j \in \overline{0, 2}$, are biorthogonal:

$$(2.9) \quad \langle f_s|f_k \rangle = 0$$

for $k \neq s = \overline{0, 2}$ at all $p(z) \in M_{\mathbb{C}}^2$.

Proof. It is easy to check that $\langle f_0|f_1 \rangle = 0$. Now, by induction, we assume that all vectors $f_j \in E^3, j = \overline{0, n}$, defined by (2.8), are biorthogonal to each other, that is $\langle f_j|f_k \rangle = 0$ for all $j \neq k = \overline{0, n-1}$. Concerning the vector $f_{n+1} \in \mathbb{E}^3$ we can calculate that

$$(2.10) \quad \begin{aligned} \langle f_{n+1}|f_k \rangle &:= \langle Q_V f_n|f_k \rangle - \\ &- \sum_{j=\overline{0, n-1}} \langle f_j|f_k \rangle \langle df_n(V)|f_j \rangle |f_j|^{-2} = \\ &= \langle Q_V f_n|f_k \rangle - \langle df_n(V)|f_k \rangle = \\ &= \langle df_n(V)|f_k \rangle - \langle df_n(V)|f_k \rangle = 0, \end{aligned}$$

for all $k = \overline{0, n-1}$. In the last case $k = n$, we have that

$$(2.11) \quad \begin{aligned} \langle f_{n+1}|f_n \rangle &= \langle Q_V f_n|f_n \rangle = \\ &= \langle df_n(V) - f_n|f_n|^{-2} \langle df_n(V)|f_n \rangle |f_n \rangle = \\ &= \langle df_n(V)|f_n \rangle - \langle df_n(V)|f_n \rangle = 0. \end{aligned}$$

Taking now into account that $\dim \mathbb{C}^3 = 3$, we naturally derive that all $f_j = 0$ for $j = \overline{3, n}$, proving completely the lemma. \square

The lemma above makes it possible to describe effectively possible deformations of the fundamental symplectic structure $\omega \in \Lambda^2(M_{\mathbb{C}}^2)$, parameterized by two parameters: tangent vector $V \in T(M)$ and a suitably defined element $f \in \mathbb{E}^3$.

3. THE DEFORMED KÄHLER FUNDAMENTAL FORM AND INDUCED MONGE-AMPÈRE EQUATION

Consider now a t -parametric deformation of the symplectic form $\omega \in \Lambda^2(M_{\mathbb{C}}^2) : \omega \rightarrow \omega_t := t d\alpha + \omega$, where $t \in [0, 1], \alpha \in \Lambda^1(M_{\mathbb{C}}^2)$ is some one-form and $\varphi_t : M_{\mathbb{C}}^2 \rightarrow M_{\mathbb{C}}^2$ - a one-parametric group of diffeomorphisms of the Kähler manifold $M_{\mathbb{C}}^2$. Now we need the following Moser's [15] theorem.

Theorem 3.1. *Let $(M^{2n}; \omega)$ be oriented symplectic manifold and some symplectic deformation $\omega \rightarrow \omega_t := \omega - \omega t \in \Lambda^2(M^{2n}), t \in [0, 1]$, with fixed two-dimensional periods, that is*

$$(3.1) \quad \int_{\sigma} \omega_t = \int_{\sigma} \omega$$

for every two-cycle $\sigma \in H_2(M^{2n}; \mathbb{R})$. Then there exists such a diffeomorphism $\varphi_t : M^{2n} \rightarrow M^{2n}$ that

$$(3.2) \quad \varphi_t^* \omega = \omega_t$$

for all $t \in [0, 1]$.

As the symplectic deformation $\omega \rightarrow \omega_t := t d\alpha + \omega \in \Lambda^2(M_{\mathbb{C}}^2)$ a priori satisfies for all $t \in [0, 1]$ the condition (3.1), as a consequence of Theorem 3.1 one easily derives by differentiation with respect to the parameter $t \in [0, 1]$ that there exists such a vector field $K \in \Gamma(T(M_{\mathbb{C}}^2))$ that the Lie derivative $L_K \omega = d\alpha$, where the one-form $\alpha \in \Lambda^1(M_{\mathbb{C}}^2)$ can be $\alpha = d^s \psi$ on the Kähler manifold $M_{\mathbb{C}}^2$. Taking into account the symplectic deformation (1.5), the latter means that

