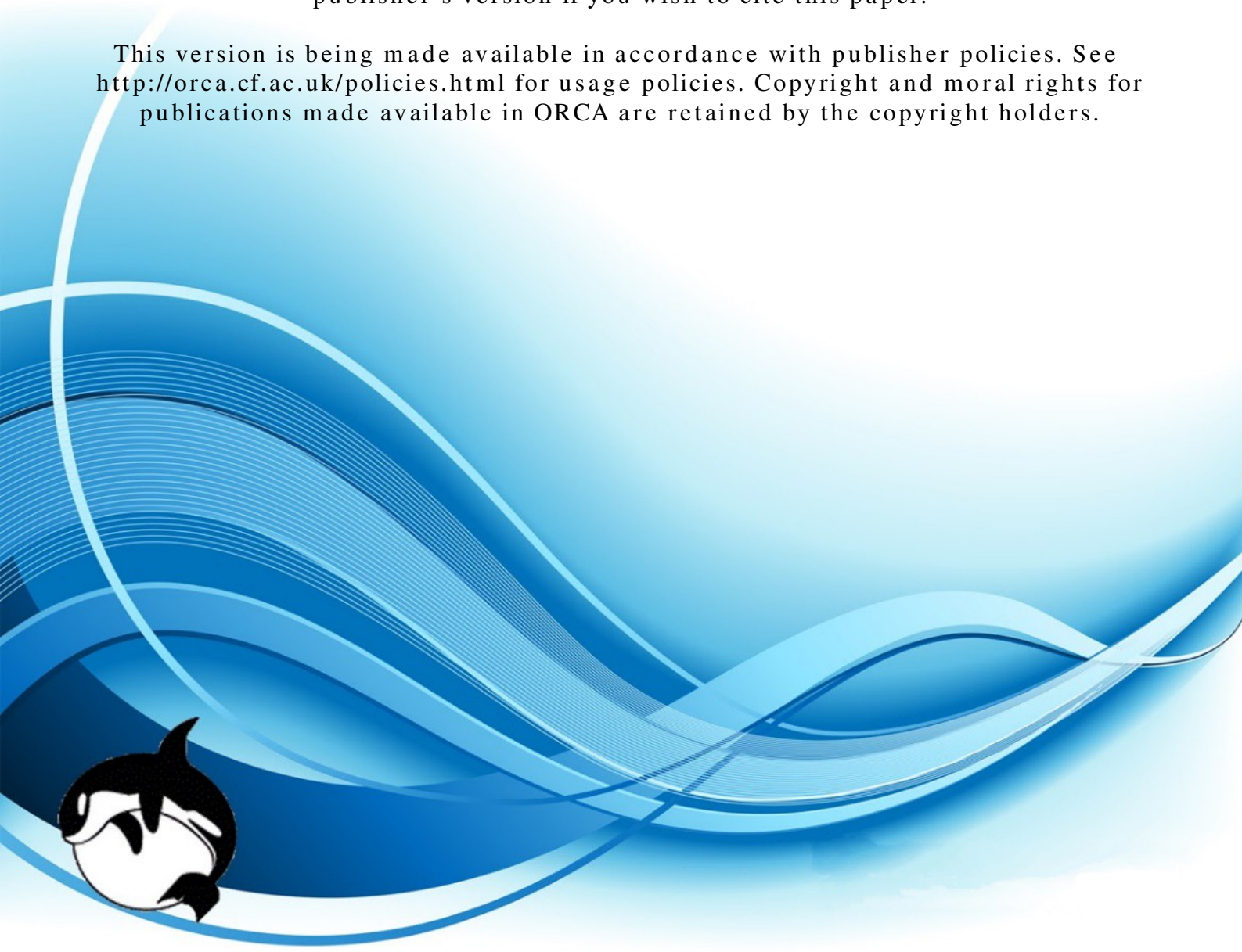

SUBFACTORS AND MATHEMATICAL PHYSICS

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This paper is dedicated to the memory of Vaughan Jones

Abstract. This paper surveys the long-standing connections and impact between Vaughan Jones's theory of subfactors and various topics in mathematical physics, namely statistical mechanics, quantum field theory, quantum information and two-dimensional conformal field theory.

1. Subfactors and mathematical physics

Subfactor theory was initiated by Vaughan Jones [69]. This led him to the study of a new type of *quantum symmetry*. This notion of quantum symmetries led to a diverse range of applications including the Jones polynomial, a completely new invariant in knot theory which led to the new field of *quantum topology*. His novel theory has deep connections to various topics in mathematical physics. This renewed interest in known connections between mathematical physics and operator algebras, and opened up totally novel frontiers. We present a survey on these interconnecting topics with emphasis on statistical mechanics and quantum field theory, particularly two-dimensional conformal field theory.

2. Subfactors and statistical mechanics

Let $N \subset M$ be a subfactor of type II_1 and $[M : N]$ its *Jones index*, which is a positive real number or infinity. That is, N and M are infinite-dimensional simple von Neumann algebras with a trace tr. We only consider the case that $[M : N]$ is finite. Vaughan [69] constructed a sequence of projections e_j, $j = 1, 2, 3, \dots$, called the *Jones projections*, and discovered the following relations:

$$\begin{cases} e_j = e_j^2 = e_j^*, \\ e_j e_k = e_k e_j, \quad j \neq k, \\ e_j e_{j\pm1} e_j = [M : N]^{-1} e_j. \end{cases} \tag{2.1}$$

Using these relations and a trace, Vaughan showed that the set of possible values of the Jones indices is exactly equal to

$$\left\{ 4\cos^2 \frac{\pi}{n} \mid n = 3, 4, 5, \dots \right\} \cup [4, \infty].$$

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Vaughan made the substitution

$$\sigma_j = te_j - (1 - e_j) \tag{2.2}$$

where $[M : N] = \lambda^{-1} = 2 + t + t^{-1}$ to yield the Artin relations of the braid group, where σ_j is the braid which interchanges the j and $j+1$ strands. Vaughan's representation came equipped with a trace tr satisfying the Markov trace property in the probabilistic sense $\mathrm{tr}(xe_j) = [M : N]^{-1}\mathrm{tr}(x)$, where x belongs to the algebra generated by $e_1, e_2, \ldots, e_{j-1}$. Any link arises as a closure of a braid by a theorem of Alexander, and two braids give the same link if and only if they are related by a series of two types of moves, known as the *Markov moves*, by a theorem of Markov. The trace property $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ and the above Markov trace property give invariance of a certain adjusted trace value of a braid under the two Markov moves. This is the Jones polynomial [70, 71] in the variable t a polynomial invariant of a link.

Evans pointed out in 1983 that these relations (2.1) appear in similar formalism to one studied by Temperley-Lieb [116] in solvable statistical mechanics. The *Yang-Baxter equation* plays an important role in subfactor theory and quantum groups. The two-dimensional Ising model assigns two possible spin values $\pm$ at the vertices of a lattice. Important generalisations include the Potts model, with Q states at each vertex, and vertex or IRF (interaction round a face) models, where the degrees of freedom are assigned to the edges of the lattice. The transfer matrix method, originated by Kramers and Wannier, assigns a matrix of Boltzmann weights to a one dimensional row lattice. The partition function of a rectangular lattice in general is then obtained by gluing together matrix products of the transfer matrix. Baxter [6] showed how to construct commuting families of transfer matrices via Boltzmann weights satisfying the Yang-Baxter Equation YBE. The YBE is an enhancement of the braid relations in (2.2), as it reduces to them in a certain limit. This commutativity permits simultaneous diagonalisation, with the largest eigenvalue being crucial for computing the free energy. The transfer matrix method transforms the classical statistical mechanical model to a one-dimensional quantum model. A conformal field theory can arise from the scaling limit of a statistical mechanical lattice model at criticality. Temperley and Lieb [116] found that the transfer matrices of the Potts model and an ice-type vertex model could both be described through generators obeying the same relations as in Vaughan's work (2.1) and in this way demonstrated equivalence of the models. Whilst the relations for the Potts model only occur when $\lambda^{-1} = Q$ is integral, the partition function is a Tutte-Whitney dichromatic polynomial. One variable is Q which can be extrapolated and the partition function is then related to the Jones polynomial on certain links associated to the lattice [70, page 108]. The ice-type representation though has a continuous parameter λ. In both cases, the Markov trace did not manifest itself.

