Sir Vaughan Jones
31 December 1952 – 6 September 2020
Elected FRS 1990

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Abstract

Vaughan Jones made fundamental contributions to mathematics and mathematical physics bringing together disparate areas of operator algebras, knots, links and low dimensional topology in mathematics, and statistical mechanics, quantum field theory and quantum information in physics, whilst opening up the new field of quantum topology. The key which unlocked all this was his seminal work on subfactor theory in von Neumann algebras of operators which led to a new invariant of links, the Jones polynomial. For this he was awarded the Fields Medal in 1990.

1 Family Background and Early Life

Vaughan Frederick Randal Jones was born on 31 January 1952 to parents James (Jim) Henry Jones and Joan Jones (née Collins) in New Zealand. Vaughan's paternal family has its roots in Wales. In October 1914, Freddie Jones, from Haverfordwest, in Pembrokeshire, enlisted at the age of 15 in the Welsh Regiment. He was shot through the chest on 9th August 1915, when coming ashore through barbed wire in the sea at Suvla Bay just north of ANZAC Cove in Gallipoli. He forever thanked a New Zealand barber, Lewis, for saving his life. After being invalided back to Wales, Freddie worked as a police constable in Pembre Munitions Factory in Carmarthenshire near the estuary of the Gwendraeth Valley and was later formally discharged from the army in October 1916. In 1919, he married Elizabeth Alice (Bessie) Butler whose Welsh speaking family were butchers in nearby Burry Port. Freddie's health was a cause of concern as a result of losing the use of one lung in Gallipoli. The medical advice was that he should move to a warmer climate or he would not live long. So on Christmas Eve 1926, Freddie sailed to New Zealand and obtained employment with the Traffic Police in Auckland. He was joined in 1927 by Bessie and their three sons Jim, Eric and Ronald. Due to Freddie's frail health, Bessie often alone maintained the family through teaching at Waiau Pu School and rearing poultry.

During the Second World War their eldest son Jim served in both the Army and the Air Force and subsequently had a career as a Sales Representative. He married Joan, an

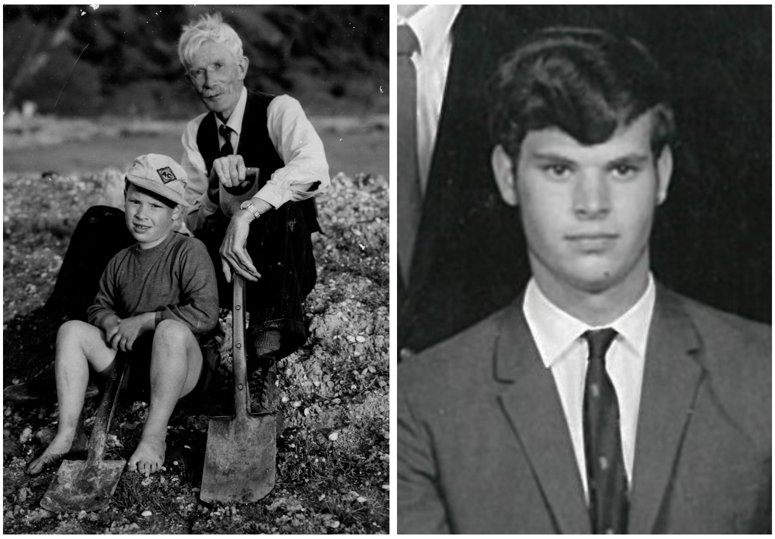

Figure 1: With family friend A.J. Reid & Auckland Grammar University Scholar 1969

accountant's assistant whose family were long established in New Zealand, in December 1945. They had two children – Tessa born 1949 in Auckland, and Vaughan born 1952 in Gisborne on the East Coast of North Island. The family moved back to Auckland when Vaughan was 1, where his school education began at the age of 5 at Ponsonby School. When his parents separated and divorced his primary education was completed between the ages of 8 and 12 at St Peter's boarding school in Cambridge in rural North Island, about 50 miles south of Auckland. He credited St Peter's as inspiring him and giving him a firm foundation in music, sport and academic studies. Indeed he proudly wore the school tie when he received the Fields medal in 1990 in Kyoto and the Rutherford Medal in 1993 from the New Zealand Government at Old Government House, Auckland. After St Peter's, Vaughan attended Auckland Grammar School until the age of sixteen from 1966 to 1969. He was in the A-stream after the entrance examination and was usually the top student in each subject. Vaughan enjoyed playing cricket and rugby, although later at University he did much better at rugby after trimming down.

2 University Education

At the age of 17 Vaughan entered Auckland University as an Auckland Grammar School Scholar and Gillies Scholar where he studied from February 1970 to December 1973, obtaining the degrees of BSc in Mathematics and Physics in 1972 and MSc in Mathematics in 1973. Amongst the lecturers who inspired him was Michael Lennon who brought von Neumann algebras to his attention particularly, directing him to read Naimark's book Normed Algebras which has a chapter on von Neumann algebras. It was during this period that a group of friends, *the boys*, emerged with Vaughan as one of the ringleaders. This friendship endured with a strong bond of loyalty during their subsequent careers worldwide. The boys included Mark Warner (FRS 2012) who left early from Auckland University to Cambridge and became a theoretical physicist at the Cavendish. They would interact socially, usually with a walk on Midsummer's Day in June. Vaughan would recently try to get Mark to test his theories, related to his Thompson group models which predicted behaviour for correlations different from conformal field theory (CFT), in the laboratory. Another of the boys was Peter Brothers who was Chief Executive at the Manukau Institute of Technology in Auckland when Vaughan took a barista course to hone his coffee-brewing skills in 2014.

After graduation Vaughan remained at the University of Auckland for half a year in early 1974 as a Lecturer teaching calculus. Unlike when at Auckland Grammar, Vaughan did not have the top grades in all his University courses, particularly in the final year, and so was left behind when scholarships were awarded for graduate study overseas. Paul Hafner, another of Vaughan's lecturers and mentors at Auckland University, recommended that he apply for a Swiss Government grant, Bundesstipendium. One was available for New Zealand students, which had been normally awarded to German language students due to a lack of science applicants. Vaughan was also aware of the monograph of the Geneva-based theoretical physicist Josef-Maria Jauch on the foundations of quantum mechanics, which interested him. He then made a successful application for a scholarship, and left New Zealand in 1974 for graduate study, funded by the Swiss Government Scholarship and a F.W.W. Rhodes Memorial Scholarship, at the University of Geneva. He initially spent three months in Fribourg improving his French. Vaughan's intention was to write a thesis in Physics with Jauch. Unfortunately, Jauch died a week after Vaughan arrived and they met only once. Vaughan remained at the Physics Department, working under the supervision of a former student of Jauch, Constantin Piron, and Jean-Pierre Eckmann. However he gradually moved in 1974–76 to work formally under the supervision of André Haefliger in Mathematics. Vaughan acted as a teaching assistant to Haefliger and followed courses on foliations and de Rham theory. In the autumn of 1975 at a conference in Strasbourg, he met Alain Connes. Vaughan was enthralled and under the informal mentorship of Connes, based at IHES in Paris, he wrote a doctoral thesis on the classification of actions of finite groups on the hyperfinite II_1 factor. Vaughan was awarded the Vacheron Constantin Prize for his thesis that became his favourite watch.

