International Journal of Non–Linear Mechanics 163 (2024) 104728

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Solutions for the unsteady motion of porous elastic solids within the context of an implicit constitutive theory

K.R. Rajagopal [a], Y. Şengül [b,*]

[a] Department of Mechanical Engineering, Texas A&M University, College Station, TX, USA
[b] Cardiff University, School of Mathematics, Senghenydd Road, Cardiff, CF24 4AG, UK

ARTICLE INFO

Keywords:
Implicit constitutive relations
Porous solids
Linearized strain
Travelling waves
Nonlinear differential equations

ABSTRACT

In this work we investigate well-posedness for the partial differential equation stemming from the balance of linear momentum for an implicit constitutive relation that can describe the response of porous elastic solids whose material moduli depend on the density. We study heteroclinic travelling waves and obtain closed form analytic solutions for the resulting ordinary differential equation. Finally, we consider some special solutions for the partial differential equation and solve the resulting equations numerically.

1. Introduction

Recently, there has been considerable interest in the use of implicit algebraic constitutive relations to describe the response of materials. By implicit "algebraic constitutive relations" we mean constitutive relations which in the case of solids represents an implicit algebraic relationship between the stress and the deformation gradient, no derivatives of the stress or the deformation gradient being involved, and in the case of fluids an implicit algebraic relationship between the stress and the symmetric part of the velocity gradient, once again with no derivatives of the stress or the symmetric part of the gradient being involved. More general implicit constitutive relations have been in place for a long time. For instance, with regard to fluids, Burgers [1] introduced a constitutive relation for viscoelastic fluids that involved the stress and its material time derivatives with the symmetric part of the velocity gradient and its material time derivative. Later, Oldroyd (1950) introduced a systematic methodology wherein he was able to establish many implicit relationships between the stress and its time derivative with the symmetric part of the velocity gradient and its time derivative. With regard to solids, implicit relationships have an even earlier reference. A classic implicit constitutive relation is a "rigid-perfectly plastic" body, wherein one can neither express the stress as a function of the strain, or the strain as a function of the stress. All constitutive relations wherein the notion of "plastic strain" is introduced is invariably an implicit constitutive relationship.[1]

Rajagopal [2,3,4] introduced implicit algebraic constitutive relations to describe the response of both solids and fluids. With regard to elastic solids, such implicit constitutive relations are able to describe the nonlinear behaviour of metallic alloys, cement concrete and rocks even when they are subject to small strains. In [4,5], it was shown that linearization under the assumption of small displacement gradient can lead to a nonlinear relationship between the linearized strain and the stress, which is an impossibility within the context of the classical Cauchy theory of elasticity. These relations can also describe the response of porous materials like sintered metals, ceramics, bone, rocks and concrete whose material moduli depend on the density and the mean normal stress (mechanical pressure[2]) (see [7,8]). Last but not least, those implicit constitutive relations present the possibility of strain-limiting behaviour as observed in biological fibres, DNA (see [9, 10]) and many other additional beneficial characteristics which allow one to describe bounded stresses at the tips of notches, crack tips and fracture (see [5,11–15]).

Viscoelastic solids described by algebraic implicit equations have been studied by Şengül and co-workers within the context of wave propagation (see [16–22]). Many of these studies were concerned with strain-limiting viscoelastic solids that allow for bounded strains at crack tips as the strains can be limited a priori. Rajagopal and Wineman [23] introduced constitutive relations for a class of viscoelastic bodies whose

* Corresponding author.
E-mail addresses: krajagopal@tamu.edu (K.R. Rajagopal), sengultezely@cardiff.ac.uk (Y. Şengül).

[1] Plasticity/inelasticity as developed by Tresca, von Mises, Prandtl, Reuss, Huber, Nadai and others wherein repeated loading and unloading of inelastic body that yields takes place automatically leads to implicit relationships between the stress and strain.

[2] The term pressure in often times confused with the Lagrange multiplier that appears due to the constraint of incompressibility, especially in nonlinear theories of fluids and solids (see [6]).

https://doi.org/10.1016/j.ijnonlinmec.2024.104728
Received 1 December 2023; Received in revised form 5 April 2024; Accepted 6 April 2024
Available online 15 April 2024

material properties are density dependent, and Bustamante and Rajagopal [24] developed a constitutive relation wherein the stress, strain and strain rate all appear linearly.

With regard to fluids, implicit algebraic constitutive relations can describe the response of fluids with pressure and shear-dependent viscosity (see [25–27]), colloidal solutions and suspensions (see [28,29]).[3] A detailed discussion of implicit algebraic constitutive relations used to describe the response of fluids can be found in the recent review article by Rajagopal [31].

