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On the Quantum Deformations of Associative Sato Grassmannian Algebras and the Related Matrix Problems

Alexander A. Balinsky [1,†,‡], Victor A. Bovdi [2,†,‡] and Anatolij K. Prykarpatski [2,3,4,*]

[1] Department of Mathematical Physics at the Mathematics Institute of the Cardiff University, Cardiff CF24 4AG, UK ; balinskya@cardiff.ac.uk
[2] Department of Mathematics, United Arab Emirates University, P.O. Box 15551, Al Ain, United Arab Emirates; vbovdi@gmail.com
[3] Department of Computer Science and Telecommunication at the Cracow University of Technology, 30-155 Kraków, Poland
[4] Department of Advanced Mathematics at the Lviv Polytechnic National University, 79000 Lviv, Ukraine
[*] Correspondence: pryk.anat@cybergal.com; Tel.: +48-535-531-185
[†] These authors contributed equally to this work.
[‡] On memoria of the prominent Ukrainian-Hungarian algebraist Adalbert Bovdi (1938–2023) with admiration.

Abstract: We analyze the Lie algebraic structures related to the quantum deformation of the Sato Grassmannian, reducing the problem to studying co-adjoint orbits of the affine Lie subalgebra of the specially constructed loop diffeomorphism group of tori. The constructed countable hierarchy of linear matrix problems made it possible, in part, to describe some kinds of Frobenius manifolds within the Dubrovin-type reformulation of the well-known WDVV associativity equations, previously derived in topological field theory. In particular, we state that these equations are equivalent to some bi-Hamiltonian flows on a smooth functional submanifold with respect to two compatible Poisson structures, generating a countable hierarchy of commuting to each other's hydrodynamic flows. We also studied the inverse problem aspects of the quantum Grassmannian deformation Lie algebraic structures, related with the well-known countable hierarchy of the higher nonlinear Schrödinger-type completely integrable evolution flows.

Keywords: Sato Grassmannians; torus diffeomorphisms; heavenly equations; co-adjoint action; Lax integrability; Lax–Sato equations; loop Lie algebra; Lie algebraic scheme; Casimir invariants; associativity; Lie–Poisson structure

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1. Introduction

Since the classical works by Gerstenhaber [1,2] on the deformations of associative algebras, investigations of the related algebraic structures were strongly stimulated by the Witten–Dijkgraaf–Verlinde–Verlinde [3,4] functional relationships, beautifully reformulated by Dubrovin [5,6] in terms of the Frobenius manifolds and their subsequent extension to F-manifolds. These results gave rise to the remarkable realization of one of Gerstenhaber's approaches [2] to deformation of associative algebras, based on the treatment of '*the set of structure constants as parameter space for the deformation theory*, taking into account that the Frobenius and F-manifolds [7–10] are characterized by the action algebra, which is defined on the tangent sheaf of these manifolds [5,6,11–15].

It was also observed [16,17] that deformations of associated algebras have much of properties that are deeply motivated by the algebraic and geometric structures, associated with the Birkhoff strata of the Sato Grassmannian Gr. These strata are, in general, specified [18,19] by means of a subset W, whose points are endowed with the corresponding infinite-dimensional linear fibers of the tautological tangent subbundle $T(W)$, determining infinite families of infinite-dimensional associative and commutative algebras. Application to the abovementioned structure constants' parameter space for the deformation of the

Sato Grassmannian gave rise to some differential-matrix compatibility relationships, whose solutions describe, in particular, an interesting class [5,20–26] of the Frobenius manifolds within Dubrovin's (related) reformulation scheme.

Moreover, as was suggested in [27], the deformation of the structure constants' parameters can naturally be generalized to the suitably interpreted quantum deformations of the corresponding quantum Sato Grassmannian, naturally characterized by means of some infinite-dimensional associative operator algebra.

Having analyzed, in detail, this deformation and the corresponding structure constants' differential-matrix relationships, we have succeeded in obtaining their Lie algebraic [28–34] reformulation by means of reducing their solution to pure linear matrix algebra equations. We also paid some attention to both the inverse problem aspects of the quantum Sato Grassmannian deformation and the associated linear matrix algebra structures, generated within the Lie algebraic scheme by the well-known countable hierarchy of the higher-order nonlinear Schrödinger completely integrable evolution flows.

2. The Sato Grassmannian Families of Classical Deformation Structures

Since the abovementioned seminal works by Sato [16,17] on the infinite-dimensional Grassmannian Gr, the investigation of algebraic curves in Gr, specified by some subspaces $W \subset Gr$, became an active research field during the past decades. In particular, it was stated in [27] that each Birkhoff stratum Σ_s of the Sato Grassmannian Gr contains a subset W_s of points, carrying the infinite-dimensional linear spaces that coincide with the fibers of the tautological subbundle $T(W_s)$, which is closed with respect to the related pointwise multiplication. This, in particular, means that, algebraically, all tangent spaces $T(W_s)$ are infinite-dimensional associative commutative algebras.

From a geometrical point of view, each fiber $T(W_s)$ is an algebraic variety and the whole $T(W_s)$ is the algebraic variety, with each finite-dimensional subvariety being a family of algebraic curves. For the big cell $\Sigma_\varnothing$, the tangent space $T(W_s)$ is the collection of families of normal rational curves, called Veronese webs of all degrees $N \in \mathbb{N}\backslash\{1\}$. For the stratum Σ_1, each fiber of $T(W_s)$ is the coordinate ring of the elliptic curve and the tangent space $T(W_1)$ is the infinite family of such rings. For the set $W_{1,2}$, the space $T(W_{1,2})$ is equivalent to the families of coordinate rings of a special spatial curve with very interesting properties. The related family of curves in $T(W_{1,2})$ contains a plane trigonal curve of genus two; moreover, it was conjectured, in [27], that the tangent space $T(W_{\overline{1,n}})$ in a higher stratum $\Sigma_{\overline{1,n}}$ for $n \geq 3$ contains a plane $(n+1, n+2)$-curve of genus $n \in \mathbb{N}\backslash\{1,2\}$.

To specify the deformation structures subject to the associative algebras related to the Sato Grassmannian Gr, we denote by $H = \mathbb{C}((\xi))$ the set of formal Laurent series for symbol ξ and by $H_+ = \mathbb{C}[\xi]$ the corresponding set of all formal polynomials. The Sato Grassmannian Gr is, by definition, the parametric space of a closed vector subspace $W \subset H$, such that the projection $W \to H_+$ is Fredholmian. Each $W \subset Gr$ possesses an algebraic basis $\{w_0(\xi), w_1(\xi), ..., w_n(\xi), ...\}$, with the basis elements $w_n(\xi) := \sum_{k=-\infty}^{n} a_k^{(n)} \xi^k$ of finite degree $n \in \mathbb{Z}$. A point on the Sato Grassmannian Gr represents a linear space, generated by thesis-basis series. The related linear bundle constructed as the disjoint union of all these linear fibers is of particular interest, as, in the well-known case of infinite-dimensional Grassmannians, such a bundle is referred to as the tautological tangent bundle $T(Gr)$ over the Sato Grassmannian.

For any subset $W \subset Gr$, the Sato Grassmannian naturally defines the corresponding tautological subbundle $T(W)$. The Grassmannian Gr proves to be a connected Banach space that exhibits [35] a stratified structure. The latter can be described by means of the subset $S \subset \mathbb{Z}$, which is bounded from below and contains all sufficiently high integers. Then, for a subset $W \subset Gr$, one naturally defines the set $S_W = \{S \subset : \deg w(\xi) = s \in \mathbb{Z}$ for any $w(\xi) \in W\}$. Moreover, for any $S \subset \mathbb{Z}$, the related subset $\Sigma_S \subset Gr$, defined as $\Sigma_S = \{W \in Gr : S_W = S\}$, is called the Birkhoff stratum, associated with the subset $S \subset \mathbb{Z}$. The closure of Σ_S, called the Birkhoff variety, is an infinite-dimensional irreducible ind-variety of the finite co-dimension $\operatorname{co\,dim} \Sigma_S = \sum_{k \in \mathbb{Z}_+} (k - s_k)$, where $S := \{s_0, s_1, ..., s_n, ...\} \subset \mathbb{Z}$, where, for some great enough $n \in \mathbb{Z}_+$, $s_n = n$ holds. In particular, if the set $S := \{0, 1, 2, ...n, ...\} \subset \mathbb{Z}_+$, the corresponding stratum has the co-dimension $\mathrm{co}\dim \Sigma_S = 0$, which is a dense open subset of the Grassmannian Gr that is called the principal stratum or the big cell. The Birkhoff stratification (described above) of the Sato Grassmannian Gr induces the stratification of the tautological tangent bundle $T(Gr)$ into subbundles $T(\Sigma_S), S \subset \mathbb{Z}$. It is also worth remarking here that, in addition to algebraic and geometric aspects of the Birkhoff stratification of the Sato Grassmannian Gr, its interesting analytical structure was also revealed. As was demonstrated in [19], the Laurent series $w_n(\xi) = \Sigma_{k=-\infty}^{n} a_k^{(n)} \xi^k$, $n \in \mathbb{Z}$, when $\xi \in \mathbb{C}$, are the boundary values of certain functions on $\mathbb{C} \backslash D_\infty$, where D_∞ is a small disk around the infinite point $\infty \in \bar{\mathbb{C}}$. This observation was formalized by Witten in [36], having suggested that the Sato Grassmannian can be viewed as the space of boundary conditions for the $\bar{\partial}-$operator, reduced on the domain D_∞. Then, as was shown in [36], the index inion of the $\bar{\partial}_W-$operator on the domain D_∞ proved to be finite, that is, $\mathrm{ind}\bar{\partial}_W = \mathrm{card}(S_W - \mathbb{N}) - \mathrm{card}(S_{\tilde{W}} - \mathbb{N})$, where, by definition, $S_{\tilde{W}} := \{-n : n \notin S_W\}$ for any $S \subset \mathbb{Z}$. Considering the principal stratum $\Sigma_\varnothing \subset Gr$, its basis is composed by the Laurent series of all nonnegative degrees $n \in \mathbb{N} : \{p_0, p_1(\xi), p_2(\xi), ..., p_n(\xi), ...\} \subset \Sigma_\varnothing$, where

$$p_n(\xi) = \xi^n + \Sigma_{k \in \mathbb{N}} a_k^{(n)} \xi^{-k}, \tag{1}$$

where coefficients $a_k^{(n)} \in \mathbb{C}, n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$. The corresponding points of $\Sigma_\varnothing$ are represented by means of the infinite-dimensional linear subspace spanned by the basis elements (1) and, moreover, the stratum $\Sigma_\varnothing$ itself is a family of such linear subspaces, parameterized by these basis elements.