It was in Switzerland during his graduate studies that Vaughan met his wife Martha (Wendy) Myers, a US citizen. Wendy held a scholarship to study at the University of Fribourg in Switzerland having applied to this programme through the Institute for International Education at the United Nations. They met in 1978 at a ski camp in Engelberg that was funded by the Confédération for scholarship students (boursiers de la Confédération), which included both Vaughan and Wendy. After her scholarship year at Fribourg ended in the summer of 1978, Wendy worked at the UN (Economic Commission for Europe) in Geneva. Vaughan and Wendy were married in 1979 in Wendy's home town of Westfield, New Jersey, USA and raised three children together, Bethany Martha, Ian Randal and Alice Collins. Their 40^{th} anniversary in 2019 was celebrated with family and friends in Geneva. Wendy studied for a two-year Master in Public Affairs at Princeton from 1980, and afterwards worked at the NY Federal Reserve Bank. Vaughan spent the first year at UCLA and then moved to UPenn to join Wendy, living at Metuchen NJ midway between New York and Philadelphia. He rode from UCLA to the East Coast on a newly acquired Kawasaki 1000 motorcycle having never owned or ridden a bike before. Wendy later obtained a PhD at Berkeley in Economics in 1996, taught at various universities in the San Francisco Bay area, Geneva and Lausanne, Switzerland, worked for the State of California and was appointed Associate Professor of Medicine, Health and Society at Vanderbilt in

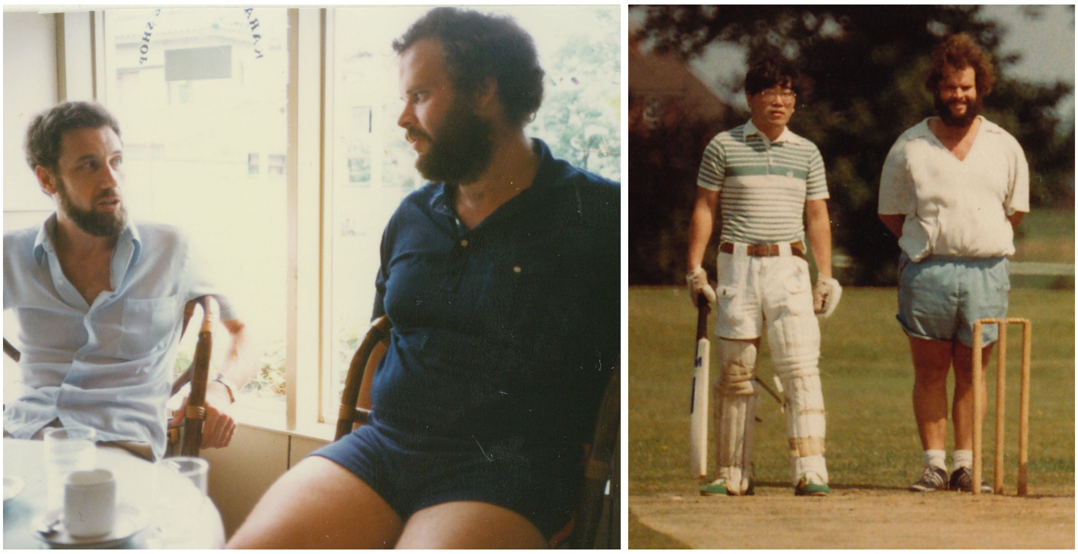

Figure 2: With Alain Connes at US-Japan Seminar on *Geometric Methods in Operator Algebras* Kyoto 1983 & with Masamichi Takesaki during Warwick Symposium 1980–81

2012 a year after Vaughan.

At their wedding, Vaughan and Wendy played a Mozart trio on the violin and flute respectively, along with flautist (and Geneva mathematician) Jean-Claude Hausmann. Music played an important part of their family and social life. As an accomplished baritone, Vaughan sang with Wendy at Geneva in the University choir. He enjoyed playing chamber music with his advisor André Haefliger and with the University Orchestra's conductor Chen Liang-Sheng. Vaughan and Wendy continued to sing in choirs, together at the Université Paris-Sud (when at the IHES) and then Vaughan alone with the Oakland Symphony Chorus in CA. They both played in the UC Berkeley Mathematics Orchestra every year at graduation ceremonies. Their enthusiasm and love of music was transmitted to their children. Ian is now a professional musician as Assistant Principal Cello at Tucson Symphony Orchestra.

Sport and mathematics were often intertwined and to be enjoyed and explored together. From Berkeley, Vaughan arranged workshops combined with skiing at Lake Tahoe for his graduate students and occasional visitors. The summer schools that he directed in New Zealand since 1994 were usually arranged to be on the coast in remote locations all over the islands such as Huia, Kaikōura, Raglan, Taipa, Waitangi. These spots enabled him to engage in his passion for wind surfing which later evolved to kite-boarding. The regular meetings in Maui, Hawaii were also run as informal events with free late afternoons for water sports or hiking. The family retreat at Bodega Bay had a welcome for students and colleagues as an informal meeting place for mathematics to be discussed and enjoyed together. Scientific visits were often aligned with rugby events. Vaughan kept in touch with his cousins and rugby in Wales during his frequent visits there. Squash and tennis were other competitive passions shared with family, friends and colleagues. Golf another as on journeys through New Zealand taking in a few courses on the way from Auckland to a workshop in a remote part of New Zealand. What became an institution were the beer and pizza sessions for informal mathematical discussions after the subfactor seminar on Fridays, beginning in Berkeley and continuing in Vanderbilt.

3 Academic Career

On the basis of his thesis, which was a stunning piece of work, Vaughan made a successful application for an E.R. Hedrick Assistant Professorship at UCLA to start in the Autumn of 1980. He moved after a year to be Assistant Professor 1981–84, and then Associate Professor 1984–85 at the University of Pennsylvania. Vaughan spent the last year of his appointment at Penn at the recently created Mathematical Sciences Research Institute (MSRI) at Berkeley. After which he was from the age of 32, a Professor at UC Berkeley for 28 years. Vaughan was from 2011 the Stevenson Distinguished Professor of Mathematics at Vanderbilt University, Nashville, Tennessee until his death. From 1992 he also had a part-time appointment as a Distinguished Alumnus Professor at the University of Auckland. Vaughan kept in contact with Europe, spending sabbaticals at the IHES during 1986–87 and 1989–90 and at Geneva in 1993–94 and 1998–99.

Vaughan had a strong commitment of service to the community. He was a member of the Executive Committee of the International Association of Mathematical Physics 1991–94, Vice-President of the American Mathematical Society 2004–06, and Vice-President of the International Mathematical Union 2014-18. He had a long term commitment through time, energy and personal funding to nurture mathematics in New Zealand, in particular through the annual summer schools and workshops that he initiated where he spent every January since 1994 as Director of the New Zealand Mathematics Research Institute NZMRI. Vaughan's reputation and contacts could bring to New Zealand other outstanding world leaders covering a broad range of topics improving networking links of the community, raising the standards of mathematics in his home country. The Jones medal of the Royal Society of New Zealand Te Apārangi is named in his honour.

The creation and flourishing of research institutes in the latter part of the 20th century with focused programmes running over many months was completely suited to Vaughan's style of research with his openness and generosity of sharing ideas. The informal discussions in the common rooms of Warwick Mathematics Research Centre (MRC), MSRI Berkeley, Isaac Newton Institute (INI) Cambridge and Hausdorff Institute Bonn were instrumental in shaping the field and future directions for many young and established researchers alike as well as for Vaughan to thrive on this social interaction.

