Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #64, 12 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

Partial Permutohedra

Roger E. Behrend*[1], Federico Castillo†[2], Anastasia Chavez‡[3], Alexander Diaz-Lopez§[4], Laura Escobar¶[5], Pamela Harris||[6], and Erik Insko**[7]

[1] *School of Mathematics, Cardiff University, Cardiff, UK*
[2] *Departamento de Matemáticas, Pontificia Universidad Católica de Chile, Santiago, Chile*
[3] *Department of Mathematics and Computer Science, Saint Mary's College of California, Moraga, CA, USA*
[4] *Department of Mathematics and Statistics, Villanova University, Villanova, PA, USA*
[5] *Department of Mathematics and Statistics, Washington University in St. Louis, St. Louis, MO, USA*
[6] *Department of Mathematical Sciences, University of Wisconsin Milwaukee, Milwaukee, WI, USA & Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA*
[7] *Department of Mathematics, Florida Gulf Coast University, Fort Myers, FL, USA*

Abstract. Partial permutohedra are lattice polytopes which were recently introduced and studied by Heuer and Striker [arXiv:2012.09901]. For positive integers m and n, the partial permutohedron $\mathcal{P}(m,n)$ is the convex hull of all vectors in $\{0,1,\ldots,n\}^m$ whose nonzero entries are distinct. We study the face lattice, volume and Ehrhart polynomial of $\mathcal{P}(m,n)$, and our methods and results include the following. For any m and n, we obtain a bijection between the nonempty faces of $\mathcal{P}(m,n)$ and certain chains of subsets of $\{1,\ldots,m\}$, thereby confirming a conjecture of Heuer and Striker. We use this characterization of faces to obtain a closed expression for the h-polynomial of $\mathcal{P}(m,n)$. For any m and n with $n \geq m-1$, we use a pyramidal subdivision of $\mathcal{P}(m,n)$ to establish a recursive formula for the normalized volume of $\mathcal{P}(m,n)$, from which we then obtain closed expressions for this volume. We also use a sculpting process (in which $\mathcal{P}(m,n)$ is reached by successively removing certain pieces from a simplex or hypercube) to obtain closed expressions for the Ehrhart polynomial of $\mathcal{P}(m,n)$ with arbitrary m and fixed $n \leq 3$, the volume of $\mathcal{P}(m,4)$ with arbitrary m, and the Ehrhart polynomial of $\mathcal{P}(m,n)$ with fixed $m \leq 4$ and arbitrary $n \geq m-1$.

*behrendr@cardiff.ac.uk. Roger E. Behrend was partially supported by Leverhulme Trust Grant RPG-2019-083.

†federico.castillo@mat.uc.cl. Federico Castillo was partially supported by FONDECYT Grant 1221133.

‡amc59@stmarys-ca.edu.

§alexander.diaz-lopez@villanova.edu.

¶laurae@wustl.edu. Laura Escobar was partially supported by NSF Grant DMS-1855598 and NSF CAREER Grant DMS-2142656.

||peharris@uwm.edu. Pamela Harris was partially supported by a Karen Uhlenbeck EDGE Fellowship.

**einsko@fgcu.edu.

Keywords: Lattice polytopes, Ehrhart polynomials, partial permutohedra, generalized permutohedra.

1 Introduction

Computing the volume of a polytope is hard, even when the complete face structure is known [8]. In fact, few exact volume formulas have been discovered in much generality. Stanley gave a notable volume formula for the regular permutohedron $\Pi(1,2,\ldots,m)$, specifically that its normalized volume is m^{m-2} [13, Example 3.1]. More generally, Postnikov [11] studied the permutohedron $\Pi(z_1,\ldots,z_m)$ (which is the convex hull of all vectors obtained by permuting the entries of an arbitrary vector $(z_1,\ldots,z_m)$ in $\mathbb{R}^m$), as well as a certain class of generalized permutohedra, and obtained three distinct formulas for the volume of $\Pi(z_1,\ldots,z_m)$ [11, Theorems 3.1, 5.1 and 17.1], each one subtle in its own way.

In this paper, we study a related family of polytopes called *partial permutohedra*, which were introduced recently by Heuer and Striker [7]. For positive integers m and n, the partial permutohedron $\mathcal{P}(m,n)$ is the convex hull of all vectors in $\{0,1,\ldots,n\}^m$ whose nonzero entries are distinct. These lattice polytopes are anti-blocking versions of certain permutohedra. We also note that for any $n \geq m$, $\mathcal{P}(m,n)$ is combinatorially equivalent to the *m-stellohedron* which, for example, has been studied in [10, Section 10.4] and has appeared recently in connection with matroid theory [4].

