

Competition for Budget-Constrained Buyers: Exploring All-Pay Auctions

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1 Introduction

This note pursues two primary objectives. First, we analyze the outcomes of an all-pay auction within a store where buyers with and without financial constraints arrive at varying rates, and where buyer types are private information. Second, we investigate the selection of an auction format (comprising first-price, second-price, and all-pay formats) in a competitive search setting, where sellers try to attract customers. We show that if the budget constraint is not too restrictive, the all-pay rule emerges as the preferred selling format in the unique symmetric equilibrium.

2 Model

Environment. The economy consists of a large number of risk-neutral buyers and sellers, with an overall buyer-seller ratio denoted by λ . Each seller has one unit of a good and aims to sell it at a price exceeding his reservation price, which is zero. Similarly, each buyer seeks to purchase one unit and is willing to pay up to their reservation price, set at one. While buyers share identical valuations of the good, their purchasing abilities vary. Specifically, a fraction σ of buyers, known as "low types", have constrained budgets and can pay only up to $b < 1$, while the remaining buyers, termed "high types", can pay up to 1. The type of buyer is private information, but the parameters λ , σ , and b are common knowledge.

The game proceeds over the course of two stages. In the first stage sellers simultaneously and independently choose an auction format m and a reserve price r_m . The set of formats consists of first-price auction, second-price auction and all-pay auction. In the second stage buyers observe sellers' selections and select one store to visit. If the customer is alone at the store then he pays the reserve price and obtains the good. If there are $n \geq 2$ buyers then bidding ensues and the winner as well as the sale price are determined based on the auction format. If trade takes place at price r then the seller realizes payoff r , the buyer realizes $1 - r$ whereas those who do not trade earn zero.

Demand Distribution. Following the competitive search literature, we focus on visiting strategies that are symmetric and anonymous on and off the equilibrium path, which, in a large market,

imply that the distribution of demand at any store is Poisson (Burdett et al., 2001; Shimer, 2005). Therefore, the probability that a seller with the terms (m, r_m) meets n customers of type $i = h, l$ is given by $z_n(x_{i,m})$ where

$$z_n(x) := \frac{e^{-x} x^n}{n!}. \quad (1)$$

Since high types and low types arrive at independent Poisson rates $x_{h,m}$ and $x_{l,m}$, the distribution of the *total demand* is also Poisson with $x_{h,m} + x_{l,m}$. Both $x_{h,m}$ and $x_{l,m}$ are endogenous and they depend on what the seller posts and how it compares with the rest of the market.

3 Auctions

Let $u_{i,m}(n)$ denote the expected utility of a type i buyer at a store that trades via rule m and has n customers, including the buyer himself. Similarly let $\pi_m(n)$ denote a store's expected profit conditional on trading via rule m and having n customers. Below we pin down these payoffs for all auction rules. (We drop the subscript m when understood.) Bidding ensues if $n \geq 2$, so consider a store with $n \geq 2$ customers. Low budget types arrive at Poisson rate x_l and high budget types arrive at rate x_h . The distribution of the number of low types, therefore, is *binomial* (n, θ) where

$$\theta = \frac{x_l}{x_h + x_l}.$$

Note that θ is the probability that a buyer is a low type and it is endogenous as it depends on the endogenous arrival rates x_h and x_l .

Remark 1 *With a first-price auction, there exists a unique symmetric equilibrium in which low type buyers bid b while high type buyers randomize within $[b, 1 - (1 - b)\theta^{n-1}]$. Similarly, with a second price auction in the unique symmetric equilibrium low type buyers bid b while high types bid their valuation, 1. Under both auction formats buyers' expected earnings are*

$$u_h(n) = \theta^{n-1} (1 - b) \quad \text{and} \quad u_l(n) = \frac{\theta^{n-1}}{n} (1 - b) \quad \text{for } n \geq 2$$

whereas the seller earns

$$\pi(n) = [\theta^n + n\theta^{n-1}(1 - \theta)] b + 1 - \theta^n - n\theta^{n-1}(1 - \theta) \quad \text{for } n \geq 2. \quad (2)$$

The remark is based on Selcuk (2017). In both auction formats, low types bid their budget b . Conversely, high types exhibit less aggressive bidding in the first-price auction compared to the second-price auction. However, their expected earnings remain equal. Moreover, both auction formats yield identical expected revenue for the seller. Given this payoff equivalence, in a competitive setting sellers and buyers are indifferent to adopting or joining either auction format. From now on we will treat first and second price auctions equally, occasionally referring to them as *standard* auctions.

Proposition 2 *All-pay auction.* If $b < \theta^{n-1}$ then a unique symmetric equilibrium exists where high types randomize in the interval $[b, 1 - \theta^{n-1} + b]$ according to cdf

$$G_h(p) = \frac{(p + \theta^{n-1} - b)^{\frac{1}{n-1}} - \theta}{1 - \theta}.$$

For low types there are two scenarios. If $b < \frac{\theta^{n-1}}{n}$ then they all bid b , but if $\frac{\theta^{n-1}}{n} \leq b < \theta^{n-1}$ then they employ a strategy $G_l(p)$ with support $[0, b]$ that has an atom at b . The size of the atom μ falls as b rises. Specifically, we have

$$G_l(p) = \begin{cases} \frac{p^{\frac{1}{n-1}}}{\theta} & \text{if } 0 \leq p < (1 - \mu)^{n-1} \theta^{n-1} \\ 1 - \mu & \text{if } (1 - \mu)^{n-1} \theta^{n-1} \leq p < b \\ 1 & \text{if } p = b \end{cases}.$$