Vaughan made many visits to programmes at the Warwick MRC starting with the programme on *Foliations* in 1979 accompanying Haefliger when he had just completed his thesis. This was followed in the summer of 1981 with *von Neumann algebras and Ergodic Theory* when he gave an early glimpse of his developing index theory and partial results, and *Operator Algebras and Applications* in 1987. MSRI had opened in 1982 at Berkeley. It was fortuitous that MSRI was running two simultaneous programmes during 1984–85 on *K-Theory, Index Theory, and Operator Algebras* and *Low Dimensional Topology*. Both areas were thriving and covering wide topics so were ideal for MSRI. At the time of planning the programme, the unexpected bridge provided by Vaughan between the two areas through his polynomial link invariant was not known. MSRI provided a home for the two programmes to cross-pollinate ideas, problems and avenues of research with Vaughan at the centre of this unpredicted interaction. This was Vaughan's first year in the City of Berkeley, on leave from UPenn, where he remained until 2011. During this time he himself was an organiser of the MSRI programme on *Operator Algebras* during 2000–01. The INI opened in July 1992 with one of the inaugural programmes on *Low*

Dimensional Topology and Quantum Field Theory. Just as at MSRI earlier, Vaughan was instrumental in linking up the disparate areas. He gave the first lecture in an event to open the Institute on *Knots*, during which he blew smoke rings from a cigar to reproduce Tait's experiments from the 19^{th} century. This alarmed the organisers, potentially setting off the alarm system and precipitating the evacuation of the building. After the dinner that evening at Trinity College hosted by the Master and Founding INI Director Sir Michael Atiyah (FRS 1962), Vaughan tested the acoustics in the dining hall with his fine baritone voice. Vaughan was an organiser of an INI programme on *Operator Algebras: Subfactors and their Applications* in 2017. He took this responsibility as always seriously. In 2020, he was again an organiser for *Quantum Symmetries* at MSRI Berkeley. When this programme was curtailed in mid March and MSRI closed due to the pandemic he isolated at Bodega Bay – but still took part in online Zoom seminars with his usual gusto. His final public seminar was in the online Harvard Picture Language Seminar on 21^{st} July 2020 on his most recent work (22) on von Neumann algebras, Hecke groups and modular forms. Vaughan returned to Nashville in August and died there on 6^{th} September.

4 Classical Symmetries of the Hyperfinite II_1 factor

During the initial years 1974–76 of his graduate studies, Vaughan moved from Physics to the Mathematics Department at Geneva. He became fascinated with von Neumann algebras and came under the influence of Alain Connes who had recently completed profound work on the hyperfinite factors (Connes (1976)). A von Neumann unital algebra of operators on a separable Hilbert space is closed under involution and the weak operator topology. They were introduced by von Neumann as a tool for studying group representations and as a mathematical framework for quantum mechanics. Commutative von Neumann algebras are measure theoretic such as the diffuse $\mathrm{L}^\infty[0, 1]$. The factors are those with trivial center or equivalently those von Neumann algebras whose non-zero representations are faithful and so are highly non-commutative. Murray and von Neumann had divided algebras into the following types. Type I factors are matrix algebras or all bounded linear operators $B(H)$ on an infinite dimensional Hilbert space H, with integral dimension for projections. Type II_1 factors are those with a finite trace – a positive linear functional tr which vanishes on commutators, where the dimensions, the values of the trace on projections, range over the continuous interval $[0, 1]$. The II_∞ factors are of the form $M \otimes B(H)$, for a II_1 factor M, have semi-finite traces and dimensions $[0, \infty]$. The remaining factors are type III. Connes refined this classification with III_λ, $\lambda \in [0, 1]$, based on Tomita-Takesaki modular theory that dynamical systems naturally arise from states for which they describe equilibrium. The hyperfinite factors are those with matricial approximations. Murray and von Neumann had shown uniqueness of the hyperfinite II_1 factor R which can be regarded as the non-commutative or quantised interval $[0, 1]$. Connes showed the equivalence of several notions including hyperfiniteness, being the range of a projection from $B(H)$ and the vanishing of certain cohomology and called these amenable factors. Connes (1976) showed uniqueness of the hyperfinite or amenable II_∞ factor as $R \otimes B(H)$, uniqueness of the hyperfinite III_λ factors for $\lambda \in (0, 1)$, and characterised hyperfinite III_0 in terms of ergodic flows. The uniqueness of the hyperfinite III_1 factor was completed by Uffe Haagerup.

As Vaughan himself pointed out, Connes described that *this gives us a beautifully*

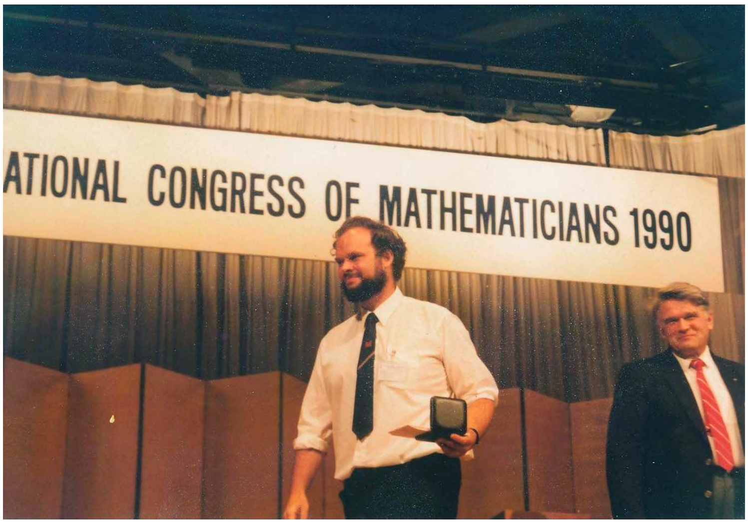

Figure 3: Receiving the Fields Medal & delivering plenary talk at ICM Kyoto 1990

bound blank book into which we can now enter many pages of mathematics. Over the course of the next four decades, Vaughan wrote many pages with elegant beauty, clarity of thought and enduring impact beyond von Neumann algebras and operator algebras. These pages brought together disparate areas of mathematics and mathematical physics. In the course of his work, Connes (1975) classified periodic automorphisms of the hyperfinite II_1 factor and also showed that the only infinite dimensional factor in R has to be isomorphic to R. This was the starting point for Vaughan. He brought novel revolutionary ideas of symmetries beyond groups and their actions.

In his thesis work at Geneva (1), Vaughan classified actions of a finite group G on the hyperfinite II_1 factor R, extending the work of Connes for cyclic actions. The complete invariant was cohomological which involved naturally the normal subgroup N where the group action was inner, with resulting 2-cocycle on N. Then the action of G on the implementing unitaries for N yields another scalar valued invariant called the characteristic invariant. Vaughan showed that the inner invariant and his characteristic invariant together formed a complete conjugacy invariant for actions of the finite group on the hyperfinite factor R. Ocneanu extended Vaughan's classification to actions of discrete amenable groups on any II_1 factor M with the property of Dusa McDuff (FRS 1994), i.e. when $M \simeq M \otimes R$. Vaughan showed in (2) non-uniqueness for actions of non-amenable groups on R. In later work with Masamichi Takesaki (4), they classified actions of compact abelian groups on hyperfinite type II factors. In ergodic theory, Vaughan and Klaus Schmidt (6) in a highly influential work on type II_1 ergodic equivalence relations showed that such a relation S is ergodic equivalent to a product relation $S \times R$ with R hyperfinite if and only if the full group of S contains non-trivial asymptotically central sequences.

