Eigenfunctions localised on a defect in high-contrast random media

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Abstract

We study the properties of eigenvalues and corresponding eigenfunctions generated by a defect in the gaps of the spectrum of a high-contrast random operator. We consider a family of elliptic operators $\mathcal{A}^\varepsilon$ in divergence form whose coefficients are random, possess double porosity type scaling, and are perturbed on a fixed-size compact domain (a defect). Working in the gaps of the limiting spectrum of the unperturbed operator $\widehat{\mathcal{A}}^\varepsilon$, we show that the point spectrum of $\mathcal{A}^\varepsilon$ converges in the sense of Hausdorff to the point spectrum of the limiting two-scale operator $\mathcal{A}^{\mathrm{hom}}$ as $\varepsilon \to 0$. Furthermore, we prove that the eigenfunctions of $\mathcal{A}^\varepsilon$ decay exponentially at infinity uniformly for sufficiently small ε. This, in turn, yields strong stochastic two-scale convergence of such eigenfunctions to eigenfunctions of $\mathcal{A}^{\mathrm{hom}}$.

Keywords: high contrast media, random media, stochastic homogenisation, defect modes, localised eigenfunctions.

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Contents

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1 Introduction

We consider a high-contrast two-phase random medium, comprising a moderately "stiff" material — the *matrix* — in which small "soft" inclusions are randomly dispersed, with a fixed size defect filled by a third phase. The spectral properties of high-contrast random media were recently studied in [9], [10]. It was shown that similarly to the periodic high-contrast setting [30], [31], under some physically natural assumptions the corresponding operators may exhibit gaps in their spectrum. It is well known that in this case a defect can induce localised modes in the gaps [6, 4, 15]. The goal of the paper is to study localised modes generated by a defect.

The existence of localised modes, i.e. finite energy wave-like solutions of equations of physics — typically of electromagnetic or elastic nature — such that almost all their energy remains in a bounded region at all times, is a classical and well-studied problem. This phenomenon is particularly important in applications, in that it can confer advantageous and desirable physical properties on the material in question. It is responsible, for instance, for the behaviour of photonic and phononic fibres [24, 20, 26].

In mathematical terms, the study of localised modes typically reduces to the analysis of an elliptic spectral problem on the cross-section of the material. The prototypical example is that of a composite medium described by an elliptic operator in divergence form with periodic coefficients perturbed on a relatively compact set. The spectral theory of operators with periodic coefficients, both in the presence and in the absence of a defect, has received considerable attention over the years. We refer the reader to [21] for a review of the subject, which focuses, in particular, on the mathematics of photonic band-gap optical materials.

In the current work we are interested in high-contrast media, namely, materials described by an elliptic operator in divergence form whose coefficients possess the so-called double porosity type scaling (see (2.8) below). Defect modes for high-contrast *periodic* media have been studied by Cherdantsev [7] and Kamotski–Smyshlyaev [19] in dimension $d \geq 2$. Somewhat stronger results were later obtained in dimension one by Cherdantsev, Cherednichenko and Cooper [8], see also [11], although it is worth mentioning that one-dimensional setting is very special, and the nature of the band-gap spectrum there is different from the one in higher dimensions. Even in the absence of high-contrast in the constituents, by carefully choosing the frequency of a Bloch wave type solution in relation to the wavenumber along the fibres one can *artificially* obtain a high-contrast operator on the cross-section of the material, see [12] for details.

The mathematical properties of high-contrast *random* media have remained, until very recently, largely unexplored. Some progress has been made in the last few years by Cherdantsev, Cherednichenko and Velčić, both in bounded domains [9] and in the whole space [10]. A summary of the results from [9, 10] will be provided later in this and the next section.

The goal of the current paper is to perform a detailed analysis of localised modes in high-contrast random media, thus extending the results of [7, 19] to the stochastic setting of [10].

Let us discuss our problem in more detail.

We begin by recalling a classical result of Figotin and Klein [15], which applies, in particular, to a general class of elliptic self-adjoint operators of the form $-\nabla \cdot A\nabla$. Figotin and Klein showed that if the operator in question has a gap in its spectrum, then one can create an eigenvalue within any specific subinterval of the gap, by an appropriate compact perturbation of the coefficients A; furthermore, the corresponding eigenfunction decays exponentially with the rate of decay proportional, loosely speaking, to the distance from the eigenvalue to the edges of the gap, with proportionality factor depending on the ellipticity constant.

The basic starting point for creating localised modes — the existence of spectral gaps — is, in general, a non-trivial problem. While for periodic operators the spectrum has a band-gap structure, the existence of gaps is not guaranteed, since all the bands may overlap. In particular, this question is open for general elliptic operators of the form $-\nabla\cdot A\nabla$ with periodic coefficients. However, in specific physically relevant examples, the existence of gaps can be shown, see, e.g., [16], and [22] for a general overview of the topic. It is well known that in homogenisation setting in the case of uniformly elliptic coefficients A, both periodic and random, the limit homogenised operator has no gaps in the spectrum. The picture is drastically different for the high-contrast setting with infinitely many gaps opening in homogenisation limit, which makes it particularly interesting for the study of localisation phenomena. One more observation we make here is that the type of the spectrum surrounding the gap is irrelevant for the basic argument one employs to argue the existence of localised modes induced by a defect, at least in the high-contrast homogenisation setting. Thus, while the problem of characterising the spectrum of an operator $-\nabla\cdot A\nabla$ with random coefficients A is wide open to date, the answer to this question has no implications for our analysis.

In order to proceed we need to preliminarily introduce some notation; rigorous definitions will be given in the next section. We denote by $\widehat{\mathcal{A}}^\varepsilon$ the unperturbed (i.e., without defect) high-contrast operator, where ε is a small parameter characterising the size of the microstructure, and by $\mathcal{A}^\varepsilon$ its perturbation by a finite size defect; $\widehat{\mathcal{A}}^{\mathrm{hom}}$ and $\mathcal{A}^{\mathrm{hom}}$ will denote the corresponding limiting two-scale operators.

For *periodic* high-contrast operators Zhikov [30, 31] showed that the spectrum of $\widehat{\mathcal{A}}^{\mathrm{hom}}$, characterised by a certain nonlinear function of the spectral parameter $\beta(\lambda)$, has infinitely many gaps. Furthermore, one has convergence of spectra $\sigma(\widehat{\mathcal{A}}^\varepsilon)\to\sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$ in the sense of Hausdorff, so that, for sufficiently small ε gaps open in the spectrum of $\widehat{\mathcal{A}}^\varepsilon$. Therefore, according to [15], by introducing a compact defect one can generate discrete spectrum in the gaps of $\widehat{\mathcal{A}}^\varepsilon$. It can be shown that one can induce localised eigenvalues in the gaps of $\sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$ via an argument analogous to that of [15]. In particular, Kamotski and Smyshlyaev in [19] provided an explicit example of a localised eigenfunction of $\mathcal{A}^{\mathrm{hom}}$ induced by a defect under an additional assumption of isotropy of the homogenised medium. Their main result, however, consists in showing that for a defect eigenvalue λ_0 in a gap of the essential spectrum of $\mathcal{A}^{\mathrm{hom}}$, there exists a defect eigenvalue λ_ε of $\mathcal{A}^\varepsilon$ converging to λ_0 with a rate of order $\varepsilon^{1/2}$. Moreover, the distance (in an appropriate sense) between the localised eigenfunction u_0 corresponding to λ_0 and the eigenspace of $\mathcal{A}^\varepsilon$ corresponding to its eigenvalues located in the vicinity of λ_0 is also of order $\varepsilon^{1/2}$. In order to prove the converse statement, i.e. that for a sequence of defect eigenvalues λ_ε of $\mathcal{A}^\varepsilon$ converging to a point λ_0 in a gap of the essential spectrum of $\mathcal{A}^{\mathrm{hom}}$ it holds that λ_0 is an eigenvalue of $\mathcal{A}^{\mathrm{hom}}$, one needs a compactness statement for the corresponding sequence of eigenfunctions u_ε. This was obtained in [7], where the author proved a uniform exponential decay of u_ε independent of ε. Note that the general results of [15] do not allow one to obtain compactness, since there the decay rate depends on the ellipticity constant, which vanishes in the high-contrast setting as $\varepsilon\to 0$. Combining the results of [7] and [19] one also gets an asymptotic one-to-one correspondence between the defect modes of $\mathcal{A}^\varepsilon$ and $\mathcal{A}^{\mathrm{hom}}$. The final remark we make about the high-contrast periodic setting is that both u_0 and u_ε (for small ε) decay exponentially as $e^{-\alpha|x|}$, where α is of order $\sqrt{|\beta(\lambda_0)|}$ with proportionality factor depending on the ellipticity constant of the homogenised coefficients. Note that the quantity $\sqrt{|\beta(\lambda_0)|}$ blows up as λ_0 to approach the left end of a spectral gap; therefore, the latter provides a much better rate of decay in the high-contrast setting than the more general result [15] of Figotin and Klein.

Whilst for *stochastic* operators the picture is to a certain extent similar, at the same time there is a number of significant technical and phenomenological differences. The spectrum of $\widehat{\mathcal{A}}^{\mathrm{hom}}$ can be characterised by a stochastic analogue of Zhikov's function $\beta(\lambda)$ — see [10] and Section 2 below — however, the limit of the spectra of $\widehat{\mathcal{A}}^\varepsilon$ is, in general, strictly larger than $\sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$ (in fact, in some examples $\sigma(\widehat{\mathcal{A}}^\varepsilon)=\mathbb{R}_+$ [10, § 5.6]). In [10] the authors show that $\sigma(\mathcal{A}^{\mathrm{hom}})\subset\lim_{\varepsilon\to0}\sigma(\widehat{\mathcal{A}}^\varepsilon)\subset\mathcal{G}$, where the set $\mathcal{G}$ is determined by a function $\beta_\infty(\lambda)$ — a modification of the function $\beta(\lambda)$ which carries an information about areas with "non-typical" distribution of inclusions in $\mathbb{R}^d$. In realistic examples, e.g. the random parking model [10, § 5.6], the set $\mathcal{G}$ can be shown to have gaps, and hence so does $\sigma(\widehat{\mathcal{A}}^\varepsilon)$ for small enough ε.

In this work we adapt the general strategy of [7] and [19] to the stochastic setting. While postponing a detailed description of our results to later section, let us briefly elaborate on the novelty and challenges posed by the random setting.

We restrict our attention to the gaps of $\mathcal{G}$ (assuming that they exist), rather than the gaps of $\sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$. Indeed, we have to do this in order to avoid the limit set of $\sigma(\widehat{\mathcal{A}}^\varepsilon)$ (note that under the assumption of finite range of dependence of the distribution of inclusions in $\mathbb{R}^d$ one has $\lim_{\varepsilon\to0}\sigma(\widehat{\mathcal{A}}^\varepsilon)=\mathcal{G}$, see [10, Theorem 5.5]). The two-scale nature of the limiting operator $\mathcal{A}^{\mathrm{hom}}$ captures the micro- and macroscopic scales of $\mathcal{A}^\varepsilon$. We perform a delicate analysis, incorporating the subtle relation between the multiple scales of the operators $\mathcal{A}^\varepsilon$ and $\mathcal{A}^{\mathrm{hom}}$, in order to establish an asymptotic one-to-one correspondence between the defect modes of the operators $\mathcal{A}^{\mathrm{hom}}$ and $\mathcal{A}^\varepsilon$ with the eigenvalues in the gaps of the set $\mathcal{G}$.

Another manifestation of the technical difficulties arising from the stochastic nature of the problem is that while for a defect eigenfunction of $\mathcal{A}^{\mathrm{hom}}$ the exponential decay rate is determined by the stochastic version of $\beta(\lambda_0)$ (in direct analogy to the periodic setting), for the defect eigenfunctions of $\mathcal{A}^\varepsilon$ we were only able to characterise the exponential decay in terms of $\beta_\infty(\lambda_0)$ (the function that carries information about the limit of the spectra of $\mathcal{A}^\varepsilon$) satisfying

$$\beta(\lambda_0)\le\beta_\infty(\lambda_0),$$

thus obtaining a slightly worse decay rate than one would hope for. At this stage, it is not entirely clear whether this result is optimal.

2 Statement of the problem

The precise mathematical formulation of our model is as follows.

We work in Euclidean space $\mathbb{R}^d$, $d\ge2$, equipped with the Lebesgue measure. For every measurable subset $A\subset\mathbb{R}^d$ we denote by $\overline{A}$ its closure, by $|A|:=\int_A dx$ its Lebesgue measure and by $\mathbb{1}_A$ its characteristic function. Furthermore, we denote by

$$\langle f\rangle_A:=\frac{1}{|A|}\int_A f(x)\,dx$$

the average over A of a function $f\in L^1(A)$. Finally, we define

$$\square_x^L:=[-L/2,L/2]^d+x \tag{2.1}$$

to be the box centred at x of side L and we set $\square^L:=\square_0^L$.

Throughout the paper, the letter C is used in estimates to denote a positive numerical constant, whose precise value is inessential and can change from line to line. Furthermore, $B_r(x)$ denotes the open ball of radius r centred at x. Finally, as per standard practice in the field, we do not distinguish in our notation L^p and Sobolev spaces of scalar, vector and matrix functions: which is which will be clear from the context.

Before rigorously introducing our probability space, let us recall the well-established notion of minimal smoothness [28, Chapter VI, Section 3.3], for the reader's convenience.

Definition 2.1 (Minimally smooth set)**.** An open set $A\subset\mathbb{R}^d$ is said to be *minimally smooth with constants* (ζ,N,M) if there exists a countable (possibly finite) family of open sets $\{U_i\}_{i\in I}$ such that

(a) each $x \in \mathbb{R}^d$ is contained in at most N of the open sets U_i;

(b) for every $x \in \partial A$ there exists $i \in I$ such that $B_\zeta(x) \subset U_i$;

(c) for every $i \in I$ the set $\partial A \cap U_i$ is, in a suitably chosen coordinate system, the graph of a Lipschitz function with Lipschitz seminorm not exceeding M.

Our probability space $(\Omega, \mathcal{F}, P)$ is defined as follows.

We define Ω to be the set of possible collections of randomly distributed inclusions in $\mathbb{R}^d$, that is, elements $\omega \in \Omega$ are subsets of $\mathbb{R}^d$ satisfying appropriate geometric conditions. Namely, we require that individual inclusions are approximately of the same size, that they do not get too close to each other and that their boundary is sufficiently regular. This is formalised by the following collection of assumptions.

Assumption 2.2. There exist constants ζ, N and M such that for all $\omega \in \Omega$ the set $\mathbb{R}^d \setminus \omega$ is connected, and ω can be written as a disjoint union

$$\omega = \bigsqcup_{k \in \mathbb{N}} \omega^k$$

of sets satisfying the following properties.

(a) For every $k \in \mathbb{N}$ the set ω^k is open and connected.

(b) There exists a family $\{\mathcal{B}_\omega^k\}_{k\in\mathbb{N}}$ of bounded open subsets of $\mathbb{R}^d$ such that for every $k \in \mathbb{N}$ we have[1]

(i) $\omega \cap \mathcal{B}_\omega^k = \omega^k$;

(ii) $\mathcal{B}_\omega^k \setminus \overline{\omega^k}$ is non-empty and minimally smooth with constants (ζ, N, M);

(iii) $\operatorname{diam} \omega^k < \frac{1}{2}$ and

$$\omega^k - D_\omega^k - d_{1/4} \subset \mathcal{B}_\omega^k - D_\omega^k - d_{1/4} \subset [-1/2, 1/2]^d,$$

where $D_\omega^k \in \mathbb{R}^d$ is defined by

$$(D_\omega^k)_j := \inf_{x \in \omega^k} x_j$$

and $d_{1/4} := (1/4, \dots, 1/4)^T \in \mathbb{R}^d$.

Remark 2.3. The above Assumption warrants a few observations.

- The quantity D_ω^k appearing in Assumption 2.2(b)(iii) represents, from a geometric point of view, the "bottom-left" corner of the smallest hypercube containing the inclusion ω^k. Its introduction serves the purpose of shifting the inclusion to the origin, so that it fits into the unit hypercube $[-1/2, 1/2]^d$.

- The precise value of the bounds on the size of ω^k and $\mathcal{B}_\omega^k$ (e.g., the requirement that they fit into a hypercube of size 1) are purely conventional. One could impose more general boundedness assumptions and recover our specific values by elementary scaling arguments.

[1] Here and further on

$$\operatorname{diam} A := \sup_{t,s \in A} |t - s|.$$

We define $\mathcal{F}$ to be the σ-algebra on Ω generated by the mappings $\pi_q : \Omega \to \{0,1\}$, where $q \in \mathbb{Q}^d$ and

$$\pi_q(\omega) := \mathbb{1}_\omega(q). \tag{2.2}$$

That is, $\mathcal{F}$ is the smallest σ-algebra that makes the mappings π_q, $q \in \mathbb{Q}^d$, measurable. There is a natural group action of $\mathbb{R}^d$ on Ω, and hence on $\mathcal{F}$, given by translations. Namely, for every $y \in \mathbb{R}^d$ the translation map $T_y : \Omega \to \Omega$ acts on $\omega \in \Omega$ as

$$\omega \mapsto T_y\omega = \{z - y \mid z \in \omega\} \subset \mathbb{R}^d.$$

We equip $(\Omega, \mathcal{F})$ with a probability measure P assumed to be invariant under translations, i.e., we assume that $P(T_yF) = P(F)$ for every $F \in \mathcal{F}$ and $y \in \mathbb{R}^d$, where $T_yF := \bigcup_{\omega\in F} T_y\omega$. It is easy to see that $(T_y)_{y\in\mathbb{R}^d}$ satisfies the following properties:

(a) $T_{y_1} \circ T_{y_2} = T_{y_1+y_2}$ for all $y_1, y_2 \in \mathbb{R}^d$, where $\circ$ stands for composition;

(b) the map $\mathcal{T} : \mathbb{R}^d \times \Omega \to \Omega$, $(y,\omega) \to T_y(\omega)$ is measurable with respect to the standard σ-algebra on the product space induced by $\mathcal{F}$ and the Borel σ-algebra on $\mathbb{R}^d$.

Finally, we assume the translation group action $(T_y)_{y\in\mathbb{R}^d}$ to be ergodic, i.e. if an element $F \in \mathcal{F}$ satisfies

$$P((T_yF \cup F) \setminus (T_yF \cap F)) = 0 \quad \text{for all} \quad y \in \mathbb{R}^d,$$

then $P(F) \in \{0,1\}$.

We adopt the standard notation

$$\mathbb{E}[\overline{f}] := \int_\Omega \overline{f}(\omega)\, dP(\omega)$$

and we denote by $L^p(\Omega)$ the usual spaces of p-integrable functions in $(\Omega, \mathcal{F}, P)$. Since $\mathcal{F}$ is, clearly, countably generated, $L^p(\Omega)$, $1 \le p < \infty$, is separable. Given $\overline{f} : \Omega \to \mathbb{R}$, we denote by $f(y,\omega) := \overline{f}(T_y\omega)$ its *realisation* or *stationary extension*. Note that if $\overline{f} \in L^p(\Omega)$, then $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d; L^p(\Omega))$ [18, Chapter 7]. In the current paper we are mostly concerned with the case $p = 2$.

Notation 2.4. Throughout our paper, unless otherwise stated, we will denote with an overline functions on Ω and we will remove the overline to denote the corresponding realisation (stationary extension). We will reserve the letter y for the extension variable. So, for example:

$$\overline{f} = \overline{f}(\omega), \quad f = f(y,\omega) := \overline{f}(T_y\omega), \quad \mathbb{E}[f] := \mathbb{E}[\overline{f}].$$

Functions on Ω may additionally depend on other variables, not necessarily in a stationary manner. For example, if $\overline{f}(x,\omega) : \mathbb{R}^d \times \Omega \to \mathbb{R}$, then $f(x,y,\omega) := \overline{f}(x, T_y\omega)$ and $\mathbb{E}[f] = \mathbb{E}[\overline{f}(x,\cdot)]$ (note that the dependence on the non-stationary variable x remains upon taking the expectation).