We expand on the work of Heuer and Striker [7] by obtaining, in Theorem 2.4, a bijection between the nonempty faces of $\mathcal{P}(m,n)$ and certain chains of subsets of $\{1,\ldots,m\}$, for any m and n, thus proving Conjecture 5.25 of [7]. This conjecture has also, very recently, been independently proved by Black and Sanyal [3, Theorem 6.5]. We then use this characterization of the faces of $\mathcal{P}(m,n)$ to obtain, in Theorem 2.5, a closed expression for the h-polynomial of $\mathcal{P}(m,n)$ with any m and n.

Our results for the volumes of partial permutohedra include the following. In Theorem 3.1, we use a technique, in which $\mathcal{P}(m,n)$ is subdivided into certain pyramids, to establish a recursive formula for the normalized volume of $\mathcal{P}(m,n)$, for any m and n with $n \geq m-1$. Using this recursion and Stong's computation for a particular case [14], we are then able to prove, in Theorem 3.3, a closed expression for the normalized volume of $\mathcal{P}(m,n)$ for the case $n \geq m-1$. For the remaining case $m-1 > n$, the combinatorial type of $\mathcal{P}(m,n)$ depends on both m and n (whereas for $n \geq m-1$ it depends only on m), which suggests that finding a completely general closed formula for the volume in this case is unlikely.

We also compute the Ehrhart polynomial of $\mathcal{P}(m,n)$ when m or n is small. One of our main techniques is based on the idea, exploited by algebraists in the days of yore, of completing the (hyper)cube. We start with a lattice polytope for which we know the volume or Ehrhart polynomial, and then carefully remove pieces until we reach the polytope of interest. This idea makes itself apparent after analyzing certain expressions, as given in Example 3.2, for the normalized volume of $\mathcal{P}(m,n)$ for small fixed m and any $n \geq m-1$. These expressions are polynomials in n, with all coefficients negative, except in the leading term which is $m!\,n^m$, the normalized volume of a hypercube in $\mathbb{R}^m$ of side-length n. This suggests that we start with a hypercube from which we can sculpt a partial permutohedron, and this is precisely what we do in Section 4 for the case $n \geq m-1$. This sculpting approach has been used recently to compute the Ehrhart polynomials of matroid polytopes starting from the hypersimplex. See [5] for sparse paving matroids, and [6] for paving matroids.

In Theorem 4.2, we provide results for the normalized volume and Ehrhart polynomial of $\mathcal{P}(m,n)$ with arbitrary m and fixed $n \leq 4$, where these are obtained by sculpting $\mathcal{P}(m,n)$ from a $\binom{n+1}{2}$-dilated standard m-simplex $\Delta_m = \{\mathbf{x} \in \mathbb{R}^m \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^{m} x_i \leq 1\}$. For example, we find that the normalized volume of $\mathcal{P}(m,2)$ is $3^m - 3$, thereby confirming Conjecture 5.30 of [7], we obtain explicit expressions for the Ehrhart polynomials of $\mathcal{P}(m,1)$, $\mathcal{P}(m,2)$ and $\mathcal{P}(m,3)$ with arbitrary m, and we obtain an explicit expression for the normalized volume of $\mathcal{P}(m,4)$ with arbitrary m.

In Theorem 4.4, we provide explicit expressions for the Ehrhart polynomials of $\mathcal{P}(m,n)$ with fixed $m \leq 4$ and arbitrary $n \geq m-1$, where these are obtained by sculpting $\mathcal{P}(m,n)$ from a hypercube $[0,n]^m$.

We end, in Conjecture 5.1, by conjecturing a closed expression for the Ehrhart polynomial of $\mathcal{P}(m,n)$ with $n \geq m-1$ which generalizes our volume and Ehrhart polynomial results for this case.

2 Faces of partial permutohedra

2.1 Description of the partial permutohedron $\mathcal{P}(m,n)$

We introduce the partial permutohedron $\mathcal{P}(m,n)$, for any positive integers m and n, using a similar approach to that used by Heuer and Striker [7, Section 5].

Definition 2.1. Let the *partial permutohedron* $\mathcal{P}(m,n)$ be the polytope given by the convex hull of all vectors in $\{0,1,\ldots,n\}^m$ whose nonzero entries are distinct.