5 Subfactors and Quantum Symmetries

Soon after completing his thesis in 1979, Vaughan returned to a problem he had already considered as a graduate student, that of the position or embedding of one II_1 factor N

in another M – what came to be known as *subfactor theory*. During 1980-81 he began, slowly at first, to unravel the mysterious rich structure of these inclusions.

He defined the index as the dimension of M as an N module and denoted this by $[M : N]$. For an outer action, of a finite group G with subgroup H on a II_1 factor N, the index of both subfactors $N^G \subset N^H$ and $N \rtimes H \subset N \rtimes G$ is the index $[G : H]$ and hence integral. Here $N \rtimes G$ denotes the crossed or semi-direct product and N^G the fixed point algebra. An outer group action on a hyperfinite factor can be recovered from the inclusion of the fixed-point algebra in the original factor as the automorphisms of the larger factor which fix the subfactor. A subfactor is then a generalisation of a group action or a quantum symmetry. Whilst dimension in a matrix algebra is discrete, the dimension in the hyperfinite factor R or indeed any type II_1 factor becomes continuous. It was therefore natural to expect that the index could take all values above 1. For a II_1 factor M, one can define matrices $\mathbb{M}_t(M)$ for any positive real t. Since $\mathbb{M}_t(R) \cong R$, Vaughan could swiftly realise any number above or at 4 as an index value for the hyperfinite factor. However, these subfactors are not irreducible.

Arriving in UCLA in the spring of 1980 on a preliminary visit from the East Coast before taking up the Hedrick Fellowship in the autumn, Vaughan told Takesaki in the car when he picked him up at the airport that there were forbidden index values. Takesaki was stunned and had to stop the car to encourage Vaughan to pursue this line as a matter of urgency. The only index values possible less than $1 + \sqrt{2}$ were 1 and 2. The expectation of the experts then was that the index values below 4 could only be $1, 2$ or 3.

It was after moving from UCLA to UPenn in 1981-82 that Vaughan made the breakthrough and proved that the index can only take discrete values below 4 at $4\cos^2(\pi/n)$ $= \{1, 2, (3 + \sqrt{5})/2, \cdots\}$ for integral $n \geq 3$ and moreover realised these values with irreducible subfactors of the hyperfinite factor. Vaughan analysed the structure of a finite index subfactor through an ingenious construction extending the subfactor to an infinite tower of subfactors of constant index:

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots \qquad [1]$$

First, taking the projection or conditional expectation $e = e_1$ of M onto N, the von Neumann algebra M_1 generated by M and e_1 is again a II_1 factor with the same index $[M_1 : M] = [M : N]$. Iterating this *basic construction* leads to the tower and a sequence of projections, i.e. self-adjoint idempotents, $\{e_1, e_2, e_3, \dots\}$, satisfying the relations

$$e_j e_k = e_k e_j, \text{ if } |j-k| \geq 2; \qquad e_j e_{j\pm1} e_j = \lambda e_j \qquad [2]$$

where $\lambda^{-1} = [M : N]$. Using these relations and the positive definiteness of the trace, Vaughan (3) obtained the restriction on the index values.

The notion of a subfactor was constituting a new kind of symmetry. This could be expressed, in the representation theory spirit of Tannaka duality, through the bimodule ${}_N M_M$. That bimodule together with its conjugate ${}_M M_N$ generate bimodules, of the four types N-N, N-M, M-N, M-M, through composition or fusion which then form this new symmetry object. The challenge then was to formulate symmetry invariants for the subfactors and to realise them or see what symmetries can act on a factor – whether hyperfinite or not.

The relative commutant algebras $A_{ij} = M_i' \cap M_j$ for $i \leq j$, where $'$ denotes the commutant, contain the algebras T_{ij} generated by $\{1, e_{i+2}, e_{i+3}, \dots, e_j\}$. Popa (1995) axiomatised the data in these coherent families of inclusions of algebras, together with the trace inherited from $\bigcup M_j$, to form the *standard invariant* $\mathcal{G}_{N\subset M}$ or a λ-lattice. For finite index subfactors, the A_{ij} are finite dimensional and so a sum of matrix algebras. The multiplicities of the embeddings of matrix algebras in one centraliser in the next define the principal graph and dual principal graph. There are only two graphs due to the inclusions in the tower [1] being of period 2, cf. Pontryagin duality for groups and hence group subfactors $N^G \subset N \subset N \rtimes G$. This stimulated discussions with Fred Goodman from 1982 at UPenn on, amongst other things, finite dimensional inclusions and their graphs and towers which eventually led to the monograph (9). Taking the A Coxeter-Dynkin diagram or graph and pairing with another Coxeter-Dynkin graph G of the same norm led to the Goodman-de la Harpe-Jones subfactor generated by embedding projections satisfying the relations [2] in a path algebra on G.

In the case of *rational* subfactors, when the number of irreducible modules is finite or equivalently the number of simple components in the relative commutants is bounded, the square of the norms of the adjacency graphs yields the index. Since graphs with norm less than 2 are known to be ADE this gives an alternative derivation of the restriction of the index values in the discrete range. Vaughan could show that for index less than 4 the subfactor is automatically rational and irreducible. The graphs themselves are not sufficient to classify the subfactor, even for rational subfactors.

A tool for constructing subfactors is through embeddings of finite dimensional algebras, with consistent traces

$$\begin{array}{ccccccccc} B_0 \subset & B_1 \subset & B_2 \subset & \cdots \subset & B_n \subset \cdots \\ \cup & \cup & \cup & & \cup \\ A_0 \subset & A_1 \subset & A_2 \subset & \cdots \subset & A_n \subset \cdots \end{array} \qquad [3]$$

For a rational hyperfinite subfactor, Popa showed that $A \subset B$, the completion of the unions, constructed from the standard invariant recovers the subfactor and that the standard invariant $\mathcal{G}_{N\subset M}$ is a complete invariant for rational subfactors of the hyperfinite factor. He also introduced a notion of amenable subfactor and showed that this is equivalent to the index being the same as the square of the norm of the standard graph. In particular, the graph A_∞ can only be the principal graph of an amenable subfactor for index precisely 4. Using a random or free probabilistic reconstruction method Popa could realise any standard invariant through a subfactor (Popa (1995)) and later Popa and Shlyakhtenko (2003) achieved this for a subfactor of the von Neumann algebra for the free group on an infinite number of generators.