We define the Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{N}$, as

$$H^s(\Omega) := \left\{ \overline{f} \in L^2(\Omega) \mid f \in H^s_{\mathrm{loc}}(\mathbb{R}^d; L^2(\Omega)) \right\}, \tag{2.3}$$

and denote

$$H^\infty(\Omega) := \bigcap_{s\in\mathbb{N}} H^s(\Omega).$$

We use a standard notation for partial derivatives $\partial^{\boldsymbol{\alpha}} := \partial^{\alpha_1}_{y_1} \cdots \partial^{\alpha_d}_{y_d}$ with a multi-index $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $\sum_{l=1}^d \alpha_l \le s$. We note that, given $\overline{f} \in H^s(\Omega)$, for every multi-index

$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, $\sum_{l=1}^d \alpha_l \le s$, the quantity $\partial^{\boldsymbol{\alpha}} f$ is the stationary extension of an element from $L^2(\Omega)$. Accordingly, the quantity $\overline{\partial^{\boldsymbol{\alpha}} f}$ is to be understood as the random variable whose stationary extension is $\partial^{\boldsymbol{\alpha}} f$. In view of the above one defines the norm on (2.3) as

$$\|\overline{f}\|^2_{H^s(\Omega)} := \sum_{|\boldsymbol{\alpha}| \le s} \|\overline{\partial^{\boldsymbol{\alpha}} f}\|^2_{L^2(\Omega)} .$$

We also define

$$C^\infty(\Omega) := \left\{ \overline{f} \in H^\infty(\Omega) \mid \overline{\partial^{\boldsymbol{\alpha}} f} \in L^\infty(\Omega) \text{ for every multi-index } \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \right\}. \quad (2.4)$$

Remark 2.5. Observe that the definition of probabilistic Sobolev spaces retraces the classical one. We refrain from introducing the stationary differential calculus on Ω more formally, as it will not be needed in this paper. We refer the interested reader to [14, Appendices A.2 and A.3] for further details.

Finally, we define

$$L_0^2(\Omega) := \left\{ \overline{f} \in L^2(\Omega) \mid \; f(\cdot, \omega)|_{\mathbb{R}^d \setminus \omega} = 0 \text{ for all } \omega \in \Omega \right\},$$

$$H_0^s(\Omega) := \left\{ \overline{f} \in H^s(\Omega) \mid \; f(\cdot, \omega)|_{\mathbb{R}^d \setminus \omega} = 0 \text{ for all } \omega \in \Omega \right\}$$

and

$$C_0^\infty(\Omega) := \left\{ \overline{f} \in C^\infty(\Omega) \mid \; f(\cdot, \omega)|_{\mathbb{R}^d \setminus \omega} = 0 \text{ for all } \omega \in \Omega \right\}.$$

The latter are spaces of functions in $L^2(\Omega)$, $H^s(\Omega)$ and $C^\infty(\Omega)$, respectively, whose realisations vanish identically outside the inclusions.

The above function spaces enjoy the following properties: (i) $H^1(\Omega)$ is separable; (ii) $H^\infty(\Omega)$ is dense in $L^2(\Omega)$; (iii) $C^\infty(\Omega)$ is dense in $L^p(\Omega)$, $1 \le p < \infty$; (iv) $C^\infty(\Omega)$ is dense in $H^s(\Omega)$, endowed with the natural Banach space structure, for every s. Furthermore, we have at our disposal the following result, whose proof may be found in [9, 10].

Theorem 2.6. *Under Assumption 2.2, $C_0^\infty(\Omega)$ is dense in $L_0^2(\Omega)$ and in $H_0^1(\Omega)$ with respect to $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$.*

A bridge between objects in physical space and objects in the abstract probability space is provided by the Ergodic Theorem, a classical result on ergodic dynamical systems that appears in the literature in various (not always equivalent) formulations. For the reader's convenience, we report here the version that we will be using in our paper, see, e.g., [25], or [1] for a more general take.

Theorem 2.7 (Birkhoff's Ergodic Theorem)**.** *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with an ergodic dynamical system $(T_y)_{y \in \mathbb{R}^d}$. Let $\overline{f} \in L^p(\Omega)$, $1 \le p < \infty$. Then we have*

$$f(\,\cdot\,/\varepsilon, \omega) = \overline{f}(T_{\cdot/\varepsilon}\omega) \rightharpoonup \mathbb{E}[\overline{f}]$$

in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ as $\varepsilon \to 0$ almost surely.

Remark 2.8. Observe that Theorem 2.7 implies

$$\lim_{R \to +\infty} \frac{1}{R^d} \int_{\square^R} f(y, \omega)\, dy = \mathbb{E}[\overline{f}]. \quad (2.5)$$

Further on in the paper, we will often apply the Ergodic Theorem in the more concrete version (2.5).

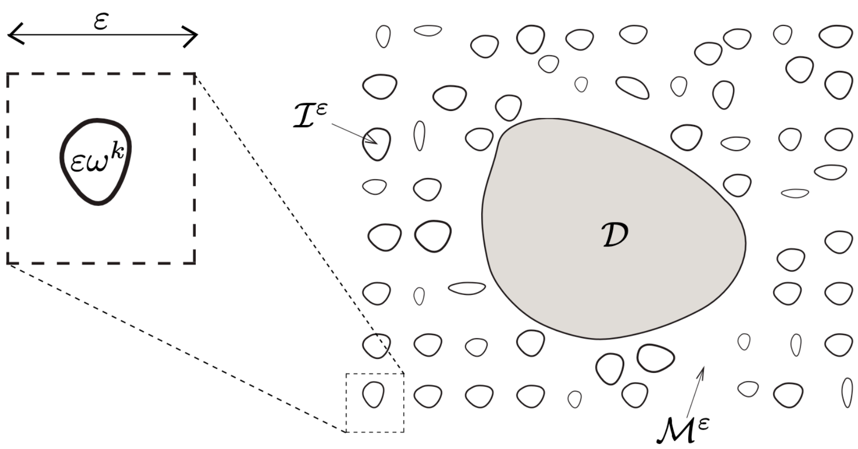

Figure 1: Mathematical model.

As shown in [10], the assumption of uniform minimal smoothness is sufficient to ensure that our inclusions possess the extension property, which will prove essential throughout our paper.

Theorem 2.9 (Extension property [10, Theorem 3.8])**.** *For every $p \geq 1$ there exists a bounded linear extension operator $E_k : W^{1,p}(\mathcal{B}^k_\omega \setminus \omega^k) \to W^{1,p}(\mathcal{B}^k_\omega)$ such that for every $u \in W^{1,p}(\mathcal{B}^k_\omega \setminus \omega^k)$ the extension $\widetilde{u} := E_k u$ satisfies the estimates*

$$\|\nabla \widetilde{u}\|_{L^p(\mathcal{B}^k_\omega)} \leq c\, \|\nabla u\|_{L^p(\mathcal{B}^k_\omega \setminus \omega^k)}\,,$$

$$\|\widetilde{u}\|_{W^{1,p}(\mathcal{B}^k_\omega)} \leq c\, \|u\|_{W^{1,p}(\mathcal{B}^k_\omega \setminus \omega^k)},$$

where the constant c depends on ζ, N, M, p but is independent of ω and k. Furthermore, if $p = 2$ the extension can be chosen to be harmonic in ω^k,

$$\Delta \widetilde{u} = 0 \ \textit{in}\ \omega^k.$$

Let $\mathcal{D} \subset \mathbb{R}^d$ be a fixed open domain with $C^{1,\alpha}$ boundary, for some $\alpha > 0$. For each $\omega \in \Omega$ and $0 < \varepsilon < 1$ put

$$\mathbb{N}^\varepsilon(\omega) := \{k \in \mathbb{N} \,|\, \varepsilon\omega^k \cap \overline{\mathcal{D}} = \emptyset\}. \tag{2.6}$$

Up to a set of measure zero, for each random set of inclusions ω we partition $\mathbb{R}^d$ into three parts: the *defect* $\mathcal{D}$, the *inclusions*

$$\mathcal{I}^\varepsilon(\omega) := \bigcup_{k \in \mathbb{N}^\varepsilon(\omega)} \varepsilon\omega^k$$

and the *matrix*

$$\mathcal{M}^\varepsilon(\omega) := \mathbb{R}^d \setminus \overline{\mathcal{I}^\varepsilon(\omega) \cup \mathcal{D}},$$

see Figure 1. Without loss of generality, we assume that $0 \in \mathcal{D}$. We shall often drop the argument ω and write simply $\mathcal{I}^\varepsilon$ and $\mathcal{M}^\varepsilon$, to keep notation light.

Remark 2.10. Note that in the above construction we discard entirely the inclusions intersecting the boundary of the defect. Though our results could be obtained without such an assumption — i.e., keeping, for those inclusions intersecting $\partial\mathcal{D}$, the portion of inclusion that lies outside the defect — we do so for two reasons. The first is that this is not unreasonable from the point of view of potential applications, where inclusions are dispersed in the medium around the defect. The second is that doing so reduces the amount of technical material needed in the proofs, thus improving clarity and readability of the paper without compromising the key ideas and techniques.

Let $1_\Omega : \mathbb{R}^d \times \Omega \to \{0,1\}$ be the stationary function defined in accordance with

$$1_\Omega(y,\omega) := \mathbb{1}_\omega(y). \tag{2.7}$$

Then, by the Ergodic Theorem we have

$$\mathbb{E}[1_\Omega] = \lim_{R\to+\infty} \frac{1}{R^d} \int_{\square^R} \mathbb{1}_\omega(y)\, dy$$

almost surely; therefore, the quantity $\mathbb{E}[1_\Omega]$ is the relative density of inclusions outside the defect.

Definition 2.11. We define $\mathcal{A}^\varepsilon(\omega)$ to be the self-adjoint linear operator in $L^2(\mathbb{R}^d)$ associated with the bilinear form

$$\int_{\mathbb{R}^d} A^\varepsilon(\,\cdot\,,\omega)\nabla u\cdot\nabla v, \quad u,v\in H^1(\mathbb{R}^d),$$

where

$$A^\varepsilon(x,\omega) := \mathbb{1}_{\mathcal{M}^\varepsilon}(x)\, A_1 + \varepsilon^2 \mathbb{1}_{\mathcal{I}^\varepsilon}(x)\, \mathrm{Id} + \mathbb{1}_{\mathcal{D}}(x)\, A_2, \tag{2.8}$$

and A_1, A_2 are positive definite symmetric matrices in $\mathrm{GL}(d,\mathbb{R})$.

In order to introduce the limiting operator we need the function spaces H and V defined as follows.

Definition 2.12 (Function space H)**.** We define

$$H := L^2(\mathbb{R}^d) + \{\overline{v}\in L^2(\mathbb{R}^d; L^2_0(\Omega)) \,|\, \overline{v}(x,\omega)=0 \text{ for } x\in\mathcal{D}\}, \tag{2.9}$$

as the space of functions of the form $u+\overline{v}$, where $u(x)\in L^2(\mathbb{R}^d)$ and $\overline{v}(x,\omega)\in L^2(\mathbb{R}^d\times\Omega)$ is a random field whose stationary extension $v(x,y,\omega)$ vanishes outside the inclusions (in y) and in the defect $\mathcal{D}$ (in x).

Definition 2.13 (Function space V)**.** We define

$$V := H^1(\mathbb{R}^d) + \{\overline{v}\in L^2(\mathbb{R}^d; H^1_0(\Omega)) \,|\, \overline{v}(x,\omega)=0 \text{ for } x\in\mathcal{D}\} \tag{2.10}$$

as the space of functions of the form $u+\overline{v}$, where $u\in H^1(\mathbb{R}^d)$ and $\overline{v}\in L^2(\mathbb{R}^d; H^1_0(\Omega))$ is a random field whose stationary extension $v(x,y,\omega)$ vanishes outside the inclusions (in y) and in the defect $\mathcal{D}$ (in x).

Clearly, $H\subset L^2(\mathbb{R}^d\times\Omega)$. Furthermore, since $L^2(\mathbb{R}^d\times\Omega)=L^2(\mathbb{R}^d;L^2(\Omega))$, by Theorem 2.6 we have that the space V is dense in H with respect to $L^2(\mathbb{R}^d\times\Omega)$ norm.

Recall, see, e.g., [18], that a vector field $p\in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ is said to be

- *potential* if there exists $\varphi\in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that $p=\nabla\varphi$;
- *solenoidal* if

$$\int_{\mathbb{R}^d} p\cdot\nabla\varphi = 0 \qquad \forall\varphi\in C_0^\infty(\mathbb{R}^d).$$

By analogy, one defines a vector field $\overline{p}\in L^2(\Omega)$ to be potential (respectively solenoidal) if for a.e. ω its realisation $y\mapsto p(y,\omega)\in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ is such. The classical Weyl's decomposition theorem holds for vector fields in $L^2(\Omega)$ as well.

Theorem 2.14 (Weyl's decomposition)**.** *The space of vector fields $L^2(\Omega)$ admits the orthogonal decomposition*

$$L^2(\Omega) := \mathcal{V}^2_{\mathrm{pot}}(\Omega)\oplus\mathcal{V}^2_{\mathrm{sol}}(\Omega)\oplus\mathbb{R}^d,$$

where

$$\mathcal{V}^2_{\mathrm{pot}}(\Omega) := \{\overline{p}\in L^2(\Omega) \,|\, \overline{p} \textit{ is potential},\ \mathbb{E}[\overline{p}]=0\},$$
$$\mathcal{V}^2_{\mathrm{sol}}(\Omega) := \{\overline{p}\in L^2(\Omega) \,|\, \overline{p} \textit{ is solenoidal},\ \mathbb{E}[\overline{p}]=0\}.$$

Let $A_1^{\mathrm{hom}} \in \mathrm{GL}(d,\mathbb{R})$ be the symmetric matrix of homogenised coefficients arising from the stiff material (matrix) defined by

$$A_1^{\mathrm{hom}}\xi\cdot\xi := \inf_{\overline{p}\in\mathcal{V}^2_{\mathrm{pot}}(\Omega)} \mathbb{E}[A_1(\xi+p)\cdot(\xi+p)(1-1_\Omega)], \qquad \xi\in\mathbb{R}^d.$$

The existence of a minimiser in $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$ for the above problem is guaranteed by [18, § 8.1] combined with the extension result [10, Lemma D.3]. Note that, in the setting of this paper, the space $\mathcal{X}$ from [10] is the closure of $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$ with respect to the seminorm $(\cdot)\mapsto\|(\cdot)(1-1_\Omega)\|_{L^2(\Omega)}$, see also Remark 2.18 below.

Remark 2.15. One can show, using Γ-convergence and Assumption 2.2, that the homogenised coefficients can be recovered almost surely from a given set of inclusions ω as

$$A_1^{\mathrm{hom}}\xi\cdot\xi = \lim_{R\to+\infty}\inf_{u\in H^1_0(\square^R)}\left\{\frac{1}{R^d}\int_{\square^R\setminus\omega}A_1(\xi+\nabla u)\cdot(\xi+\nabla u)\right\}, \qquad \xi\in\mathbb{R}^d.$$

Definition 2.16. We define the limiting operator $\mathcal{A}^{\mathrm{hom}}$ to be the self-adjoint linear operator in H associated with the bilinear form

$$\int_{\mathbb{R}^d\setminus\mathcal{D}}A_1^{\mathrm{hom}}\nabla u_0\cdot\nabla\varphi_0+\int_{\mathcal{D}}A_2\nabla u_0\cdot\nabla\varphi_0+\int_{\mathbb{R}^d\setminus\mathcal{D}}\mathbb{E}\left[\nabla_y u_1\cdot\nabla_y\varphi_1\right]dx,$$
$$u_0+\overline{u_1},\ \varphi_0+\overline{\varphi_1}\in V.$$

(See also [9, Remark 4.6] and [30].)

In an analogous manner, one can define the *unperturbed operators* $\widehat{\mathcal{A}}^\varepsilon$ and $\widehat{\mathcal{A}}^{\mathrm{hom}}$, i.e. the corresponding operators in the absence of the defect, as follows.

Definition 2.17 (Unperturbed operators)**.** We define $\widehat{\mathcal{A}}^\varepsilon$ to be the self-adjoint linear operator in $L^2(\mathbb{R}^d)$ associated with the bilinear form

$$\varepsilon^2\int_{\varepsilon\omega}\nabla u\cdot\nabla v+\int_{\mathbb{R}^d\setminus\overline{\varepsilon\omega}}A_1\nabla u\cdot\nabla v, \qquad u,v\in H^1(\mathbb{R}^d).$$

Let

$$\widehat{H}:=L^2(\mathbb{R}^d)+L^2(\mathbb{R}^d;L^2_0(\Omega)), \qquad \widehat{V}:=H^1(\mathbb{R}^d)+L^2(\mathbb{R}^d;H^1_0(\Omega)).$$

The spaces $\widehat{H}$ and $\widehat{V}$ are the analogues of H and V defined in (2.9), (2.10) obtained by formally setting $\mathcal{D}=\emptyset$.

We define $\widehat{\mathcal{A}}^{\mathrm{hom}}$ to be the self-adjoint linear operator in $\widehat{H}$ associated with the bilinear form

$$\int_{\mathbb{R}^d}A_1^{\mathrm{hom}}\nabla u_0\cdot\nabla\varphi_0+\int_{\mathbb{R}^d}\mathbb{E}\left[\nabla_y u_1\cdot\nabla_y\varphi_1\right]dx, \quad u_0+\overline{u_1},\ \varphi_0+\overline{\varphi_1}\in\widehat{V}.$$

The spectra of the unperturbed operators $\widehat{A}^\varepsilon$ and $\widehat{\mathcal{A}}^{\mathrm{hom}}$ were studied in [9, 10]. We summarise below the results therein which are relevant to our work, referring the reader to the original papers for further details and a more complete picture. Before doing so, for the convenience of the reader, let us briefly explain how the setting of [9, 10] relates to the setting of our paper, as we will be referring to results from [9, 10] several times further on.

Remark 2.18. In [9, 10] the probability framework is introduced as follows. One starts with an abstract probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P})$, $\widetilde{\mathcal{F}}$ being countably generated, equipped with an ergodic dynamical system $(\widetilde{T}_y)_{y\in\mathbb{R}^d}$, $\widetilde{T}_y:\widetilde{\Omega}\to\widetilde{\Omega}$ that satisfies the following properties:

(a) $\widetilde{T}_{y_1}\circ\widetilde{T}_{y_2}=\widetilde{T}_{y_1+y_2}$ for all $x,y\in\mathbb{R}^d$;

(b) $\widetilde{P}(\widetilde{T}_y F) = \widetilde{P}(F)$ for all $y \in \mathbb{R}^d$, $F \in \widetilde{\mathcal{F}}$;

(c) the map $\widetilde{\mathcal{T}} : \mathbb{R}^d \times \widetilde{\Omega} \to \widetilde{\Omega}$, $(y, \widetilde{\omega}) \to \widetilde{T}_y(\widetilde{\omega})$ is measurable with respect to the standard σ-algebra on the product space induced by $\widetilde{\mathcal{F}}$ and the Borel σ-algebra on $\mathbb{R}^d$.

One then fixes a 'model inclusion' $\widetilde{\mathcal{O}} \subset \widetilde{\Omega}$ (a given subset of the abstract probability space), and to each $\widetilde{\omega} \in \widetilde{\Omega}$ one associates the set of inclusions $\widetilde{\mathcal{O}}_{\widetilde{\omega}}$ (a subset of $\mathbb{R}^d$) defined as

$$\widetilde{\mathcal{O}}_{\widetilde{\omega}} := \{x \in \mathbb{R}^d : \widetilde{T}_x \widetilde{\omega} \in \widetilde{\mathcal{O}}\},$$

with the assumption that for a.e. $\widetilde{\omega} \in \widetilde{\Omega}$ the set $\widetilde{\mathcal{O}}_{\widetilde{\omega}}$ satisfies Assumption 2.2.

It is easy to see that every such abstract framework can be realised as the more 'concrete' probability space $(\Omega, \mathcal{F}, P)$ set out above as follows. We identify $\widetilde{\omega} \in \widetilde{\Omega}$ with its realisation $\widetilde{\mathcal{O}}_{\widetilde{\omega}}$ and define

$$\Omega := \{\omega := \widetilde{\mathcal{O}}_{\widetilde{\omega}} \mid \widetilde{\omega} \in \widetilde{\Omega}\}\,.$$

Next, with $\mathcal{F}$ being the σ-algebra on Ω introduced earlier in this section, on $(\Omega, \mathcal{F})$ we define the probability measure

$$P(F) := \widetilde{P}(\widetilde{F}),$$

where

$$\widetilde{F} = \{\widetilde{\omega} \in \widetilde{\Omega} : \omega \in F\}.$$

This is simply the push-forward of the probability measure $\widetilde{P}$ by the mapping $\widetilde{\omega} \mapsto \omega$, which can be easily seen to be measurable.

Note that the counterpart of $\widetilde{\mathcal{O}}$ in $(\Omega, \mathcal{F}, P)$ is

$$\mathcal{O} := \pi_0^{-1}(1),$$

that is, the set of random collections of inclusions containing the origin. Recall that the map π_0 is defined in accordance with (2.2).

Given a self-adjoint operator $\mathcal{A}$, we denote its spectrum by $\sigma(\mathcal{A})$. Furthermore, we denote by $\sigma_d(\mathcal{A})$ its discrete spectrum, i.e. isolated eigenvalues of finite multiplicity, by $\sigma_{\mathrm{ess}}(\mathcal{A}) := \sigma(\mathcal{A}) \setminus \sigma_d(\mathcal{A})$ its essential spectrum, and by $\sigma_p(\mathcal{A})$ its point spectrum.