Now, we proceed to studying special points in the stratum $\Sigma_\varnothing$, satisfying some specified properties imposed on the corresponding fibers:

$$p_j(\xi) \circ p_k(\xi) = \sum_{l \in \mathbb{Z}_+} C_{jk}^l p_l(\xi) \tag{2}$$

for $j, k \in \mathbb{Z}_+$ imposed on the basis elements (1) and defined by some structure coefficients $C_{jk}^l \in \mathbb{C}, j, k, l \in \mathbb{Z}_+$. Under the conditions in (2), the tangent subbundle $T(W_\varnothing)$ was characterized in [13] by the following proposition.

Proposition 1. *The subbundle $T(W_\varnothing)$ is an infinite family of infinite-dimensional commutative associative algebra, specified by the structure coefficient matrices*

$$C_j = \{C_{jk}^l = C_{kj}^l \in \mathbb{C} : k, l \in \mathbb{Z}_+\}, \tag{3}$$

that satisfy the following commutative conditions:

$$[C_j, C_k] = 0 \tag{4}$$

for all $j, k \in \mathbb{Z}_+$.

The associative and commutative algebraic structure (constructed above) on the subbundle $T(W_\varnothing)$ can be deformed by means of an infinite parametric set $t = (t_0, t_1, t_2, \ldots) \in \mathbb{R}^\infty$, making use of an analytical construction, as devised in works [14,15]. This construction, called the co-isotropic deformation, as applied to the algebraic variety $T(W_{J_\varnothing})$, consists [13] in defining the co-isotropic submanifold $\Gamma_\varnothing \subset W_\varnothing \times \mathbb{R}^\infty$, endowed with the canonical Poisson bracket $\{\cdot\,,\cdot\}$, subject to the variables $(p, t) \in \Gamma_\varnothing$, such that the corresponding skew-orthogonal complement $\Gamma_\varnothing^\perp \subset \Gamma_\varnothing$. The abovementioned co-isotropic submanifold $\Gamma_\varnothing$ is defined as the zero-locus of the following determining relationships:

$$\Gamma_\varnothing = \{f_{jk}(\xi, t) = p_j(\xi, t) \circ p_k(\xi, t) - \sum_{k \in \mathbb{Z}_+} C_{jk}^l(t) p_l(\xi, t) = 0 : \quad j, k \in \mathbb{Z}_+\} \tag{5}$$

for which the canonical Poisson brackets

$$\{f_{jk}, f_{lm}\}|_{\Gamma_\theta} = 0 \tag{6}$$

for all j,k and $l,m \in \mathbb{Z}_+$. The conditions in (5) and (6) are geometrically equivalent to the closedness of the ideal $J_\emptyset := \langle f_{jk} : j,k \in \mathbb{Z}_+\rangle \subset T(W_\emptyset)$, that is, $\{J_\emptyset, J_\emptyset\} \subset J_\emptyset$.

A slightly scrutinized analysis of the conditions in (6), subject to the canonical Poisson bracket, gives rise to the following matrix relationships:

$$\partial C_j(t)/\partial t_k = \partial C_k(t)/\partial t_j, \qquad [C_j(t), C_k(t)] = 0 \tag{7}$$

for all $j,k \in \mathbb{Z}_+$, which, under some conditions on $\mathbb{R}^\infty$, belong to t-dependence imposed on the matrices' structures $C_j(t) \in \mathrm{End}\,\mathbb{E}^\infty, j \in \mathbb{Z}_+$, which reduce [13,37,38] to the well-known WDVV associativity equations, describing the well-known Frobenius manifolds. Namely, if to put, by definition, that the matrices $C_j(t) \in \mathrm{End}\,\mathbb{E}^n$, $j = \overline{1,N}$ nontrivially depend on $t \in \mathbb{R}^N$, the system of differential matrix equations was proven [5,9] to be equivalent to the next compatible system of $\mathbb{R} \ni \lambda$-parametric parallel transporting equations:

$$\partial x/\partial t_k + \lambda C_k(t)x = 0 \tag{8}$$

on a vector $x \in T(\mathbb{T}^N)$, satisfied for all $k = \overline{1,N}$ and $\lambda \in \mathbb{R}$. Moreover, as was demonstrated in the work [31,33,34], the system (8) is compatible iff there exists such a countable set of generating matrices $l_j \in \mathrm{End}\,\mathbb{E}^N$, $j = \overline{-1,0} \cup \mathbb{N}$, that, for each $k = \overline{0,N-1}$, the following set of linear recurrent differential algebraic relationships holds:

$$\begin{aligned} l_{-1}\mathrm{tr}C_k + l_{-1}C_k + C_k^\intercal l_{-1} &= 0, \\ \partial l_{-1}/\partial t_k + l_0\mathrm{tr}C_k + l_0C_k + C_k^\intercal l_0 &= 0, \\ \partial l_0/\partial t_k + l_1\mathrm{tr}C_k + l_1C_k + C_k^\intercal l_1 &= 0, \\ \partial l_1/\partial t_k + l_2\mathrm{tr}C_k + l_2C_k + C_k^\intercal l_2 &= 0, \\ \partial l_2/\partial t_k + l_3\mathrm{tr}C_k + l_3C_k + C_k^\intercal l_3 &= 0... \end{aligned} \tag{9}$$

For the case $N = 3$, the matrices $C_k \in \mathrm{End}\,\mathbb{E}^3, k = \overline{0,2}$, are given, owing to [5,9] by the following expressions:

$$C_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_1 = \begin{pmatrix} 0 & 1 & 0 \\ b & a & 1 \\ c & b & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 & 1 \\ c & b & 0 \\ b^2 - ac & c & 0 \end{pmatrix}, \tag{10}$$

and satisfy the differential algebraic relationships

$$\partial C_1/\partial t_2 = \partial C_2/\partial t_1, [C_1, C_2] = 0, \tag{11}$$

equivalent to such evolution differential relationships:

$$a_{t_2} = b_{t_1}, b_{t_2} = c_{t_1}, c_{t_2} = (b^2 - ac)_{t_1} \tag{12}$$

on a functional manifold $\tilde{M} \subset C^1(\mathbb{R};\mathbb{R}^3)$ with respect to the evolution variable $t_2 \in \mathbb{R}$ and spatial variable $t_1 \in \mathbb{R}$. Since $C_2 = C_1^2 - aC_1 - bI$, the additional commuting condition $[C_1, C_2] = 0$ is satisfied automatically. The resulting evolution flow (12) with respect to the variable $t_2 \in \mathbb{R}$ proved to be [39] a Hamiltonian system. This can be easily stated if to make use of the gradient-holonomic scheme devised in [28,40] and calculate a countable series of conservation laws for the evolution flow (12).

To demonstrate this in more detail, let us preliminarily rewrite the flow (12) on the functional manifold $\tilde{M}$, as the following dynamical system:

$$\left.\begin{pmatrix} \partial p/\partial t_2 \\ \partial q/\partial t_2 \\ \partial r/\partial t_2 \end{pmatrix} = \begin{pmatrix} \partial q/\partial s \\ r \\ (\partial^2 q/\partial s)^2 - \partial r/\partial t_1 \partial^2 p/\partial s^2, \end{pmatrix}\right\} := K[p,q,r] \tag{13}$$

with respect to the evolution parameter $t_2 \in \mathbb{R}$ on a suitably chosen smooth functional manifold $M \subset C^2(\mathbb{R}/\{2\pi\mathbb{Z}\};\mathbb{R}^3)$ subject to the following Backlund-type transformation:

$$\tilde{M} \ni (a,b,c) \to (\partial^2 p/\partial s^2, \partial^2 q/\partial s^2, \partial r/\partial s) \in M, \tag{14}$$

where we put, by definition, the differentiation $\partial/\partial t_1 := \partial/\partial s$, subject to the spatial variable $s \in \mathbb{R}/\{2\pi\mathbb{Z}\}$. To construct a countable hierarchy of conservation laws to the dynamical system (13), it is necessary to construct an asymptotic $|\lambda| \to \infty$ solution to the linear Noether–Lax equation:

$$\partial\varphi/\partial t_2 + K'^{,*}\varphi = 0, \tag{15}$$

where $\varphi \in T^*(M)$ satisfies the symmetry condition $\varphi'[p,q,r] = \varphi'^{,*}[p,q,r]$ for all $(p,q,r)^\intercal \in M$, and the mapping $K'^{,*} : T^*(M) \to T^*(M)$ equals

$$K'^{,*} = \begin{pmatrix} 0 & 0 & -\partial^2/\partial s^2 \circ \partial r/\partial s \\ -\partial & 0 & 2\,\partial^2/\partial s^2 \circ \partial^2 q/\partial s^2 \\ 0 & 1 & \partial/\partial s \circ \partial^2 p/\partial s^2 \end{pmatrix}, \tag{16}$$

