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On the equivalence of the proportional and damped trend order-up-to policies: An eigenvalue analysis

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ABSTRACT

We investigate the equivalence of the order-up-to (OUT) replenishment policy with damped trend forecasting (OUT-DT) to the proportional OUT (POUT) policy via an eigenvalue (zero-pole) analysis. We show when the damped trend forecasting parameters are selected from a specific region of the parameter space, the *Bullwhip Avoidance* ($\mathcal{BA}$) region, the OUT-DT policy potentially possesses some desirable characteristics. Under an independent and identically distributed (i.i.d.) random demand (the simplest random demand) we show the OUT-DT policy: (a) can eliminate the bullwhip effect, (b) has the same dynamic response as the POUT policy (because of the superposition principle, this holds true for all demands, not just i.i.d. demands), (c) forecasting parameters can be set so as to minimize the sum of the inventory and order variances, and (d) has a similar order and inventory variance as the POUT policy when non-linear constraints are present. We investigate the case when the forecasting parameters are selected from the $\mathcal{BA}$ region and correlated demand with one auto-regressive term, one integrated term, and two moving average terms, ARIMA(1,1,2), is present. We reveal the effect of finite lead times on the bullwhip effect (order variance) using an approach based on the order of the eigenvalues (the zeros and poles). We reveal either: (a) the bullwhip effect is always present and always increasing in the lead time or (b) a smoothing effect can be present with short lead times (and the order variance may even be decreasing in the lead time) but the bullwhip effect may return with long lead times.

1. Introduction

We study the dynamic behaviour of the order-up-to (OUT) policy when the damped trend (DT) forecasting method is used to predict the lead time demand. In particular, we investigate the *Bullwhip* and net stock variance amplification (*NSAmp*) generated by this replenishment system. *Bullwhip* is usually defined as the ratio of the variance of the replenishment orders, $\mathbb{V}[o_t]$, to the variance the demand, $\mathbb{V}[d_t]$; $\mathbb{V}[\cdot]$ is the variance operator. *NSAmp* is the ratio of the variance of the net stock levels, $\mathbb{V}[ns_t]$, to the variance of the demand:

$$Bullwhip = \frac{\mathbb{V}[o_t]}{\mathbb{V}[d_t]} \quad \text{and} \quad NSAmp = \frac{\mathbb{V}[ns_t]}{\mathbb{V}[d_t]}. \tag{1}$$

The bullwhip effect is an important phenomenon in supply chains, describing how demand fluctuations are amplified as orders pass echelon-to-echelon up the supply chain. Bullwhip creates the undesirable consequences of excessive capacity investment, inefficient transport use, labour idling and over-time, and increased safety stock requirements (Lee et al., 1997). *NSAmp* is related to the popular safety stock and fill-rate concepts in the inventory literature (Zipkin, 2000; Disney et al., 2015). It is commonly understood that a small *NSAmp* ratio leads to high levels of customer service and increases the effectiveness of inventory investments in supply chains (Costantino et al., 2016; Khosroshahi et al., 2016; Lin et al., 2017). Both the *Bullwhip* and *NSAmp* metrics are prevalent in studies that investigate the costs in, and dynamics of, supply chains and are highly interrelated as production will eventually (after the lead time) become inventory and the inventory deviations from a target safety stock are added to production (Shaban et al., 2019; Ponte et al., 2022).

Li et al. (2014) show the bullwhip effect can be avoided by using unconventional (with negative forecasting parameters) DT forecasting parameters within the OUT-DT policy. This analysis was extended by Li and Disney (2018) where the inventory implications of the OUT-DT policy were explored. By investigating the relationship between stability and invertibility, Li and Disney (2018) showed the DT stability region is the same as its invertability region, further justifying the use of unconventional DT parameter values. Li and Disney (2018) also

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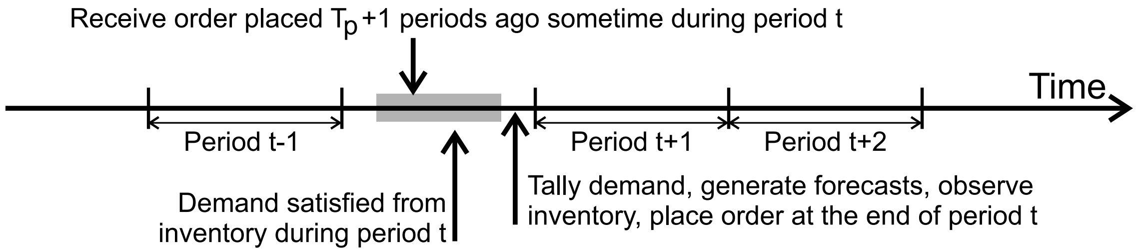


Fig. 1. Sequence of event in the OUT policy.

characterized the frequency response of the inventory levels maintained by the OUT-DT policy, finding good inventory control when the forecasting parameters were selected from within an identified *Bullwhip Avoidance* ($\mathcal{BA}$) region. The $\mathcal{BA}$ region is a set of DT parameters where it is possible some demand processes result in the OUT-DT order variance being less than the demand variance. Similar bullwhip avoidance behaviour has been recorded in the proportional order-up-to (POUT) policy literature. The POUT policy contains a proportional feedback controller to effectively balance the trade-off between inventory and capacity costs, Chen and Disney (2007).

First, we study how the OUT-DT policy reacts to the i.i.d. demand process. Herein, we relax the unit lead-time assumption in Li et al. (2014) and investigate the behaviour of the OUT-DT policy for i.i.d. demand with arbitrary lead times, and obtain expressions for the order and inventory variances. The superposition principle means, in linear systems, all demand time series can be made up of scaled and delayed impulses. The unit impulse is closely related to the i.i.d. demand process and understanding the OUT-DT policy's response to the unit impulse allows one to gain structural insight for all demand processes. We study the eigenvalues (the poles and zeros) of the OUT-DT policy and show when they are equivalent to the eigenvalues of the POUT policy. If the two systems (OUT-DT and POUT) have the same eigenvalues, they will have the same dynamic response to all demand processes, Nise (2004). We verify this claim numerically for the first-order auto-regressive (AR(1)) demand process and the auto-regressive, integrated, moving average (with one auto-regressive. term, one integrative term, and two moving average terms, ARIMA(1,1,2)) demand process. That the OUT-DT and POUT policies are equivalent is an interesting and practically useful insight as it means we can incorporate the POUT policy into an enterprise resource planning (ERP) system without having to create User-Defined Functions in the production planning module. Instead, we can obtain the same dynamic response in the production orders and inventory levels by setting the parameters in the forecasting module appropriately. Potentially this offers an easier implementation route for the POUT policy.

Secondly, we study the OUT-DT policy's behaviour when ARIMA (1,1,2) demand is present. DT is known to produce minimum mean squared error (MMSE) forecasts of demand k-periods ahead for the ARIMA(1,1,2) demand process, Roberts (1982). However, in principle, DT can be used to forecast any demand process (just as exponential smoothing is optimal for the IMA(0,1,1) demand processes but can be used, rightly or wrongly, to forecast other demand processes). As ARIMA(1,1,2) demand is non-stationary, our preferred method for obtaining variance expressions, Tsypkin's Relation (Tsypkin, 1964), is not valid as the both demand and order variance is infinite. However, we are able to obtain the inventory variance expression and we adapt the approach of Gaalman et al. (2022) to investigate whether the difference between the order and demand variance is increasing in the lead time for ARIMA(1,1,2) demand.

1.1. Structure of our paper

The paper is organized as follows. After reviewing relevant literature in Section 2, we present the OUT policy and the POUT policy in Sections 3 and 4 respectively. We revisit the concepts of stability and invertibility of the DT method in Section 5. We study the OUT-DT policy under i.i.d. demand and compare it to the POUT policy in Section 6. Section 7 studies the OUT-DT policy under non-stationary ARIMA(1,1,2) demand where we reveal how the bullwhip effect is influenced by the lead time. Section 8 includes additional numerical analysis to verify the previous analysis on the OUT-DT and POUT policies and ARIMA(1,1,2) demand. Section 9 concludes.

2. Related literature

The DT forecasting method, often attributed to Gardner and McKenzie (1985), is an exponential smoothing based forecasting method based on three steps. The first step produces an exponential smoothing forecast of the level of the demand. The second step produces an exponential smoothing forecast of the rate of change in the demand, the trend. The third step produces a future projection, the forecast. The nature of the future projection depends on the observed demand and γ, the damping parameter. *Damped* future projections eventually flatten out to a constant level, $0 < \gamma < 1$. The future projection could also exhibit linear ($\gamma = 1$) or exponential ($\gamma > 1$) growth. The future projections could also oscillate over time when $-1 < \gamma < 0$, which may be useful for negatively correlated demand. DT is a generalization of Holt's method which has linear future projections, Roberts (1982) and Gardner and McKenzie (1985) argue that sustained linear future projections is unrealistic and demand is more likely to flatten out, suggesting $0 < \gamma < 1$ should be used.

We have elected to study the DT forecasting method as it is relatively under-studied in the supply chain dynamics literature and the M3 competition showed DT outperformed many other forecasting methods, Makridakis and Hibon (2000). Only a few methods requiring significant additional effort (and cost) were able to consistently produce forecasts with better accuracy than DT. However, those improvements are small and, in most cases, not statistically significant, Makridakis and Hibon (2000). Gardner and McKenzie (2011) show DT forecasting is also a generalization of eleven other forecasting methods, that can be accessed by setting the forecasting parameters to specific values. Using the monthly industry series from the M3 competition data, Petropoulos et al. (2019) explored the implications of various forecasting methods on both order and inventory variance in the OUT policy and confirmed DT's robust inventory performance.

Dejonckheere et al. (2003) found, for all lead times and all possible demand processes, the OUT policy with exponential smoothing forecasts always created the bullwhip effect. Li et al. (2014) showed the OUT policy with Holt's forecasts also created bullwhip for all lead times and all demand processes, but DT does not always create bullwhip. In addition, Li et al. (2014) proved the bullwhip effect cannot be avoided when the forecasting parameters were selected from the [0,1] region, i.e. when $0 \leq \{\alpha, \beta, \gamma\} \leq 1$. This was later verified in an empirical analysis by Chiang et al. (2016) on the monthly unit sales of Chevrolet automobiles from 1991–2008, where they found the DT parameters values in the [0,1] range always generated the bullwhip effect.

A considerable amount of literature studies the bullwhip effect under a finite variance stationary demand. For instance, Gaalman

(2006) considers the closely related stationary ARMA(2,2) demand process where it was possible to obtain both demand and order variance expressions. Bullwhip investigations for non-stationary demand is more challenging as the demand and order variances are infinite. Gaalman et al. (2022) present a novel method to determine whether the bullwhip effect generated by OUT policy reacting the ARMA(p, q) demand process is increasing in the lead time or not. The influence of the lead time on the bullwhip effect was shown to depend upon the order of the eigenvalues.

A stream of research has shown the POUT policy is effective at smoothing the bullwhip effect. The POUT is an implementable and linear generalization of the OUT policy, often employed in the real world (Potter and Disney, 2010; Cannella and Ciancimino, 2010; Disney et al., 2013). Unlike the OUT policy, where the complete discrepancy between the target and actual inventory (and the target and actual WIP) is corrected in each order, the POUT policy's order only corrects a fraction of the discrepancy between the target and actual inventory (and WIP) in each order. One of the earliest studies of the POUT policy with discrete control theory (z-transform) was conducted by Deziel and Eilon (1967). John et al. (1994) provides one of the first studies with continuous control theory (Laplace transform). The behaviour of the POUT policy was studied by Gaalman and Disney (2009) under ARMA(2,2) demand with arbitrary lead times. They show the bullwhip effect can be avoided by carefully calibrating the feedback controller. Li and Disney (2017) find, when retailers adopt the POUT policy and supply advanced order information to the manufacturer, the bullwhip effect, the supply chain inventory costs, and the manufacturer's MRP nervousness can all be reduced. We are unaware of any paper considering the OUT-DT policy, or indeed a bullwhip study, with an ARIMA(1,1,2) demand process.

Recent studies have revealed the superiority of POUT on bullwhip and inventory performance in closed-loop supply chains, Cannella et al. (2021). They find when tuning the feedback controller based on the cost structure and the average return rate, the POUT policy brings larger cost savings than the OUT policy in a hybrid manufacturing/remanufacturing system. The POUT policy is also studied in other contexts; for example, in a dual sourcing setting by Boute et al. (2022), with machine learning by Priore et al. (2019) and with low volume intermittent demand by Rostami-Tabar and Disney (2023). For a complete review of the applications of proportional control to operations and supply chain management, we refer readers to Lin et al. (2017) and Ivanov et al. (2018).