We denote by $-\Delta_\Omega$ the positive definite self-adjoint operator on $L^2_0(\Omega)$ associated with the bilinear form

$$\mathbb{E}\left[\nabla u \cdot \nabla v\right], \qquad \overline{u}, \overline{v} \in H^1_0(\Omega).$$

For $\lambda \notin \sigma(-\Delta_\Omega)$ let $\overline{b}_\lambda \in H^1_0(\Omega)$ be the solution to $(-\Delta_\Omega - \lambda)\overline{b}_\lambda = 1$, which in weak formulation reads

$$\mathbb{E}\left[\nabla b_\lambda \cdot \nabla v - \lambda\, b_\lambda v\right] = \mathbb{E}[v] \qquad \text{for all} \quad \overline{v} \in H^1_0(\Omega). \tag{2.11}$$

Observe that in physical space we have

$$b_\lambda(y, \omega) = \sum_k b^k_\lambda(y, \omega) \tag{2.12}$$

where b^k_λ is the unique solution in $H^1_0(\omega^k)$ of $(-\Delta - \lambda) b^k_\lambda = 1$, extended by zero in $\mathbb{R}^d \setminus \omega^k$. The following spectral decomposition holds:

$$b^k_\lambda(\cdot, \omega) = \sum_j \frac{\int_{\omega^k} \phi^k_j}{\nu^k_j - \lambda} \phi^k_j, \tag{2.13}$$

where ν^k_j and ϕ^k_j are the eigenvalues (repeated according to their multiplicity) and orthonormalised eigenfunctions of $-\Delta$ with the Dirichlet boundary condition on ω^k.

Definition 2.19. We define the function $\beta : \mathbb{R} \setminus \sigma(-\Delta_\Omega) \to \mathbb{R}$ by setting

$$\beta(\lambda) := \lambda + \lambda^2\, \mathbb{E}[\overline{b}_\lambda], \tag{2.14}$$

or, equivalently,

$$\beta(\lambda) := \lambda + \lambda^2\, \mathbb{E}\left[(-\Delta_\Omega - \lambda)^{-1}\,\overline{1_\Omega}\right],$$

where 1_Ω is defined by (2.7).

This function is the direct analogue of $\beta(\lambda)$ introduced in [30] in the periodic setting, and we will refer to it as *Zhikov's β-function*.

The spectrum of the unperturbed limiting operator can be characterised purely in terms of $\beta(\lambda)$ and $\sigma(-\Delta_\Omega)$.

Theorem 2.20 (Spectrum of $\widehat{\mathcal{A}}^{\mathrm{hom}}$ [9, 10])**.**

(a) The spectrum of $-\Delta_\Omega$ is positive, detached from 0, and it is given by

$$\sigma(-\Delta_\Omega) = \overline{\bigcup_{k\in\mathbb{N}} \sigma(-\Delta_{\omega^k})} \qquad \text{for a.e. } \omega,$$

where Δ_{ω^k} is the Laplacian on ω^k with Dirichlet boundary condition.

(b) The spectrum of $\widehat{\mathcal{A}}^{\mathrm{hom}}$ is given by

$$\sigma(\widehat{\mathcal{A}}^{\mathrm{hom}}) = \sigma(-\Delta_\Omega) \cup \overline{\{\lambda \,|\, \beta(\lambda) \ge 0\}}.$$

(c) The point spectrum of $\widehat{\mathcal{A}}^{\mathrm{hom}}$ coincides with the set of eigenvalues of $-\Delta_\Omega$ whose eigenfunctions have zero mean,

$$\sigma_p(\widehat{\mathcal{A}}^{\mathrm{hom}}) = \{\lambda \in \sigma_p(-\Delta_\Omega) \,|\, \exists\, \overline{f} \in H_0^1(\Omega) \;:\; -\Delta_\Omega \overline{f} = \lambda \overline{f},\ \mathbb{E}[\overline{f}] = 0\}.$$

Note that $\sigma_{\mathrm{ess}}(\widehat{\mathcal{A}}^{\mathrm{hom}}) = \sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$. This is due to the fact that the point spectrum of $\widehat{\mathcal{A}}^{\mathrm{hom}}$ always has infinite multiplicity, cf. the proof of [10, Proposition 4.13].

In the case when $\mathbb{R}^d$ is replaced by a bounded domain $S \subset \mathbb{R}^d$ with Lipschitz boundary, it was shown in [9] that the spectrum of $\widehat{\mathcal{A}}^\varepsilon$ converges in the sense of Hausdorff to the spectrum of $\widehat{\mathcal{A}}^{\mathrm{hom}}$. However, in the whole space setting this is not the case. The set of limit points of the spectrum of $\widehat{\mathcal{A}}^\varepsilon$ as $\varepsilon \to 0$ (which will be referred to as *limiting spectrum*) is, in general, strictly larger that the spectrum of $\widehat{\mathcal{A}}^{\mathrm{hom}}$, see [10, Section 5].

In order to characterise the limiting spectrum of $\widehat{\mathcal{A}}^\varepsilon$ one needs a "global" analogue of Zhikov's β-function, which, loosely speaking, knows about the distribution of inclusions around each point in $\mathbb{R}^d$. To this end for $\lambda \notin \sigma(-\Delta_\Omega)$ we introduce the quantity

$$\ell_{\lambda,L}(x,\omega) := \frac{1}{L^d} \int_{\square_x^L} \left(\lambda + \lambda^2 b_\lambda(y,\omega)\right) dy, \tag{2.15}$$

where $L > 0$. Recall that $\square_x^L$ is defined by (2.1).

Definition 2.21. For $\lambda \notin \sigma(-\Delta_\Omega)$ we define

$$\beta_\infty(\lambda,\omega) := \liminf_{L\to+\infty} \sup_{x\in\mathbb{R}^d} \ell_{\lambda,L}(x,\omega)$$

One can show that the function $(\lambda,\omega) \mapsto \beta_\infty(\lambda,\omega)$ is deterministic almost surely [10, Proposition 5.11], i.e. $\beta_\infty(\lambda,\omega) =: \beta_\infty(\lambda)$ almost surely.

Theorem 2.22 (Limiting spectrum [10, Theorem 5.2])**.** *The limiting spectrum of the family of operators $\widehat{\mathcal{A}}^\varepsilon$ is a subset of*

$$\mathcal{G} := \sigma(-\Delta_\Omega) \cup \overline{\{\, \lambda \in \mathbb{R} \,|\, \beta_\infty(\lambda) \geq 0\}} \tag{2.16}$$

almost surely. Namely, for every sequence of elements $\lambda_\varepsilon \in \sigma(\widehat{\mathcal{A}}^\varepsilon)$ such that $\lim_{\varepsilon\to 0} \lambda_\varepsilon = \lambda_0$ we have $\lambda_0 \in \mathcal{G}$.

It is not difficult to see using the Ergodic Theorem that

$$\beta(\lambda) = \lim_{L\to+\infty} \ell_{\lambda,L}(0,\omega)$$

almost surely. Therefore $\beta(\lambda) \leq \beta_\infty(\lambda)$, and one has the following inclusions:

$$\sigma(\widehat{\mathcal{A}}^{\mathrm{hom}}) \subset \lim_{\varepsilon\to 0} \sigma(\widehat{\mathcal{A}}^\varepsilon) \subset \mathcal{G}.$$

Under an additional assumption of that the range of correlation of the distribution of inclusions in the physical space is finite, see [10, Assumption 5.4] for a precise formulation, the equality

$$\lim_{\varepsilon\to 0} \sigma(\widehat{\mathcal{A}}^\varepsilon) = \mathcal{G} \tag{2.17}$$

holds almost surely.

The operators $\mathcal{A}^\varepsilon$ and $\mathcal{A}^{\mathrm{hom}}$ are a perturbation of $\widehat{\mathcal{A}}^\varepsilon$ and $\widehat{\mathcal{A}}^{\mathrm{hom}}$, respectively, in that their coefficients differ on a relatively compact subset of $\mathbb{R}^d$, the defect $\mathcal{D}$. Even though this substantially modifies the domains of our operators, their essential spectra are stable under the introduction of a defect. This is formalised by the following

Theorem 2.23. *We have*

(i) $\sigma_{\mathrm{ess}}(\mathcal{A}^\varepsilon) = \sigma_{\mathrm{ess}}(\widehat{\mathcal{A}}^\varepsilon)$ *for every* ε *and*

(ii) $\sigma_{\mathrm{ess}}(\mathcal{A}_{\mathrm{hom}}) = \sigma_{\mathrm{ess}}(\widehat{\mathcal{A}}_{\mathrm{hom}})$.

Property (i) is a well known classical result, see, e.g., [15, Theorem 1]. For the sake of completeness, in Appendix A we provide a direct self-contained proof of this fact based on Weyl's criterion for the essential spectrum. Property (ii) for the two-scale operators can be established by retracing the arguments presented in [7, Theorem 7.1] for the periodic case, upon making appropriate changes to the "microscopic" part of the limiting operator in order to accommodate the stochastic setting.

Therefore, the presence of a defect only affects the discrete spectrum of the operators in question. In particular, if the spectra of $\widehat{\mathcal{A}}^\varepsilon$ and $\widehat{\mathcal{A}}^{\mathrm{hom}}$ have gaps, eigenvalues in the gaps — often known as *defect modes* — may appear as a result.

It is natural to ask the following questions.

Question 1. Suppose we have an eigenvalue $\lambda_0 \in \sigma_d(\mathcal{A}^{\mathrm{hom}})$ in a gap of $\mathcal{G}$, hence due to the defect. Is it true that there exist eigenvalues $\lambda_\varepsilon \in \sigma_d(\mathcal{A}^\varepsilon)$ such that $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$?

Question 2. Suppose we have a sequence of eigenvalues $\lambda_\varepsilon \in \sigma_d(\mathcal{A}^\varepsilon)$ converging to some λ_0 in a gap of $\mathcal{G}$ as $\varepsilon \to 0$. Is it true that $\lambda_0 \in \sigma_d(\mathcal{A}^{\mathrm{hom}})$?

Question 3. Suppose that the answer to Question 1 or 2 is affirmative. What can we say about the convergence of the corresponding eigenfunctions?

The main goal of this paper is to provide a rigorous answer to Questions 1, 2 and 3.

Remark 2.24. Another natural question to ask is what if an eigenvalue of $\mathcal{A}^{\mathrm{hom}}$ is in $\mathcal{G}$, i.e. $\lambda_0 \in \sigma_d(\mathcal{A}^{\mathrm{hom}}) \cap \mathcal{G}$? Can one prove that there exists a sequence of localised modes λ_ε of $\mathcal{A}^\varepsilon$ such that $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$? It was shown in [10, Proposition 5.11] that $\beta_\infty(\lambda)$ is continuous and strictly increasing on every interval contained in $\mathbb{R}^+ \setminus \sigma(-\Delta_\Omega)$. This implies that the part of the set $\mathcal{G}$ contained in the gaps of $\sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$ is a union of intervals of positive length (i.e. it contains no isolated points). Let us assume for simplicity that the equality (2.17) holds. Then for $\lambda_0 \in \mathcal{G} \setminus \sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$ and $\delta > 0$ one obviously has that

$$\dim \operatorname{ran} \widehat{E}^\varepsilon_{(\lambda_0-\delta,\lambda_0+\delta)} \to +\infty \quad \text{as} \quad \varepsilon \to 0,$$

where $\widehat{E}^\varepsilon_I$ denotes the spectral projection onto the interval $I \subset \mathbb{R}$ associated with the unperturbed operator $\widehat{\mathcal{A}}^\varepsilon$. Therefore, the usual strategy of seeking defect modes with standard tools of functional analysis utilised in the present work would no longer be useful in the set $\mathcal{G} \setminus \sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$. Indeed, such problem is related to or — when $(\lambda_0-\delta,\lambda_0+\delta) \cap \sigma(\mathcal{A}^\varepsilon) = (\lambda_0-\delta,\lambda_0+\delta)$ — is the problem of embedded eigenvalues, whose analysis is extremely challenging in general. For this reason, we refrain from studying defect modes in $\mathcal{G} \setminus \sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$ here, with the plan to perform this delicate analysis elsewhere.

3 Main results

In what follows we always assume that the set $\mathcal{G}$ defined in accordance with (2.16) has gaps, namely, it does not coincide with the whole positive real line. Note that this assumption does not yield an empty set of operators, as there are explicit examples — e.g., the random parking model or randomly scaled inclusions — for which it is satisfied, see [10, § 5.6]. More generally, consider a model with identical inclusions such that for a fixed (large enough) L the number of inclusions contained in any cube $\square^L_x$, $x \in \mathbb{R}^d$, is at least 1. Then it is not difficult to see from the definition of $\beta_\infty(\lambda)$, taking into account the spectral decomposition (2.13), that $\mathcal{G}$ has infinitely many gaps.

Our main results can be summarised in the form of three theorems stated in this section.

Theorem 3.1. *Let $\lambda_0 \in \mathbb{R} \setminus \mathcal{G}$ be an eigenvalue of $\mathcal{A}^{\mathrm{hom}}$. Then almost surely*

$$\lim_{\varepsilon\to 0} \operatorname{dist}(\lambda_0, \sigma_d(\mathcal{A}^\varepsilon)) = 0. \tag{3.1}$$

In order to state our second theorem, we need to recall the notion of two-scale convergence, adapted to the stochastic setting.

Definition 3.2 (Stochastic two-scale convergence [32])**.** Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\mathbb{R}^d)$. We say that $\{u^\varepsilon\}$ weakly stochastically two-scale converges to $\overline{u} \in L^2(\mathbb{R}^d \times \Omega)$ (for a given $\omega_0 \in \Omega$) and write $u^\varepsilon \xrightharpoonup{2} \overline{u}$ if

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} u^\varepsilon(x)\, f(x, x/\varepsilon, \omega_0)\, dx = \mathbb{E}\left[\int_{\mathbb{R}^d} \overline{u}\overline{f}\right] \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes C^\infty(\Omega). \tag{3.2}$$

We say that $\{u^\varepsilon\}$ strongly stochastically two-scale converges to $\overline{u} \in L^2(\mathbb{R}^d \times \Omega)$ and write $u^\varepsilon \xrightarrow{2} \overline{u}$ if it satisfies (3.2) and

$$\lim_{\varepsilon\to 0} \|u^\varepsilon\|_{L^2(\mathbb{R}^d)} = \|\overline{u}\|_{L^2(\mathbb{R}^d\times\Omega)}.$$

Some properties of stochastic two-scale convergence are provided in Appendix B.

The statement of the following theorem is true almost surely.

Theorem 3.3. *Let $\{\lambda_\varepsilon\}$, $\lambda_\varepsilon \in \sigma_d(\mathcal{A}^\varepsilon) \cap (\mathbb{R} \setminus \mathcal{G})$, be a sequence of eigenvalues of $\mathcal{A}^\varepsilon$ in the gaps of the limiting spectrum such that*

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon = \lambda_0 \notin \mathcal{G}. \tag{3.3}$$

Denote by $\{u^\varepsilon\}$ a sequence of corresponding normalised eigenfunctions,

$$\mathcal{A}^\varepsilon u^\varepsilon = \lambda_\varepsilon u^\varepsilon, \qquad \|u^\varepsilon\|_{L^2(\mathbb{R}^d)} = 1. \tag{3.4}$$

Then we have the following.

(a) The eigenfunctions u^ε are uniformly exponentially decaying at infinity, namely, for every

$$0 < \alpha < \sqrt{\frac{|\beta_\infty(\lambda_0)|}{\gamma}} \tag{3.5}$$

there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we have

$$\|e^{\alpha |x|} u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C,$$

where C is a constant uniform in ε, the quantity $\gamma := \max_{\lambda \in \sigma(A_1)} \lambda$ is the largest eigenvalue of the matrix A_1, and β_∞ is the function introduced in Definition 2.21.

(b) The limit λ_0 is an (isolated) eigenvalue of $\mathcal{A}^{\mathrm{hom}}$,

$$\lambda_0 \in \sigma_d(\mathcal{A}^{\mathrm{hom}}).$$

Furthermore, possibly up to extracting a subsequence, the sequence $\{u^\varepsilon\}$ strongly stochastically two-scale converges to an eigenfunction $\overline{u}^0$ of $\mathcal{A}^{\mathrm{hom}}$ corresponding to the eigenvalue λ_0.

Remark 3.4. We prove the exponential decay for the eigenfunctions whose eigenvalues converge to a point in a gap of the set $\mathcal{G}$. Note that the gaps of $\mathcal{G}$ are given by $\{\lambda : \beta_\infty(\lambda) < 0\}$. In fact, we expect that the range of admissible α's in (3.5) to extend (similarly to the periodic setting [7]) up until $\sqrt{|\beta(\lambda_0)|/\gamma}$; recall that $\beta(\lambda)$ characterises the gaps of $\sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$ as $\{\lambda : \beta(\lambda) < 0\}$. This conjecture is supported by the notions of statistically relevant and irrelevant spectra introduced in [10]. Indeed, the statistically relevant limiting spectrum of $\widehat{\mathcal{A}}^\varepsilon$ coincides with $\sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$, while the quasimodes of $\widehat{\mathcal{A}}^\varepsilon$ corresponding to the spectrum contained in the gaps of $\sigma(\widehat{\mathcal{A}}^{\mathrm{hom}})$ have most of their energy "far away" from the origin (for small enough ε). Therefore, the eigenfunctions of $\mathcal{A}^\varepsilon$ localised on the defect should not "feel" the statistically irrelevant spectrum. Unfortunately, by means of the techniques developed in the current paper we were unable to replace β_∞ with β in (3.5). Doing so would require a new set of tools which we plan to develop elsewhere, see also Remark 2.24.

The next theorem, whose statement is true almost surely, establishes an "asymptotic" one-to-one correspondence between the defect eigenvalues and eigenfunctions of $\mathcal{A}^\varepsilon$ and $\mathcal{A}^{\mathrm{hom}}$ as $\varepsilon \to 0$.

Theorem 3.5. *(a) Suppose there exists a sequence of positive real numbers $\{\varepsilon_n\}$ such that*

(i) $\lim_{n \to +\infty} \varepsilon_n = 0$ and

(ii) for every n there exist (at least) m eigenvalues $\lambda_{\varepsilon_n,1} \le \dots \le \lambda_{\varepsilon_n,m}$ of $\mathcal{A}^{\varepsilon_n}$, with account of multiplicity, satisfying

$$\lim_{n \to +\infty} \lambda_{\varepsilon_n,j} = \lambda_0 \in \mathbb{R} \setminus \mathcal{G}, \qquad j = 1, \dots, m.$$

Then, λ_0 is an eigenvalue of $\mathcal{A}^{\mathrm{hom}}$ of multiplicity at least m.

(b) Let $\lambda_0 \in \mathbb{R} \setminus \mathcal{G}$ be an eigenvalue of $\mathcal{A}^{\mathrm{hom}}$ of multiplicity m. Then, in any neighbourhood of λ_0 for sufficiently small ε there exist at least m distinct (with account of multiplicity) eigenvalues $\lambda_{\varepsilon,j} \in \sigma_d(\mathcal{A}^\varepsilon)$, $j = 1, \dots, m$, such that

$$\lim_{\varepsilon \to 0} \lambda_{\varepsilon,j} = \lambda_0, \qquad j = 1, \dots, m.$$

Remark 3.6. In the deterministic periodic setting [19] Kamotski and Smyshlyaev obtained quantitative results on the rate of convergence of the eigenvalues. Combined with the results of [7], the results from [19] imply (in our notation)

$$|\lambda_{\varepsilon,j} - \lambda_0| \le C\varepsilon^{1/2}, \quad j = 1, \dots, m,$$

and a similar statement with $\varepsilon^{1/2}$ convergence in L^2-norm can be formulated for the corresponding eigenfunctions, see [19, Theorem 7.1] for details. In the current paper, we are unable to formulate similar results, because a quantitative theory for the homogenisation corrector and other mathematical quantities appearing in (4.29) in the high-contrast stochastic setting is not yet available. This notwithstanding, as soon as one is able to quantitatively describe the convergence of the objects on the right-hand side of (4.54), the latter provides a starting point for the analysis of rate of convergence for the eigenvalues and eigenfunctions.

Remark 3.7. An explicit example was constructed in [19] in the case of the defect being a ball, $\mathcal{D} = B_R(0)$, and under the assumption of isotropy of the matrix homogenised coefficients A_1^{hom}, i.e. $A_1^{\mathrm{hom}} = a_1^{\mathrm{hom}}\mathrm{Id}$. The example shows that by changing the radius of the ball one can induce localised modes for $\mathcal{A}^{\mathrm{hom}}$ anywhere in the gaps of its essential spectrum. Note that the assumption of isotropy of A_1^{hom} is more natural in the stochastic setting. Indeed, it is not difficult to see that whenever the probability space $(\Omega, \mathcal{F}, P)$ and the dynamical system $(T_y)_{y\in\mathbb{R}^d}$ are stationary under the rotations in $\mathbb{R}^d$, the matrix of homogenised coefficients A_1^{hom} is isotropic. Hence, the argument of [19] is applicable in the stochastic setting, thus providing an example of $\mathcal{A}^{\mathrm{hom}}$ with eigenvalues in the gaps of the essential spectrum.

