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Dispersive transverse waves for a strain-limiting continuum model

HA Erbay
Department of Natural and Mathematical Sciences, Faculty of Engineering, Ozyegin University, Istanbul, Turkey.

KR Rajagopal
Department of Mechanical Engineering, Texas A&M University, College Station, TX, USA

G Saccomandi
Dipartimento di Ingegneria, Università degli Studi di Perugia, Perugia, Italy

Y Şengül
School of Mathematics, Cardiff University, Cardiff, UK

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Abstract
It is well known that propagation of waves in homogeneous linearized elastic materials of infinite extent is not dispersive. Motivated by the work of Rubin, Rosenau, and Gottlieb, we develop a generalized continuum model for the response of strain-limiting materials that are dispersive. Our approach is based on both a direct inclusion of Rivlin–Ericksen tensors in the constitutive relations and writing the linearized strain in terms of the stress. As a result, we derive two coupled generalized improved Boussinesq-type equations in the stress components for the propagation of pure transverse waves. We investigate the traveling wave solutions of the generalized Boussinesq-type equations and show that the resulting ordinary differential equations form a Hamiltonian system. Linearly and circularly polarized cases are also investigated. In the case of unidirectional propagation, we show that the propagation of small-but-finite amplitude long waves is governed by the complex Korteweg–de Vries (KdV) equation.

Keywords
Implicit constitutive theory, strain-limiting model, improved Boussinesq equations, traveling wave solutions, dispersive transverse waves

1. Introduction

As Rajagopal [1] mentions, nonlinear relationships between the Cauchy stress and the linearized strain appear in continuum mechanics, especially in the studies of inelastic materials. However, classical Cauchy elasticity

Corresponding author:
Yasemin Şengül, School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF24 4AG, UK.
Email: sengultezely@cardiff.ac.uk

is unable to give a complete reasoning to such situations since the linearized strain is not an objective quantity. On the contrary, the implicit constitutive theoretical approach introduced by Rajagopal [2, 3] allows these unexplained phenomena to be clearly understood.

The idea behind Rajagopal's pioneering work is that starting with an implicit constitutive relation, one is able to express the strain as a function of the stress rather than vice versa without contradicting the principle of causality. As a result, after linearization, it is possible to maintain nonlinear relationship between the linearized strain and the stress which is not possible in Cauchy elasticity when a constitutive relation giving the stress explicitly as a function of the strain is adopted. This gives rise to *strain-limiting theories* which are introduced and studied extensively by Rajagopal [1, 4, 5] and Rajagopal and Saccomandi [6].

In the absence of dissipative and dispersive effects, the propagation of transverse waves in various materials was investigated from the point of view of hyperbolic nonlinear systems in the literature. For transverse waves propagating in homogeneous isotropic elastic solids, Carroll [7] found a class of exact solutions which are global in space and time, proving that not all solutions blow-up (see also Saccomandi and Vitolo [8]). On the contrary, the propagation of transverse waves in an infinite elastic medium involving dispersive effects still remains to be investigated.

Rubin et al. [9] introduced the idea of modifying the free energy and the stress so that dispersive effects are modeled without altering the usual restrictions on the unmodified constitutive relations obeying the first and second laws of thermodynamics. The interesting feature of this theory is that the constitutive equation proposed in Rubin et al. [9] is a simple material *à la Noll* [10], and no additional boundary conditions with respect to the classical elastic theory are needed. This is a major advantage with respect to dispersive theories based on the second gradient or microstructure where additional boundary conditions are necessary.

Following this approach, Destrade and Saccomandi [11, 12] considered the problem of wave propagation in an isotropic elastic solid by taking the dispersive effects into account and determined a class of global in time and space solutions, the structure of the traveling waves in the incompressible and unconstrained case, and some asymptotic model equations. We note that the strain-limiting approach is not considered in these works. It is also worth remarking that in constitutive relations wherein the strain is expressed as a nonlinear function of the stress, it is possible to generate "stress waves" (see Kannan et al. [13]).

In this paper, we incorporate analysis of transverse wave propagation in an isotropic homogeneous medium with dispersion within the context of the strain-limiting theory for incompressible material response. In section 2, we first propose an model including dispersive effects and investigate the corresponding strain-limiting approximation. This results in a constitutive relation including the linearized strain, its second-order time derivative, and the Cauchy stress. In the same section, using this constitutive equation, we also derive the coupled system of nonlinear wave equations governing the propagation of pure transverse waves. Also, in section 5, we look at the exact solutions of this partial differential equations system, including the Carroll solutions. Section 3 is devoted to the traveling wave solutions of this coupled system where the resulting ordinary differential equation system is a Hamiltonian system. Linearly and circularly polarized cases are also studied in this section. Finally, in section 4, we derive the equations governing the propagation of unidirectional long waves and obtain the complex modified KdV equation.

2. Basic equations

In this section, we introduce the dispersive model we want to study by modifying the general implicit constitutive modeling approach with the addition of Rivlin–Ericksen tensors. Following Rajagopal's approach, we look at the strain-limiting behavior by linearizing the strain.