which is the adjoint Frechet derivative of the vector field mapping $K : M \to T(M)$ with respect to the standard convolution form $(\cdot|\cdot)$ on the Euclidean product $T^*(M) \times T(M)$. Simple enough, yet slightly cumbersome, calculations give rise to the following analytical expressions:

$$\begin{aligned} &\varphi_1 = (1,0,0)^\intercal, \varphi_2 = (q_{sss}q_s + q_{ss}^2, -r_s - q_s p_{sss} - q_{ss}p_{ss}, q_s)^\intercal, \\ &\varphi_3 = (r_s - 2p_{ss}q_{ss} - 2p_s q_{sss}, p_{sss}p_s + p_{ss}^2 + q_{ss}, 0)^\intercal, \quad \ldots \end{aligned} \tag{17}$$

generating, via the Volterra homotopy formula $H_j = \int_0^1 d\mu(\varphi_j[\mu p, \mu q, \mu r]|(p,q,r)^\intercal), j = \overline{1,3}$, such conservation laws as

$$\begin{aligned} &H_1 = \int dsp, \quad H_2 = \int ds(qr_s - q_s^2 p_{ss}/2), \\ &H_3 = \int ds\Big(p\,r_s + \; p_s^2 q_{ss}/2 - q_s^2/2\Big) \end{aligned} \tag{18}$$

for the evolution flow (13). The latter are represented as a Hamiltonian system on the functional manifold M, if there exists [28,40,41] a conservation law $H \in \mathcal{D}(M)$, allowing the following convolutional representation: $H = (\psi|(p_s,q_s,q_s)^\intercal)$, where the covector $\psi \in T^*(M)$ satisfies the corresponding Noether–Lax condition:

$$\partial\psi/\partial t_2 + K'^{,*}\psi = \operatorname{grad}\mathcal{L} \tag{19}$$

on the manifold M for some smooth functional $\mathcal{L} \in \mathcal{D}(M)$. Then, the corresponding symplectic operator $\vartheta^{-1} : T(M) \to T^*(M)$ is given by means of the following operator expression:

$$\vartheta^{-1} = \psi' - \psi'^{,*}, \tag{20}$$

whose inverse is the related Poisson operator $\vartheta : T^*(M) \to T(M)$ on the functional manifold M, which means that the dynamical system (13) is Hamiltonian and representable [28,42] in the following canonical form:

$$K = -\vartheta \operatorname{grad} H, \tag{21}$$

where the Hamiltonian function $H = (\psi|K) - \mathcal{L} \in \mathcal{D}(M)$. It is easy to check that the following convolutional representation holds:

$$\begin{gathered} H_3 = \int ds\big(p\, r_s + \ p_s^2 q_{ss}/2 - q_s^2/2\big) = \\ = ((p_s q_{ss}, p_s p_{ss} - 1/2q_s, p)^\intercal \,|(p_s, q_s, q_s)^\intercal) := (\psi|(p_s, q_s, q_s)^\intercal) \end{gathered} \tag{22}$$

providing the covector $\psi = (p_s q_{ss}, p_s p_{ss} - 1/2q_s, p)^\intercal \in T^*(M)$. The latter generates, owing to Expression (20), the symplectic operator

$$\vartheta^{-1} = \begin{pmatrix} q_{ss} \circ \partial/\partial s + \partial/\partial s \circ q_{ss} & -p_{ss} \circ \partial/\partial s & -1 \\ -\partial/\partial s \circ p_{ss} & -\partial/\partial s & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{23}$$

whose inverse Poisson operator equals

$$\vartheta = \begin{pmatrix} 0 & -p_{ss} \circ \partial/\partial s & 1 \\ 0 & -(\partial/\partial s)^{-1} & -p_{ss} \\ -1 & p_{ss} & q_{ss} \circ \partial/\partial s + \partial/\partial s \circ q_{ss} + p_{ss} \circ \partial/\partial s \circ p_{ss} \end{pmatrix}. \tag{24}$$

The corresponding Hamiltonian function is given, respectively, by the functional expression

$$H = \int ds(q r_s - q_s^2 p_{ss}/2), \tag{25}$$

exactly coinciding with the conservation law $H_2 \in \mathcal{D}(M)$ found above. Having now returned to the previous variables (14), one obtains the next Poisson operator:

$$\begin{gathered} \tilde{\vartheta} := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a \\ 1 & -a & a^2 + 2b \end{pmatrix} \frac{\partial^3}{\partial s^3} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2a_s \\ 0 & -a_s & 3(b_s + a a_s) \end{pmatrix} \frac{\partial^2}{\partial s^2} + \\ + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_{ss} + a_s^2 + a a_{ss} \end{pmatrix} \frac{\partial}{\partial s}, \end{gathered} \tag{26}$$

on the functional manifold $\tilde{M}$, coinciding with the one constructed before in [39], with respect to which the initial evolution flow (12) is of Hamiltonian form:

$$\begin{pmatrix} \partial a/\partial t_2 \\ \partial b/\partial t_2 \\ \partial c/\partial t_2 \end{pmatrix} = \begin{pmatrix} b_s, \\ c_s \\ (b^2 - ac)_s \end{pmatrix} = -\tilde{\vartheta} \operatorname{grad} \tilde{H}_2, \tag{27}$$

where the Hamiltonian function $\tilde{H}_2 = \int ds\Big[a\Big((\partial/\partial s)^{-1} b\Big)^2/2 - \Big((\partial/\partial s)^{-1} b\Big)c\Big] \in \mathcal{D}(\tilde{M})$. As was observed in the abovementioned above inspiring work [39], the Hamiltonian system looks strongly simplified in the vector eigenvalue variable $u = (u_1, u_2, u_3)^\intercal \in \bar{M} \subset C^2(\mathbb{R}; \mathbb{R}^3)$ of the matrix $C_1 \in \operatorname{End} \mathbb{E}^3$:

$$\det(C_1 - u_j I) = 0 \sim u_j^3 - a u_j^2 - 2b u_j - c = 0, \tag{28}$$

where

$$a = \sum_{j=1}^{3} u_j, b = -\frac{1}{2} \sum_{j<k=1}^{3} u_j u_k, c = \prod_{j=1}^{3} u_j, \tag{29}$$

which is the unique invariant hydrodynamical densities of the Hamiltonian system (27). Namely, in the u-variables, the latter is representable as the equivalent Hamiltonian flow:

$$\begin{pmatrix} \partial u_1/\partial t_2 \\ \partial u_2/\partial t_2 \\ \partial u_3/\partial t_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (u_2u_3 - u_1u_2 - u_1u_3)_s \\ (u_1u_3 - u_2u_1 - u_2u_3)_s \\ (u_1u_2 - u_1u_3 - u_2u_3)_s \end{pmatrix} = -\bar{\eta}\,\mathrm{grad}\,\bar{H}(u), \tag{30}$$

where $\bar{\vartheta} : T^*(\bar{M}) \to T(\bar{M})$ is the corresponding Poisson operator:

$$\bar{\eta} = \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \frac{\partial}{\partial s} + \frac{\partial}{\partial s} \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \tag{31}$$

on the submanifold $\bar{M}$, and $\bar{H} = -\int ds(u_1u_2u_3)$—the Hamiltonian function. Returning to the earlier topic, owing to the Backlun-type mappings (29) to the variables of the functional manifold $\tilde{M}$, one easily obtains the following Poisson operator:

$$\begin{aligned} \tilde{\eta} := \tfrac{1}{2} \begin{pmatrix} -3 & a & 2b \\ a & 2b & 3c \\ 2b & 3c & 4(b^2 - ac) \end{pmatrix} \tfrac{\partial}{\partial s} + \tfrac{\partial}{\partial s} \tfrac{1}{2} \begin{pmatrix} -3 & a & 2b \\ a & 2b & 3c \\ 2b & 3c & 4(b^2 - ac) \end{pmatrix} + \\ + \begin{pmatrix} 0 & a_s/2 & b_{t_1} \\ -a_s/2 & 0 & -c_s/2 \\ -b_s & c_s/2 & 0 \end{pmatrix}, \end{aligned} \tag{32}$$

representing the evolution flow (27) as the Hamiltonian system

$$\begin{pmatrix} \partial a/\partial t_2 \\ \partial b/\partial t_2 \\ \partial c/\partial t_2 \end{pmatrix} = \begin{pmatrix} b_s, \\ c_s \\ (b^2 - ac)_s \end{pmatrix} = -\tilde{\eta}\,\mathrm{grad}\,\tilde{H}_1 \tag{33}$$

with the Hamiltonian function $\tilde{H}_1 = -\int dsc \in \mathcal{D}(\tilde{M})$. Taking into account that the Poisson operators $\tilde{\vartheta}$ and $\eta : T^*(\tilde{M}) \to T(\tilde{M})$ are compatible on the submanifold $\tilde{M}$, one can construct the related symmetry recursion operator $\tilde{\Phi} = \tilde{\eta}\tilde{\vartheta}^{-1} : T(\tilde{M}) \to T(\tilde{M})$ and construct the infinite countable hierarchy of commuting both to each other and to (33), i.e., dispersive Hamiltonian systems

$$(\partial a/\partial t_j, \partial b/\partial t_j, \partial c/\partial t_j)^\intercal := \tilde{\Phi}^j\,(\partial a/\partial s, \partial b/\partial s, \partial c/\partial s)^\intercal \tag{34}$$

for all $j \in \mathbb{Z}_+$. The results presented above can be formulated as the following theorem.