3. The order-up-to policy

The OUT replenishment policy is frequently applied in industry, especially in high volume settings (Li and Disney, 2017; Huang et al., 2021). We follow the assumptions and notation used in Li et al. (2014) except now we consider a general lead time $T_p \in \mathbb{N}_0$, rather than $T_p = 1$ as in Li et al. (2014). The sequence of events within each time period is illustrated in Fig. 1. First, the order placed $T_p + 1$ periods ago is received sometime during the period; demand is satisfied from finished goods inventory (we prefer to use the moniker *net stock* as negative inventory represents a back-order) during the period, ns_t. Demand is tallied, future forecasts determined, and replenishment orders o_t are generated at the end of the period with

$$o_t = \hat{d}_{t+T_p+1|t} + ns^* - ns_t + \sum_{i=1}^{T_p} \left(\hat{d}_{t+i|t} - o_{t-i}\right). \tag{2}$$

Here, $\hat{d}_{t+T_p+1|t}$ is a forecast of demand in the period $t + T_p + 1$, made (and using only information available) at time t. That is, $\hat{d}_{t+T_p+1|t}$ is a forecast of demand in the period after the lead time. $\sum_{i=1}^{T_p} o_{t-i}$ is the work-in-progress. The time-varying target work-in-progress, $\sum_{i=1}^{T_p} \hat{d}_{t+i|t}$, is the sum of the demand forecasts made at time t for the periods from $t + 1$ to $t + T_p$. The target net stock (ns^*) is a safety stock used to ensure a strategic level of inventory availability; $ns^* = \sqrt{\mathbb{V}[ns_t]}F^{-1}[p]$. Here $F^{-1}[p]$ is the inverse of the cumulative inventory distribution evaluated at the target availability p (Hosoda and Disney, 2009). If per unit, per period, inventory holding (h) and backlog costs (b) exist, then $p = b/(b + h)$, and ns^* minimizes inventory costs, Churchman et al. (1957). The inventory balance equation,

$$ns_t = ns_{t-1} - d_t + o_{t-T_p-1}, \tag{3}$$

completes the OUT policy specification.

In order to preserve linearity of the system and to allow for a tractable analysis, the following assumptions are made: Negative demand quantities indicate that customers are free to return products to suppliers (this can become negligible when the mean demand is sufficiently larger than the standard deviation of demand). Negative orders indicate that finished goods are disassembled into raw material (this can become negligible when the mean of the orders is sufficiently larger than the standard deviation of the orders). There are no capacity constraints in the system unless specified, and unmet demand is backlogged. We refer readers to Disney et al. (2021) and Wang et al. (2022) for more discussion on these factors. Boute et al. (2022) provide the following generic z-transform transfer function for the orders in the OUT policy,

$$\frac{O(z)}{\epsilon(z)} = \frac{D(z)}{\epsilon(z)}\left(1 + \sum_{k=1}^{T_p} \frac{\hat{D}_k(z)}{\epsilon(z)}\left(1 - \frac{1}{z}\right)\right), \tag{4}$$

where z is the z-transform operator, Nise (2004), $D(z)/\epsilon(z)$ is the transfer function of the demand generation process (in relation to the white noise input ϵ_t) and $\hat{D}_k(z)/\epsilon(z)$ is the transfer function of the k-periods ahead forecast. These will be defined in later sections.

A general form of the net stock transfer function, $NS(z)/\epsilon(z)$, is given by

$$\frac{NS(z)}{\epsilon(z)} = \frac{z}{z-1}\left(z^{-T_p-1}\frac{O(z)}{\epsilon(z)} - \frac{D(z)}{\epsilon(z)}\right). \tag{5}$$

Here, $z/(z-1)$ is the z-transform of the integration operator and z^{-T_p-1} is the z-transform of the delay operator. By convention, lower case letters are used for variables in the time domain and equivalent upper case letters are used for the corresponding variables in the frequency domain.

4. The proportional order-up-to (POUT) policy under i.i.d. demand

The POUT policy is the optimal linear replenishment rule for minimizing the weighted sum of order and inventory variance (Chen and Disney, 2007) and is an appropriate benchmark for this study. The POUT policy (Boute et al., 2009) is defined as

$$o_t = \hat{d}_{t+T_p+1|t} + \frac{1}{T_i}\left(ns^* - ns_t + \sum_{i=1}^{T_p}\left(\hat{d}_{t+i|t} - o_{t-i}\right)\right). \tag{6}$$

T_i is a proportional feedback controller with which we can alter the dynamic behaviour of the OUT policy, Disney and Towill (2003). When an i.i.d. demand is present, MMSE forecasts of future demands are given by

$$\forall i, \quad \hat{d}_{t+i|t} = \hat{d}_{t+1|t} = \mu_d. \tag{7}$$

Eqs. (6) and (3) can be converted into a block diagram (see Fig. 2) which can be rearranged using common control theory techniques (we refer interested readers to Nise (2004) for more on this aspect) to yield the following z-transform of the POUT policy,

$$\left.\frac{O(z)}{\epsilon(z)}\right|_{\text{POUT}} = \frac{\frac{1}{T_i}z}{z - \frac{T_i-1}{T_i}} \tag{8}$$

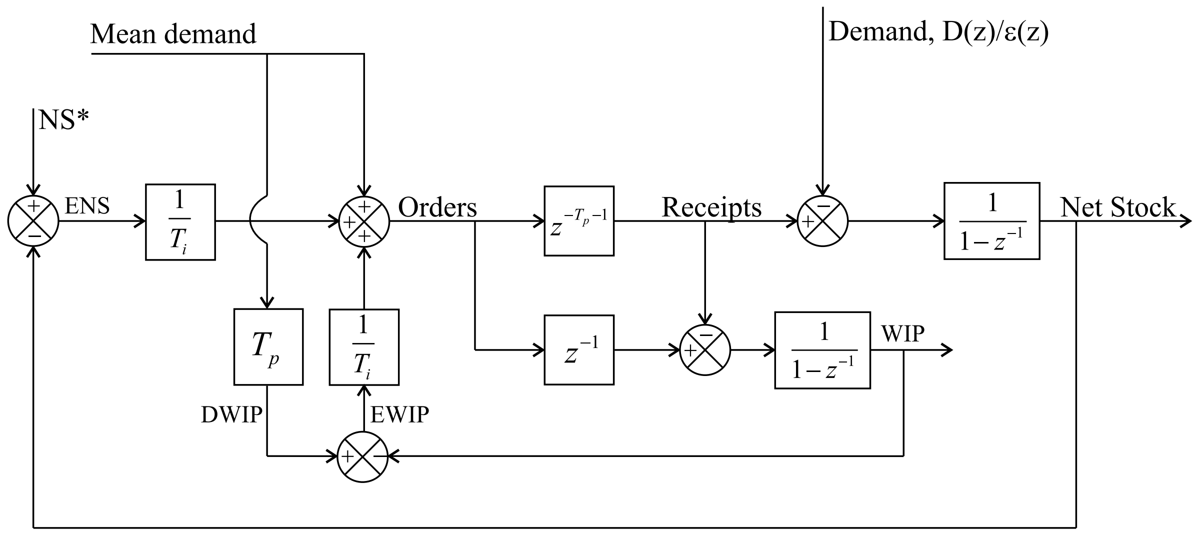


Fig. 2. Block diagram of the POUT policy under i.i.d. demand with MMSE forecasts.

which is in zero-pole form. The transfer function of the net stock levels maintained by the POUT policy can be written as

$$\left.\frac{NS(z)}{\epsilon(z)}\right|_{\text{POUT}} = \frac{\sum_{i=0}^{T_p-1} z^i + z^{T_p} T_i}{z^{T_p-1}(T_i-1) - z^{T_p} T_i} = \frac{\frac{1-z^{T_p}}{1-z} + z^{T_p} T_i}{z^{T_p-1}(T_i-1) - z^{T_p} T_i}. \tag{9}$$

The lead time T_p clearly has an impact on the order of the net stock transfer function.

4.1. Stability of the POUT policy

Stability is concerned with a system's response to a bounded system input. If the system produces a bounded output, the system is considered to be stable. If the system's response diverges exponentially, or oscillates with ever increasing amplitude, the system is unstable. For stability, the eigenvalues (the zeros and poles) of the POUT policy must lie within the unit circle in the complex plane, Nise (2004) and Jury (1974). Eq. (8) shows the POUT policy has one real zero at $\lambda_1^\theta = 0$ and one real pole at $\lambda_1^\phi = (T_i-1)/T_i$. The pole is inside the unit circle in the complex plane if $T_i > 0.5$, indicating the stability criteria, Disney (2008).

4.2. Variance ratio analysis of the POUT policy under i.i.d. demand

For a linear system reacting to an i.i.d. random (white noise) input ϵ_t, the long-run variance of the system's output x_t, can be calculated via Tsypkin's Relation, (Disney and Towill, 2003):

$$\frac{\mathbb{V}[\text{System output, } x_t]}{\mathbb{V}[\text{White noise input, } \epsilon_t]} = \sum_{t=0}^{\infty} (\tilde{x}_t)^2 \tag{10}$$

where $\tilde{x}_t$ is the response of the system when demand is given by the unit impulse (Dirac delta) function; i.e. $\epsilon_t = 1$ if $t = 0$, $\epsilon_t = 0$ otherwise, Gaalman et al. (2022). Consider first the *Bullwhip* ratio. The relevant system output is the orders o_t, those impulse response $\tilde{o}_t$ can be obtained by taking the inverse z-transform of (8),

$$\tilde{o}_t = Z^{-1}\left[\frac{\frac{1}{T_i} z}{z - \frac{T_i-1}{T_i}}\right] = \frac{1}{T_i}\left(\frac{T_i-1}{T_i}\right)^t. \tag{11}$$

Using (11) in (10) yields the *Bullwhip* ratio for the POUT policy under i.i.d. demand[1]:

$$Bullwhip = \frac{\mathbb{V}[o_t]}{\mathbb{V}[d_t]} = \sum_{t=0}^{\infty} (\tilde{o}_t)^2 = \sum_{t=0}^{\infty}\left(\frac{1}{T_i}\left(\frac{T_i-1}{T_i}\right)^t\right)^2 = \frac{1}{2T_i-1}. \tag{12}$$

Note, *Bullwhip* $= 1$ when $T_i = 1$, *Bullwhip* is decreasing convex in T_i, and *Bullwhip* $= 0$ when $T_i \to \infty$.[2] Notice that, under i.i.d. demand the *Bullwhip* produced by the POUT policy is independent of the lead time. The *NSAmp* ratio can be obtained by first taking the inverse z-transform of (9) to yield the net stock impulse response,

$$\tilde{ns}_t = Z^{-1}\left[\frac{\sum_{i=0}^{T_p-1} z^i + z^{T_p} T_i}{z^{T_p-1}(T_i-1) - z^{T_p} T_i}\right] = \begin{cases} -1 & \text{if } t \le T_p, \\ -\left(\frac{T_i-1}{Ti}\right)^{t-T_p} & \text{if } t > T_p. \end{cases} \tag{13}$$

Using (13) in (10) provides *NSAmp* for the POUT policy under i.i.d. demand:

$$\begin{aligned} NSAmp = \frac{\mathbb{V}[ns_t]}{\mathbb{V}[d_t]} &= \sum_{t=0}^{T_p}(-1)^2 + \sum_{t=T_p+1}^{\infty}\left(-\left(\frac{T_i-1}{Ti}\right)^{t-T_p}\right)^2 \\ &= 1 + T_p + \frac{(T_i-1)^2}{2T_i-1}. \end{aligned} \tag{14}$$

NSAmp is convex in T_i, with an asymptote to infinity when $T_i \downarrow 0.5$, and is increasing in T_i when $T_i > 1$; a global minimum of *NSAmp* $= 1 + T_p$ exists at $T_i = 1$. The POUT policy represents the optimal linear quadratic regulator for minimizing the sum of the inventory and order variances, Chen and Disney (2007). The aim of the next section is to determine if the OUT-DT policy can match, or better, this performance.