Our study in this paper concerns the investigation of implicit constitutive relations developed to describe the response of porous solids whose material moduli depend on the density, a class of constitutive relations recently introduced by Rajagopal [7]. This constitutive relation has been studied by Murru and Rajagopal [32], Murru and Rajagopal [33], Vajipeyajula et al. [34,35], and a further simplification of the constitutive relation by thresholding as carried out by Itou et al. [14,15], ErbaŞ et al. [36]. We study the propagation of waves in a body described by the constitutive relation in [7] (see also [37] for a related problem). While studying problems concerned with implicit equations, since the stress and strain are implicitly related, one cannot in general substitute the expression for the stress into the balance of linear momentum and obtain a single equation for the displacement field and then solve for the displacement field. However, when one makes specific assumptions for the nature of the stress field and displacement field, the general system might be reducible to a single equation, for instance if the stress and the displacement field on one space variable and time, as in the case of our study. It is imperative to recognize that such a simplification is not always possible. Moreover, it is also possible that in the process of obtaining a single equation we might increase the order of the equation and we might not have an adequate number of boundary or initial conditions. In problems involving infinite domains, such a difficulty might not arise. Kannan et al. [38] studied the problem of wave propagation by considering the system of the balance of linear momentum and the constitutive relation simultaneously. Here, in this study, we are able to reduce the problem to a single equation and then investigate the possibility of travelling wave solutions as well as some special solutions in such a material.

The structure of this note is as follows. In Section 3 we will derive the partial differential equation for stress that is going to be studied in this work using the constitutive relation and the equation of motion. In Section 4 we will investigate travelling wave solutions for this equation which as a result of considering the travelling wave variable reduces to an ordinary differential equation. In Section 5 we study special solutions for the partial differential equation appealing to a similarity transformation as a result of which we obtain two ordinary differential equations and solve using Matlab.

2. Kinematics

In this section we shortly introduce the notation we use throughout the paper. Let $\mathbf{u}(\mathbf{x}, t)$ be the displacement of the body at the current position $\mathbf{x} \in \mathbb{R}^3$ of a particle $\mathbf{X}$ in the reference configuration at time t. That is, $\mathbf{u} = \mathbf{x} - \mathbf{X}$. We can define the deformation of the body, which is assumed to be stress free initially, as $\chi(\mathbf{X}, t)$ so that the deformation gradient is defined as $\mathbf{F} = \partial \chi / \partial \mathbf{X}$. Having the deformation gradient, we can ensure existence of positive definite, symmetric tensors $\mathbf{U}$ and $\mathbf{V}$, and a rotation $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R},$$

where $\mathbf{U}$ is *the right* and $\mathbf{V}$ is *the left Cauchy–Green stretch tensor*. Moreover, we know that each of these decompositions is unique and

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}, \quad \mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T,$$

[3] Pruša and Rajagopal [30] developed very general constitutive relations for fluids wherein one has an implicit functional relationship between the history of the stress and the history of the relative deformation gradient.

where $\mathbf{B}, \mathbf{C}$ are called *the right* and *the left Cauchy–Green deformation tensors*, respectively. Under the assumption that

$$\max_{\mathbf{x},t} |\nabla \mathbf{u}| = O(\delta), \qquad \delta \ll 1, \tag{1}$$

one can obtain the linearized strain as

$$\epsilon = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right], \tag{2}$$

where $\nabla \mathbf{u} := \partial \mathbf{u} / \partial \mathbf{x}$ and $|\cdot|$ stands for the usual trace norm.

3. Derivation of the equation

As described by Rajagopal [7], for isotropic elastic materials, modelling via implicit constitutive theory begins with the relation

$$\mathrm{f}(\rho, \mathbf{T}, \mathbf{B}) = 0, \tag{3}$$

where ρ is the mass density and $\mathbf{T}$ is the Cauchy stress. The representation of f in terms of the material moduli and the principle invariants (see [19] for a derivation using the Cayley–Hamilton Theorem) is given by

$$\begin{aligned} \mathrm{f}(\rho, \mathbf{T}, \mathbf{B}) = {} & \chi_0 \mathbf{I} + \chi_1 \mathbf{T} + \chi_2 \mathbf{B} + \chi_3 \mathbf{T}^2 + \chi_4 \mathbf{B}^2 \\ & + \chi_5(\mathbf{T}\mathbf{B} + \mathbf{B}\mathbf{T}) + \chi_6(\mathbf{T}^2\mathbf{B} + \mathbf{B}\mathbf{T}^2) \\ & + \chi_7(\mathbf{B}^2\mathbf{T} + \mathbf{T}\mathbf{B}^2) + \chi_8(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2), \end{aligned} \tag{4}$$

where the χ's are depending on ρ and the scalar invariants of $\mathbf{T}$ and $\mathbf{B}$ expressible in terms of

$$\mathrm{tr}\,\mathbf{T}, \mathrm{tr}\,\mathbf{B}, \mathrm{tr}\,\mathbf{T}^2, \mathrm{tr}\,\mathbf{B}^2, \mathrm{tr}\,\mathbf{T}^3, \mathrm{tr}\,\mathbf{B}^3, \mathrm{tr}\,\mathbf{T}\mathbf{B}, \mathrm{tr}\,\mathbf{T}^2\mathbf{B}, \mathrm{tr}\,\mathbf{T}\mathbf{B}^2, \mathrm{tr}\,\mathbf{T}^2\mathbf{B}^2.$$

Depending on the properties of f, it might be possible by (3) and (4) to write the stress as a function of the strain, as it is done in classical elasticity, or the strain as a function of the stress, as expected by the causality argument.