Our paper is structured as follows.

Section 4 is concerned with the proof of our first main result, Theorem 3.1. A series of lemmata leads up to the key technical estimate (4.28), which constitutes the central ingredient of the proof.

The subsequent two sections are concerned with the proof of Theorem 3.3. In Section 5 we show that the exponential decay of eigenfunctions corresponding to eigenvalues $\lambda_\varepsilon \in \mathcal{A}^\varepsilon$, $\lambda_\varepsilon \to \lambda_0 \notin \mathcal{G}$, is *uniform in* ε. This allows us to prove, in Section 6, that such eigenfunctions strongly stochastically two-scale converge to an eigenfunction of the limiting operator $\mathcal{A}^{\mathrm{hom}}$.

Section 7 contains the proof of Theorem 3.5, which follows from the results of Section 6.

The paper is complemented by two appendices. Appendix B contains some background material on (stochastic) two-scale convergence, whereas Appendix A provides an alternative self-contained proof of the stability of the essential spectrum of our operators when coefficients are perturbed in a relatively compact region.

4 Approximating eigenvalues of $\mathcal{A}^{\mathrm{hom}}$ in the gaps of $\mathcal{G}$

Assume that $\lambda_0 \in \mathbb{R} \setminus \mathcal{G}$ is an eigenvalue of $\mathcal{A}^{\mathrm{hom}}$ due to the defect. The task at hand is to show that, as ε tends to zero, there are eigenvalues of $\mathcal{A}^\varepsilon$ arbitrarily close to λ_0. This will be achieved by first constructing approximate eigenfunctions — *quasimodes* — for $\mathcal{A}^\varepsilon$ starting from an eigenfunction of $\mathcal{A}^{\mathrm{hom}}$ and then arguing that this implies the existence of genuine eigenvalues $\lambda_\varepsilon \in \sigma_d(\mathcal{A}^\varepsilon)$ close to λ_0 for sufficiently small ε. In doing so, we will follow a strategy proposed by Kamotski and Smyshlyaev [19] (who followed, in turn, a general strategy found, e.g., in [29]) in the periodic case, suitably adapted to our setting.

4.1 Proof of Theorem 3.1

Let $u^0(x,\omega) = u_0(x) + \overline{u}_1(x,\omega) \in V$ be an eigenfunction of $\mathcal{A}^{\mathrm{hom}}$ corresponding to λ_0, i.e. satisfying the system of equations

$$\int_{\mathbb{R}^d\setminus\mathcal{D}} A_1^{\mathrm{hom}}\nabla u_0\cdot\nabla\varphi_0 + \int_{\mathcal{D}} A_2\nabla u_0\cdot\nabla\varphi_0 = \lambda_0\int_{\mathbb{R}^d}(u_0+\mathbb{E}[\overline{u}_1])\,\varphi_0 \quad \forall\varphi_0\in H^1(\mathbb{R}^d), \tag{4.1a}$$

$$\mathbb{E}\left[(\nabla_y u_1(x,\cdot,\cdot)\cdot\nabla_y\varphi_1(x,\cdot,\cdot))\,1_\Omega\right] = \lambda_0\,\mathbb{E}[1_\Omega(u_0(x)+\overline{u}_1(x,\cdot)\overline{\varphi}_1)] \quad \forall\overline{\varphi_1}\in H_0^1(\Omega),\,\forall x\in\mathbb{R}^d\setminus\mathcal{D}. \tag{4.1b}$$

We assume that

$$\|u_0\|_{L^2(\mathbb{R}^d)} = 1.$$

It is easy to see that

$$\overline{u}_1(x,\omega) := \lambda_0\,u_0(x)\,\overline{b}_{\lambda_0}(\omega), \tag{4.2}$$

solves (4.1b), where $\overline{b}_{\lambda_0}$ is defined as in (2.11), (2.12). Substituting (4.2) into (4.1a) we obtain the macroscopic equation on u_0:

$$\int_{\mathbb{R}^d\setminus\mathcal{D}} A_1^{\mathrm{hom}}\nabla u_0\cdot\nabla\varphi_0 + \int_{\mathcal{D}} A_2\nabla u_0\cdot\nabla\varphi_0 = \beta(\lambda_0)\int_{\mathbb{R}^d\setminus\mathcal{D}} u_0\,\varphi_0 + \lambda_0\int_{\mathcal{D}} u_0\,\varphi_0 \qquad \forall\varphi_0\in H^1(\mathbb{R}^d). \tag{4.3}$$

For $0<\varepsilon\le 1$, consider the function

$$u^\varepsilon := u^0(x,x/\varepsilon,\omega) = \begin{cases} u_0(1+\lambda_0\,b^\varepsilon_{\lambda_0}) & \text{if } x\in\mathcal{I}^\varepsilon(\omega),\\ u_0 & \text{otherwise,}\end{cases} \tag{4.4}$$

where

$$b^\varepsilon_{\lambda_0}(x) := b_{\lambda_0}(x/\varepsilon,\omega)$$

is the ε-realisation of b_{λ_0}. For the remainder of this section we will drop λ_0 from the notation for b_{λ_0} and $b^\varepsilon_{\lambda_0}$ to keep it light. We will revert back to this notation later, however, when the dependence on λ_0 becomes important.

Though the function $\overline{b}$ is square integrable in probability space, its ε-realisation in physical space b^ε is only in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. This notwithstanding, the function u^ε is square integrable, because a) u_0 is exponentially decaying at infinity, see the proof of [4, Theorem VI], b) $u_0\in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ by [17, Theorem 8.24]. Also note that since the boundary of $\mathcal{D}$ is assumed to be $C^{1,\alpha}$, the results from [23] imply that

$$u_0\in H^1(\mathbb{R}^d)\cap W^{1,\infty}(\mathbb{R}^d). \tag{4.5}$$

In fact, one can specify the rate of the exponential decay of u_0 using the following direct argument dating back to Agmon [2], [3]. Consider the one-parameter family of functions $\psi_R : \mathbb{R}^d\to\mathbb{R}$,

$$\psi_R := \chi_R\,e^{2\theta|x|} + (1-\chi_R)\,e^{2\theta R}, \tag{4.6}$$

where χ_R is the characteristic function of the ball $B_R(0)$ and θ is a fixed positive constant. It is easy to see that the function ψ_R satisfies

$$|\nabla\psi_R^{1/2}|^2 \le \theta^2\psi_R. \tag{4.7}$$

Setting $\varphi_0 = \psi_R u_0$ in (4.3) and making use of the simple algebraic identity

$$|A^{1/2}\nabla(\psi_R^{1/2}u_0)|^2 = |A^{1/2}\nabla(\psi_R^{1/2})|^2(u_0)^2 + A\nabla u_0\cdot\nabla(\psi_R u_0) \quad \forall A = A^T\in GL(d,\mathbb{R}),$$

we arrive at

$$\int_{\mathbb{R}^d\setminus\mathcal{D}} |(A_1^{\mathrm{hom}})^{1/2}\nabla(\psi_R^{1/2}u_0)|^2 + \int_{\mathcal{D}} |A_2^{1/2}\nabla(\psi_R^{1/2}u_0)|^2 + \int_{\mathbb{R}^d\setminus\mathcal{D}} \big[-\beta(\lambda_0)\psi_R - |(A_1^{\mathrm{hom}})^{1/2}\nabla(\psi_R^{1/2})|^2\big](u_0)^2 = \int_{\mathcal{D}} |A_2^{1/2}\nabla(\psi_R^{1/2})|^2(u_0)^2 + \lambda_0\int_{\mathcal{D}}\psi_R(u_0)^2. \quad (4.8)$$

It is easy to see that for

$$0<\theta<\sqrt{\frac{|\beta(\lambda_0)|}{\gamma^{\mathrm{hom}}}}, \quad (4.9)$$

where γ^{hom} is the greatest eigenvalue of the matrix A_1^{hom}, one has the bound

$$-\beta(\lambda_0)\psi_R - |(A_1^{\mathrm{hom}})^{1/2}\nabla(\psi_R^{1/2})|^2 \ge C\psi_R.$$

Since the right-hand side in (4.8) is bounded uniformly in R, we see that $\psi_R(u_0)^2$ is summable in $\mathbb{R}^d$ for any R. Finally, utilising the Harnack inequality for solutions of elliptic equations, see e.g. [17, Theorem 8.17], we arrive at the pointwise estimate

$$|u_0(x)| \le Ce^{-\theta|x|}. \quad (4.10)$$

Moreover, the same bound (with the same θ) holds for the gradient of u_0 by arguing along the lines of [17, Theorem 3.9]:

$$|\nabla u_0(x)| \le Ce^{-\theta|x|}. \quad (4.11)$$

A similar strategy, although considerably more technical, since it will involve a delicate two-scale analysis, will be used in the proof of the exponential decay of the defect modes of $\mathcal{A}^\varepsilon$, see Theorem 5.1.

Lemma 4.1. *We have*

$$\|u_0\, b^\varepsilon\|_{L^2(\mathbb{R}^d)} + \|b^\varepsilon\,\nabla u_0\|_{L^2(\mathbb{R}^d)} + \varepsilon\,\|u_0\,\nabla b^\varepsilon\|_{L^2(\mathbb{R}^d)} + \varepsilon\,\|\nabla u_0\cdot\nabla b^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C, \quad (4.12)$$

where the constant C depends on λ_0 but is independent of ε.

Proof. By partitioning the ball $B_R(0)$ into spherical shells of thickness 1, for every $R\in\mathbb{N}$ we get

$$\|u_0\, b^\varepsilon\|^2_{L^2(B_R(0))} = \sum_{n=1}^R \|u_0\, b^\varepsilon\|^2_{L^2(B_n(0)\setminus B_{n-1}(0))}. \quad (4.13)$$

Arguing as in [10, Lemma D.8] we obtain

$$\|b^\varepsilon\|_{L^2(B_n(0)\setminus B_{n-1}(0))} + \varepsilon\|\nabla b^\varepsilon\|_{L^2(B_n(0)\setminus B_{n-1}(0))} \le C\, n^{\frac{d-1}{2}}. \quad (4.14)$$

Then utilising (4.10) in conjunction with (4.14) and (4.13), we arrive at

$$\|u_0\, b^\varepsilon\|_{L^2(B_R(0))} \le C\left(R^{\frac{d-1}{2}}e^{-\theta R}+1\right) \le C. \quad (4.15)$$

As the constant C on the right-hand side of (4.15) is independent of R, the latter implies

$$\|u_0\, b^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C. \quad (4.16)$$

With account of (4.5) and (4.11) analogous arguments yield

$$\|b^\varepsilon\,\nabla u_0\|_{L^2(\mathbb{R}^d)} \le C, \quad \varepsilon\,\|u_0\,\nabla b^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C, \quad \varepsilon\,\|\nabla u_0\cdot\nabla b^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C. \quad (4.17)$$

Formulae (4.16) and (4.17) imply (4.12). ☐

One may be tempted to regard u^ε as an approximate eigenfunction of $\mathcal{A}^\varepsilon$ for small ε, however, u^ε is not in the domain of $\mathcal{A}^\varepsilon$. Instead, we regard u^ε as an approximate eigenfunction of the resolvent operator $(\mathcal{A}^\varepsilon+1)^{-1}$.

Define $\widehat{u}^\varepsilon$ to be the solution of

$$(\mathcal{A}^\varepsilon+1)\widehat{u}^\varepsilon=(\lambda_0+1)u^\varepsilon. \tag{4.18}$$

Elementary results from spectral theory of self-adjoint operators give us

$$\begin{aligned}\operatorname{dist}((\lambda_0+1)^{-1},\sigma((\mathcal{A}^\varepsilon+1)^{-1}))&\le\frac{\|(\mathcal{A}^\varepsilon+1)^{-1}u^\varepsilon-(\lambda_0+1)^{-1}u^\varepsilon\|_{L^2(\mathbb{R}^d)}}{\|u^\varepsilon\|_{L^2(\mathbb{R}^d)}}\\&=(\lambda_0+1)^{-1}\frac{\|\widehat{u}^\varepsilon-u^\varepsilon\|_{L^2(\mathbb{R}^d)}}{\|u^\varepsilon\|_{L^2(\mathbb{R}^d)}}.\end{aligned} \tag{4.19}$$

Since, clearly, $\|u^\varepsilon\|_{L^2(\mathbb{R}^d)}\ge c>0$ uniformly in ε, cf. (4.4), formula (4.19) implies that proving Theorem 3.1 reduces to proving the following.

Theorem 4.2. *For every $\delta>0$ there exists $\varepsilon_0(\delta)>0$ such that for every $\varepsilon\le\varepsilon_0(\delta)$ we have*

$$\|\widehat{u}^\varepsilon-u^\varepsilon\|_{L^2(\mathbb{R}^d)}\le\delta,$$

where $\widehat{u}^\varepsilon$ and u^ε are defined by (4.18) and (4.4), respectively, which implies, by (4.18),

$$\|(\mathcal{A}^\varepsilon-\lambda_0)\widehat{u}^\varepsilon\|_{L^2(\mathbb{R}^d)}\le|\lambda_0+1|\delta. \tag{4.20}$$

Indeed, suppose we have proved Theorem 4.2. Then formulae (4.19)–(4.20) imply

$$\lim_{\varepsilon\to0}\operatorname{dist}(\lambda_0,\sigma(\mathcal{A}^\varepsilon))=0. \tag{4.21}$$

Now, since λ_0 is in the gaps of $\mathcal{G}$, by Theorem 2.22 there exists $\rho>0$ such that, for sufficiently small ε,

$$(\lambda_0-\rho,\lambda_0+\rho)\cap\sigma(\widehat{\mathcal{A}}^\varepsilon)=\emptyset. \tag{4.22}$$

Due to the stability of the essential spectrum under the introduction of a defect — see Theorem A.1 — formula (4.22) implies

$$(\lambda_0-\rho,\lambda_0+\rho)\cap\sigma_{\mathrm{ess}}(\mathcal{A}^\varepsilon)=\emptyset. \tag{4.23}$$

By combining (4.23) and (4.21) we arrive at (3.1).

The remainder of this section is devoted to the proof of Theorem 4.2.

4.2 A key technical estimate

Let $a^\varepsilon:H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)\to\mathbb{R}$ be the positive definite symmetric bilinear form associated with the operator $\mathcal{A}^\varepsilon+\mathrm{Id}$,

$$a^\varepsilon(u,v):=\int_{\mathcal{D}}A_2\nabla u\cdot\nabla v+\int_{\mathcal{M}^\varepsilon}A_1\nabla u\cdot\nabla v+\varepsilon^2\int_{\mathcal{I}^\varepsilon}\nabla u\cdot\nabla v+\int_{\mathbb{R}^d}uv. \tag{4.24}$$

To begin with we modify u^ε by adding the first order homogenisation corrector. Let $\overline{p}_j\in\mathcal{V}^2_{\mathrm{pot}}(\Omega)$ be the solution to the problem

$$\mathbb{E}\left[(A_1(e_j+p_j)\cdot\nabla\varphi)(1-1_\Omega)\right]=0\qquad\forall\overline{\varphi}\in C^\infty(\Omega),$$

where e_j, $j=1,\dots,d$, is the standard basis in $\mathbb{R}^d$. The existence of $\overline{p}_j\in\mathcal{V}^2_{\mathrm{pot}}(\Omega)$ is guaranteed by [18, § 8.1] combined with the extension result [10, Lemma D.3].

Then for every L and ε there exists $N_j^\varepsilon \in H^1(B_L(0))$ such that

$$\nabla N_j^\varepsilon(x) = p_j(x/\varepsilon, \omega) \qquad x \in B_L(0). \tag{4.25a}$$

One can choose N_j^ε so that

$$\int_{B_L(0)} N_j^\varepsilon = 0. \tag{4.25b}$$

By the Ergodic Theorem 2.7 we have the convergence

$$\nabla N_j^\varepsilon \rightharpoonup \mathbb{E}[\overline{p}_j] = 0 \text{ weakly in } L^2(B_L(0)).$$

The latter implies the bound

$$\|\nabla N_j^\varepsilon\|_{L^2(B_L(0))} \le C, \tag{4.25c}$$

and, together with (4.25b) — convergence to zero of the corrector:

$$\lim_{\varepsilon\to 0} \|N_j^\varepsilon\|_{L^2(B_L(0))} = 0. \tag{4.25d}$$

We call the functions N_j^ε the *first order homogenisation correctors*. Note that, in general, N_j^ε can not be represented as the realisation of a function defined in probability space Ω.

Let $\eta : \mathbb{R}^d \to [0,1]$ be an infinitely smooth function such that $\eta|_{B_{1/2}(0)} = 1$ and $\operatorname{supp}\eta \subset B_1(0)$. We denote $\eta_L := \eta(\,\cdot/L)$, thus having

$$|\nabla \eta_L(x)| \le \frac{C}{L} \qquad \forall x \in \mathbb{R}^d.$$

For every sufficiently small $\rho > 0$ put

$$\mathcal{D}^\rho := \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \mathcal{D}) < \rho\} \tag{4.26}$$

and let ξ_ρ be an infinitely smooth cut-off function satisfying

(i) $0 \le |\xi_\rho(x)| \le 1$ for all $x \in \mathbb{R}^d$,

(ii) $\operatorname{supp}\xi_\rho \subset \mathbb{R}^d \setminus \mathcal{D}^\rho$,

(iii) $\xi_\rho(x) = 1$ for $x \in \mathbb{R}^d \setminus \mathcal{D}^{2\rho}$,

(iv) $|\nabla \xi_\rho(x)| \le \frac{C}{\rho}$ for all $x \in \mathbb{R}^d$.

We define $u_{LC}^\varepsilon \in H^1(\mathbb{R}^d)$ as

$$\begin{aligned} u_{LC}^\varepsilon &:= u^\varepsilon + \eta_L \xi_\rho N_j^\varepsilon\, \partial_j u_0 \\ &= u_0 + \begin{cases} \lambda_0\, b^\varepsilon\, u_0 + \eta_L \xi_\rho N_j^\varepsilon\, \partial_j u_0 & \text{if } x \in \mathcal{I}^\varepsilon(\omega), \\ 0 & \text{if } x \in \mathcal{D}, \\ \eta_L \xi_\rho N_j^\varepsilon\, \partial_j u_0 & \text{if } x \in \mathcal{M}^\varepsilon(\omega), \end{cases} \end{aligned} \tag{4.27}$$

The quantity $\rho > 0$ is here a free (small) parameter, which will be specified later as a suitable function of ε. Here and further on we adopt Einstein's summation convention and assume that L is sufficiently large, so that $\operatorname{supp}(1 - \eta_L) \subset \operatorname{supp}\xi_\rho$.

The two cut-off functions in (4.27) serve the purpose of ensuring that $u_{LC}^\varepsilon \in H^1(\mathbb{R}^d)$; more precisely,

- η_L guarantees that $\eta_L N_j^\varepsilon$ is L^2-summable,

- ξ_ρ cuts u_0 away from $\partial\mathcal{D}$, for we do not know whether the second derivatives of u_0 are bounded in L^∞-norm in a neighbourhood of $\partial\mathcal{D}$, cf. (4.5).

The following lemma is the key technical result of this section.

Lemma 4.3. *We have*

$$|a^\varepsilon(u^\varepsilon_{LC}-\widehat{u}^\varepsilon,v)|\le C\,\mathcal{C}(\varepsilon,\rho,L)\sqrt{a^\varepsilon(v,v)}\qquad \forall v\in H^1(\mathbb{R}^d),\tag{4.28}$$

where C is a constant independent of ε,ρ and L,

$$\begin{aligned}\mathcal{C}(\varepsilon,\rho,L)&:=L^{\frac{d-1}{2}}e^{-\theta L}+\varepsilon+\rho^{1/2}+\varepsilon\big\|\textstyle\sum_j|\nabla N_j^\varepsilon|\big\|_{L^2(B_L(0))}+\|B^\varepsilon\|_{L^2(B_L(0))}\\&+\Big(\sup_j\|\nabla\partial_j u_0\|_{L^\infty(\mathbb{R}^d\setminus\mathcal{D}^\rho)}+\frac{1}{L}+\frac{1}{\rho}\Big)\Big(\big\|\textstyle\sum_j|N_j^\varepsilon|\big\|_{L^2(B_L(0))}+\big\|\textstyle\sum_j|G_j^\varepsilon|_F\big\|_{L^2(B_L(0))}\Big)\end{aligned}\tag{4.29}$$

and

- *θ is as in (4.9)–(4.11);*
- *$\mathcal{D}^\rho$ is defined in accordance with (4.26);*
- *$\{N_j^\varepsilon\}_{j=1}^d$ are the first order correctors (4.25a)–(4.25d);*
- *$\{G_j^\varepsilon\}_{j=1}^d$ are matrix-valued functions in $H^1(B_L(0);\mathbb{R}^{n\times n})$ satisfying*[2]

$$\begin{gathered}\partial_l(G_j^\varepsilon)_{lk}=\Big(\mathbb{1}_{\mathcal{M}^\varepsilon}A_1(e_j+\nabla N_j^\varepsilon)-A_1^{\mathrm{hom}}e_j\Big)_k\qquad \text{in } B_L(0),\ k=1,\dots,d,\\ \lim_{\varepsilon\to0}\||G_j^\varepsilon|_F\|_{L^2(B_L(0))}=0;\end{gathered}\tag{4.30}$$

- *B^ε is a vector-valued function in $H^1(B_L(0);\mathbb{R}^d)$ satisfying*

$$\operatorname{div}B^\varepsilon=(1-\mathbb{1}_\mathcal{D})(\mathbb{E}[\bar b]-b^\varepsilon)\qquad \text{in } B_L(0),\tag{4.31}$$

$$\int_{B_L(0)}B^\varepsilon=0,\qquad \lim_{\varepsilon\to0}\|B^\varepsilon\|_{L^2(B_L(0))}=0.\tag{4.32}$$

Remark 4.4. Note that the constant C in (4.28) depends on λ_0 through the L^∞-norm of u_0 and ∇u_0, and $\mathcal{C}(\varepsilon,\rho,L)$ depends on λ_0 through θ, B^ε and $\nabla\partial_j u_0, j=1,\dots,d$.