Pimsner and Popa [102] discovered that the inverse of the index namely $[M : N]^{-1}$ is the best constant $c \geq 0$ for which $E_N(x^*x) \geq cx^*x$, for all $x \in M$, which they called the *probabilistic index*. Here E_N is the conditional expectation of M onto N which gives rise to the first Jones projection in the tower. This was key to creating the link with the theory of Doplicher-Haag-Roberts, by Longo [91] identifying statistical dimension with the Jones index, and by Fredenhagen, Rehren and Schroer [45, 46], in the late 80's, and also key to calculating all of the subsequent entropy quantities/invariants related to subfactors, including the calculation of the

entropy of the shift on the Jones projections and the calculation of the Connes-Stormer entropy [26], $H(M|N) = \ln([M : N])$, for irreducible subfactors.

For a subfactor $N \subset M$ with finite Jones index, we have the *Jones tower construction*

$$N \subset M \subset M_1 \subset M_2 \subset \cdots ,$$

where M_k is generated by M and $e_1, e_2, \ldots, e_k$. The basic construction from $N \subset M$ to $M \subset M_1$ and its iteration to give the Jones tower of II_1 factors has a fundamental role in subfactor theory and applications in mathematical physics. The *higher relative commutants* $M_j' \cap M_k$, $j \leq k$, give a system of *commuting squares of inclusions* of finite dimensional C^*-algebras with a trace, an object denoted by $\mathcal{G}_{N\subset M}$ and called the *standard invariant* of $N \subset M$. This exceptionally rich mathematical structure encodes algebraic and combinatorial information about the subfactor, a key component of which is a connected, possibly infinite bipartite graph $\Gamma_{N\subset M}$, of Cayley type, called the *principal graph* of $N \subset M$, with a canonical weight vector $\vec{v}$, whose entries are square roots of indices of irreducible inclusions in the Jones tower. The weighted graph $(\Gamma, \vec{v})$ satisfies the Perron-Frobenius type condition $\Gamma^t\Gamma(\vec{v}) = [M : N]\vec{v}$, and also $\|\Gamma\|^2 \leq [M : N]$.

Of particular relevance to mathematical physics is when $N \subset M$ has *finite depth*, corresponding to the graph Γ being finite, in which case the weights $\vec{v}$ give the (unique) Perron-Frobenius eigenvector, entailing $\|\Gamma\|^2 = [M : N]$. Finite depth is automatic when the index $[M : N]$ is less than 4, where indeed all bipartite graphs are finite and have norms of the form $2\cos^2(\pi/n)$, $n \geq 3$.

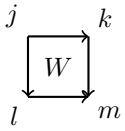
The objects $\mathcal{G}_{N\subset M}$ have been axiomatised in a number of ways, by Ocneanu with paragroups and connections [97] in the finite depth case, then in the general case by Popa with λ-lattices [107] and by Vaughan with planar algebras [80].

By Connes fundamental result in [22], the *hyperfinite* II_1 factor R, obtained as an inductive limit of finite dimensional algebras, is the unique amenable II_1 factor, so in particular all its finite index subfactors are isomorphic to R. In a series of papers [105, 106, 108, 109], Popa identified the appropriate notion of amenability for inclusions of II_1 factors $N \subset M$ and for the objects $\mathcal{G}_{N\subset M}$, in several equivalent ways, one of which being the Kesten-type condition $\|\Gamma_{N\subset M}\|^2 = [M : N]$. He proved the important result that for hyperfinite subfactors $N \subset M$ satisfying this amenability condition, $\mathcal{G}_{N\subset M}$ is a complete invariant. In other words, whenever $M \simeq R$ and $\|\Gamma_{N\subset M}\|^2 = [M : N]$ (in particular if $N \subset M$ has finite depth), $N \subset M$ can be recovered from the data encoded by the sequence of commuting squares in the Jones tower.

Constructions of interesting commuting squares are related to statistical mechanics through the Yang-Baxter equation and an *IRF* , vertex or spin model [72]. (See the monograph of Baxter [6] for this type of statistical mechanical models. Also see [75] for a general overview by Vaughan on this type of relations.) We choose one edge each from the four diagrams for the four inclusions so that they make a closed square. Then we have an assignment of a complex number to each such square. Ocneanu [97] gave a combinatorial characterisation of this assignment of complex numbers under the name of a *paragroup* and a *flat connection*. We also assign a complex number, called a *Boltzmann weight*, to each square arising from a finite graph in the theory of IRF or vertex models and we have much similarity between the two notions. The simplest example corresponds to the *Ising model* built on the Coxeter-Dynkin diagram A_3 and a more general case corresponds to

the Andrews-Baxter-Forrester model [1] related to the quantum groups $U_q(sl_2)$ for $q = \exp(2\pi i/l)$ a root of unity. These fundamental examples correspond to the subfactors generated by the Jones projections alone and the graphs for these cases are the *Coxeter-Dynkin diagrams* of type A_n. Others related to the quantum groups $U_q(sl_n)$ have been studied in [67, 28].

We give a typical example of a flat connection as follows. Fix one of the Coxeter-Dynkin diagrams of type A_n, D_{2n}, E_6 or E_8 and use it for the four diagrams. Let h be its Coxeter number and set $\varepsilon = \sqrt{-1}\exp(\pi\sqrt{-1}/2h)$. We write μ_j for the Perron-Frobenius eigenvector entry for a vertex j for the adjacency of the diagram. Then the flat connection is given as in Fig.1 and is essentially a normalisation of the braid element (2.2):



$$= \delta_{kl}\varepsilon + \sqrt{\frac{\mu_k\mu_l}{\mu_j\mu_m}}\delta_{jm}\bar{\varepsilon}$$

FIGURE 1. A flat connection on the Coxeter-Dynkin diagram

The index value given by this construction is $4\cos^2(\pi/h)$. If the graph is A_n, then the vertices are labeled with $j = 1, 2, \ldots, n$ and the Perron-Frobenius eigenvector entry for the vertex j is given by $\sin(j\pi/(n+1))$. The value in Fig.1 in this case is essentially the same as what the Andrews-Baxter-Forrester model gives at a limiting value and it also arises from a specialisation of the *quantum* $6j$*-symbols* for $U_q(sl_2)$ at a root of unity in the sense that two of the "$6j$"s are chosen to be the fundamental representation of $U_q(sl_2)$. These are also related to IRF models by Roche in [113]. These subfactors for the Dynkin diagrams A_n are the ones constructed by Vaughan [69] as $N = \langle e_2, e_3, \ldots \rangle$ and $M = \langle e_1, e_2, e_3, \ldots \rangle$ with the above relations (2.1) with $[M : N] = 4\cos^2(\pi/(n+1))$.