In this parameter region equilibrium payoffs are given by

$$u_h(n) = \theta^{n-1} - b, \text{ while } u_l(n) = 0 \text{ if } b \geq \theta^{n-1}/n \text{ and } u_l(n) = \frac{\theta^{n-1}}{n} - b \text{ otherwise.}$$

If $\theta^{n-1} \leq b$ then in any symmetric equilibrium the seller extracts the entire surplus while all bidders earn zero i.e. $\pi(n) = 1$, and $u_h(n) = u_l(n) = 0$. The following strategies constitute such an outcome: low types continuously randomize in $[0, \theta^{n-1}]$ whereas high types continuously randomize in $[\theta^{n-1}, 1]$ according to density functions

$$G_l(p) = \frac{p^{\frac{1}{n-1}}}{\theta} \quad \text{and} \quad G_h(p) = \frac{p^{\frac{1}{n-1}} - \theta}{1 - \theta}. \quad (3)$$

In the all-pay format, participants are obliged to pay their bid regardless of whether they win or lose. As a result, compared to first or second price auctions, buyers bid smaller amounts. Bid amounts decrease particularly among low types. Whereas in first and second price auctions, low types would bid their entire budget b , they now engage in randomization within a lower range. High types also reduce their bids, though the effect on them is less pronounced.

Despite the lower bids, the seller manages to collect a higher amount of revenue than first or second price auctions. For instance when $b \geq \theta^{n-1}$, i.e. when the budget is not too small, we see that all-pay auctions yield $\pi(n) = 1$ whereas first and second price auctions yield a strictly smaller $\pi(n)$. Again, this is owing to the auction format allowing the seller to collect all the bids. Absent budget constraints, revenue equivalence among fixed-price, second-price and all-pay auctions is well known in the literature. We show that with financially constrained buyers, the all-pay format revenue-dominates the first or second price formats. Che and Gale (1996) prove a similar result when the budget distribution is continuous. We show that the result remains valid under a discrete distribution. This advantage will play a key role when sellers pick an auction format to compete for buyers.

Figure of cdfs here

The amount of bids depend on θ , the percentage of low types expected to be present in the auction. This parameter is endogenous and critically impacts the auction process. For instance suppose a seller attracts high types only. Then $\theta = 0$, and per the Proposition, and they randomize in the interval $[0, 1]$ according to $G = p^{\frac{1}{n-1}}$. Baye et al. (1996) obtain this result in a setting with no budget constraints (see Theorem 1 therein).

4 Competitive Search

Let s represent standard (first or second price) formats and ap represent the all pay format. A type i buyer's expected utility from visiting a store competing with rule $m = s, ap$ is given by

$$U_{i,m}(r_m, x_{h,m}, x_{l,m}) = \sum_{n=0}^{\infty} z_n(x_{h,m} + x_{l,m}) u_{i,m}(n+1), \quad (4)$$

With probability $z_n(\cdot)$ the buyer finds $n = 0, 1, \dots$ other customers at the same store; so, in total there are $n+1$ customers (including himself) and the expected utility corresponding to this scenario is $u_{i,m}(n+1)$. Now consider a seller who competes with rule m . His expected profit is given by

$$\Pi_m(r_m, x_{h,m}, x_{l,m}) = \sum_{n=1}^{\infty} z_n(x_{h,m} + x_{l,m}) \pi_m(n).$$

With probability $z_n(\cdot)$ the seller gets n customers and the corresponding payoff associated with this scenario is $\pi_m(n)$. The following Lemma links Π to the U_i s.

Lemma 3 *The following relationship holds both for first and second price formats as well as for the all-pay format:*

$$\Pi_m = 1 - z_0(x_{h,m} + x_{l,m}) - x_{h,m}U_{h,m} - x_{l,m}U_{l,m}, \quad \text{where } m = s, ap. \quad (5)$$

The expression $1 - z_0(x_{h,m} + x_{l,m})$ can be interpreted as the expected revenue. It is the value created by a sale (one), multiplied by the probability of trading. The expression $x_{h,m}U_{h,m} + x_{l,m}U_{l,m}$ can be interpreted as the expected cost. The seller promises a payoff $U_{h,m}$ to each high type and $U_{l,m}$ to each low type customer. On average he gets $x_{h,m}$ high type and $x_{l,m}$ low type customers; so the total cost equals to $x_{h,m}U_{h,m} + x_{l,m}U_{l,m}$. The profit Π_m is simply the difference between the revenue and the cost.

Following the competitive search, let Ω_i denote the maximum expected utility ("market utility") a type i customer can obtain in the entire market.¹ For now we treat Ω_i as given, subsequently it will be determined endogenously. Consider an individual seller who advertises (m, r_m) and suppose that

¹The market utility approach greatly facilitates the characterization of equilibrium and, therefore, is standard in the directed search literature. For an extended discussion see Galenianos and Kircher (2012).

high and low type buyers respond to this advertisement with arrival rates $x_{h,m} \geq 0$ and $x_{l,m} \geq 0$. These rates satisfy

$$x_{i,m} \begin{cases} > 0 & \text{if } U_{i,m}(r_m, x_{h,m}, x_{l,m}) = \Omega_i \\ = 0 & \text{if } U_{i,m}(r_m, x_{h,m}, x_{l,m}) < \Omega_i \end{cases} . \quad (6)$$

In words, the tuple $(r_m, x_{h,m}, x_{l,m})$ must generate an expected utility of at least Ω_h for high type customers, else they will stay away ($x_{h,m} = 0$) and at least Ω_l for low type customers, else they will stay away ($x_{l,m} = 0$). When competing for buyers, each seller chooses a rule $m = s, ap$ and a price $r_m \in [0, 1]$ but realizes that expected demands $x_{h,m}$ and $x_{l,m}$ are determined via (6).

