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Novel Study on Active Reliable PID Controller Design Based on Probability Density Evolution Method and Interval-oriented Sequential Optimization Strategy

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ABSTRACT

Vibration active control is a significant problem in practical engineering, but the influence of hybrid uncertainties in practical engineering cannot be ignored. For the uncertainties with adequate data, probability theory can be used while for the uncertainties with insufficient data, probability theory is limited and interval quantification can be used. Taking the widely used proportional-integral-differential (PID) control as an example, a novel study on controller design based on probability density evolution method and interval-oriented sequential optimization strategy considering interval and random uncertainties is proposed. Firstly, under a certain sample point of interval uncertainties, probability density evolution equation is utilized to calculate the probability density function of uncertain response considering only random uncertainties. In this way the probabilistic reliability corresponding to the sample point can be obtained. Then the hybrid reliability is defined as the average probability within the interval uncertainty and the adaptive dichotomy method is utilized to choose proper sample points of interval uncertainties, in which the reliability curve and the interval uncertainty axis is regarded as the judgement of convergence. Next the hybrid reliability is introduced to the optimization as a constraint and the PID gains reliable design optimization is established to balance the energy-consuming and safety. Finally, the sequential optimization strategy is adopted to improve the efficiency of optimization. In every iteration step of the optimization, the constraint of deterministic optimal design is adjusted according to the static reliability at the moment before the passage or reaching the maximum value. Three numerical examples are presented to demonstrate the accuracy and practicability of the proposed framework.

Keywords: Vibration active control; hybrid uncertainties; probability density evolution equation; the adaptive dichotomy method; the reliable design; the sequential optimization

Nomenclature

Symbols	
$\mathbf{M}$	The mass matrix of the dynamic system
$\mathbf{P}$	The damping matrix of the dynamic system
$\mathbf{K}$	The stiffness matrix of the dynamic system
$\mathbf{u}(t)$	The active control force vector
$\mathbf{f}(t)$	The external disturbance force vector
$\mathbf{B}_s$	The position matrix of active control forces
$\mathbf{E}_s$	The position matrix of external disturbance forces
$\mathbf{x}(t)$	The displacement of the system
$\dot{\mathbf{x}}(t)$	The velocity of the system
$\ddot{\mathbf{x}}(t)$	The accelerate of the system
$\mathbf{Z}(t)$	The state vector of the state space
$\mathbf{y}(t)$	The output vector, namely the response of the system
K_p	The proportional gain
K_I	The integral gain
K_D	The differential gain
$\mathbf{e}(t)$	The error signal
y^{cr}	The allowable value of the displacement response
$g(t)$	The extreme state function
t_0	The initial moment of the analyzed time period
t_f	The final moment of the analyzed time period
J	The optimization objective
n_{Θ}	The amount of random uncertainties
n_b	The amount of interval uncertainties
$\mathbf{b}$	The interval uncertainties
Θ	The random uncertainties
t	Time
θ	The argument of Θ
$p_{\Theta}(\theta)$	The probability density function (PDF) of Θ
N_b	The amount of sample points chosen in the interval uncertainty
$p_{Y\Theta}(y, \theta, b_j, t)$	The joint PDF of $y(t)$ and Θ under the sample point b_j
$\dot{y}(\theta, b_j, t)$	The velocity under the sample point b_j
$\delta(\cdot)$	The Dirac function
y_0	The initial condition of $y(t)$

Ω_θ	The distribution domain of random uncertainties
Ω_s	The safe domain
$R_p(b_j, t)$	The probabilistic reliability under sample point b_j
$H[\bullet]$	The Heaviside function
$R_p(b_j)$	The probabilistic reliability of $R_p(b_j, t)$ at the final moment t_f
R	Then the hybrid reliability at the final moment
ε_s	The convergence criterion of the adaptive dichotomy method
N_s	The total amount of trapezoids in the adaptive dichotomy method
S_i	The area of the i -th trapezoid
R^{cr}	The allowable value of the reliability
$R[\bullet]$	The offset of the parameter
t_s	The moment of calculating the offset Δy_i^{cr} in the sequential optimization
$p_Y(y, b^c, t)$	The PDF of the response at a specified moment t_s
η	The static reliability at the moment t_s
y_{cal}^{cr}	The allowable value of the displacement response in the sequential optimization
ε_R	The convergence criterion of the reliability in optimization
N	The amount of samples in the specific numerical solution of PDEM
Superscripts	
$(\bullet)^I$	The interval parameter
$(\bullet)^c$	The center value of interval parameter
$(\bullet)^U$	The upper bound of interval parameter
$(\bullet)^L$	The lower bound of interval parameter
$\overline{(\bullet)}$	The upper bound caused by random uncertainties
$\underline{(\bullet)}$	The lower bound caused by random uncertainties
Subscripts	
$(\bullet)_i$	Count quantity
$(\bullet)_j$	Count quantity
l	Count quantity
$(\bullet)_{\max}$	The maximum value considering the sign of the vector
$(\bullet)_{\min}$	The minimum value considering the sign of the vector