Theorem 1. *The WDVV associativity equations in (11) are equivalent to the bi-Hamiltonian systems in (27) and (33) on a smooth functional submanifold $\tilde{M} \subset C^2(\mathbb{R}; \mathbb{R}^3)$, subject to two compatible Poisson operators: (26) and (32). This compatible Poissonian pair generates an associated countable hierarchy of commuting to each other Hamiltonian flows (34) of dispersive type.*

Remark here that, in general, solutions to System (9) for $N > 4$ are not available in a compact analytical form and their analysis still needs very sophisticated algebraic tools and analytic techniques. Moreover, this leaves very interesting aspects of constructing reasonable superalgebraic analogs [43] of the WDVV associativity equations and the related Dubrovin-type super-algebraic connections (8).

3. The Sato Grassmannian Families and Generalized Quantum Deformation Structures

Proceed now to a quantum deformation of the Sato Grassmannian Gr, specified by the following condition: each closed subspace $W \subset Gr$ possesses an algebraic pseudo-differential basis $\{\hat{w}_0(\xi), \hat{w}_1(\xi), \hat{w}_2(\xi), \ldots\}$ that consists of the pseudo-differential operator elements $\hat{w}_n(\xi) := \sum_{i=-\infty}^{n} a_i^{(n)} \xi^i \in \mathrm{PDO}(\xi)$, $n \in \mathbb{Z}_+$, with the differentiation symbol ξ naturally acting on the dense subspaces of the smooth functions $C^\infty(\mathbb{R};\mathbb{C})$. Then, one can construct, analogically, the principal quantum stratum $\hat{\sum_{\emptyset}} \subset \hat{Gr}$ endowed with the Laurent-type of nonnegative degree pseudo-differential basis $\{\hat{p}_0(\xi), \hat{p}_1(\xi), \ldots, \hat{p}_n(\xi), ..\} \in \sum_{\emptyset}$, where

$$\hat{p}_n(\xi) = \xi^n + \sum_{k=1}^{\infty} a_k^{(n)} \xi^{-k} \tag{35}$$

with the coefficients $a_k^{(n)} \in \mathbb{C}^\infty(\mathbb{R}^N;\mathbb{C})$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$.

Now, let us consider special points of the quantum stratum $\hat{\sum_{\emptyset}}$, satisfying, on its fiber, the following finite quantum algebraic multiplicative property:

$$\hat{p}_j(\xi) \circ \hat{p}_k(\xi) = \sum_{l=0}^{\infty} C_{jk}^l(t) \hat{p}_l(\xi) \tag{36}$$

for all $j, k \in \mathbb{Z}_+$, defined by the corresponding structure constants $C_{jk}^l(t) \in C^\infty(\mathbb{R}^N;\mathbb{C})$ for j, k and $l \in \mathbb{Z}_+$. In order to define the quantum deformation coefficients, we impose on the basis operations in (36) the following canonical commutation relationships:

$$[\hat{p}_j, \hat{p}_k] = 0 = [\hat{t}_j, \hat{t}_k], \qquad [\hat{p}_j, \hat{t}_k] = \hbar \delta_{jk} \tag{37}$$

for all $j, k \in \overline{0, N-1}$, where "$\hbar$" denotes the so-called classical Planck constant. For the multiplication property (37) to be realized, one needs to restrict the operator relationships

$$\hat{f}_{jk}(\xi) = -\hat{p}_j(\xi) \circ \hat{p}_k(\xi) + \sum_{l=0}^{N-1} C_{jk}^l(t) \hat{p}_l(\xi) \tag{38}$$

upon the kernel subspace $H_N \subset H$, where

$$\hat{f}_{jk}(\xi) H_N = 0 \tag{39}$$

for all $j, k = \overline{0, N-1}$. A naturally imposed condition [27], which should be a priori satisfied, following from the conditions in (36) consists in the commutation relationships

$$[\hat{f}_{jk}(\xi), \hat{f}_{lm}(\xi)] H_N = 0 \tag{40}$$

for all j, k and $l, m = \overline{0, N-1}$, which is equivalent to the co-isotropy condition (6) used before. As was stated in [27], the condition in (40) is equivalent to the associativity conditions

$$[(\hat{p}_j(\xi)\hat{p}_k(\xi))\hat{p}_l(\xi) - \hat{p}_j(\xi)(\hat{p}_k(\xi)\hat{p}_l(\xi))] H_N = 0 \tag{41}$$

for all j, k and $l = \overline{0, N-1}$, reducing the next structure constants' equations to

$$\hbar \partial C_j / \partial t_k - \hbar \partial C_k / \partial t_j + [C_j, C_k] = 0 \tag{42}$$

for all $j, k = \overline{0, N-1}$. The system of differential relationships (42) on the structure matrices $C_j \in C^\infty(\mathbb{R}^N; \mathrm{End}\, \mathbb{E}^N)$, $j = \overline{0, N-1}$, can be described effectively by means of the Lie algebraic methods [28,31,41,44], allowing one to represent their t-evolution as some special flows on the orbits of the co-adjoint action of a suitably constructed affine Lie algebra of vector fields on the torus $\mathbb{T}_{\mathbb{C}}^N$, which is briefly described below.

4. Quantum Deformations and the Related Lie Algebraic Structures

To study the algebraic properties of the structure constants Equation (42), we make use of the Lie algebraic approach devised before in [28,31,33,34], within which we consider the linear diffeomorphism loop group $\tilde{G} := \mathrm{Diff}(\mathbb{T}^n_{\mathbb{C}}), n \in \mathbb{N}$, of the torus $\mathbb{T}^n_{\mathbb{C}} \simeq \mathbb{T}^n \otimes \mathbb{C}$, consisting of the set of smooth linear mappings $\{\mathbb{C} \supset \mathbb{S}^1 \to G = \mathrm{Diff}(\mathbb{T}^n)\}$, extended, respectively, holomorphically from the circle $\mathbb{S}^1 \subset \mathbb{C}$, both on the set $\mathbb{D}_+$ of the internal points of $\mathbb{S}^1$ and on the set $\mathbb{D}^1$ of the external points $\lambda \in \mathbb{C}\backslash\overline{\mathbb{D}}^1_+$ so that $\lim_{\lambda\to\infty}\tilde{g}(\lambda) = Id \in G$ for any $\tilde{g}(\lambda) \in \tilde{G}, \lambda \in \mathbb{D}_-$. The corresponding diffeomorphism loop Lie algebra splitting $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, where, by definition, $\tilde{\mathcal{G}}_+ := \widetilde{diff}_+(\mathbb{T}^n) \subset \Gamma(\mathbb{T}^n_{\mathbb{C}})$ is the Lie subalgebra, consisting of affine vector fields on the torus $\mathbb{T}^n_{\mathbb{C}}$, which suitably holomorphic on the disc $\mathbb{D}^1_+$ and $\tilde{\mathcal{G}}_- := \widetilde{diff}_+(\mathbb{T}^n) \subset \Gamma(\mathbb{T}^n_{\mathbb{C}})$, consisting of affine vector fields on the torus $\mathbb{T}^n_{\mathbb{C}}$, which are suitably holomorphic on the disc $\mathbb{D}^1_- \subset \mathbb{C}$. The adjoint space $\tilde{\mathcal{G}}^* = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}^*_- \subset \Lambda^1(\mathbb{T}^n_{\mathbb{C}})$, where the space $\tilde{\mathcal{G}}^*_+$ consists of the affine differential forms on the torus $\mathbb{T}^n_{\mathbb{C}}$, and the space $\tilde{\mathcal{G}}^*_-$ consists of the affine differential forms on the torus $\mathbb{T}^n_{\mathbb{C}}$, tending to zero as $|\lambda| \to \infty$ and defined subject to the following nondegenerate convolution on the product $\tilde{\mathcal{G}}^* \times \tilde{\mathcal{G}}$:

$$(\tilde{l}|\tilde{a}) := \mathrm{res}_{\lambda\in\mathbb{C}} \int_{\mathbb{T}^n} \langle l|a\rangle d^n x, \tag{43}$$

for any affine vector field $\tilde{a} := \langle a(\lambda)x|\partial/\partial x\rangle \in \tilde{\mathcal{G}}$ and affine differential form $\tilde{l} := \langle a(\lambda)x|dx\rangle \in \tilde{\mathcal{G}}^*$ on $\mathbb{T}^n_{\mathbb{C}}$, depending linearly on the torus coordinate vector $x \in \mathbb{T}^n$ and $\lambda \in \mathbb{C}$, where, by definition, $\langle . \mid .\rangle$ is the usual bilinear form on the Euclidean space $\mathbb{E}^n$ and $\partial/\partial x := (\partial/\partial x_1, \partial/\partial x_1, \ldots, \partial/\partial x_n)^\intercal$ is the usual gradient operator.

