



Bounds for Schrödinger Operators on the Half-Line Perturbed by Dissipative Barriers

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Abstract. We consider Schrödinger operators of the form $H_R = -d^2/dx^2 + q + i\gamma\chi_{[0,R]}$ for large $R > 0$, where $q \in L^1(0, \infty)$ and $\gamma > 0$. Bounds for the maximum magnitude of an eigenvalue and for the number of eigenvalues are proved. These bounds complement existing general bounds applied to this system, for sufficiently large R .

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1. Introduction

There has recently been a surge of interest concerning bounds for the magnitude of eigenvalues and the number of eigenvalues of Schrödinger operators with complex potentials. In this paper, we consider Schrödinger operators of the form

$$H_R = -\frac{d^2}{dx^2} + q + i\gamma\chi_{[0,R]} \quad \text{on } L^2(0, \infty) \quad (R > 0), \quad (1)$$

endowed with a Dirichlet boundary condition at 0, where $\gamma > 0$ and the *background potential* $q \in L^1(0, \infty)$ (which may be complex-valued) are regarded as fixed parameters. Perturbations of the form $i\gamma\chi_{[0,R]}$ are referred to as *dissipative barriers* and arise in spectral approximation, where they can be utilised as part of numerical schemes for the computation of eigenvalues [2, 22–24, 31, 34]. Our aim is to prove estimates for the magnitude and number of eigenvalues of H_R for large R .

1.1. Existing Bounds for the Magnitude and Number of Eigenvalues

Let us first discuss some relevant existing results concerning the eigenvalues of (non-self-adjoint) Schrödinger operators and apply them to operators of the form H_R .

In [1], Abramov, Aslanyan and Davies investigated bounds for complex eigenvalues of Schrödinger operators, in particular obtaining a bound

[1, Theorem 4] for Schrödinger operator on $L^2(\mathbb{R})$ with a potential $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Such magnitude bounds were later generalised to include more general potentials, higher dimensions and more general geometries [6, 8, 10–12, 15, 17, 20, 21, 29]. The work most relevant to this paper was undertaken by Frank, Laptev and Seiringer [14], where they show that any eigenvalue λ of a Schrödinger operator $-d^2/dx^2 + V$ on $L^2(\mathbb{R}_+)$, endowed with a Dirichlet boundary condition at 0, satisfies

$$\sqrt{|\lambda|} \leq \|V\|_{L^1}. \tag{2}$$

Note that the right hand side of the bound presented in [14] depends on $\arg \lambda$ and is sharper than (2). An application of this result to operators of the form H_R gives an estimate $\sqrt{|\lambda_R|} = O(R)$ as $R \rightarrow \infty$ for any eigenvalue λ_R of H_R .

Proving bounds for the number of eigenvalues of a Schrödinger operator is often regarded a more difficult problem. A sufficient condition for the potential V to ensure that the number of eigenvalues of a Schrödinger operator on $L^2(\mathbb{R}_+)$ is finite is the *Naimark condition* [25]:

$$\exists a > 0 : \int_0^\infty e^{at}|V(t)| dt < \infty. \tag{3}$$

There exist other such sufficient conditions and it is known that the number of eigenvalues may not be finite for certain potentials decaying only sub-exponentially [3, 27, 28].

Quantitative bounds for the number of eigenvalues of a Schrödinger operator on $L^2(\mathbb{R}^d)$ were proved by Stepin in [32, 33] for dimensions $d = 1, 3$. Bounds for arbitrary odd dimensions were later proved by Frank, Laptev and Safronov in [13], which give better large R estimates when applied to operators H_R of the form (1). Frank et al. [13, Theorem 1.1] states that the number of eigenvalues N (counting algebraic multiplicity) of a Schrödinger operator $-d^2/dx^2 + V$ on $L^2(\mathbb{R}_+)$ endowed with a Dirichlet boundary condition at 0 satisfies

$$N \leq \frac{1}{\varepsilon^2} \left(\int_0^\infty e^{\varepsilon t}|V(t)| dt \right)^2. \tag{4}$$

for any $\varepsilon > 0$. With the assumption that the background potential q satisfies the Naimark condition, applying this inequality to H_R with $\varepsilon = 1/R$ gives an estimate $N(H_R) = O(R^4)$ as $R \rightarrow \infty$ for the number of eigenvalues (counting algebraic multiplicities) $N(H_R)$ of H_R .

Additionally, Korotyaev has proved in [19, Theorem 1.6] a bound specific to Schrödinger operators with compactly supported potentials: the number of eigenvalues N of a Schrödinger operator $-d^2/dx^2 + V$ on $L^2(\mathbb{R}_+)$ endowed with a Dirichlet boundary condition at 0, with $V \in L^1(\mathbb{R}_+)$ and $\text{supp} V \subseteq [0, Q]$, satisfies

$$N \leq C_1 + C_2 Q \|V\|_{L^1} \tag{5}$$

where $C_1, C_2 > 0$ are some numerical constants. With the assumption that the background potential q is compactly supported, applying this inequality

to H_R gives an estimate $N(H_R) = O(R^2)$ as $R \rightarrow \infty$. We mention also other estimates for numbers of eigenvalues in [5, 18, 30].

1.2. Summary of Results

Table 1 summarises our results for the large R behaviour of the eigenvalues of H_R and compares them to the application of the existing results to operators of the form H_R .

Let $H_R^{(0)}$ denote the operator H_R for the case $q \equiv 0$. The semi-infinite strip

$$\Gamma_\gamma := (0, \infty) + i(0, \gamma) \subset \mathbb{C} \tag{6}$$

plays an important role throughout the paper and has the property that its closure $\bar{\Gamma}_\gamma$ is equal to the numerical range of the operator $H_R^{(0)}$ for any $R > 0$. An open ball in \mathbb{C} of radius $r > 0$ about a point $z_0 \in \mathbb{C}$ is denoted by $B_r(z_0)$. Note that in this paper we make no attempt to optimise numerical constants.

Our first result gives a uniform in R enclosure for the eigenvalues of H_R :

- (A) (Theorem 4(a)) There exists $X = X(q, \gamma) > 0$ such that, for any $R > 0$, the eigenvalues of H_R lie in $B_X(0) \cup \Gamma_\gamma$.

In particular, the imaginary and negative real components of the eigenvalues are bounded independently of R .

Our next result is a bound for the magnitude of eigenvalues of H_R for sufficiently large R . The bound gives the estimate $\sqrt{|\lambda_R|} = O(R/\log R)$ as $R \rightarrow \infty$ for any eigenvalue λ_R of H_R providing a logarithmic improvement to the application of the result [14] of Frank, Laptev and Seiringer to this system.

- (B) (Theorem 4(b)) There exists $R_0 = R_0(q, \gamma) > 0$ such that for every $R \geq R_0$, any eigenvalue λ of H_R in Γ_γ satisfies

$$\sqrt{|\lambda - i\gamma|} \leq \frac{5\gamma R}{\log R}. \tag{7}$$

The first inequality in (9) shows that the estimate $\sqrt{|\lambda_R|} = O(R/\log R)$ is in fact sharp. (B) is obtained by considering an analytic function whose zeros are the eigenvalues of H_R and applying large- $|\lambda|$ Levinson asymptotics. The enclosure that results from combining (A) and (B) is illustrated in Fig. 1.

The fact that large eigenvalues of H_R for large R must be contained in the numerical range of $H_R^{(0)}$ and the right hand side of inequality (7) is independent of q indicates that the effect of the background potential q on the large eigenvalues is dominated by effect of the dissipative barrier $i\gamma\chi_{[0,R]}$ for large R .

Our first estimate for the number of eigenvalues $N(H_R)$ for H_R is for the case that the background potential q is compactly supported. It gives the estimate $N(H_R) = O(R^2/\log R)$ as $R \rightarrow \infty$, which offers a logarithmic improvement to the application of the result [19, Theorem 1.6] of Korotyaev to this system.