2.1. Kinematics and implicit modeling

Let $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ be the current position of a particle $\boldsymbol{X}$ in the reference configuration that is assumed to be stress-free, and $\boldsymbol{\chi}$ is the one-to-one mapping parametrized in time t which gives the motion. The displacement $\boldsymbol{u}$ and the deformation gradient $\boldsymbol{F}$ are defined through $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ and $\boldsymbol{F} = \partial\boldsymbol{\chi}/\partial\boldsymbol{X}$. The left Cauchy–Green strain tensor is defined as $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$. Moreover, $\boldsymbol{D}$ is the symmetric part of the velocity gradient given by $\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, where the superimposed dot denotes the material time derivative, and $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the Rivlin–Ericksen tensors (Rivlin and Ericksen [14]) given by:

$$\boldsymbol{A}_1 = 2\boldsymbol{D}, \quad \boldsymbol{A}_2 = \dot{\boldsymbol{A}}_1 + \boldsymbol{A}_1\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A}_1.$$

A body is said to be Cauchy elastic, if the Cauchy stress is given explicitly as a function of the deformation gradient. Here, we generalize this idea and start with an implicit constitutive relation between the stress and the kinematic variables as proposed by Rajagopal [12]. We consider a special class of implicit models $\mathcal{F}(\boldsymbol{T},\boldsymbol{B},\boldsymbol{A}_1,\boldsymbol{A}_2)=0$, where $\boldsymbol{T}$ is the Cauchy stress tensor. To be more precise, we consider relations of the form:

$$\boldsymbol{B}+\alpha[\boldsymbol{A}_2-\boldsymbol{A}_1^2]=\beta_0\boldsymbol{I}+\beta_1\boldsymbol{T}+\beta_2\boldsymbol{T}^2, \tag{1}$$

where α is a non-negative constant, and $\beta_i=\beta_i(I_1,I_2,I_3)$, $i=0,1,2$, are the functions of the invariants:

$$I_1=\mathrm{tr}\,\boldsymbol{T},\quad \mathrm{I}_2=\frac{1}{2}\mathrm{tr}\,\boldsymbol{T}^2,\quad \mathrm{I}_3=\frac{1}{3}\mathrm{tr}\,\boldsymbol{T}^3.$$

When $\boldsymbol{T}=\boldsymbol{0}$, we require that the body be undeformed, i.e., $\boldsymbol{F}=\boldsymbol{B}=\boldsymbol{I}$ and $\boldsymbol{A}_1=\boldsymbol{A}_2=\boldsymbol{0}$. This forces the condition:

$$\beta_0(0,0,0)=1. \tag{2}$$

2.2. Strain-limiting approximation

Within the context of the new generalized elasticity introduced by Rajagopal [3, 4], an approximation based on linearization under the assumption that the displacement gradient is small leads to constitutive relations wherein the strains are bounded. Following this approach, the first simplification of equation (1) is obtained using the approximations:

$$\boldsymbol{B}\approx\boldsymbol{I}+2\boldsymbol{\epsilon},\quad \boldsymbol{A}_1\approx2\boldsymbol{\epsilon}_t,\quad \boldsymbol{A}_2\approx2\boldsymbol{\epsilon}_{tt},$$

where the linearized strain is given by:

$$\boldsymbol{\epsilon}=\frac{1}{2}\left[\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}}+\left(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}}\right)^T\right].$$

Such approximations are possible if there exists a number $\delta<<1$ such that maximum of the norms of the gradients of $\boldsymbol{u},\boldsymbol{u}_t$, and $\boldsymbol{u}_{tt}$ is of order δ for any $\boldsymbol{X}$ and any t in $\mathcal{B}\times\mathbb{R}$, where $\mathcal{B}$ is the region of the Euclidean space occupied by the body in the reference configuration.

The second simplification is to consider $\beta_2\equiv0$, and the last one is to consider incompressible materials so that the only admissible motions are isochoric, i.e., $\mathrm{tr}\,\boldsymbol{\epsilon}\equiv0$. This gives $3(\beta_0-1)+\beta_1\mathrm{tr}\,\boldsymbol{T}=0$. Noting also that $\boldsymbol{A}_1^2$ is of order δ^2, from equation (1), we obtain:

$$2\boldsymbol{\epsilon}+2\alpha\boldsymbol{\epsilon}_{tt}=\beta_1(I_1,I_2,I_3)\left[\boldsymbol{T}-\frac{1}{3}(\mathrm{tr}\,\boldsymbol{T})\boldsymbol{I}\right]. \tag{3}$$

Finally, we assume that the only constitutive function remaining is:

$$\beta_1=\beta_1(\mathrm{tr}\boldsymbol{S}^2)\quad\text{where}\quad \boldsymbol{S}=\boldsymbol{T}-\frac{1}{3}(\mathrm{tr}\,\boldsymbol{T})\boldsymbol{I}.$$

This is just for the simplicity of expressions since by the above definition of $\boldsymbol{S}$, we have:

$$\mathrm{tr}\,\boldsymbol{S}^2=2I_2-\frac{1}{3}I_1^2=\mathrm{tr}\,\boldsymbol{T}^2-\frac{1}{3}(\mathrm{tr}\,\boldsymbol{T})^2.$$