$$(3.3) \quad L_K \omega = di_K \omega = dd^s \psi,$$

from which one ensues the equivalence

$$(3.4) \quad i_K \omega = d^s \psi \mod d\Lambda^0(M_{\mathbb{C}}^2)$$

on the Kähler manifold $M_{\mathbb{C}}^2$. The written above mod-equivalence can be easily omitted, if to take into account that the vector field $K \in \Gamma(T(M_{\mathbb{C}}^2))$ is taken to be equivalent to the naturally related set $\mathcal{H} = \{H \in \Gamma(T(M_{\mathbb{C}}^2)) : L_H \omega = 0\}$ of the Hamiltonian vector fields on the manifold $M_{\mathbb{C}}^2$, reducing our problem (1.4) to the following slightly simpler form

$$(3.5) \quad i_K \omega = d^s \psi.$$

Now the initial problem reduces to the next two tasks: the first one assumes solving the equation (3.5) with respect to the corresponding two-form $\psi \in \Lambda^2(M_{\mathbb{C}}^2)$, and the second one is a description of vector fields $K \in \Gamma(T(M_{\mathbb{C}}^2))$ on $M_{\mathbb{C}}^2$, for which the two-form $\omega + di_K \omega \in \Lambda^2(M_{\mathbb{C}}^2)$ generates a Hermitian structure $h : T(M_{\mathbb{C}}^2) \times T(M_{\mathbb{C}}^2) \rightarrow \mathbb{C}$ on the Kähler manifold $M_{\mathbb{C}}^2$.

First of all, take now into account the Kodaira theorem [3, 27] that any two-form $\psi \in \Lambda^2(M_{\mathbb{C}}^2)$ on the complex manifold $M_{\mathbb{C}}^2$ satisfies the conditions $\psi = *\psi \mod d\Lambda^1(M_{\mathbb{C}}^2)$ and $\psi \wedge \omega = 0$. The latter makes it possible upon applying to this condition the operation $\alpha \otimes i_K$ to obtain the identity

$$(3.6) \quad \alpha(K) = -(\alpha \wedge \psi | d\psi)_s$$

for any real one-form $\alpha = \bar{\alpha} \in \Lambda^1(M_{\mathbb{C}}^2)$. The identity (3.6) is equivalent to a representation of the searched for vector field $K \in \Gamma(T(M_{\mathbb{C}}^2))$ as some solvable quadratic differential expression on the two-form $\psi \in \Lambda^2(M_{\mathbb{C}}^2)$. Thereby, we can formulate the obtained above result as the following proposition.

Proposition 3.2. *Any symplectic deformation (1.5) of the symplectic structure $\omega \in \Lambda^2(M_{\mathbb{C}}^2)$ on the complex manifold $M_{\mathbb{C}}^2$ is generated by the real vector fields $K \in \Gamma(T(M_{\mathbb{C}}^2))$ on $M_{\mathbb{C}}^2$, satisfying the scalar quadratic functional identity (3.6).*

Recall now that the constructed above symplectic deformation (1.5) of the symplectic structure $\omega \in \Lambda^2(M_{\mathbb{C}}^2)$ should generate an Hermitian metric on our Kähler manifold $M_{\mathbb{C}}^2 = P_2(\mathbb{C})$, what imposes natural constraints on the generating vector field $K \in \Gamma(T(M_{\mathbb{C}}^2))$ on $M_{\mathbb{C}}^2$, some of which were described in Proposition 3.2. To proceed to their more detail analysis, observe that the Levi-Civita connection $\nabla : T(M_{\mathbb{C}}^2) \rightarrow T(M_{\mathbb{C}}^2)$, corresponding to the fundamental symplectic structure $\omega \in \Lambda^2(M_{\mathbb{C}}^2)$, leaves invariant the related complex-structure $J : T(M_{\mathbb{C}}^2) \rightarrow T(M_{\mathbb{C}}^2)$, naturally extended from the complexified tangent space $T(M^4) \otimes_{\mathbb{R}} \mathbb{C}$ on the whole $T(M_{\mathbb{C}}^2)$, that is $\nabla J = 0$ on the complex Kähler manifold $M_{\mathbb{C}}^2$. Taking now into account that the related compatible Hermitian metric $g : T(M_{\mathbb{C}}^2) \times T(M_{\mathbb{C}}^2) \rightarrow \mathbb{C}$ on the Kähler manifold $M_{\mathbb{C}}^2$ satisfies the determining relationship