TABLE 1. A summary of the large R asymptotic estimates for the eigenvalues of H_R implied by our results compared to estimates obtained by applying various results in the literature

	Literature	Our results
Magnitude bound	$\sqrt{ \lambda_R } = O(R)$ Frank, Laptev, Seiringer (2011)	$\sqrt{ \lambda_R } = O(R/\log R)$ Theorem 4
Number of eigenvalues (q compactly supported)	$N(H_R) = O(R^2)$ Korotyaev (2020)	$N(H_R) = O(R^2/\log R)$ Theorem 8
Number of eigenvalues (Naimark condition)	$N(H_R) = O(R^4)$ Frank, Laptev, Safronov (2016)	$N(H_R) = O(R^3/(\log R)^2)$ Theorem 10

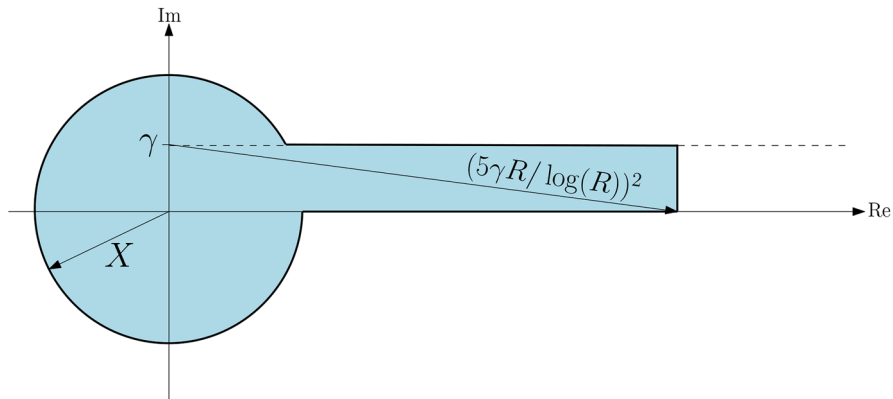


FIGURE 1. Illustration the enclosure for the eigenvalues of H_R provided by Theorem 4

- (C) (Theorem 8) If q is compactly supported then there exists $R_0 = R_0(q, \gamma) > 0$ such that for every $R \geq R_0$,

$$N(H_R) \leq \frac{11}{\log 2} \frac{\gamma R^2}{\log R}.$$

The second inequality in (9) shows that the estimate $N(H_R) = O(R^2/\log R)$ is sharp. The proof consists in an application of Jensen’s formula.

The case in which the background potential q merely satisfies the Naimark condition requires more sophisticated techniques compared to the compactly supported case. Our result gives the estimate $N(H_R) = O(R^3/(\log R)^2)$ as $R \rightarrow \infty$, providing a more significant improvement to the application of the result [13, Theorem 1.1] of Frank, Laptev and Safronov to this system, which

gives $N(H_R) = O(R^4)$. The reasons for the more significant improvement are discussed below.

(D) (Theorem 10) If there exists $a > 0$ such that

$$\int_0^\infty e^{4at}|q(t)| dt < \infty.$$

then there exists $R_0 = R_0(q, \gamma) > 0$ such that for every $R \geq R_0$,

$$N(H_R) \leq C \frac{\sqrt{X} + a}{a^2} \frac{\gamma^2 R^3}{(\log R)^2} \tag{8}$$

where $X = X(q, \gamma) > 0$ is the constant appearing in (A) and $C = 88788$.

The proof of (D) involves first obtaining a bound which counts the number of zeros in a strip for an arbitrary analytic function in the upper half plane (Proposition 9). This bound can be applied to the estimation of $N(H_R)$ thanks to the uniform in R enclosure (A), which implies that the square-roots of the eigenvalues of H_R are contained in a strip, uniformly in R . Without the uniform enclosure, we would have to use the magnitude bound (B) in place of the uniform enclosure with which the best we could obtain is inequality (8) with \sqrt{X} replaced by $O(R/\log R)$, giving the large R estimate $N(H_R) = O(R^4/(\log R)^3)$. This indicates that the more significant improvement in (D) is due to the combination of a bound for the quantity $\Im\sqrt{\lambda}$ of the eigenvalues λ with the bound Proposition 9 for analytic functions.

Operators of the form $H_R^{(0)}$, corresponding to the special case $q = 0$, have been studied by Bögli and Štampach in [4], by Golinskii in [16] and by Cuenin in [7]. A consequence of [7, Theorem 4] is that there exists constants $C_1, C_2 > 0$ such that for all large enough $R > 0$,

$$\sup_{\lambda \in \sigma(H_R^{(0)})} \sqrt{|\lambda|} \geq C_1 \frac{R}{\log(R)} \quad \text{and} \quad N(H_R^{(0)}) \geq C_2 \frac{R^2}{\log(R)}. \tag{9}$$

Note that although this result was formulated for the Schrödinger operator on \mathbb{R} , it applies to Schrödinger operators on \mathbb{R}_+ endowed with a Dirichlet or Neumann boundary condition since the author constructs both odd and even eigenfunctions of $H_R^{(0)}$ in [7, Section 7.1]. As already mentioned, the inequalities (9) show that Theorem 4(b) and Theorem 8 provide optimal large R estimates.

The reader is referred to [31, Section 5] for numerical illustrations of the eigenvalues of operators of the form H_R for large R .

1.3. Notations and Conventions

Throughout the paper, $C > 0$ denotes a constant, whose dependencies are generally indicated, that may change from line to line. $\psi'(x, \lambda)$ will denote $\frac{d}{dx} \psi(x, \lambda)$ throughout. The branch cut of $\sqrt{\cdot}$ is made along $\sigma_e(H_R) = [0, \infty)$, so that $\Im\sqrt{z} \geq 0$ for all $z \in \mathbb{C}$. $N(H_R)$ shall denote the number of eigenvalues of H_R , counting algebraic multiplicities (as above). Finally, note that f_R will always denote an analytic function but will be redefined in each section.

2. Magnitude Bound

Since $q \in L^1(0, \infty)$, we can employ Levinson’s asymptotic theorem which states that the solution space of the Schrödinger equation $-u'' + qu = \lambda u$ on $[0, \infty)$ is spanned by solutions ψ_+ and ψ_- , which admit the decomposition [26, Appendix II, Theorems 1 and 3] [9, Theorem 1.3.1]:

$$\begin{aligned} \psi_{\pm}(x, \lambda) &= e^{\pm i\sqrt{\lambda}x}(1 + E_{\pm}(x, \lambda)) \\ \psi'_{\pm}(x, \lambda) &= \pm i\sqrt{\lambda}e^{\pm i\sqrt{\lambda}x}(1 + E_{\pm}^d(x, \lambda)) \quad (x \in [0, \infty), \lambda \in \mathbb{C} \setminus \{0\}). \end{aligned} \tag{10}$$

Here, E_{\pm} and E_{\pm}^d are some functions such that,

$$|E_{\pm}(x, \lambda)| + |E_{\pm}^d(x, \lambda)| \rightarrow 0 \quad \text{as } x \rightarrow \infty \tag{11}$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$, and

$$|E_{\pm}(x, \lambda)| + |E_{\pm}^d(x, \lambda)| \leq \frac{C(q)}{\sqrt{|\lambda|}} \tag{12}$$

for all $x \in [0, \infty)$ and $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$.

While the error $E_+(x, \lambda)$ tends to 0 as $x \rightarrow \infty$ uniformly for $\lambda \in \mathbb{C} \setminus B_{\delta}(0)$, $\delta > 0$, the error E_- does not have this property. For this reason, we will need to utilise large- $|\lambda|$ asymptotics of ψ_{\pm} in this section.