1 Introduction

Vibration active control is a significant problem in practical engineering [1]. Unexpected vibration may deteriorate the performance of the system or even cause the damage of the system. Active vibration control can suppress the vibration and noise without obviously increasing the weight of the structure compared with passive vibration control [2], which meets the lightweight design requirements in practical engineering, especially in aerospace field. There are various mature empirical methods or analytical method of designing an active controller, such as Proportional-Integral-Differential (abbreviated as PID) control [3], pole assignment control [4, 5] and optimal control [6, 7]. Above these methods optimal control is a common method to guide the design of controller [8]. In an optimal control, the controller is designed to minimum or maximum the objective index while meet the response limit or the control force limit or other constraints.

However, there are various uncertainties in practical engineering [9], such as the noises in the sensors' signals, the dispersions of material properties and measurement errors, thus the real system has errors compared with ideal model [10]. The optimal controllers designed according to the deterministic systems may not meet constraints because of these uncertainties. The uncertainties should be considered when designing a controller for a system in practical engineering. The traditional robust control [11] can guarantee the safety of an uncertain system, but the close-loop system is often too conservative with large energy-consume [12]. Thus, the design according to the reliability (namely 'reliable design') [13] is proposed to balance the safety and energy-consuming of the uncertain system [14-16].

For some uncertainties, their statistical data is adequate. In this situation, these uncertainties can be quantified with probability density function (abbreviated as PDF) or characteristic function (abbreviated as CF) can be obtained [17-20], then the response and the reliability correspondingly can be calculated with probability theory [21-24]. The Monte Carlo simulation (abbreviated as MCS) defines the failure possibility as the ratio of samples to the total samples [25]. MCS requires adequate large amount of sample data, which limits its application in practical engineering. Cornell proposed the first-order second moment (abbreviated as FOSM) method [26], in which the reliability index is established under normal distribution and linear extreme state function. However, FOSM faces lack-of-invariant problem, thus the advanced first-order second moment (abbreviated as AFOSM) is proposed [27], in which the non-linear extreme state function is expanded at the design point. Rice firstly proposed

the first-passage theory [28] to calculate the probabilistic reliability of the stochastic process response. Davenport [29] solved the PDF of the extremum value of a stationary response random process with the assumption that the number of the passage is Poisson distribution. Vanmarcke [30] solved the PDF of the extremum value with the assumption of two-state Markov process. Li and Chen [31] proposed the probability density evolution method based on the principle of probability conservation and obtain decoupled probability density evolution method (abbreviated as PDEM). In this situation, the dynamic reliability can be solved by introducing absorb boundary condition [32] and integral the joint PDF, or construct a virtual random process [33] and transform the calculation of reliability into one-dimensional integration problem. Pang and Xu et al. [34, 35] performed stochastic seismic analysis to high concrete-faced rockfill dams considering the uncertainty from ground motions with PDEM, which can analyze strongly nonlinear and other geotechnical engineering structures with satisfactory accuracy. Based on this, a novel evaluating index called the average large failure probability [36] is proposed which takes time histories into account and has a good applicability to seismic performance assessment [37]. Meng et al. [38] proposed a semi-analytical extreme value method to improve the computational efficiency of extreme value method by the expansion optimal linear estimation method to simulate the dependence among different time instants.

However, for some uncertainties of dynamic systems in practical engineering [39], there samples are insufficient and their PDF cannot be accurately obtained, thus probabilistic reliability is limited in this situation. But their boundaries are easier to be obtained, thus these uncertainties are quantified with non-probabilistic interval forms [40, 41]. The non-probabilistic interval quantification of uncertainty is firstly proposed by Ben-Haim and Elishakoff [42, 43], correspondingly the reliability considering interval uncertainties is developed in recent years [44]. Mahadevan and Dey [45] calculated the time-dependent reliability of brittle structures with adaptive MCS. Wang et al [46, 47] performed the time-dependent reliability of interval uncertain systems in which the first-passage theory is used and correlation of time moments is taken into consideration. Chang et al. [48] proposed a time-dependent model to estimate the safety of mechanical structures with interval errors, in which the time-dependent extreme state function is transformed into time-independent one. Yang et al. [49] used time-dependent reliability to evaluate the safety of beam structures considering the time dependent deterioration of structure resistance and the load time effect.