Let us now consider the set $\mathrm{I}(\tilde{\mathcal{G}})$ of Casimir invariant smooth functionals $h_j : \tilde{\mathcal{G}}^*_- \to \mathbb{C}$, $j \in \mathbb{Z}_+$, defined by the co-adjoint Lie subalgebra $\tilde{\mathcal{G}}$ action

$$ad^*_{\mathrm{grad} h_j(\tilde{l})}\tilde{l} = 0 \tag{44}$$

at a seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$, which can be rewritten in the following differential–functional form:

$$\langle\frac{\partial}{\partial x} \circ |\varphi^{(j)}(l)x\rangle lx + \langle lx|\frac{\partial}{\partial x}\varphi^{(j)}(l)x\rangle = 0, \tag{45}$$

where, by definition, $\tilde{l} = \langle lx|dx\rangle$, $\mathrm{grad} h_j(\tilde{l}) := \langle\varphi^{(j)}(l)x|\frac{\partial}{\partial x}\rangle$, $\varphi^{(j)}(l) \sim \lambda^j \sum_{k\in\mathbb{Z}_+} \varphi_{(k)}\lambda^{-k}$, as $|\lambda| \to \infty$ for all $j \in \mathbb{Z}_+$. Then, the classical Adler–Kostant–Symes algebraic scheme, applied to a suitably chosen seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$, gives rise to the evolution flows

$$\hbar\partial\tilde{l}/\partial t_j := -ad^*_{\mathrm{grad} h_j(\tilde{l})_+}\tilde{l}, \tag{46}$$

commuting to each other for all $j \in \mathbb{Z}_+$ and generating completely integrable Hamiltonian systems on the adjoint space $\tilde{\mathcal{G}}^*_-$. The latter makes it possible to construct a formal enough, yet regular, algorithmic approach to describing matrix structure constants (Equation (42)), specified by the affine diffeomorphism loop Lie algebra $\tilde{\mathcal{G}}_+$ and a seed element $\tilde{l} \in \tilde{\mathcal{G}}^*_+$ of the form

$$\tilde{l} = \sum_{j\in\mathbb{N}} \lambda^{-j}\langle l_j x|dx\rangle \tag{47}$$

where $l_j \in \mathrm{End}\ \mathbb{E}^n, j \in \mathbb{N}$, generated by the corresponding Casimir functionals $h_j \in \mathrm{I}(\tilde{\mathcal{G}}^*_+), j \in \mathbb{Z}_+$. Moreover, taking into account that the flows in (3) on $\tilde{\mathcal{G}}^*_+$ are commuting to each other, one easily states the following proposition.

Proposition 2. *The infinite hierarchy of the linear vector flows*

$$\hbar\frac{\partial x}{\partial\tau_k} = \sum_{j=0}^{k} \lambda^{k-j}\varphi_{(j)}x \tag{48}$$

on the torus $\mathbb{T}^n_{\mathbb{C}}$, as well as the related infinite hierarchy of the augmented vector fields

$$\Phi_k := \hbar\partial/\partial\tau_k + \sum_{j=0}^{k} \lambda^{k-j}\langle\varphi_{(j)}x)|\partial/\partial x\rangle \tag{49}$$

on $\mathbb{T}^n_{\mathbb{C}} \times \mathbb{R}^\infty$ are commuting to each other:

$$[\Phi_s, \Phi_k] = 0 \tag{50}$$

for all $s, k \in \mathbb{Z}_+$. Moreover, the matrix coefficients $\varphi_{(j)} \in \mathrm{End}\,\mathbb{E}^n$, $j \in \mathbb{Z}_+$, satisfy differential algebraic relationships

$$\begin{array}{c} \hbar\frac{\partial\varphi_{(0)}}{\partial\tau_j} = 0, \hbar\frac{\partial\varphi_{(1)}}{\partial\tau_1} - \left[\varphi_{(0)}, \varphi_{(2)}\right] = 0, \\ \hbar\frac{\partial\varphi_{(2)}}{\partial\tau_1} - \hbar\frac{\partial\varphi_{(1)}}{\partial\tau_2} - \left[\varphi_{(1)}, \varphi_{(2)}\right] = 0, ..., \end{array} \tag{51}$$

and so on.

Let us now assume that a taken seed element (47) additionally generates, as $|\lambda| \to \infty$, the set of Casimir functional gradient elements

$$\mathrm{grad} h_s^{(1)}(\tilde{l}) \sim \lambda \sum_{j\in\mathbb{Z}_+} \lambda^{-j}\langle\varphi_{(j)}^{(s)} x|\partial/\partial x\rangle, \tag{52}$$

where coefficients $\varphi_{(j)}^{(s)} \in \mathrm{End}\mathbb{E}^n, j \in \mathbb{Z}_+, s = \overline{1,n}$, satisfy the following matrix relationships:

$$\begin{array}{c} l_1\, \mathrm{tr}\varphi_{(0)}^{(s)} + \left(l_1\varphi_{(0)}^{(s)} + \varphi_{(0)}^{(s)\intercal} l_1\right) = 0, \\ l_1 \mathrm{tr}\varphi_{(1)}^{(s)} + \left(l_1\varphi_{(1)}^{(s)} + \varphi_{(1)}^{(s)\intercal} l_1\right) + l_2 \mathrm{tr}\varphi_{(0)}^{(s)} + \left(l_2\varphi_{(0)}^{(s)} + \varphi_{(0)}^{(s)\intercal} l_2\right) = 0, \\ l_1 \mathrm{tr}\varphi_{(2)}^{(s)} + \left(l_1\varphi_{(2)}^{(s)} + \varphi_{(2)}^{(s)\intercal} l_{-1}\right) + l_2 \mathrm{tr}\varphi_{(1)}^{(s)} + \\ + \left(l_2\varphi_{(1)}^{(s)} + \varphi_{(1)}^{(s)\intercal} l_2\right) + l_3 \mathrm{tr}\varphi_{(0)}^{(s)} + \left(l_3\varphi_{(0)}^{(s)} + \varphi_{(0)}^{(s)\intercal} l_3\right) = 0, \\ l_1 \mathrm{tr}\varphi_{(3)}^{(s)} + \left(l_1\varphi_{(3)}^{(s)} + \varphi_{(3)}^{(s)\intercal} l_1\right) + l_2 \mathrm{tr}\varphi_{(2)}^{(s)} + \left(l_2\varphi_{(2)}^{(s)} + \varphi_{(2)}^{(s)\intercal} l_2\right) + \\ + l_3 \mathrm{tr}\varphi_{(1)}^{(s)} + \left(l_3\varphi_{(1)}^{(s)} + \varphi_{(1)}^{(s)\intercal} l_3\right) + l_4 \mathrm{tr}\varphi_{(0)}^{(s)} + \left(l_4\varphi_{(0)}^{(s)} + \varphi_{(0)}^{\intercal(s)} l_4\right) = 0, ... \end{array} \tag{53}$$

and so on. Now consider the flows (48) for $k = 1$ on the torus $\mathbb{T}^n$ with respect to the evolution parameters $\tau_1^{(s)} := t_s \in \mathbb{R}, s = \overline{1,n}$, generated by different solutions (52) to the determining Equation (2) as $|\lambda| \to \infty$:

$$\hbar\frac{\partial x}{\partial t_s} = (\lambda C_s + T_s)x, \tag{54}$$

where $x \in \mathbb{T}^n$ and matrices $C_s := \varphi_{(1)}^{(s)}, T_s := \varphi_{(0)}^{(s)} \in \mathrm{End}\,\mathbb{E}^n$, $s = \overline{1,n}$. As the flows (54) are also commuting to each other, being generated, owing to (46), by the related Casimir functionals $h_1^{(s)} \in \mathrm{I}(\tilde{\mathcal{G}}_-^*), s = \overline{1,n}$, the following differential matrix relationships hold:

$$\begin{array}{c} [C_s, C_k] - \hbar\frac{\partial C_k}{\partial t_s} - \hbar\frac{\partial C_s}{\partial t_k} = 0, \;\; [T_s, T_k] = 0, \\ [T_s, C_k] + [C_s, T_k] - \hbar\frac{\partial C_k}{\partial t_s} - \hbar\frac{\partial C_s}{\partial t_k} = 0..., \end{array} \tag{55}$$

for all $s,k=\overline{1,n}$. Moreover, the matrices T_s, C_s and $l_j \in \mathrm{End}\,\mathbb{E}^n$, $j\in\mathbb{N}, s=\overline{1,n}$, satisfy the next supplementing hierarchy of matrix equations:

$$\begin{aligned} l_1 \mathrm{tr} C_s + \left(l_1 C_s + C_s^{\intercal} l_1\right) + l_2 \mathrm{tr} T_s + \left(l_2 T_s + T_s^{\intercal} l_2\right) = 0, \\ \hbar \tfrac{\partial l_1}{\partial t_k} = l_1 \mathrm{tr} C_k + l_1 C_k + C_k^{\intercal} + l_2 \mathrm{tr} T_k + l_2 T_k + T_k^{\intercal} l_2, \\ \hbar \tfrac{\partial l_2}{\partial t_k} = l_2 \mathrm{tr} C_k + l_2 C_k + C_k^{\intercal} l_2, l_1 \mathrm{tr} T_s + l_1 T_s + T_s^{\intercal} l_1 = 0, \ldots \end{aligned} \tag{56}$$

for the seed matrices $l_j \in \mathrm{End}\,\mathbb{E}^n, j\in\mathbb{N}$, where the Casimir functional–gradient matrices $T_s \in \mathrm{End}\,\mathbb{E}^n, s=\overline{1,n}$, are constant, not depending on the evolution parameters $t_s\in\mathbb{R}$, $s=\overline{1,n}$.

It is now easy to observe that the first line of the matrix differential–algebraic relationships (55) coincides exactly with the quantum structure constants' deformation equations in (42), thus solving, in part, the problem posed earlier, subject to the quantum Sato Grassmannian generalization modulo-determining abovementioned matrices $T_s, C_s \in \mathrm{End}\,\mathbb{E}^n$, $s=\overline{1,n}$, satisfying the determining algebraic Equations (56). The latter algebraic matrix problem proves, up to date, to be complicated enough to require one to develop more sophisticated algebraic–analytic tools and computational techniques. Nonetheless, to more deeply understand the quantum deformation structure of the Sato Grassmannians, we analyze below the inverse problem related with the problem under regard, which consists in determining a suitable seed element $\tilde{l}\in\tilde{\mathcal{G}}_-^*$, generating a priori an infinite hierarchy of linear vector fields on a torus that commute to each other and are related with suitably defined Lax-type integrable [28,41,45] dynamical systems on smooth functional manifolds.