5. Damped trend forecasting

We now turn our attention to the DT forecasting mechanism. Gardner and McKenzie (1985) provide the following recurrence form of the DT forecasting method:

$$\hat{a}_t = \alpha d_t + (1-\alpha)\left(\hat{a}_{t-1} + \gamma \hat{b}_{t-1}\right), \tag{15}$$

$$\hat{b}_t = \beta\left(\hat{a}_t - \hat{a}_{t-1}\right) + (1-\beta)\gamma \hat{b}_{t-1}, \tag{16}$$

$$\hat{d}_{t+k|t} = \hat{a}_t + \varphi[k]\,\hat{b}_t. \tag{17}$$

Here, $\hat{d}_{t+k|t}$ is the forecast of the demand k periods ahead, d_{t+k}, made at time t. $\hat{d}_{t+k|t}$ is the sum of a level, $\hat{a}_t$, and a trend, $\hat{b}_t$, component and

$$\varphi[k] = \sum_{i=1}^{k} \gamma^i = \frac{\gamma\left(1-\gamma^k\right)}{1-\gamma}. \tag{18}$$

[1] As demand is i.i.d., $d_t = \epsilon_t$ and $\mathbb{V}[d_t] = \mathbb{V}[\epsilon_t]$.

[2] Note, ↑ means *approach from below*, ↓ means *approach from above*, and → means *tends to*.

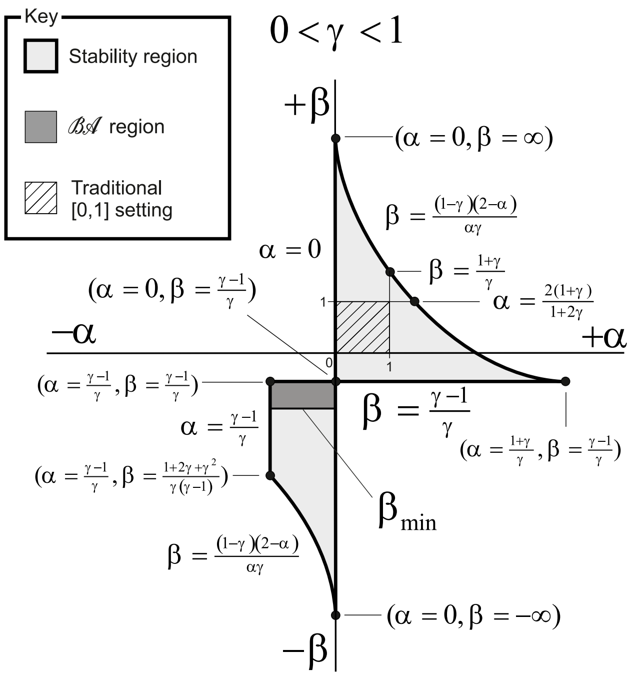


Fig. 3. Characterization of the DT parameter space when $0<\gamma<1$.
Source: Adapted from Li and Disney (2018).

$\{\alpha,\beta,\gamma\}$ are the DT forecasting parameters. α is a smoothing constant applied to the level $\hat{a}_t$, β is a smoothing constant applied to the trend $\hat{b}_t$, and γ shapes the forecasts as they are projected into the future. The trend is damped when $0<\gamma<1$, although γ can take on other values. If $\gamma=0$, there is no trend and the forecasting system acts as exponential smoothing would react. If $\gamma=1$, the model is equivalent to Holt's method (with linear future forecast projections). When $\gamma>1$, the forecasts exhibit exponential growth; $-1<\gamma<0$ can be used to create future projections which contain period-to-period oscillations.

Li et al. (2014) provide the following second order transfer function of (17), the k-period ahead DT forecast,

$$\frac{\hat{D}_k(z)}{\epsilon(z)}=\frac{z^2\alpha(1+\beta\varphi[k])-z\alpha(\gamma(1-\beta)+\beta\varphi[k])}{z^2-z(1+\gamma-\alpha(1+\beta\gamma))-\gamma(\alpha-1)}. \tag{19}$$

5.1. Invertibility and stability of damped trend forecasting

The concept of invertibility is concerned with the ability to identify the demand process structure from past demand observations. Invertibility is related to linear moving average (MA) models or the MA part of auto-regressive integrated moving average (ARIMA) models, Box et al. (2008). All exponential smoothing forecasting methods (of which DT is one) can be converted into an equivalent ARIMA model. If the MA part of an ARIMA model can be expressed as an auto-regressive (AR) model of infinite order, the model is deemed invertible and implies all relevant state variables are directly observable, Box et al. (2008). For the DT model, the invertibility region is the same as the stability region, Li and Disney (2018).

The DT stability region has been studied by others. Gardner and McKenzie (1985, p. 1239) provided a stability region for DT, but it is only valid for $0\leq\gamma\leq1$; they do however acknowledge that stable parameters exist outside of their stated stability region. Hyndman et al. (2008a, p. 412) studied the stability of the state space representation of the ETS(A, $\mathrm{A_d}$, N) model when $0<\gamma\leq1$ only. The ETS(A, $\mathrm{A_d}$, N) model is equivalent to DT after a suitable change of notation. Hyndman et al. (2008b) also provided stability boundaries under the condition that $0<\gamma\leq1$. Li et al. (2014, pp. 5–6) studied the stability of DT via Jury's Inners approach (Jury, 1974) and visualized the complete stability boundaries for all γ. For convenience we repeat them here: When $\gamma\neq0$, the following relations must be satisfied for stability,

$$\gamma-1<\alpha\gamma<\gamma+1 \quad \text{and} \quad \alpha(\gamma-1)<\alpha\beta\gamma<(2-\alpha)(\gamma+1). \tag{20}$$

When $\gamma=0$, $0<\alpha<2$ is required for stability. Eq. (20) is equivalent to the result of Hyndman et al. (2008a) when $0<\gamma\leq1$ and their trend smoothing parameter is set to $\alpha\beta\gamma$.

Eq. (20) offers a much wider range of values to the parameter set $\{\alpha,\beta,\gamma\}$, compared to the traditional $[0,1]$ interval suggested in the literature, see for example Winters (1960) and Gardner (1990). Commercial software such as SAP and Forecast Pro® also selects α and β between 0 and 1 (SAP, 2016; Stellwagen and Goodrich, 2011), SAS/ETS® considers $0<\gamma<1$, $0<\alpha<2$, and $0<\gamma\beta<(4/\alpha-2)$, SAS (2018). We emphasize, these do not include $\{\alpha,\beta\}<0$ and are only part of the complete stability region identified in (20) and characterized (when $0<\gamma<1$) in Fig. 3.

6. The OUT policy under i.i.d. demand with damped trend forecasting

To study *Bullwhip* and *NSAmp* behaviour of the OUT-DT policy, we will need the transfer function of the replenishment orders and the net stock levels. The transfer function of the orders can be obtained by using $D[z]/\epsilon[z]=1$ (the transfer function for i.i.d. demand) and (19) (the transfer function for DT forecasts) in (4) and simplifying:

$$\frac{O(z)}{\epsilon(z)}=1+\frac{\alpha(z-1)\left(\beta\zeta(z-1)+\left(T_p+1\right)(z-(1-\beta)\gamma)\right)}{z^2+z(\alpha\beta\gamma+\alpha-\gamma-1)+(1-\alpha)\gamma}, \tag{21}$$

where

$$\zeta=\Phi\left[T_p\right]+\varphi\left[T_p+1\right]. \tag{22}$$

Recall, $\varphi[k]$ was given in (18), and

$$\Phi\left[T_p\right]=\sum_{k=1}^{T_p}\varphi[k]=\frac{\gamma\left(\gamma^{T_p+1}+T_p(1-\gamma)-\gamma\right)}{(1-\gamma)^2}. \tag{23}$$

Eq. (21) concurs with the transfer function reported in Li et al. (2014) who conducted a frequency analysis of the OUT-DT policy. The net stock transfer function can be found by substituting (21) into (5) and simplifying to yield,

$$\frac{NS(z)}{\epsilon(z)}=\frac{\alpha\left(\beta\zeta(z-1)+\left(T_p+1\right)(z-(1-\beta)\gamma)\right)}{z^{T_p}\left(z^2+z(\alpha\beta\gamma+\alpha-\gamma-1)+(1-\alpha)\gamma\right)}-\frac{z^{-T_p}(z^{1+T_p}-1)}{z-1}. \tag{24}$$

These transfer functions will be used to study the *Bullwhip* and *NSAmp* behaviour under i.i.d. demand in Section 6.1; Section 6.2 considers the equivalence of the dynamic response of OUT-DT and POUT policies.

6.1. OUT-DT variance ratio analysis under i.i.d. demand

Although DT is the optimal forecast for ARIMA(1,1,2) demand, it is insightful to first investigate its performance under i.i.d. demand. We use the same procedure outlined in Section 4.2 to obtain the variance ratios. We take the inverse z-transform of (21), sum its square via (10), and divide by the demand variance, to provide the following expression for the *Bullwhip* ratio when i.i.d. demand is forecasted via the DT method:

$$\frac{\mathbb{V}[o_t]}{\mathbb{V}[d_t]}=\frac{\begin{pmatrix}2\alpha^2\beta\zeta(1+\gamma(2-3\beta(1-\gamma)+\gamma)+2T_p(1-\gamma)(1+(1-\beta)\gamma))+4\alpha^2\beta^2\zeta^2(1-\gamma)+\\ 2-2\gamma^2-2\alpha^3(\beta\gamma+\gamma+1)(\beta\zeta+T_p+1)(\beta(\gamma-\zeta)-(1-\beta)\gamma T_p)+\\ \alpha\left(4\beta\left(1-\gamma^2\right)\zeta+\gamma(\beta\gamma-\beta-\gamma+2)+4\left(1-\gamma^2\right)T_p+3\right)+\\ \alpha^2\gamma^2\left(3+\beta+2T_p(\beta-T_p+1)+2\beta^2(T_p+1)^2\right)+\\ \alpha^2(\gamma(2(2-\beta)(T_p+1)-1)+2T_p(T_p+1))\end{pmatrix}}{(1-\gamma(1-\alpha))((2-\alpha)(\gamma+1)-\alpha\beta\gamma)}. \tag{25}$$

Li and Disney (2018) identified a region of the parametric plane where it is possible for the OUT-DT to avoid creating bullwhip for any lead time. The region was specified by:

$$\left.\begin{array}{l} 0<\gamma<1, \\ \left(\alpha_{\min}=\frac{\gamma-1}{\gamma}\right)<\alpha<0, \\ \left(\beta_{\min}=\frac{-(T_p+1)(\gamma+1)(1-\gamma)^2}{(\gamma-\gamma^3)T_p+\gamma^2\left(2\gamma^{T_p+1}-\gamma-2\right)+\gamma}\right)\leq\beta\leq\left(\beta_{\max}=\frac{\gamma-1}{\gamma}\right). \end{array}\right\} \tag{26}$$

The lower bound, $\beta_{\min}$ in (26), is increasing in the lead time T_p. When (26) holds, we say the parameter set is a member of the *Bullwhip Avoidance* ($\mathcal{BA}$) area, $\{\alpha,\beta,\gamma\}\in\mathcal{BA}$. The area was found by Li and Disney (2018) in a frequency response analysis that considered how the harmonic frequencies in demand were amplified by the OUT-DT policy. Note, (26) does not guarantee *Bullwhip* <1, only that there exists a demand pattern that could have *Bullwhip* <1. In this study, we sharpen and refine the analysis of Li and Disney (2018) by focusing on the characterizations of *Bullwhip* and *NSAmp* within the $\mathcal{BA}$ region under i.i.d. demand and ARIMA(1,1,2) demand.

Studying the *Bullwhip* ratio of the OUT-DT policy (25), we found it has no stationary points within the stability region and is always differentiable within the $\mathcal{BA}$ region. Thus, any local minima and maxima must exist on the boundaries of the $\mathcal{BA}$ region. The same properties were found in the *NSAmp* ratio. Taking each boundary into consideration, we find a minimal *Bullwhip* of

$$Bullwhip=\frac{\mathbb{V}[o_t]}{\mathbb{V}[d_t]}=\frac{\left(\begin{array}{c}1+\gamma\left(2\alpha^2\gamma^2\left(\gamma^{T_p}-1\right)\left(\gamma^{T_p+1}-1\right)+\right.\\ \left.\alpha(\gamma-1)\left(\gamma\left(2(\gamma+1)\gamma^{T_p}-3\right)-1\right)+(\gamma-1)\gamma-1\right)\end{array}\right)}{(\gamma-1)^2(1+(1-\alpha)\gamma)} \tag{27}$$

exists when $\beta\uparrow\beta_{max}$. Eq. (27) has a value between 0 and 1, $\forall\gamma$ when $\alpha\in\mathcal{BA}$ and $T_p\in\mathbb{N}^0$. That $\mathbb{V}[o_t]/\mathbb{V}[d_t]<1$ implies that a smoothing effect is present under i.i.d. demand. It is also interesting that *Bullwhip* $\to 1$ when $\alpha\uparrow 0$. This suggests when capacity costs dominate, a small negative α and/or a small negative β should be adopted due to its bullwhip avoidance behaviour.