Lemma 4 *Fix some arrival rates x_h, x_l and a reserve price r . The all-pay format generates strictly lower payoffs for buyers than first-price or second-price formats. Equivalently, it generates strictly higher profits for the seller.*

Proof of Lemma 4. Let $x = x_h + x_l$ and $\theta = x_l/x$, therefore $x_l = \theta x$ and $x_h = (1 - \theta)x$. Recall that under standard auctions

$$u_h(n) = \theta^{n-1}(1-b) \quad \text{and} \quad u_l(n) = \frac{\theta^{n-1}}{n}(1-b) \quad \text{for } n \geq 2,$$

whereas with all pay auctions

$$u_h(n) = \max(\theta^{n-1} - b, 0) \quad \text{and} \quad u_l(n) = \max(\theta^{n-1}/n - b, 0) \quad \text{for } n \geq 2.$$

Further recall that

$$U_i = z_0(x)(1-r) + \sum_{n=1}^{\infty} z_n(x) u_i(n+1).$$

Noting that r and $z_n(x)$ s are the same across all rules, the claim that the all-pay format generates strictly lower payoffs for buyers than first-price or second-price formats entails showing that

$$\theta^{n-1}(1-b) >? \max(\theta^{n-1} - b, 0) \quad \text{and} \quad \frac{\theta^{n-1}}{n}(1-b) >? \max\left(\frac{\theta^{n-1}}{n} - b, 0\right).$$

It is straightforward to check that both inequalities hold. The claim that sellers earn more under all pay auction follows from the fact that

$$\Pi = 1 - z_0(x) - x_l U_l - x_h U_h.$$

Since x s are controlled for, the fact that U s are smaller with all pay auctions implies that the Π is higher. ■

Proposition 5 . *A first or second price auction equilibrium (in which all sellers adopt these rules) fails to exist. In such a scenario a seller can unilaterally switch to all pay auctions and earn more.*

Proof of Proposition 5. Consider an equilibrium in which sellers adopt a standard (first or second price) format. Symmetry in buyers' visiting strategies implies that each seller receives λ_l low

types, λ_h high types, and it total $\lambda_l + \lambda_h = \lambda$ customers. Recall that the percentage of low types is σ . The expected payoffs and profits are

$$U_{h,s} = z_0(\lambda)(1 - r_s) + z_0(\lambda_h)(1 - z_0(\lambda_l))(1 - b) \text{ and}$$

$$U_{l,s} = z_0(\lambda)(1 - r_s) + z_0(\lambda_h) \frac{1 - z_0(\lambda_l) - z_1(\lambda_l)}{\lambda_l} (1 - b).$$

and therefore

$$\Pi_s = 1 - z_0(\lambda) - \lambda(1 - \sigma)U_{h,s} - \sigma\lambda U_{l,s}.$$

Now consider a seller who switches to all pay auctions. We will show that this seller can keep providing high types with $U_{h,s}$ and low types with $U_{l,s}$ yet can earn more.

Once the seller makes such a switch, there are three key parameters: the reserve price r_a , the total demand, x , and the composition of demand, θ . The fact that these parameters are potentially different than r_s , λ and σ makes the comparison difficult. To get around this issue, we will control for λ , and show that there there exists a reserve price $\hat{r} < r_s$ and $\hat{\theta} > \sigma$ ensuring that $U_{l,a} = U_{l,s}$ and $U_{h,a} = U_{h,s}$, i.e. the seller attracts λ customers in total (albeit with a different composition θ and reserve r) and provides low types the same payoff $U_{l,s}$ and high types with the same payoff.

Per Lemma 4 we have

$$U_{h,a}(r, \lambda, \theta) < U_{h,s}(r_s, \lambda, \sigma) \quad \text{and} \quad U_{l,a}(r, \lambda, \theta) < U_{l,s}(r_s, \lambda, \sigma) \text{ when } r = r_s \text{ and } \theta = \sigma.$$

Note that both $U_{h,a}$ and $U_{l,a}$ fall in r . In addition under all pay auctions

$$u_h = \max(\theta^{n-1} - b, 0) \quad \text{and} \quad u_l = \max(\theta^{n-1}/n - b, 0)$$

which means that both $U_{h,a}$ and $U_{l,a}$ rise in θ . It follows that there exists some $\hat{\theta} > \sigma$ and $\hat{r} < r_s$ satisfying $U_{l,a} = U_{l,s}$ and $U_{h,a} = U_{h,s}$. Now, the seller's payoff is

$$\Pi_a = 1 - z_0(\lambda) - \lambda(1 - \hat{\theta})U_{h,s} - \hat{\theta}\lambda U_{l,s}$$

Since $U_{h,s} > U_{l,s}$ and $\hat{\theta} > \sigma$, we have $\Pi_a > \Pi_s$, i.e. deviation is profitable. ■

In words, the seller attracts the same total demand λ , yet the buyer mix shifts with $\hat{\theta} > \sigma$ implying an increase in low-type buyers and a decrease in high-type ones. High types are paid higher utility than low types, so this shift benefits the seller, enabling him to generate more earnings.

Proposition 6 *If $b > \sigma$ then in the unique symmetric equilibrium all sellers select the all-pay auction format.*

The proof outlines the all-pay equilibrium: Sellers set $r = 0$, and all buyers, both low and high types, achieve the identical expected utility $z_0(\lambda)$. Remarkably, this is the same outcome in a model with homogeneous buyers (with no budget constraints). Due to the structure of the all-pay auction, participants place bids of small amounts, and therefore, those with lower financial capabilities still

engage and, on expected terms, earn as much as they would without any budget constraints.