There is combinatorial data in constructing a consistent sequence of squares as in [3] to form a subfactor in the limit, which Ocneanu (1988) introduced and called a *connection* on the graphs of multiplicities of embeddings of matrix algebras. This is essentially the $6j$-symbols for tensoring with a basic bimodule. If the connection satisfies an integrability condition, called flatness, then the principal graphs of the resulting subfactor are those of the graphs in [3]. By Weyl duality, the fixed point algebra of $\otimes^m \mathrm{End}(\mathbb{C}^n)$ under the product adjoint action of $SU(n)$ is generated by a representation of the symmetric group S_m on the tensor factors. Deforming this, representations of the Hecke algebra appear as the centraliser of representations of the quantum group $SU(n)_q$, which is a certain deformation of the group $SU(n)$. The algebra $\mathrm{Alg}\{1, e_1, e_2, e_3 \dots\}$ is then this representation of the Hecke algebra for $n = 2$ and so is intrinsically related to quantum $SU(2)$. Using further

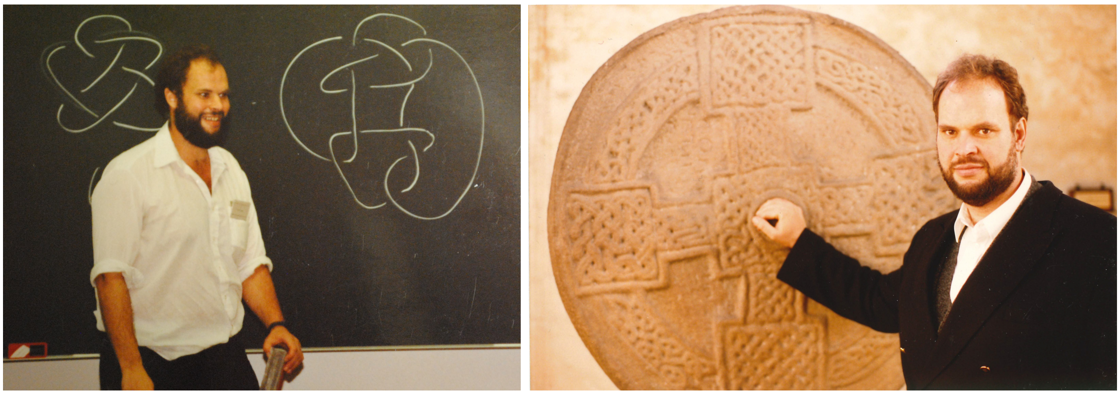

Figure 4: Newton Institute 1992 & with Cross of Conbelin, Margam Stones Museum 1993

representations of the Hecke algebras, Hans Wenzl, Vaughan's graduate student at UPenn, could construct analogous hyperfinite subfactors associated to $SU(n)$ at roots of unity. Feng Xu, Vaughan's graduate student at Berkeley, constructed standard invariants from quantum groups and so by Popa's construction this yielded subfactors.

What is fundamentally novel is that subfactor theory provides a framework for new integrable models beyond those arising from the Yang-Baxter Equation (YBE) in statistical mechanics or from quantum groups.

6 Planar Algebras and Classification

Vaughan found a novel formulation of encoding the subfactor data through his theory of two dimensional planar algebras. Through a variety of associated tools introduced by Vaughan, planar algebras became a significant tool in the classification of subfactors and their standard invariants. Classification involves realising or showing existence of some candidates as well as eliminating other potential candidates. The planar algebra framework subsequently had unexpected applications in mathematical physics and representation theory amongst other things.

The higher relative commutants in the tower [1] can be organised through Frobenius reciprocity to allow a multitude of operations beyond composition and tensor product. The data is built on discs having an outer boundary with marked points, and with holes or inner boundaries again which have marked points. This is the planar operad. The elements of the operad can be combined by inserting such punctured discs in the holes of another. The planar algebra over the planar operad is then a sequence of algebras $\mathcal{P}_n$ which can be multiplied in two dimensions. With a further notion of a binary shading in the diagrams, related to labelings from N or M and corresponding to the relative commutants $\mathcal{P}_{n;+} = M_0' \cap M_n$, $\mathcal{P}_{n;-} = M_1' \cap M_{n+1}$, $\mathcal{P}_{0;-\pm} \simeq \mathbb{C}$, and a diagrammatic positive trace one has the concept of a *subfactor planar algebra*. Conversely, using the construction theorem of Popa (1995), Vaughan could show that subfactor planar algebras give rise to subfactors.

Kauffman (1987) had found a diagrammatic description of the Temperley-Lieb algebra [2] which then has a natural home in the planar algebra formalism. Vaughan successfully analysed and exploited the structure of Temperley-Lieb annular planar algebras as modules over subfactor planar algebras (13) using rotational operators to unveil hidden data and employing the cellular theory of Graham and Lehrer (1998). Another tool introduced by Vaughan was *quadratic tangles* which in some cases can produce extremely strong constraints (14). A further technique to construct a subfactor planar algebra is embedding in a graph planar algebra, which he showed exists but is too large in low dimension to itself represent a subfactor (12). A generator-relation approach and Yang-Baxter relation algebras coming from integrability and ideas in statistical mechanics were also novel techniques in classifying and constructing standard invariants (e.g. (20)) introduced by Vaughan and Dietmar Bisch and developed by his student Zhengwei Liu in his Vanderbilt thesis.

Under index 4, there is an ADE classification of subfactors where A refers to the original discrete series constructed by Vaughan (3), D their orbifolds under the $\mathbb{Z}_2$ symmetry coming from the center of $SU(2)$ with E_7 and D_{odd} disallowed for various combinatorial reasons. At index 4 there is an affine ADE classification, corresponding to subgroups G of $SU(2)$ together with some group cohomological $H^3(G, \mathbb{T})$ data derived from multiplicity of actions as in Connes (1975) and the thesis of Vaughan (1). Between 4 and 5 the only non-rational principal graph possible is that of A_∞, and the first realisation was with a non-flat connection on E_{10} at index $(2.00659)^2 \simeq 4.02642$ constructed by Ocneanu, Haagerup and Schou. Indeed Haagerup (1994) was responsible for the classification between 4 and $(5 + \sqrt{13})/2 \simeq 4.30278$ with no rational subfactor before $(5 + \sqrt{13})/2$ and explicitly constructing a biunitary flat connection for a subfactor at that index. The Haagerup system is based on a quadratic extension of the $\mathbb{Z}_3$ fusion rules. Haagerup left open whether certain series of graphs with larger norms could be principal or dual principal graphs, which came under intense scrutiny in the years following. Asaeda and Haagerup (1998) realised the index value $(5 + \sqrt{17})/2 \simeq 4.56155$ by again finding a connection using a subfusion ring based on $\mathbb{Z}_3$. Vaughan's student Emily Peters, in her Berkeley PhD thesis, gave an alternative planar algebra construction of the existence of the Haagerup subfactor. The only other construction of the Haagerup subfactor is through Masaki Izumi finding solutions of the fusion rules (as endomorphisms which yield bimodules). This method is suitable for certain quadratic extensions of groups as in the Haagerup fusion rules $\rho^2 = 1 + \sum_{g \in \mathbb{Z}_3} g\rho$ and near groups $\rho^2 = (\sum_g g) + n\rho$, $g\rho = \rho$. The methods of Peters were elaborated (Bigelow et al (2012)) to construct the extended Haagerup subfactor at index value $8/3 + (2/3)Re \sqrt[3]{(13(-5 - 3i\sqrt{3}))} \simeq 4.37720$.

A culmination of more than 10 years of work 1995–2005 involving Vaughan and others, primarily his students, led to the classification of standard invariants up to index 5 which is summarised in (18) with planar algebras being a significant tool. Apart from possibly A_∞, between 4 and 5, there are exactly 10 standard invariants corresponding to the Haagerup subfactor, the Asaeda-Haagerup subfactor, the extended Haagerup subfactor, a GHJ subfactor at index $3 + \sqrt{3} \simeq 4.73205$ from the pair A_{11} and E_6 and Izumi-Xu at index $(5 + \sqrt{21})/2 \simeq 4.79129$ derived from a GHJ style $G_{2,3}$ and E_6 or from a near group $\mathbb{Z}_3$ quadratic system – with some multiplicities due to taking the dual or opposite subfactor in some cases. There are seven subfactor planar algebras at index 5, all arising from finite groups and subgroups of index 5. On top of combinatorial arguments and extensive computer calculations, various obstructions from graph considerations (triple point obstructions) and number theoretic were employed. The index must be an algebraic integer, and dimensions in fusion categories must be cyclotomic integers. Large families of potential graphs are eliminated which was also part of the original strategy of Haagerup.