Using Tsypkin's relation, the *NSAmp* ratio can be obtained from (24). We proceed in exactly the same manner: first take the inverse z-transform of (24) to obtain the impulse response and then sum the squared impulse response over $t=0$ to infinity to yield the variance of the net stock levels. The only complication is the need to specify a lead time to be able to take the inverse z-transform. Repeating the procedure for $T_p=\{0,1,2\ldots\}$ and using induction provides

$$\frac{\mathbb{V}[ns_t]}{\mathbb{V}[d_t]}=1+T_p+ \frac{\alpha\left(\begin{array}{c}(\gamma-1)^2\left(-T_p^2\right)((\beta-1)\gamma+1)\left(\gamma\left((\alpha-1)\gamma^2-\alpha-2\alpha\beta\gamma+\gamma+1\right)-1\right)+\\ \gamma\left(\alpha+\beta(\gamma+1)(\gamma-1)^4-\gamma^3((\gamma-4)\gamma+5)+\alpha\gamma\left(\gamma((\gamma-2)(\gamma-1)\gamma+2)+\right.\right.\\ 2\beta^2\gamma^2\left(\gamma^{T_p}-1\right)\left(\gamma^{T_p+2}+1-2\gamma\right)+\beta(\gamma-1)^2\left(\gamma\left(-\gamma+2(\gamma+1)\gamma^{T_p}-4\right)+\right.\\ \left.\left.\left.1\right)-3\right)-4+5\gamma\right)-2(\gamma-1)T_p((\beta-1)\gamma+1)\left(\gamma\left(\alpha-\left((\gamma-2)\gamma^2\right)+\right.\right.\\ \left.\left.\alpha\gamma\left(\beta+(\gamma-1)\gamma+\beta\gamma\left((\gamma+1)\gamma^{T_p}-3\right)-1\right)-2\right)+1\right)+1\end{array}\right)}{(\gamma-1)^4((1-\alpha)\gamma-1)(\gamma(\alpha\beta+\alpha-2)+\alpha-2)}. \tag{28}$$

A minimal *NSAmp* of $1+T_p$ occurs when $\alpha\uparrow 0$. This means when i.i.d. demand is present, *NSAmp* $\geq 1+T_p$ in the OUT-DT system for $\{\alpha,\beta,\gamma\}\in\mathcal{BA}$. Further, when the inventory variance is minimized, *Bullwhip* $=1$. We can conclude, when inventory costs are significantly larger than capacity costs and i.i.d. demand is present, $\alpha=0$ is recommended. Setting $\alpha=0$ results in MMSE forecasts of demand (that is, all future forecasts equal the mean of the i.i.d. demand).

Fig. 4 illustrates *Bullwhip* and *NSAmp* via contour plots of the $\{\alpha,\beta\}$ plane for various lead times. When $\beta\downarrow\beta_{min}$ and $\alpha\downarrow\alpha_{min}$, both *Bullwhip* and *NSAmp* ratios increase dramatically. Furthermore, the *Bullwhip* value is rather insensitive to the lead time, similar to the behaviour of the POUT policy, while the lead time has a significant influence on *NSAmp*. These imply, in the long lead time cases, $\beta\downarrow\beta_{min}$ and $\alpha\downarrow\alpha_{min}$ needs to be avoided; the top right quarter of the $\mathcal{BA}$ plane is always superior.

Note, when $\beta\uparrow\beta_{max}$, $\alpha\downarrow\alpha_{min}$ and $\gamma\downarrow 0$, the bullwhip effect reaches a global minimum (of $\mathbb{V}[o_t]/\mathbb{V}[d_t]=0$), but *NSAmp* $\to\infty$. In other words, the order variance can be reduced to zero for i.i.d. demand at the cost of increased inventory variance, indicating that a trade-off exists, just as it does for the POUT policy. When the business objective is to reduce both inventory and capacity costs, a γ close to 0 and a small negative β are recommended as it is guaranteed to eliminate the bullwhip effect; we can then optimize α to minimize the sum of inventory and capacity costs.

Inspired by the Linear Quadratic Gaussian (LQG) approach (with a linear system, a quadratic objective function, and a Gaussian input signal), Fig. 5 considers the trade-off between *NSAmp* and *Bullwhip* as a convex combination of a single weight, w, via the following objective function,

$$J=(1-w)\mathbb{V}[ns_t]+w\mathbb{V}[o_t]. \tag{29}$$

Holt et al. (1955) also assumed costs to the linear weighted sum of the variances. This cost function is also similar to the inventory holding and backlog cost and installed capacity and over time approach of Boute et al. (2022), except there the objective function reduces to a weighted sum of the net stock and order standard deviations. When the weight w increases (in other words, when more emphasis is placed on the order variance (bullwhip costs) rather than on the net stock variance (inventory costs)) the optimal solution moves from the right of the parametric plane (see Fig. 5a) to the top the parametric plane (see Fig. 5c). This further verifies our recommendations that $\alpha=0$ can be used to minimize inventory costs, while $\beta\uparrow\beta_{max}$ is required if it is important to minimize the costs associated with the bullwhip effect.

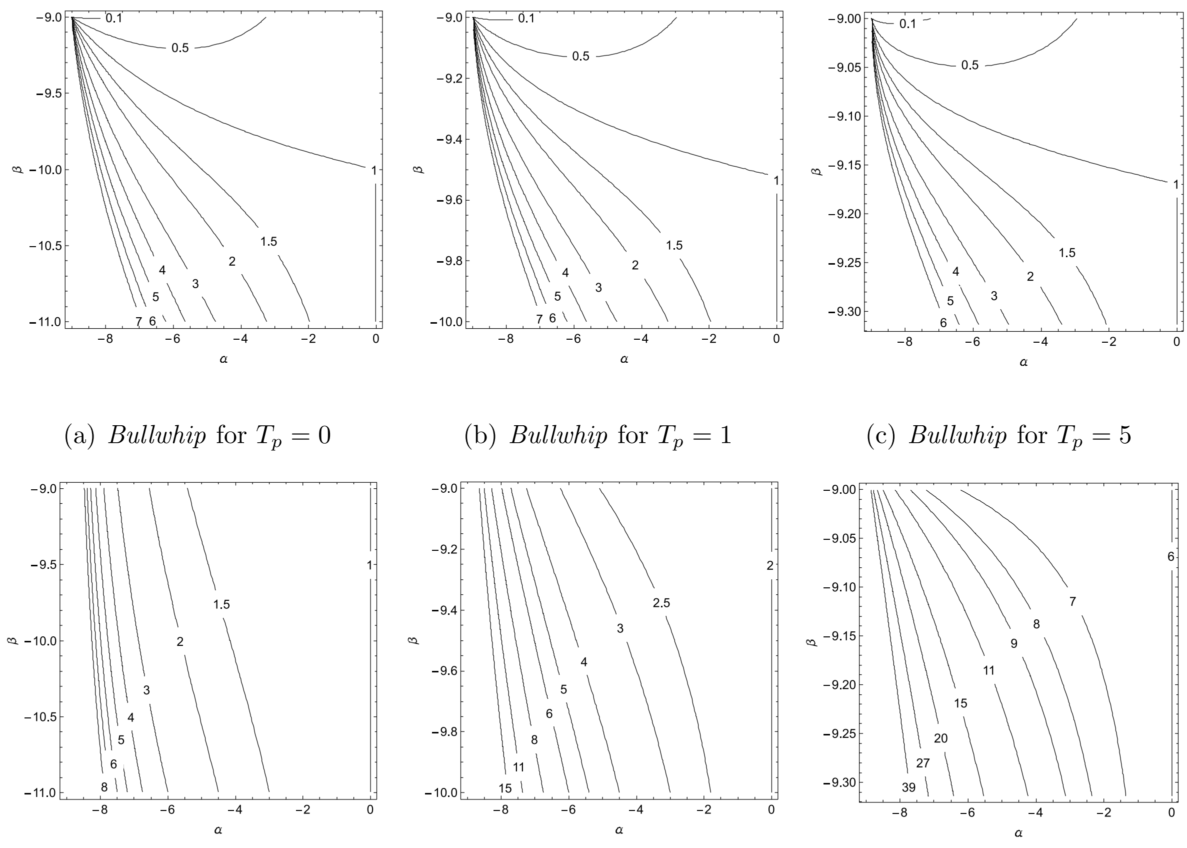
6.2. Comparison between the OUT-DT policy and the proportional OUT policy

Fig. 6a shows the inventory and order impulse responses for $\gamma\downarrow 0$, $\beta\uparrow\beta_{max}$, and $\alpha_{min}<\alpha<0$. They are quite different to the impulse responses from the conventional $[0,1]$ parameter region as shown in Fig. 6b. Our recommended $\mathcal{BA}$ region produces a smoothed, damped, and exponential increasing (or decreasing) impulse response, rather than an under-damped oscillatory response (seen in Fig. 6b). These are desirable dynamic properties which further strengthens our argument for selecting parameters from the $\mathcal{BA}$ region. *Bullwhip* and *NSAmp* ratios are also noted in Fig. 6 where we can see *Bullwhip* $=0.4$ and *NSAmp* $=1.25$ for the $\mathcal{BA}$ system in Fig. 6a. Both these measures dominate the non-$\mathcal{BA}$ settings in Fig. 6b where *Bullwhip* $=2.21$ and *NSAmp* $=1.31$.

That the OUT-DT policy is capable of eliminating bullwhip effect without using a proportional controller in the inventory position feedback loop is astonishing. Visually, the character of the impulse response in Fig. 6a is similar to the POUT policies (Dejonckheere et al., 2003; Gaalman and Disney, 2009) and the closely related automatic pipeline inventory and order based production control system (Disney and Towill, 2003). These insights motivate us to investigate the similarity between the OUT-DT policy and the POUT policy. In this section, we compare the poles and zeros of POUT and OUT-DT policies and show when they are equivalent. To avoid lengthy equations, we assume $\{\alpha,\beta,\gamma\}$ are selected from the $\mathcal{BA}$ region where $\beta\uparrow\beta_{\max}=(\gamma-1)/\gamma$.

Writing (21), the OUT-DT order transfer function, in pole-zero form to match (8) gives

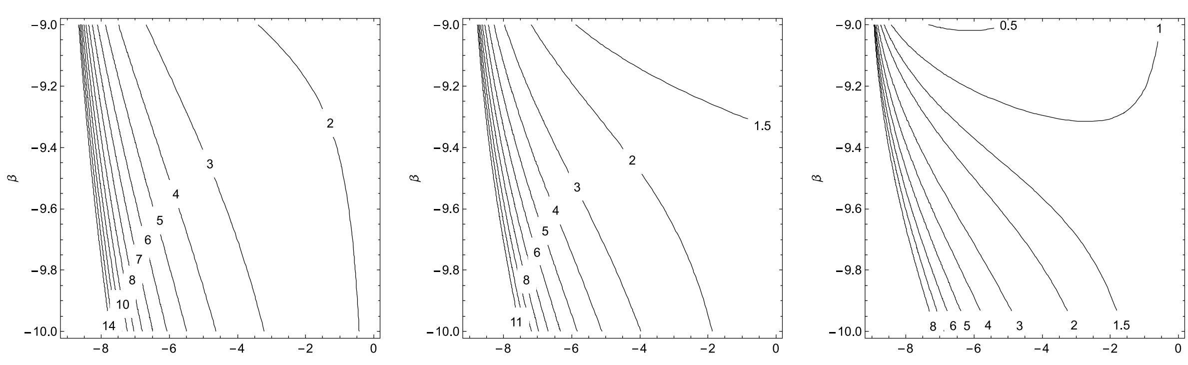
$$\frac{O(z)}{\epsilon(z)}=\frac{\frac{\alpha\gamma-\alpha\gamma^{T_p+2}-\gamma+1}{1-\gamma}\left(z-\frac{\gamma\left(\alpha\gamma-\alpha\gamma^{T_p+1}-\gamma+1\right)}{\alpha\gamma-\alpha\gamma^{T_p+2}-\gamma+1}\right)}{z-(1-\alpha)\gamma}. \tag{30}$$



(a) *Bullwhip* for $T_p = 0$ (b) *Bullwhip* for $T_p = 1$ (c) *Bullwhip* for $T_p = 5$

(d) *NSAmp* for $T_p = 0$ (e) *NSAmp* for $T_p = 1$ (f) *NSAmp* for $T_p = 5$

Fig. 4. *Bullwhip* and *NSAmp* in the OUT-DT policy when $\gamma = 0.1$.