5 Conclusion

In an all-pay auction, every participant must pay their bids, irrespective of winning or losing. This setup encourages lower bids, opening avenues for individuals with limited budgets to participate. Lower bids don't necessarily translate to reduced revenues since the seller gathers all bids, not just the winning one. In fact, we demonstrate that in a similar context, the all-pay format outperforms both first and second-price formats in terms of revenue. This difference gives the all-pay rule a competitive edge over other formats in a model where stores compete for customers.

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6 Appendix

Proof of the Proposition 2. We focus on symmetric mixed strategies where bidders of the same type pick the same cumulative distribution function (cdf) $G_i(p) : [\underline{s}_i, \bar{s}_i] \rightarrow [0, 1]$. A point p is an ‘increasing point’ of G_i if G_i is not constant in an ε neighborhood of p_i i.e. if for each $\varepsilon > 0$ the probability of having a value in $(p - \varepsilon, p + \varepsilon)$ is positive. Conversely, p is a ‘constant point’ if G_i is constant in an ε neighborhood of p . Note that if there is an atom at p then, by definition, p is an increasing point. If the pair (G_h, G_l) corresponds to an equilibrium, then type i buyers earn their equilibrium payoff u_i at each increasing point p of G_i . Similarly they earn an expected payoff that is less than or equal to u_i at each constant point p of G_i (Hillman and Samet, 1987; Baye et al., 1996). Said differently, in a mixed strategy equilibrium players must be indifferent to all increasing points and they must weakly prefer increasing points over constant points in the support of their bidding distribution. These claims are immediate from the definition of a mixed strategy equilibrium; Hillman and Samet (1987) provide a more formal discussion (see Proposition 2 therein).

The equilibrium cdf G_h cannot have a mass point anywhere on its support $[\underline{s}_h, \bar{s}_h]$. A mass point means tying with other bidders in which case the surplus is divided among the tying bidders via random rationing. A financially unconstrained bidder can always beat the tie and improve his payoff by placing a bid that is slightly above the mass point, which is inconsistent with G_h being an equilibrium distribution. The argument applies to the entire support including the upper bound \bar{s}_h : If $\bar{s}_h < 1$ then there is room to beat a potential tie at \bar{s}_h . If $\bar{s}_h = 1$ then a mass at 1 would result in a negative payoff. In either case there cannot be a mass at \bar{s}_h . To sum up G_h is continuous on its support with no jumps (it does not have flat spots either, but more on this below).

The equilibrium cdf G_l cannot have an atom anywhere below b for the same reason above, however it may have an atom at b . There are three scenarios for G_l :

- L1 — The entire mass is at b .
- L2 — Partial mass at b with a continuous tail below b .
- L3 — No mass at b , so the entire cdf is continuous.

Before we move on, the following expression will be useful. Assuming that p is not a mass point, the expected payoff associated with bidding p is given by

$$\begin{aligned} EU(p) &= \sum_{i=0}^{n-1} \binom{n-1}{i} [\theta G_l(p)]^i [(1-\theta) G_h(p)]^{n-1-i} - p \\ &= [\theta G_l(p) + (1-\theta) G_h(p)]^{n-1} - p. \end{aligned} \tag{7}$$

This is true for both types of buyers. Indeed buyers have identical valuations for the item, so both types earn the same expected payoff $EU(p)$, assuming, of course, $p \leq b$. If $p > b$ then low types are sure to be outbid; the expected payoff for high types can be obtained by substituting $G_l(p) = 1$ into above.

- **Region 1:** Suppose that $b < \frac{\theta^{n-1}}{n}$

1A. In this parameter region high types are guaranteed to receive a payoff $\theta^{n-1} - b$ whereas low

types are guaranteed to receive $\frac{\theta^{n-1}}{n} - b$, both of which are positive because $b < \frac{\theta^{n-1}}{n}$. Low types cannot bid more than their budget b , so if a high type buyer bids slightly more than b , then even if he loses against all other high types, he can still win the item with probability θ^{n-1} (everyone else is a low type) and would obtain a payoff $\theta^{n-1} - b$. Similarly if a low type bids b , then in the worst case scenario he loses against all high types and ties with every other low type, so his payoff is at least $\frac{\theta^{n-1}}{n} - b$. It follows that $u_h \geq u_l > 0$.

1B. We rule out scenarios L2 and L3, which leaves L1 as the only possible scenario for G_l . In scenarios L2 and L3 the cdf G_l is assumed to be continuous over some interval $[\underline{s}_l, \bar{s}_l]$ where $0 \leq \underline{s}_l < \bar{s}_l \leq b$. Recall that G_h is also continuous over $[\underline{s}_h, \bar{s}_h]$, so there are three possibilities: either $\underline{s}_h > \underline{s}_l$ or $\underline{s}_l > \underline{s}_h$ or $\underline{s}_l = \underline{s}_h$. Consider the first one, i.e. suppose $\underline{s}_h > \underline{s}_l$. This implies $u_l = -\underline{s}_l$. To see why, note that \underline{s}_l is an increasing point of G_l and in a mixed strategy equilibrium any increasing point, including \underline{s}_l , must yield the equilibrium payoff to the bidder. If the buyer bids $p = \underline{s}_l$ then he is sure to lose the item as everyone else is sure to bid more than \underline{s}_l (recall that $\underline{s}_l < \underline{s}_h$). His payoff, therefore, is $-\underline{s}_l \leq 0$ because in the all-pay auction all participants forfeit their bids (if the lower bound \underline{s}_l is set to be zero then the resulting payoff is zero, but if \underline{s}_l is positive then the payoff is negative). This, of course, contradicts the fact that $u_l > 0$. Now suppose $\underline{s}_l > \underline{s}_h$. This implies $u_h = -\bar{s}_h \leq 0$, which contradicts $u_h > 0$. Finally if $\underline{s}_l = \underline{s}_h$ then $u_h = u_l = -\underline{s}_l \leq 0$, which again is a contradiction. In words, if both cdfs have continuous bits then the one with the lower bound on the far left is bound to yield at most a zero payoff. It follows that G_l cannot have a continuous part over some interval below b ; the entire mass must be at point b .