It follows from the definition that $\mathcal{P}(m,n)$ is a lattice polytope. As noted in [7, Remark 5.5], it has dimension m. From [7, Proposition 5.7], its vertices are the vectors in $\mathbb{R}^m$ with entries of zero in any $m-k$ positions, and with the other k entries being $n, n-1, \ldots, n-k+1$ in any order, where k ranges from 0 to $\min(m,n)$. Its facet description is given in

Theorems 5.10 and 5.11 in [7] as

$$\mathcal{P}(m,n) = \left\{ \mathbf{x} \in \mathbb{R}^m \;\middle|\; \begin{array}{rcll} 0 & \leq & x_i, & \text{for all } i \in [m], \\ \sum_{i \in S} x_i & \leq & \binom{n+1}{2} - \binom{n+1-|S|}{2}, & \text{for all nonempty } S \subseteq [m] \\ & & & \text{with } |S| \leq n-1 \text{ or } |S| = m \end{array} \right\}, \quad (2.1)$$

where the inequalities correspond to distinct facets, and where $\binom{n+1-|S|}{2}$ is taken to be 0 if $n+1-|S| \leq 1$ (which occurs if $|S| = m \geq n$). Also, $[m]$ denotes the set $\{1, \dots, m\}$.

In [7, Theorem 5.27], it is shown that $\mathcal{P}(m,n)$ is a projection of the partial permutation polytope $\mathrm{PPerm}(m,n)$ (or polytope of $m \times n$ doubly substochastic matrices), which can be defined as the convex hull in $\mathbb{R}^{mn}$ of all $m \times n$ matrices in which each entry is in $\{0,1\}$ and each row and column contains at most one 1. We remark that $\mathrm{PPerm}(m,n)$ can be regarded as the matching polytope of the complete bipartite graph $K_{m,n}$, and it is thus of interest in the context of combinatorial optimization (see, for example, [12, Chapters 18 and 25]). The polytope $\mathrm{PPerm}(m,m)$ was studied by Kohl, Olsen and Sanyal [9, Section 5].

2.2 The faces of $\mathcal{P}(m,n)$

The whole face structure of $\mathcal{P}(m,n)$ was conjectured in [7, Conjecture 5.25], and we prove this conjecture in Theorem 2.4. Recently, this was independently proved by Black and Sanyal [3, Theorem 6.5] in the context of monotone path polytopes of polymatroids. To characterize the faces of $\mathcal{P}(m,n)$, we need some further definitions, as follows.

Definition 2.2. The *Boolean lattice* $\mathcal{B}_m$ is the poset consisting of subsets $A \subseteq [m]$, ordered by inclusion, where $A \in \mathcal{B}_m$ has rank $|A|$, the cardinality of A. A *chain* C in $\mathcal{B}_m$ is a nonempty ordered collection $C = (A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_\ell)$ of subsets $A_i \in \mathcal{B}_m$. We say that a rank i is *missing* from a chain C in $\mathcal{B}_m$ if there is no subset of rank i in C and there is a subset of rank greater than i in C.

Definition 2.3. Let $\mathcal{C}(m,n)$ denote the set of all chains $(A_1 \subsetneq \cdots \subsetneq A_\ell)$ in $\mathcal{B}_m$ which satisfy the following:

- If $A_1 \neq \varnothing$, then $|A_\ell \setminus A_1| \leq n-1$.
- If $A_1 = \varnothing$ and $\ell \geq 2$, then $|A_\ell \setminus A_2| \leq n-1$.

Our theorem for the face structure of $\mathcal{P}(m,n)$, with any m and n, is as follows.

Theorem 2.4. *Given a chain $C = (A_1 \subsetneq \cdots \subsetneq A_\ell)$ in $\mathcal{C}(m,n)$, let $\mathcal{F}_C$ be the intersection of $\mathcal{P}(m,n)$ with the following hyperplanes:*

(i) $\{\mathbf{x} \in \mathbb{R}^m \mid x_i = 0\}$*, for all* $i \in [m] \setminus A_\ell$*.*

(ii) $\left\{\mathbf{x} \in \mathbb{R}^m \,\middle|\, \sum_{i \in A_\ell \setminus A_j} x_i = \binom{n+1}{2} - \binom{n+1-|A_\ell \setminus A_j|}{2}\right\}$*, for all* $2 \leq j \leq \ell - 1$*, and also for* $j = 1$ *unless* $A_1 = \varnothing$ *and* $|A_\ell| \geq n$*.*