Lemma 1. $\lambda \in \mathbb{C} \setminus [0, \infty)$ with $\lambda \neq i\gamma$ is an eigenvalue of H_R if and only if $f_R(\lambda) = 0$, where

$$\begin{aligned} f_R(\lambda) &:= \psi_-(0, \lambda - i\gamma) \left(\sqrt{\lambda} - \sqrt{\lambda - i\gamma} + \mathcal{E}_1(R, \lambda) \right) e^{i\sqrt{\lambda - i\gamma}R} \\ &\quad - \psi_+(0, \lambda - i\gamma) \left(\sqrt{\lambda} + \sqrt{\lambda - i\gamma} + \mathcal{E}_2(R, \lambda) \right) e^{-i\sqrt{\lambda - i\gamma}R}. \end{aligned}$$

Here, $\mathcal{E}_1, \mathcal{E}_2$ are defined, for any $R > 0$ and $\lambda \in \mathbb{C} \setminus \{0, i\gamma\}$, by

$$\begin{aligned} \mathcal{E}_1(R, \lambda) &= \sqrt{\lambda} \left(E_+(R, \lambda - i\gamma) + E_+^d(R, \lambda) + E_+(R, \lambda - i\gamma)E_+^d(R, \lambda) \right) \\ &\quad - \sqrt{\lambda - i\gamma} \left(E_+^d(R, \lambda - i\gamma) + E_+(R, \lambda) + E_+^d(R, \lambda - i\gamma)E_+(R, \lambda) \right), \end{aligned} \tag{13}$$

$$\begin{aligned} \mathcal{E}_2(R, \lambda) &= \sqrt{\lambda} \left(E_+^d(R, \lambda) + E_-(R, \lambda - i\gamma) + E_+^d(R, \lambda)E_-(R, \lambda - i\gamma) \right) \\ &\quad + \sqrt{\lambda - i\gamma} \left(E_+(R, \lambda) + E_-^d(R, \lambda - i\gamma) + E_+(R, \lambda)E_-^d(R, \lambda - i\gamma) \right) \end{aligned} \tag{14}$$

and, for some $\mathcal{C}_1 = \mathcal{C}_1(q, \gamma) > 0$, satisfy

$$|\mathcal{E}_1(R, \lambda)| + |\mathcal{E}_2(R, \lambda)| \leq \mathcal{C}_1 \tag{15}$$

for all $R > 0$ and all $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1 + \gamma$. Furthermore, $f_R, \mathcal{E}_1(R, \cdot)$ and $\mathcal{E}_2(R, \cdot)$ are analytic on $\mathbb{C} \setminus ([0, \infty) \cup (i\gamma + [0, \infty)))$.

Proof. Let $\lambda \in \mathbb{C} \setminus [0, \infty)$ with $\lambda \neq i\gamma$. λ is an eigenvalue of H_R if and only if there a solution to the boundary value problem

$$-\psi'' + (q + i\gamma\chi_{[0,R]})\psi = \lambda\psi \text{ on } [0, \infty), \psi(0) = 0, \psi \in L^2(0, \infty). \tag{16}$$

Any solution to (16) on $[0, R]$ must be of the form $C_1\psi_1(\cdot, \lambda)$, where

$$\psi_1(x, \lambda) := \psi_-(0, \lambda - i\gamma)\psi_+(x, \lambda - i\gamma) - \psi_+(0, \lambda - i\gamma)\psi_-(x, \lambda - i\gamma) \tag{17}$$

and $C_1 \in \mathbb{C}$ is independent of x . Any solution to the boundary value problem (16) on $[R, \infty)$ must be of the form $C_2\psi_+(x, \lambda)$, where $C_2 \in \mathbb{C}$ is independent of x . Hence λ is an eigenvalue if and only if there exists $C_1, C_2 \in \mathbb{C} \setminus \{0\}$ independent of x such that the function

$$x \mapsto \begin{cases} C_1\psi_1(x, \lambda) & \text{if } x \in [0, R) \\ C_2\psi_+(x, \lambda) & \text{if } x \in [R, \infty) \end{cases}$$

is continuously differentiable which holds if and only if

$$if_R(\lambda)e^{i\sqrt{\lambda}R} \equiv \psi_1(R, \lambda)\psi'_+(R, \lambda) - \psi'_1(R, \lambda)\psi_+(R, \lambda) = 0. \tag{18}$$

The required expression for f_R holds by a direct computation, using expressions (10) for ψ_{\pm} .

For $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1 + \gamma$ we have $|\lambda| \geq 1$ and $|\lambda - i\gamma| \geq 1$. Therefore, estimates (12) apply to all the terms in (13) and (14) involving E_{\pm} or E_{\pm}^d . The $O(1/\sqrt{|\lambda|})$ decay of the terms involving E_{\pm} or E_{\pm}^d as $|\lambda| \rightarrow \infty$ cancel the growth of the square roots hence estimate (15) holds. Finally, $f_R, \mathcal{E}_1(R, \cdot)$ and $\mathcal{E}_2(R, \cdot)$ are analytic on $\mathbb{C} \setminus ([0, \infty) \cup (i\gamma + [0, \infty)))$ because $\sqrt{\cdot}, E_{\pm}(R, \cdot)$ and $E_{\pm}^d(R, \cdot)$ are analytic on $\mathbb{C} \setminus [0, \infty)$. \square

In the special case $q \equiv 0$, f_R is denoted by $f_R^{(0)}$ and we have that:

$$\lambda \in \mathbb{C} \setminus [0, \infty) \text{ is an eigenvalue of } H_R^{(0)} \text{ if and only if } f_R^{(0)}(\lambda) = 0.$$

The terms E_{\pm} and E_{\pm}^d in Levinson's asymptotic theorem are simply zero for this case, so

$$f_R^{(0)}(\lambda) = \left(\sqrt{\lambda} - \sqrt{\lambda - i\gamma}\right)e^{i\sqrt{\lambda-i\gamma}R} - \left(\sqrt{\lambda} + \sqrt{\lambda - i\gamma}\right)e^{-i\sqrt{\lambda-i\gamma}R}. \tag{19}$$

Lemma 2. *There exists a constant $\mathcal{C}_2 = \mathcal{C}_2(q, \gamma) > 0$ such that*

$$|f_R(\lambda) - f_R^{(0)}(\lambda)| \leq \mathcal{C}_2 e^{\Im\sqrt{\lambda-i\gamma}R}$$

for all $R > 0$ and all $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1 + \gamma$.

Proof. By a direct computation, using Lemma 1 and the fact that

$$\psi_{\pm}(0, \lambda - i\gamma) = 1 + E_{\pm}(0, \lambda - i\gamma),$$

we have

$$\begin{aligned} (f_R(\lambda) - f_R^{(0)}(\lambda))e^{i\sqrt{\lambda-i\gamma}R} &= E_-(0, \lambda - i\gamma)[*]\sqrt{\lambda} - \sqrt{\lambda - i\gamma}e^{2i\sqrt{\lambda-i\gamma}R} \\ &\quad - E_+(0, \lambda - i\gamma)[*]\sqrt{\lambda} + \sqrt{\lambda - i\gamma} \\ &\quad + (1 + E_-(0, \lambda - i\gamma))\mathcal{E}_1(R, \lambda)e^{2i\sqrt{\lambda-i\gamma}R} \\ &\quad - (1 + E_+(0, \lambda - i\gamma))\mathcal{E}_2(R, \lambda). \end{aligned} \tag{20}$$

Each term on the right hand side of (20) is bounded uniformly for all $R > 0$ and all $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1 + \gamma$; this follows using the boundedness for \mathcal{E}_1 and \mathcal{E}_2 proved in Lemma 1 as well as the large- $|\lambda|$ asymptotics of $E_{\pm}(0, \lambda - i\gamma)$ in (12). In particular, inequality (12) implies that $E_{\pm}(0, \lambda - i\gamma) = O(1/\sqrt{|\lambda|})$ as $|\lambda| \rightarrow \infty$, balancing the growth of the factors $\sqrt{\lambda} \pm \sqrt{\lambda - i\gamma}$ in the first two terms of (20). \square