After linearizing the strain under the assumption (1), $\mathbf{B}$ becomes $\mathbf{B} = \mathbf{I} + 2\epsilon + O(\delta^2)$, where ϵ is the linearized strain as defined in (2). As a result, (3) and (4) lead to

$$\epsilon + \hat{\chi}_0 \mathbf{I} + \hat{\chi}_1 \mathbf{T} + \hat{\chi}_2 \mathbf{T}^2 + \hat{\chi}_3(\mathbf{T}\epsilon + \epsilon\mathbf{T}) + \hat{\chi}_4(\mathbf{T}^2\epsilon + \epsilon\mathbf{T}^2) = \mathbf{0}, \tag{5}$$

where the $\hat{\chi}_i$'s for $i = 0, 1, 2$, are scalar-valued functions depending at most linearly on ϵ and arbitrarily on the invariants of $\mathbf{T}$, while for $i = 3, 4$, they are functions of the invariants of $\mathbf{T}$ only (cf. [39]). Obviously, this is a nonlinear relationship between the linearized strain and the stress, which is the novelty of this approach observed by Rajagopal. Moreover, this phenomenon is observed in the response of various materials including metallic alloys and concrete (see e.g., [7] and references therein), and the classical linearized theory is incapable of describing the observed nonlinear behaviour.

Returning to (4), as explained by Rajagopal [7], implicit constitutive theory clearly allows the material moduli to depend on the density, the invariants of the stress, and the invariants of the linearized strain, as well as mixed invariants. However, this dependence has to be very special due to the requirement that the constitutive relation is linear in the linearized strain. Therefore, since the density also depends on the $\mathrm{tr}\,\epsilon$, the requirement that our model depends linearly on ϵ restricts the way the coefficients in (5) depend on the density and $\mathrm{tr}\,\epsilon$. Moreover, under the restriction that both the stress and the linearized strain should appear at most linearly in the constitutive relation, we obtain (see equation (7) in [7])

$$\begin{aligned} & \epsilon + A_1 \mathbf{T} + A_2(\mathrm{tr}\epsilon)\mathbf{T} + A_3(\mathrm{tr}\mathbf{T})\mathbf{I} + A_4(\mathrm{tr}\epsilon)\mathbf{I} + A_5(\mathrm{tr}\mathbf{T})\epsilon \\ & + A_6(\mathrm{tr}\mathbf{T})(\mathrm{tr}\epsilon)\mathbf{I} + A_7(\epsilon\mathbf{T} + \mathbf{T}\epsilon) + A_8 \mathrm{tr}(\epsilon\mathbf{T})\mathbf{I} = \mathbf{0}, \end{aligned} \tag{6}$$

where A_i, $i = 1, \ldots, 8$ are constants. Under assumption (1), one can approximate the balance of mass as

$$\rho_R = \rho(1 + \mathrm{tr}\,\epsilon), \tag{7}$$

where ρ_R and ρ are the densities in the reference and deformed configurations, respectively. This allows us to replace $\mathrm{tr}\epsilon$ by a ratio of the current and the reference densities. This implies that the coefficients in (6) are density-dependent material moduli, which brings us to modelling of porous materials as required.

In particular, we consider a special subclass of (6), which is the constitutive relation given in [7] (equation (22)), namely,

$$\epsilon = B_1(1+\lambda_2(\mathrm{tr}\epsilon))\mathbf{T} + B_2(1+\lambda_3(\mathrm{tr}\epsilon))(\mathrm{tr}\mathbf{T})\mathbf{I}, \tag{8}$$

where ϵ is the linearized strain as before, $\mathbf{T}$ is the Cauchy stress, and B_1, B_2 and λ_2, λ_3 are constants. It is important to note that when $\lambda_2 = \lambda_3 = 0$ and

$$B_1 = \frac{1+\nu}{E}, \quad B_2 = -\frac{\nu}{E}, \tag{9}$$

where E is the Young's modulus and ν is the Poisson's ratio, the classical linearized elastic model is recovered. Here, we work in one space dimension so that the constitutive relation (8) takes the form

$$\epsilon = B_1(1+\lambda_2\epsilon)\sigma + B_2(1+\lambda_3\epsilon)\sigma, \tag{10}$$

where σ stands for the one-dimensional stress. Defining $\alpha = (B_1\lambda_2 + B_2\lambda_3)$ with B_1 and B_2 as in (9), we obtain the system, given below, consisting of the balance of linear momentum and the reduced constitutive relation, under the assumption that gravity can be ignored:

$$\begin{aligned} \rho u_{tt} &= \sigma_x, \\ \epsilon &= \frac{\sigma}{E} + \alpha\,\epsilon\,\sigma, \end{aligned} \tag{11}$$

where $\epsilon = \epsilon(x,t)$ is the linearized strain, $\sigma = \sigma(x,t)$ is the stress, and $u = u(x,t)$ is the displacement. From the second equation in (11) we can deduce that

$$\epsilon = \frac{\sigma}{E}\frac{1}{(1-\alpha\sigma)}, \tag{12}$$

since E is a non-zero constant.