1C. Given that the entire mass of G_l is placed at b the lower bound of G_h cannot be below b , i.e. we must have $\underline{s}_h \geq b$. Indeed if $\underline{s}_h < b$ then $u_h = -\underline{s}_h \leq 0$, a contradiction. Since all low types bid b and high types are sure to outbid the low types ($\underline{s}_h \geq b$) the equilibrium payoff of a low type equals to $u_l = \frac{\theta^{n-1}}{n} - b$.

1D. As discussed earlier, G_h cannot have an atom, i.e. there are no jumps. We now argue that it cannot have intermittent flat spots either. By contradiction, suppose G_h is constant at some interval $(a_1, a_2) \subset [\underline{s}_h, \bar{s}_h]$. Both a_1 and a_2 are increasing points of the distribution function G_h hence they both must deliver the same payoff u_h . Since G_h is flat in this interval the probability of winning the auction is the same at both points (notice also $G_l = 1$ at both points). This, however, means that the player gets a lower payoff at a_2 than a_1 since $a_2 > a_1$; so, he cannot be indifferent; a contradiction.

1E. We can now characterize G_h . We established that G_h is monotonically increasing on its support $[\underline{s}_h, \bar{s}_h]$. The expected payoff associated with bidding any $p \in [\underline{s}_h, \bar{s}_h]$ is given by

$$EU(p) = [\theta + (1 - \theta) G_h(p)]^{n-1} - p,$$

which is obtained by substituting $G_l(p) = 1$ into (7) (since $\underline{s}_h \geq b$ we have $G_l(p) = 1$ for all $p \geq \underline{s}_h$). High types must earn their equilibrium payoff u_h at any increasing point of G_h . Since G_h is monotonically increasing we must have

$$EU(p) = u_h \text{ for all } p \in [\underline{s}_h, \bar{s}_h].$$

Substituting for $EU(p)$ and solving for G_h we have

$$G_h(p) = \frac{(p + u_h)^{\frac{1}{n-1}} - \theta}{1 - \theta}.$$

We know $G_h(\underline{s}_h) = 0$ and $G_h(\bar{s}_h) = 1$, hence $u_h = \theta^{n-1} - \underline{s}_h$ and $u_h = 1 - \bar{s}_h$. Recall that $u_h \geq \theta^{n-1} - b$ and that $\underline{s}_h \geq b$. This means $\underline{s}_h = b$ and $u_h = \theta^{n-1} - b$ and therefore $\bar{s}_h = 1 - \theta^{n-1} + b$. Substituting for u_h yields the expression of G_h in the body of the proposition.

- **Region 2:** Suppose that $\frac{\theta^{n-1}}{n} < b < \theta^{n-1}$

2A. Per the discussion in 1A, high types are still guaranteed to receive a positive payoff, $\theta^{n-1} - b$; however, low types are no longer guaranteed to receive a positive payoff i.e. $u_h > 0$, $u_l \geq 0$.

2B. There are three scenarios for G_l which are outlined in L1, L2 and L3. Suppose L1 is valid, i.e. low types bid b for sure. Then $\underline{s}_h \geq b$ and the equilibrium payoff of a low type equals to $u_l = \frac{\theta^{n-1}}{n} - b$ (see 1C). This, however, is negative because $\frac{\theta^{n-1}}{n} < b$; a contradiction. Since L1 is ruled out, we have either L2 or L3. In either case G_l is continuous over some interval $[\underline{s}_l, \bar{s}_l]$ where $0 \leq \underline{s}_l < \bar{s}_l \leq b$. Recall that G_h is continuous on its support $[\underline{s}_h, \bar{s}_h]$, so there are three possibilities: either $\underline{s}_h > \underline{s}_l$ or $\underline{s}_l > \underline{s}_h$ or $\underline{s}_l = \underline{s}_h$. Per the discussion in 1B, we cannot have $\underline{s}_l > \underline{s}_h$ or $\underline{s}_l = \underline{s}_h$ because in either case $u_h \leq 0$, a contradiction. So we must have $\underline{s}_h > \underline{s}_l$. This implies $u_l = -\underline{s}_l \leq 0$. Clearly if $\underline{s}_l > 0$ then u_l is negative; hence we must have $\underline{s}_l = 0$ and therefore $u_l = 0$.

2C. We will show that $\underline{s}_h \geq \bar{s}_l$, i.e. the supports of G_h and G_l cannot overlap. Suppose they do, i.e. suppose that $\underline{s}_h < \bar{s}_l$ so that $G_h(\bar{s}_l) > 0$. Pick some point $p \in [\underline{s}_h, \bar{s}_l]$ at which G_h is increasing (since $G_h(\bar{s}_l) > 0$ such a point must exist). Recall that the expected payoff associated with bidding p is equal to $EU(p)$, which is given by (7), and it is the same for both types of buyers. Since p is an increasing point of G_h we have $EU(p) = u_h$. Now p is either an increasing point of G_l or it is a constant point of G_l . The first case implies $EU(p) = u_l$ whereas the second one implies $EU(p) \leq u_l$. In either case we have a contradiction since $u_h > 0$ and $u_l = 0$. It follows that the supports of G_h and G_l cannot overlap, so we must have $\underline{s}_h \geq \bar{s}_l$.