Remark 4.5. It is straightforward to see from (4.25c), (4.25d), (4.30), and (4.32) that

$$\lim_{L\to\infty}\lim_{\rho\to0}\lim_{\varepsilon\to0}\mathcal{C}(\varepsilon,\rho,L)=0.\tag{4.33}$$

Proof. Let v be an arbitrary element in $H^1(\mathbb{R}^d)$. Consider the quantity

$$a^\varepsilon(u^\varepsilon_{LC},v)=\int_\mathcal{D}A_2\nabla u_0\cdot\nabla v+\varepsilon^2\int_{\mathcal{I}^\varepsilon}\nabla u^\varepsilon_{LC}\cdot\nabla v+\int_{\mathcal{M}^\varepsilon}A_1\nabla u^\varepsilon_{LC}\cdot\nabla v+\int_{\mathbb{R}^d}u^\varepsilon_{LC}\,v.\tag{4.34}$$

For convenience, let us denote

$$\mathcal{T}_0^\varepsilon:=\varepsilon^2\int_{\mathcal{I}^\varepsilon}\nabla u^\varepsilon_{LC}\cdot\nabla v\tag{4.35}$$

and

$$\mathcal{T}_1^\varepsilon:=\int_{\mathcal{M}^\varepsilon}A_1\nabla u^\varepsilon_{LC}\cdot\nabla v.\tag{4.36}$$

[2]Here $|\cdot|_F$ denotes the Frobenius matrix norm.

The first step of the proof consists in analysing (4.35) and (4.36). To this end, let us denote by $\widetilde{v}^\varepsilon$ an extension of $v|_{\mathbb{R}^d\setminus\mathcal{I}^\varepsilon}$ into $\mathcal{I}^\varepsilon$ satisfying

$$\widetilde{v}^\varepsilon|_{\mathbb{R}^d\setminus\mathcal{I}^\varepsilon} = v|_{\mathbb{R}^d\setminus\mathcal{I}^\varepsilon}\,, \qquad \Delta\widetilde{v}^\varepsilon|_{\mathcal{I}^\varepsilon} = 0, \tag{4.37a}$$

$$\|\nabla\widetilde{v}^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C\|\nabla v\|_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}, \qquad \|\widetilde{v}^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C'\|v\|_{H^1(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}, \tag{4.37b}$$

whose existence is guaranteed by Theorem 2.9. Note that the constants C, C' in (4.37b) are independent of v and ε. It is easy to see that the functions v and $\widetilde{v}^\varepsilon$ satisfy the estimate

$$\|\widetilde{v}^\varepsilon\|_{L^2(\mathbb{R}^d)} + \|\nabla\widetilde{v}^\varepsilon\|_{L^2(\mathbb{R}^d)} + \varepsilon\|\nabla v\|_{L^2(\mathbb{R}^d)} \le C\sqrt{a^\varepsilon(v,v)}. \tag{4.38}$$

It is also easy to see that the function $v_0^\varepsilon := v - \widetilde{v}^\varepsilon \in H_0^1(\mathcal{I}^\varepsilon)$ satisfies

$$\|v_0^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C\varepsilon\|\nabla v_0^\varepsilon\|_{L^2(\mathbb{R}^d)}, \qquad \|\nabla v_0^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C\|\nabla v\|_{L^2(\mathbb{R}^d)}. \tag{4.39}$$

We are now in a position to examine (4.35). We have

$$\begin{aligned}
\mathcal{T}_0^\varepsilon =&\varepsilon^2\int_{\mathcal{I}^\varepsilon}\nabla\left(u_0(1+\lambda_0\,b^\varepsilon)+\eta_L\,\xi_\rho\,N_j^\varepsilon\,\partial_j u_0\right)\cdot\nabla v\\
=&\varepsilon^2\int_{\mathcal{I}^\varepsilon}(1+\lambda_0\,b^\varepsilon)\nabla u_0\cdot\nabla v + \varepsilon^2\int_{\mathcal{I}^\varepsilon}\lambda_0\,u_0\nabla b^\varepsilon\cdot\nabla(v_0^\varepsilon+\widetilde{v}^\varepsilon)\\
&+\varepsilon^2\int_{\mathcal{I}^\varepsilon}\left[N_j^\varepsilon\,(\eta_L\xi_\rho\nabla\partial_j u_0+\xi_\rho\,(\partial_j u_0)\nabla\eta_L+\eta_L(\partial_j u_0)\nabla\xi_\rho)+\eta_L\,\xi_\rho\,(\partial_j u_0)\nabla N_j^\varepsilon\right]\cdot\nabla v.
\end{aligned}\tag{4.40}$$

Integrating by parts and using the identity

$$-\varepsilon^2\Delta b^\varepsilon = \lambda_0 b^\varepsilon + 1$$

we obtain

$$\varepsilon^2\int_{\mathcal{I}^\varepsilon}\lambda_0\,u_0\nabla b^\varepsilon\cdot\nabla v_0^\varepsilon = \lambda_0\int_{\mathcal{I}^\varepsilon}u_0(\lambda_0\,b^\varepsilon+1)\,v_0^\varepsilon - \varepsilon^2\lambda_0\int_{\mathcal{I}^\varepsilon}\nabla u_0\cdot\nabla b^\varepsilon\,v_0^\varepsilon.$$

In view of Lemma 4.1 and (4.39), we have

$$\left|\varepsilon^2\int_{\mathcal{I}^\varepsilon}(1+\lambda_0\,b^\varepsilon)\nabla u_0\cdot\nabla v\right| \le C\varepsilon^2\|\nabla v\|_{L^2(\mathbb{R}^d)},$$

$$\left|\varepsilon^2\lambda_0\int_{\mathcal{I}^\varepsilon}\nabla u_0\cdot\nabla b^\varepsilon\,v_0^\varepsilon\right| \le C\varepsilon\|v_0^\varepsilon\|_{L^2(\mathbb{R}^d)},$$

$$\left|\varepsilon^2\lambda_0\int_{\mathcal{I}^\varepsilon}u_0\nabla b^\varepsilon\cdot\nabla\widetilde{v}^\varepsilon\right| \le C\varepsilon\|\nabla\widetilde{v}^\varepsilon\|_{L^2(\mathbb{R}^d)},$$

$$\begin{aligned}
&\left|\varepsilon^2\int_{\mathcal{I}^\varepsilon}N_j^\varepsilon\,(\eta_L\xi_\rho\nabla\partial_j u_0+\xi_\rho\,(\partial_j u_0)\nabla\eta_L+\eta_L(\partial_j u_0)\nabla\xi_\rho)\cdot\nabla v\right|\\
&\qquad\le C\varepsilon^2\left(\sup_j\|\nabla\partial_j u_0\|_{L^\infty(\mathbb{R}^d\setminus\mathcal{D}^\rho)}+\frac{1}{L}+\frac{1}{\rho}\right)\left\|\textstyle\sum_j|N_j^\varepsilon|\right\|_{L^2(B_L(0))}\|\nabla v\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

$$\left|\varepsilon^2\int_{\mathcal{I}^\varepsilon}\eta_L\,\xi_\rho\,(\partial_j u_0)\nabla N_j^\varepsilon\cdot\nabla v\right| \le C\varepsilon^2\left\|\textstyle\sum_j|\nabla N_j^\varepsilon|\right\|_{L^2(B_L(0))}\|\nabla v\|_{L^2(\mathbb{R}^d)}.$$

Hence, using (4.38) and (4.39) we can recast (4.40) as

$$\mathcal{T}_0^\varepsilon = \lambda_0\int_{\mathcal{I}^\varepsilon}u_0(\lambda_0\,b^\varepsilon+1)\,v_0^\varepsilon + \mathcal{R}_0^\varepsilon, \tag{4.41}$$

with

$$|\mathcal{R}_0^\varepsilon| \le C\varepsilon \left[1 + \left(\sup_j \|\nabla \partial_j u_0\|_{L^\infty(\mathbb{R}^d\setminus\mathcal{D}^\rho)} + \frac{1}{L} + \frac{1}{\rho}\right) \left\|\sum_j |N_j^\varepsilon|\right\|_{L^2(B_L(0))} + \left\|\sum_j |\nabla N_j^\varepsilon|\right\|_{L^2(B_L(0))}\right] \sqrt{a^\varepsilon(v,v)}.$$

Let us move on to $\mathcal{T}_1^\varepsilon$. By adding and subtracting

$$\int_{\mathbb{R}^d\setminus\mathcal{D}} \xi_\rho\, A_1^{\mathrm{hom}} \nabla u_0 \cdot \nabla \widetilde{v}^\varepsilon$$

from the right-hand side of (4.36), we obtain

$$\begin{aligned}
\mathcal{T}_1^\varepsilon = &\int_{\mathbb{R}^d\setminus\mathcal{D}} \xi_\rho\, A_1^{\mathrm{hom}} \nabla u_0 \cdot \nabla \widetilde{v}^\varepsilon + \int_{S_1^\varepsilon} (1-\xi_\rho) A_1 \nabla u_0 \cdot \nabla \widetilde{v}^\varepsilon \\
&+ \int_{\mathbb{R}^d} \eta_L\, \xi_\rho \left(\mathbb{1}_{\mathcal{M}^\varepsilon} A_1(e_j + \nabla N_j^\varepsilon) - A_1^{\mathrm{hom}} e_j\right) \partial_j u_0 \cdot \nabla \widetilde{v}^\varepsilon \\
&+ \int_{\mathbb{R}^d} (1-\eta_L)\xi_\rho \left(\mathbb{1}_{\mathcal{M}^\varepsilon} A_1 - A_1^{\mathrm{hom}}\right) \nabla u_0 \cdot \nabla \widetilde{v}^\varepsilon \\
&+ \int_{\mathcal{M}^\varepsilon} \eta_L\, \xi_\rho\, N_j^\varepsilon\, \nabla \partial_j u_0 \cdot \nabla \widetilde{v}^\varepsilon + \int_{\mathcal{M}^\varepsilon} N_j^\varepsilon\, \partial_j u_0 \nabla(\eta_L \xi_\rho) \cdot \nabla \widetilde{v}^\varepsilon.
\end{aligned} \tag{4.42}$$

By [10, Corollary D.5], for every $j = 1, \dots, d$ there exists a skew-symmetric matrix-valued function $G_j^\varepsilon \in H^1(B_L(0); \mathbb{R}^{d\times d})$) such that

$$\lim_{\varepsilon\to 0} \||G_j^\varepsilon|_F\|_{L^2(B_L(0))} = 0$$

and

$$\partial_l (G_j^\varepsilon)_{lk} = \left(\mathbb{1}_{\mathcal{M}^\varepsilon} A_1(e_j + \nabla N_j^\varepsilon) - A_1^{\mathrm{hom}} e_j\right)_k, \qquad x \in B_L(0).$$

Integrating by parts and using the skew-symmetry of G_j^ε, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \eta_L\, \xi_\rho \left(\mathbb{1}_{\mathcal{M}^\varepsilon} A_1(e_j + \nabla N_j^\varepsilon) - A_1^{\mathrm{hom}} e_j\right) \partial_j u_0 \cdot \nabla \widetilde{v}^\varepsilon \\
&\qquad = \int_{\mathbb{R}^d} \eta_L\, \xi_\rho\, \partial_l (G_j^\varepsilon)_{lk}\, \partial_j u_0\, \partial_k \widetilde{v}^\varepsilon = -\int_{\mathbb{R}^d} (G_j^\varepsilon)_{lk}\, \partial_k \widetilde{v}^\varepsilon\, \partial_l(\eta_L\, \xi_\rho\, \partial_j u_0).
\end{aligned} \tag{4.43}$$

Recalling (4.5), (4.11), using (4.43) and (4.38) we can rewrite (4.42) as

$$\mathcal{T}_1^\varepsilon = \int_{\mathbb{R}^d} \xi_\rho\, A_1^{\mathrm{hom}} \nabla u_0 \cdot \nabla \widetilde{v}^\varepsilon + \mathcal{R}_1^\varepsilon, \tag{4.44}$$

with

$$|\mathcal{R}_1^\varepsilon| \le C\Big[e^{-\theta L} + |\mathcal{D}^{2\rho} \setminus \mathcal{D}|^{1/2} + \left(\sup_j \|\nabla \partial_j u_0\|_{L^\infty(\mathbb{R}^d\setminus\mathcal{D}^\rho)} + \frac{1}{L} + \frac{1}{\rho}\right)\left(\|\textstyle\sum_j |N_j^\varepsilon|\|_{L^2(B_L(0))} + \|\textstyle\sum_j |G_j^\varepsilon|_F\|_{L^2(B_L(0))}\right)\Big] \sqrt{a^\varepsilon(v,v)}.$$

It is straightforward to see that

$$\left|\int_{\mathbb{R}^d\setminus\mathcal{D}} (1-\xi_\rho)\, A_1^{\mathrm{hom}} \nabla u_0 \cdot \nabla \widetilde{v}^\varepsilon\right| \le C |\mathcal{D}^{2\rho} \setminus \mathcal{D}|^{1/2} \sqrt{a^\varepsilon(v,v)}.$$

Clearly, $|\mathcal{D}^{2\rho} \setminus \mathcal{D}|^{1/2} \le C\rho^{1/2}$. Hence, substituting (4.41), (4.35) and (4.44), (4.36) into (4.34), we arrive at

$$a^\varepsilon(u^\varepsilon_{LC}, v) = \int_{\mathbb{R}^d\setminus\mathcal{D}} A_1^{\mathrm{hom}} \nabla u_0 \cdot \nabla \widetilde{v}^\varepsilon + \lambda_0 \int_{\mathcal{I}^\varepsilon} u_0(\lambda_0 b^\varepsilon + 1)\, v_0^\varepsilon \\ + \int_{\mathcal{D}} A_2 \nabla u_0 \cdot \nabla v + \int_{\mathbb{R}^d} u^\varepsilon_{LC} v + \mathcal{R}^\varepsilon, \quad (4.45)$$

where

$$|\mathcal{R}^\varepsilon| \le C \Bigg[\varepsilon + e^{-\theta L} + \rho^{1/2} + \varepsilon \left\| \textstyle\sum_j |\nabla N_j^\varepsilon| \right\|_{L^2(B_L(0))} + \\ \left(\sup_j \|\nabla \partial_j u_0\|_{L^\infty(\mathbb{R}^d\setminus\mathcal{D}^\rho)} + \frac{1}{L} + \frac{1}{\rho} \right) \left(\left\| \textstyle\sum_j |N_j^\varepsilon| \right\|_{L^2(B_L(0))} + \left\| \textstyle\sum_j |G_j^\varepsilon|_F \right\|_{L^2(\square_L)} \right) \Bigg] \sqrt{a^\varepsilon(v,v)}. \quad (4.46)$$

Taking into account (4.3) we rewrite (4.45) as

$$a^\varepsilon(u^\varepsilon_{LC}, v) = \beta(\lambda_0) \int_{\mathbb{R}^d\setminus\mathcal{D}} u_0\, \widetilde{v}^\varepsilon + \lambda_0 \int_{\mathcal{I}^\varepsilon} u_0(\lambda_0 b^\varepsilon + 1)\, v_0^\varepsilon \\ + \lambda_0 \int_{\mathcal{D}} u_0\, v + \int_{\mathbb{R}^d} u^\varepsilon_{LC} v + \mathcal{R}^\varepsilon$$

Then utilising the identity

$$a^\varepsilon(\widehat{u}^\varepsilon, v) = (\lambda_0 + 1) \int_{\mathbb{R}^d} u^\varepsilon v,$$

which follows from (4.18), and recalling (2.14), (4.4) and (4.27), we arrive at

$$a^\varepsilon(u^\varepsilon_{LC} - \widehat{u}^\varepsilon, v) = \int_{\mathbb{R}^d\setminus\mathcal{D}} \lambda_0^2\, u_0(\mathbb{E}[\overline{b}] - b^\varepsilon)\widetilde{v}^\varepsilon + \int_{\mathbb{R}^d} \eta_L\, \xi_\rho\, N_j^\varepsilon\, (\partial_j u_0)\, v + \mathcal{R}^\varepsilon. \quad (4.47)$$

It remains to estimate the two integrals on the right-hand side of (4.47). By the Ergodic Theorem, the sequence of functions $(1 - \mathbb{1}_{\mathcal{D}})(\mathbb{E}[\overline{b}] - b^\varepsilon)$ converges to zero weakly in $L^2(B_L(0))$ as $\varepsilon \to 0$. Therefore, [10, Lemma D.6] ensures the existence of a sequence of functions $B^\varepsilon \in H^1(B_L(0); \mathbb{R}^d)$ satisfying (4.31)–(4.32). The introduction of the cut-off function $(1 - \mathbb{1}_{\mathcal{D}})$ will allow us to extend the integrals to the whole of $\mathbb{R}^d$ and avoid dealing with boundary terms when integrating by parts.

Let us decompose the first integral on the right-hand side of (4.47) as

$$\int_{\mathbb{R}^d\setminus\mathcal{D}} \lambda_0^2\, u_0(\mathbb{E}[\overline{b}] - b^\varepsilon)\widetilde{v}^\varepsilon = \int_{\mathbb{R}^d\setminus\mathcal{D}} \lambda_0^2\, \eta_L\, u_0(\mathbb{E}[\overline{b}] - b^\varepsilon)\widetilde{v}^\varepsilon + \int_{\mathbb{R}^d\setminus\mathcal{D}} \lambda_0^2\, (1-\eta_L)\, u_0(\mathbb{E}[\overline{b}] - b^\varepsilon)\widetilde{v}^\varepsilon.$$

Utilising (4.39), integrating by parts and resorting once again to (4.38), we obtain

$$\left| \int_{\mathbb{R}^d\setminus\mathcal{D}} \lambda_0^2 \eta_L\, u_0(\mathbb{E}[\overline{b}] - b^\varepsilon)\widetilde{v}^\varepsilon \right| = \left| \lambda_0^2 \int_{\mathbb{R}^d} B^\varepsilon \cdot \nabla(\eta_L u_0 \widetilde{v}^\varepsilon) \right| \le C \|B^\varepsilon\|_{L^2(B_L(0))} \sqrt{a^\varepsilon(v,v)}. \quad (4.48)$$

Arguing as in Lemma 4.1, it is easy to see that

$$\|u_0(\mathbb{E}[\overline{b}] - b^\varepsilon)\|_{L^2(\mathbb{R}^d\setminus B_{L/2}(0))} \le C L^{\frac{d-1}{2}} e^{-\theta L}.$$

Therefore we obtain

$$\left| \int_{\mathbb{R}^d\setminus\mathcal{D}} \lambda_0^2\, (1-\eta_L)\, u_0(\mathbb{E}[\overline{b}] - b^\varepsilon)\widetilde{v}^\varepsilon \right| \le C L^{\frac{d-1}{2}} e^{-\theta L} \sqrt{a^\varepsilon(v,v)}. \quad (4.49)$$

Formulae (4.48) and (4.49) imply

$$\left| \int_{\mathbb{R}^d \setminus \mathcal{D}} \lambda_0^2 \, u_0(\mathbb{E}[\overline{b}] - b^\varepsilon) \widetilde{v}^\varepsilon \right| \le C \left(\|B^\varepsilon\|_{L^2(B_L(0))} + L^{\frac{d-1}{2}} e^{-\theta L} \right) \sqrt{a^\varepsilon(v,v)}. \tag{4.50}$$

The estimate of the second integral on the right-hand side of (4.47) is straightforward:

$$\left| \int_{\mathbb{R}^d} \eta_L \, \xi_\rho \, N_j^\varepsilon \, (\partial_j u_0) \, v \right| \le C \left\| \sum_j |N_j^\varepsilon| \right\|_{L^2(B_L(0))} \sqrt{a^\varepsilon(v,v)}. \tag{4.51}$$

Finally, combining (4.47), (4.46), (4.50) and (4.51) we arrive at (4.28)–(4.29). $\square$

4.3 Proof of Theorem 4.2

We are now in a position to prove Theorem 4.2.