At this juncture, we make a further approximation that greatly simplifies the governing equations of the problem under consideration. While the norm of the linearized strain in $O(\delta)$, $\delta \ll 1$, the product $\lambda_i(\mathrm{tr}\epsilon)$, $i = 1,2$, is comparable to 1, that is $\lambda_i(\mathrm{tr}\epsilon)$, $i = 1,2$, cannot be neglected with respect to 1. This is because λ_i are large. On the other hand, since $(\mathrm{tr}\epsilon) = O(\delta)$, $\delta \ll 1$, it can be neglected in comparison to 1 and thus the balance of mass leads to the density ρ being approximated by ρ_R which is assumed to be a constant in the reference configuration (as in the case of linearized elasticity).

For convenience, we non-dimensionalize (11) and consider the one given in terms of the dimensionless quantities instead. Assuming enough regularity so that differentiation in the first equation in (11) is possible, we can use (12) to replace $u_x = \epsilon$ giving a single partial differential equation as

$$\rho_R\left(\frac{\sigma}{E}\frac{1}{(1-\alpha\sigma)}\right)_{tt} = \sigma_{xx}. \tag{13}$$

Since E and α are constants, they do not depend on time so that we can rewrite (13) as

$$\sigma_{tt} = \frac{E}{\rho_R}\sigma_{xx}(1-\alpha\sigma)^2 - 2\alpha\frac{\sigma_t^2}{1-\alpha\sigma}. \tag{14}$$

Firstly, it is worth noting that Eq. (14) is an equation of motion given in terms of the stress, rather than the displacement. While this is interesting in its own right, it also prevents one from having any sort of intuition about possible expressions for the energy of the system, dissipation, and the like. Similarly, in [38], the balance of linear momentum was coupled to a nonlinear constitutive relation between the linearized strain and the stress, and the coupling was eliminated leading to a higher order nonlinear hyperbolic equation for the stress. However, in [38], authors preferred investigating the coupled system rather than this higher order equation for the stress. In this work, we follow the other route.

Moreover, Eq. (14) is in the form $\sigma_{tt} = \mathcal{F}(\sigma_{xx}, \sigma, \sigma_t)$ with nonlinear $\mathcal{F}$ for which there is no systematic way of proving existence of solutions. It is worth noting that when $\alpha = 0$, Eq. (14) becomes a wave equation for the stress, which is expected since in this case we have classical linearized elastic body. However, due to the fact that the unknown is the stress, we have a stress wave equation. Moreover, the interesting case to consider is when α is not zero, which is the case we investigate here. Therefore, for the rest of the paper, we assume that E and α are non-zero constants.

4. Travelling wave solutions

In this section we focus on travelling wave solutions of (14) corresponding to the heteroclinic connections between two constant states (defined as K^+ and K^- below). We define $\xi = x - ct$ as the travelling wave variable, where c is the wave speed. This reduces (14) to a nonlinear ordinary differential equation for σ. Since in this case $\sigma = \sigma(\xi)$, differentiating with respect to time t and space variable x in (14) leads to

$$c^2\sigma'' = \frac{E}{\rho_R}\sigma''(1-\alpha\sigma)^2 - 2\alpha c^2\frac{(\sigma')^2}{1-\alpha\sigma}. \tag{15}$$

Clearly, $\sigma(\xi) \equiv$ constant is a solution to (15). For boundary conditions, we will assume

$$\lim_{\xi\to+\infty}\sigma(\xi) = K^+, \quad \lim_{\xi\to-\infty}\sigma(\xi) = K^-,$$

where $K^+ \neq K^-$. Assuming that $\sigma' \neq 0$, we can rewrite Eq. (15) to get

$$\frac{\sigma''}{\sigma'} = -\frac{2\alpha c^2\sigma'}{(1-\alpha\sigma)\left(c^2 - \frac{E}{\rho_R}(1-\alpha\sigma)^2\right)}.$$

For the right-hand side, using partial fractions, we obtain

$$\frac{\sigma''}{\sigma'} = \frac{\alpha c\sigma'}{(\alpha\sigma-1)\sqrt{E/\rho_R}}\left\{\frac{1}{(\alpha\sigma-1)+\frac{c}{\sqrt{E/\rho_R}}} - \frac{1}{(\alpha\sigma-1)-\frac{c}{\sqrt{E/\rho_R}}}\right\}.$$