5. The Quantum Grassmannian Deformation Structure: The Nonlinear Schrödinger Hierarchy Inverse Problem Aspects

Let us begin by recalling the classical Zakharov–Shabat result [41,45] about the differential–algebraic completely integrable Nonlinear Schrödinger-type equation:

$$\begin{aligned} u_{t_2} - (u_{t_1 t_1} - 2u^2 v)/2 = 0, \\ v_{t_2} + (v_{t_1 t_1} - 2uv^2)/2 = 0 \end{aligned} \tag{57}$$

and all their commuting to each other symmetries:

$$\begin{aligned} u_{t_3} = u_{t_1 t_1 t_1}/4 - 3uvu_{t_1}/2, \\ v_{t_3} = v_{t_1 t_1 t_1}/4 - 3uvv_{t_1}/2, \end{aligned} \tag{58}$$

$$\begin{aligned} u_{t_4} &= -u_{t_1 t_1 t_1 t_1}/8 + 3vu_{t_1}^2/4 + uvu_{t_1 t_1} - 3u^3v^2/4 + uv_{t_1}u_{t_1}/2 + u^2 v_{t_1 t_1}/4, \\ v_{t_4} &= v_{t_1 t_1 t_1 t_1}/8 + 3vu_{t_1}^2/4 - uvv_{t_1 t_1} + 3u^2v^3/4 - uv_{t_1}u_{t_1}/2 - v^2 u_{t_1 t_1}/4, \ldots \end{aligned}$$

and so on with respect to evolution parameters $t_j\in\mathbb{R}, j\in\mathbb{N}$, considered as evolution flows on the jet-manifold $J^\infty(\mathbb{R}^\infty;\mathbb{C}^2)$. The flows in (58) arise [41,45–47] as the compatibility conditions for the following affine evolution flows:

$$\frac{\partial x}{\partial t_1} = \begin{pmatrix} -\lambda & u \\ v & \lambda \end{pmatrix} x, \frac{\partial x}{\partial t_2} = \begin{pmatrix} -\lambda^2 + \frac{1}{2}uv & \lambda u - \frac{1}{2}u_{t_1} \\ \lambda v + \frac{1}{2}v_{t_1} & \lambda^2 - \frac{1}{2}uv \end{pmatrix} x, \tag{59}$$

$$\frac{\partial x}{\partial t_3} = \begin{pmatrix} -\lambda^3 + \frac{1}{2}\lambda uv + \frac{1}{4}(vu_{t_1} - uv_{t_1}) & \lambda^2 u - \lambda\frac{1}{2}u_{t_1} + \frac{1}{4}(u_{t_1 t_1} - 2u^2 v) \\ \lambda^2 v + \frac{1}{2}\lambda v_{t_1} + \frac{1}{4}(v_{t_1 t_1} - 2uv^2) & \lambda^3 - \frac{1}{2}\lambda uv - \frac{1}{4}(vu_{t_1} - uv_{t_1}) \end{pmatrix} x,$$

$$\frac{\partial x}{\partial t_4} = \begin{pmatrix} \begin{array}{c} -\lambda^4 + (2uv\lambda^2 + 2u\,v_{t_1}\lambda - \\ -u_{t_1}v\lambda)/4 + (u\,v_{t_1 t_1} - u_{t_1}v_{t_1} - \\ -3u^2v^2 + v\,u_{t_1 t_1})/8 \end{array} & \begin{array}{c} u\lambda^3 - (2u_{t_1}\lambda^2 + 2u^2 v \\ -u_{t_1 t_1}\lambda)/4 + (6vuu_{t_1} - u_{t_1 t_1 t_1})/8 \end{array} \\ \begin{array}{c} v\lambda^3 + (2v_{t_1}\lambda^2 + 2v_{t_1 t_1}\lambda - \\ -2uv^2\lambda)/4 + (v_{t_1 t_1 t_1} - 6uvv_{t_1})/8 \end{array} & \begin{array}{c} \lambda^4 - (2uv\lambda^2 + u\,v_{t_1}\lambda \\ -u_{t_1}v\lambda)/4 - (u\,v_{t_1 t_1} - u_{t_1}v_{t_1} - \\ -3u^2v^2 + v\,u_{t_1 t_1})/8 \end{array} \end{pmatrix} x, \ldots$$

on the torus $\mathbb{T}^2_{\mathbb{C}}$ with respect to the real temporal parameters $t_j \in \mathbb{R}, j \in \mathbb{N},$, depending on the parameter $\lambda \in \mathbb{C}$. These evolution flows generate the following equivalent hierarchy:

$$\begin{aligned}
X^{(t_1)} &= \frac{\partial}{\partial t_1} + (-\lambda x_1 + u x_2)\frac{\partial}{\partial x_1} + (v x_1 + \lambda x_2)\frac{\partial}{\partial x_2}, \\
X^{(t_2)} &= \frac{\partial}{\partial t_2} + [(-\lambda^2 + \frac{1}{2}uv)x_1 + (\lambda u - \frac{1}{2}u_{t_1})x_2]\frac{\partial}{\partial x_1} + \\
&+ [(\lambda v - \frac{1}{2}v_{t_1})x_1 - \frac{1}{4}(\lambda^2 - \frac{1}{2}uv)x_2]\frac{\partial}{\partial x_2}, \\
X^{(t_3)} &= \frac{\partial}{\partial t_3} + [((-\lambda^3 + \frac{1}{2}\lambda uv + \frac{1}{4}(uv_{t_1} - vu_{t_1}))x_1 + \\
&+ ((\lambda^2 u + \frac{i}{2}u_{t_1}) - \frac{1}{4}(u_{t_1t_1} - 2u^2 v))x_2]\frac{\partial}{\partial x_1} + \\
&+ [(\lambda^2 v - \frac{1}{2}\lambda v_{t_1}) + \frac{1}{4}(v_{t_1t_1} - 2uv^2))x_1 + \\
&+ ((\lambda^3 - \frac{1}{2}\lambda uv) - \frac{1}{4}(uv_{t_1} - vu_{t_1}))x_2]\frac{\partial}{\partial x_2},
\end{aligned} \tag{60}$$

of affine vector fields on the augmented torus $\mathbb{T}^2_{\mathbb{C}} \times \mathbb{R}^3$, commuting to each other, that is,

$$\left[X^{(t_1)}, X^{(t_2)}\right] = 0, \left[X^{(t_2)}, X^{(t_3)}\right] = 0, \left[X^{(t_3)}, X^{(t_1)}\right] = 0 \tag{61}$$

for all $t_1, t_2, t_3 \in \mathbb{R}$ and $x \in \mathbb{T}^2$. The latter makes it possible to describe mathematical properties of this countable hierarchy of commuting to each other vector fields within the Lie algebraic approach, devised before in [31,33,34] and, in part, formulated in the preceding section.

Namely, let $\mathbb{T}^2_{\mathbb{C}} := \mathbb{T}^2 \otimes \mathbb{C}$ and $\tilde{G} := \widetilde{Diff}(\mathbb{T}^2_{\mathbb{C}})$ denote the linear loop torus diffeomorphism group [35] of smooth mappings $\{\mathbb{C}^1 \supset \mathbb{S}^1 \to Diff(\mathbb{T}^2_{\mathbb{C}})\}$ of the unit circle $\mathbb{S}^1 \subset \mathbb{C}^1$, holomorphically extended to the inner part $\mathbb{D}_+ \subset \mathbb{C}^1$ of this circle $\mathbb{S}^1$ and to its outer part $\mathbb{D}_- \subset \mathbb{C}^1$, under the condition that, for any $\tilde{g}(\lambda), \lambda \in \mathbb{D}_-$, $\lim_{\lambda \to \infty} \tilde{g}(\lambda) = \mathrm{Id} \in Diff(\mathbb{T}^2_{\mathbb{C}})$. Then, the affine loop Lie algebra $\tilde{\mathcal{G}} := \widetilde{diff}(\mathbb{T}^2_{\mathbb{C}})$ splits as the direct sum $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ of the subalgebras, holomorphic, respectively, in the inner $\mathbb{D}_+$ and outer $\mathbb{D}_+$ parts of the unit circle $\mathbb{S}^1 \subset \mathbb{C}^1$. Consider also a countable hierarchy of smooth Casimir invariants $h^{(j)} \in \mathrm{I}(\tilde{\mathcal{G}}^*), j \in \mathbb{Z}_+$, on the adjoint space $\tilde{\mathcal{G}}^* \simeq \Lambda^1(\mathbb{T}^2_{\mathbb{C}})$ with respect to the bilinear form (43) on $\tilde{\mathcal{G}}^* \times \tilde{\mathcal{G}}$; assume that their gradients $\mathrm{grad}\, h^{(j)}(\tilde{l}) := \langle \mathrm{grad}\, h^{(j)}(l)x|\partial/\partial x\rangle \in \tilde{\mathcal{G}}, j \in \mathbb{Z}_+$, when calculated at the seed element $\tilde{l} = \langle lx|dx\rangle \in \tilde{\mathcal{G}}^*$ and projected on the subalgebra $\tilde{\mathcal{G}}_+$, coincide, respectively, as $|\lambda| \to \infty$ with the following matrix expressions:

$$\mathrm{grad}\, h_+^{(1)}(l) = \begin{pmatrix} -\lambda & u \\ v & \lambda \end{pmatrix}, \quad h_+^{(2)}(l) = \begin{pmatrix} -\lambda^2 + \frac{1}{2}uv & \lambda u - \frac{1}{2}u_{t_1} \\ \lambda v + \frac{1}{2}v_{t_1} & \lambda^2 - \frac{1}{2}uv \end{pmatrix}, \tag{62}$$