2.3. Propagation of nonlinear transverse waves

We consider the deformation given by:

$$x_1=X_1+u(X_3,t),\quad x_2=X_2+v(X_3,t),\quad x_3=X_3, \tag{4}$$

where u and v are the transverse displacement components. In this case, the stress distribution becomes:

$$\boldsymbol{T}=\left[-p\sum_{i=1}^{3}\boldsymbol{e}_i\otimes\boldsymbol{e}_i\right]+T_{13}(\boldsymbol{e}_1\otimes\boldsymbol{e}_3+\boldsymbol{e}_3\otimes\boldsymbol{e}_1)+T_{23}(\boldsymbol{e}_2\otimes\boldsymbol{e}_3+\boldsymbol{e}_3\otimes\boldsymbol{e}_2), \tag{5}$$

where $T_{13} = T_{13}(x_3, t)$ and $T_{23} = T_{23}(x_3, t)$ are the transverse stress components and $p = -\mathrm{tr}\,\boldsymbol{T}/3$ is the mechanical pressure (in general, the Lagrange multiplier that enforces the constraint of incompressibility need not be the mechanical pressure (see Rajagopal [15])). In the absence of body forces, the balance equations $\rho \boldsymbol{u}_{tt} = \mathrm{div}\,\boldsymbol{T}$ reduce to:

$$\rho u_{tt} = \frac{\partial T_{13}}{\partial x_3} - p_{x_1}, \quad \rho v_{tt} = \frac{\partial T_{23}}{\partial x_3} - p_{x_2}, \quad \frac{\partial T_{33}}{\partial x_3} = 0, \tag{6}$$

where ρ is the constant density function. The constitutive equation (3) now reads:

$$u_z + \alpha u_{ttz} = \beta(\Omega^2) T_{13}, \quad v_z + \alpha v_{ttz} = \beta(\Omega^2) T_{23},$$

where $\Omega^2 = T_{13}^2 + T_{23}^2$, $z = x_3$ and $\beta = \beta_1$. Assuming:

$$p(x_1, x_2, z, t) = f_1(t)x_1 + f_2(t)x_2 + q(z, t),$$

where $f_1(t)$ and $f_2(t)$ are the arbitrary functions and the function $q(z, t)$ is determined by the last equation in equation (6), we obtain:

$$\rho u_{ttz} = \frac{\partial^2 T_{13}}{\partial z^2}, \quad \rho v_{ttz} = \frac{\partial^2 T_{23}}{\partial z^2}.$$

Equivalently, introducing the notation $U = u_z$ and $V = v_z$, we have:

$$\rho U_{tt} = \frac{\partial^2 T_{13}}{\partial z^2}, \quad \rho V_{tt} = \frac{\partial^2 T_{23}}{\partial z^2}, \tag{7}$$

$$U + \alpha U_{tt} = \beta(\Omega^2) T_{13}, \quad V + \alpha V_{tt} = \beta(\Omega^2) T_{23}. \tag{8}$$

Systems (7) and (8) are equivalent to the two coupled nonlinear wave equations:

$$T_{13,zz} + \alpha T_{13,zztt} = \rho \left[\beta(\Omega^2) T_{13}\right]_{tt}, \quad T_{23,zz} + \alpha T_{23,zztt} = \rho \left[\beta(\Omega^2) T_{23}\right]_{tt}, \tag{9}$$

or introducing the complex unknown $T = T_{13} + iT_{23}$ to a single complex equation:

$$T_{zz} + \alpha T_{zztt} = \rho[\beta(\Omega^2) T]_{tt}, \tag{10}$$

where we have used the complex representation $T = \Omega \exp(i\theta)$ to rewrite the dependence of the constitutive function β.

If we linearize equation (9) about the stress-free state, we get two decoupled linear dispersive wave equations. In such a case, the linear dispersion relation is given by:

$$\omega^2 = \frac{k^2}{\rho\beta(0) + \alpha k^2}, \tag{11}$$

where ω and k represent the frequency and the wave number, respectively. Equation (11) shows that α characterizes the dispersive nature of the transverse waves. Henceforth, we assume that $\beta(0) > 0$ to avoid singularities in equation (11). Moreover, it is clear from equation (11) that for large values of k, the frequency remains bounded. Relation (11) is the same as the linear dispersion relation of the improved Boussinesq equations (see, for example, Makhankov [16]). So, we can think of equations (9) as a generalized form of the two coupled improved Boussinesq equations.

3. Traveling wave solutions

In this section, we look for traveling wave solutions for equation (9) in the form $T_{13} = f(\xi)$, $T_{23} = g(\xi)$, where $\xi = \frac{1}{\sqrt{\alpha c^2}}(z - ct)$ is the traveling wave coordinate. In this case, $T_{13}^2 + T_{23}^2 = f^2 + g^2 = \Omega^2$, and the equations in equation (9) become two coupled ordinary differential equations given by:

$$f'' + f^{(\mathrm{iv})} = \rho c^2 \left(\beta(\Omega^2) f\right)''$$
$$g'' + g^{(\mathrm{iv})} = \rho c^2 \left(\beta(\Omega^2) g\right)''.$$

Integrating twice, while assuming $f, f', \ldots$ and $g, g', \ldots$ are all converging to 0 as $\xi \to \pm\infty$, we obtain the system:

$$f'' + \left[1 - \rho c^2 \beta(\Omega^2)\right] f = 0,$$
$$g'' + \left[1 - \rho c^2 \beta(\Omega^2)\right] g = 0. \tag{12}$$

Multiplying the first equation by f' gives:

$$\frac{d}{d\xi}(f')^2 + \frac{d}{d\xi} f^2 - \rho c^2 \beta(\Omega^2) \frac{d}{d\xi} f^2 = 0. \tag{13}$$

A similar equation also holds for g, namely:

$$\frac{d}{d\xi}(g')^2 + \frac{d}{d\xi} g^2 - \rho c^2 \beta(\Omega^2) \frac{d}{d\xi} g^2 = 0. \tag{14}$$

Assume existence of a potential function $\psi(\Omega^2) = \int_0^{\Omega^2} \beta(s) ds$ such that $\psi' = \beta$ and $\psi(0) = 0$. Therefore, adding equations (13) and (14) gives:

$$\frac{d}{d\xi}\left[(f')^2 + (g')^2 + f^2 + g^2 - \rho c^2 \psi(\Omega^2)\right] = 0.$$