There are various uncertainties in practical engineering, and their proper quantification methods are different because of sample data, correlation and other factors, namely there are

hybrid uncertainties [50, 51]. For these uncertain systems with hybrid uncertainties, the reliability is also hybrid reliability. Wen et al. [52] proposed a hybrid reliability index considering both probabilistic and fuzzy uncertainties. Ni and Qiu [53] proposed a hybrid reliability model considering probabilistic and fuzzy uncertainties which can handle either linear or non-linear state functions. Shi et al. [54] proposed two indices to measure safety degree of time-dependent structure from different aspects considering probabilistic and fuzzy inputs.

However, the research on the controller reliable design of the vibration active control dynamic system with hybrid uncertainties is still preliminary. Moreover, the traditional double-loop optimization strategy for solving hybrid reliable design problem is low-efficiency, thus new strategy is required. Taking the widely-used PID controller design as an example, a PID controller hybrid reliability-based sequential optimization (HRBSO) strategy considering both random and interval uncertainties is proposed. The novelties and contributions mainly lie in three aspects: (1) A hybrid reliability considering both random and interval uncertainties is proposed and the adaptive dichotomy method is proposed to perform the detailed calculation with enough accuracy and efficiency; (2) The reliable optimization equation of PID gains considering the above hybrid reliability is proposed, which can balance the energy-consuming and safety of hybrid uncertain systems; (3) The sequential optimization strategy is used to decouple the PID gains optimization and hybrid reliability evaluation. In this way the reliable optimization can be performed with high efficiency.

This manuscript is organized as follows. Section 2 establish the PID gains optimization equation of a deterministic vibration active control system. Section 3 uses PDEM to calculate the probabilistic reliability under a certain sample of interval uncertainty, then the adaptive dichotomy method is used to choose proper sample points of interval uncertainty and hybrid reliability is calculated as the average probabilistic reliability over the interval uncertainties. Section 4 establishes the PID gains optimization equation of a hybrid uncertain system, in which the hybrid reliability is regarded as a constraint. Then the sequential optimization strategy is introduced to improve the efficiency of optimization. Three examples in Section 5 are presented to prove the accuracy and practicability of the proposed framework.

2 The PID gains optimal design of a deterministic system

For an n degree-of-freedom (abbreviated as DOF) system in practical engineering with p active close-loop control forces $\mathbf{u}(t)$ and q external disturbance loads $\mathbf{f}(t)$, the dynamic

equation is:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{P}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}_s \mathbf{u}(t) + \mathbf{E}_s \mathbf{f}(t) \quad (1)$$

in which $\mathbf{M}, \mathbf{P}, \mathbf{K}$ are all the n -dimensional matrices with the meanings of the mass matrix, the damping matrix and the stiffness matrix, respectively. $\mathbf{x}(t)$, $\dot{\mathbf{x}}(t)$ and $\ddot{\mathbf{x}}(t)$ are the displacement, the velocity and the accelerate of the system, respectively. $\mathbf{u}(t)$ is the $p \times 1$ vector of control force. $\mathbf{f}(t)$ is the $q \times 1$ vector of disturbance loads. $\mathbf{B}_s$ is the $n \times p$ position matrix corresponding to $\mathbf{u}(t)$ and $\mathbf{E}_s$ is the $n \times q$ position matrix corresponding to $\mathbf{f}(t)$. The state space of the system in Eq. (1) can be written as:

$$\begin{aligned} \dot{\mathbf{Z}}(t) &= \mathbf{A}\mathbf{Z}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{E}\mathbf{f}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{Z}(t) \end{aligned} \quad (2)$$

in which $\mathbf{Z}(t) = [\mathbf{x}(t) \ \dot{\mathbf{x}}(t)]^T$, $\mathbf{A} = \begin{bmatrix} \mathbf{O}_{n \times n} & \mathbf{I}_{n \times n} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{P} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \mathbf{O}_{n \times p} \\ \mathbf{M}^{-1}\mathbf{B}_s \end{bmatrix}$, $\mathbf{E} = \begin{bmatrix} \mathbf{O}_{n \times q} \\ \mathbf{M}^{-1}\mathbf{E}_s \end{bmatrix}$, $\mathbf{C}$ is the $s \times n$ output matrix of the state space, and $\mathbf{y}(t)$ is the s -dimensional output vector. $\mathbf{O}$ means the zero matrix and $\mathbf{I}$ means the unit matrix.

For a single-output system, the relationship between the PID control force and the noise signal is

$$u(t) = -K_p \mathbf{C}\mathbf{Z}(t) - K_I \int_0^t \mathbf{C}\mathbf{Z}(\tau) d\tau - K_D \frac{d(\mathbf{C}\mathbf{Z}(t))}{dt} \quad (3)$$

in which K_p , K_I , K_D are the proportional gain, the integral gain and the differential gain of the controller, respectively.