$$(3.7) \quad g(X, Y) := \omega(X, JY),$$

as well as the Levi-Civita connection ∇ - invariance

$$(3.8) \quad Kg(X, Y) = g(\nabla_K X, Y) + g(X, \nabla_K Y)$$

for any vector fields $X, Y \in \Gamma(T(M_{\mathbb{C}}^2))$ along the vector field $K \in \Gamma(T(M_{\mathbb{C}}^2))$ on $M_{\mathbb{C}}^2$ one can derive by means of easy, yet slightly cumbersome calculations, the following identity:

$$(3.9) \quad (L_K \omega)(X, JY) = g((\nabla_K - L_K)X, Y) + g(X, (\nabla_K + JL_K J)Y).$$

The expression (3.9), considered jointly with (3.7), gives rise to the deformed metric $g_K : T(M_{\mathbb{C}}^2) \times T(M_{\mathbb{C}}^2) \rightarrow \mathbb{C}$ on the Kähler manifold $M_{\mathbb{C}}^2$, defined by means of the expression

$$(3.10) \quad g_K(X, Y) := g(X, Y) + (L_K \omega)(X, Y),$$

which should be for all $Y = X \in \Gamma(T(M_{\mathbb{C}}^2))$ positive definite, imposing suitable constraints on the vector field $K \in \Gamma(T(M_{\mathbb{C}}^2))$ on $M_{\mathbb{C}}^2$.

To specify this metric positivity constraint on the vector field $K \in \Gamma(T(M_{\mathbb{C}}^2))$, satisfying the additional quadratic relationship (3.6), we need, preliminarily, to construct the related Levi-Civita connection $d_{\mathcal{A}} : \Gamma(T(M_{\mathbb{C}}^2)) \rightarrow \Gamma(T^*(M_{\mathbb{C}}^2) \otimes T(M_{\mathbb{C}}^2)) \simeq \Gamma(\text{End}(T(M_{\mathbb{C}}^2)))$ on sections of $\Gamma(T(M_{\mathbb{C}}^2))$:

$$(3.11) \quad d_{\mathcal{A}}X := dX + \mathcal{A}^{(1)}X,$$

where $\mathcal{A}^{(1)} \in \Gamma(\text{End}(T(M_{\mathbb{C}}^2))) \otimes \Lambda^1(M_{\mathbb{C}}^2)$ is the corresponding connection matrix, which is compatible with the constructed above Fubini-Study metric (2.6). Since the latter is representable on sections $X, Y \in \Gamma(T(M_{\mathbb{C}}^2))$ as

$$(3.12) \quad g(X, Y) = \langle hX | Y \rangle,$$

where the Hermitian matrix $h \in \text{End}(T(M_{\mathbb{C}}^2))$ satisfies the following differential relationship:

$$(3.13) \quad dh = h\mathcal{A}^{(1)} + \bar{\mathcal{A}}^{(1)\dagger}h,$$

its solution provides the related curvature matrix two-form

$$(3.14) \quad \Omega_{\mathcal{A}} = d\mathcal{A}^{(1)} + \mathcal{A}^{(1)} \wedge \mathcal{A}^{(1)},$$

whose matrix trace

$$(3.15) \quad \omega_{\mathcal{A}} := \frac{i}{2} \text{tr } \Omega_{\mathcal{A}}$$

is a priori closed and belongs to the Chern class, that is $\omega_{\mathcal{A}} \in c_1(M_{\mathbb{C}}^2)$. Moreover, the following proposition holds.

Proposition 3.3. *The closed two-form $\omega_{\mathcal{A}} \in \Lambda^2(M_{\mathbb{C}}^2)$ proves to be nondegenerate and defines an equivalent to $\omega \in c_1(M_{\mathbb{C}}^2)$ fundamental two-form, generating a compatible Hermitian metric on the Kähler manifold $M_{\mathbb{C}}^2$.*