Now, integrating, we obtain:

$$(f')^2 + (g')^2 + f^2 + g^2 - \rho c^2 \psi(\Omega^2) = C, \tag{15}$$

where C is a constant. Since $\psi(0) = 0$ and $f, g, f', g' \to 0$ as $\xi \to \infty$, we find $C = 0$. Defining the vector $\gamma = (f, g)$, we can rewrite equation (15) as:

$$|\gamma'| + |\gamma| - \rho c^2 \psi(|\gamma|) = 0. \tag{16}$$

We define the following Hamiltonian:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left(q_1^2 + q_2^2\right) - \frac{1}{2}\rho c^2 \psi(\Omega^2). \tag{17}$$

It is easy to see that the Hamiltonian system:

$$q_j' = \frac{\partial H}{\partial p_j}, \quad p_j' = -\frac{\partial H}{\partial q_j}, \quad j = 1, 2, \tag{18}$$

is equivalent to equation (12) for $(q_1, q_2, p_1, p_2) = (f, g, f', g')$. One can immediately see that the equilibrium points of this Hamiltonian system are given as the point $(0, 0, 0, 0)$ and the points in the set:

$$\mathcal{S} = \{(q_{1*}, q_{2*}, 0, 0) : \rho c^2 \beta(\Omega_*^2) = 1 \ \text{ and } \ (q_{1*}, q_{2*}) \neq (0, 0)\},$$

for $\Omega_*^2 = q_{1*}^2 + q_{2*}^2$. As a special case, if we take $\beta(\Omega^2) = \frac{1}{\mu}(1 - \kappa \Omega^2)$, where μ and κ are the positive constants, q_{1*} and q_{2*} satisfy the following equation:

$$q_{1*}^2 + q_{2*}^2 = \frac{1}{\kappa \rho c^2}\left(\rho c^2 - \mu\right).$$

If we consider the constant μ as the shear modulus in the linear case, then $\mu/\rho = c_T^2$ gives the speed of the transverse waves in a linearized elastic medium. Therefore, when $c^2 > c_T^2$, there are infinitely many equilibrium points in $\mathcal{S}$. Another special case can be given as $\beta(\Omega^2) = 1/\big(\mu\sqrt{1+\kappa\Omega^2}\big)$, where μ and κ are the positive constants. In this case, we obtain:

$$q_{1*}^2 + q_{2*}^2 = \frac{1}{\kappa}\left[\left(\frac{\rho c^2}{\mu}\right)^2 - 1\right] = \frac{1}{\kappa}\left(\frac{c^4}{c_T^4} - 1\right).$$

When $c^2 > c_T^2$, there are infinitely many equilibrium points in $\mathcal{S}$. It is worth noting that there is an important difference between the physical implications of these two special cases of β. Namely, in the latter case, the strain stays bounded as the stress becomes arbitrarily large, while in the former case, this is not true.

We now study the nature of the equilibrium points $(q_{1e}, q_{2e}, 0, 0)$ (henceforth $(q_{1e}, q_{2e}) = (0, 0)$ for the first equilibrium point, and $(q_{1e}, q_{2e}) = (q_{1*}, q_{2*})$ for the equilibrium points in $\mathcal{S}$). We first rewrite the Hamiltonian system (18) as:

$$q_1' = p_1, \quad q_2' = p_2, \quad p_1' = \varphi q_1, \quad p_2' = \varphi q_2, \tag{19}$$

where $\varphi = \rho c^2 \psi'(\Omega^2) - 1$. Considering linearization around the equilibrium points, we study the behavior of linear perturbations $(\tilde{q}_1, \tilde{q}_2, \tilde{p}_1, \tilde{p}_2)$. As a result of linearization, the first two equations in equation (19) keep their forms for the perturbations. For the latter two, we first consider the expansion of the function φ about the equilibrium points and obtain:

$$\varphi = \varphi_e + 2\rho c^2 \psi''(\Omega_e^2) q_{1e}\tilde{q}_1 + 2\rho c^2 \psi''(\Omega_e^2) q_{2e}\tilde{q}_2 + \dots,$$

where φ_e represents the value of φ at an equilibrium point and is given by $\varphi_e = \rho c^2 \psi'(\Omega_e^2) - 1$ with $\Omega_e^2 = q_{1e}^2 + q_{2e}^2$. Note that $\varphi_e q_{1e} = 0$ and $\varphi_e q_{2e} = 0$. So, neglecting the higher-order terms, the linearized form of the last two equations of equation (19) becomes:

$$\begin{aligned}
\tilde{p}_1' &= \big(\varphi_e + 2\rho c^2 \psi''(\Omega_e^2) q_{1e}^2\big)\tilde{q}_1 + \big(2\rho c^2 \psi''(\Omega_e^2) q_{1e} q_{2e}\big)\tilde{q}_2 \\
\tilde{p}_2' &= \big(2\rho c^2 \psi''(\Omega_e^2) q_{1e} q_{2e}\big)\tilde{q}_1 + \big(\varphi_e + 2\rho c^2 \psi''(\Omega_e^2) q_{2e}^2\big)\tilde{q}_2.
\end{aligned}$$