$$\mathrm{grad}\, h_+^{(3)}(l) = \begin{pmatrix} -\lambda^3 + \frac{1}{2}\lambda uv + \frac{1}{4}(vu_{t_1} - uv_{t_1}) & \lambda^2 u - \lambda\frac{1}{2}u_{t_1} + \frac{1}{4}(u_{t_1t_1} - 2u^2 v) \\ \lambda^2 v + \frac{1}{2}\lambda v_{t_1} + \frac{1}{4}(v_{t_1t_1} - 2uv^2) & \lambda^3 - \frac{1}{2}\lambda uv - \frac{1}{4}(vu_{t_1} - uv_{t_1}) \end{pmatrix},$$

$$\mathrm{grad}\, h^{(4)}(l) = \begin{pmatrix} \begin{matrix} -\lambda^4 + (2uv\lambda^2 + 2u\, v_{t_1}\lambda - \\ -u_{t_1}v\lambda)/4 + (u\, v_{t_1t_1} - u_{t_1}v_{t_1} - \\ -3u^2v^2 + v\, u_{t_1t_1})/8 \end{matrix} & \begin{matrix} u\lambda^3 - (2u_{t_1}\lambda^2 + 2u^2 v \\ -u_{t_1t_1}\lambda)/4 + (6vuu_{t_1} - u_{t_1t_1t_1})/8 \end{matrix} \\ \begin{matrix} v\lambda^3 + (2v_{t_1}\lambda^2 + 2v_{t_1t_1}\lambda - \\ -2uv^2\lambda)/4 + (v_{t_1t_1t_1} - 6uvv_{t_1})/8 \end{matrix} & \begin{matrix} \lambda^4 - (2uv\lambda^2 + u\, v_{t_1}\lambda \\ -u_{t_1}v\lambda)/4 - (u\, v_{t_1t_1} - u_{t_1}v_{t_1} - \\ -3u^2v^2 + v\, u_{t_1t_1})/8 \end{matrix} \end{pmatrix}, \ldots$$

and so on. Here, by definition, we put

$$\mathrm{grad}\, h_+^{(j)}(\tilde{l}) := \left(\lambda^j\, \mathrm{grad}\, h(\tilde{l})\right)|_+, \tag{63}$$

and denote by $h(\tilde{l}) \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ the related Casimir functional on $\tilde{\mathcal{G}}^*$, generated by a suitably chosen seed element

$$\tilde{l} := \sum_{j \in \mathbb{N},} \lambda^{-j} \langle l_j x | dx \rangle, \tag{64}$$

specified at any $m \in \mathbb{Z}_+$ by some matrix elements $l_j \in \mathrm{End}\,\mathbb{C}^n$, $j \in \mathbb{N}$, satisfying the determining algebraic relationship

$$ad^*_{\mathrm{grad}\, h(\tilde{l})} \tilde{l} = 0. \tag{65}$$

The gradient elements in (63), owing to the classical Adler–Kostant–Souriau scheme [28,41,42,44], generate the following vector fields that commute to each other:

$$\partial \tilde{l} / \partial t_s := ad^*_{\mathrm{grad}\, h_+^{(s)}(\tilde{l})} \tilde{l} \tag{66}$$

on the adjoint space $\tilde{\mathcal{G}}^*_-$, reducing to some differential-matrix equations on the matrix elements $l_j \in \mathrm{End}\,\mathbb{C}^n, j \in \mathbb{N}$:

$$\begin{gathered}
\partial l_0 / \partial t_s = l_0 \mathrm{tr} \varphi_{(s)} + (l_0 \varphi_{(s)} + \varphi^{\intercal}_{(s)} l_0) + \ldots + \\
+ l_s \mathrm{tr} \varphi_{(0)} + (l_s \varphi_{(0)} + \varphi^{\intercal}_{(0)} l_{s+1}), \\
\\
\partial l_1 / \partial t_s = l_1 \mathrm{tr} \varphi_{(s)} + (l_1 \varphi_{(s)} + \varphi^{\intercal}_{(s)} l_1) + \ldots \\
+ l_{s+1} \mathrm{tr} \varphi_{(0)} + (l_{s+1} \varphi_{(0)} + \varphi^{\intercal}_{(0)} l_{s+1}), \\
\\
\partial l_2 / \partial t_s = l_2 \mathrm{tr} \varphi_{(s)} \ + (l_2 \varphi_{(s)} + \varphi^{\intercal}_{(s)} l_2) + \\
+ l_3 \mathrm{tr} \varphi_{(s-1)} \ + (l_3 \varphi_{(s-1)} + \varphi^{\intercal}_{(s-1)} l_3) + \\
+ \ldots + l_{2+s} \mathrm{tr} \varphi_{(0)} \ + (l_{2+s} \varphi_{(0)} + \varphi^{\intercal}_{(0)} l_{2+s}), \\
\ldots \\
\partial l_3 / \partial t_s = l_3 \mathrm{tr} \varphi_{(s)} \ + (l_3 \varphi_{(s)} + \varphi^{\intercal}_{(s)} l_3) + l_4 \mathrm{tr} \varphi_{(s-1)} \ + \\
+ (l_4 \varphi_{(s-1)} + \varphi^{\intercal}_{(s-1)} l_4) + \ldots + (l_{s+3} \varphi_{(0)} + \varphi^{\intercal}_{(0)} l_{s+3}),
\end{gathered} \tag{67}$$

and so on with respect to evolution parameters $t_s \in \mathbb{R}, s \in \mathbb{Z}_+$, where we put, by definition, the asymptotic expansion

$$\mathrm{grad}\, h(\tilde{l}) \sim \sum_{j \in \mathbb{Z}_+} \lambda^{-j} \langle \varphi_{(j)}(l) x | \partial / \partial x \rangle \tag{68}$$

as $|\lambda| \to \infty$ for some matrix elements $\varphi_{(j)} \in \mathrm{End}\,\mathbb{C}^2, j \in \mathbb{Z}_+$. Having substituted Expansion (68) into Expression (65), the corresponding determining differential matrix relationship ensues:

$$\langle \frac{\partial}{\partial x} \circ | \varphi(l) x \rangle l x + \ \langle l x | \frac{\partial}{\partial x} \varphi(l) x \rangle = 0, \tag{69}$$

where, we put, by definition, $\mathrm{grad}\, h(\tilde{l}) := \langle \varphi(l) x | \partial / \partial x \rangle \in \tilde{\mathcal{G}}$ and $\tilde{l} := \langle l x | dx \rangle \in \tilde{\mathcal{G}}^*$. As a result of simple calculations, one easily obtains an infinite recurrent hierarchy of matrix algebraic relationships:

$$\sum_{j \in \mathbb{N}} l_j \mathrm{tr} \varphi_{(s-j)} + \sum_{s \in \mathbb{N}} l_{\ j}\ \varphi_{(s-j)} + \sum_{s \in \mathbb{N}} \varphi^{\intercal}{}_{(s-j)} l_j = 0 \tag{70}$$

for any $s \in \mathbb{Z}_+$, where, by definition, the standard matrix trace is denoted as $tr\varphi_{(j)}$, $j \in \mathbb{Z}_+$, and whose solution, that is, an infinite set of matrices $\{l_j \in \mathrm{End}\,\mathbb{C}^n : j \in \mathbb{N}\}$, gives rise to the searched seed element ub (64).

We now proceed to solving the system of matrix algebraic equations (70) for the case $m = 3$, which reduces to the following compatible matrix algebraic relationships:

$$\begin{gathered} l_1 \mathrm{tr}\varphi_{(0)} + (l_1\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_1) = 0, \\ l_1 \mathrm{tr}\varphi_{(1)} + (l_1\varphi_{(1)} + \varphi_{(1)}^{\intercal} l_1) + l_2 \mathrm{tr}\varphi_{(0)} + (l_2\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_2) = 0, \\ l_1 \mathrm{tr}\varphi_{(2)} + (l_1\varphi_{(2)} + \varphi_{(2)}^{\intercal}\, l_1) + l_2 \mathrm{tr}\varphi_{(1)} + \\ + l_2 \mathrm{tr}\varphi_{(1)} + \varphi_{(1)}^{\intercal} l_2) + l_3 \mathrm{tr}\varphi_{(0)} + (l_3\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_3) = 0, \\ l_1 \mathrm{tr}\varphi_{(4)} + (\varphi_{(4)}^{\intercal} l_1 + \varphi_{(4)}^{\intercal} l_1) + l_2 \mathrm{tr}\varphi_{(3)} + \\ + (l_2\varphi_{(3)} + \varphi_{(3)}^{\intercal} l_2) + l_3 \mathrm{tr}\varphi_{(2)} + (l_3\varphi_{(2)} + \varphi_{(2)}^{\intercal} l_3) + \\ + l_4 \mathrm{tr}\varphi_{(1)} + (l_4\varphi_{(1)} + \varphi_{(1)}^{\intercal} l_4) + l_5 \mathrm{tr}\varphi_{(0)} + (l_5\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_5), \dots \end{gathered} \tag{71}$$

and so on for the unknown matrices $l_j \in \mathrm{End}\,\mathbb{E}^2, j \in \mathbb{N}$. In addition, as the expressions in (62) belong to the Lie algebra $sl(2;\mathbb{R})$, we have to put $tr\varphi_{(j)} = 0$ for all $j \in \mathbb{Z}_+$, thus reducing (71) to the next algebraic relationships:

$$\begin{gathered} (l_1\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_1) = 0, \\ (l_1\varphi_{(1)} + \varphi_{(1)}^{\intercal} l_1) + + (l_2\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_2) = 0, \\ (l_1\varphi_{(2)} + \varphi_{(2)}^{\intercal}\, l_1) + (l_2\varphi_{(1)} + \varphi_{(1)}^{\intercal} l_2) + (l_3\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_3) = 0, \\ (\varphi_{(4)}^{\intercal} l_1 + \varphi_{(4)}^{\intercal} l_1) + (l_2\varphi_{(3)} + \varphi_{(3)}^{\intercal} l_2) + (l_3\varphi_{(2)} + \varphi_{(2)}^{\intercal} l_3) + \\ + (l_4\varphi_{(1)} + \varphi_{(1)}^{\intercal} l_4) + (l_5\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_5), \dots \end{gathered} \tag{72}$$