(iii) $\left\{\mathbf{x} \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^m x_i = \binom{n+1}{2}\right\}$*, if* $A_1 = \varnothing$ *and* $|A_\ell| \geq n$*.*

Then the following is a bijection:

$$\begin{aligned} \mathcal{C}(m,n) &\longrightarrow \{\textit{nonempty faces of } \mathcal{P}(m,n)\}, \\ C &\longmapsto \mathcal{F}_C. \end{aligned}$$

Moreover, this bijection maps chains with k *missing ranks to faces of dimension* k*, for each* $k = 0, \ldots, m$*.*

Our proof of Theorem 2.4 involves verifying directly that the stated mapping is well-defined, injective and surjective.

2.3 The h-polynomial of $\mathcal{P}(m,n)$

We now consider the h-polynomial of $\mathcal{P}(m,n)$. Given a d-dimensional polytope P and $0 \leq i \leq d$, let $f_i(P)$ denote the number of i-dimensional faces of P. The *f-polynomial* of P is then defined as $f_P(t) = \sum_{i=0}^d f_i(P)\, t^i$, and the *$h$-polynomial* of P is defined as $h_P(t) = f_P(t-1)$.

Since Eulerian polynomials will play a role in the h-polynomial of $\mathcal{P}(m,n)$, we proceed to introduce them. For a positive integer m, let $A_m(t)$ denote the Eulerian polynomial for $\mathfrak{S}_m$, i.e., $A_m(t) = \sum_{i=0}^{m-1} A(m,i)\, t^i$, where the Eulerian number $A(m,i)$ is the number of permutations in $\mathfrak{S}_m$ with exactly i descents. Also, let $A(0,i) = \delta_{0,i}$ and $A_0(t) = 1$.

Our theorem for the h-polynomial of $\mathcal{P}(m,n)$, with any m and n, is as follows.

Theorem 2.5. *The h-polynomial of* $\mathcal{P}(m,n)$ *is*

$$h_{\mathcal{P}(m,n)}(t) = 1 + \sum_{i=0}^{n-1} \sum_{j=1}^{m-i} \binom{m}{i} A_i(t)\, t^j. \tag{2.2}$$

This is equivalent to the recurrence relation and initial condition

$$h_{\mathcal{P}(m,n+1)}(t) = h_{\mathcal{P}(m,n)}(t) + \binom{m}{n} A_n(t) \sum_{i=1}^{m-n} t^i, \quad h_{\mathcal{P}(m,1)}(t) = \sum_{i=0}^{m} t^i. \tag{2.3}$$

Our proof of Theorem 2.5 uses the characterization of faces provided by Theorem 2.4 to verify (2.3).

3 Volume of $\mathcal{P}(m,n)$ with $n \geq m-1$

Let $v(m,n)$ denote the normalized volume of $\mathcal{P}(m,n)$. Heuer and Striker [7, Figure 6] used SageMath to compute $v(m,n)$ for $m,n \leq 7$. In this section, we consider $v(m,n)$ for $n \geq m-1$. The first result is a recursive formula, and is proved by subdividing $\mathcal{P}(m,n)$ into pyramids, for which the apex is the origin and the base is a facet that does not contain the origin. The same method was used to obtain Theorem 3.1 for the case $n = m-1$ in [1, Theorem 4.1] and [14, Problem 12191].

Theorem 3.1. *For any m and n with $n \geq m-1$, the normalized volume of $\mathcal{P}(m,n)$ is given recursively by*

$$v(m,n) = (m-1)! \sum_{k=1}^{m} k^{k-2} \frac{v(m-k,n-k)}{(m-k)!} \left(kn - \binom{k}{2}\right) \binom{m}{k}, \tag{3.1}$$

with the initial condition $v(0,n) = 1$.