Recall that Γ_γ is an open strip defined by Eq. (6). We shall need the following elementary inequalities:

Lemma 3. (a) *If $\lambda \in \Gamma_\gamma \cup [0, \infty)$ then*

$$|\sqrt{\lambda} + \sqrt{\lambda - i\gamma}| \leq \frac{\gamma}{\sqrt{|\lambda - i\gamma|}} \text{ and } |\sqrt{\lambda} - \sqrt{\lambda - i\gamma}| \geq \sqrt{|\lambda - i\gamma|}.$$

(b) *If $\lambda \in \mathbb{C} \setminus (\Gamma_\gamma \cup [0, \infty))$ then*

$$|\sqrt{\lambda} + \sqrt{\lambda - i\gamma}| \geq \sqrt{|\lambda|} \text{ and } |\sqrt{\lambda} - \sqrt{\lambda - i\gamma}| \leq \frac{\gamma}{\sqrt{|\lambda|}}.$$

(c) *If $\lambda \in \Gamma_\gamma$ then*

$$\Im \sqrt{\lambda - i\gamma} \leq \frac{1}{\sqrt{2}} \frac{\gamma}{\sqrt{|\lambda - i\gamma|}}.$$

Proof. (a) If $\lambda \in \Gamma_\gamma \cup [0, \infty)$ then

$$\operatorname{sgn} \Re \sqrt{\lambda - i\gamma} = -\operatorname{sgn} \Re \sqrt{\lambda}, \quad |\Re \sqrt{\lambda - i\gamma}| \geq \Im \sqrt{\lambda - i\gamma} \text{ and } |\Re \sqrt{\lambda}| \geq \Im \sqrt{\lambda}$$

so

$$\begin{aligned} |\sqrt{\lambda} - \sqrt{\lambda - i\gamma}|^2 &= (\Re \sqrt{\lambda})^2 + (\Re \sqrt{\lambda - i\gamma})^2 + (\Im \sqrt{\lambda})^2 + (\Im \sqrt{\lambda - i\gamma})^2 \\ &\quad - 2\Re \sqrt{\lambda} \Re \sqrt{\lambda - i\gamma} - 2\Im \sqrt{\lambda} \Im \sqrt{\lambda - i\gamma} \\ &\geq |\lambda - i\gamma|. \end{aligned} \tag{21}$$

The inequality for $\sqrt{\lambda} + \sqrt{\lambda - i\gamma}$ follows from the identity

$$\sqrt{\lambda} + \sqrt{\lambda - i\gamma} = \frac{i\gamma}{\sqrt{\lambda} - \sqrt{\lambda - i\gamma}}. \tag{22}$$

(b) If $\lambda \in i\gamma + \mathbb{C}_+ \cup [0, \infty)$ or $\lambda \in \mathbb{C}_-$ then, similarly to (21),

$$\operatorname{sgn} \Re \sqrt{\lambda} = \operatorname{sgn} \Re \sqrt{\lambda - i\gamma} \Rightarrow |\sqrt{\lambda} + \sqrt{\lambda - i\gamma}|^2 \geq |\lambda|.$$

If $\lambda \in (-\infty, 0] + i[0, \gamma]$ then $|\Re \sqrt{\lambda}| \leq \Im \sqrt{\lambda}$ and $|\Re \sqrt{\lambda - i\gamma}| \leq \Im \sqrt{\lambda - i\gamma}$ so

$$|\sqrt{\lambda} + \sqrt{\lambda - i\gamma}|^2 \geq |\lambda - i\gamma| + |\lambda| \geq |\lambda|$$

hence the inequality for $\sqrt{\lambda} + \sqrt{\lambda - i\gamma}$ holds. The inequality for $\sqrt{\lambda} - \sqrt{\lambda - i\gamma}$ follows from (22).

(c) Let $\lambda \in \Gamma_\gamma$ and let $z = \lambda - i\gamma$. Then $|\Im z| \leq \gamma$ so

$$2(\Im \sqrt{z})^2 = |z| - \Re z = \frac{(\Im z)^2}{|z| + \Re z} \leq \frac{\gamma^2}{|z|}.$$

□

Using the function f_R for the eigenvalues of H_R , combined with the large- $|\lambda|$ asymptotics of ψ_\pm , we can estimate the location of the eigenvalues of H_R :

Theorem 4. (a) *There exists $X = X(q, \gamma) > 0$ such that, for any $R > 0$, the eigenvalues of H_R lie in $B_X(0) \cup \Gamma_\gamma$.*

- (b) *There exists $R_0 = R_0(q, \gamma) > 0$ such that for every $R \geq R_0$, any eigenvalue λ of H_R in Γ_γ satisfies*

$$\sqrt{|\lambda - i\gamma|} \leq \frac{5\gamma R}{\log R}. \tag{23}$$

Proof. (a) Let $R > 0$. H_R has no eigenvalues in $[0, \infty)$ (indeed, this follows from the Levinson asymptotic formulas (10)) so it suffices to show that any zero of f_R in $\mathbb{C} \setminus (\Gamma_\gamma \cup [0, \infty))$ must lie in an open ball in the complex plane, whose radius is independent of R . Let $\lambda \in \mathbb{C} \setminus (\Gamma_\gamma \cup [0, \infty))$ be such that $|\lambda| \geq X$, where $X = X(q, \gamma) > 0$ is a large enough constant to be further specified. Let $X > 0$ be large enough so that $|\lambda| \geq 1 + \gamma$. By the expression for f_R in Lemma 1,

$$\begin{aligned} |f_R(\lambda)e^{i\sqrt{\lambda-i\gamma}R}| &\geq \left| \left| \psi_+(0, \lambda - i\gamma) \left(\sqrt{\lambda} + \sqrt{\lambda - i\gamma} + \mathcal{E}_2(R, \lambda) \right) \right| \right. \\ &\quad \left. - \left| \psi_-(0, \lambda - i\gamma) \left(\sqrt{\lambda} - \sqrt{\lambda - i\gamma} + \mathcal{E}_1(R, \lambda) \right) \right| e^{-2\Im\sqrt{\lambda-i\gamma}R} \right|. \end{aligned} \tag{24}$$

By the boundedness of \mathcal{E}_1 and E_- (Lemma 1 and estimates (12)), as well an inequality in Lemma 3(b), there exists $C_1 = C_1(q, \gamma) > 0$ such that

$$\left| \psi_-(0, \lambda - i\gamma) \left(\sqrt{\lambda} - \sqrt{\lambda - i\gamma} + \mathcal{E}_1(R, \lambda) \right) \right| e^{-2\Im\sqrt{\lambda-i\gamma}R} \leq C_1. \tag{25}$$

Let $\delta > 0$. Recall that $|\mathcal{E}_2(R, \lambda)| \leq C_1$, where $C_1 > 0$ is the constant appearing in Lemma 1. Let $X > 0$ be large enough such that $|\psi_+(0, \lambda - i\gamma)| \geq \frac{1}{2}$ and

$$\sqrt{|\lambda|} \geq 2(C_1 + \delta) + C_1.$$

Then, using Lemma 3(b),

$$\left| \psi_+(0, \lambda - i\gamma) \left(\sqrt{\lambda} + \sqrt{\lambda - i\gamma} + \mathcal{E}_2(R, \lambda) \right) \right| \geq \frac{1}{2} \left| \sqrt{|\lambda|} - C_1 \right| \geq C_1 + \delta. \tag{26}$$

Combining (24), (25) and (26), we have

$$|f_R(\lambda)| \geq \delta > 0.$$

Consequently, λ is not an eigenvalue of H_R proving that there are no eigenvalues of H_R in $\mathbb{C} \setminus \Gamma_\gamma$ with magnitude greater than X .