We can write the linearized form of the Hamiltonian system (19) as:

$$\begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_3 & 0 & 0 \\ a_3 & a_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix}.$$

where:

$$\begin{aligned}
a_1 &= \varphi_e + 2\rho c^2 \psi''(\Omega_e^2) q_{1e}^2, \\
a_2 &= \varphi_e + 2\rho c^2 \psi''(\Omega_e^2) q_{2e}^2, \\
a_3 &= 2\rho c^2 \psi''(\Omega_e^2) q_{1e} q_{2e}.
\end{aligned}$$

Denoting the coefficient matrix on the right-hand side by M, we have:

$$\begin{aligned}
\det M = \det \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} &= a_1 a_2 - a_3^2 \\
&= \big(\varphi_e + 2\rho c^2 \psi''(\Omega_e^2) q_{1e}^2\big)\big(\varphi_e + 2\rho c^2 \psi''(\Omega_e^2) q_{2e}^2\big) - \big(2\rho c^2 \psi''(\Omega_e^2) q_{1e} q_{2e}\big)^2 \\
&= \varphi_e^2 + 2\rho c^2 \varphi_e \psi''(\Omega_e^2)\big(q_{1e}^2 + q_{2e}^2\big) = \varphi_e^2,
\end{aligned}$$

where we used the fact that $\varphi_e q_{1e} = 0$ and $\varphi_e q_{2e} = 0$. Now, let us consider each equilibrium point:

(1) when $(q_{1e}, q_{2e}) = (0, 0)$, then $\det M = \varphi_e^2 > 0$ if $\varphi_e \neq 0$, i.e., $\rho c^2 \psi'(0) - 1 \neq 0$. Therefore, the equilibrium point $(0, 0, 0, 0)$ is a center point.

If $\rho c^2\psi'(0) = 1$, on the contrary, then $\det M = 0$, and hence, the equilibrium point $(0, 0, 0, 0)$ is a cusp.

(2) when $(q_{1e}, q_{2e}) = (q_{1*}, q_{2*})$, then we have $\rho c^2\psi'(\Omega_*^2) = 1$, and hence, $\det M = 0$ and the equilibrium points $(q_{1*}, q_{2*}, 0, 0)$ are cusps.

In order to write the Hamiltonian in polar coordinates, we define:

$$f = \Omega\cos\theta \quad \text{and} \quad g = \Omega\sin\theta. \tag{20}$$

Since $f, g \to 0$ when $\xi \to \pm\infty$ holds, we observe that the circularly polarized transverse waves (for which $\Omega = \Omega_0 = \text{constant} \neq 0$) do not propagate in the strain-limiting medium that we consider. In this case, equation (20) leads to $q_1^2 + q_2^2 = f^2 + g^2 = \Omega^2$ and $p_1^2 + p_2^2 = (\Omega')^2 + \Omega^2(\theta')^2$. The Hamiltonian (17), thus, becomes:

$$H = \frac{1}{2}\big((\Omega')^2 + (\Omega\theta')^2\big) + \frac{1}{2}\Omega^2 - \frac{1}{2}\rho c^2\psi(\Omega^2). \tag{21}$$

We define the new variables:

$$q_\Omega = \Omega, \quad q_\theta = \theta, \quad p_\Omega = \Omega', \quad p_\theta = \Omega^2\theta'.$$

Then, equation (21) can be expressed as:

$$H = \frac{1}{2}\left(p_\Omega^2 + \frac{p_\theta^2}{q_\Omega^2}\right) + \frac{1}{2}q_\Omega^2 - \frac{1}{2}\rho c^2\psi(q_\Omega^2).$$

In this case, the Hamiltonian system (18) can be written as:

$$\begin{aligned}
q'_\Omega &= \frac{\partial H}{\partial p_\Omega} = p_\Omega,\\
q'_\theta &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{q_\Omega^2},\\
p'_\Omega &= -\frac{\partial H}{\partial q_\Omega} = -\frac{p_\theta^2}{q_\Omega^3} - q_\Omega\Big[1 - \rho c^2\psi'(q_\Omega^2)\Big]\\
p'_\theta &= -\frac{\partial H}{\partial q_\theta} = 0.
\end{aligned}$$

The third equation is equivalent to:

$$\Omega'' + \Omega\Big[(\theta')^2 + 1 - \rho c^2\psi'(\Omega^2)\Big] = 0, \tag{22}$$

and the fourth one is:

$$\big(\Omega^2\theta'\big)' = 0. \tag{23}$$

In the linearly polarized case, i.e., when $\theta = \theta_0$ is a constant, we have equation (23) is trivially satisfied and equation (22) leads to:

$$\Omega'' + \Omega\big[1 - \rho c^2\psi'(\Omega^2)\big] = 0. \tag{24}$$

The existence of a pulse-like traveling wave solution of equation (24) depends on the nature and the number of zeros of the antiderivative of the second term. In the following, we discuss the existence of solitary wave solutions for two special forms of $\psi'(\Omega^2)$.