and so on. Also, take into account that matrices $\varphi_{(j)} \in \mathrm{End}\,\mathbb{C}^2, j \in \mathbb{Z}_+$, characterizing the differential–algebraic nonlinear Schrödinger Equation (57), are given by the following (62) matrix expressions:

$$\varphi_{(0)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi_{(1)} = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad \varphi_{(2)} = \begin{pmatrix} \frac{1}{2}uv & -\frac{1}{2}u_{t_1} \\ \frac{1}{2}v_{t_1} & -\frac{1}{2}uv \end{pmatrix}, \tag{73}$$

$$\varphi_{(3)} = \begin{pmatrix} \frac{1}{4}(vu_{t_1} - uv_{t_1}) & \frac{1}{4}(u_{t_1t_1} - 2u^2v) \\ \frac{1}{4}(v_{t_1t_1} - 2uv^2) & -\frac{1}{4}(vu_{t_1} - uv_{t_1}) \end{pmatrix},$$

$$\varphi_{(4)} = \begin{pmatrix} (uv_{t_1t_1} - u_{t_1}v_{t_1} - 3u^2v^2 + vu_{t_1t_1})/8 & (6uvu_{t_1} - u_{t_1t_1t_1})/8 \\ (v_{t_1t_1t_1} - 6uvv_{t_1})/8 & (-uv_{t_1t_1} + u_{t_1}v_{t_1} + 3u^2v^2 - vu_{t_1t_1})/8 \end{pmatrix}, \dots \tag{74}$$

and so on, a priori satisfying the differential-matrix relationships (51). The matrix elements $l_j \in \mathrm{End}\,\mathbb{C}^2$, $j \in \overline{-3,0} \cup \mathbb{N}$, respectively, satisfy the infinite hierarchy of the following differential-matrix relationships:

$$\begin{gathered} -\partial l_1/\partial t_s = (l_1\varphi_{(s)} + \varphi_{(s)}^{\intercal} l_1) + (l_2\varphi_{(s-1)} + \varphi_{(s-1)}^{\intercal} l_2) + \\ + (l_3\varphi_{(s-2)} + \varphi_{(s-2)}^{\intercal} l_3) + (l_4\varphi_{(s-3)} + \varphi_{(s-3)}^{\intercal} l_4) + \\ + (l_5\varphi_{(s-4)} + \varphi_{(s-4)}^{\intercal} l_5) + \dots + (l_{s+1}\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_{s+1}) + \\ -\partial l_2/\partial t_s = (l_2\varphi_{(s)} + \varphi_{(s)}^{\intercal} l_2) + (l_3\varphi_{(s-1)} + \varphi_{(s-1)}^{\intercal} l_3) + \\ + (l_4\varphi_{(s-2)} + \varphi_{(s-2)}^{\intercal} l_4) + \dots + (l_{s+2}\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_{s+2}), \\ -\partial l_3/\partial t_s = (l_3\varphi_{(s)} + \varphi_{(s)}^{\intercal} l_3) + (l_4\varphi_{(s-1)} + \varphi_{(s-1)}^{\intercal} l_4) + \\ + \dots + (l_{s+3}\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_{s+3}), \quad \dots \end{gathered} \tag{75}$$

easily ensuing from (67). Subject to the unknown matrices $l_j \in \operatorname{End} \mathbb{C}^2$, $j \in \mathbb{N}$, we solve recurrently the algebraic matrix relationships (72) jointly with the differential matrix equalities

$$\begin{aligned} -\partial l_1/\partial t_1 &= (l_1\varphi_{(1)} + \varphi_{(1)}^{\intercal} l_1) + (l_2\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_2), \\ -\partial l_2/\partial t_1 &= (l_2\varphi_{(1)} + \varphi_{(1)}^{\intercal} l_2) + (l_3\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_3), \\ -\partial l_3/\partial t_1 &= (l_3\varphi_{(1)} + \varphi_{(1)}^{\intercal} l_3) + (l_4\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_4), \\ -\partial l_4/\partial t_1 &= (l_4\varphi_{(1)} + \varphi_{(1)}^{\intercal} l_4) + (l_5\varphi_{(0)} + \varphi_{(0)}^{\intercal} l_5), \ldots \end{aligned} \tag{76}$$

and so on, following from (75) at $s = 1$, and obtain the following matrix expressions:

$$\begin{aligned} & l_1 = \begin{pmatrix} 0 & \alpha - \eta_1 \\ \alpha + \eta_1 & 0 \end{pmatrix}, l_2 = \begin{pmatrix} \alpha v & \beta - \eta_2 \\ \beta + \eta_2 & -\alpha u \end{pmatrix}, \\ & l_3 = \begin{pmatrix} \alpha v_{t_1}/2 + \beta v & -\alpha uv/2 + \gamma - \eta_3 \\ -\alpha uv/2 + \gamma + \eta_3 & \alpha u_{t_1}/2 - \beta u \end{pmatrix}, \\ & l_4 = \begin{pmatrix} \alpha v_{t_1 t_1}/4 + \beta v_{t_1}/2 - \alpha u v^2/2 + \gamma v & -\alpha(u v_{t_1} - v u_{t_1})/4 - \eta uv/2 \\ -\alpha(u v_{t_1} - v u_{t_1})/4 - \eta uv/2 & \alpha u^2 v/2 - \alpha u_{t_1 t_1}/4 + \beta u_{t_1}/2 - \gamma u \end{pmatrix}, \ldots \end{aligned} \tag{77}$$

and so on, where α, β, γ and $\eta_j \in \mathbb{R}$, $j = \overline{1,3}$, are arbitrary constant parameters.

Recall now that, following the Adler–Kostant–Souriau scheme [28,41,42,44], the constructed evolution flows (66) are Hamiltonian flows that commute to each other with respect to the standard Lie–Poisson bracket:

$$\{\gamma(\tilde{l}), \mu(\tilde{l})\} := (\tilde{l} | [\operatorname{grad}_+ \gamma(\tilde{l}), \operatorname{grad}_+ \mu(\tilde{l})] - [\operatorname{grad}_- \gamma(\tilde{l}), \operatorname{grad}_- \mu(\tilde{l})]) \tag{78}$$

for arbitrary smooth functionals $\gamma, \mu \in D(\tilde{\mathcal{G}}_+^*)$ on the adjoint space $\tilde{\mathcal{G}}_+^*$ and generated by the corresponding Casimir invariant functionals, where $(\ldots)_\pm$ denotes the projection upon the loop Lie subalgebras $\tilde{\mathcal{G}}_\pm$. Moreover, we observe that the first two affine linear flows of the countable hierarchy (48) satisfy the non-commutative quantum deformation type relationships (51), subject to the two-dimensional Sato Grassmannian associative algebra. The result obtained above, solving the corresponding inverse problem for the infinite commuting to each other hierarchy of affine vector fields (59) on the torus $\mathbb{T}^2$, can be reformulated as the following theorem.

Theorem 2. *The infinite completely integrable nonlinear Schrödinger-type hierarchy of evolution flows (58) is equivalent to a hierarchy of commuting to each other orbit flows generated by the co-adjoint action of a special loop diffeomorphism group of the torus $\mathbb{T}^2$ on the adjoint space $\tilde{\mathcal{G}}_-^*$ to its affine loop Lie algebra $\tilde{\mathcal{G}} \simeq \widetilde{diff}(\mathbb{T}^2)$. The related first two affine linear flows of (59) on the torus $\mathbb{T}^2$ describe a non-commutative quantum deformation of the two-dimensional Sato Grassmannian associative algebra.*

Proof. Taking into account Proposition 2 and the linear differential–algebraic relationships (51), one obtains the corresponding matrix structure constants, specified by means of the first two affine linear flows on the torus $\mathbb{T}^2$, determining a suitable quantum deformation of the Sato Grassmannian associative algebra. □

It is worth remarking here that the iterative differential-matrix scheme described above, based on Relationships (72) and (75) and applied to the the suitably reduced generating matrix seed element $l_1 \in \operatorname{End} \mathbb{E}^2$, gives rise to the well-known countable KP-hierarchy of completely integrable Hamiltonian systems. A similar statement can also be proven for the case of a matrix seed element $l_1 \in \operatorname{End} \mathbb{E}^3$, generating the well-known countable hierarchy of integrable two-component Manakov-type Hamiltonian systems.

6. Conclusions

Inspired by recent investigations of the Sato Grassmannian and its deep connections with description of the Frobenius-type manifolds, initiated by B. Dubrovin, we analyzed these within a special Adler–Kostant–Symes approach to construction of infinite hierarchies of integrable matrix flows as co-adjoint orbits of a special subgroup of the loop diffeomorphism group of tori. The studied affine Lie subalgebras of linear vector fields on tori made it possible, in part, to describe some kinds of Frobenius manifolds within the Dubrovin-type reformulation of the well-known WDVV associativity equations, previously derived in topological field theory. Based on studying a related Lax-type spectral problem, we have stated that these equations are equivalent to some bi-Hamiltonian flows on a smooth functional submanifold with respect to two compatible Poisson structures, generating a countable hierarchy of hydrodynamic flows that commute to each other. We also studied the inverse problem aspects of the quantum Sato Grassmannian structure constants' deformations, related to the well-known countable hierarchy of the higher-order nonlinear Schrödinger-type completely integrable evolution flows.

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