- (b) Let $R \geq R_0$, where $R_0 = R_0(q, \gamma) > 0$ is a large enough constant to be further specified. Let $\lambda \in \Gamma_\gamma$ be such that

$$\sqrt{|\lambda - i\gamma|} \log |\lambda - i\gamma| \geq 8\gamma R. \tag{27}$$

We aim to prove that λ is not an eigenvalue of H_R .

Using the expression (19) for $f_R^{(0)}$,

$$\frac{|f_R^{(0)}(\lambda)|}{|\lambda - i\gamma|^{1/4}} e^{-\Im\sqrt{\lambda-i\gamma}R} \geq \left| \frac{|\sqrt{\lambda} - \sqrt{\lambda - i\gamma}|}{|\lambda - i\gamma|^{1/4}} e^{-2\Im\sqrt{\lambda-i\gamma}R} - \frac{|\sqrt{\lambda} + \sqrt{\lambda - i\gamma}|}{|\lambda - i\gamma|^{1/4}} \right|$$

Using the inequality (27) and Lemma 3(c), λ satisfies

$$e^{-2\Im\sqrt{\lambda-i\gamma}R} \geq e^{-\sqrt{2}\gamma R/\sqrt{|\lambda-i\gamma|}} \geq e^{-\frac{\sqrt{2}}{8}\log|\lambda-i\gamma|} = \frac{1}{|\lambda-i\gamma|^{\sqrt{2}/8}}. \tag{28}$$

Ensure $R_0 > 0$ is large enough so that $|\lambda-i\gamma|^{1/4} \geq 2|\lambda-i\gamma|^{\sqrt{2}/8}$. Then, using Lemma 3(a),

$$\frac{|\sqrt{\lambda}-\sqrt{\lambda-i\gamma}|}{|\lambda-i\gamma|^{1/4}} \geq |\lambda-i\gamma|^{1/4} \geq 2|\lambda-i\gamma|^{\sqrt{2}/8}. \tag{29}$$

Ensure also that $R_0 > 0$ is large enough so that $|\lambda-i\gamma| \geq \gamma^{4/3}$. Combining (29) with (28) and using Lemma 3(a) again,

$$\frac{|\sqrt{\lambda}-\sqrt{\lambda-i\gamma}|}{|\lambda-i\gamma|^{1/4}} e^{-2\Im\sqrt{\lambda-i\gamma}R} \geq 2 \geq 1 + \frac{|\sqrt{\lambda}+\sqrt{\lambda-i\gamma}|}{|\lambda-i\gamma|^{1/4}}.$$

and hence

$$|f_R^{(0)}(\lambda)| \geq |\lambda-i\gamma|^{1/4} e^{\Im\sqrt{\lambda-i\gamma}R}. \tag{30}$$

In particular, $f_R^{(0)}(\lambda) \neq 0$.

Recall that $\mathcal{C}_2 = \mathcal{C}_2(q, \gamma) > 0$ denotes the constant appearing in Lemma 2. Ensure that $R_0 > 0$ is large enough so that $|\lambda| \geq 1 + \gamma$ and $|\lambda-i\gamma|^{1/4} \geq 2\mathcal{C}_2$. By (30) and Lemma 2,

$$|f_R(\lambda) - f_R^{(0)}(\lambda)| \leq \mathcal{C}_2 e^{\Im\sqrt{\lambda-i\gamma}R} \leq \frac{1}{2} |\lambda-i\gamma|^{1/4} e^{\Im\sqrt{\lambda-i\gamma}R} \leq \frac{1}{2} |f_R^{(0)}(\lambda)|$$

therefore $f_R(\lambda) \neq 0$ and, consequently, λ is not an eigenvalue of H_R . This proves that any eigenvalue of H_R must satisfy

$$\sqrt{|\lambda-i\gamma|} \log \sqrt{|\lambda-i\gamma|} \leq 4\gamma R. \tag{31}$$

Let W denote the Lambert- W -function (also known as the product log function). W satisfies

$$W(x) = \log\left(\frac{x}{W(x)}\right) \quad \text{and} \quad y \log y = x \iff y = \frac{x}{W(x)} \quad (x > 0, y > 0).$$

Hence (31) can be written as

$$\sqrt{|\lambda-i\gamma|} \leq \frac{4\gamma R}{W(4\gamma R)} = \frac{4\gamma R}{\log(4\gamma R) - \log(W(4\gamma R))}$$

from which (23) follows. □

Remark 1. The constant $X = X(q, \gamma) > 0$ in Theorem 4(a) satisfies

$$X = O(\|q\|_{L^1}^3) \quad \text{as} \quad \|q\|_{L^1} \rightarrow \infty.$$

This can be seen by noting that $E_{\pm}(R, \lambda), E_{\pm}^d(R, \lambda) = O(\|q\|_{L^1})$ (see [9, Chapter 1.4]), $\mathcal{C}_1 = O(\|q\|_{L^1}^2)$ and $C_1 = O(\|q\|_{L^1}^3)$.

3. Number of Eigenvalues

In this section, we estimate the number of eigenvalues for H_R , for which we necessarily need to add additional assumptions on the background potential q .

3.1. Preliminaries

Let ψ_{\pm} denote the solutions (10) for the Schrödinger equation and

$$\varphi(x, z) := \psi_+(x, z^2) \quad (x \in [0, \infty), z \in \mathbb{C}_+).$$

φ is commonly referred to as the *Jost solution*. For each $R > 0$, define function $f_R : \mathbb{C}_+ \rightarrow \mathbb{C}$ by

$$if_R(z)e^{izR} = \theta(R, z)\varphi'(R, z) - \theta'(R, z)\varphi(R, z) \quad (z \in \mathbb{C}_+).$$

where, for any $z \in \mathbb{C}$, $\theta(\cdot, z)$ is defined as the solution to the initial value problem

$$-\theta'' + q\theta = (z^2 - i\gamma)\theta, \theta(0) = 0, \theta'(0) = 1. \tag{32}$$

By the same arguments as in Lemma 1, we have the following.

Lemma 5. f_R is analytic on \mathbb{C}_+ and any $z \in \mathbb{C}_+$ satisfies

$$f_R(z) = 0 \iff z^2 \text{ is an eigenvalue of } H_R. \tag{33}$$

φ can be decomposed in a similar way to ψ_{\pm} ,

$$\begin{aligned} \varphi(x, z) &= e^{izx}(1 + E(x, z)) \\ \varphi'(x, z) &= iz e^{izx}(1 + E^d(x, z)) \end{aligned} \quad (x \in [0, \infty), z \in \mathbb{C}_+) \tag{34}$$

for some functions E and E^d whose properties will be later specified, for the different assumptions on the background potential q that we consider. We shall need the following facts concerning f_R and θ . Note that in Lemma 6, \mathcal{E}_1 and \mathcal{E}_2 are defined in a different way than in Lemma 1.