Example 3.2. For $1 \leq m \leq 7$ and $n \geq m-1$, Theorem 3.1 gives the following expressions for $v(m,n)$:

$$\begin{array}{lrrrrrrrr}
v(1,n) = & & n, & & & & & & \\
v(2,n) = & -1 & & +2n^2, & & & & & \\
v(3,n) = & -6 & -9n & & +6n^3, & & & & \\
v(4,n) = & -54 & -96n & -72n^2 & & +24n^4, & & & \\
v(5,n) = & -840 & -1350n & -1200n^2 & -600n^3 & & +120n^5, & & \\
v(6,n) = & -21150 & -30240n & -24300n^2 & -14400n^3 & -5400n^4 & & +720n^6, & \\
v(7,n) = & -782460 & -1036350n & -740880n^2 & -396900n^3 & -176400n^4 & -52920n^5 & & +5040n^7.
\end{array}$$

Note that the expression $v(2,n) = 2n^2 - 1$ was obtained in [7, Theorem 5.29].

Using the recursion of (3.1), and building on ideas from [1] and [14, Problem 12191], we are able to obtain the following explicit expressions for $v(m,n)$.

Theorem 3.3. *For any m and n with $n \geq m-1$, the normalized volume of $\mathcal{P}(m,n)$ is explicitly*

$$v(m,n) = -\frac{m!}{2^m} \sum_{0 \leq i \leq k \leq m} \binom{m}{k} \binom{k}{i} (2i-3)!!\,(2n)^{m-k}, \tag{3.2}$$

or equivalently

$$v(m,n) = -\frac{m!}{2^m} \sum_{i=0}^{m} \binom{m}{i} (2i-3)!!\,(2n+1)^{m-i}. \tag{3.3}$$

Remark 3.4. We note that, by regarding n as fixed and defining $V(m,n)$ to be the RHS of (3.2) or (3.3) for any nonnegative integer m (with the restriction $n \geq m-1$ no longer applying) we have the generating function expression

$$\sum_{m=0}^{\infty} \frac{V(m,n)\, z^m}{(m!)^2} = \sqrt{1-z}\, e^{(n+1/2)z}. \tag{3.4}$$

Furthermore, by defining $W(m,n) = V(m,n)/m!$, we have the simple recurrence relation

$$W(m,n) = (m+n-1)\, W(m-1,n) - (m-1)(n+1/2)\, W(m-2,n), \tag{3.5}$$

with initial conditions $W(0,m) = 1$ and $W(1,m) = n$.

4 The Ehrhart polynomial of some partial permutohedra

We now shift our focus to the computation of the Ehrhart polynomial of $\mathcal{P}(m,n)$ for small fixed values of m or n.

4.1 Ehrhart polynomials

We begin by recalling some basic facts about Ehrhart polynomials. For a lattice polytope $\mathcal{P} \subseteq \mathbb{R}^m$, the function $|t\,\mathcal{P} \cap \mathbb{Z}^m|$ of a positive integer variable t (where $t\,\mathcal{P}$ denotes the t-th dilate $\{t\,\mathbf{x} \mid \mathbf{x} \in \mathcal{P}\}$ of $\mathcal{P}$) is known to agree with a polynomial $\mathrm{Ehr}(\mathcal{P}) \in \mathbb{Q}[t]$ of degree $\dim(\mathcal{P})$, called the *Ehrhart polynomial* of $\mathcal{P}$. Furthermore, the coefficient of the leading term of $\mathrm{Ehr}(\mathcal{P})$ is the relative volume of $\mathcal{P}$, where this is a normalized volume if $\mathcal{P}$ is non-full-dimensional, see [2, Corollary 3.20]. For the polytopes $\mathcal{P}(m,n)$ considered here, the normalized volume is $m!$ times the relative volume.

4.2 The sculpting strategy

All of our remaining proofs follow the same sculpting strategy. We start with a well-known polytope and remove other known polytopes by adding inequalities, until we obtain the desired polytope $\mathcal{P}(m,n)$. More precisely, to compute the volume or Ehrhart polynomial of $\mathcal{P}(m,n)$, we create a sequence of lattice polytopes $\mathcal{P}_1, \ldots, \mathcal{P}_k = \mathcal{P}(m,n)$, where $\mathcal{P}_1$ is either the $\binom{n+1}{2}$-dilated standard m-simplex $\binom{n+1}{2}\,\Delta_m$ (in Theorem 4.2) or the m-cube $[0,n]^m$ of side-length n (in Theorem 4.4). We then obtain $\mathcal{P}_{i+1}$ from $\mathcal{P}_i$ by adding inequalities to $\mathcal{P}_i$, i.e., by taking an intersection of $\mathcal{P}_i$ with closed halfspaces, and thus removing some pieces from $\mathcal{P}_i$.

A simple example which illustrates this idea is as follows.