(a) $w = 0.1$ (b) $w = 0.5$ (c) $w = 0.9$

Fig. 5. Contour plots for the weighted convex sum of the order and inventory variances given by (29), maintained by the OUT-DT policy when $\gamma = 0.1$ and $T_p = 1$. Note, when $w \to 0$ inventory costs dominate, when $w \to 1$ capacity costs dominate.

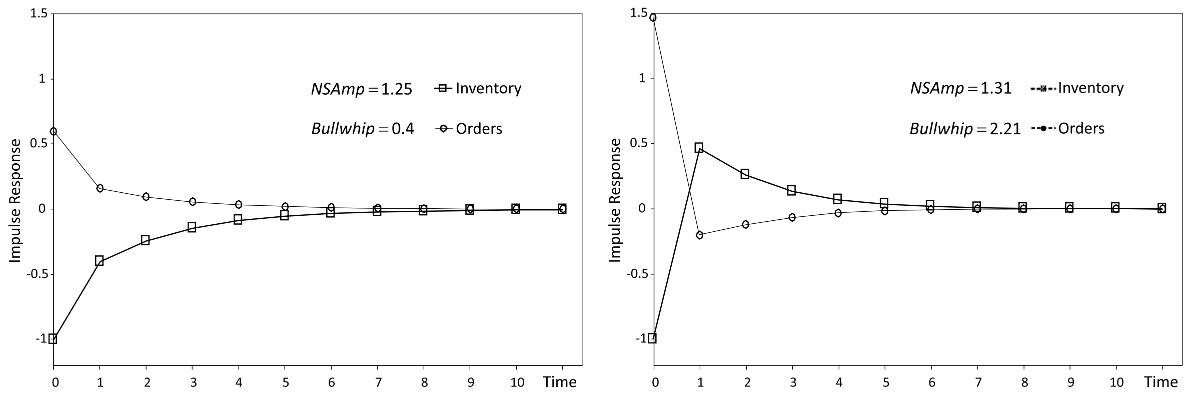
When

$$\alpha = \frac{T_i(\gamma - 1) + 1}{T_i\gamma}, \quad \beta \uparrow \beta_{\max} = (\gamma - 1)/\gamma, \quad \text{and} \quad \gamma \downarrow 0, \tag{31}$$

we may re-write (30) into the following form

$$\frac{O(z)}{\epsilon(z)} = \frac{\frac{1-(1-T_i)\gamma^{T_p+1}-T_i\gamma^{T_p+2}}{T_i(1-\gamma)}\left(z - \frac{\gamma\left(1-(1-T_i)\gamma^{T_p}-T_i\gamma^{T_p+1}\right)}{1-(1-T_i)\gamma^{T_p+1}-T_i\gamma^{T_p+2}}\right)}{z - \frac{T_i-1}{T_i}}. \tag{32}$$

Both (8) and (32) are first-order systems with a single pole at $z = (T_i - 1)/T_i$ and have a geometrically decreasing impulse responses when $T_i > 1$. When $\gamma = 0$, the order transfer function of the OUT-DT policy has a zero at $z = 0$. Although γ cannot be 0 if we wish to select the DT parameters from the $\mathcal{BA}$ region, the zeros of the order transfer functions in OUT-DT and POUT policies can be very close to each other if $\gamma \downarrow 0$ (as when $\gamma > 0$, the $\mathcal{BA}$ region exists). Therefore, by letting $\alpha = (T_i(\gamma - 1) + 1)/(T_i\gamma)$, $\beta \uparrow \beta_{max}$ and $\gamma \downarrow 0$, the order transfer function in both the OUT-DT and POUT policies will have, for all intents and purposes, identical poles and zeros. If the poles



(a) $\alpha = -2, \beta = -4, \gamma = 0.2$.   (b) $\alpha = 0.4, \beta = 0.8, \gamma = 0.2$.

Fig. 6. The impulse response of inventory and order in the OUT-DT system, $T_p = 0$. Panel a: A damped impulse response. Panel b: An undamped impulse response.

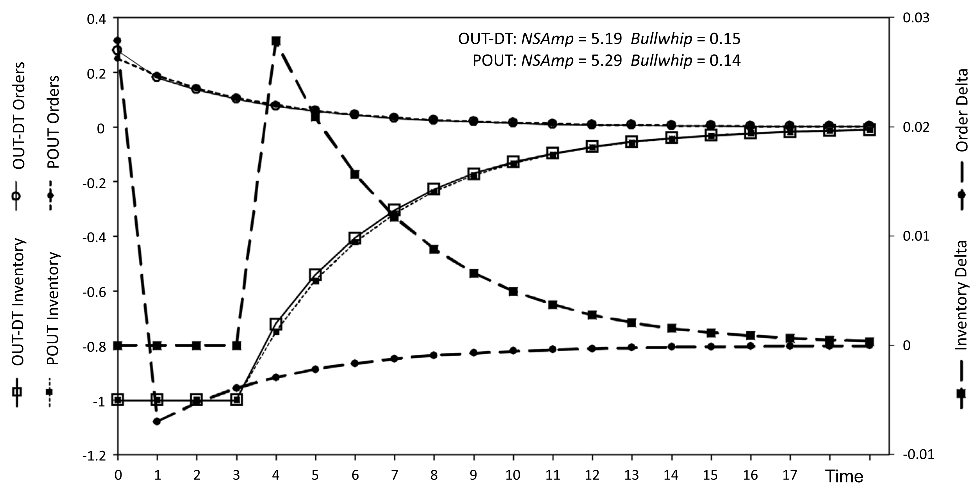


Fig. 7. Impulse responses of the POUT (with $T_i = 4$) and OUT-DT (with $\{\gamma = 0.1, \alpha = -6.5, \beta = -9\}$) systems and their differences when $T_p = 3$. Note: Inventory Delta $= ns_t|_{\text{OUT-DT}} - ns_t|_{\text{POUT}}$, Order Delta $= o_t|_{\text{OUT-DT}} - o_t|_{\text{POUT}}$.

and zeros are identical, the order transfer functions are identical, both systems respond to demand in exactly the same way, and their order and inventory responses will be identical. Fig. 7 provides an example of the system impulse response when $T_p = 3$. Fig. 7 confirms the order and inventory impulse responses in the OUT-DT system approximate the POUT's system responses. Note, we could have set γ closer to zero in Fig. 7 and this would have resulted in a pair of impulse responses that are indistinguishable from each other. However, we elected to use $\gamma = 0.1$ to demonstrate how small the discrepancy is.

It is interesting to note we can always set the eigenvalues of the OUT-DT policy arbitrarily close to the eigenvalues of the POUT policy. In this case, the two systems will have the same dynamic response and the same variance ratios. Thus, as the POUT policy's *Bullwhip* ratio (under i.i.d. demand) is independent of the lead time, so can be the OUT-DT policy's *Bullwhip* ratio.

The golden ratio in production and inventory control. Assuming the objective function stated in (29), Disney et al. (2004) studied the POUT system and found the optimal T_i is given by

$$T_i^* = \frac{w - 1 - \sqrt{1 + 2w - 3w^2}}{2w - 2}, \tag{33}$$

which is easily recognized as having the same form as the golden ratio when $w = 1/2$. Substituting (33) into (31), we obtain the optimal forecasting parameters to use in OUT-DT policy for minimizing the objective function (29) are

$$\alpha = 1 - \frac{w - \sqrt{1 + 2w - 3w^2} + 1}{2w\gamma}, \quad \beta \uparrow \beta_{max}, \quad \text{and} \quad \gamma \downarrow 0. \tag{34}$$

In summary, we have shown the OUT-DT replenishment policy has (almost) the same dynamic behaviour as the POUT policy but without introducing proportional feedback controllers into the replenishment policy.

6.3. Impact of capacity constraints

Having established the OUT-DT policy can be made to have the same response as the POUT policy, this section will consider if this equivalence holds when we have the following non-linear capacity constraint on the orders,

$$o_t = \min\left[\hat{d}_{t+T_p+1|t} + ns^* - ns_t + \sum_{i=1}^{T_p}\left(\hat{d}_{t+i|t} - o_{t-i}\right), k\right]. \tag{35}$$

Here $\min[x, k]$ returns the minimum of x and k where k is the capacity limit (the maximum quantity that can be produced). Studying this type of policy is very difficult; only limited theoretical progress has been made in the literature. For example, see Disney et al. (2021) for issues with non-linear effects placed on the inventory levels and Wang et al.

(2022) for results on non-linear effect placed on the production orders. Hence, we will proceed via a simulation analysis.

We set up a Monte Carlo simulation (one in Excel and another in Python to verify our work), 10 000 time periods in length, and average the results from 1000 replications. Fig. 8 shows how order and inventory variances of the two replenishment policies change with capacity constraints and the value of T_i. Here, the α in the OUT-DT policy adapts to the POUT policy's T_i value using (31). Fig. 9 illustrates the performance of the OUT-DT and POUT systems via the order and inventory variances under different capacity constraints and lead times.

Both figures show the variance of the orders and the variance of the inventory maintained by the OUT-DT is very similar to the POUT policy. When the capacity constraint is tight (that is, when k is close to, but higher than the mean demand, μ), the order variance is small and the inventory variance is very high. When the capacity constraint is sufficiently large, the capacity limit no longer constrains the production and the order and inventory variances matches those predicted by (12), (14), (27) and (28). The sum of an increasing concave function (the order variance) and a decreasing convex function (the inventory variance) results in a convex weighted sum of the two variances. This was also noted by Ponte et al. (2017). The natural consequence is that the capacity constraint can also be used as a mechanism to balance the trade-off between the order and inventory variances.

We find when capacity constraint is tight, the order variability in the OUT-DT policy is slightly higher than the POUT policy, see panels A1-A4 in Fig. 8, a larger value of the feedback controller T_i, increases the difference between the two systems' order variances. The two systems' order variance are the same when $T_i = 1$, see panel A5 in Fig. 8, or when the capacity constraint is sufficiently large.

Fig. 9 shows both policies' order variance and inventory variance under capacity constraints and for different lead times. The trade-off between the order and inventory variances exists for any lead time. The marginally higher OUT-DT's order variances with tight capacity constraints are also observed under different lead times. Thus, we conclude whatever the feedback controller value the POUT policy adopts, there is always an equivalent OUT-DT policy, reacting to i.i.d. demand similarly to the POUT policy, with or without capacity constraints, and for all lead times.

As the order and inventory variances for the POUT and DT-OUT policies, in both the linear and non-linear setting, are similar we can be confident the dynamic behaviour of an ERP system can be manipulated by altering how forecasts are made without any unpredicted or unusual consequences. We do not need to incorporate the POUT policy into the ERP system's planning book. The damped trend forecasting mechanism offers an easy implementation route as changes to the planning book can be done by the commercial forecasting department alone.

7. The OUT policy under ARIMA(1,1,2) demand with damped trend forecasting

In Section 6 we revealed the OUT-DT system can avoid the bullwhip effect when an i.i.d. demand is present. The DT forecasting system is not an optimal forecasting system for the i.i.d. demand process; DT is the optimal forecasting method for predicting ARIMA(1,1,2) demand, Gardner and McKenzie (1985). In Section 7.1 we explore the ARIMA(1,1,2) demand process; in Section 7.2 we investigate the bullwhip-lead time behaviour of the OUT-DT policy.