Lemma 6. Suppose that, for each $R > 0$, $\varphi(R, \cdot)$ and $\varphi'(R, \cdot)$ admits an analytic continuation from \mathbb{C}_+ into some open $U \subset \mathbb{C}$. Then, f_R admits analytic continuation into U . Furthermore, for each $R > 0$ and $z \in U \setminus \{\pm\sqrt{i\gamma}\}$,

$$\begin{aligned} f_R(z)u(z) &= \psi_-(0, z^2 - i\gamma) \left(z - \sqrt{z^2 - i\gamma} + \mathcal{E}_1(R, z) \right) e^{i\sqrt{z^2 - i\gamma}R} \\ &\quad - \psi_+(0, z^2 - i\gamma) \left(z + \sqrt{z^2 - i\gamma} + \mathcal{E}_2(R, z) \right) e^{-i\sqrt{z^2 - i\gamma}R} \end{aligned} \tag{35}$$

where

$$\begin{aligned} u(z) &:= \psi_-(0, z^2 - i\gamma)\psi'_+(0, z^2 - i\gamma) - \psi_+(0, z^2 - i\gamma)\psi'_-(0, z^2 - i\gamma), \tag{36} \\ \mathcal{E}_1(R, z) &:= z(E_+(R, z^2 - i\gamma) + E^d(R, z) + E_+(R, z^2 - i\gamma)E^d(R, z)) \\ &\quad - \sqrt{z^2 - i\gamma}(E_+^d(R, z^2 - i\gamma) + E(R, z) + E_+^d(R, z^2 - i\gamma)E(R, z)) \end{aligned} \tag{37}$$

and

$$\begin{aligned} \mathcal{E}_2(R, z) &:= z(E^d(R, z) + E_-(R, z^2 - i\gamma) + E^d(R, z)E_-(R, z^2 - i\gamma)) \\ &\quad + \sqrt{z^2 - i\gamma}(E(R, z) + E_-^d(R, z^2 - i\gamma) + E(R, z)E_-^d(R, z^2 - i\gamma)). \end{aligned} \tag{38}$$

Proof. Analytic continuation holds by the fact that $\theta(R, \cdot)$ is entire [35, Lemma 5.7] for each $R > 0$. If $z \neq \pm\sqrt{i\gamma}$ then the functions $\psi_{\pm}(\cdot, z^2 - i\gamma)$ span the solution space of the Schrödinger equation $-\psi'' + q\psi = (z^2 - i\gamma)\psi$ so

$$\begin{aligned} \theta(R, z) &= \frac{\psi_-(0, z^2 - i\gamma)\psi_+(R, z^2 - i\gamma) - \psi_+(0, z^2 - i\gamma)\psi_-(R, z^2 - i\gamma)}{\psi_-(0, z^2 - i\gamma)\psi'_+(0, z^2 - i\gamma) - \psi_+(0, z^2 - i\gamma)\psi'_-(0, z^2 - i\gamma)} \\ &= \frac{\psi_1(R, z^2)}{u(z)} \end{aligned}$$

where ψ_1 denotes the function defined by (17) in Lemma 1. The lemma follows by a direct computation, similar to one in Lemma 1. \square

Lemma 7. *For any $x \in [0, \infty)$ and $z \in \mathbb{C} \setminus \{\pm i\gamma\}$, the solution θ to the initial value problem (32) satisfies the inequality*

$$|\theta(x, z)| + |\theta'(x, z)| \leq (1 + x)e^{|\Im\sqrt{z^2 - i\gamma}|x} \exp\left(\int_0^x (1 + t)|q(t)| dt\right).$$

Proof. Let $\mu = \mu(z) := \sqrt{z^2 - i\gamma}$. θ and θ' satisfy the integral equations

$$\theta(x, z) = \frac{\sin(\mu x)}{\mu} + \int_0^x \frac{\sin(\mu(x - t))}{\mu} q(t)\theta(t, z) dt$$

and

$$\theta'(x, z) = \cos(\mu x) + \int_0^x \cos(\mu(x - t))q(t)\theta(t, z) dt$$

hence satisfy the integral inequality

$$|\theta(x, z)| + |\theta'(x, z)| \leq (1 + x)e^{|\Im\mu|x} [*] 1 + \int_0^x e^{-|\Im\mu|t} |q(t)| (|\theta(t, z)| + |\theta'(t, z)|) dt$$

where we used the fact that $|\sin(\mu x)| |\mu|^{-1} \leq xe^{|\Im\mu|x}$ and $|\cos(\mu x)| \leq e^{|\Im\mu|x}$. The result follows from an application of Grönwall’s Lemma. \square

3.2. Compactly Supported Potentials

Assumption 1. q is compactly supported, that is, there exists $Q > 0$ such that

$$\text{supp } q \subset [0, Q].$$

If Assumption 1 holds and then the Jost solution φ satisfies

$$\varphi(R, z) = e^{izR} \quad (R > Q, z \in \mathbb{C}) \tag{39}$$

hence, for each $x \in [0, \infty)$, $\varphi(x, \cdot)$ can be analytically continued to \mathbb{C} . Consequently, for $R > Q$, f_R can be analytically continued to \mathbb{C} and can be written as

$$f_R(z) = z\theta(R, z) + i\theta'(R, z) \quad (z \in \mathbb{C}). \tag{40}$$

Theorem 8. *Suppose that Assumption 1 holds. Then there exists $R_0 = R_0(q, \gamma) > 0$ such that for every $R \geq R_0$,*

$$N(H_R) \leq \frac{11}{\log 2} \frac{\gamma R^2}{\log R}. \tag{41}$$

Proof. Let $z_0 \in \mathbb{C}_+$ be such that

$$\psi_+(0, z_0^2 - i\gamma) \neq 0, \quad \Im\sqrt{z_0^2 - i\gamma} \geq 1 \quad \text{and} \quad \sqrt{|z_0^2 - i\gamma|} \leq 2. \quad (42)$$

In fact, by choosing z_0 to be the minimiser of some suitable total order on \mathbb{C} in the set of points that maximise $z \mapsto \psi_+(0, z^2 - i\gamma)$ while satisfying the latter two inequalities of (42), z_0 can be determined uniquely by q and γ , $z_0 = z_0(q, \gamma)$. Define $r = r(R) > 0$ by

$$\frac{r}{2} = \gamma^{1/2} + |z_0| + \frac{5\gamma R}{\log R}. \quad (43)$$

By the triangle inequality,

$$|z - z_0| \leq |z| + |z_0| \leq \sqrt{|z^2 - i\gamma| + \gamma} + |z_0| \leq \sqrt{|z^2 - i\gamma|} + \gamma^{1/2} + |z_0| \quad (44)$$

so,

$$S_R := \{*\}z \in \mathbb{C}_+ : \sqrt{|z^2 - i\gamma|} \leq \frac{5\gamma R}{\log R} \subseteq \overline{B}_{r/2}(z_0). \quad (45)$$

Let $R > Q > 0$ be large enough so that estimate (23) of Theorem 4(b) holds. Since the zeros of f_R in \mathbb{C}_+ have a bijective correspondence with the eigenvalues of H_R , the set S_R contains all the zeros of f_R in \mathbb{C}_+ and hence the number of eigenvalues of H_R is bounded by the number of zeros for f_R in the ball $\overline{B}_{r/2}(z_0)$,

$$N(H_R) \leq |f_R^{-1}\{0\} \cap \overline{B}_{r/2}(z_0)|. \quad (46)$$

Since f_R is entire, Jensen’s formula gives us

$$|f_R^{-1}\{0\} \cap \overline{B}_{r/2}(z_0)| \leq \frac{1}{\log 2} \log \left| \frac{1}{f_R(z_0)} \sup_{|z-z_0|=r} |f_R(z)| \right|. \quad (47)$$

Since $R > Q$, the terms $\mathcal{E}_1(R, z)$ and $\mathcal{E}_2(R, z)$, defined by (37) and (38) respectively, vanish. Hence, by Lemma 6 and the fact that $\Im\sqrt{z_0^2 - i\gamma} \geq 1$,

$$\begin{aligned} |f_R(z_0)u(z_0)| &\geq |\psi_+(0, z_0^2 - i\gamma)| \left(z_0 + \sqrt{z_0^2 - i\gamma} \right) |e^R \\ &\quad - |\psi_-(0, z_0^2 - i\gamma)(z_0 - \sqrt{z_0^2 - i\gamma})| |e^{-R}. \end{aligned}$$

Note that $\Im\sqrt{z_0^2 - i\gamma} \geq 1$ implies that $z_0 \neq \pm\sqrt{i\gamma}$ so Lemma 6 is indeed applicable here. Then, since $\psi_+(0, z_0^2 - i\gamma) \neq 0$, $\sqrt{|z_0^2 - i\gamma|} \leq 2$ and $z_0 = z_0(q, \gamma)$,

$$|f_R(z_0)| \geq C(q, \gamma) \quad (48)$$

for large enough R .