2D. Now we will rule out scenario L3. Again, by contradiction, suppose L3 is valid, i.e. G_l is atomless and does not jump anywhere on its support $[0, \bar{s}_l]$. In 2C we proved that $\underline{s}_h \geq \bar{s}_l$, hence $G_h(p) = 0$ for all $p \in [0, \bar{s}_l]$. Furthermore, per the discussion in 1D, G_l cannot have intermittent flat spots either; hence G_l is monotonically increasing on its support. The expected payoff associated with bidding any $p \in [0, \bar{s}_l]$ is given by

$$EU(p) = [\theta G_l(p)]^{n-1} - p,$$

which is obtained by substituting $G_h(p) = 0$ into (7). Low types must earn their equilibrium payoff $u_l = 0$ at any increasing point of G_l . Since G_l is monotonically increasing we must have $EU(p) = u_h$ for all $p \in [\underline{s}_h, \bar{s}_h]$. Substituting for $EU(p)$ and solving for G_l we have

$$G_l(p) = \frac{p^{\frac{1}{n-1}}}{\theta}. \quad (8)$$

Note that $G_l(\bar{s}_l)$ must be equal to 1; however this is impossible because $\bar{s}_l \leq b$ and $G_l(b) = \frac{b^{\frac{1}{n-1}}}{\theta} < 1$ since $b < \theta^{n-1}$. Without a mass point at b , the function G_l cannot be a valid cdf.

2E. Since L3 is ruled out, the only possible scenario is L2 where G_l has some partial mass μ at point b while the remaining mass is spread over some interval starting at the lower bound $\underline{s}_l = 0$. Per the discussion in 2D, at any point of increase in the region where G_l is atomless low types must earn $u_l = 0$, which implies $G_l(p) = \frac{p^{\frac{1}{n-1}}}{\theta}$. Buyers should get the same payoff $u_l = 0$ at the mass point b as well. The expected payoff associated with bidding b is given by

$$\theta^{n-1} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{\mu^i}{i+1} (1-\mu)^{n-i-1} - b = \frac{\theta^{n-1} [1 - (1-\mu)^n]}{\mu n} - b.$$

For this to be equal to zero, μ must solve

$$\frac{bn}{\theta^{n-1}} - \frac{1 - (1-\mu)^n}{\mu} = 0$$

It is straightforward to show that so as long as $\frac{\theta^{n-1}}{n} < b < \theta^{n-1}b$ there exists a unique $\mu \in (0, 1)$ satisfying above. The upper bound of the atomless portion of G_l , call it \bar{p} , satisfies $G_l(\bar{p}) = 1 - \mu$ hence

$$\bar{p} = \theta^{n-1} (1 - \mu)^{n-1}.$$

An argument similar to the one in 1D reveals that G_l cannot have flat spots in the region $(0, \bar{p})$. So G_l monotonically rises in $[0, \bar{p}]$, has mass point at b and it is flat in between.

2F. We now characterize G_h . We know $\underline{s}_h \geq \bar{s}_l = b$, so for any $p \geq \underline{s}_h$ we have $G_l(p) = 1$. At any point of increase in the support of G_h we must have $EU(p) = u_h$ where

$$EU(p) = [\theta + (1-\theta)G_h(p)]^{n-1} - p,$$

which is obtained by substituting $G_l(p) = 1$ into (7). It follows that

$$G_h(p) = \frac{(p + u_h)^{\frac{1}{n-1}} - \theta}{1 - \theta}.$$

Since $G_h(\underline{s}_h) = 0$ we have $\underline{s}_h = \theta^{n-1} - u_h$. Since $u_h \geq \theta^{n-1} - b$ and $\underline{s}_h \geq b$ we have $\underline{s}_h = b$ and $u_h = \theta^{n-1} - b$ and therefore $\bar{s}_h = 1 - \theta^{n-1} + b$. Per the discussion in 1D, G_h cannot have intermittent flat spots anywhere in its support.

- **Region 3:** Suppose $\theta^{n-1} \leq b$.

3A. Since $\theta^{n-1} \leq b$, per 1A, buyers are no longer guaranteed to receive a positive payoff. In regions 1 and 2 at least one of the equilibrium payoffs was positive, and this feature played a key role in establishing the uniqueness of the equilibrium. Without this information proving uniqueness becomes a challenge; so, instead of attempting to characterize the equilibrium, we focus on expected payoffs

and prove that in any symmetric equilibrium the seller extracts the entire surplus, i.e. $u_l = u_h = 0$. We then verify that the strategy profile in (3) constitutes such an equilibrium.

3B. Consider a symmetric mixed strategy equilibrium with cdfs G_h, G_l and associated equilibrium payoffs u_h, u_l . WLOG let $u_h \geq u_l$, so there are three possibilities: (i) $u_h \geq u_l > 0$ or (ii) $u_h > u_l = 0$ or (iii) $u_h = u_l = 0$. We will rule out (i) and (ii). To start, suppose $u_h \geq u_l > 0$. Recall that G_h is atomless whereas there are three scenarios for G_l . Per 1B, if both cdfs have continuous bits then the one with the lower bound on the far left will yield at most a zero payoff. So, if u_l is positive then G_l cannot have a continuous part over some interval below b ; the entire mass must be at point b . This scenario is analyzed in 1C and the equilibrium payoff of a low type equals to $u_l = \frac{\theta^{n-1}}{n} - b$, which is negative since $\theta^{n-1} \leq b$; a contradiction.