We first consider $\beta(\Omega^2) = \psi'(\Omega^2) = \frac{1}{\mu}(1 - \kappa\Omega^2)$, where μ and κ are the positive constants. The phase portrait of equation (24) corresponding to this form of β with $\rho = c = 1$, $\mu = 0.5$, and $\kappa = 0.06$ is shown in Figure 1(a). In this case, the antiderivative $\Pi(\Omega)$ of the second term is:

$$\Pi(\Omega) = \frac{1}{2}\Omega^2 - \frac{\rho c^2}{2\mu}\left(\Omega^2 - \frac{1}{2}\kappa\Omega^4\right),$$

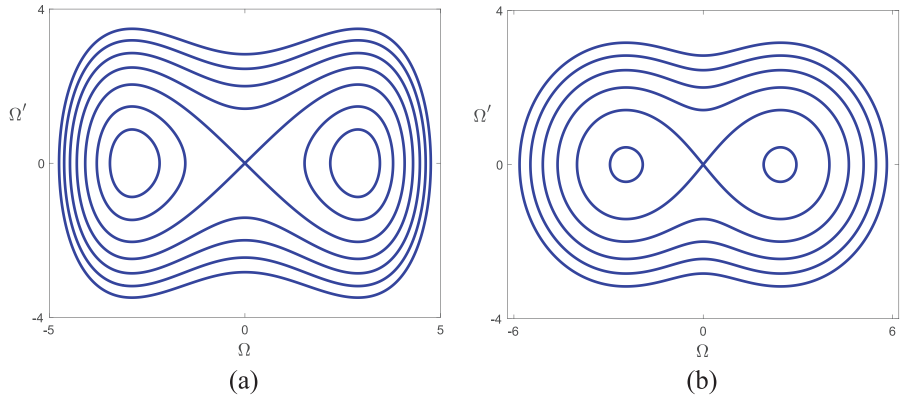

Figure 1. Phase portraits of equation (24) for two different forms of $\beta(\Omega^2)$, where $\rho = c = 1$ and $\mu = 0.5$. (a) $\beta(\Omega^2) = \frac{1}{\mu}(1-\kappa\Omega^2)$ and $\kappa = 0.06$. (b) $\beta(\Omega^2) = 1/\big(\mu\sqrt{1+\kappa\Omega^2}\big)$ and $\kappa = 0.5$.

which satisfies $\Pi(0) = 0$. It is clear that $\Omega = 0$ is a double root, and the two other roots are:

$$\pm\sqrt{\frac{2}{\kappa}\left(1-\frac{\mu}{\rho c^2}\right)},$$

with $\rho c^2 > \mu$. This implies that a pulse-like traveling wave exists if $c^2 > c_T^2$, i.e., for supersonic waves. In this case, the exact solution of equation (24) can be found explicitly as the solitary wave given by:

$$\Omega(\xi) = A\,\mathrm{sech}(B\xi), \quad A = \sqrt{\frac{2}{\kappa}\left(1-\frac{\mu}{\rho c^2}\right)}, \quad B = \sqrt{\frac{\rho c^2}{\mu} - 1}.$$

The graph of the solitary wave solution can be seen in Figure 2(a), where only half of the profile is shown due to symmetry with respect to the origin. Recall that $f(\xi) = \Omega(\xi)\cos\theta_0$ and $g(\xi) = \Omega(\xi)\sin\theta_0$.

As the second special form, we take $\beta(\Omega^2) = \psi'(\Omega^2) = 1/\big(\mu\sqrt{1+\kappa\Omega^2}\big)$, where μ and κ are the positive constants. In this case, the phase portrait of equation (24) corresponding to $\rho = c = 1$, $\mu = 0.5$, and $\kappa = 0.5$ is shown in Figure 1(b). The antiderivative $\Pi(\Omega)$ of the second term is:

$$\Pi(\Omega) = \frac{1}{2}\Omega^2 - \frac{\rho c^2}{\mu\kappa}\left(\sqrt{1+\kappa\Omega^2} - 1\right),$$

which satisfies $\Pi(0) = 0$. This function has $\Omega = 0$ as a double root, and the two other roots are:

$$\pm 2\sqrt{\frac{\rho c^2}{\mu\kappa}\left(\frac{\rho c^2}{\mu} - 1\right)},$$

with $\rho c^2 > \mu$. Clearly, a pulse-like traveling wave exists if $c^2 > c_T^2$. However, since it cannot be calculated explicitly, we solve equation (24) numerically using the ode45 package of MATLAB. In Figure 2(b), we present again the half of the profile corresponding to the numerical solution.

4. Unidirectional waves

In this section, we derive the equations governing the propagation of unidirectional long waves. In order to get the appropriate scaling, we consider the Taylor series expansion of the linear dispersion relation $\omega = \pm k/\sqrt{\rho\beta(0) + \alpha k^2}$ obtained in equation (11) in terms of wavenumber k. Taking the $+$ sign and using the notation $c_0 = 1/\sqrt{\rho\beta(0)}$, we obtain:

$$\omega = c_0 k - \frac{1}{2}\alpha c_0^3 k^3 + \ldots, \tag{25}$$

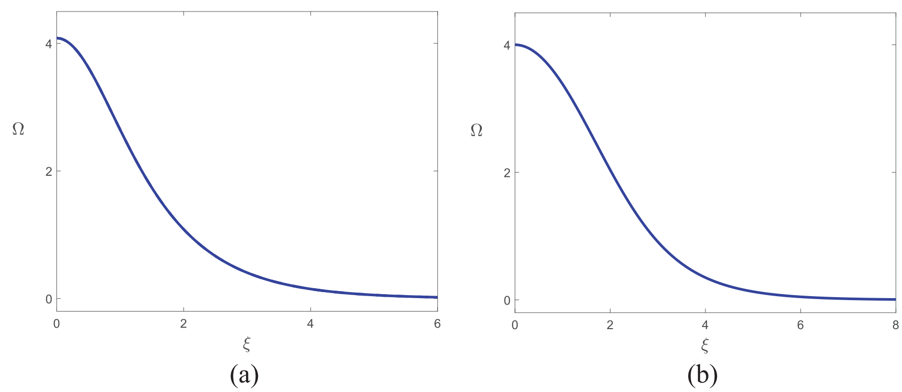

Figure 2. Solitary wave solutions of equation (24) for two different forms of $\beta(\Omega^2)$, where $\rho = c = 1$ and $\mu = 0.5$. Due to symmetry only, half of the profile is shown. (a) $\beta(\Omega^2) = \frac{1}{\mu}(1 - \kappa\Omega^2)$ and $\kappa = 0.06$. (b) $\beta(\Omega^2) = 1/\big(\mu\sqrt{1+\kappa\Omega^2}\big)$ and $\kappa = 0.5$.