Proof of Theorem 4.2. By choosing $v = u^\varepsilon_{LC} - \widehat{u}^\varepsilon$ in (4.28) one gets

$$\|u^\varepsilon_{LC} - \widehat{u}^\varepsilon\|_{L^2(\mathbb{R}^d)} \le \sqrt{a^\varepsilon(u^\varepsilon_{LC} - \widehat{u}^\varepsilon, u^\varepsilon_{LC} - \widehat{u}^\varepsilon)} \le C\, \mathcal{C}(\varepsilon, \rho, L). \tag{4.52}$$

Comparing (4.4) and (4.27) we have

$$\|u^\varepsilon - u^\varepsilon_{LC}\|_{L^2(\mathbb{R}^d)} \le C \left\| \sum_j |N_j^\varepsilon| \right\|_{L^2(B_L(0))}. \tag{4.53}$$

Therefore, from (4.52) and (4.53) we obtain

$$\|\widehat{u}^\varepsilon - u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C \left(\mathcal{C}(\varepsilon, \rho, L) + \left\| \sum_j |N_j^\varepsilon| \right\|_{L^2(B_L(0))} \right). \tag{4.54}$$

The result follows immediately from (4.25d) and (4.33) by choosing in (4.27) sufficiently large L and sufficiently small ρ. $\square$

Remark 4.6. Observe that, unlike previous related works [19, 7], we assume the boundary of the defect to be only $C^{1,\alpha}$. As a result, we do not have L^∞ control of the components of the Hessian of u_0. However, if one knew that

$$u_0 \in H^1(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d), \tag{4.55}$$

then from (4.29) and (4.54) it would follow that

$$\begin{aligned} \|\widehat{u}^\varepsilon - u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C \Bigg[& L^{\frac{d-1}{2}} e^{-\theta L} + \varepsilon + |\mathcal{D}^{2\rho} \setminus \mathcal{D}|^{1/2} + \varepsilon \left\| \sum_j |\nabla N_j^\varepsilon| \right\|_{L^2(B_L(0))} + \|B^\varepsilon\|_{L^2(B_L(0))} \\ & + \left(1 + \frac{1}{\rho}\right) \left(\left\| \sum_j |N_j^\varepsilon| \right\|_{L^2(B_L(0))} + \left\| \sum_j |G_j^\varepsilon|_F \right\|_{L^2(B_L(0))} \right) \Bigg]. \end{aligned}$$

In particular, this would mean that one can explicitly specify the rate of convergence of ρ to zero in terms of N_j^ε and G_j^ε only, and quantitative estimates for the latter and the term B^ε would translate, when available, into quantitative estimates for the convergence of eigenfunctions. See also Remark 3.6.

The property (4.55) is certainly guaranteed if the defect $\mathcal{D}$ has smooth boundary (this is the assumption made in [19]), as can be shown by adapting the argument from [13, Theorem 5.3.8] to the case of smooth interface. In fact, one can show that (4.55) holds under a much weaker regularity assumption on $\partial\mathcal{D}$, namely, that $\partial\mathcal{D}$ is of $C^{2,\alpha}$ class for $\alpha > 0$. In this case one can show that each component of u_0 is in $C^{2,\alpha'}(\mathbb{R}^d)$ for some α', $0 < \alpha' < \alpha$, by pushing the argument of [23, Theorem 1.1] to higher regularity and combining it with the technique of difference quotients (after localising and flattening the boundary).

5 Uniform exponential decay of eigenfunctions

The current section and the next are devoted to proving Theorem 3.3.

We assume (3.3) and (3.4). It is well known that the eigenfunctions u^ε decay exponentially, namely, for a fixed ε there exist $\alpha(\varepsilon), c(\varepsilon) > 0$ such that

$$\|e^{\alpha(\varepsilon)|x|}u^\varepsilon\|_{L_2(\mathbb{R}^d)} \le c(\varepsilon),$$

see, e.g., [4, 15]. The goal of this section is to show that the exponential decay of the family $\{u^\varepsilon\}$ is, in fact, *uniform in* ε. More precisely, we will prove the following theorem, which, in particular, implies Theorem 3.3 part (a).

Theorem 5.1. *Let $\{\lambda_\varepsilon\}$ and $\{u^\varepsilon\}$ satisfy (3.3) and (3.4). Let $\widetilde{u}^\varepsilon$ be an extension*[3] *of $u^\varepsilon|_{\mathbb{R}^d\setminus\mathcal{I}^\varepsilon}$ into $\mathcal{I}^\varepsilon$ satisfying*

$$\widetilde{u}^\varepsilon|_{\mathbb{R}^d\setminus\mathcal{I}^\varepsilon} = u^\varepsilon|_{\mathbb{R}^d\setminus\mathcal{I}^\varepsilon}, \qquad \Delta\widetilde{u}^\varepsilon|_{\mathcal{I}^\varepsilon} = 0, \tag{5.1a}$$

$$\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C_1\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}, \tag{5.1b}$$

$$\|\widetilde{u}^\varepsilon\|_{H^1(\mathbb{R}^d)} \le C_2\,\|u^\varepsilon\|_{H^1(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}, \tag{5.1c}$$

where C_1 and C_2 are constants independent of ε. Then, for every $0 < \alpha < \sqrt{|\beta_\infty(\lambda_0)|/\gamma}$ there exist $\varepsilon_0 = \varepsilon_0(\omega) > 0$ and $C > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we have

$$\|e^{\alpha|x|}u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C \tag{5.2}$$

and

$$\|e^{\alpha|x|}\widetilde{u}^\varepsilon\|_{H^1(\mathbb{R}^d)} \le C, \tag{5.3}$$

where $\gamma := \max_{\lambda\in\sigma(A_1)}\lambda$.

5.1 Preparatory lemmata

Before addressing the proof of Theorem 5.1, which relies on a version of Agmon's operator positivity method [2, 3] (whose basic idea is illustrated by the argument leading to the bound (4.10)) adapted to our setting, in the spirit of [7], we need to state and prove a few preparatory lemmata.

Lemma 5.2. *For every $\delta > 0$ there exists $L = L(\omega) > 0$ such that*

$$\frac{1}{L^d}\int_{\square_x^L}\big(\lambda + \lambda^2\, b_\lambda(y,\omega)\big)\, dy \le \beta_\infty(\lambda) + \delta \qquad \forall x \in \mathbb{R}^d \tag{5.4}$$

almost surely.

Proof. For $n \in \mathbb{N}$ let us put

$$\overline{\ell}_{\lambda,n}(\omega) := \sup_{y\in\mathbb{R}^d}\ell_{\lambda,n}(y,\omega), \tag{5.5}$$

where $\ell_{\lambda,n}(x,\omega)$ is defined in accordance with (2.15). Then, by [10, Lemma 5.9], for every $\delta > 0$ there exists $N = N(\omega) \in \mathbb{N}$ such that for every $n > N$ we have

$$\frac{1}{n^d}\int_{\square_x^n}\big(\lambda + \lambda^2\, b_\lambda(y,\omega)\big)\, dy \le \overline{\ell}_{\lambda,n}(\omega) \le \beta_\infty(\lambda,\omega) + \delta, \qquad \forall x \in \mathbb{R}^d.$$

The fact that β_∞ is deterministic almost surely completes the proof. $\square$

[3] The existence of such an extension is guaranteed by Theorem 2.9.

Lemma 5.3. *For every λ_0 and $\delta > 0$ such that*

$$(\lambda_0 - \delta, \lambda_0 + \delta) \cap \sigma(-\Delta_\Omega) = \emptyset$$

there exists $C_{\lambda_0,\delta} > 0$ such that, almost surely, if $u \in H_0^1(\omega^k)$, $k \in \mathbb{N}$, is a solution of

$$-\Delta u = \lambda u + 1$$

for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, then

$$\|u\|_{L^\infty(\omega^k)} \le C_{\lambda_0,\delta}, \tag{5.6}$$

uniformly in k and ω.

Proof. The estimate (5.6) can be obtained by means of a Moser's iteration argument retracing — with minor modifications — the proof of [5, Theorem 2.6], with account of [10, Lemma D.8] and Assumption 2.2. That (5.6) is also uniform in k and ω is guaranteed by Assumption 2.2. Indeed, upon extending u by zero outside ω^k and translating ω^k so that it fits within the unit cube $[-1/2, 1/2]^d$, performing the above argument one only requires the Poincaré inequality and Sobolev embedding theorems applied in the unit cube. □

Corollary 5.4. *The realisation $b^\varepsilon_{\lambda_\varepsilon}$ satisfies the estimate*

$$\|b^\varepsilon_{\lambda_\varepsilon}\|_{L^\infty(\omega^k)} \le C, \tag{5.7}$$

uniformly in ε, ω and k.

Proof. The estimate (5.7) follows in a straightforward manner from Lemma 5.3. □

Recall the function ψ_R defined by (4.6). In the next section we will need the following technical estimates, which are required for fiddling with the exponential term ψ_R in the variational formulation of the eigenvalue problem (3.4) with test function of the form $\psi_R \widetilde{u}^\varepsilon$, and which follow from the basic observation that for every (scaled) extension domain $\varepsilon\mathcal{B}^k_\omega$ one has, by Assumption 2.2,

$$\sup_{\varepsilon\mathcal{B}^k_\omega} \psi_R^{1/2} \le e^{\alpha\varepsilon\sqrt{d}} \inf_{\varepsilon\mathcal{B}^k_\omega} \psi_R^{1/2}. \tag{5.8}$$

Lemma 5.5. *(a) For every $k \in \mathbb{N}^\varepsilon(\omega)$ and $R > 0$ the following estimate holds:*

$$\begin{aligned}\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}&\|\nabla(\psi_R\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\omega^k)} \\ &+ \|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}\|\nabla(\psi_R\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\omega^k)} + \|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}\|\psi_R\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)} \\ &\qquad \le C\left(\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon\mathcal{B}^k_\omega)} + \|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon\mathcal{B}^k_\omega\setminus\varepsilon\omega^k)}\right).\end{aligned} \tag{5.9}$$

(b) Let $\square^L_x$ be defined as in (2.1) and suppose that $L > 2$. Then for every $x \in \mathbb{R}^d$ we have

$$\begin{aligned}\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\square^L_x)}&\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\square^L_x)} \\ &\le C\left(\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon\square^{2L}_x)} + \|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon\square^{2L}_x\setminus\mathcal{D}^\varepsilon)}\right).\end{aligned} \tag{5.10}$$

The constant C in (5.10) is independent of L, R and ε.

Proof. (a) From (4.7) we have

$$\begin{aligned}\|\nabla(\psi_R\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\omega^k)} &\le \|\nabla(\psi_R^{1/2})\,\psi_R^{1/2}\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)} + \|\psi_R^{1/2}\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\omega^k)} \\ &\le C\Big(\sup_{\varepsilon\omega^k}\psi_R^{1/2}\Big)\left[\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)} + \|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\omega^k)}\right].\end{aligned} \tag{5.11}$$

From (5.8) it easily follows that

$$\Big(\sup_{\varepsilon\omega^k}\psi_R^{1/2}\Big)\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}\le C\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}. \tag{5.12}$$

Combining (5.11) and (5.12) and using some elementary algebra we get

$$\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}\|\nabla(\psi_R\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\omega^k)}\le C\left(\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\varepsilon\omega^k)}+\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\varepsilon\omega^k)}\right). \tag{5.13}$$

Finally, employing the bounds (5.8) and (5.1b), we obtain

$$\begin{aligned}\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\omega^k)}&\le C\Big(\sup_{\varepsilon\omega^k}\psi_R^{1/2}\Big)\left(\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}+\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\mathcal{B}^k_\omega\setminus\varepsilon\omega^k)}\right)\\&\le C\left(\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}+\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)-(\nabla\psi_R^{1/2})\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\mathcal{B}^k_\omega\setminus\varepsilon\omega^k)}\right)\\&\le C\left(\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\mathcal{B}^k_\omega)}+\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\mathcal{B}^k_\omega\setminus\varepsilon\omega^k)}\right).\end{aligned} \tag{5.14}$$

Combining (5.13) with (5.14) we obtain the desired bound for the first term on the left-hand side of (5.9). The remaining two terms can be easily estimated in a similar manner.

(b) Note that for every inclusion $\varepsilon\omega^k$ that has non-empty intersection with the cube $\varepsilon\square_x^L$ the corresponding extension domain is contained in the larger cube: $\varepsilon\mathcal{B}^k_\omega\subset\varepsilon\square_x^{2L}$. Then the bound (5.10) follows by applying the inequality (5.14) on each inclusion satisfying $\varepsilon\omega^k\cap\varepsilon\square_x^L\neq\emptyset$. $\square$

5.2 Proof of Theorem 5.1

The variational formulation of the eigenvalue problem (3.4) reads

$$\int_{\mathcal{D}}A_2\nabla u^\varepsilon\cdot\nabla v+\varepsilon^2\int_{\mathcal{I}^\varepsilon}\nabla u^\varepsilon\cdot\nabla v+\int_{\mathcal{M}^\varepsilon}A_1\nabla u^\varepsilon\cdot\nabla v=\lambda_\varepsilon\int_{\mathbb{R}^d}u^\varepsilon v,\qquad\forall v\in H^1(\mathbb{R}^d). \tag{5.15}$$

By choosing $v=u^\varepsilon$ in (5.15) and using the positive-definiteness of A_1 and (3.3), we obtain the *a priori* estimate

$$\|\nabla u^\varepsilon\|_{L^2(\mathcal{D})}+\varepsilon\|\nabla u^\varepsilon\|_{L^2(\mathcal{I}^\varepsilon)}+\|\nabla u^\varepsilon\|_{L^2(\mathcal{M}^\varepsilon)}\le C, \tag{5.16}$$

which implies, in particular,

$$\|u^\varepsilon\|_{H^1(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}\le C. \tag{5.17}$$

While one does not have a uniform H^1-bound for the eigenfunction u^ε, whose gradient in the inclusions is of order ε^{-1}, one has a uniform H^1-bound for the extension $\widetilde{u}^\varepsilon$. Indeed, from (5.1c) we immediately get

$$\|\widetilde{u}^\varepsilon\|_{H^1(\mathbb{R}^d)}\le C. \tag{5.18}$$

Consider the test function

$$v=\psi_R\,\widetilde{u}^\varepsilon\in H^1(\mathbb{R}^d), \tag{5.19}$$

where ψ_R is define in accordance with (4.6). Plugging (5.19) into (5.15) and using (5.1a), we obtain

$$\begin{aligned}\int_{\mathcal{D}}A_2\nabla\widetilde{u}^\varepsilon\cdot\nabla(\psi_R\widetilde{u}^\varepsilon)+\varepsilon^2\int_{\mathcal{I}^\varepsilon}\nabla u^\varepsilon\cdot\nabla(\psi_R\widetilde{u}^\varepsilon)+\int_{\mathcal{M}^\varepsilon}A_1\nabla\widetilde{u}^\varepsilon\cdot\nabla(\psi_R\widetilde{u}^\varepsilon)&\\=\lambda_\varepsilon\int_{\mathcal{M}^\varepsilon\cup\mathcal{D}}\psi_R\,(\widetilde{u}^\varepsilon)^2+\lambda_\varepsilon\int_{\mathcal{I}^\varepsilon}u^\varepsilon\psi_R\widetilde{u}^\varepsilon.&\end{aligned} \tag{5.20}$$

With the help of the algebraic identity

$$|A^{1/2}\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)|^2=|A^{1/2}\nabla(\psi_R^{1/2})|^2(\widetilde{u}^\varepsilon)^2+A\nabla\widetilde{u}^\varepsilon\cdot\nabla(\psi_R\widetilde{u}^\varepsilon)\quad\forall A=A^T\in GL(d,\mathbb{R}),$$

we can recast (5.20) as

$$\int_{\mathcal{D}} |A_2^{1/2}\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)|^2 + \int_{\mathcal{M}^\varepsilon} |A_1^{1/2}\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)|^2 + \varepsilon^2\int_{\mathcal{I}^\varepsilon} \nabla u^\varepsilon\cdot\nabla(\psi_R\widetilde{u}^\varepsilon) - \int_{\mathcal{M}^\varepsilon}|A_1^{1/2}\nabla(\psi_R^{1/2})|^2(\widetilde{u}^\varepsilon)^2$$
$$= \lambda_\varepsilon\int_{\mathcal{D}}\psi_R\,(\widetilde{u}^\varepsilon)^2 + \int_{\mathcal{D}}|A_2^{1/2}\nabla(\psi_R^{1/2})|^2(\widetilde{u}^\varepsilon)^2 + \lambda_\varepsilon\int_{\mathcal{M}^\varepsilon}\psi_R\,(\widetilde{u}^\varepsilon)^2 + \lambda_\varepsilon\int_{\mathcal{I}^\varepsilon} u^\varepsilon\psi_R\widetilde{u}^\varepsilon. \quad (5.21)$$

In view of (5.18) and (4.7), the first two integrals on the right-hand side of (5.21) can be estimated as follows:

$$\lambda_\varepsilon\int_{\mathcal{D}}\psi_R\,(\widetilde{u}^\varepsilon)^2 + \int_{\mathcal{D}}|A_2^{1/2}\nabla(\psi_R^{1/2})|^2(\widetilde{u}^\varepsilon)^2 \le C(1+\alpha^2\operatorname{tr}(A_2))\sup_{x\in\mathcal{D}} e^{2\alpha|x|}. \quad (5.22)$$

Furthermore,

$$\int_{\mathcal{M}^\varepsilon}|A_1^{1/2}\nabla(\psi_R^{1/2})|^2(\widetilde{u}^\varepsilon)^2 \le \gamma\alpha^2\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}, \quad (5.23)$$

where $\gamma := \max_{\lambda\in\sigma(A_1)}\lambda$ is the greatest eigenvalue of the positive-definite matrix A_1.

In view of (5.22), (5.23), and the positive definiteness of the matrices A_1 and A_2, we can turn the identity (5.21) into the inequality

$$c\,\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)} + \varepsilon^2\int_{\mathcal{I}^\varepsilon}\nabla u^\varepsilon\cdot\nabla(\psi_R\widetilde{u}^\varepsilon) - \gamma\alpha^2\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}$$
$$\le C + \lambda_\varepsilon\int_{\mathcal{M}^\varepsilon}\psi_R\,(\widetilde{u}^\varepsilon)^2 + \lambda_\varepsilon\int_{\mathcal{I}^\varepsilon}u^\varepsilon\psi_R\widetilde{u}^\varepsilon, \quad (5.24)$$

where positive constants c and C are independent of ε and R.

The next step consists in estimating the remaining integrals in (5.24). To this end, we put

$$\mathcal{I}_0 := \left|\varepsilon^2\int_{\mathcal{I}^\varepsilon}\nabla u^\varepsilon\cdot\nabla(\psi_R\widetilde{u}^\varepsilon)\right|, \quad (5.25)$$

$$\mathcal{I}_1 := \lambda_\varepsilon\int_{\mathcal{M}^\varepsilon}\psi_R\,(\widetilde{u}^\varepsilon)^2 + \lambda_\varepsilon\int_{\mathcal{I}^\varepsilon}u^\varepsilon\psi_R\widetilde{u}^\varepsilon, \quad (5.26)$$

and analyse $\mathcal{I}_0$ and $\mathcal{I}_1$ separately.