The same formula as in Fig.1 for the Coxter-Dynkin diagrams D_{2n+1} and E_7 almost gives a flat connection, but the flatness axiom fails. There are corresponding subfactors but they have principal graphs A_{4n-1} and D_{10} respectively. Nevertheless, the diagrams D_{2n+1} and E_7 have interesting interpretations in connection with non-local extensions of *conformal nets* $SU(2)_k$, as explained below.

The relations (2.1) of the Jones projections e_j are reminiscent of the defining relations of the *Hecke algebra* $H_n(q)$ of type A with complex parameter q, which is the free complex algebra generated by $1, g_1, g_2, \ldots, g_{n-1}$ satisfying

$$\begin{cases} g_j g_{j+1} g_j = g_{j+1} g_j g_{j+1}, \\ g_j g_k = g_k g_j, \quad \text{for } j \neq k, \\ g_j^2 = (q-1)g_j + q. \end{cases}$$

This similarity was exploited to construct more examples of subfactors with index values $\dfrac{\sin^2(k\pi/l)}{\sin^2(\pi/l)}$ with $1 \leq k \leq l-1$ in the early days of subfactor theory by Wenzl in a University of Pennsylvania thesis supervised by Vaughan [127]. He constructed representations ρ of $H_\infty(q) = \bigcup_{n=1}^{\infty} H_n(q)$ with roots of unity $q = \exp(2\pi i/l)$ and $l = 4, 5, \ldots$ such that $\rho(H_n(q))$ is always semi-simple and gave a subfactor as

$\rho(\langle g_2, g_3, \dots \rangle)'' \subset \rho(\langle g_1, g_2, \dots \rangle)''$ using a suitable trace. The index values converge to k^2 as $l \to \infty$. When $k = 2$, these subfactors are the ones constructed by Vaughan for the Coxeter-Dynkin diagram A_{l-1}. This construction is also understood in the context of IRF models [28, 67] related to $SU(k)$. The relation between the Hecke algebras and the quantum groups $U_q(sl_n)$ is a "quantum" version of the classical *Weyl duality*. This duality also connects this Jones-Wenzl approach based on statistical mechanics and type II_1 factors with the Jones-Wassermann approach based on quantum field theory and type III_1 factors which is explained below.

It is important to have a *spectral parameter* for the Boltzmann weights satisfying the Yang-Baxter equation in *solvable lattice models*, but we do not have such a parameter for a flat connection initially in subfactor theory. We usually obtain a flat connection by a certain specialisation of a spectral parameter for a Boltzmann weight. Vaughan proposed "Baxterization" in [73] for the converse direction in the sense of introducing a parameter for analogues of the Boltzmann weights in subfactor theory. This is an idea to obtain a physical counterpart from a subfactor, and we discuss a similar approach to construct a conformal field theory from a given subfactor at the end of this article. It should be noted that to rigorously construct a conformal field theory at criticality is a notoriously difficult problem – even for the Ising model, see e.g. [114].

The finite depth condition means that we have a finite graph in this analogy to solvable lattice models. Even from a set of algebraic or combinatorial data similar to integrable lattice models involving infinite graphs, one sometimes constructs a corresponding subfactor. A major breakthrough of Popa [104] was to show that the Temperley-Lieb-Jones lattice is indeed a standard invariant showing for the first time that for any index greater than 4 that there exist subfactors with just the Jones projections as the higher relative commutants. Then, introducing tracial amalgamated free products, Popa [107] could show existence in full generality. These papers [104, 107] led to important links with free probability theory, leading to more sophisticated free random models to prove that certain amalgamated free products are free group factors and adapted, by Ueda [122], to prove similar existence/reconstruction statements for actions of quantum groups. Popa and Shlyakhtenko [110] showed that any λ-lattice acts on the free group factor $L(\mathbb{F}_\infty)$. This involved a new construction of subfactors from λ-lattices, starting from a commuting square of semifinite von Neumann algebras, each one a direct sum of type I_∞ factors with a semifinite trace, and with free probability techniques showing that the factors resulting from this construction are ∞-amplifications of $L(\mathbb{F}_\infty)$. The von Neumann algebras resulting in these constructions are not hyperfinite. A new proof using graphical tools, probabilistic methods and planar algebras was later found by Guionnet-Jones-Shlyakhtenko [59]. Moreover they and Zinn-Justin [61] use matrix model computations in loop models of statistical mechanics and graph planar algebras to construct novel matrix models for Potts models on random graphs. This is based on the *planar algebra* machinery developed by Vaughan [80] for understanding higher relative commutants of subfactors. In [60] Guionnet-Jones-Shlyakhtenko explicitly show that it is the same construction as in the Popa-Shlyakhtenko [110] paper. The paper [80] has been published only very recently in the Vaughan Jones memorial special issue after his passing away, but its preprint version appeared in 1999 and has been highly influential. Note also that Kauffman [82, 83] had found a diagrammatic construction of the Jones polynomial directly related to the Potts

model based on a diagrammatic presentation of the Temperley-Lieb algebra which then has a natural home in the planar algebra formalism. The polynomial was understood by Reshetikhin-Turaev in [112] in the context of representations of the quantum groups $U_q(sl_2)$ [33, 66].

3. Subfactors and quantum field theory

Witten [128] gave a new interpretation of the Jones polynomial based on quantum field theory, the Chern-Simons gauge field theory, and generalised it to an invariant of a link in a compact 3-manifold. However, it was not clear why we should have a *polynomial* invariant in this way. Taking an empty link, yields an invariant of a compact 3-manifold. Witten used a path integral formulation and was not mathematically rigorous. A mathematically well-defined version based on combinatorial arguments using *Dehn surgery* and the *Kirby calculus* has been given by Reshetikhin and Turaev [112]. In the case of an empty link, we realise a 3-manifold from a *framed link* with the Dehn surgery, make a weighted sum of invariants of this link using representations of a certain quantum group at a root of unity and prove that this weighted sum is invariant under the Kirby moves. Two framed links give homeomorphic manifolds if and only if they are related with a series of Kirby moves. For the quantum group $U_q(sl_2)$, the link invariant is the *colored Jones polynomial*. A *color* is a representation of the quantum group and labels a connected component of a link. This actually gives a $(2+1)$-dimensional *topological quantum field theory* in the sense of Atiyah [5], which is a certain mathematical axiomatisation of a quantum field theory based on topological invariance. Roughly speaking, we assign a finite dimensional Hilbert space to each closed 2-dimensional manifold, and also assign a linear map from one such Hilbert space to another to a cobordism so that this assignment is functorial. It is also easy to extend this construction from quantum groups to general *modular tensor categories* as we explain below.

A closely related, but different, $(2+1)$-dimensional topological quantum field theory has been given by Turaev and Viro [121]. In this formulation, one triangulates a 3-manifold, considers a weighted sum of quantum $6j$-symbols arising from a quantum group depending on the triangulation, and proves that this sum is invariant under the *Pachner moves*. Two triangulated manifolds are homeomorphic to each other if and only if we obtain one from the other with a series of Pachner moves. This has been generalised to another $(2+1)$-dimensional topological quantum field theory using *quantum* $6j$*-symbols* arising from a subfactor by Ocneanu. (See [42, Chapter 12].) Here we only need a *fusion category* structure which we explain below, and no *braiding*. This is different from the above Reshetikhin-Turaev case. For a given fusion category, we apply the Drinfel′d center construction, a kind of "quantum double" construction, to get a modular tensor category with a non-degenerate braiding. This construction was developed in subfactor theory by Ocneanu [97] through an asymptotic inclusion, by Popa [106] through a symmetric enveloping algebra, through the Longo-Rehren subfactor [94] and Izumi [64, 65] and in a categorical setting by Müger [96]. We then apply the Reshetikhin-Turaev construction to the double. We can also apply the Turaev-Viro-Ocneanu construction to the original fusion category, and these two procedures give the same topological quantum field theory [87]. In particular, if we start with $U_q(sl_2)$ at a root of unity, the Turaev-Viro invariant of a closed 3-manifold is the square of the absolute value of the Reshetikhin-Turaev invariant of the same 3-manifold.