Return now to the deformed metric $g_K : T(M_{\mathbb{C}}^2) \times T(M_{\mathbb{C}}^2) \rightarrow \mathbb{C}$ on the Kähler manifold $M_{\mathbb{C}}^2$, defined by the expression (3.10) and depending on the covariant derivative $\nabla_K : T(M_{\mathbb{C}}^2) \rightarrow T(M_{\mathbb{C}}^2)$, whose action on $X \in \Gamma(T(M_{\mathbb{C}}^2))$ can be now rewritten as

$$(3.16) \quad \nabla_K(X) = i_K dX + (i_K \mathcal{A}^{(1)})X.$$

Taking into account (3.16), we obtain the following linear mappings

$$(3.17) \quad \begin{aligned} \nabla_K - L_K + I/2 &= K_* + i_K \mathcal{A}^{(1)} + I/2, \\ \nabla_K + J L_K J + I/2 &= -J K_* J + 2i_K \mathcal{A}^{(1)} + J(i_K \mathcal{A}^{(1)})J + I/2, \end{aligned}$$

entering the deformed metric (3.10), where we made use of the following covariant and Lie derivatives

$$(3.18) \quad \nabla_K J = J_* K + [J, i_K \mathcal{A}^{(1)}], \quad L_K J = J_* K + [J, K_*],$$

respectively, and denoted by dash "r" the corresponding tangent mapping, making the following tangent vector bundle diagrams

$$(3.19) \quad \begin{array}{ccc} T(M_{\mathbb{C}}^2) & \xrightarrow{K_*} & T(T(M_{\mathbb{C}}^2)) \\ \downarrow & & \downarrow \\ M_{\mathbb{C}}^2 & \xrightarrow{K} & T(M_{\mathbb{C}}^2) \end{array}, \quad \begin{array}{ccc} T(M_{\mathbb{C}}^2) & \xrightarrow{J_*} & T(\text{End}(T(M_{\mathbb{C}}^2))) \\ \downarrow & & \downarrow \\ M_{\mathbb{C}}^2 & \xrightarrow{J} & \text{End}(T(M_{\mathbb{C}}^2)) \end{array}$$

commutative. Whence, the deformed metric (3.10) finally reduces to the following bilinear symmetric expression

$$(3.20) \quad \begin{aligned} g_K(X, Y) &:= g((K_* + i_K \mathcal{A}^{(1)} + I/2)X, JY) + \\ &\quad + g(X, (-J K_* J + 2i_K \mathcal{A}^{(1)} + J(i_K \mathcal{A}^{(1)})J + I/2)Y) \end{aligned}$$

on the product $T(M_{\mathbb{C}}^2) \times T(M_{\mathbb{C}}^2)$. The obtained result we can formulate as the following preliminary theorem.

Theorem 3.4. *The deformed metric (3.10) is correctly defined on the complex Kähler manifold $M_{\mathbb{C}}^2$ as a bilinear symmetric form (3.11) on the product $T(M_{\mathbb{C}}^2) \times T(M_{\mathbb{C}}^2)$, whose positive definiteness depends uniquely on a choice of the vector field $K : M_{\mathbb{C}}^2 \rightarrow T(M_{\mathbb{C}}^2)$.*

A detail Hermitian analysis and application of the deformed metric expression (3.20) to solving the Monge–Ampère type equation (1.4) is postponed for another work under preparation.

4. CONCLUSION

We analyzed the cohomology structure of the fundamental two-form deformation related with a modified Monge–Ampère type on the complex Kähler manifold $P_2(\mathbb{C})$. Based on the Levi-Civita connection jointly with the related vector field deformation of the fundamental two-form there is constructed a hierarchy of bilinear symmetric forms on the tangent bundle to the Kähler manifold $P_2(\mathbb{C})$, generating on it Hermitian metric and suitable solutions to the studied Monge–Ampère type equation. There is also generalized the classical fundamental two-form construction on the complex Kähler manifold $P_2(\mathbb{C})$, its relation to the Hermitian metric deformations is discussed.

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MATHEMATICS INSTITUTE AT THE CARDIFF UNIVERSITY, CARDIFF CF24 4AG, GREAT BRITAIN
E-mail address: BalinskyA@cardiff.ac.uk

¹⁾DEPARTMENT OF COMPUTER SCIENCE AND TELECOMMUNICATION AT THE CRACOW UNIVERSITY OF TECHNOLOGY, KRAKÓW 30-155, POLAND, ²⁾INSTITUTE OF APPLIED MATHEMATICS AND FUNDAMENTAL SCIENCES, LVIV POLYTECHNICAL NATIONAL UNIVERSITY, LVIV 79013, UKRAINE

E-mail address: pryk.anat@cybergal.com (★02.01.1938–†04.12.2020)