We are now in a position to take the integral of both sides with respect to ξ. Calculating the integral of the right-hand side we obtain

$$\log|\sigma'(\xi)/C_1| = \log\left|\frac{(\alpha\sigma-1)}{(\alpha\sigma-1)+\frac{c}{\sqrt{E/\rho_R}}}\right| + \log\left|\frac{(\alpha\sigma-1)}{(\alpha\sigma-1)-\frac{c}{\sqrt{E/\rho_R}}}\right|,$$

where $C_1 > 0$ is the constant of integration. This is equivalent to

$$\sigma'(\xi) = C_1\left(1 - \frac{c^2\rho_R}{E(\alpha\sigma(\xi)-1)^2}\right). \tag{16}$$

Now, using the boundary conditions given by $\sigma'(\xi) \to 0$ as $\xi \to \pm\infty$, we obtain $c^2 = \frac{E}{\rho_R}(\alpha K^+ - 1)^2 = \frac{E}{\rho_R}(\alpha K^- - 1)^2$ leading to

$$K^+ = K^- \quad \text{or} \quad K^+ + K^- = \frac{2}{\alpha}. \tag{17}$$

Since we are looking for heteroclinic travelling waves, the first option in (17) does not apply in our situation. Therefore, we will only consider the second option. Also, we can calculate the wave speed as

$$c = \sqrt{\frac{1}{2}\left(\frac{E}{\rho_R}(\alpha K^+ - 1)^2 + \frac{E}{\rho_R}(\alpha K^- - 1)^2\right)}.$$

Considering the values of $\sigma(0)$ which will be determined later, we can easily find from (16) that

$$C_1 = \sigma'(0)\frac{E(\alpha\sigma(0)-1)^2}{E(\alpha\sigma(0)-1)^2 - c^2\rho_R}.$$

Denoting $\sigma'(\xi) = \frac{1}{\alpha}\frac{d}{d\xi}(\alpha\sigma(\xi)-1)$, Eq. (16) is equivalent to the equation given by

$$z'(\xi) = C_1\alpha - \frac{C_1\alpha c^2\rho_R}{Ez^2(\xi)},$$

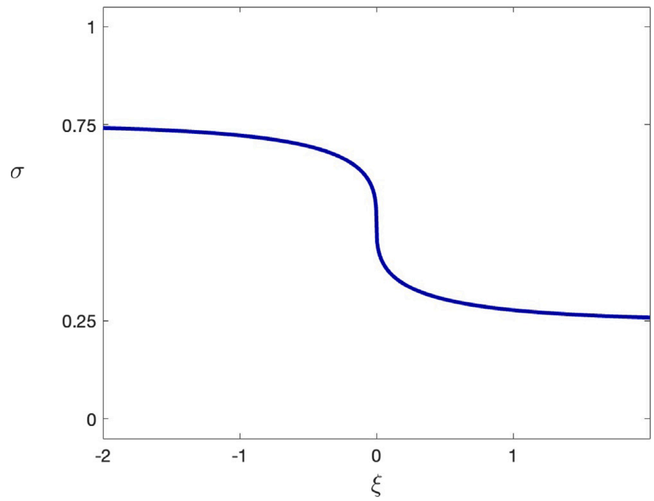

Fig. 1. Travelling wave profile $\sigma(\xi)$ given in (18) when the choices of $\sigma'(0) = 0.1, \rho_R = 1\ \mathrm{kg/m^3}, \alpha = 2\,\mathrm{m^2/N}, E = 0.01\,\mathrm{Pa}, K^- = 0.75$ and $K^+ = 0.25$ are made.

where we denote $z = (\alpha\sigma - 1)$. Separating the variables, we obtain

$$\frac{z^2 dz}{z^2 - \frac{c^2\rho_R}{E}} = C_1 \alpha d\xi.$$

Integrating both sides with respect to the corresponding variables gives

$$z - \frac{c}{\sqrt{E/\rho_R}} \tanh^{-1}\left(\frac{\sqrt{E/\rho_R}}{c} z\right) = C_1\alpha\xi + C_2,$$

where C_2 is another constant of integration. Recalling that $z = \alpha\sigma - 1$, we find the travelling wave profile as

$$\sigma(\xi) - \frac{c}{\sqrt{E/\rho_R}\alpha} \tanh^{-1}\left(\frac{\sqrt{E/\rho_R}}{c}(\alpha\sigma(\xi) - 1)\right) = C_1\xi + \frac{C_2 + 1}{\alpha}. \tag{18}$$

This gives the solution of (15) in closed form (see Fig. 1 for the travelling profile corresponding to (18)).