By expression (40) for f_R and the estimates in Lemma 7 for θ and θ' , for all $z \in \partial B_r(z_0)$,

$$|f_R(z)| \leq C(q)(1 + R)(1 + |z|)e^{|\sqrt{z^2 - i\gamma}|R}. \quad (49)$$

Furthermore, by the triangle inequality and expression (43) for r , for all $z \in \partial B_r(z_0)$,

$$\sqrt{|z^2 - i\gamma|} \leq |z - z_0| + |z_0| + \gamma^{1/2} = 3\gamma^{1/2} + 3|z_0| + \frac{10\gamma R}{\log R}. \tag{50}$$

Noting that for $z \in \partial B_r(z_0)$, the factor $(1 + |z|)$ in (49) is $o(R)$, combining (46)–(50) gives us

$$N(H_R) \leq \frac{1}{\log 2} \left(\log o(R^2) + (3\gamma^{1/2} + 3|z_0|)R + \frac{10\gamma R^2}{\log R} \right)$$

as $R \rightarrow \infty$. Estimate (41) follows. □

3.3. Exponentially Decaying Potentials

Assumption 2. (Naimark Condition) There exists $a > 0$ such that

$$\int_0^\infty e^{4at}|q(t)| dt < \infty.$$

If Assumption 2 is satisfied then for each $x > 0$ the functions $\varphi(x, \cdot)$ and $\varphi'(x, \cdot)$ admit analytic continuations from \mathbb{C}_+ into $\{\Im z > -2a\}$. For each $x > 0$, the functions E and E^d appearing in the decomposition (34) of the Jost solution φ satisfy

$$|E(x, z)| + |E^d(x, z)| \leq C(q) \quad \text{if } \Im z \geq -a \tag{51}$$

and

$$|E(x, z)| + |E^d(x, z)| \leq \frac{C(q)}{|z|} \quad \text{if } \Im z \geq -a \text{ and } |z| \geq 1. \tag{52}$$

See [25, Theorem 2.6.1] and [32, Lemma 1] for proofs of the above claims.

The next proposition allows us to utilise the uniform enclosure of Theorem 4(a) in the estimation of the number of eigenvalues of H_R .

Proposition 9. *Suppose that f is an analytic function defined on an open neighbourhood of the closed semi-disc $D_r := \overline{B}_r(0) \cap \overline{\mathbb{C}}_+$ for some $r > 0$. Let α and β be any numbers in the interval $(0, 1)$ satisfying*

$$\beta \left(\frac{1 - \alpha}{\alpha + \beta} \right)^2 > \frac{Y}{\eta} \tag{53}$$

and let $N(\alpha r)$ denote the number of zeros in the region

$$D_{\alpha r, \eta, Y} := \{z \in \mathbb{C} : \eta \leq \Im z \leq Y, |z| \leq \alpha r\} \tag{54}$$

where $Y, \eta > 0$ are given parameters satisfying $\eta < Y < r$. Then,

$$N(\alpha r) \leq \frac{2}{\log \Lambda(r)} \log \left(\frac{1}{\min\{\beta, 1 - \beta\}} \frac{\sup_{z \in \partial D_r} |f(z)|}{|f(i\beta r)|} \right) \tag{55}$$

where

$$\Lambda(r) := \frac{1 + \frac{4\beta\eta}{(\alpha + \beta)^2} \frac{1}{r}}{1 + \frac{4Y}{(1 - \alpha)^2} \frac{1}{r}}. \tag{56}$$

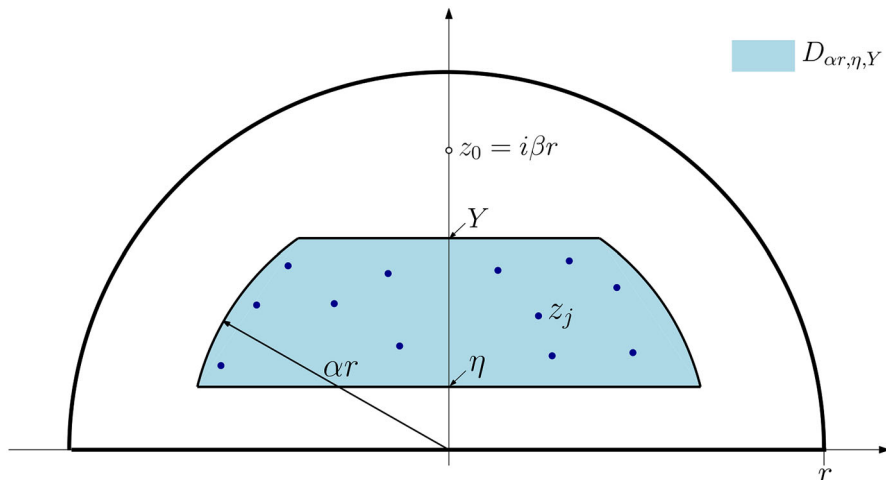


FIGURE 2. Illustration for the setup of Proposition 9

Remark 2. One can always guarantee that condition (53) for α and β is satisfied by choosing, for instance,

$$\alpha = \beta = \frac{1}{4} \frac{\eta}{2Y + \eta}. \tag{57}$$

Proof of Proposition 9. Let $\{z_j\}_{j=1}^{N(\alpha r)}$ denote the set of zeros of f in the set $D_{\alpha r, \eta, Y}$ and consider the Blaschke product

$$b(z) := \prod_j \frac{z - \bar{z}_j}{z - z_j} \equiv \prod_j b_j(z).$$

Note that higher multiplicity zeros of f are repeated in the set $\{z_j\}$ accordingly. Let $z_0 := i\beta r$. The function $f(z)b(z)$ is analytic on an open neighbourhood of D_r so by Cauchy’s formula,

$$\frac{1}{2\pi i} \oint_{\partial D_r} \frac{f(z)b(z)}{z - z_0} dz = f(z_0)b(z_0). \tag{58}$$

Observing that $|z - z_0| \geq \min\{\beta, 1 - \beta\}r$ for all $z \in \partial D_r$, it holds that

$$\frac{1}{2\pi} \oint_{\partial D_r} \frac{|dz|}{|z - z_0|} \leq \frac{1}{\min\{\beta, 1 - \beta\}}$$

which can be used to estimate the integral in (58) to get

$$\prod_j \frac{|b_j(z_0)|}{\sup_{z \in \partial D_r} |b_j(z)|} \leq \frac{\sup_{z \in \partial D_r} |f(z)|}{|f(z_0)|} \frac{1}{\min\{\beta, 1 - \beta\}}. \tag{59}$$

By a direct computation, we have

$$|b_j(z)| = \sqrt{1 + \frac{4\Im z \Im z_j}{|z - z_j|^2}}.$$

Since

$$\Im z_0 = \beta r, \quad \Im z_j \geq \eta, \quad |z_0 - z_j| \leq (\alpha + \beta)r,$$

giving us a lower bound for $|b_j(z_0)|$, and since for any $z \in \mathbb{C}$ with $|z| = r$

$$\Im z \leq r, \quad \Im z_j \leq Y, \quad |z - z_j| \geq (1 - \alpha)r,$$

giving us an upper bound for $|b_j(z)|$, we have

$$\frac{|b_j(z_0)|}{|b_j(z)|} \geq \Lambda(r)^{1/2} \tag{60}$$

for any $z \in \partial D_r$ with $|z| = r$. Furthermore, if $z \in \mathbb{R}$ then $|b_j(z)| = 1$ so (60) in fact holds for every $z \in \partial D_r$. Combining (60) with (59) gives us

$$\Lambda(r)^{N(\alpha r)/2} \leq \frac{1}{\min\{\beta, 1 - \beta\}} \frac{\sup_{z \in \partial D_r} |f(z)|}{|f(z_0)|}. \tag{61}$$

If hypothesis (53) for α and β holds then $\Lambda(r) > 1$ so we can take the logarithm of both sides of (61) and rearrange to obtain inequality (55). \square

Theorem 10. *Suppose that Assumption 2 holds. Then there exists $R_0 = R_0(q, \gamma) > 0$ such that for every $R \geq R_0$,*

$$N(H_R) \leq C \frac{\sqrt{X} + a}{a^2} \frac{\gamma^2 R^3}{(\log R)^2} \tag{62}$$

where $C = 88788$ and $X = X(q, \gamma) > 0$ is the constant appearing in Theorem 4(a).