Figure 5: With Sorin Popa at NZMRI Summer Workshop on Operator Algebras, Nelson 2001 & honorary degree Rome 2006

Then examples constructed for the survivors such as the planar algebra construction of the extended Haagerup, for which there is no other construction at present. The first composite index value beyond 4, namely $2 \times (3+\sqrt{5})/2 \simeq 5.2360$ invokes the analysis of (11) on the free composition of A_3 at index 2 and A_4 at index $(3+\sqrt{5})/2$. With few exceptions, such as the Haagerup subfactor, which had been constructed by combinatorial tours de force, the known subfactor invariants seemed to be group theoretical or quantum group theoretical.

7 Knots, Links and Topology

The algebraic relations [2] between the projections $\{e_i\}$ in the tower of subfactors led to representations of the braid group. Vaughan initially made the substitution

$$\sigma_j = te_j - (1 - e_j)$$

where $[M : N] = \lambda^{-1} = 2 + t + t^{-1}$ to yield the following Artin relations of the braid group, where σ_j is the braid which interchanges the j and $j+1$ strands:

$$\sigma_j\sigma_k = \sigma_k\sigma_j, \text{ if } |j-k| \geq 2; \qquad \sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}. \tag{4}$$

This is a unitary representation in the discrete range of the index $[M : N] = 4\cos^2(\pi/n)$ where $t = e^{\pm 2\pi i/n}$. However, Vaughan's subfactor analysis was providing much more than this – because his trace has a Markov property in a probabilistic sense

$$\mathrm{tr}(xe_m) = \lambda \mathrm{tr}(x) \quad \text{for } x \in \mathrm{Alg}\{1, e_1, \ldots e_{m-1}\}. \tag{5}$$

Vaughan was aware of these braid representations from conversations after giving talks in UCLA and Geneva in 1982 and worked on implications of this over the next few years (7). Alexander's theorem shows that every link is the closure of a braid and Markov's theorem tells us that two braids give rise to the same link if and only if they can be transformed to each other via Markov moves. The trace property $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ shows that evaluation of the trace on a braid is unchanged by the first Markov move of conjugating a braid. Vaughan had studied since 1982 the monograph of Joan Birman on braids and links. In May 1984, when living in New Jersey and working at UPenn, he visited her twice at Columbia University, New York. The first meeting was not conclusive, but by the second meeting a week later on 14th May, Vaughan had realised how to achieve invariance under the second Markov move, allowing increasing or decreasing the number of strings in the braid. The probabilistic property [5] of Vaughan's trace compensated exactly:

$$\mathrm{tr}(xt^{1/2}\sigma_m) = (-t^{1/2} - t^{-1/2})^{-1}\,\mathrm{tr}(x) \quad \text{for } x \text{ a braid on } m \text{ strands.}$$

Indeed, by a clever renormalisation of the trace and all braid generators σ_j to $t^{1/2}\sigma_j$, he could define

$$V_L(t) = (-t^{1/2} - t^{-1/2})^{m-1} tr(b)$$

for a braid b on m strings, which is also invariant under the second Markov move and so only depended on the link which is the closure of the braid. This was a completely new invariant of knots and links – the *Jones polynomial*. An elementary computation with the braid σ_1^3 showed that it could distinguish a trefoil from its mirror image.

A two variable polynomial HOMFLYPT which contained both the Alexander and Jones polynomials was soon discovered independently by several groups of mathematicians (Freyd et al (1985)). Vaughan had found a skein relation for his polynomial so together with a corresponding skein relation of John Conway (FRS 1981) for the Alexander polynomial it was natural to look for a two variable polynomial satisfying a skein relation. This was the approach of Lickorish and Millet. The Temperley-Lieb algebra [2] and hence the Jones polynomial appears from a Hecke algebra representation associated to $SU(2)$, with the two variable polynomial arising using Markov style traces (8) on the Hecke algebras.

Turaev and Viro (1992) produced invariants of 3-manifolds by using $6j$-symbols from quantum group representations to assign weights for building blocks of tetrahedra. Reshetikin and Turaev (1991) found invariants of 3-manifolds through Dehn surgery and invariants of framed links. Here, Kirby moves describe when two links provide the same 3-manifold. If the tensor category is modular, i.e. provided with a non-degenerate braiding, then the Reshetikhin-Turaev is essentially the square of the Turaev-Viro invariant. The Jones polynomial also inspired the homology of Khovanov (1998). This is a categorification of the state summation model, replacing polynomials with graded vector spaces, whose Euler characteristic is the Jones polynomial. It is still unknown if the Jones polynomial determines the unknot which Khovanov homology does. Bar-Natan, using Vaughan's planar algebra formalism, found an efficient method for the relevant homology computations. The N-coloured Jones polynomial is obtained by cabling a link with N components and using the $N+1$ dimensional representation of $SU(2)$. The interest in the volume conjecture that the growth rate of the coloured Jones polynomial is given by the hyperbolic volume of the knot complement is another example of the influence of Vaughan's seminal work. The modern field of *quantum topology* is based on his work.

8 Subfactors and Statistical Mechanics

The algebraic relations [2] between the Jones projections in the subfactor tower also appeared in work of Neville Temperley (Rumford Medal 1992) and Elliott Lieb (ForMemRS 2013) in solvable statistical mechanics (Temperley and Lieb (1971)). This observation triggered enduring bilateral applications and symbiosis between subfactors, the Jones polynomial, statistical mechanics and quantum field theory and boosted by the planar algebra formalism.

The two-dimensional Ising model assigns two possible spin values $\pm$ at the vertices of a lattice. Important generalisations include the Potts model, with Q states at each vertex, and vertex models, where (with possibly confusing terminology) the degrees of freedom are assigned to the edges of the lattice. The transfer matrix method, originated by Kramers and Wannier, assigns a matrix of Boltzmann weights to a one dimensional row lattice. The partition function of a rectangular lattice in general is then obtained by gluing together matrix products of the transfer matrix. Temperley and Lieb (1971) found that the transfer matrices of the Potts model and an ice-type vertex model, could both be described through generators obeying the same relations as in Vaughan's work [2] and in this way demonstrated their equivalence. Rodney Baxter (FRS 1982) showed how to construct commuting families of transfer matrices via Boltzmann weights satisfying the YBE developed by him and Chen-Ning Yang (ForMemrRS 1992). This permits simultaneous diagonalisation, with the largest eigenvalue being crucial for computing the free energy. The YBE is an enhancement of the braid relations in [4], as it reduces to them in a certain limit. Pimsner and Popa (1986) rediscovered the representation of the Temperley-Lieb-Jones relations in the infinite tensor product of 2×2 matrices which Temperley and Lieb had for the ice-type model. These relations between subfactor theory of Vaughan, the Pimsner-Popa representation and the Temperley-Lieb relations were already noticed in the autumn of 1983 and needed a deeper understanding.