Another connection of subfactors to quantum field theory is through *algebraic quantum field theory*, which is a bounded operator algebraic formulation of quantum field theory. The usual ingredients for describing a quantum field theory are as follows.

(1) A spacetime, such as the 4-dimensional *Minkowski space*.
(2) A spacetime symmetry group, such as the *Poincaré group*.
(3) A Hilbert space of states, including the *vacuum*.
(4) A projective unitary representation of the spacetime symmetry group on the Hilbert space of states.
(5) A set of quantum fields, that is, operator-valued distributions defined on the spacetime acting on the Hilbert space of states.

An ordinary distribution assigns a number to each test function. An operator-valued distribution assigns a (possibly unbounded) operator to each test function. The *Wightman axioms* give a direct axiomatisation using these and they have a long history of research, but it is technically difficult to handle operator-valued distributions, so we have a different approach based on bounded linear operators giving observables. Let O be a region within the spacetime. Take a quantum field φ and a test function f supported on O. The self-adjoint part of $\langle \varphi, f \rangle$ is an observable in O which could be unbounded. Let $A(O)$ denote the von Neumann algebra generated by spectral projections of such self-adjoint operators. This passage from operator-valued distributions to von Neumann algebras is also used in the construction of a *conformal net* from a *vertex operator algebra* by Carpi-Kawahigashi-Longo-Weiner [20] which we explain below. Note that a von Neumann algebra contains only bounded operators.

Locality is an important axiom arising from the Einstein causality which says that if two regions are spacelike separated, observables in these regions have no interactions, hence the corresponding operators commute. In terms of the von Neumann algebras $A(O)$, we require that $[A(O_1), A(O_2)] = 0$, if O_1 and O_2 are spacelike separated, where the Lie bracket means the commutator. This family of von Neumann algebras parameterised by spacetime regions is called a *net of operator algebras*. Algebraic quantum field theory gives an axiomatisation of a net of operator algebras, together with a projective unitary representation of a spacetime symmetry group on the Hilbert space of states including the vacuum. A main idea is that it is not each von Neumann algebra but the relative relations among these von Neumann algebras that contains the physical contents of a quantum field theory. In the case of two-dimensional conformal field theory, which is a particular example of a quantum field theory, each von Neumann algebra $A(O)$ is always a hyperfinite type III_1 factor, which is unique up to isomorphism and is the *Araki-Woods factor* of type III_1 . Thus the isomorphism class of a single von Neumann algebra contains no physical information. Each local algebra of a conformal net is a factor of type III_1 by [58, Proposition 1.2]. It is also hyperfinite because it has a dense subalgebra given as an increasing union of type I algebras, which follows from the split property shown in [95, Theorem 5.4].

Fix a net $\{A(O)\}$ of von Neumann algebras. It has a natural notion of a representation on another Hilbert space without the vacuum vector. The action of these von Neumann algebras on the original Hilbert space itself is a representation and it is called the *vacuum representation*. We also have natural notions of *unitary equivalence* and *irreducibility* of representations. The unitary equivalence

class of an irreducible representation of the net $\{A(O)\}$ is called a *superselection sector*. We also have a direct sum and irreducible decomposition for representations. If we have two representations of a group, it is very easy to define their tensor product representation, but it is not clear at all how to define a tensor product representation of two representations of a single net of operator algebras. Doplicher-Haag-Roberts gave a proper definition of the tensor product of two representations [30, 31]. Under a certain natural assumption, each representation has a representative given by an endomorphism of a single algebra $A(O)$ acting on the vacuum Hilbert space for some fixed O. This endomorphism contains complete information about the original representation. For two such endomorphisms ρ and σ, the composed endomorphism $\rho\sigma$ also corresponds to a representation of the net $\{A(O)\}$. This gives a correct notion of the tensor product of two representations. Furthermore, it turns out that the two compositions $\rho\sigma$ and $\sigma\rho$ of endomorphisms give unitarily equivalent representations. If the spacetime dimension is higher than 2, this commutativity of the tensor product is similar to unitary equivalence of $\pi_1 \otimes \pi_2$ and $\pi_2 \otimes \pi_1$ for two representations π_1 and π_2 of the same group. The representations now give a *symmetric monoidal C^*-category*, where a representation gives an object, an intertwiner gives a morphism, and the above composition of endomorphisms gives the tensor product structure. This category produces a compact group from the new duality of Doplicher-Roberts [32]. Here an object of the category is an endomorphism and a morphism in $\mathrm{Hom}(\rho, \sigma)$ is an intertwiner, that is, an element in

$$\{T \in A(O) \mid T\rho(x) = \sigma(x)T \text{ for all } x \in A(O)\}.$$

In other words, the Doplicher-Roberts duality gives an abstract characterisation of the representation category of a compact group among general tensor categories. The vacuum representation plays the role of the trivial representation of a group, and the *dual representation* of a net of operator algebras corresponds to the dual representation of a compact group. This duality is related to the classical *Tannaka duality*, but gives a duality more generally for abstract tensor categories.

Using the structure of a symmetric monoidal C^*-category, we define a *statistical dimension* of each representation, which turns out to be a positive integer or infinity [30, 31]. That the Jones index value takes on only discrete values below 4 is reminiscent of this fact that a statistical dimension can take only integer values. Longo [91, 92] showed that the statistical dimension of the representation corresponding to an endomorphism ρ of $A(O)$ is equal to the square root of the Jones index $[A(O) : \rho(A(O))]$. This opened up a wide range of new interactions between subfactor theory and algebraic quantum field theory.

Generalizing the notion of a superselection sector, Longo [91, 92] introduced the notion of a *sector*, the unitary equivalence class of an endomorphism of a factor of type III, inspired by Connes theory of correspondences, based on the equivalences between Hilbert bimodules, endomorphisms and positive definite functions on doubles [23] [24, VB], [25] and see e.g. Popa [103] for developments. He defined a *dual sector* using the *canonical endomorphism* which he had introduced based on the modular conjugation in Tomita-Takesaki theory. Note that in a typical situation of a subfactor $N \subset M$, these von Neumann algebras are isomorphic, so we have an endomorphism ρ of M onto N. Then we have the dual endomorphism $\bar{\rho}$, and the irreducible decompositions of $\rho\bar{\rho}\rho\bar{\rho}\cdots\bar{\rho}$ give objects of a *tensor category*, where the morphisms are the intertwiners of endomorphisms and the tensor product

operation is composition of endomorphisms. If we have finitely many irreducible endomorphisms arising in this way, which is equivalent to the finite depth condition, our tensor category is a *fusion category*, where we have the dual object for each object and we have only finitely many irreducible objects up to isomorphisms. The higher relative commutants $M' \cap M_k$ are described as intertwiner spaces like $\mathrm{End}(\rho\bar{\rho}\rho\bar{\rho}\cdots\bar{\rho})$ or $\mathrm{End}(\rho\bar{\rho}\rho\bar{\rho}\cdots\rho)$.