for which the phase $kx - \omega t$ becomes:

$$kx - \omega t = k\big(x - c_0 t\big) + \frac{1}{2}\alpha c_0^3 k^3 t + ... \tag{26}$$

Assuming that k is of order $\epsilon^{1/2}$, we introduce the slow variables ξ and τ in the form:

$$\xi = \epsilon^{1/2}\big(x - c_0 t\big), \qquad \tau = \epsilon^{3/2} t. \tag{27}$$

Equation (9) takes the following form in terms of ξ, t:

$$\begin{aligned} &T_{13,\xi\xi} + \alpha\big[\epsilon c_0^2 T_{13,\xi\xi\xi\xi} - 2\epsilon^2 c_0 T_{13,\tau\xi\xi\xi} + \epsilon^3 T_{13,\tau\tau\xi\xi}\big] \\ &\quad = c_0^2\rho\big[\beta(\Omega^2)T_{13}\big]_{\xi\xi} - 2\epsilon c_0\rho\big[\beta(\Omega^2)T_{13}\big]_{\tau\xi} + \epsilon^2 c_0\rho\big[\beta(\Omega^2)T_{13}\big]_{\tau\tau}. \end{aligned} \tag{28}$$

$$\begin{aligned} &T_{23,\xi\xi} + \alpha\big[\epsilon c_0^2 T_{23,\xi\xi\xi\xi} - 2\epsilon^2 c_0 T_{23,\tau\xi\xi\xi} + \epsilon^3 T_{23,\tau\tau\xi\xi}\big] \\ &\quad = c_0^2\rho\big[\beta(\Omega^2)T_{23}\big]_{\xi\xi} - 2\epsilon c_0\rho\big[\beta(\Omega^2)T_{23}\big]_{\tau\xi} + \epsilon^2 c_0\rho\big[\beta(\Omega^2)T_{23}\big]_{\tau\tau}. \end{aligned} \tag{29}$$

We now consider the following series expansion about the stress-free state:

$$\begin{aligned} T_{13} &= \epsilon^{1/2}\Big[P + \epsilon P_1 + \epsilon^2 P_2 + \cdots\Big]. \\ T_{23} &= \epsilon^{1/2}\Big[Q + \epsilon Q_1 + \epsilon^2 P_2 + \cdots\Big], \end{aligned}$$

from which we have:

$$\begin{aligned} \Omega^2 &= \epsilon\Big[P^2 + Q^2 + 2\epsilon(PP_1 + QQ_1) + \cdots\Big]. \\ \beta(\Omega^2) &= \beta(0) + \epsilon\beta'(0)(P^2 + Q^2) + \cdots . \end{aligned}$$

When we substitute the above series expansions into equations (28) and (29), we get a hierarchy of equations. The zeroth-order equations are:

$$[1 - c_0^2\rho\beta(0)]P_{\xi\xi} = 0, \qquad [1 - c_0^2\rho\beta(0)]Q_{\xi\xi} = 0. \tag{30}$$

Since $c_0^2\rho = 1/\beta(0)$, the zeroth-order equations are identically satisfied and P, Q remain arbitrary. If we use again that $c_0^2\rho = 1/\beta(0)$, the first-order equations of the hierarchy become the two coupled MKdV (Modified Korteweg–de Vries) equations:

$$P_\tau + aP_{\xi\xi\xi} + b\big[(P^2 + Q^2)P\big]_\xi = 0, \tag{31}$$

$$Q_\tau + aQ_{\xi\xi\xi} + b\big[(P^2 + Q^2)Q\big]_\xi = 0, \tag{32}$$

where the constants a and b are defined by:

$$a = \frac{\alpha c_0}{2\rho\beta(0)}, \qquad b = -\frac{c_0\beta'(0)}{2\beta(0)}. \tag{33}$$

We now consider waves that travel to the right for which $c_0 > 0$. Since α and $\beta(0)$ are positive, we have $a > 0$. For both of the special cases of β we considered in the previous section, we obtain $\beta'(0) < 0$ implying $b > 0$ ($\beta'(0) = -\kappa/\mu$ and $\beta'(0) = -\kappa/2\mu$, respectively).

If we define the complex quantity w as $w = P + iQ$, equations (31) and (32) can be rewritten as a single equation:

$$w_\tau + aw_{\xi\xi\xi} + b\left(|w|^2 w\right)_\xi = 0. \tag{34}$$

This equation is known as the CMKdV (Complex Modified Korteweg–de Vries) equation, and it has been derived to model the propagation of small-but-finite amplitude transverse waves in many different areas of continuum mechanics [11, 17–19]. The above equation has the following solitary wave solution:

$$w(\xi, \tau) = \sqrt{\frac{2c_s}{b}}\,\mathrm{sech}\left[\sqrt{\frac{c_s}{a}}(\xi - c_s\tau - \xi_0)\right]e^{i\theta_0},$$

which represents a traveling wave with speed c_s and initially at ξ_0, where c_s, ξ_0, and θ_0 are the constants.

5. Special cases

Equation (9) is symmetric, and we can use this property to determine some remarkable class of exact solutions.