Let us denote

$$u_0^\varepsilon := u^\varepsilon - \widetilde{u}^\varepsilon \in H_0^1(\mathcal{I}^\varepsilon). \quad (5.27)$$

On account of (5.15) and (5.1a), the function $u_0^\varepsilon\in H_0^1(\mathcal{I}^\varepsilon)$ satisfies the (rescaled) equation

$$(-\Delta_y - \lambda_\varepsilon)u_0^\varepsilon(\varepsilon y) = \lambda_\varepsilon\widetilde{u}^\varepsilon(\varepsilon y), \qquad y\in\omega^k, \quad k\in\mathbb{N}^\varepsilon(\omega) \quad (5.28)$$

(recall the notation (2.6)). Since $\lambda_0\notin\sigma(-\Delta_y)$ by assumption — see also Lemma 5.3 — and $\lambda_\varepsilon\to\lambda_0$ as ε tends to zero, for sufficiently small ε the norm of the resolvent $(-\Delta_y-\lambda_\varepsilon)^{-1}$ is bounded above uniformly in ε. Consequently, (5.28) gives us

$$\|u_0^\varepsilon\|_{L^2(\mathcal{I}^\varepsilon)} \le C\|\widetilde{u}^\varepsilon\|_{L_2(\mathcal{I}^\varepsilon)}. \quad (5.29)$$

Multiplying (the unscaled version of) (5.28) by $\widetilde{u}_0^\varepsilon$, integrating by parts and using (5.29), we obtain

$$\|\varepsilon\nabla u_0^\varepsilon\|_{L^2(\mathcal{I}^\varepsilon)} + \|u_0^\varepsilon\|_{L^2(\mathcal{I}^\varepsilon)} \le C\|\widetilde{u}^\varepsilon\|_{L^2(\mathcal{I}^\varepsilon)}. \quad (5.30)$$

We are ready to estimate (5.25). Utilising (5.30), we have

$$\begin{aligned}\mathcal{I}_0 \le C\varepsilon^2 &\sum_{k\in\mathbb{N}^\varepsilon(\omega)} \|\nabla(u_0^\varepsilon+\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\omega^k)}\|\nabla(\psi_R\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\omega^k)}\\ \le& C\varepsilon \sum_{k\in\mathbb{N}^\varepsilon(\omega)}\left(\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}+\varepsilon\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}\right)\|\nabla(\psi_R\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\omega^k)}.\end{aligned} \tag{5.31}$$

Applying Lemma 5.5 to the right-hand side of (5.31) we obtain

$$\mathcal{I}_0 \le C\varepsilon\left(\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}+\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}\right). \tag{5.32}$$

Let us move on to $\mathcal{I}_1$. Substituting (5.27) into (5.26) we obtain

$$\mathcal{I}_1 = \lambda_\varepsilon\int_{\mathcal{I}^\varepsilon\cup\mathcal{M}^\varepsilon}\psi_R\,(\widetilde{u}^\varepsilon)^2+\int_{\mathcal{I}^\varepsilon}u_0^\varepsilon\psi_R\widetilde{u}^\varepsilon. \tag{5.33}$$

We further decompose u_0^ε as $u_0^\varepsilon=u_{0,1}^\varepsilon+u_{0,2}^\varepsilon$, where $u_{0,1}^\varepsilon$ and $u_{0,2}^\varepsilon$ are defined to be the solutions of

$$-\Delta_y u_{0,1}^\varepsilon(\varepsilon y)-\lambda_\varepsilon u_{0,1}^\varepsilon(\varepsilon y)=\lambda_\varepsilon\langle\widetilde{u}^\varepsilon(\varepsilon y)\rangle_{\omega^k},\qquad y\in\omega^k,\ k\in\mathbb{N}^\varepsilon(\omega), \tag{5.34}$$

and

$$-\Delta_y u_{0,2}^\varepsilon(\varepsilon y)-\lambda_\varepsilon u_{0,2}^\varepsilon(\varepsilon y)=\lambda_\varepsilon\left[\widetilde{u}^\varepsilon(\varepsilon y)-\langle\widetilde{u}^\varepsilon(\varepsilon y)\rangle_{\omega^k}\right],\qquad y\in\omega^k,\ k\in\mathbb{N}^\varepsilon(\omega), \tag{5.35}$$

respectively. Formula (5.35), combined with the Poincaré inequality and the uniform (in ε) boundedness of the resolvent $(-\Delta_y-\lambda_\varepsilon)^{-1}$, implies

$$\|u_{0,2}^\varepsilon(\varepsilon y)\|_{L^2(\omega^k)}\le C\|\widetilde{u}^\varepsilon(\varepsilon y)-\langle\widetilde{u}^\varepsilon(\varepsilon y)\rangle_{\omega^k}\|_{L^2(\omega^k)}\le C\|\nabla_y\widetilde{u}^\varepsilon(\varepsilon y)\|_{L^2(\omega^k)},$$

$$\|u_{0,2}^\varepsilon\|_{L^2(\varepsilon\omega^k)}\le C\varepsilon\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\omega^k)}.$$

Arguing as above and resorting to Lemma 5.5, we thus obtain

$$\left|\int_{\mathcal{I}^\varepsilon}u_{0,2}^\varepsilon\psi_R\widetilde{u}^\varepsilon\right|\le C\varepsilon\left(\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}+\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}\right). \tag{5.36}$$

Furthermore, formula (5.34) implies

$$u_{0,1}^\varepsilon=\lambda_\varepsilon\, b_{\lambda_\varepsilon}^\varepsilon\sum_{k\in\mathbb{N}^\varepsilon(\omega)}\mathbb{1}_{\varepsilon\omega^k}\,\langle\widetilde{u}^\varepsilon\rangle_{\varepsilon\omega^k}. \tag{5.37}$$

Therefore, on account of (5.36) and (5.37), we can estimate (5.33) as

$$\begin{aligned}\mathcal{I}_1\le\int_{\mathcal{I}^\varepsilon\cup\mathcal{M}^\varepsilon}(\lambda_\varepsilon+\lambda_\varepsilon^2\,\mathbb{1}_\varepsilon^{\mathrm{out}}\,b_{\lambda_\varepsilon}^\varepsilon)\psi_R(\widetilde{u}^\varepsilon)^2+C\varepsilon\left(\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}+\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}\right)\\ +\lambda_\varepsilon^2\sum_{k\in\mathbb{N}^\varepsilon(\omega)}\int_{\varepsilon\omega^k}b_{\lambda_\varepsilon}^\varepsilon\left(\langle\widetilde{u}^\varepsilon\rangle_{\varepsilon\omega^k}-\widetilde{u}^\varepsilon\right)\psi_R\widetilde{u}^\varepsilon,\end{aligned} \tag{5.38}$$

where

$$\mathbb{1}_\varepsilon^{\mathrm{out}}:=\sum_{k\in\mathbb{N}^\varepsilon(\omega)}\mathbb{1}_{\varepsilon\omega^k}$$

is the characteristic function of the set of ε-scaled inclusions disjoint from $\overline{\mathcal{D}}$.

By means of the bound (5.7), the Poincaré inequality and Lemma 5.5, we get

$$\left|\lambda_\varepsilon^2\sum_{k\in\mathbb{N}^\varepsilon(\omega)}\int_{\varepsilon\omega^k}b_{\lambda_\varepsilon}^\varepsilon\left(\langle\widetilde{u}^\varepsilon\rangle_{\varepsilon\omega^k}-\widetilde{u}^\varepsilon\right)\psi_R\widetilde{u}^\varepsilon\right|\le C\varepsilon\left(\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}+\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}\right),$$

so that (5.38) turns into

$$\mathcal{I}_1 \le \int_{\mathcal{I}^\varepsilon \cup \mathcal{M}^\varepsilon} (\lambda_\varepsilon + \lambda_\varepsilon^2\, \mathbb{1}_\varepsilon^{\mathrm{out}}\, b_{\lambda_\varepsilon}^\varepsilon)\psi_R(\widetilde{u}^\varepsilon)^2 + C\varepsilon \left(\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)} + \|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}\right). \quad (5.39)$$

Next we analyse the integral on the right-hand side of (5.39). Suppose

$$\alpha \in (0, \sqrt{|\beta_\infty(\lambda_0)|/\gamma}).$$

Let us tile $\mathbb{R}^d$ with hypercubes of size L,

$$\mathbb{R}^d = \bigcup_{\xi \in L\,\mathbb{Z}^d} \square_\xi^L,$$

where the value of L is chosen in accordance with Lemma 5.2 for some δ satisfying

$$\alpha < \sqrt{\frac{|\beta_\infty(\lambda_0)| - \delta}{\gamma}}, \quad (5.40)$$

and define

$$J_{\mathrm{bd}}(\varepsilon) = \{\xi \in L\,\mathbb{Z}^d \mid \varepsilon\square_\xi^L \cap \partial\mathcal{D} \ne \emptyset \quad \text{or} \quad \varepsilon\square_\xi^L \cap (\varepsilon\omega \setminus \mathcal{I}^\varepsilon) \cap (\mathbb{R}^d \setminus \overline{\mathcal{D}}) \ne \emptyset\},$$

$$J_{\mathrm{out}}(\varepsilon) = \{\xi \in L\,\mathbb{Z}^d \mid \varepsilon\square_\xi^L \subset \mathbb{R}^d \setminus \overline{\mathcal{D}}, \quad \xi \notin J_{\mathrm{bd}}(\varepsilon)\}.$$

The set $J_{\mathrm{bd}}(\varepsilon)$ labels hypercubes that either intersect the boundary of $\mathcal{D}$ or have non-empty intersection with one of the inclusions that we have discarded at the very beginning, on the grounds that they had non-empty intersection with $\overline{\mathcal{D}}$, cf. Remark 2.10.

To begin with, let us estimate the contribution from hypercubes labelled by $J_{\mathrm{bd}}(\varepsilon)$. In light of Corollary 5.4, we have

$$\left| \sum_{\xi \in J_{\mathrm{bd}}(\varepsilon)} \int_{(\mathcal{I}^\varepsilon \cup \mathcal{M}^\varepsilon)\cap \varepsilon\square_\xi^L} (\lambda_\varepsilon + \lambda_\varepsilon^2\, \mathbb{1}_\varepsilon^{\mathrm{out}}\, b_{\lambda_\varepsilon}^\varepsilon)\psi_R(\widetilde{u}^\varepsilon)^2 \right| \le C \sum_{\xi \in J_{\mathrm{bd}}(\varepsilon)} \int_{\varepsilon\square_\xi^L} (\widetilde{u}^\varepsilon)^2. \quad (5.41)$$

From (5.18) we have that $\{\widetilde{u}^\varepsilon\}$ is uniformly bounded in $H^1(\mathbb{R}^d)$, hence weakly compact, namely, up to a subsequence,

$$\widetilde{u}^\varepsilon \rightharpoonup u_0 \quad \text{in} \quad H^1(\mathbb{R}^d) \quad (5.42)$$

for some $u_0 \in H^1(\mathbb{R}^d)$. Note that at this stage we are not yet in a position (nor need) to show that u_0 is the macroscopic component of an eigenfunction of $\mathcal{A}^{\mathrm{hom}}$ corresponding to λ_0; this fact will be established in Section 6. Let U be an open bounded set such that $\overline{\mathcal{D}} \subset U$. By the Rellich–Kondrachov theorem, up to a subsequence,

$$\widetilde{u}^\varepsilon \to u_0 \quad \text{in} \quad L^2(U). \quad (5.43)$$

For small enough ε we have

$$\bigcup_{\xi \in J_{\mathrm{bd}}(\varepsilon)} \varepsilon\square_\xi^L \subset U,$$

and

$$\left| \bigcup_{\xi \in J_{\mathrm{bd}}(\varepsilon)} \varepsilon\square_\xi^L \right| = o(1) \quad \text{as} \quad \varepsilon \to 0. \quad (5.44)$$

Then from (5.43) and (5.44) one easily gets

$$\sum_{\xi \in J_{\mathrm{bd}}(\varepsilon)} \int_{\varepsilon\square_\xi^L} (\widetilde{u}^\varepsilon)^2 = o(1) \quad \text{as} \quad \varepsilon \to 0.$$

Let us then estimate the contribution to the integral on the right-hand side of (5.39) from hypercubes disjoint from $\mathcal{D}$ and labelled by $J_{\mathrm{out}}(\varepsilon)$. Resorting to Corollary 5.4, the Poincaré inequality, Lemma 5.5, and Lemma 5.2, we get

$$\begin{aligned}
\sum_{\xi\in J_{\mathrm{out}}(\varepsilon)} \int_{(\mathcal{I}^\varepsilon\cup\mathcal{M}^\varepsilon)\cap\varepsilon\square_\xi^L} (\lambda_\varepsilon+\lambda_\varepsilon^2\,\mathbb{1}_\varepsilon^{\mathrm{out}}\,b_{\lambda_\varepsilon}^\varepsilon)\psi_R(\widetilde{u}^\varepsilon)^2 &= \sum_{\xi\in J_{\mathrm{out}}(\varepsilon)} \int_{\varepsilon\square_\xi^L} (\lambda_\varepsilon+\lambda_\varepsilon^2\,b_{\lambda_\varepsilon}^\varepsilon)\psi_R(\widetilde{u}^\varepsilon)^2 \\
&= \sum_{\xi\in J_{\mathrm{out}}(\varepsilon)} \int_{\varepsilon\square_\xi^L} \left(\lambda_\varepsilon+\lambda_\varepsilon^2\,b_{\lambda_\varepsilon}^\varepsilon\right)\psi_R^{1/2}\,\widetilde{u}^\varepsilon\left(\psi_R^{1/2}\,\widetilde{u}^\varepsilon-\langle\psi_R^{1/2}\,\widetilde{u}^\varepsilon\rangle_{\varepsilon\square_\xi^L}\right) \\
&+ \sum_{\xi\in J_{\mathrm{out}}(\varepsilon)} \int_{\varepsilon\square_\xi^L} \left(\lambda_\varepsilon+\lambda_\varepsilon^2\,b_{\lambda_\varepsilon}^\varepsilon\right)\langle\psi_R^{1/2}\,\widetilde{u}^\varepsilon\rangle_{\varepsilon\square_\xi^L}\left(\psi_R^{1/2}\,\widetilde{u}^\varepsilon-\langle\psi_R^{1/2}\,\widetilde{u}^\varepsilon\rangle_{\varepsilon\square_\xi^L}\right) \\
&+ \sum_{\xi\in J_{\mathrm{out}}(\varepsilon)} \int_{\varepsilon\square_\xi^L} \left(\lambda_\varepsilon+\lambda_\varepsilon^2\,b_{\lambda_\varepsilon}^\varepsilon\right)\langle\psi_R^{1/2}\,\widetilde{u}^\varepsilon\rangle_{\varepsilon\square_L(\xi)}\langle\psi_R^{1/2}\,\widetilde{u}^\varepsilon\rangle_{\varepsilon\square_\xi^L} \\
&\le C\sum_{\xi\in J_{\mathrm{out}}(\varepsilon)} \|\psi_R^{1/2}\,\widetilde{u}^\varepsilon\|_{L^2(\varepsilon\square_\xi^L)}\|\varepsilon\nabla(\psi_R^{1/2}\,\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon\square_\xi^L)} \\
&+ \sum_{\xi\in J_{\mathrm{out}}(\varepsilon)} \langle\lambda_\varepsilon+\lambda_\varepsilon^2\,b_{\lambda_\varepsilon}^\varepsilon\rangle_{\varepsilon\square_\xi^L}\langle\psi_R^{1/2}\,\widetilde{u}^\varepsilon\rangle_{\varepsilon\square_\xi^L}\int_{\varepsilon\square_\xi^L}\psi_R^{1/2}\,\widetilde{u}^\varepsilon \\
&\le C\varepsilon\left(\|\psi_R^{1/2}\,\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}+\|\nabla(\psi_R^{1/2}\,\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}\right) \\
&- (|\beta_\infty(\lambda_0)|-\delta)\sum_{\xi\in J_{\mathrm{out}}(\varepsilon)}\int_{\varepsilon\square_\xi^L}\psi_R^{1/2}\,\widetilde{u}^\varepsilon\langle\psi_R^{1/2}\,\widetilde{u}^\varepsilon\rangle_{\varepsilon\square_\xi^L} \\
&\le (C\varepsilon-|\beta_\infty(\lambda_0)|+\delta)\|\psi_R^{1/2}\,\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}+C\varepsilon\|\nabla(\psi_R^{1/2}\,\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}\,.
\end{aligned} \tag{5.45}$$

Combining (5.41) and (5.45), we can estimate the integral on the RHS of (5.39) to obtain

$$\mathcal{I}_1 \le (C\varepsilon-|\beta_\infty(\lambda_0)|+\delta)\|\psi_R^{1/2}\,\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}+C\varepsilon\|\nabla(\psi_R^{1/2}\,\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}+o(1)\,. \tag{5.46}$$

Finally, substituting (5.25), (5.32) and (5.26), (5.46) into (5.24), we arrive at

$$(c-C\varepsilon)\,\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}+(|\beta_\infty(\lambda_0)|-\delta-\gamma\alpha^2-C\varepsilon)\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\le C+o(1).$$

Hence, in view of (5.40), for sufficiently small ε we get

$$\|\nabla(\psi_R^{1/2}\widetilde{u}^\varepsilon)\|^2_{L^2(\mathbb{R}^d\setminus\mathcal{I}^\varepsilon)}+\|\psi_R^{1/2}\widetilde{u}^\varepsilon\|^2_{L^2(\mathbb{R}^d)}\le C, \tag{5.47}$$

where C is a constant independent of ε and R.

Now, formulae (5.47) and (5.1b) imply

$$\|e^{\alpha|x|}\widetilde{u}^\varepsilon\|_{H^1(B_R(0))}\le C,$$

uniformly in ε and R. Letting R tend to $+\infty$ we obtain (5.3).

It remains only to translate (5.3) into an L^2 estimate for our original family of eigenfunctions u^ε. In view of the decomposition $u^\varepsilon=\widetilde{u}^\varepsilon+u_0^\varepsilon$, we have

$$\begin{aligned}
\|e^{\alpha|x|}u^\varepsilon\|_{L^2(\mathbb{R}^d)} &\le \|e^{\alpha|x|}\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^d)}+\|e^{\alpha|x|}u_0^\varepsilon\|_{L^2(\mathbb{R}^d)} \\
&\le \|e^{\alpha|x|}\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^d)}+\sum_{k\in\mathbb{N}^\varepsilon(\omega)}\|e^{\alpha|x|}u_0^\varepsilon\|_{L^2(\varepsilon\omega^k)}.
\end{aligned} \tag{5.48}$$

The right-hand side of (5.48) can then be estimated by resorting to (5.3), (5.29) and (5.12), to obtain (5.2).

6 Strong stochastic two-scale convergence

Using the results of Section 5, we will show in this section that our sequence of eigenfunctions $\{u^\varepsilon\}$, up to a subsequence, strongly stochastically two-scale converges to an eigenfunction $\overline{u}^0 \in H$ of the operator $\mathcal{A}^{\mathrm{hom}}$, thus completing the proof of Theorem 3.3.

We will do this in two steps: first we will show that the sequence $\{u^\varepsilon\}$ strongly stochastically two-scale converges, up to a subsequence, to some limit function $\overline{u}^0(x,\omega) = u_0(x)+\overline{u}_1(x,\omega) \in H$, and then we will argue that $\overline{u}^0$ is an eigenfunction of $\mathcal{A}^{\mathrm{hom}}$ corresponding to λ_0.

Recall the decomposition $u^\varepsilon = \widetilde{u}^\varepsilon + u_0^\varepsilon$, where $\widetilde{u}^\varepsilon$ is the extension from Theorem 5.1 and u_0^ε is defined in accordance with (5.27).

We have that $\widetilde{u}^\varepsilon$ weakly converges in $H^1(\mathbb{R}^d)$ to some function u_0, see (5.42). Because $\mathbb{R}^d$ is not bounded, one cannot immediately argue strong convergence in $L^2(\mathbb{R}^d)$. However, this can be achieved with the help of Theorem 5.1.

Lemma 6.1. *We have*

$$\widetilde{u}^\varepsilon \to u_0 \quad in \quad L^2(\mathbb{R}^d). \tag{6.1}$$

Proof. For a given $n \in \mathbb{N}$, we have that $\{\widetilde{u}^\varepsilon\}$ is weakly compact in $H^1(B_n(0))$, hence strongly compact in $L^2(B_n(0))$. A standard diagonalisation argument gives us that, up to a subsequence, $\widetilde{u}^\varepsilon \to u_0$ in $L^2(B_n(0))$ for every $n \in \mathbb{N}$. The H^1-exponential decay (5.3) of $\widetilde{u}^\varepsilon$ implies

$$\|\widetilde{u}^\varepsilon\|_{H^1(\mathbb{R}^d\setminus B_n(0))} \le Ce^{-n\alpha}.$$

For every $\delta > 0$, one can choose n sufficiently large and ε sufficiently small so that

$$\|\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^d\setminus B_n(0))} \le \delta, \quad \|u_0\|_{L^2(\mathbb{R}^d\setminus B_n(0))} \le \delta, \quad \|\widetilde{u}^\varepsilon - u_0\|_{L^2(B_n(0))} \le \delta.$$

Hence

$$\|\widetilde{u}^\varepsilon - u_0\|_{L^2(\mathbb{R}^d)} \le \|\widetilde{u}^\varepsilon - u_0\|_{L^2(B_n(0))} + \|\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^d\setminus B_n(0))} + \|u_0\|_{L^2(\mathbb{R}^d\setminus B_n(0))} \le 3\delta,$$

which infers (6.1). $\square$

Lemma 6.2. *There exists $\overline{u}_1 \in L^2(\mathbb{R}^d; H_0^1(\Omega))$ such that*

$$u_0^\varepsilon \xrightarrow{2} \overline{u}_1.$$

Furthermore, the function $\overline{u}_1$ satisfies

$$\mathbb{E}\left[(\nabla_y u_1 \cdot \nabla_y \psi - \lambda_0\, u_1\, \psi)\, 1_\Omega\right] = \lambda_0(1 - \mathbb{1}_{\mathcal{D}})u_0\, \mathbb{E}\left[\psi\, 1_\Omega\right], \qquad \forall \overline{\psi} \in H_0^1(\Omega),\ \forall x \in \mathbb{R}^d. \tag{6.2}$$

Proof. Weak convergence $u_0^\varepsilon \overset{2}{\rightharpoonup} \overline{u}_1$ to a function satisfying (6.2) follows from a straightforward adaptation of [9, Proposition 4.1] to the case at hand, with account of the fact that u_0^ε satisfies

$$\varepsilon^2 \int_{\mathcal{I}^\varepsilon} \nabla u_0^\varepsilon \cdot \nabla\varphi - \lambda_\varepsilon \int_{\mathcal{I}^\varepsilon} u_0^\varepsilon \varphi = \lambda_\varepsilon \int_{\mathcal{I}^\varepsilon} \widetilde{u}^\varepsilon \varphi \qquad \forall \varphi \in H_0^1(\mathcal{I}^\varepsilon), \tag{6.3}$$

and that $\lambda_\varepsilon \mathbb{1}_{\mathcal{I}^\varepsilon}\widetilde{u}^\varepsilon \overset{2}{\rightharpoonup} \lambda_0\, 1_\Omega\, \mathbb{1}_{\mathbb{R}^d\setminus\mathcal{D}}\, u_0$. (Clearly, $\overline{u}_1 = 0$ in $\mathcal{D}\times\Omega$.)