In our setting, for a factor M, we have the *standard representation* of M on the Hilbert space $L^2(M)$, the completion of M with respect to a certain inner product, and this $L^2(M)$ also has a right multiplication by M based on Tomita-Takesaki theory. For an endomorphism ρ of M, we have a new M-M bimodule structure on $L^2(M)$ by twisting the right action of M by ρ. In this setting, all M-M bimodules arise in this way, and we have a description of the above tensor category in terms of bimodules. Here the tensor product operation is given by a *relative tensor product* of bimodules over M. For type II_1 factors, we need to use this bimodule description to obtain the correct tensor category structures. It is more natural to use type II_1 factors in statistical mechanics, and it is more natural to use type III_1 factors in quantum field theory, but they give rise to equivalent tensor categories, so if we are interested in tensor category structure, including braiding, this difference between type II_1 and type III_1 is not important.

4. Subfactors and conformal field theory

A two-dimensional conformal field theory is a particular example of a quantum field theory, but it is a rich source of deep interactions with subfactor theory, so we treat this in an independent section.

We start with the $(1+1)$-dimensional Minkowski space and consider quantum field theory with *conformal symmetry*. We restrict a quantum field theory onto two light rays $x = \pm t$ and compactify a light ray by adding a point at infinity. The resulting S^1 is our "spacetime" now, though space and time are mixed into one dimension, and our symmetry group for S^1 is now $\mathrm{Diff}(S^1)$, the orientation preserving *diffeomorphism group* of S^1. Our spacetime region is now an *interval* I, a non-empty, non-dense open connected subset of S^1. For each such an interval I, we have a corresponding von Neumann algebra $A(I)$ acting on a Hilbert space H of states containing the vacuum vector. *Isotony* means that we have $A(I_1) \subset A(I_2)$ if we have $I_1 \subset I_2$. *Locality* now means that $[A(I_1), A(I_2)] = 0$, if $I_1 \cap I_2 = \emptyset$. Note that spacelike separation gives this very simple disjointness. Our spacetime symmetry group now is $\mathrm{Diff}(S^1)$, and we have a projective unitary representation U on H. Conformal covariance asks for $U_g A(I) U_g^* = A(gI)$ for $g \in \mathrm{Diff}(S^1)$. *Positivity of the energy* means that the restriction of U to the subgroup of rotations of S^1 gives a one-parameter unitary group and its generator is positive. In this setting, a family $\{A(I)\}$ of von Neumann algebras satisfying these axioms is called a *conformal net*.

A representation theory of a conformal net in the style of Doplicher-Haag-Roberts now gives a *braiding* due to the low-dimensionality of the "spacetime" S^1. This is a certain form of the non-trivial commutativity of endomorphisms up to inner automorphisms. That is, two representations give two endomorphisms λ, μ of a single von Neumann algebra $A(I_0)$ for some fixed interval I_0, and we have a unitary $\varepsilon(\lambda, \mu) \in A(I)$ satisfying $\mathrm{Ad}(\varepsilon(\lambda, \mu))\lambda\mu = \mu\lambda$. This unitary $\varepsilon(\lambda, \mu)$, sometimes called a *statistics operator*, arises from the monodromy of moving an interval in S^1

to a disjoint one and back, and satisfies various compatibility conditions such as *braiding-fusion equations* for intertwiners as in [45, 52, 92]. Switching two tensor components corresponds to switching two wires of a braid. For two wires, we have an overcrossing and an undercrossing. They correspond to $\varepsilon(\lambda,\mu)$ and $\varepsilon(\mu,\lambda)^*$. In particular, if we fix an irreducible endomorphism and use it for both λ and μ, we have a unitary representation of the *braid group* B_n for every n. In the case of a higher-dimensional Minkowski space, $\varepsilon(\lambda,\mu)$ gives a so-called *degenerate* braiding, like the case of a group representation where we easily have unitary equivalence of $\pi\otimes\sigma$ and $\sigma\otimes\pi$ for two representations π and σ, but we now have a braiding in a more non-trivial way on S^1. It was proved by Kawahigashi-Longo-Müger in [86] that if we have a certain finiteness of the representation theory of a conformal net, called *complete rationality*, then the braiding of its representation category is *non-degenerate*, and hence it gives rise to a *modular tensor category* by definition. A modular tensor category is also expected to be useful for *topological quantum computations* as in the work of Freedman-Kitaev-Larsen-Wang [48]. This is a hot topic in quantum information theory and many researchers work on topological quantum information using the Jones polynomial and its various generalisations.

It is a highly non-trivial task to construct examples of conformal nets. The first such attempt started in a joint project of Vaughan and Wassermann trying to construct a subfactor from a *positive energy representation* of a loop group. Wassermann [125] then constructed conformal nets arising from positive energy representations of the loop groups of $SU(N)$ corresponding to the Wess-Zumino-Witten models $SU(N)_k$, where k is a positive integer called a *level*. These examples satisfy complete rationality as shown by Xu in [132]. The conformal nets corresponding to $SU(2)_k$ give unitary representations of the braid groups B_n which are the same as the one given by Vaughan from the Jones projections e_j. Wassermann's construction has been generalised to other Lie groups by Loke, Toledano Laredo and Verrill in dissertations supervised by him, [90, 120, 124], see also [126]. Loke worked with projective unitary representations of $\mathrm{Diff}(S^1)$ and obtained the *Virasoro nets*.

A relative version $\{A(I)\subset B(I)\}$ of a conformal net for intervals $I\subset S^1$ called a *net of subfactors* has been given in [94]. Suppose that $\{A(I)\}$ is completely rational. Assuming that we know the representation category of $\{A(I)\}$, we would like to know that of $\{B(I)\}$. The situation is similar to a group inclusion $H\subset G$ where we know representation theory of H and would like to know that of G. In the group representation case for $H\subset G$, we have a *restriction* of a representation of G to H and an *induction* of a representation of H to G. In the case of a net of subfactors, the restriction of a representation of $\{B(I)\}$ to $\{A(I)\}$ is easy to define, but the induction procedure is more subtle. Our induction procedure is now called *α-induction*, first defined by Longo-Rehren in [94] and studied by Xu [129], Böckenhauer-Evans [8, 9, 10, 11], and Böckenhauer-Evans-Kawahigashi [12, 13], also in connection to Ocneanu's graphical calculus on Coxeter-Dynkin diagrams in the last two papers. (In these two papers, this α-induction is studied in the more general context of abstract modular tensor categories of endomorphisms rather than conformal field theory. For an A-A bimodule X, then the tensor product $X\otimes B$ can be regarded as a B-B module if one uses the braiding to let B act on the left.) Take a representation of λ of $\{A(I)\}$ which is given as an endomorphism of $A(I_0)$ for some fixed interval I_0. Then using the braiding on the representation category of $\{A(I)\}$, we define an endomorphism $\alpha^{\pm}_{\lambda}$ of $B(I_0)$ where $\pm$ represents a choice of a positive

or negative braiding, $\varepsilon^{\pm}(\lambda,\theta)$, where θ represents the dual canonical endomorphism of the subfactor $A(I) \subset B(I)$. This nearly gives a representation of $\{B(I)\}$, but not exactly. It turns out that the irreducible endomorphisms arising both from a positive induction and a negative one exactly correspond to those arising from irreducible representations of $\{B(I)\}$. The braiding of the representation category of $\{A(I)\}$ gives a finite dimensional unitary representation of $SL(2,\mathbb{Z})$ through the so-called *S- and T-matrices*. Böckenhauer-Evans-Kawahigashi [12] showed that the matrix $Z_{\lambda,\mu} = \langle \alpha^+_\lambda, \alpha^-_\mu \rangle$, where λ, μ label irreducible representations of $\{A(I)\}$ and the symbol $\langle \cdot, \cdot \rangle$ counts the number of common irreducible endomorphisms including multiplicities, satisfies the following properties:

(1) We have $Z_{\lambda,\mu} \in \{0, 1, 2, \dots\}$.
(2) We have $Z_{0,0} = 1$, where the label 0 denotes the vacuum representation.
(3) The matrix Z commutes with the image of the representation of $SL(2,\mathbb{Z})$.