7.1. Eigenvalue analysis of ARIMA(1,1,2) demand

The ARIMA(1,1,2) demand process is given by,

$$d_t = d_{t-1} + \phi_1(d_{t-1} - d_{t-2}) - \theta_1\epsilon_{t-1} - \theta_2\epsilon_{t-2} + \epsilon_t. \tag{36}$$

Gardner and McKenzie (1985) also show the damped trend forecast produces a MMSE forecast of ARIMA(1,1,2) demand when

$$\left.\begin{aligned} \theta_1 &= 1 + \gamma - \alpha - \alpha\beta\gamma, \\ \theta_2 &= \gamma(\alpha - 1), \\ \phi_1 &= \gamma. \end{aligned}\right\} \tag{37}$$

Given a set of ARIMA(1,1,2) parameters, perhaps identified from a real time series, we can solve the simultaneous equations in (37) for the damped trend parameters:

$$\left.\begin{aligned} \alpha &= \tfrac{\theta_2+\phi_1}{\phi_1}, \\ \beta &= \tfrac{\phi_1^2-\theta_2-\theta_1\phi_1}{\theta_2\phi_1+\phi_1^2}, \\ \gamma &= \phi_1. \end{aligned}\right\} \tag{38}$$

Later, we will exploit the eigenvalues of the ARIMA(1,1,2) demand process to make some bullwhip predictions. The eigenvalues can be identified from the z-transform transfer function of the ARIMA(1,1,2) demand process,

$$\frac{D_{ARIMA(1,1,2)}(z)}{\epsilon(z)} = \frac{z^2 - z\theta_1 - \theta_2}{z^2 - z(1+\phi_1) + \phi_1}. \tag{39}$$

Eq. (39) has the following eigenvalues:

$$\begin{aligned} &\lambda_1^\theta = \frac{1}{2}\left(\theta_1 - \sqrt{\theta_1^2 + 4\theta_2}\right), \quad \lambda_2^\theta = \frac{1}{2}\left(\theta_1 + \sqrt{\theta_1^2 + 4\theta_2}\right), \\ &\lambda_1^\phi = \phi_1, \quad \text{and} \quad \lambda_2^\phi = 1. \end{aligned} \tag{40}$$

Here, $\lambda_{\{1,2\}}^\theta$ are the zeros, the roots of the numerator of (39) w.r.t. z; $\lambda_{\{1,2\}}^\phi$ are the poles, the roots of the denominator of (39) w.r.t. z. Note, the largest pole, $\lambda_2^\phi = 1$, implying the system is non-stationary. The impulse response, while it does not escape to infinity or oscillate with ever increasing amplitude, it does not return to zero. Rather it has an off-set, implying the demand has infinite variance, see (10). The order variance is also infinite, so the *Bullwhip* ratio $\mathbb{V}[o_t]/\mathbb{V}[d_t]$ is undefined; later we will show that the positivity of the difference $\mathbb{V}[o_t] - \mathbb{V}[d_t]$ can be used to determine if the infinite order variance is larger than the infinite demand variance. The variance of the net stock levels is finite and is given by:

$$\begin{aligned} \frac{\mathbb{V}[ns_t]}{\mathbb{V}[\epsilon_t]} =& (-1)^2 + \sum_{t=1}^{T_p} (\tilde{ns}_t)^2 \\ =& \frac{T_p^3(\theta_1+\theta_2-1)^2}{3(\phi_1-1)^2} + \frac{T_p^2(\theta_1+\theta_2-1)(\phi_1(3\theta_1+\theta_2-5)-\theta_1+\theta_2+3)}{2(\phi_1-1)^3} + \\ & \frac{\begin{pmatrix} T_p(\phi_1^2(13\theta_1^2+\theta_1(8\theta_2-44)+(\theta_2-14)\theta_2+37)+ \\ 4\phi_1(2\theta_1(\theta_2+5)-2\theta_1^2+(\theta_2-2)\theta_2-11)+\theta_1^2-4\theta_1\theta_2-8\theta_1+ \\ \theta_2^2+10\theta_2+13)-12\phi_1^{T_p+1}(\phi_1(\theta_1-\phi_1)+\theta_2)(\theta_1+T_p(\theta_1+\theta_2-1)-2) \end{pmatrix}}{6(\phi_1-1)^4} + \\ & \frac{\begin{pmatrix} \phi_1(\theta_1^2\phi_1^3-2\theta_1^2\phi_1-2\theta_1\theta_2-4\theta_1\phi_1^3+2\theta_1\phi_1^2+4\theta_1\phi_1-\theta_2^2\phi_1+2\theta_2\phi_1^3- \\ 4\theta_2\phi_1^2+4\theta_2+\phi_1^{2T_p+1}(\phi_1(\theta_1-\phi_1)+\theta_2)^2+5\phi_1^3-4\phi_1^2-5\phi_1+4)-1 \end{pmatrix}}{(\phi_1-1)^5(\phi_1+1)}. \end{aligned} \tag{41}$$

The variance of the net stock levels maintained by (correctly specified) the OUT-DT policy under ARIMA(1,1,2) demand is increasing in the lead time T_p and always greater than $\mathbb{V}[\epsilon_t]$. However, $NSAmp = 0$ under ARIMA(1,1,2) demand as the demand variance is infinite.

7.2. Bullwhip-lead time behaviour of the OUT-DT policy under ARIMA (1,1,2) demand

The ARIMA(1,1,2) demand process is non-stationary and as such the demand and order variances are infinite; the *Bullwhip* ratio does not exist. However, Gaalman et al. (2022) has investigated how the bullwhip produced by the OUT policy is affected by the lead time. Their study contains two important innovations that allow us to gain some

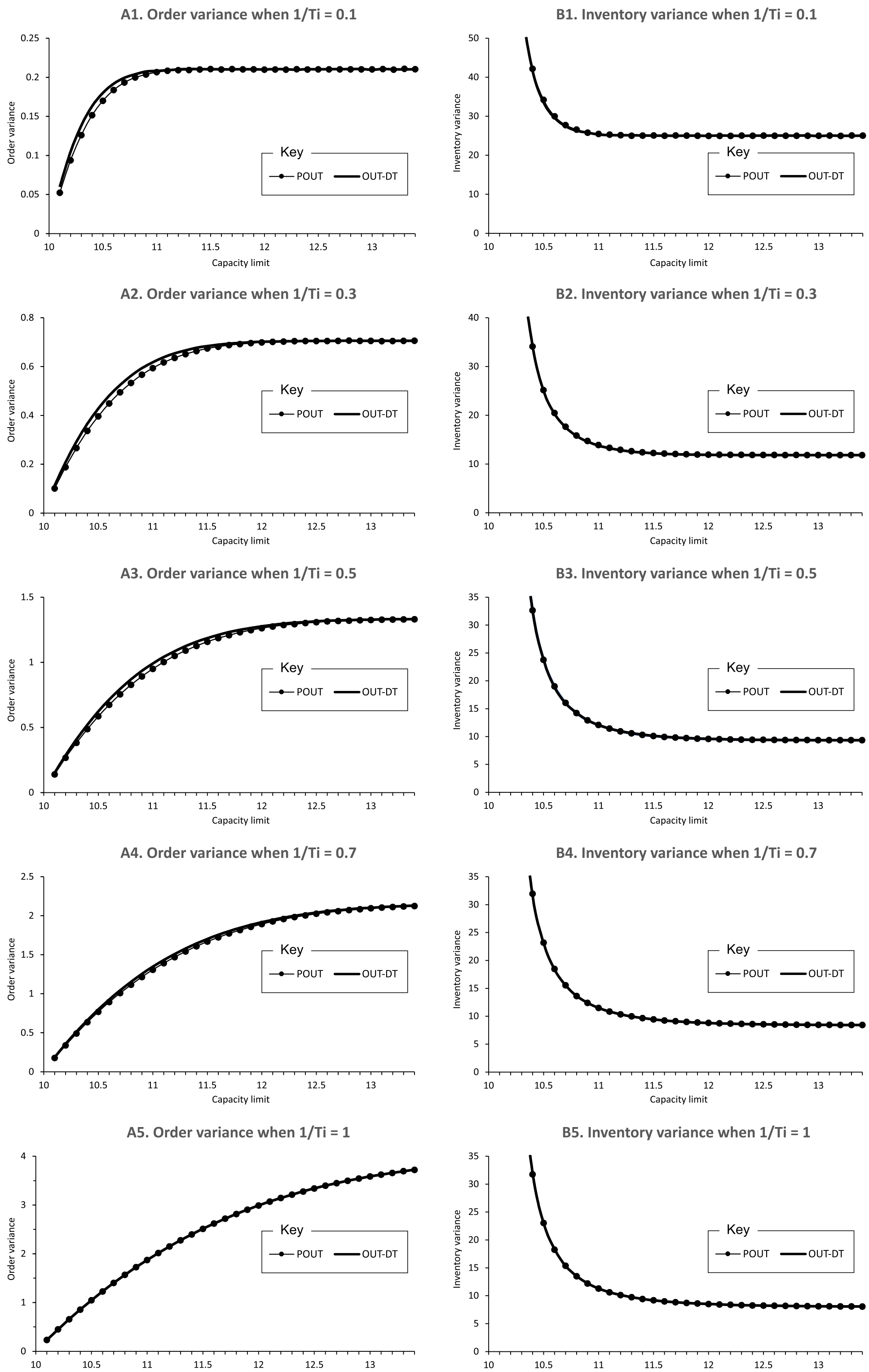


Fig. 8. Order (left column) and inventory (right column) variances under different capacity constraints and different feedback controller values (rows). Note, in all panels: $T_p = 1, \gamma = 0.000001$, i.i.d. demand with $\mu = 10, \sigma = 2$.

insight into bullwhip behaviour, despite the non-stationary nature of demand. First, they show how one may use the difference between the order variance and the demand variance to determine whether a bullwhip effect is present or not under non-stationary demand. As the difference between the two infinite variances is finite, the positivity of this difference can indicate whether a bullwhip effect is present or not. That is, they show bullwhip effect is present if $CB[T_p] = (\mathbb{V}(o_t) - \mathbb{V}(d_t))/\mathbb{V}(\epsilon_t) > 0$, where $CB[T_p]$ is the *critical bullwhip* metric for lead time T_p.

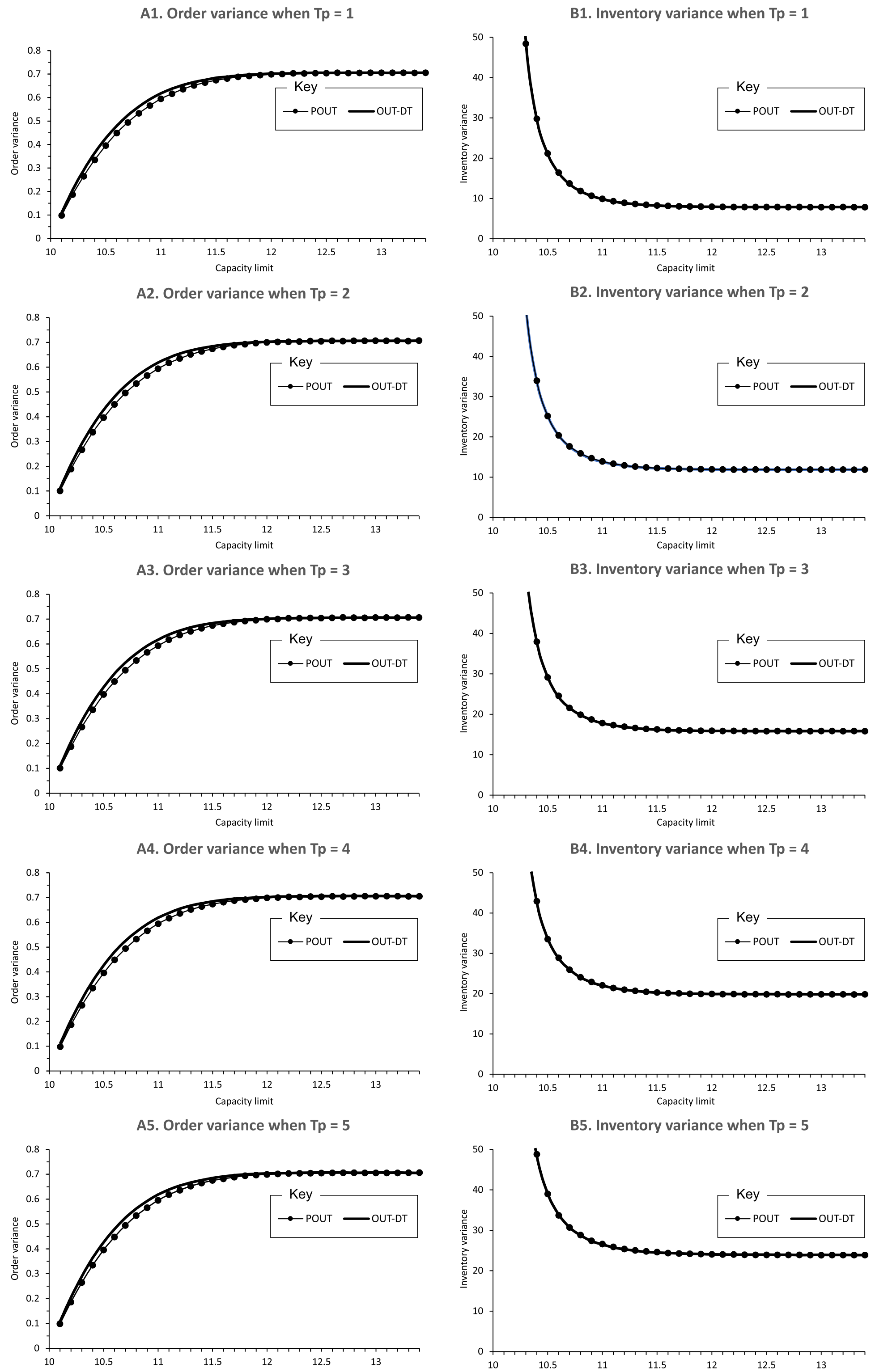


Fig. 9. Order (left column) and inventory (right column) variances under different capacity constraints and different lead times (rows). Note, in all panels: $T_i = 3, \gamma = 0.000001$, i.i.d. demand with $\mu = 10, \sigma = 2$.