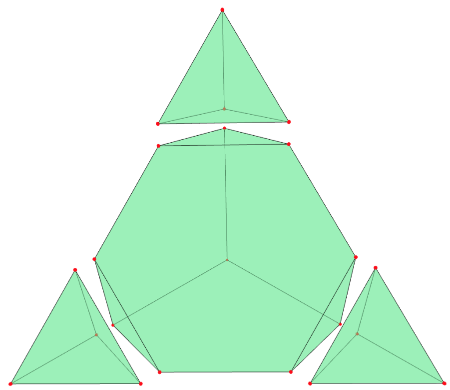

Figure 1: Illustration of $P(3,2)$ as $3\,\Delta_3$, from which three copies of Δ_3 have been removed.

Example 4.1. Figure 1 shows the partial permutohedron $\mathcal{P}(3,2) = \{\mathbf{x} \in \mathbb{R}^3 \mid 0 \le x_i \le 2 \text{ for all } i \in [3],\ x_1 + x_2 + x_3 \le 3\}$ as the 3-dilated standard 3-simplex $3\,\Delta_3$, from which three copies of the standard 3-simplex Δ_3 have been removed.

Further illustrations of the sculpting process are given in Figure 2 (for $\mathcal{P}(3,3)$) and Figure 3 (for $\mathcal{P}(3,n)$ with $n \ge 2$).

4.3 New results

We have the following formulas for $\mathcal{P}(m,n)$ with arbitrary m and $n \le 4$. The expression for $v(m,2)$ was conjectured in [7, Conjecture 5.30]. Our sculpting methods become harder as n grows, and for $\mathcal{P}(m,4)$ we have only computed the volume.

Theorem 4.2. *The following statements are true.*

1. *For any m, $\mathcal{P}(m,1)$ is the standard m-simplex Δ_m, and so $\mathcal{P}(m,1)$ has Ehrhart polynomial* $\mathrm{Ehr}(\mathcal{P}(m,1)) = \binom{t+m}{m}$ *and normalized volume $v(m,1) = 1$.*

2. *For any m, the Ehrhart polynomial of $\mathcal{P}(m,2)$ is*

$$\mathrm{Ehr}(\mathcal{P}(m,2)) = \binom{3t+m}{m} - m\binom{t+m-1}{m}, \tag{4.1}$$

and thus, taking $m!$ times the coefficient of t^m in $\mathrm{Ehr}(\mathcal{P}(m,2))$*, the normalized volume of $\mathcal{P}(m,2)$ is*

$$v(m,2) = 3^m - m.$$

3. *For any m, the Ehrhart polynomial of $\mathcal{P}(m,3)$ is*

$$\operatorname{Ehr}(\mathcal{P}(m,3)) = \binom{6t+m}{m} - m\binom{3t+m-1}{m} - \binom{m}{2}\left(\binom{t+m-1}{m} + (m-2)\binom{t+m-2}{m}\right), \quad (4.2)$$

and thus, taking $m!$ times the coefficient of t^m in $\operatorname{Ehr}(\mathcal{P}(m,3))$, the normalized volume of $\mathcal{P}(m,3)$ is

$$v(m,3) = 6^m - m\,3^m - (m-1)\binom{m}{2}.$$

4. *For any m, the normalized volume of $\mathcal{P}(m,4)$ is*

$$v(m,4) = 10^m - m\,6^m - \frac{m(m-1)(m-3)}{6}\,3^m - (3m^2-6m+1)\binom{m}{3}. \quad (4.3)$$

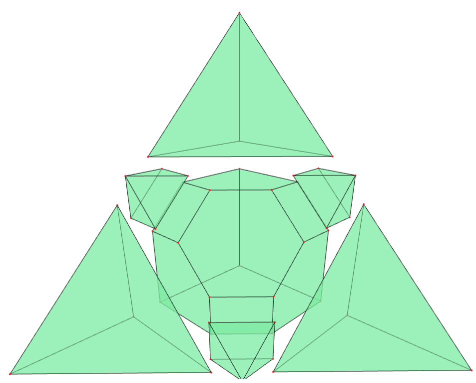

Figure 2: The sculpting of $\mathcal{P}(3,3)$ (in the center), as used in our proof of (4.2) with $m=3$.

Having calculated $v(m,n)$ for $n \le 4$, an obvious open problem remains.