Such a matrix Z is called a *modular invariant*, because $PSL(2,\mathbb{Z})$ is called the *modular group*. For a given completely rational conformal net (or more generally, a given modular tensor category), we have only finitely many modular invariants. Modular invariants naturally appear as *partition functions* in 2-dimensional conformal field theory and they have been classified for several concrete examples since Cappelli-Itzykson-Zuber [17] for the $SU(2)_k$ models and the *Virasoro nets* with $c < 1$, where c is a numerical invariant called the *central charge*. It takes a positive real value, and if $c < 1$, then it is of the form $1 - 6/m(m+1)$, $m = 3, 4, 5, \dots$ by Friedan-Qiu-Shenker [51] and Goddard-Kent-Olive [55]. This number arises from a projective unitary representation of $\mathrm{Diff}(S^1)$ and its corresponding unitary representation of the *Virasoro algebra*, a central extension of the complexification of the Lie algebra arising from $\mathrm{Diff}(S^1)$. Note that some modular invariants defined by the above three properties do not necessarily correspond to *physical* ones arising as partition functions in conformal field theory. Modular invariants arising from α-induction are physical in this sense.

The action of the A-A system on the A-B sectors (obtained by decomposing $\{\iota\lambda = \alpha^{\pm}_\lambda \iota : \lambda \in A\text{-}A\}$ into irreducibles where $\iota : A \to B$ is the inclusion) gives naturally a representation of the fusion rules of the Verlinde ring: $G_\lambda G_\mu = \sum N^\nu_{\lambda\mu} G_\nu$, with matrices $G_\lambda = [G^b_{\lambda a} : a, b \in A\text{-}B \text{ sectors}]$. Consequently, the matrices G_λ will be described by the same eigenvalues but with possibly different multiplicities. Böckenhauer-Evans-Kawahigashi [13] showed that these multiplicities are given exactly by the diagonal part of the modular invariant: $\mathrm{spectrum}(G_\lambda) = \{S_{\lambda\kappa}/S_{0\kappa} : \text{with multiplicity } Z_{\kappa\kappa}\}$. This is called a *nimrep* – a non-negative integer matrix representation. Thus a physical modular invariant is automatically equipped with a compatible nimrep whose spectrum is described by the diagonal part of the modular invariant. The case of $SU(2)$ is just the A-D-E classification of Cappelli-Itzykson-Zuber [17] with the A-B system yielding the associated (unextended) Coxeter-Dynkin graph. Since there is an A-D-E classification of matrices of norm less than 2, we can recover independently of Cappelli-Itzykson-Zuber [17] that there are unique modular invariants corresponding to the three exceptional E graphs.

If we use only positive α-inductions for a given modular tensor category, we still have a fusion category of endomorphisms, but no braiding in general. This is an example of a *module category*. For the tensor category $Rep(G)$ of representations of a finite group G, all module categories are of the form $Rep(H, \chi)$ for the projective

representations with 2-cocycle χ for a subgroup H [98]. For this reason, module categories have also been called *quantum subgroups*. Such categories have been studied in a more general categorical context by Ostrik in [100]. However, Carpi, Gaudio, Giorgetti and Hillier [19], have shown that for unitary fusion categories, such as those that occur in subfactor theory or arise from loop groups, that all module categories are equivalent to unitary ones. For the conformal nets corresponding to $SU(2)_k$, the module categories or quantum subgroups are labeled with all the Coxeter-Dynkin diagrams A_n, D_n and $E_{6,7,8}$. Here there is a coincidence with the affine A-D-E classification of finite subgroups of $SU(2)$. Di Francesco and Zuber [28] were motivated to try to relate $SU(3)$ modular invariants with subgroups of $SU(3)$. There is a partial match but this is not helpful. In general whilst the number of finite subgroups of $SU(n)$ grows with n, the number of exceptional modular invariants, beyond the obvious infinite series, does not.

If we have a net of subfactors $\{A(I) \subset B(I)\}$ with $\{A(I)\}$ being a completely rational conformal net, then the restriction of the vacuum representation of $\{B(I)\}$ to $\{A(I)\}$ gives a *local Q-system* in the sense of Longo [93]. This notion is essentially the same as a *commutative Frobenius algebra*, a special case of an *algebra in a tensor category*, in the algebraic or categorical literature. This Q-system is a triple consisting of an object and two intertwiners. Roughly speaking, the object gives $B(I)$ as an $A(I)$-$A(I)$ bimodule and the intertwiners give the multiplicative structure on $B(I)$. Our general theory of α-induction shows that the corresponding modular invariant Z for the modular tensor category of representations of $\{A(I)\}$ recovers this object. Since we have only finitely many modular invariants for a given modular tensor category, we have only finitely many objects for a local Q-system. It is known that each object has only finitely many local Q-system structures, and we thus have only finitely many local Q-systems, which means that we have only finitely many possibilities for extensions $\{B(I)\}$ for a given $\{A(I)\}$.

For some concrete examples of $\{A(I)\}$, we can classify all possible extensions. In the case of the $SU(2)_k$ nets, such extensions were studied in the context of α-induction in [12] by Böckenhauer-Evans-Kawahigashi and it was shown in [84] by Kawahigashi-Longo that they exhaust all possible extensions. (A similar classification based on quantum groups was first given in [88].) They correspond to the Coxeter-Dynkin diagrams A_n, D_{2n}, E_6 and E_8. The A_n cases are the $SU(2)_k$ nets themselves, the D_{2n} cases are given by *simple current extensions* of order 2, and the E_6 and E_8 cases are given by *conformal embeddings* $SU(2)_{10} \subset SO(5)_1$ and $SU(2)_{28} \subset (G_2)_1$, respectively. These correspond to *type I extensions* in Itzykson-Zuber [17], Böckenhauer-Evans [11]. Type II extensions corresponding to D_{2n+1} and E_7 arise from extensions of the $SU(2)_k$ nets *without* locality. In general [11] for a physical modular invariant Z there are by Böckenhauer-Evans local chiral extensions $N(I) \subset M_+(I)$ and $N(I) \subset M_-(I)$ with local Q-systems naturally associated to the vacuum column $\{Z_{\lambda,0}\}$ and vacuum row $\{Z_{0,\lambda}\}$ respectively. These extensions are indeed maximal and should be regarded as the subfactor version of left- and right maximal extensions of the chiral algebra. The representation theories or modular tensor categories of $M_\pm$ are then identified. For example, the E_7 conformal net or module category is a then a twist or auto-equivalence on the left and right local D_{10} extensions which form the type I parents. This reduces the analysis to understanding first local extensions and then classifying auto-equivalences to identify the two left and right local extensions. For $SU(2)$ there are only three

exceptional modular invariants $E_{6,7,8}$, and in general one expects, e.g. [99], for a WZW model that there are only a finite number of exceptionals beyond the infinite series of the trivial, orbifolds and their conjugates. Schopieray [115] using α-induction found bounds for levels of exceptional invariants for rank 2 Lie groups, and Gannon [57] extended this for higher rank with improved lower bounds using Galois transformations as a further tool. Edie-Michell has undertaken extensive studies of auto-equivalences [34]. The realisation by Evans-Pugh [43] of $SU(3)$-modular invariants as full CFT's, announced in [97], is based on the classification of Gannon [53] of $SU(3)$ modular invariants, and the classification by Evans-Pugh of full $SO(3)$ theories or $SO(3)$ module categories is in [44].