At criticality and in a continuum limit, these models can be described by a CFT. The Virasoro algebra describes the infinite dimensional conformal symmetry in two dimensions in terms of a central extension of the infinitesimals of the diffeomorphism group of the circle. It is characterised by a number, the central charge. Using work of Kac on Virasoro representation theory, Friedan, Qiu and Shenker showed that the only allowed values of the central charge below 1 in unitary theories are $\{1 - 6/(m(m+1)); m = 2, 3\}$. Thus the central charge displays a dichotomy between discrete and continuous values similar to Vaughan's subfactor indices. The mystery was deepened further by the subsequent ADE classification of the $SU(2)$ CFTs by Cappelli, Itzykson and Zuber. From 1983-84 onwards, the puzzle was to connect all these pieces to form a coherent picture, with further perspective coming from Chern-Simons quantum field theory.

Kauffman (1987) found a diagrammatic state summation model for the Jones polynomial directly related to the Potts model. This model was key to the resolution by Kauffman, Murasugi, Thistlethwaite and Munasco of the Tait link conjectures which had been open since formulated at the end of the 19^{th} century. The work of Peter Guthrie Tait was stimulated by Lord Kelvin (FRS 1851) who incorrectly proposed that atoms be modelled on links in an aether. The new sounder connections between knots and physics, stimulated by Vaughan's link invariant, were only starting to appear. Much more was to come.

The combinatorial data which describes the standard invariant $\mathcal{G}$ of a subfactor, essen-

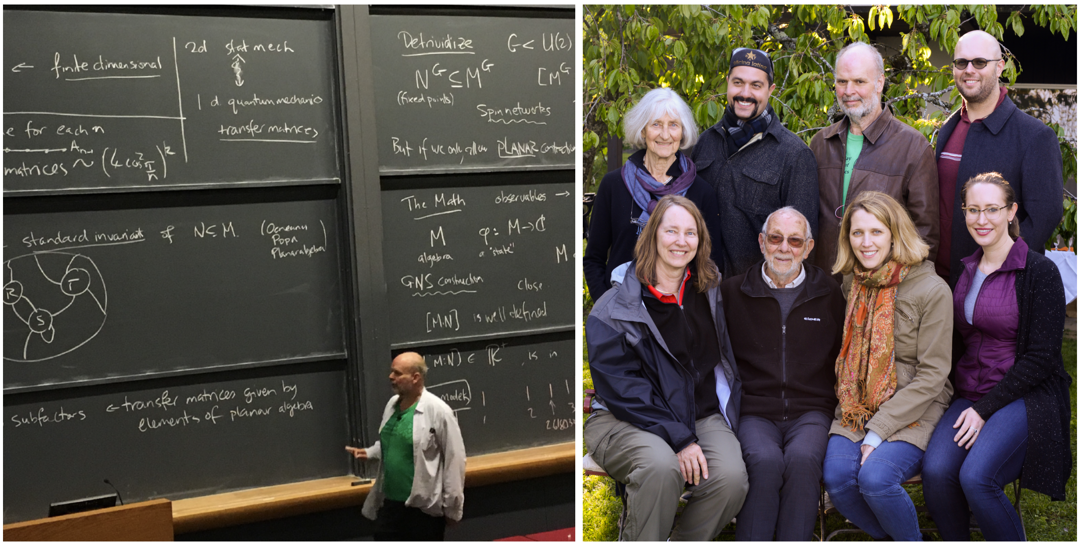

Figure 6: Speaking at the *Subfactors, Quantum Field Theory, and Quantum Information* meeting Harvard 2018 & at Geneva May 2019 – Front row: Wendy, André Haefliger, Bethany Ghassemi, Alice; Back Row: Hermina Haefliger, Arash Ghassemi, Vaughan, Ian

tially 6j-symbols, can arise as the Boltzmann weights of a solvable statistical mechanical model at the critical temperature, which satisfied the YBE. The principal machine to provide solutions to the YBE was quantum groups. This then raises the question of whether subfactors give rise to physical integrable statistical mechanical models through the enhancement of an insertion of a spectral parameter – a process which Vaughan termed *Baxterization*. The two variable polynomial of Kauffman is related to the orthogonal groups $SO(n)$ in the same way that the Jones polynomial is related to $SU(2)$ and the HOMFLYPT polynomial to $SU(n)$. This polynomial, was pursued (10) for spin models which Vaughan showed could be Baxterized, with Jaeger showing that an integrable model could be found over the Higman-Sims sporadic group.

Free probability and planar algebra techniques, in particular interpreting the semicircular law as a trace on an algebra of diagrams, have been applied by Vaughan et al (16) to reconstruct subfactors from standard invariants giving an alternative proof of Popa (1995). Moreover (17) use matrix model computations in loop models of statistical mechanics and graph planar algebras to construct novel matrix models for Potts models on random graphs.

9 Subfactors and Quantum Field Theory

A fuller understanding of the Jones polynomial and connection with physics came through quantum field theory in particular Chern-Simons three dimensional gauge theories and conformal field theory.

Tsuchiya and Kanie (1988) found that the braid representations of Vaughan were appearing through the monodromy representations from the conformal blocks of correlation functions of a two dimensional CFT for Wess-Zumino-Witten WZW models based on the loop groups of $SU(n)$. In the autumn of 1986, Michael Atiyah ran a weekly seminar in

Oxford on the Jones polynomial seeking to understand these connections with physics at which Vaughan spoke twice at the end of term. This fuelling by Atiyah came to fruition at the the International Association of Mathematical Physics IAMP Congress in Swansea in July 1988 — in particular at a dinner at Annie's restaurant attended by Michael Atiyah, Phil Nelson, Graeme Segal (FRS 1982) and Edward Witten (ForMemRS 1999). The next morning, Witten through Chern-Simons gauge theory presented a physical three dimensional interpretation of the Jones polynomial. The conformal blocks could be recovered with the Tsuchiya-Kanie picture of the braid representations as well as the original Jones polynomial at certain discrete values. This 3-dimensional formulation was also a spark for the 3-dimensional topological quantum invariants of Reshetikin and Turaev (1991) and Turaev and Viro (1992).

Vaughan and Antony Wassermann proposed to relate subfactor theory with CFT by constructing loop group subfactors $\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I^c} SU(n))'$ for positive energy representations λ of the loop group of $SU(n)$ at level k and restricting loops which lived on a subinterval I of the circle or its complement I^c. Here $'$ denotes as usual the commutant, so that the double commutant $\pi_\lambda(L_I SU(n))''$ is the von Neumann algebra generated by $\pi_\lambda(L_I SU(n))$. Wassermann (1998) went on to analyse these inclusions and found that they reproduced the Verlinde ring. For $SU(2)$ at level k these are given by the representations λ of $SU(2)$ described by the Coxeter-Dynkin diagram A_{k+1}. That inclusion could be described by $\lambda N \subset N$ for the hyperfinite III_1 factor and a braided system of endomorphisms $\{\lambda\}$ of N which form a modular tensor category whose modular data agrees with the loop group of $SU(n)$ at level k. In algebraic quantum field theory, Doplicher, Haag and Roberts DHR had introduced a theory of superselection sectors of endomorphisms on local algebras. In this theory a statistical dimension was assigned to endomorphims. Longo showed that this statistical dimension of an endomorphism μ on a local algebra $N = N(I)$ is the square root of the Jones index $[M : \mu(N)]$. What the subfactor machinery initiated by Vaughan did was to provide substantial examples for the DHR setting of algebraic quantum field theory. The localised interchange of support regions produces braided statistics in low dimension rather than the symmetric group in higher.