That the convergence is, in fact, strong can be established by means of a general argument due to Zhikov [30, 31]. Namely, let $z^\varepsilon \in H_0^1(\mathcal{I}^\varepsilon)$ be the solution of

$$\varepsilon^2 \int_{\mathcal{I}^\varepsilon} \nabla z^\varepsilon \cdot \nabla\varphi - \lambda_\varepsilon \int_{\mathcal{I}^\varepsilon} z^\varepsilon \varphi = \lambda_\varepsilon \int_{\mathcal{I}^\varepsilon} u_0^\varepsilon \varphi \qquad \forall \varphi \in H_0^1(\mathcal{I}^\varepsilon). \tag{6.4}$$

Then, the same adaptation of [9, Proposition 4.1] gives us

$$z^\varepsilon \stackrel{2}{\rightharpoonup} \overline{z}(x,\omega) \in L^2(\mathbb{R}^d; H_0^1(\Omega)),$$

where $\overline{z}$ solves

$$\mathbb{E}[(\nabla_y z \cdot \nabla_y \psi - \lambda_0\, z\psi)\, 1_\Omega] = \lambda_0 \mathbb{E}[\overline{u}_1 \overline{\psi}\, \overline{1_\Omega}] \qquad \forall \overline{\psi} \in H_0^1(\Omega). \tag{6.5}$$

By using z^ε and u_0^ε as test functions in (6.3) and (6.4), respectively, and equating the right-hand sides, we get

$$\lim_{\varepsilon\to 0} \int_{\mathcal{I}^\varepsilon} (u_0^\varepsilon)^2 = \lim_{\varepsilon\to 0} \int_{\mathcal{I}^\varepsilon} \widetilde{u}^\varepsilon z^\varepsilon = \mathbb{E}\left[\int_{\mathbb{R}^d} u_0\, \overline{z}\, dx\right]. \tag{6.6}$$

Finally, by using $\overline{z}$ and $\overline{u}_1$ as test functions in (6.2) and (6.5), respectively, we arrive at

$$\mathbb{E}\left[\int_{\mathbb{R}^d} u_0\, \overline{z}\, dx\right] = \mathbb{E}\left[\int_{\mathbb{R}^d} \overline{u}_1^2\, dx\right]. \tag{6.7}$$

Formulae (6.6) and (6.7) give us strong stochastic two-scale convergence of u_0^ε. □

Lemma 6.3. *The function $\overline{u}^0 = u_0 + \overline{u}_1$ is an eigenfunction of $\mathcal{A}^{\mathrm{hom}}$ corresponding to the eigenvalue λ_0, i.e. it satisfies (4.1a)–(4.1b).*

Proof. The strategy of the proof is quite standard and consists in choosing appropriate test functions in (5.15) and passing to the limit. Recall that the functions u^ε, $\widetilde{u}^\varepsilon$ and u_0^ε satisfy the estimates (5.1b), (5.1c), (5.16) and (5.30).

Lemmata 6.1 and 6.2, together with Theorem B.1, imply that, up to extracting subsequences, we have

$$\nabla \widetilde{u}^\varepsilon \stackrel{2}{\rightharpoonup} \nabla u_0 + \overline{q}, \qquad \varepsilon \nabla u_0^\varepsilon \stackrel{2}{\rightharpoonup} \overline{\nabla_y u_1}. \tag{6.8}$$

for some $\overline{q} \in L^2(\mathbb{R}^d; \mathcal{V}^2_{\mathrm{pot}}(\Omega))$.

Consider a test function

$$v_1(x) f(x/\varepsilon, \omega), \qquad v_1 \in C_0^\infty(\mathbb{R}^d \setminus \mathcal{D}),\ \overline{f} \in C_0^\infty(\Omega). \tag{6.9}$$

Substituting (6.9) into (5.15) we obtain

$$\begin{aligned} \varepsilon^2 \int_{\mathcal{I}^\varepsilon} & \nabla \widetilde{u}^\varepsilon \cdot \nabla(v_1 f(x/\varepsilon,\omega)) \\ & + \varepsilon^2 \int_{\mathcal{I}^\varepsilon} \nabla u_0^\varepsilon \cdot (f(x/\varepsilon,\omega)\nabla v_1) + \varepsilon \int_{\mathcal{I}^\varepsilon} v_1\, \nabla u_0^\varepsilon \cdot \nabla_y f(y,\omega)|_{y=x/\varepsilon} \\ & = \lambda_\varepsilon \int_{\mathbb{R}^d} u^\varepsilon\, v_1 f(x/\varepsilon,\omega). \end{aligned} \tag{6.10}$$

It is easy to see that

$$\varepsilon^2 \int_{\mathcal{I}^\varepsilon} \nabla \widetilde{u}^\varepsilon \cdot \nabla(v_1 f(x/\varepsilon,\omega)) + \varepsilon^2 \int_{\mathcal{I}^\varepsilon} \nabla u_0^\varepsilon \cdot (f(x/\varepsilon,\omega)\nabla v_1) \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

whereas by (6.8) we have

$$\varepsilon \int_{\mathcal{I}^\varepsilon} v_1\, \nabla u_0^\varepsilon \cdot \nabla_y f(y,\omega)|_{y=x/\varepsilon} \to \mathbb{E}\left[\int_{\mathbb{R}^d} v_1\, \nabla_y u_1 \cdot \nabla_y f\right] \quad \text{as} \quad \varepsilon \to 0. \tag{6.11}$$

Formulae (6.10)–(6.11) give us (4.1b), with $\overline{\varphi}_1(x,\omega) = v_1(x)\overline{f}(\omega)$.

Consider a test function

$$\varepsilon v_1(x) f(x/\varepsilon, \omega), \qquad v_1 \in C_0^\infty(\mathbb{R}^d \setminus \mathcal{D}),\ \overline{f} \in C^\infty(\Omega).$$

Substituting (6.9) into (5.15), passing to the limit for $\varepsilon \to 0$ and using (6.8), we obtain

$$\mathbb{E}\left[(A_1(\nabla u_0 + q) \cdot \nabla_y f)(1 - 1_\Omega)\right] = 0 \qquad \forall \overline{f} \in C^\infty(\Omega). \tag{6.12}$$

Note that the latter is the corrector equation, which implies that $\overline{q}$ is the corrector associated with ∇u_0, in particular, one has

$$\mathbb{E}\left[A_1(\nabla u_0 + q)(1 - 1_\Omega)\right] = A_1^{\mathrm{hom}} \nabla u_0.$$

Finally, substituting the test function $v_0 \in C_0^\infty(\mathbb{R}^d)$ into (5.15) and passing to the limit as $\varepsilon \to 0$ with account of (6.8) and (6.12), one arrives at (4.1a).

A standard density argument combined with the uniqueness of the solution of (4.1a)–(4.1b) completes the proof. ☐

By combining Lemma 6.1, Theorem B.1, Lemma 6.2 and Lemma 6.3 we obtain Theorem 3.3(b).

7 Asymptotic spectral completeness

In conclusion, adapting an argument from [7], we show that strong stochastic two-scale convergence of eigenfunctions implies that the multiplicity of eigenvalues is preserved in the limit $\varepsilon \to 0$, thus proving Theorem 3.5. The latter establishes an asymptotic one-to-one correspondence between eigenvalues and eigenfunctions of $\mathcal{A}^\varepsilon$ and $\mathcal{A}^{\mathrm{hom}}$ as $\varepsilon \to 0$.

Lemma 7.1. *Let E_I^ε be the spectral projection onto the interval $I \subset \mathbb{R}$ associated with the operator $\mathcal{A}^\varepsilon$. Suppose that $u \in L^2(\mathbb{R}^d)$, $\|u\|_{L^2(\mathbb{R}^d)} = 1$, satisfies*

$$\|(\mathcal{A}^\varepsilon - \lambda_0)u\|_{L^2(\mathbb{R}^d)} \le \rho$$

for some $\rho \ge 0$. Then for every $\delta > 0$

$$\|E^\varepsilon_{(\lambda_0-\delta,\lambda_0+\delta)} u\|_{L^2(\mathbb{R}^d)} \ge 1 - 2\frac{\rho}{\delta}.$$

Proof. The claim follows immediately from the Spectral Theorem. ☐

Proof of Theorem 3.5. (a) For each ε_n let $u_j^{\varepsilon_n}$, $j = 1, \dots, m$, a family of orthonormal eigenfunctions associated with $\lambda_{\varepsilon_n,j}$, $j = 1, \dots, m$. Theorem 3.3 tells us that $\lambda_0 \in \sigma_d(\mathcal{A}^{\mathrm{hom}})$. Furthermore, there exist $\overline{u}_j^0$, $j = 1, \dots, m$, eigenfunctions of $\mathcal{A}^{\mathrm{hom}}$ corresponding to λ_0 such that, up to extracting a subsequence,

$$u_j^{\varepsilon_n} \overset{2}{\to} \overline{u}_j^0 \quad \text{as} \quad n \to +\infty.$$

The strong stochastic two-scale convergence of the eigenfunctions implies

$$\delta_{jk} = (u_j^{\varepsilon_n}, u_k^{\varepsilon_n})_{L^2(\mathbb{R}^d)} \to (\overline{u}_j^0, \overline{u}_k^0)_H \quad \text{as} \quad n \to +\infty,$$

namely, the eigenfunctions $\overline{u}_j^0$, $j = 1, \dots, m$, are orthonormal. Here and further on in the proof δ_{jk} denotes the Kronecker delta.

(b) Let $\overline{u}_j^0$, $j = 1, \dots, m$, be an orthonormal basis for the λ_0-eigenspace of $\mathcal{A}^{\mathrm{hom}}$. Then, by Theorem 4.2, for every $\rho > 0$ one can construct m normalised quasimodes $\widehat{u}_j^\varepsilon$ for $\mathcal{A}^\varepsilon$,

$$\|(\mathcal{A}^\varepsilon - \lambda_0)\widehat{u}_j^\varepsilon\|_{L^2(\mathbb{R}^d)} \le \rho, \qquad j = 1, \dots, m,$$

for every $0 < \varepsilon \le \varepsilon_0(\rho)$. It is not difficult to see that the functions $\widehat{u}_j^\varepsilon$ are linearly independent and satisfy

$$(\widehat{u}_j^\varepsilon, \widehat{u}_k^\varepsilon)_{L^2(\mathbb{R}^d)} \to \delta_{jk} \quad \text{as} \quad \varepsilon \to 0.$$

Lemma 7.1 then implies that for every $0 < \delta < \operatorname{dist}(\lambda_0, \mathcal{G})$ we have

$$\dim \operatorname{ran} E^\varepsilon_{(\lambda_0-\delta,\lambda_0+\delta)} \ge m,$$

where ran stands for range. This concludes the proof. □

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Appendix A Stability of the essential spectrum

Theorem A.1. *Let $T_i, i = 1, 2,$ be the non-negative self-adjoint operators on $L^2(\mathbb{R}^d)$ uniquely defined by the densely defined bilinear forms*

$$\int_{\mathbb{R}^d} a_i \nabla u \cdot \nabla v, \quad u, v \in H^1(\mathbb{R}^d),$$

where a_i, $i = 1, 2$, are measurable matrix-valued functions satisfying

$$0 < \nu|\xi|^2 \le a_i\xi \cdot \xi \le \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbb{R}^d \tag{A.1}$$

for some constant ν. Assume that a_1 and a_2 differ only in a relatively compact set, i.e. $\operatorname{supp}(a_1 - a_2) \subset B_R$ *for sufficiently large R. Then*

$$\sigma_{\mathrm{ess}}(T_1) = \sigma_{\mathrm{ess}}(T_2).$$

Proof. Let $\lambda \in \sigma_{\mathrm{ess}}(T_1)$ and φ_n be a corresponding Weyl sequence for T_1. Namely, $\|\varphi_n\|_{L^2(\mathbb{R}^d)} = 1$, $\varphi_n \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^d)$, and

$$T_1\varphi_n = \lambda\varphi_n + h_n, \tag{A.2}$$

where $\|h_n\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$. From (A.1) it follows that φ_n is bounded in $H^1(\mathbb{R}^d)$ and, hence, converges to zero weakly in $H^1(B_R)$ and strongly in $L^2(B_R)$ for any $R > 0$. Multiplying (A.2) by $\eta_{2R}\varphi_n$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^d} \eta_{2R}\, a_1\nabla\varphi_n \cdot \nabla\varphi_n + \int_{\mathbb{R}^d} \varphi_n a_1 \nabla\eta_{2R} \cdot \nabla\varphi_n = \lambda \int_{\mathbb{R}^d} \eta_{2R}|\varphi_n|^2 + \int_{\mathbb{R}^d} h_n\eta_{2R}\varphi_n.$$

From the latter and (A.1) one easily concludes that

$$\|\nabla\varphi_n\|^2_{L^2(B_R)} \le \frac{C}{R}\|\varphi_n\|_{L^2(B_{2R})}\|\nabla\varphi_n\|_{L^2(B_{2R})} + \lambda\|\varphi_n\|^2_{L^2(B_{2R})} + \|h_n\|_{L^2(B_{2R})}\|\varphi_n\|_{L^2(B_{2R})},$$

and, therefore, $\|\nabla\varphi_n\|_{L^2(B_R)} \to 0$ for any $R > 0$. It follows that there exists a subsequence n_k such that

$$\|\varphi_{n_k}\|_{H^1(B_k)} \to 0 \text{ as } k \to \infty. \tag{A.3}$$

Next we follow the general argument utilised in Section 4 (and initially devised in [19]). Define $\psi_k := (1-\eta_k)\varphi_{n_k}$ and $\widehat{\psi}_k := (\lambda+1)(T_2+I)^{-1}\psi_k$. We claim that $\widehat{\psi}_k$ is a Weyl sequence for the operator T_2 corresponding to λ. Clearly,

$$(T_2-\lambda)\widehat{\psi}_k = (\lambda+1)(\psi_k-\widehat{\psi}_k).$$

Thus, in order to prove the claim, it is sufficient to show that $\|\psi_k-\widehat{\psi}_k\|_{L^2(\mathbb{R}^d)}$ vanishes in the limit. (Indeed, this would also imply via (A.3) that $\|\widehat{\psi}_k\|_{L^2(\mathbb{R}^d)} \to 1$ and $\widehat{\psi}_k \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^d)$.) After observing that

$$\|\psi_k-\widehat{\psi}_k\|^2_{L^2(\mathbb{R}^d)} \le \tau_2(\psi_k-\widehat{\psi}_k,\psi_k-\widehat{\psi}_k), \tag{A.4}$$

where $\tau_i, i=1,2,$ denotes the bilinear form of the operator T_i+I, we focus on the right hand side of the latter.

For an arbitrary $v\in H^1(\mathbb{R}^d)$ from the definition of $\widehat{\psi}_k$ we have

$$\tau_2(\psi_k-\widehat{\psi}_k,v) = \tau_2(\psi_k,v)-(\lambda+1)(\psi_k,v).$$

Since for sufficiently large k the defect $\operatorname{supp}(a_1-a_2)$ is contained in $B_{k/2}$, one has

$$\begin{aligned}\tau_2(\psi_k,v)-(\lambda+1)(\psi_k,v) &= \tau_1(\varphi_{n_k},v)-(\lambda+1)(\varphi_{n_k},v)\\ &-\int_{\mathbb{R}^d}\eta_k a_1\nabla\varphi_{n_k}\cdot\nabla v-\int_{\mathbb{R}^d}\varphi_{n_k}a_1\nabla\eta_k\cdot\nabla v+\lambda\int_{\mathbb{R}^d}\eta_k\varphi_{n_k}v.\end{aligned}$$

We estimate the last three terms as follows:

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\eta_k a_1\nabla\varphi_{n_k}\cdot\nabla v\right| &\le C\|\nabla\varphi_{n_k}\|_{L^2(B_k)}\|\nabla v\|_{L^2(B_k)}\,,\\ \left|\int_{\mathbb{R}^d}a_1\varphi_{n_k}\nabla\eta_k\cdot\nabla v\right| &\le \frac{C}{k}\|\varphi_{n_k}\|_{L^2(B_k)}\|\nabla v\|_{L^2(B_k)}\,,\\ \left|\lambda\int_{\mathbb{R}^d}\eta_k\varphi_{n_k}v\right| &\le \|\varphi_{n_k}\|_{L^2(B_k)}\|v\|_{L^2(B_k)}\,.\end{aligned}$$

We conclude that

$$|\tau_2(\psi_k-\widehat{\psi}_k,v)| \le C\|\varphi_{n_k}\|_{H^1(B_k)}\|v\|_{H^1(B_k)} \le C\|\varphi_{n_k}\|_{H^1(B_k)}\sqrt{\tau_2(v,v)}\,, \tag{A.5}$$

where the last inequality follows from (A.1). Finally, replacing v with $\psi_k-\widehat{\psi}_k$ in (A.5) and taking into account (A.3), we conclude from (A.4) that

$$\|\psi_k-\widehat{\psi}_k\|_{L^2(\mathbb{R}^d)} \to 0.$$

The theorem is proved. □

Appendix B Properties of two-scale convergence

In this appendix we summarise, in the form of a theorem, some properties of stochastic two-scale convergence used in our paper. We refer the reader to [32] for further details, see also [9].

Theorem B.1. *Stochastic two-scale convergence enjoys the following properties.*

(i) Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\mathbb{R}^d)$. Then there exists $\overline{u}\in L^2(\mathbb{R}^d\times\Omega)$ such that, up to extracting a subsequence, $u^\varepsilon \overset{2}{\rightharpoonup} \overline{u}$.

(ii) If $u^\varepsilon \overset{2}{\rightharpoonup} \overline{u}$, then $\|\overline{u}\|_{L^2(\mathbb{R}^d\times\Omega)} \le \liminf_{\varepsilon\to0}\|u^\varepsilon\|_{L^2(\mathbb{R}^d)}$.

(iii) *If* $u^\varepsilon \to \overline{u}$ *in* $L^2(\mathbb{R}^d)$, *then* $u^\varepsilon \xrightharpoonup{2} \overline{u}$.

(iv) *Let* $\{v^\varepsilon\}$ *be a uniformly bounded sequence in* $L^\infty(\mathbb{R}^d)$ *such that* $v^\varepsilon \to v$ *in* $L^1(\mathbb{R}^d)$, $\|v\|_{L^\infty(\mathbb{R}^d)} < +\infty$. *Suppose that* $\{u^\varepsilon\}$ *is a bounded sequence in* $L^2(\mathbb{R}^d)$ *such that* $u^\varepsilon \xrightharpoonup{2} \overline{u}$ *for some* $\overline{u} \in L^2(\mathbb{R}^d \times \Omega)$. *Then* $v^\varepsilon u^\varepsilon \xrightharpoonup{2} v\overline{u}$.

(v) *Let* $\{u^\varepsilon\}$ *be a bounded sequence in* $H^1(\mathbb{R}^d)$. *Then, there exist* $u_0 \in H^1(\mathbb{R}^d)$ *and* $\overline{p} \in L^2(\mathbb{R}^d; \mathcal{V}^2_{\mathrm{pot}}(\Omega))$ *such that, up to extracting a subsequence,*

$$u^\varepsilon \rightharpoonup u_0 \quad in \quad H^1(\mathbb{R}^d),$$

$$\nabla u^\varepsilon \xrightharpoonup{2} \nabla u_0 + \overline{p}.$$

(vi) *Let* $\{u^\varepsilon\}$ *be a bounded sequence in* $L^2(\mathbb{R}^d)$ *such that* $\{\varepsilon \nabla u^\varepsilon\}$ *is also bounded in* $L^2(\mathbb{R}^d)$. *Then there exists* $\overline{u} \in L^2(\mathbb{R}^d; H^1(\Omega))$ *such that, up to extracting a subsequence,*

$$u^\varepsilon \xrightharpoonup{2} \overline{u},$$

$$\varepsilon \nabla u^\varepsilon \xrightharpoonup{2} \overline{\nabla_y u}.$$

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