For a general conformal net, we always have a subnet generated by the projective unitary representation of $\mathrm{Diff}(S^1)$, which is called the *Virasoro net*, so a conformal net is always an extension of the Virasoro net. Through a unitary representation of the Virasoro algebra, a conformal net has a numerical invariant c, the central charge. The Virasoro net is completely rational if $c < 1$, so the above classification scheme applies to this case, and we have a complete classification of conformal nets with $c < 1$ by Kawahigashi-Longo in [84], where they are shown to be in a bijective correspondence with the type I modular invariants of Cappelli-Itzykson-Zuber in [17]. Four of exceptional modular invariants involving the Dynkin diagrams E_6 and E_8 give exceptional conformal nets. Three of them are given by the *coset construction*, but the other one gives a new example. Similarity between discreteness of the Jones index values below 4 and discreteness of the central charge value below 1 has been pointed out since the early days of subfactor theory [74], and we have an *A-D-E* classification of subfactors with index below 4 as in Popa [105] (also see [97, 42]) and an *A-D-E* classification of the modular invariants of the Virasoro minimal models of Capelli-Itzykson-Zuber [17]. We then have natural understanding of classification of conformal nets with $c < 1$ in this context.

K-theory has had a role in relating subfactor theory with statistical mechanics and conformal field theory. The phase transition in the two dimensional Ising model is analysed through an analysis of the ground states of the one dimensional quantum system arising from the transfer matrices. This is manifested by Araki-Evans through a jump in the Atiyah-Singer mod-2 index of Fredholm operators [3]. Here Kramers-Wannier high-temperature duality is effected by the shift endomorphism ρ on the corresponding Jones projections $e_j \to e_{j+1}$ which leads, Evans [35], to the Ising fusion rules $\rho^2 = 1 + \sigma$, where σ is the symmetry formed from interchanging $+$ and $-$ states, see also Evans-Gannon [41]. The tensor category of the Verlinde ring of compact Lie groups, or doubles of finite groups has been described by Freed-Hopkins-Teleman [47] through the twisted equivariant K-theory of the group acting on itself by conjugation. This has allowed the interchange of ideas between the subfactor approach and a K-theory approach to conformal field theory, employing α-induction and modular invariants as bi-variant Kasparov KK-elements by Evans-Gannon [37, 38, 41]. In a similar spirit, regarding K-theory in terms of projective modules, a finitely generated modular tensor category can be realised by Aaserud-Evans [2] as C^*-Hilbert modules. This applies to the modular tensor categories of Temperley-Lieb-Jones associated to quantum $SU(2)$, or more generally those of loop groups – as well as quantum doubles such as that of the Haagerup subfactor which we will focus on in the final section. This also gives a

framework for braided tensor categories acting on some C^*-algebras as a quantum symmetry.

5. Vertex operator algebras

We have another, more algebraic, mathematical axiomatisation for a chiral conformal field theory, namely, a *vertex operator algebra*. Since a conformal net and a vertex operator algebra are both mathematical formulations of the same physical theory, they naturally have close relations. We now explain those here.

A quantum field on the "spacetime" S^1 is an operator-valued distribution on S^1, so it has a Fourier expansion with operator coefficients. In this axiomatisation, we have a $\mathbb{C}$-vector space V which is a space of finite energy vectors and is supposed to give the Hilbert space of states after completion. For each vector $u \in V$, we have a formal series $Y(u,z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$ with a formal variable z and linear operators u_n on V, which corresponds to the Fourier expansion of a quantum field acting on the completion of V. This correspondence from a vector to a formal series is called the *state-field correspondence*. We have two distinguished vectors, the *vacuum vector* and the *Virasoro vector*. The Fourier coefficients of the latter give the Virasoro algebra. The locality axiom in this setting says that for $u, v \in V$, we have a sufficiently large positive integer N satisfying $(z-w)^N[Y(u,z), Y(v,w)] = 0$. Roughly speaking, this means $Y(u,z)Y(v,w) = Y(v,w)Y(u,z)$ for $z \neq w$.

The origin of this notion of a vertex operator algebra is as follows. A classical *elliptic modular function*

$$j(\tau) = 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2},$$

where $\operatorname{Im}\tau > 0$ and $g_2(\tau)$ and $g_3(\tau)$ are defined by the *Eisenstein series*, has the following Fourier expansion with $q = \exp(2\pi i\tau)$.

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots.$$

McKay noticed that the coefficient 196884 is very close to 196883, which is the dimension of the lowest-dimensional non-trivial irreducible representation of the *Monster group*. Recall that the Monster group is the largest among the 26 sporadic finite simple groups in terms of its order which is around 8×10^{53}. It turns out that we have a similar relation $21493760 = 1 + 196883 + 21296876$, where 21296876 is the dimension of the next lowest-dimensional irreducible representation of the Monster group. Based on this and many other pieces of information on modular functions, Conway-Norton [27] made the following *Moonshine conjecture*.

(1) There is a graded $\mathbb{C}$-vector space $V = \bigoplus_{n=0}^{\infty} V_n$ with *some algebraic structure* whose automorphism group isomorphic to the Monster group.
(2) For any element g in the Monster group, $\sum_{n=0}^{\infty} \operatorname{Tr}(g|_{V_n})q^{n-1}$ is the *Hauptmodul* for a genus 0 subgroup of $SL(2,\mathbb{R})$, where $g|_{V_n}$ is the linear action of an automorphism g on V_n.

Frenkel-Lepowsky-Meurman [49] gave a precise definition of a *certain algebraic structure* as a vertex operator algebra, constructed the *Moonshine vertex operator algebra* $V^\natural$, and proved that its automorphism group is exactly the Monster group. They first constructed a vertex operator algebra from the *Leech lattice*, an exceptional 24-dimensional lattice giving the densest sphere packing in dimension 24, and applied the *twisted orbifold construction* for the order two automorphism

of the vertex operator algebra arising from the multiplication by -1 on the Leech lattice to obtain $V^\natural$. Borcherds [14] next proved the remaining part of the Moonshine conjecture. The construction of a vertex operator algebra from an even lattice has an operator algebraic counterpart for a conformal net given in Dong-Xu [29]. The operator algebraic counterpart of the Moonshine vertex operator algebra has been constructed as the *Moonshine net* in Kawahigashi-Longo [85]. Frenkel-Zhu gave a construction of vertex operator algebras from affine Kac-Moody and Virasoro algebras in [50], and this corresponds to the construction of conformal nets of Wassermann [125], Loke, Toledano Laredo and Verrill.

We have constructions of new examples of vertex operator algebras or conformal nets from known ones as follows.

(1) A tensor product
(2) Coset construction
(3) Orbifold construction
(4) An extension using a Q-system

In the operator algebraic setting, the *coset construction* gives a relative commutant $A(I)' \cap B(I)$ for an inclusion $\{A(I) \subset B(I)\}$ of conformal nets of infinite index. The *orbifold construction* gives a fixed point conformal subnet given by an automorphic action of a finite group. These constructions for conformal nets have been studied by Xu in [130] and [131], respectively. The extension of a local conformal net using a Q-system was first studied by Kawahigashi-Longo in [84] for constructing exceptional conformal nets, and this was extended by Xu in [133]. The vertex operator algebra counterpart has been studied by Huang-Kirillov-Lepowsky in [63]. Xu has shown that various subfactor techniques are quite powerful even for purely algebraic problems in vertex operator algebras.