5. Special solutions to the equation of motion

In this section, we will investigate Eq. (14) as a nonlinear partial differential equation. Similar to the argument in the previous section, since α is a constant, we have $\sigma_t = \frac{1}{\alpha}\frac{\partial}{\partial t}(\alpha\sigma - 1)$ and $\sigma_{tt} = \frac{1}{\alpha}\frac{\partial^2}{\partial^2 t}(\alpha\sigma - 1)$. These lead to rewriting (14) as

$$\left(\alpha\sigma - 1\right)_{tt} = \frac{E}{\rho_R}\left(\alpha\sigma - 1\right)_{xx}\left(\alpha\sigma - 1\right)^2 + 2\frac{\left(\alpha\sigma - 1\right)_t^2}{\left(\alpha\sigma - 1\right)}. \tag{19}$$

Now, defining $\eta(x,t) := \alpha\sigma(x,t) - 1$, we can rewrite (19) as

$$\eta_{tt} = \frac{E}{\rho_R}\eta_{xx}\eta^2 + 2\frac{\eta_t^2}{\eta}. \tag{20}$$

We will consider special solutions to Eq. (20) by assuming that the variables of η can be separated, that is, $\eta(x,t) = \mathcal{X}(x)\mathcal{T}(t)$. In this case, Eq. (20) yields

$$\mathcal{X}\mathcal{T}'' = \frac{E}{\rho_R}\mathcal{T}^3\mathcal{X}^2\mathcal{X}'' + 2\frac{(\mathcal{X}\mathcal{T}')^2}{\mathcal{X}\mathcal{T}},$$

which is equivalent to

$$\frac{\mathcal{T}''\mathcal{T} - 2(\mathcal{T}')^2}{\mathcal{T}^4} = \frac{E}{\rho_R}\mathcal{X}\mathcal{X}''.$$

Clearly, this leads to

$$\mathcal{X}\mathcal{X}'' = \frac{\lambda\rho_R}{E}, \qquad \frac{\mathcal{T}''\mathcal{T} - 2(\mathcal{T}')^2}{\mathcal{T}^4} = \lambda, \tag{21}$$

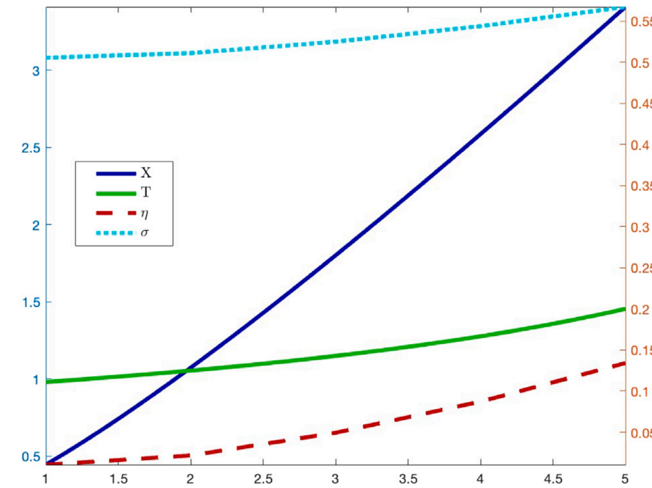

Fig. 2. Functions $\mathcal{X}$ and $\mathcal{T}$ in Eq. (21) are obtained using the package ode45 in Matlab by making the choices $\lambda = 0.001$, $\rho_R = 1\ \mathrm{kg}/m^3$, and $E = 0.01\,\mathrm{Pa}$. since Eqs. (21) are second order ordinary differential equations, some initial conditions on the first derivatives are required to be put into the code. The conditions chosen for this figure were $\mathcal{X}(0) = 0.1$, $\mathcal{X}'(0) = 0$ and $\mathcal{T}(0) = 0.1$, $\mathcal{T}'(0) = 0.01$. Once $\mathcal{X}$ and $\mathcal{T}$ are found, η and σ are graphed using their respective definitions.

for some constant λ. Solving these ordinary differential equations in Matlab we obtain the solution functions $\mathcal{X}$ and $\mathcal{T}$. From these, one can find η and by using the definition of $\eta = \alpha\sigma - 1$, one can find σ (see Fig. 2).

6. Concluding remarks

Understanding the response of bodies undergoing unsteady motions has important technological relevance. For instance, wave propagation is a common approach to non-destructively test materials. Another important application is elastography, a methodology used to determine the location of disease in soft tissues, bones, etc., wherein an increase in the local stiffness is used as a sign of disease in that location. While palpation, the use of one's hands, is often employed to detect locally stiff areas, a more sophisticated method is the use of ultrasound. In view of the constitutive relation studied in this paper being appropriate for porous elastic bodies such as soft tissues, bones, etc., a systematic study of the unsteady motion of bodies described seems to make eminent sense and this study is a consequence of such a determination. Unlike classical compressible Cauchy elastic bodies wherein one has the conservation of mass and the balance of linear momentum to solve for the density and the displacement, in the case of the constitutive relation considered in this paper, in general, we have a system of partial differential equations consisting of the conservation of mass, balance of linear momentum and the constitutive relation that need to be solved simultaneously for the density, displacement and the stress. However, for the specific one-dimensional problem under consideration, the system of equations can be reduced to a single partial differential equation for the one-dimensional stress in one space variable and time, allowing one to consider the possibility of "travelling stress waves" in the material, similar to the study by Kannan et al. [38] wherein they investigated the possibility of travelling waves for a partial equation in space and time for the stress that stems from a constitutive relation having a nonlinear relationship between the linearized strain and the stress. An interesting feature of the partial differential equation for the stress is that both the first and second derivative of the stress appear in the equation. The presence of the first derivative of the stress does not imply any dissipative mechanism at play as the body in question is elastic.