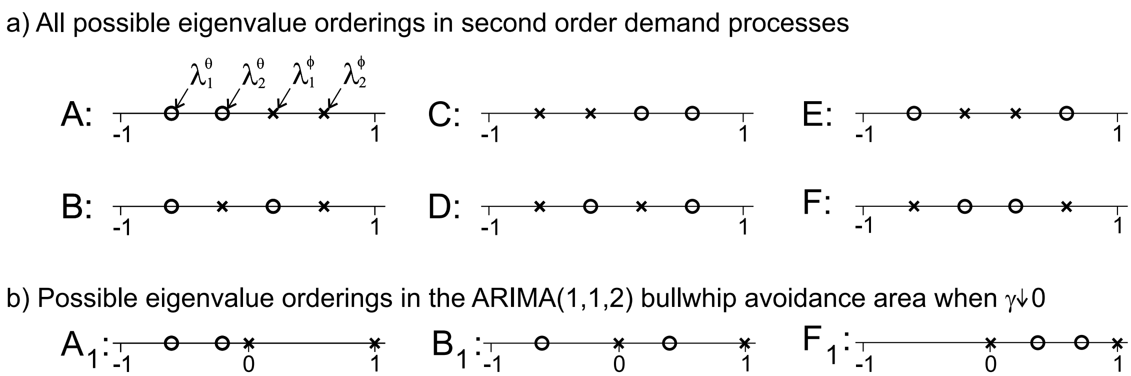


Fig. 10. Possible eigenvalue orderings. Panel (a) All possible eigenvalue ordering for second order demand processes. Panel (b) Possible eigenvalue orderings under ARIMA(1,1,2) with parameters selected from the $\mathcal{BA}|_{\gamma\downarrow 0}$ region.
Source: Adapted from Gaalman et al. (2022).

Using the demand impulse responses and Tsypkin's Relation, they show the critical bullwhip metric $CB[T_p]$ becomes

$$
\begin{aligned}
CB[T_p] &= \left(\sum_{j=0}^{T_p+1} \tilde{d}_j\right)^2 - \sum_{t=0}^{T_p+1} \tilde{d}_t^2 \\
&= \frac{2\alpha^2\beta^2\gamma^5\left(\gamma^{T_p}-1\right)\left(\gamma^{T_p+1}-1\right)}{(\gamma-1)^4(\gamma+1)} - \frac{2\alpha T_p(\gamma(\beta-1)+1)}{\gamma-1} - \\
&\quad \frac{\alpha^2 T_p((\beta-1)\gamma+1)\left(\gamma\left(\beta(1-3\gamma+2\gamma^{T_p+2})+\gamma-2\right)+1\right)}{(\gamma-1)^3} + \\
&\quad \frac{\alpha\left(\alpha((\beta-1)T_p\gamma+T_p)^2+2\gamma\left(\beta\left(\gamma^{T_p+2}-2\gamma+1\right)+\gamma-2\right)+2\right)}{(\gamma-1)^2}
\end{aligned}
\tag{42}
$$

when the OUT-DT policy is used to place replenishment orders. It is profound that the question of whether bullwhip exists or not depends on only the demand process and its parameters up until the period after the lead time. Gaalman et al. (2022) also reveal that if the demand impulse was always positive ($\forall t,\ \tilde{d}_t > 0$) the bullwhip effect is always produced by the OUT-DT policy and that it increases in the lead time. We repeat their results in our notation to make this paper self-contained:

Theorem 1 (*Necessary-Sufficient Condition for a Bullwhip Effect that is Increasing the Lead Time, Gaalman et al. (2022)*). *$CB[T_p]$ is always positive and increasing in the lead time $\forall T_p$ iff $\{\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_{T_p+1}\} > 0$.*

Proof. We refer readers to Gaalman et al. (2022) for the proof of Theorem 1. □

Theorem 1 shows that bullwhip is always present and always increasing in the lead time if, and only if, the demand impulse response $\tilde{d}$ is positive for all t; that is, $CB[T_p]$ is increasing in T_p iff $\forall t,\ \tilde{d}_t > 0$. There is one important subtlety to consider that Gaalman et al. (2022) capture in the following Corollary.

Corollary 1 (*Necessary-Sufficient Condition for an Order Variance that is Increasing in the Lead Time, Gaalman et al. (2022)*). *The order variance is increasing in the lead time T_p iff $\{\tilde{d}_2, \ldots, \tilde{d}_{T_p+1}\} > 0$ and $\tilde{d}_1 > -1$.*

Proof. We refer readers to Gaalman et al. (2022) for the proof of Corollary 1.

Corollary 1 shows that an order variance (which may be less than the demand variance when $T_p = 0$ if $-1 < \tilde{d}_1 < 0$) is increasing in the lead time if $\forall t > 1,\ \tilde{d}_t > 0$. Thus, the bullwhip effect may not be present with short lead times but may re-emerge as the lead time increases.

The positivity of the demand impulse can be determined by the order (location) of the eigenvalues (poles and zeros) of the demand process. Eq. (39) gave the transfer function of the ARIMA(1,1,2) demand; it is a second order transfer function with six possible eigenvalue orderings (Cases $A, B, \ldots, F$, see Fig. 10). Then, within each ordering, a further three sub-cases were present depending on how many poles are positive or negative. Assume, from (40), the larger pole $\lambda_2^{\phi} \uparrow 1$ (i.e. ϕ is very slightly smaller than unity) to ensure stability. This implies that cases C, D, and E cannot exist when we have ARIMA(1,1,2) demand, as those cases have a zero above the largest pole.

Let $\mathcal{BA}|_{\gamma\downarrow 0}$ denote the $\mathcal{BA}$ area defined by a small positive γ, i.e. $\gamma \downarrow 0$, the $\mathcal{BA}$ area where the OUT-DT policy mimics the POUT policy. In the $\mathcal{BA}|_{\gamma\downarrow 0}$ area, the smaller pole lies at $\lambda_1^{\phi} =\downarrow 0$. That is, the smaller pole is positive, implying that the $\mathcal{BA}|_{\gamma\downarrow 0}$ region contains only the sub-cases A_1, B_1, and F_1 which have two positive poles, see Gaalman et al. (2022). In cases A and F, as the two zeros do not have a poles between them, Cases A and F can have complex zeros and we need to consider when the real part of the largest zero is negative for case A and when the real part of the smallest zero is positive for case F.

Case A_1 exists if the largest zero λ_2^{θ} is less than $\lambda_1^{\phi} =\downarrow 0$. This is equivalent to

$$\theta_1 < 0 \wedge \theta_2 < 0, \tag{43}$$

where $\wedge$ is the logical *and* operator. When case A_1 exists, the demand impulse, $\tilde{d}_{t+1} > 0$, and the bullwhip is always present and it increases in the lead time (Gaalman et al., 2022). In the $\mathcal{BA}|_{\gamma\downarrow 0}$ region, the $\theta_1 < 0$ constraint is equivalent to

$$\beta \in (-\infty, (1-\alpha+\gamma)/(\alpha\gamma)). \tag{44}$$

The topological spaces in (44) and (26) are disjoint, thus case A_1 cannot exist in the $\mathcal{BA}|_{\gamma\downarrow 0}$ area.

Case B_1 exists if the largest zero λ_2^{θ} is greater than $\lambda_1^{\phi} =\downarrow 0$ and the smallest zero λ_1^{θ} is less than $\lambda_1^{\phi} =\downarrow 0$. This is equivalent to $\theta_2 > 0$. When case B_1 exists, the demand impulse, $\tilde{d}_{t+1} > 0$ and the bullwhip is always present and increases in the lead time (Gaalman et al., 2022). The constraint that $\theta_2 > 0$ is equivalent to $\gamma(\alpha-1) > 0$. However, this is not possible in the $\mathcal{BA}|_{\gamma\downarrow 0}$ region as $\alpha < 0$ and $\gamma > 0$.

Case F_1 exists if the smallest zero λ_1^{θ} is greater than $\lambda_1^{\phi} =\downarrow 0$, which is equivalent to

$$\theta_1 > 0 \wedge \theta_2 < 0. \tag{45}$$

Case F_1 exists in the $\mathcal{BA}|_{\gamma\downarrow 0}$ region as it is the logical complement of case A_1 and B_1.

Gaalman et al. (2022) showed the case F demand impulse has two essential characters:

- **Case** F_{1a}: If $\tilde{d}_1 = 1 + \phi_1 - \theta_1 < 0$ then the demand impulse is initially negative (i.e. $CB[0] < 0$) indicating bullwhip is not

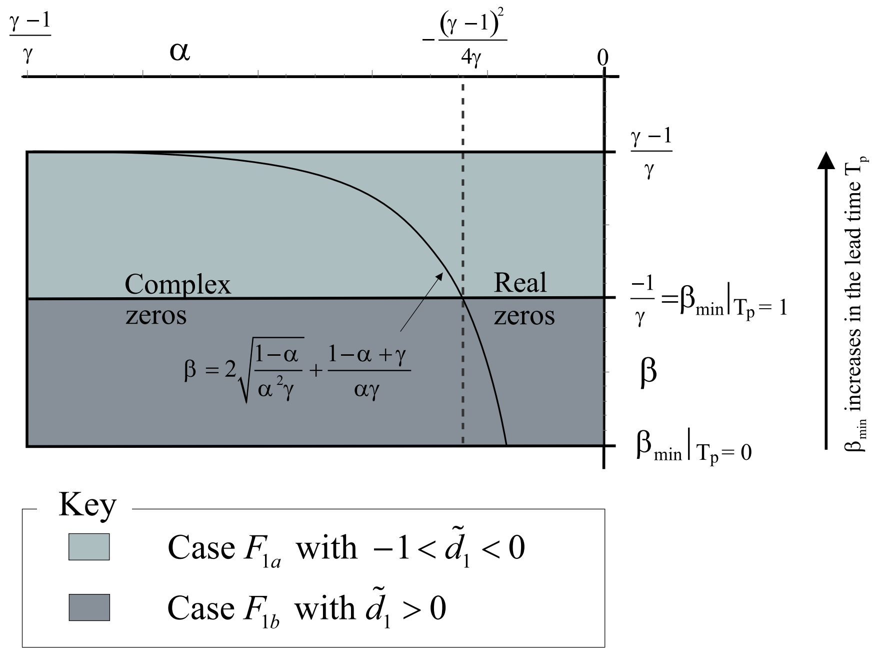


Fig. 11. Areas of increasing bullwhip over the lead time, Case F_1, in the $\mathcal{BA}$ region.

present when the lead time $T_p = 0$. Gaalman et al. (2022) show when t becomes sufficiently large the demand impulse response turns, and remains, positive after one change of sign. That is, $\tilde{d}_{t+1} > 0$ when $t \geq \tau$, where

$$\tau = \frac{\ln(|r_2/r_1|)}{\ln(|\lambda_1^\phi/\lambda_2^\phi|)} \tag{46}$$

and

$$r_1 = \frac{(\lambda_1^\phi - \lambda_1^\theta)(\lambda_1^\phi - \lambda_2^\theta)}{(\lambda_1^\phi - \lambda_2^\phi)} \quad \text{and} \quad r_2 = \frac{(\lambda_2^\phi - \lambda_1^\theta)(\lambda_2^\phi - \lambda_2^\theta)}{(\lambda_2^\phi - \lambda_1^\phi)}. \tag{47}$$

The order variance can be increasing in the lead time if $\tilde{d}_1 \geq -1$ and $\forall t \geq 2$, $\tilde{d}_t > 0$, Gaalman et al. (2022). However, when $\tilde{d}_1 \geq -1$ and $1 < t < \tau$, $\tilde{d}_t < 0$, the order variance is initially less than the demand variance and is decreasing in the lead time until $T_p \geq \tau$ at which point the order variance will then increase in the lead time.