From the above results, it is clear that we have close connections between conformal nets and vertex operator algebras, as expected, but it is more desirable to have a direct construction of one from the other. The relation between the two should be like the one between Lie groups and Lie algebras, and the former should be given by "exponentiating" the latter. Such a construction was first given in Carpi-Kawahigashi-Longo-Weiner [20]. That is, we have a construction of a conformal net from a vertex operator algebra with *strong locality* and we also recover the original vertex operator algebra from this conformal net. (Note that we obviously need *unitarity* for a vertex operator algebra for such construction, since we need a nice positive definite inner product on V. This unitarity is a part of the strong locality assumption. There are many vertex operator algebras without unitarity, and they may be related to operator algebras through different routes such as planar algebras.) In addition to an abstract definition of strong locality, concrete sufficient conditions for this have been also given in [20]. This correspondence between vertex operator algebras and conformal nets has been vastly generalised recently in Gui [57], Raymond-Tanimoto-Tener [111] and Tener [117, 118, 119] including identification of their representation categories, and this is a highly active area of research today. Some of them started from dissertations supervised by Vaughan.

6. Other directions in conformal field theory

The classification of subfactors with index less than 4 has an A-D-E pattern. That is, the flat connections given in Fig.1 give a complete list of hyperfinite II_1 subfactors with index less than 4; see the review [81]. It naturally has connections

to many other topics in mathematics and physics where A-D-E patterns appear. At the index value equal to 4, we still have a similar A-D-E classification based on extended Dynkin diagrams due to Popa [105]. They correspond to subgroups of $SU(2)$, and the extended Coxeter-Dynkin diagrams appear through the *McKay correspondence*. These subfactors arise as simultaneous fixed point algebras of actions of a subgroup of $SU(2)$ on

$$\mathbb{C} \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots \subset M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$$

with infinite tensor products of the adjoint actions, possibly with extra cohomological twists as in the classification of periodic actions by Connes [21] and for finite group actions by Vaughan in his thesis [68] on the hyperfinite II_1 factor. The subfactors with index less than 4 can be regarded as "quantum" versions of this construction.

We have a quite different story above the index value 4. Haagerup searched for subfactors of finite depth above index value 4 and found several candidates of the principal graphs. The smallest index value among them is $(5+\sqrt{13})/2$ and he proved this index value is indeed attained by a subfactor, which is now called the *Haagerup subfactor* [4]. A similar method also produced the *Asaeda-Haagerup subfactor* in [4]. New constructions of the Haagerup subfactor were given in Izumi [65] and in Peters [101]. The latter is based on the planar algebra machinery, and has been extended to the construction of the *extended Haagerup subfactor*. Today we have a complete classification of subfactors of finite depth with index value between 4 and 5 as reviewed in [81], and we have five such subfactors (after identifying $N \subset M$ and $M \subset M_1$): the Haagerup subfactor, the Asaeda-Haagerup subfactor, the extended Haagerup subfactor, the Goodman-de la Harpe-Jones [56] subfactor and the Izumi-Xu subfactor. The latter two are now understood as arising from conformal embeddings $SU(2) \to E_6$ and $G_2 \to E_6$. If a subfactor arises from a connection on a finite graph Γ, it may not have principal graph or standard invariant based on Γ as happens with D_{2n+1} or E_7. Any graph whose norm squared is in the range $(4,5)$ but is not one of the five allowed values can only have A_∞ as principal graph just like what happens with E_{10} by unpublished work of Ocneanu, Haagerup and Schou.

The fusion categories arising from the Haagerup subfactor do not have a braiding, but their *Drinfel′d center* always gives a modular tensor category. Izumi gave a new construction of the Haagerup subfactor and computed the S- and T-matrices of its Drinfel′d center in [64, 65], using endomorphisms of the Cuntz algebra. It is an important problem whether an arbitrary modular tensor category is realised as the representation category of a conformal net or not, and this particular case of the Drinfel′d center of the fusion category of the Haagerup has caught much attention. Note that all the known constructions [4, 65, 101] of the Haagerup subfactor are based on algebraic or combinatorial computations. There is little conceptual understanding of this subfactor and its double and it is not clear at all whether they are related to statistical mechanics or conformal field theory. Evidence in the positive direction has been given by Evans-Gannon in [36, 41]. They found characters for the representation of the modular group $SL(2,\mathbb{Z})$ arising from the braiding and showing that this modular data, their S and T matrices and fusion rules have a simple expression in terms of a grafting of the double of the dihedral group S_3 and $SO(13)_2$, or indeed the orbifolds of two Potts models or quadratic (Tambara-Yamagami) systems based on $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_{13}$ respectively.

Information about a conformal field theory from the scaling limit of a statistical mechanical model may be detected from the underlying statistical mechanical system. Cardy [18] argued from conformal invariance for a critical statistical system, that the central charge c may be computed from the asymptotics of the partition function and transfer matrices on a periodic rectangular lattice. This has been well studied for the ABF, Q-state Potts models for Q=2,3,4 and certain ice-type models; see [42, pages 453–454]. In this spirit, numerical computations have been made using transfer matrices built from associator or certain $6j$ symbols for a Haagerup system, though not the double. These give a value of $c = 2$ (or around 2) [62, 123, 89]. However the results shown there and these methods do not show that if there is a CFT at $c = 2$ (or around 2) that it is not a known one and that if there is a CFT that its representation theory is related to the representation theory of the double of the Haagerup. Recall what we described in a preceding paragraph, that a subfactor constructed from a graph may not reproduce the graph through its invariants.

The first non-trivial reconstructions of conformal field theories were achieved by Evans-Gannon for the twisted doubles of finite groups and the orbifolds of Potts models [40, 41]. Whilst von Neumann algebras and subfactors are inherently unitary, non unitary theories have been analysed by Evans-Gannon from ideas derived from subfactors. This includes the Leavitt path algebras to replace Cuntz algebras in constructing non-unitary tensor categories of algebra endomorphisms which do not necessarily preserve the $*$-operation [39]. These and non-unitary planar algebras could also be a vehicle to understand non-semisimple and logarithmic conformal field theories.

In attempting to construct a conformal net realizing a given modular tensor category, a natural idea is to construct algebras as certain limit through finite dimensional approximations. We then use lattice approximation of the circle S^1, but diffeomorphism symmetry is lost in this finite dimensional approximation, so it is a major problem how to recover diffeomorphism symmetry. Vaughan studied this problem, used *Thompson's groups* as approximations of $\mathrm{Diff}(S^1)$, and obtained various interesting representations of Richard Thompson's groups [15, 76, 77]. Though he proved in [77] that translation operators arising as a limit of translations for the n-chains do not extend to a translation group that is strongly continuous at the origin, these representations are interesting in their own right. The clarity of his formalism and analysis, led to concise and elegant proofs of the previously difficult facts that the Thompson groups did not have Kazhdan's property T and with his Berkeley student Arnaud Brothier [16] that the Thompson's group T does have the Haagerup property. New results also followed – certain wreath products of groups have the Haagerup property by taking the group of fractions of group labelled forests. Taking a functor from binary forests to Conway tangles, replacing a fork by an elementary tangle, Vaughan could show that every link arises in this way from the fraction of a pair of forests just as braids yield all links through taking their closures — providing another unexpected bridge with knots and links [79]. He further studied related problems on scale invariance of transfer matrices on *quantum spin chains*, introduced two notions of scale invariance and weak scale invariance, and gave conditions for transfer matrices and nearest neighbour Hamiltonians to be scale invariant or weakly scale invariant [78].

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