Open Problem 4.3. *Find $v(m,n)$ for all m,n with $n > 4$.*

We now return to $\mathcal{P}(m,n)$ in the case $n \ge m-1$. It follows from Postnikov's results on generalized permutohedra [11], and an interpretation of partial permutohedra as certain generalized permutohedra, that in this case $\operatorname{Ehr}(\mathcal{P}(m,n))$ is a polynomial in n of degree m and that $\mathcal{P}(m,n)$ is Ehrhart positive (i.e., that each coefficient of $\operatorname{Ehr}(\mathcal{P}(m,n))$, as a polynomial in t, is positive). We obtain explicit expressions for the Ehrhart polynomial of $\mathcal{P}(m,n)$ for fixed $m \le 4$ and arbitrary $n \ge m-1$, from which this Ehrhart positivity can also be easily verified.

Theorem 4.4. *The following statements are true.*

1. *For $m = 1$ and any n, $\mathcal{P}(1,n)$ is the line segment $[0,n]$, and so the Ehrhart polynomial of $\mathcal{P}(1,n)$ is*
$$\mathrm{Ehr}(\mathcal{P}(1,n)) = nt + 1. \tag{4.4}$$

2. *For $m = 2$ and any n, the Ehrhart polynomial of $\mathcal{P}(2,n)$ is*
$$\mathrm{Ehr}(\mathcal{P}(2,n)) = (n^2 - 1/2)\, t^2 + (2n - 1/2)\, t + 1, \tag{4.5}$$
where this can be obtained, as a simple application of the sculpting process, by constructing $\mathcal{P}(2,n)$ as a square $[0,n]^2$, from which a half-open triangle
$$\mathit{ConvexHull}(\{(n-1,n),\,(n,n),\,(n,n-1)\}) \setminus \mathit{ConvexHull}(\{(n-1,n),\,(n,n-1)\})$$
has been removed.

3. *For any $n \geq 2$, the Ehrhart polynomial of $\mathcal{P}(3,n)$ is*
$$\mathrm{Ehr}(\mathcal{P}(3,n)) \\ = \Big(n^3 - 3n/2 - 1\Big)\, t^3 + \Big(3n^2 - 3n/2 - 3/2\Big)\, t^2 + (3n - 3/2)\, t + 1. \tag{4.6}$$

4. *For any $n \geq 3$, the Ehrhart polynomial of $\mathcal{P}(4,n)$ is*
$$\mathrm{Ehr}(\mathcal{P}(4,n)) = \Big(n^4 - 3n^2 - 4n - 9/4\Big)\, t^4 + \Big(4n^3 - 3n^2 - 6n - 5/2\Big)\, t^3 \\ + \Big(6n^2 - 6n - 9/4\Big)\, t^2 + (4n - 3)\, t + 1. \tag{4.7}$$

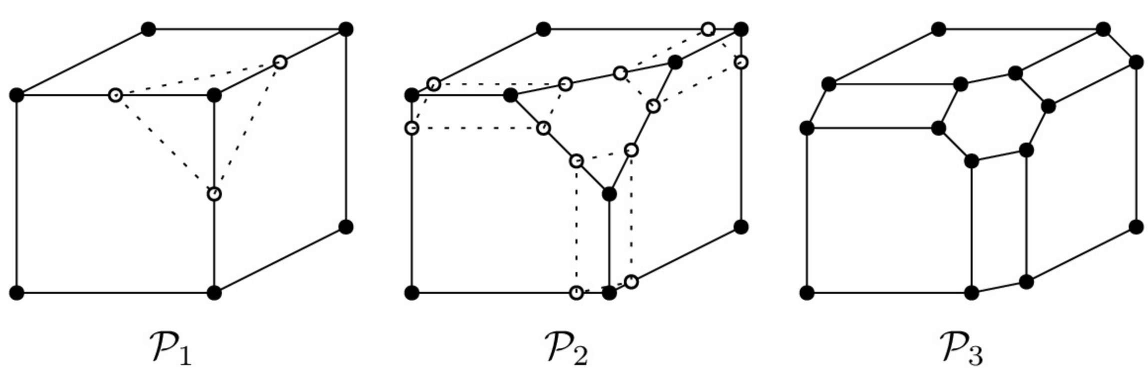

Figure 3: The sculpting of $\mathcal{P}(3,n)$, as used in our proof of (4.6).

5 Further directions

The expressions in (3.3) and (4.4)–(4.7) lead us to the following conjecture of a completely explicit formula for $\mathrm{Ehr}(\mathcal{P}(m,n))$ with any $n \geq m-1$.