In addition to obtaining travelling wave solutions for the stress by simplifying the partial differential equations by appealing to a similarity transformation, we are able to introduce a new variable which reduces the partial differential equation to an ordinary differential equation which is amenable to the determination of the solution using Matlab. In future work, we plan to investigate the possibility of other unsteady motions of bodies described by this constitutive relation (8). In our study, the body is assumed to be homogeneous, the density in the unstressed and unstrained reference configuration is constant, changes in density in the body being a consequence of the deformation. Of particular interest is the generalization involving inhomogeneous bodies, as material such as soft tissues and bones are inhomogeneous.

The constitutive relation considered in this paper admits the possibility for the material moduli to depend on the density. Recently, Jayavel et al. [40] have studied a generalization of the constitutive relation (8) wherein the material moduli depend on both the density and mechanical pressure (mean value of the stress), a constitutive relation particularly well suited to describe the response of bones. Allowing for the body to be inhomogeneous they consider uniaxial extension and compression of such bodies and are able to capture the distinct behaviour exhibited by bones in tension and compression. We plan to study unsteady problems within the context of both homogeneous and inhomogeneous bodies when the material moduli depend on both the density and the mechanical pressure.

CRediT authorship contribution statement

K.R. Rajagopal: Writing – review & editing, Writing – original draft, Project administration, Methodology, Investigation. **Y. Şengül:** Writing – review & editing, Writing – original draft, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

References

[1] J.M. Burgers, Mechanical considerations–model systems–phenomenological theories of relaxation and of viscosity, in: J.M. Burgers (Ed.), First Report on Viscosity and Plasticity, Nordemann Publishing Company, New York, 1935.

[2] K.R. Rajagopal, On implicit constitutive theories, Appl. Math. 48 (2003) 279–319.

[3] K.R. Rajagopal, On implicit constitutive theories for fluids, J. Fluid Mech. 550 (2006) 243–249.

[4] K.R. Rajagopal, The elasticity of elasticity, Z. Angew. Math. Phys. 58 (2007) 309–317.

[5] K.R. Rajagopal, J.R. Walton, Modeling fracture in the context of a strain-limiting theory of elasticity: A single anti-plane shear crack, Int. J. Fract. 169 (1) (2011) 39–48.

[6] K.R. Rajagopal, Remarks on the notion of "pressure", Int. J. Non-Linear Mech. 71 (2015) 165–172.

[7] K.R. Rajagopal, An implicit constitutive relation for describing the small strain response of porous elastic solids whose material moduli are dependent on the density, Maths. Mech. Solids 26 (8) (2021) 1138–1146.

[8] P. Alagappan, J. Arumugam, K.R. Rajagopal, A note on the response of elastic bodies whose material moduli depend on the density and the mechanical pressure, Appl. Eng. Sci. 16 (2023) 100162.

[9] A.D. Freed, K.R. Rajagopal, A promising approach for modeling biological fibers, Acta Mech. 227 (2016) 1609–1619.

[10] M. Bulíček, J. Málek, K.R. Rajagopal, E. Süli, On elastic solids with limiting small strain: Modelling and analysis, EMS Surv. Math. Sci. 1 (2) (2014) 283–332.

[11] V. Kulvait, J. Málek, K.R. Rajagopal, Anti-plane stress state of a plate with a V-notch for a new class of elastic solids, Int. J. Fract. 179 (1–2) (2013) 59–73.

[12] V. Kulvait, J. Málek, K.R. Rajagopal, The state of stress and strain adjacent to notches in a new class of nonlinear elastic bodies, J. Elasticity 135 (1–2) (2019) 375–397, 3940374.

[13] K. Gou, M. Mallikarjuna, K.R. Rajagopal, J.R. Walton, Modeling fracture in the context of a strain-limiting theory of elasticity: A single plane-strain crack, Internat. J. Engrg. Sci. 88 (2015) 73–82.

[14] H. Itou, V.A. Kovtunenko, K.R. Rajagopal, On an implicit model linear in both stress and strain to describe the response of porous solids, J. Elasticity 144 (2021) 107–118.

[15] H. Itou, V.A. Kovtunenko, K.R. Rajagopal, Investigation of implicit constitutive relations in whichboth the stress and strain appear linearly, adjacent to non-penetrating cracks, Math. Models Methods Appl. Sci. 32 (7) (2022) 1475–1492.

[16] H.A. Erbay, Y. Şengül, Traveling waves in one-dimensional nonlinear models of strain-limiting viscoelasticity, Int. J. Non-Linear Mech. 77 (2015) 61–68.