Proof. Let $\tilde{f}_R(z) := f_R(z - ia)$ and let $\alpha, \beta > 0$ satisfy Eq. (57) of Remark 2 with $\eta = a$ and $Y = \sqrt{X} + a$ where $X = X(q, \gamma)$ is the constant appearing in Theorem 4. Then hypothesis (53) of Proposition 9 is satisfied. Note that with this choice of β we have $\beta < 1/2$, so,

$$\min\{\beta, 1 - \beta\} = \beta. \tag{63}$$

The zeros of \tilde{f}_R in $\{\Im z > a\}$ have a bijective correspondence to eigenvalues of H_R given by

$$(z - ia)^2 \in \sigma_d(H_R) \iff \Im z > a \text{ and } \tilde{f}_R(z) = 0. \tag{64}$$

Assuming without loss of generality that $X \geq \gamma$, the square root of the enclosure $B_X(0) \cup \Gamma_\gamma$ is contained in the strip $\{0 \leq \Im w \leq \sqrt{X}\} \subset \mathbb{C}$. Then by the uniform enclosure of Theorem 4(a), the zeros of \tilde{f}_R in $\{\Im z > a\}$ are contained in the strip $\{a \leq \Im z \leq \sqrt{X} + a\}$. By the triangle inequality and the magnitude bound of Theorem 4(b), any zero z of \tilde{f}_R with $\Im z > a$ satisfies

$$|z| \leq \gamma^{1/2} + a + \sqrt{|(z - ia)^2 - i\gamma|} \leq \alpha r \tag{65}$$

where $r = r(R)$ is defined by

$$\alpha r = \gamma^{1/2} + a + \frac{5\gamma R}{\log R}. \tag{66}$$

Hence the zeros of \tilde{f}_R in $\{\Im z > a\}$ are contained in $D_{\alpha r, \eta, Y}$.

Applying Proposition 9, we get an estimate for the number of eigenvalues of H_R ,

$$N(H_R) = |\tilde{f}_R^{-1}\{0\} \cap D_{\alpha r, \eta, \mathcal{Y}}| \leq \frac{2}{\log \Lambda(r)} \log \left(\frac{1}{\beta} \frac{\sup_{z \in \partial D_r} |\tilde{f}_R(z)|}{|\tilde{f}_R(i\beta r)|} \right) \tag{67}$$

where

$$\Lambda(r) = \frac{1 + C_1/r}{1 + C_2/r} \tag{68}$$

for some constants $C_1 > C_2 > 0$ depending only on X and a . The remainder of the proof consists in estimating the right hand side of (67).

Let $z_R := i\beta r(R) - ia$. By Lemma 6,

$$\begin{aligned} |f_R(z_R)u(z_R)| &\geq |\psi_+(0, z_R^2 - i\gamma)(z_R + \sqrt{z_R^2 - i\gamma + \mathcal{E}_2(R, z_R)})| \\ &\quad - |\psi_-(0, z_R^2 - i\gamma)(z_R - \sqrt{z_R^2 - i\gamma + \mathcal{E}_1(R, z_R)})| \end{aligned} \tag{69}$$

for large enough R . By estimates (12) for E_\pm and E_\pm^d , and the corresponding estimates (52) for E and E^d ,

$$|u(z_R)| + |\psi_-(0, z_R^2 - i\gamma)| + |\mathcal{E}_1(R, z_R)| + |\mathcal{E}_2(R, z_R)| \leq C(q, \gamma) \tag{70}$$

and

$$|\psi_+(0, z_R^2 - i\gamma)| \geq C(q, \gamma) \tag{71}$$

for large enough R . By Lemma 3,

$$\lim_{R \rightarrow \infty} |z_R + \sqrt{z_R^2 - i\gamma}| = \infty \quad \text{and} \quad \lim_{R \rightarrow \infty} |z_R - \sqrt{z_R^2 - i\gamma}| = 0. \tag{72}$$

Combining (69) with (70), (71) and (72) gives us

$$|\tilde{f}_R(i\beta r)| = |f_R(z_R)| \geq 1 \tag{73}$$

for large enough R .

The factor involving $\Lambda(r)$ on the right hand side of (67) can be estimated using the expression (68) for Λ and the inequality $\log x \geq (x - 1)/(x + 1)$ ($x \geq 1$),

$$\log \Lambda(r) \geq \frac{\Lambda(r) - 1}{\Lambda(r) + 1} = \frac{(C_1 - C_2)/r(R)}{2 + (C_1 + C_2)/r(R)} \geq \frac{C_1 - C_2}{3r(R)} \tag{74}$$

for large enough R .

The function \tilde{f}_R is estimated from above using the bound in Lemma 7 for θ and θ' and the uniform bounds (51) for $E(R, \cdot)$ and $E^d(R, \cdot)$,

$$|\tilde{f}_R(z)| \leq C(q)(1 + R)(1 + |z|)e^{aR}e^{|\sqrt{(z-ia)^2 - i\gamma}|R} \quad (z \in \overline{\mathbb{C}}_+). \tag{75}$$

Using the expression (66) for r , for any $z \in \partial D_r$ we have

$$\sqrt{|(z - ia)^2 - i\gamma|} \leq \gamma^{1/2} + a + |z| \leq O(1) + \frac{5\gamma R}{\alpha \log R} \tag{76}$$

as $R \rightarrow \infty$. Combining (67) with (73), (74), (75) and (76), noting that $|z| = o(R)$ for $z \in \partial D_R$ and $\beta^{-1} = O(1)$, gives

$$N(H_R) \leq \frac{6}{C_1 - C_2} \left(O(1) + \frac{5\gamma R}{\alpha \log R} \right) \left(O(R) + \frac{5\gamma R^2}{\alpha \log R} \right)$$

as $R \rightarrow \infty$ and so

$$N(H_R) \leq \frac{151\gamma^2 R^3}{(C_1 - C_2)\alpha^2(\log R)^2} \tag{77}$$

for large enough R .

Finally, we put the constant into a more illuminating form. By the definition (56) of Λ in Proposition 9,

$$C_1 = \frac{\eta}{\alpha} \quad \text{and} \quad C_2 = \frac{4Y}{(1 - \alpha)^2}. \tag{78}$$

Since $\frac{\eta}{12Y} \leq \alpha \leq \frac{\eta}{8Y}$, we have

$$(C_1 - C_2)\alpha^2 = \eta\alpha - \frac{4Y\alpha^2}{(1 - \alpha)^2} \geq \frac{\eta^2}{12Y} - \frac{4Y\alpha^2}{(1 - \frac{\eta}{8Y})^2} \tag{79}$$

and since $0 \leq \eta/Y \leq 1$, we have

$$\frac{\alpha^2}{(1 - \frac{\eta}{8Y})^2} = \frac{\eta^2}{64Y^2} \frac{1}{(1 + \frac{\eta}{2Y})^2(1 - \frac{\eta}{8Y})^2} \leq \frac{\eta^2}{49Y^2} \tag{80}$$

Combining (79) and (80), we have

$$(C_1 - C_2)\alpha^2 \geq \frac{1}{588} \frac{\eta^2}{Y}. \tag{81}$$

which gives estimate (62) when substituted into (77), with $Y = \sqrt{X} + a$ and $\eta = a$. □

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