Conjecture 5.1. *We conjecture that, for any m and n with $n \geq m-1$, the Ehrhart polynomial of $\mathcal{P}(m,n)$ is explicitly*

$$\mathrm{Ehr}(\mathcal{P}(m,n)) = \frac{1}{2^m}\sum_{i=0}^{\lfloor m/2\rfloor}\sum_{j=2i}^{m}(-1)^{i+1}\binom{m}{m-j,\, j-2i,\, i,\, i}\, i!\,(2j-4i-3)!!\, t^{j-i}\,(2nt+t+2)^{m-j}. \tag{5.1}$$

In terms of generating functions, we have

$$\sum_{m=0}^{\infty}\frac{E(m,n,t)\,(2z)^m}{m!} = \sqrt{1-2tz}\, e^{(2nt+t+2)z-tz^2}, \tag{5.2}$$

where $E(m,n,t)$ is the RHS of (5.1) for any nonnegative integer m, with n and t fixed.

Conjecture 5.1 can be seen to generalize Theorem 3.3. Specifically, since $v(m,n)/m!$ is the leading coefficient of $\mathrm{Ehr}(\mathcal{P}(m,n))$, as a polynomial in t, and since the degree of this polynomial is m (as $\dim\mathcal{P}(m,n) = m$), it follows that $v(m,n)/m!$ is given by $\lim_{T\to 0}\left(T^m\,\mathrm{Ehr}(\mathcal{P}(m,n))|_{t=1/T}\right)$. Now (5.2) gives

$$\sum_{m=0}^{\infty}\frac{E(m,n,1/t)\,(2tz)^m}{m!} = \sqrt{1-2z}\, e^{(2n+1+2t)z-tz^2},$$

and taking $t\to 0$ on both sides then gives (3.4).

It can also easily be checked that (5.1) reproduces the expressions (4.4)–(4.7) for $\mathrm{Ehr}(\mathcal{P}(m,n))$ with $m \leq 4$ and $n \geq m-1$.

Acknowledgements

The authors thank Spencer Backman, Jessica Striker and Shaun Sullivan for helpful exchanges. We thank the American Institute of Mathematics for research support through a SQuaRE grant.

References

[1] A. Amanbayeva and D. Wang. “The convex hull of parking functions of length n”. 2021. arXiv:2104.08454.

[2] M. Beck and S. Robins. *Computing the continuous discretely*. Vol. 61. Springer, 2007.

[3] A. E. Black and R. Sanyal. "Flag polymatroids". 2022. arXiv:2207.12221.

[4] C. Eur, J. Huh, and M. Larson. "Stellahedral geometry of matroids". 2022. arXiv:2207.10605.

[5] L. Ferroni. "Matroids are not Ehrhart positive". *Adv. Math.* **402** (2022), Paper No. 108337, 27. DOI.

[6] D. Hanely, J. L. Martin, D. McGinnis, D. Miyata, G. D. Nasr, A. R. Vindas-Meléndez, and M. Yin. "Ehrhart theory of paving and panhandle matroids". 2022. arXiv:2201.12442.

[7] D. Heuer and J. Striker. "Partial permutation and alternating sign matrix polytopes". 2020. arXiv:2012.09901.

[8] L. Khachiyan. "Complexity of polytope volume computation". *New trends in discrete and computational geometry*. Vol. 10. Algorithms Combin. Springer-Verlag, Berlin, 1993, pp. 91–101. DOI.

[9] F. Kohl, M. Olsen, and R. Sanyal. "Unconditional reflexive polytopes". *Discrete & Computational Geometry* **64** (2020), pp. 427–452.

[10] A. Postnikov, V. Reiner, and L. Williams. "Faces of generalized permutohedra". *Doc. Math.* **13** (2008), pp. 207–273.

[11] A. Postnikov. "Permutohedra, associahedra, and beyond". *Int. Math. Res. Not.* **2009**.6 (2009), pp. 1026–1106. DOI.

[12] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. A*. Paths, flows, matchings, Chapters 1–38. Springer-Verlag, Berlin, 2003, pp. xxxviii+647.

[13] R. P. Stanley. "Decompositions of rational convex polytopes". *Ann. Discrete Math.* **6** (1980), pp. 333–342.

[14] D. H. Ullman et al. "Problems and Solutions". *The American Mathematical Monthly* **129 (3)** (2022), pp. 285–294. DOI.