[17] H.A. Erbay, Y. Şengül, A thermodynamically consistent stress-rate type model of one-dimensional strain-limiting viscoelasticity, Z. Angew. Math. Phys. 71 (3) (2020) 94, 4106810.

[18] Y. Şengül, One-dimensional strain-limiting viscoelasticity with an arctangent type nonlinearity, Appl. Eng. Sci. 7 (2021) 100058.

[19] Y. Şengül, Viscoelasticity with limiting strain, Disc. Cont. Dyn. Syst. S 14 (1) (2021) 57–70.

[20] Y. Şengül, Global existence of solutions for the one-dimensional response of viscoelastic solids within the context of strain limiting theory, in: M.I. Español, M. Lewicka, L. Scardia, A. Schlömerkemper (Eds.), in: Research in Mathematics of Materials Science, vol. 31, Association for Women in Mathematics Series Springer, 2022, pp. 319–332.

[21] M. Bulíček, V. Patel, E. Süli, Y. Şengül, Existence of large-data global weak solutions to a model of a strain-limiting viscoelastic body, Comm. Pure Appl. Math. 20 (5) (2021) 1931–1960, http://dx.doi.org/10.3934/cpaa.2021053.

[22] M. Bulíček, V. Patel, E. Süli, Y. Şengül, Existence and uniqueness of global weak solutions to strain-limiting viscoelasticity with Dirichlet boundary data, SIAM J. Math. Anal. 54 (6) (2022) 6186–6222.

[23] K.R. Rajagopal, A. Wineman, A note on viscoelastic bodies whose material properties depend on the density, Math. Mech. Solids 26 (11) (2021) 1726–1731.

[24] R. Bustamante, K.R. Rajagopal, A note on constitutive relationships for viscoelastic bodies with density-dependent material properties, Mech. Time-Depend. Mater. (2023).

[25] J. Málek, K.R. Rajagopal, Mathematical properties of the solutions to the equations governing the flow of fluids with pressure and shear rate dependent viscosities, in: S. Friedlander, D. Serre (Eds.), in: Handbook of Mathematical Fluid Dynamics, vol. 4, Elsevier, 2007, pp. 407–444.

[26] J. Málek, J. Nečas, K.R. Rajagopal, Global analysis of the flows of fluids with pressure-dependent viscosities, Arch. Ration. Mech. Anal. 165 (2002) 243–263.

[27] M. Franta, J. Málek, K.R. Rajagopal, On steady flows of fluids with pressure-and shear-dependent viscosities, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2005) 651–670.

[28] C. Le Roux, K.R. Rajagopal, Shear flows of a new class of power-law fluids, Appl. Math. 58 (2013) 153–177.

[29] T. Pérlacová, V. Pruša, Tensorial implicit constitutive relations in mechanics of incompressible non-Newtonian fluids, J. Non-Newton. Fluid Mech. 216 (2015) 13–21.

[30] V. Pruša, R.K. Rajagopal, A note on the modeling of incompressible fluids with material moduli dependent on the mean normal stress, Int. J. Non-Linear Mech. 52 (2013) 41–45.

[31] K.R. Rajagopal, A review of implicit algebraic constitutive relations for describing the response of nonlinear fluids, Comptes Rendus Mécanique (2023).

[32] P. Murru, K.R. Rajagopal, Stress concentration due to the bi-axial deformation of a plate of a porous elastic body with a hole, ZAMM Z. Angew. Math. Mech. 101 (11) (2021) e202100103.

[33] P. Murru, K.R. Rajagopal, Stress concentration due to the presence of a hole within the context of elastic bodies, Mater. Des. Process. Commun. 3 (5) (2021) e219.

[34] B. Vajipeyajula, P. Murru, K.R. Rajagopal, Stress concentration due to the presence of a rigid elliptical inclusion in porous elastic solids described by a new class of constitutive relations, J. Elasticity 154 (2023) 255–273.

[35] B. Vajipeyajula, P. Murru, K.R. Rajagopal, Stress concentration due to an elliptic hole in a porous elastic plate, Math. Mech. Solids 28 (3) (2023) 854–869.

[36] B. ErbaŞ, J. Kaplunov, K.R. Rajagopal, 2D asymptotic analysis of a thin elastic beam with density-dependent generalized Young's modulus, in: Advanced Structured Materials, vol. 195, Springer, 2023.

[37] H.A. Erbay, K.R. Rajagopal, G. Saccomandi, Y. Şengül, Dispersive transverse waves for a strain-limiting continuum model, Math. Mech. Solids (2023).

[38] K. Kannan, K.R. Rajagopal, G. Saccomandi, Unsteady motions of a new class of elastic solids, Wave Motion 51 (2014) 833–843.

[39] K.R. Rajagopal, A note on the linearization of the constitutive relations of non-linear elastic bodies, Mech. Res. Comm. 93 (2018) 132–137.

[40] A. Jayavel, P. Alagappan, J. Bird, M. Moreno, K.R. Rajagopal, A new constitutive relation to describe the response of bones, Int. J. Non-Linear Mech. (2023) submitted for publication.