5.1. Carroll's solutions

A simple but general and remarkable set of exact solutions is provided considering the ansatz:

$$T_{13} = A\cos(kz - \omega t), \quad T_{23} = \pm A\sin(kz - \omega t), \tag{35}$$

where A, ω , and k are the constants. Introducing equation (35) into the equations (9), we obtain the relationship:

$$c^2 = \frac{1}{\rho\beta(A^2) + \alpha k^2}, \tag{36}$$

where $c = \omega/k$ is the speed of propagation.

5.2. Separable solutions

Let us consider the case where the amplitude and the phase in the complex representation $T = \Omega\exp(i\theta)$ are the separable solutions, i.e., $\Omega(z, t) = \Omega_1(z) + \Omega_2(t)$ and $\theta = \theta_1(z) + \theta_2(t)$. Special and remarkable cases of this decomposition are the generalized oscillatory stress waves:

$$T_{13} = \phi(z)\cos(\omega t) + \zeta(z)\sin(\omega t), \quad T_{23} = \phi(z)\sin(\omega t) - \zeta(z)\cos(\omega t). \tag{37}$$

In this case, we have indeed $\Omega^2(z) = \phi^2 + \zeta^2$. Introducing equation (37) in equation (9), we got the reduction of these partial differential equations to a set of two ordinary differential equations:

$$\begin{aligned}
&(1 - \alpha\omega^2)\frac{d^2\phi}{dz^2} + \rho\omega^2\beta(\Omega^2)\phi = 0, \\
&(1 - \alpha\omega^2)\frac{d^2\zeta}{dz^2} + \rho\omega^2\beta(\Omega^2)\zeta = 0.
\end{aligned} \tag{38}$$

This set of equations may be reduced to the problem of motion of a particle in a central force field as in Carroll [20] but in a more direct way since the first integral:

$$\zeta \frac{d\phi}{dz} - \phi \frac{d\zeta}{dz} = \text{constant},$$

of the angular momentum is in our case immediate. Therefore, the usual transformation:

$$\phi = \Omega \cos\theta, \quad \zeta = \Omega \sin\theta,$$

allows the transformation of equation (38) to:

$$\begin{aligned} &\frac{d^2\Omega}{dz^2} - \Omega \left(\frac{d\theta}{dz}\right)^2 = -\frac{\rho\omega^2}{1-\alpha\omega^2}\beta(\Omega^2)\Omega, \\ &\Omega \frac{d^2\theta}{dz^2} + 2\frac{d\Omega}{dz}\frac{d\theta}{dz} = 0. \end{aligned} \tag{39}$$

The role of the dispersive term in equation (39) is fundamental. Indeed, if $1 - \alpha\omega^2 > 0$, the potential is attractive but clearly at high frequency, it is repulsive.

Another possibility are the generalized oscillatory shear stress waves where, unlike the previous case, $\Omega = \Omega(t)$ is a function of t instead of z, and:

$$T_{13} = \Omega(t)\cos(kz), \quad T_{23} = \Omega(t)\sin(kz), \tag{40}$$

so that $\beta = \beta(\Omega^2)$. Introducing equation (40) in equation (9), we got the reduction:

$$\alpha \frac{d^2\Omega}{dt^2} + \Omega + \frac{\rho}{k^2}\frac{d^2}{dt^2}\left(\Omega\beta(\Omega^2)\right) = 0. \tag{41}$$

Introducing the notation $\Gamma(\Omega) = \frac{\rho}{k^2}\Omega\beta(\Omega^2)$, we rewrite the last equation as:

$$\left(\alpha + \Gamma'(\Omega)\right)\frac{d^2\Omega}{dt^2} + \Gamma''(\Omega)\left(\frac{d\Omega}{dt}\right)^2 + \Omega = 0.$$

This is an autonomous equation, and the usual transformations $d\Omega/dt = \pi(\Omega)$ and $q(\Omega) = \pi^2$ reduce it to the linear equation for q given as:

$$\frac{1}{2}\left(\alpha + \Gamma'(\Omega)\right)\frac{dq}{d\Omega} + \Gamma''(\Omega)q + \Omega = 0.$$

The general solution of the linear equation is the energy integral of equation (41):

$$\frac{1}{2}\left(\alpha + \Gamma'(\Omega)\right)\left(\frac{d\Omega}{dt}\right)^2 = E - \alpha\Omega^2 - \frac{1}{2}U(\Omega^2),$$

where $\Omega\beta = dU/d\Omega$ and E is an integration constant.

6. Concluding remarks

In this short note, we have shown that it is possible to generalize to strain-limiting materials a number of classical solutions in the field of nonlinear elastic theory. Although the mathematical structure of the solutions that have been found is similar to their counterparts in the classical field from a physical point of view, it must be noted that there are fundamental differences. Indeed, in our case, the unknown components are the stress components and the corresponding strain is infinitesimal.

In this way, we were able to generalize the strain-limiting models proposed by Rajagopal and co-workers in elasticity to the dispersive case, determining classes of explicit solutions of direct physical interest and some general model equations in the long wave approximation.

Our results show that even in the case where there are similarities with the classical theory of Cauchy elasticity, the strain-limiting models introduce mathematical subtleties and modified physical interpretation that make the new theory particularly interesting on both levels.

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ORCID iDs

KR Rajagopal https://orcid.org/0000-0002-4636-5131
G Saccomandi https://orcid.org/0000-0001-7987-8892
Y Şengül https://orcid.org/0000-0001-5923-3173

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