There is a threshold, denoted by an inverse function of β, where $d_n = 0$,

$$\beta[n] = -1 \Big/ \sum_{j=1}^{n} \gamma^j = \frac{1-\gamma}{\gamma(\gamma^n - 1)}, n \in \mathbb{N}^+, \tag{48}$$

where $n = \lceil \tau \rceil - 1$. Eq. (48) reveals that when $\beta > \beta[n]$ there will be n periods with a decreasing order variance until $T_p = n$ before there being one period of no change in the order variance after which the order variance will increase in the lead time. When $\beta = \beta[1] = -1/\gamma$, the order variance equals the demand variance for $T_p = 0$, and will be increasing in the lead time. For $\beta = \beta[n]$, $n \neq 1$, the order variance will not change when T_p is between $n-2$ and $n-1$. Before $T_p = n-2$ there is an decreasing order variance and after $T_p = n-1$ the order variance is increasing in the lead time. In Case F_{1a}, the bullwhip effect does not exist when $T_p < n$, but will emerge eventually. The lead time where the bullwhip effect appears is determined by the parameter values in the ARIMA(1,1,2) demand process.

- **Case** F_{1b}: The demand impulse is always positive if $\tilde{d}_1 > 0$, which is equivalent to $\beta < -1/\gamma$. In this sub-case, *Bullwhip* is always increasing in the lead time.

From Theorem 1 and Corollary 1, it is easy to show $\tilde{d}_1 > 0$ when

$$0 < \gamma < 1, \quad \left(\alpha_{\min} = \frac{\gamma-1}{\gamma}\right) < \alpha < 0, \quad \text{and} \quad \beta < -1/\gamma. \tag{49}$$

Thus, when (49) holds, Case F_{1b} is present. Furthermore, $-1 < \tilde{d}_1 < 0$ occurs when

$$\begin{aligned} &0 < \gamma < 1, \quad \left(\alpha_{\min} = \frac{\gamma-1}{\gamma}\right) < \alpha < 0, \quad \text{and} \\ &-1/\gamma < \beta < \left(\beta_{\max} = \frac{\gamma-1}{\gamma}\right) \end{aligned} \tag{50}$$

holds, indicating that Case F_{1a} is present. Together, this means in the $\mathcal{BA}$ region, either the bullwhip effect is always present and increases in the lead time (case F_{1b}) or the bullwhip effect is initially not present with a short lead time (and may even decrease in the lead time) but with a long enough lead time, the order variance will become increasing in the lead time (case F_{1a}). Fig. 11 maps these possibilities to the $\mathcal{BA}$ region.

8. Numerical confirmation of main results

In this section we will confirm our main results via a numerical investigation. In Table 1, we detail the order response to i.i.d., AR(1), and two different ARIMA(1,1,2) demand processes. For each demand process we compare the POUT order's response to the OUT-DT order's response. We also quantify the variance of the orders, the variance of the inventory and the variance of the demand when they exist. In all cases the OUT-DT policy uses $\alpha = -0.65, \beta = -99, \gamma = 0.01$; the POUT policy always uses $T_i = 1.081081$, to reflect the equivalence in (31). The lead time is always $T_p = 3$.

Consider first, the i.i.d. demand. As $\gamma = 0.01$ the zeros of the transfer function are very close together (closer than the zeros in the impulse response detailed in Fig. 7), and the two impulse responses

Table 1
Comparison of the POUT and OUT-DT responses to various different demand processes. The superposition principle means the two policies respond to different demands in a similar manner. Notes: Lead time, $T_p = 3$, POUT policy uses $T_i = 1.081081$, DT-OUT policy uses $\alpha = -0.65, \beta = -99, \gamma = 0.01$.

Demand	i.i.d.		AR(1) ($\phi = 0.5$)		ARIMA(1,1,2)[a]		ARIMA(1,1,2)[b]	
Impulse response	POUT	OUT-DT	POUT	OUT-DT	POUT	OUT-DT	POUT	OUT-DT
$t = 0$	0.9250	0.9343	0.9250	0.9343	0.9250	0.9343	0.9250	0.9343
$t = 1$	0.0694	0.0607	0.5319	0.5279	0.0093	0.0000	0.3719	0.3663
$t = 2$	0.0052	0.0046	0.2711	0.2685	0.0001	0.0000	0.3528	0.3526
$t = 3$	0.0004	0.0003	0.1360	0.1346	0.0000	0.0000	0.3716	0.3718
$t = 4$	0.0000	0.0000	0.0680	0.0673	0.0000	0.0000	0.3913	0.3915
$t = 5$	0.0000	0.0000	0.0340	0.0337	0.0000	0.0000	0.4091	0.4093
$t = 6$	0.0000	0.0000	0.0170	0.0168	0.0000	0.0000	0.4252	0.4254
$t = 7$	0.0000	0.0000	0.0085	0.0084	0.0000	0.0000	0.4397	0.4398
$t = 8$	0.000	0.0000	0.0043	0.0042	0.0000	0.0000	0.4527	0.4528
$t = 9$	0.0000	0.0000	0.0021	0.0021	0.0000	0.0000	0.4644	0.4646
$t = 10$	0.0000	0.0000	0.0011	0.0011	0.0000	0.0000	0.4750	0.4751
$t = 11$	0.0000	0.0000	0.0005	0.0005	0.0000	0.0000	0.4845	0.4846
$t = 12$	0.0000	0.0000	0.0003	0.0003	0.0000	0.0000	0.4931	0.4931
$\mathbb{V}$[Orders]	0.86047	0.87671	1.23669	1.24794	0.85571	0.87300	∞	∞
$\mathbb{V}$[Inventory]	4.00565	4.00433	11.2029	11.1767	3.62032	3.62023	∞	∞
$\mathbb{V}$[Demand]	1.00000	1.00000	1.33333	1.33333	1.00423	1.00423	∞	∞
$CB[T_p]$	−0.13953	−0.12329	−0.09661	−0.08536	−0.14851	−0.13123	−0.12841	−0.11325
Bullwhip	0.86047	0.87671	0.92754	0.93598	0.85211	0.86932	Undefined	Undefined

[a] ARIMA(1,1,2) demand parameters: $\phi_1 = 0.01, \theta_1 = 1.075, \theta_2 = -0.075$, a correctly identified ARIMA(1,1,2) process.
[b] ARIMA(1,1,2) demand parameters: $\phi_1 = 0.9, \theta_1 = 1.573, \theta_2 = -0.63$, an incorrectly identified ARIMA(1,1,2) demand process.

are very similar. The i.i.d. demand has unit variance and as the impulse responses return to zero over time, the variance of the orders, inventory, and demand exist. $CB[T_p]$ is the difference between the order and demand variance, a negative number indicates the order variance is smaller then the demand variance. The *Bullwhip* ratio can be calculated in this case, and is less than unity, also indicating that the order variance is less than the demand variance for both the POUT policy and the OUT-DT policy. The difference between the POUT and OUT-DT bullwhip ratios is less than 1.9%, confirming that policies with near identical eigenvalues have a very similar dynamic responses.

The second demand process we studied is an AR(1) demand with an auto-regressive co-efficient of $\phi = 0.5$. Although we have only detailed the order's impulse response for first 12 time periods, the impulse responses return to zero over time. The demand is stationary and has a finite variance of 4/3. The POUT and OUT-DT policy have almost identical dynamic responses and summary statics. The differences in the two bullwhip ratios is less than 1%. That the AR(1) responses of the two systems (POUT and OUT-DT) are similar is a direct consequence of the superposition principle in linear systems. The superposition principle states that any demand process can be constructed from scaled and delayed unit impulses. Furthermore, the dynamic response to that arbitrary demand, in a linear system, can be constructed from scaled and delayed impulse responses. Thus, if the impulse response is similar in two linear systems, so will their dynamic response to any other demand process.

The third demand process is an ARIMA(1,1,2) demand process that matches the DT parameters used in the OUT-DT policy. Recall $\{\alpha, \beta, \gamma\}$ can be mapped to $\{\phi_1, \theta_1, \theta_2\}$ via the relations in (37) and (38). As $\beta = \beta_{\max} = (\gamma - 1)/\gamma$ then the demand impulse returns to zero and the demand, order, and inventory variances exist and are finite. This demand process is rather special; the ARIMA(1,1,2) is non-stationary in general, but on the $\beta_{\max}$ boundary of the $\mathcal{BA}$ region, the ARIMA(1,1,2) is stationary.

The forth demand process is an ARIMA(1,1,2) demand process which is not matched to the DT forecasting parameters in the OUT-DT policy. The demand process is non-stationary and does not return to zero over time (there is an off-set of $\alpha(\gamma(1-\beta)-1)/(1-\gamma)$). Due to this off-set, the demand and order variances are infinite. However, there is a finite difference between the two infinite variance as indicated by the finite $CB[T_p]$ metric. $CB[T_p] < 0$ indicating that the order variance is smaller than the demand variance. Notice as this ARIMA(1,1,2) is misspecified by the DT forecasting method here, the inventory levels do not return to zero and inventory variance is infinite. That is, (41) does not hold here (it does hold for the third demand process though, which is correctly specified).

9. Concluding remarks

The invertibility and the stability regions of the DT forecasting mechanism offered some theoretical support for exploring the performance of the OUT-DT policy over a wider range of parameter values than is usually recommended in the literature. While other research has chosen DT parameter values from the $[0, 1]$ interval, our work shows that if unconventional $\{\alpha, \beta, \gamma\}$ values are selected the bullwhip effect can be avoided without unduly increasing the *NSAmp* ratio; these results hold for all lead times. This is interesting as closely related forecasting methods such as exponential smoothing and Holt's method are known to create bullwhip for all lead times and all demand processes.

We studied the OUT-DT policy reacting to i.i.d. random demand and identified the variance ratios for *Bullwhip* and *NSAmp*. The OUT-DT policy was shown to have nearly identical poles and zeros to the well established POUT policy. The POUT policy, with its proportional feedback controller, has long been known to avoid the bullwhip effect while maintaining reasonable inventory control. The OUT-DT policy has no such proportional feedback controller; yet despite this, it is able to perform – for all practical purposes – identically to the POUT policy. This provides a new opportunity to reduce the bullwhip effect in practical situations. With only a change in the forecasting software one can obtain a smooth production rate without the need to make changes to an ERP system's planning book (which we have done in the past using User-Defined Functions, a rather complicated programming task as it requires extra rows in the MRP table and, of course, user training). This has practically important managerial implications as it allows the change to be easily implemented in only the forecasting module of popular ERP systems. In large ERP settings, access to different parts of the ERP system is often limited to users from individual departments within the company. Having only a need to alter the forecasting parameters in a forecasting module is likely to require only local changes within the forecasting/planning department. This also calls for ERP and forecasting software providers to allow parameters to be set in this unconventional region.

DT is structurally optimal for forecasting the non-stationary ARIMA (1,1,2) demand process; the ARIMA(1,1,2) demand has infinite variance and because of this, so does the order variance. However, we were able to obtain an expression for the net stock variance and were able to adapt the eigenvalue ordering approach of Gaalman et al. (2022) to investigate how the bullwhip effect (via the order variance) was influenced by the lead time. Within the $\mathcal{BA}|_{\gamma\downarrow 0}$ area we found the bullwhip effect either exists for all lead times or it is initially not present with short lead times, but reappears when the lead time becomes sufficiently long. Near the point of minimum order variance, the order variance was found to be smaller than the demand variance with lead time $T_p = 0$. In some cases the order variance was found to be decreasing in the lead time (with short lead times), but the order variance always becomes increasing in the lead time when the lead time is long enough. These facts emphasize the need for lead time reduction in well-designed supply chains.

9.1. Further work

Future research could be directed to expanding our work on the equivalence of OUT-DT and POUT policies by considering further the impact of non-linearity's such as forbidden returns, minimum order quantities, and non-negative inventory constraints. Furthermore, studying the use of the DT forecasting method in the POUT policy, or even the full-state order-up-to policy, Gaalman (2006), would be interesting. Finally, we should remember that we have only studied a subset of the possible ARIMA(1,1,2) demands; those ARIMA(1,1,2) demands in the $\mathcal{BA}$ region. Understanding the nature of the OUT-DT policy for all ARIMA(1,1,2) demands is worthy of consideration.

Data availability

No data was used for the research described in the article.

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