3C. Suppose $u_h > u_l = 0$. The cdf G_l cannot have the entire mass at b (3B). So either it has a partial mass at b or it is atomless everywhere. Suppose it has a partial mass at b . Then, per 2C, $\underline{s}_h \geq \bar{s}_l = b$ (cdfs cannot overlap). For a high type the expected payoff associated with bidding \underline{s}_h equals to $EU(\underline{s}_h) = \theta^{n-1} - b$, which is less than or equal to zero since $\theta^{n-1} \leq b$. The lower bound \underline{s}_h is an increasing point of G_h , so we must have $EU(\underline{s}_h) = u_h$. This, however, is a contradiction because $u_h > 0$ but $EU(\underline{s}_h) \leq 0$. The final scenario for G_l is where it is atomless everywhere; so, suppose this is the case. Per 2B, 2C and 2D, G_l is given by (8) and it must be monotonically increasing on its support $[0, \bar{s}_l]$ with no flat spots. Solving $G_l(\bar{s}_l) = 1$ yields $\bar{s}_l = \theta^{n-1}$. Recall that $\underline{s}_h \geq \bar{s}_l$, so $\underline{s}_h \geq \theta^{n-1}$. It follows that $EU(\underline{s}_h) \leq 0$, which, again is a contradiction since we must have $EU(\underline{s}_h) = u_h > 0$.

3D. Arguments in 3B and 3C imply that we must have $u_l = u_h = 0$. What remains to be done is to characterize equilibrium strategies G_l and G_h delivering these payoffs. For the purpose of the paper the fact that any symmetric equilibrium yields zero payoffs is sufficient, so we refrain from attempting to characterize all possible combinations of G_l and G_h ; instead we will verify that the specific forms of G_l and G_h , given by (3), indeed correspond to an equilibrium and they yield zero payoffs. So suppose all players adopt the cdfs in (3) and consider a potential deviation by a low type buyer who picks a different cdf, say, $\tilde{G} : [0, b] \rightarrow [0, 1]$. His expected payoff \tilde{u} is given by

$$\tilde{u} = \int_0^b \{[\theta G_l(p) + (1 - \theta) G_h(p)]^{n-1} - p\} d\tilde{G}(p).$$

Observe that $G_l(p) = 1$ for $p \geq \theta^{n-1}$ and $G_h(p) = 0$ for $p \leq \theta^{n-1}$; therefore

$$\tilde{u} = \int_0^{\theta^{n-1}} \{\theta^{n-1} G_l^{n-1}(p) - p\} d\tilde{G}(p) + \int_{\theta^{n-1}}^b \{[\theta + (1 - \theta) G_h(p)]^{n-1} - p\} d\tilde{G}(p).$$

After substituting for $G_l(p)$ and $G_h(p)$, which are given by (3), the expressions inside the curly brackets vanish, thus $\tilde{u} = 0$ irrespective of \tilde{G} , i.e. if everyone else sticks to (3) then a low type cannot earn anything but zero irrespective of the cdf he picks. A similar argument applies to high types as well. This completes the proof. ■

Proof of Lemma 3. Start with standard auctions (denoted with subscript s). If a buyer is alone at

an auction store then he obtains the item by paying the reserve price i.e. $u_{h,s}(1) = u_{l,s}(1) = 1 - r_s$. If $n \geq 2$ then we know that under both auction formats

$$u_{h,s}(n) = \theta^{n-1}(1-b) \text{ and } u_{l,s}(n) = \frac{\theta^{n-1}}{n}(1-b), \text{ where } \theta = \frac{x_{l,s}}{x_{h,s} + x_{l,s}}.$$

Substituting these expressions into (4) yields

$$\begin{aligned} U_{h,s} &= z_0(x_{h,s} + x_{l,s})(1 - r_s) + \sum_{n=1}^{\infty} z_n(x_{h,s} + x_{l,s})\theta^n(1-b) \text{ and} \\ U_{l,s} &= z_0(x_{h,s} + x_{l,s})(1 - r_s) + \sum_{n=1}^{\infty} z_n(x_{h,s} + x_{l,s})\frac{\theta^n}{n+1}(1-b). \end{aligned}$$

After substituting for θ and re-arranging we get

$$U_{h,s} = z_0(x_{h,s} + x_{l,s})(1 - r_s) + z_0(x_{h,s})(1 - z_0(x_{l,s}))(1 - b) \text{ and} \quad (9)$$

$$U_{l,s} = z_0(x_{h,s} + x_{l,s})(1 - r_s) + z_0(x_{h,s})\frac{1 - z_0(x_{l,s}) - z_1(x_{l,s})}{x_{l,s}}(1 - b). \quad (10)$$

With all pay auctions, assuming $b > \theta$, we have

$$u_{h,ap}(1) = u_{l,ap}(1) = 1 - r_{ap} \text{ and } u_{h,ap}(n) = u_{l,ap}(n) = 0 \text{ for } n \geq 2.$$

Substituting this into (4) yields

$$U_{h,ap} = U_{l,ap} = z_0(x_{h,ap} + x_{l,ap})(1 - r_{ap}). \quad (11)$$

Now we can link these to the profit functions. Start with standard auctions. If a single customer is present then the reserve price is charged, i.e. $\pi_s(1) = r_s$. If $n \geq 2$ then both auction formats yield the same $\pi_s(n)$, given by (2); hence

$$\Pi_s = z_1(x_{h,s} + x_{l,s})r_s + \sum_{n=2}^{\infty} z_n(x_{h,s} + x_{l,s})\{[\theta^n + n\theta^{n-1}(1-\theta)]b + 1 - \theta^n - n\theta^{n-1}(1-\theta)\}.$$

After substituting for z_n and θ and simplifying we have

$$\begin{aligned} \Pi_a &= z_1(x_{h,s} + x_{l,s})r_s + 1 - z_0(x_{h,s}) - z_1(x_{h,s}) \\ &\quad + b\{z_0(x_{h,s}) + z_1(x_{h,s}) - z_0(x_{h,s} + x_{l,s}) - z_1(x_{h,s} + x_{l,s})\}. \end{aligned} \quad (12)$$

Note that $z_1(x) = xz_0(x)$ and $z_0(x+y) = z_0(x)z_0(y)$. It follows that

$$\begin{aligned} x_{h,s}U_{h,s} + x_{l,s}U_{l,s} &= z_0(x_{h,s}) + z_1(x_{h,s}) - z_0(x_{h,s} + x_{l,s}) - z_1(x_{h,s} + x_{l,s})r_s + \\ &\quad - b\{z_0(x_{h,s}) + z_1(x_{h,s}) - z_0(x_{h,s} + x_{l,s}) - z_1(x_{h,s} + x_{l,s})\}, \end{aligned}$$

where $U_{h,s}$ and $U_{l,s}$ are given by (9) and (10). The expected profit Π_a is given by (12). A term by term comparison reveals that

$$\Pi_a = 1 - z_0(x_{h,s} + x_{l,s}) - x_{h,s}U_{h,s} - x_{l,s}U_{l,s},$$

confirming the validity of the relationship under auctions.