The subfactor framework, with their quantised symmetries and conformal nets $N(I)$, $I \subset S^1$, of von Neumann algebras are then a natural framework to study two dimensional CFT. The net carries a Verlinde ring as a braided tensor category, the representation theory of the net. The full two dimensional CFT can be described through the gluing of two local extensions which involves the modular invariant partition function – a positive integer matrix $\{Z_{\lambda,\mu}\}$ which commutes with the modular group representation of $SL(2, \mathbb{Z})$ arising from the braiding. The ADE classification for $SU(2)$ of Cappelli-Itzykson-Zuber is then natural through these extensions as module categories. Just as classical symmetries or groups are studied via their representation theories, then so are quantum symmetries via their module categories. The key tool which subfactor theory provided was that of α-induction with the modular invariant recovered as $Z_{\lambda\mu} = \langle \alpha_\lambda, \alpha_\mu \rangle$, Böckenhauer et al (2000) building on work of Ocneanu, Longo-Rehren and Xu. The conformal net representation of the $SU(2)$ Verlinde ring is then the A-series of Jones-Wassermann with trivial modular invariant. The D series of modular invariants arise from orbifolds with only D_{even} providing local extensions. The GHJ subfactors could now be understood by Xu through conformal inclusions including the E_6 and E_8 subfactors from the local extensions $SU(2)_{10} \to SO(5)_1$ and $SU(2)_{28} \to G_{2,1}$ respectively providing the E_6 and E_8 modular invariants. The E_7 modular invariant arises from a non-local extension through a twist on the orbifold D_{10}.

Vaughan's braid representations and his polynomial have found applications in quantum computing. Freedman et al (2003) constructed a model of quantum computation based on Topological Quantum Field Theory and Chern-Simons Theory implicitly implying an efficient quantum algorithm for approximating the Jones polynomial at $e^{2\pi i/5}$. Vaughan et al (15) use a different route avoiding TQFT to connect quantum computation and the Jones polynomial, presenting an explicit algorithm which works for all roots of unity.

10 The Thompson Groups

The question of whether there is a CFT behind subfactors had been apparent since 1985. Vaughan (19) came to model the diffeomorphism group of the circle, a fundamental part of a CFT, with groups of transformations which were discrete approximations — the groups of Richard Thompson. This procedure could not be made rigorous but these ideas led to unexpected progress in mathematics and physics including spin offs with knots.

The relation between the discrete index values under 4 and discrete central charge under 1 had been understood, via Virasoro conformal nets of factors, as cosets of $SU(2)$ theories (Kawahigashi and Longo (2004)). The power of the quantum symmetry subfactor formulation is that it permits the wondrous possibility of constructing new exotic CFT beyond the known, well-studied ones arising from loop groups, doubles of finite groups or natural constructions such as cosets. The classification programme on subfactors and standard invariants was throwing up examples which were potentially exotic. In particular, there was the question of whether there is a CFT underlying the Haagerup subfactor. Evans and Gannon (2011) produced evidence that the double of the Haagerup systems described the representation theory of a CFT. They found characters for the representation of the modular group $SL(2,\mathbb{Z})$ arising from the braiding and showing that this data had a simple expression in terms of a grafting of the double of the dihedral group S_3 and $SO(13)_2$ or indeed the orbifolds of two Potts models or quadratic (Tambara-Yamagami) systems based on $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_{13}$ respectively.

This reinvigorated Vaughan's quest to produce a CFT from subfactors through the use of planar algebras. Planar algebras at a rudimentary level describe how local partition functions of statistical mechanical models combine in the plane. Inserting one planar diagram systematically in another by surrounding it with an annulus which increases the number of marked outer labels and iterating suggests a procedure to take a continuum limit with the diffeomorphism group of the circle being approximated by homeomorphisms of the circle, which are piecewise linear with slope powers of 2 except possibly at exceptional rational dyadic points. This is the Thompson group of type T. Larger partition functions are obtained by encircling an operad with an annulus replacing each marked point on the original outer boundary with two new ones on the outer boundary of the annulus.

The Thompson group can be thought of as the group of fractions of rooted binary forests. Vaughan realised actions of the fractions through replacing a diagram such as a trident with an operator R distributed around an annulus. Applying the functor to algebraic and indeed operator algebraic data, he found new ways of constructing representations of the Thompson groups. The clarity of his formalism and analysis, led Vaughan and his

Berkeley student Arnaud Brothier (21) to show that the Thompson group T did not have the Haagerup property. New results also followed - certain wreath products of groups have the Haagerup property by taking the group of fractions of group labelled forests. Taking a functor from binary forests to Conway tangles, replacing a fork by an elementary tangle, provided Vaughan with an unexpected bridge with knots. Indeed he could show that every link arises in this way from the fraction of a pair of forests — just as braids yield all links through taking their closures.

Concluding Remarks

Vaughan had a distinctive and personal style of conducting research and sharing ideas, not only with his own graduate students, more than 30 in total, but with the wider mathematical and physical community. His warmth and generosity led him to thrive on social interaction, and for the mathematical community to significantly benefit from his openness in sharing ideas, both orally and in written correspondence, through every stage of development from speculation and conjecture of the way forward to discussing and explaining results. Scientific activity was often interwoven with his other passions in music and sport. Vaughan's analysis of the subfactors of von Neumann algebras of operators, with the novel notions and tools of quantum symmetry, planar algebras and his new link invariant led to unexpected connections and changing the landscape in a myriad of fields, whilst creating new ones, in mathematics and physics.

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Figure 1 left is courtesy of Wendy Jones, right Auckland Grammar School; Figure 2 left Wendy Jones, right Klaus Schmidt; Figure 3 left Inter. Math. Union, right Hiroaki Kono originally published in Sugaku Seminar-Special Edition ICM90, Nippon Hyoronsha Co.,Ltd. Publi.; Figure 4 left INI Cambridge, right David Evans; Figure 5 left Wendy Jones, right Roberto Longo; Figure 6 left Arthur Jaffe, right Nikita Nikolaev.

AWARDS AND RECOGNITION

Fellow of the Royal Society (1990)

Fields Medal (1990)

New Zealand Government Science Medal (now Rutherford Medal) (1991)

Honorary Fellow of the Royal Society of New Zealand Te Apārangi (1991)

Honorary DSc, University of Auckland (1992)

Corresponding Member, Australian Academy of Sciences (1992)

American Academy of Arts and Sciences (1993)

Honorary DSc, University of Wales (1993)

US National Academy of Sciences (1999)

Onsager Medal (2000)

Foreign Member to the Norwegian Royal Society of Letters and Sciences (2001)

DCNZM Distinguished Companion of the New Zealand Order of Merit (2002), redesignated KNZM Knight Companion (2009)

Honorary Member of the London Mathematical Society (2002)

Doctorat Honoris Causa, Université du Littoral, Côte d'Opale (2002)

Laurea Honoris Causa, Tor Vergata University of Rome (2006)

Prix Mondial Nessim Habif (2007)

Honorary Fellow of the Learned Society of Wales (2018)

ICCM International Cooperation Award (2019)

Author Profile

David E Evans

He studied mathematics at Oxford obtaining a DPhil under the supervision of Brian Davies FRS in 1975. After postdoctoral stints in Dublin, Oslo, UCLA, Copenhagen and Ottawa he was appointed lecturer then reader at Warwick, followed by chairs at Swansea and Cardiff and a year as a guest professor at Kyoto. He has worked on operator algebras, subfactors and K-theory in particular applications in statistical mechanics and conformal field theory, including co-authoring the monograph (Evans and Kawahigashi (1998)) on Quantum Symmetries.

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