As for all pay auctions, suppose $b > \theta$. Then we will prove that the seller's expected earning is $\pi(n) = 1$. First we calculate the expected bids from low and high types. For low types this is given by

$$\int_0^{\theta^{n-1}} pdG_l(p) = pG_l(p)|_0^{\theta^{n-1}} - \int_0^{\theta^{n-1}} G_l(p) dp = \frac{\theta^{n-1}}{n}.$$

For high types

$$\int_{\theta^{n-1}}^1 pdG_h(p) = pG_h(p)|_{\theta^{n-1}}^1 - \int_{\theta^{n-1}}^1 G_h(p) dp = \frac{1 - \theta^n}{(1 - \theta)n}.$$

The seller collects all these bids, thus

$$\begin{aligned} \pi(n) &= \sum_{j=0}^n \binom{n}{j} (1 - \theta)^j \theta^{n-j} \left[(n - j) \frac{\theta^{n-1}}{n} + j \frac{1 - \theta^n}{(1 - \theta)n} \right] \\ &= \theta^{n-1} \sum_{j=0}^n \binom{n}{j} (1 - \theta)^j \theta^{n-j} + \frac{1 - \theta^{n-1}}{(1 - \theta)n} \sum_{j=0}^n \binom{n}{j} (1 - \theta)^j \theta^{n-j} j \end{aligned}$$

The first term involving the summation sign is equal to 1, as it represents the sum of the probabilities under *binomial* $(n, 1 - \theta)$. The second term involving the summation sign is equal to $n(1 - \theta)$, as it is the expected value under *binomial* $(n, 1 - \theta)$. Therefore

$$\pi(n) = \theta^{n-1} \times 1 + \frac{1 - \theta^{n-1}}{(1 - \theta)n} \times n(1 - \theta) = 1$$

The expected profit of an all pay seller is easier to calculate. We have

$$\pi_{ap}(1) = r \quad \text{and} \quad \pi_{ap}(n) = 1 \quad \text{for } n \geq 2$$

hence

$$\Pi_{ap} = z_1(x_{h,ap} + x_{l,ap})r + 1 - z_0(x_{h,ap} + x_{l,ap}) - z_1(x_{h,ap} + x_{l,ap}). \quad (13)$$

It is easy to verify that

$$\Pi_{ap} = 1 - z_0(x_{h,ap} + x_{l,ap}) - x_{h,ap}U_{h,ap} - x_{l,ap}U_{l,ap},$$

where $U_{h,ap}$ and $U_{l,ap}$ are given by (11). This completes the proof. ■

Proof of Proposition 6. The proof involves two steps. First we will characterize the outcome

where all sellers compete via all pay auctions. Then we will show that no seller has a profitable deviation by selecting 1st or 2nd price auctions.

As for the first task, conjecture an outcome where sellers compete with all pay auctions. Symmetry in buyers' visiting strategies implies that $x_{l,a} = \lambda_l$ and $x_{h,a} = \lambda_h$; so, the total demand at each store is $\lambda_l + \lambda_h = \lambda$. When $b > \sigma$, the expected utility of a high type as well as a low type is equal to

$$U_{h,ap} = U_{l,ap} = z_0(\lambda)(1 - r).$$

A seller solves

$$\max_{\lambda} 1 - z_0(\lambda) - \lambda\Omega,$$

which yields $z_0(\lambda) = \Omega$. It follows that

$$z_0(\lambda)(1 - r) = z_0(\lambda)$$

implying that the equilibrium reserve price is $r^* = 0$ and the seller earns

$$\Pi = 1 - z_0(\lambda) - z_1(\lambda).$$

We now show that a seller cannot do better by switching to a first or second price format. At all-pay stores we have $U_h = U_l = \Omega$, whereas at the deviating store we have $U'_h > U'_l$. There are three scenarios, therefore:

- The store attracts low types, while high types stay away. This requires $U'_l = \Omega > U'_h$, but contradicts the fact that $U'_h > U'_l$.
- The store attracts both types. This requires $U'_l = \Omega$ and $U'_h = \Omega$, but contradicts $U'_h > U'_l$.
- The store attracts high types, while low types stay away, i.e. $U'_h = \Omega > U'_l$. This is feasible, so we focus on this scenario.

The seller solves

$$\max_x \Pi' = 1 - z_0(x) - xU'_h \quad \text{s.t.} \quad U'_h = \Omega$$

The first order condition implies $z_0(x) = \Omega$. When $x_l = 0$, under the first-price or second-price formats we have

$$U'_h = z_0(x)(1 - r)$$

Solving $z_0(x)(1 - r) = z_0(x)$ implies that $r = 0$. Furthermore since $U'_h = \Omega = z_0(\lambda)$ implies $x = \lambda$. Consequently, the seller earns as much as Π , which does not present an incentive to deviate; hence the all pay equilibrium remains. Its uniqueness follows from Proposition 5. ■