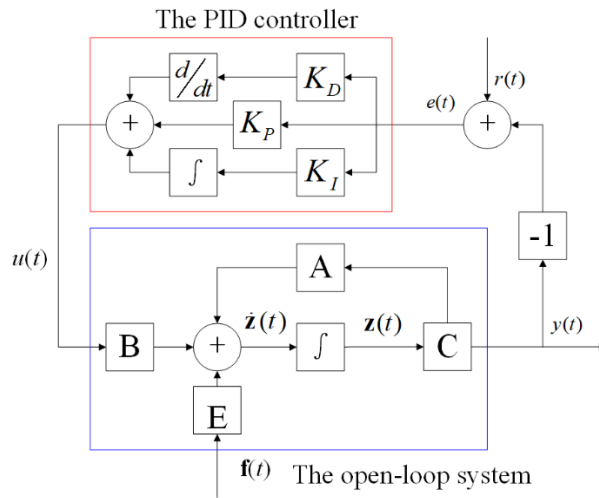


Fig. 1 The diagram of the close-loop system with one PID controller

When designing a PID controller by optimization, the PID gains should achieve the

requirements with the least energy. Without losing generality, only the displacement requirement in a single-output system is considered in the later analyses and the constraint in the deterministic optimal design is:

$$g(t) = y^{cr} - \max(|y_{\max}|, |y_{\min}|) \quad t_0 \leq t \leq t_f \quad (4)$$

in which y^{cr} is the allowable value of the displacement response. t_0 and t_f are the initial and final moment of the analyzed time period $[t_0, t_f]$. $g(t)$ is the extreme state function and $g(t) < 0$ means the failure of the system. For $y(t)$ there are a maximum absolute value of peaks $y_{\max}$ and a maximum absolute value of troughs $y_{\min}$.

Similarly, denoting the maximum absolute peak value and maximum absolute trough value of the control force $u(t)$ during the time period $[t_0, t_f]$ as $u_{\max}$ and $u_{\min}$, then the optimization objective can be defined as the maximum absolute value and amplitude of $u_{\max}$ and $u_{\min}$.

$$J = \max(|u_{\max}|, |u_{\min}|) \quad (5)$$

Based on the above analyses, the optimization function of the deterministic system is summarized as

$$\begin{aligned} & \text{find } K_p, K_I, K_D \\ & \min J = \max(|u_{\max}|, |u_{\min}|) \\ & \text{s.t. } g(t) = y^{cr} - \max(|y_{\max}|, |y_{\min}|) \geq 0 \\ & \quad K_{p\min} \leq K_p \leq K_{p\max} \\ & \quad K_{I\min} \leq K_I \leq K_{I\max} \\ & \quad K_{D\min} \leq K_D \leq K_{D\max} \end{aligned} \quad (6)$$

in which $K_{p\max}$, $K_{I\max}$, $K_{D\max}$, $K_{p\min}$, $K_{I\min}$ and $K_{D\min}$ are the boundaries of the design domain of corresponding PID gains, respectively. These boundaries are preset to improve the optimization efficiency. Eq. (6) is a deterministic optimization equation. It means that the optimal PID gains should make the objective smallest, namely the maximum absolute value of the control force is the smallest, while the deterministic response should not exceed the allowable value y^{cr} .

3 The hybrid reliability evaluation based on the probability density evolution method and the adaptive dichotomy method

3.1 The probabilistic reliability evaluation based on the probability density evolution method

Without loss of generality, for a single-output dynamic system with n_{Θ} random uncertainties and n_b interval uncertainties, the state space of the system in Eq. (2) can be rewritten as:

$$\begin{aligned}\dot{\mathbf{Z}}(\boldsymbol{\theta}, \mathbf{b}, t) &= \mathbf{A}(\boldsymbol{\theta}, \mathbf{b})\mathbf{Z}(\boldsymbol{\theta}, \mathbf{b}, t) + \mathbf{B}(\boldsymbol{\theta}, \mathbf{b})u(\boldsymbol{\theta}, \mathbf{b}, t) + \mathbf{E}(\boldsymbol{\theta}, \mathbf{b})\mathbf{f}(\boldsymbol{\theta}, \mathbf{b}, t) \\ y(\boldsymbol{\theta}, \mathbf{b}, t) &= \mathbf{C}(\boldsymbol{\theta}, \mathbf{b})\mathbf{Z}(\boldsymbol{\theta}, \mathbf{b}, t)\end{aligned}\quad (7)$$

in which $\mathbf{b}$ is the vector of interval uncertainties and $\boldsymbol{\Theta}$ is the vector of random uncertainties with its PDF $p_{\Theta}(\boldsymbol{\theta})$, in which $\boldsymbol{\theta}$ is the argument of $\boldsymbol{\Theta}$.

The interval uncertainties can be quantified with interval mathematics

$$\mathbf{b} = [\mathbf{b}^L \quad \mathbf{b}^U] = [\mathbf{b}^c - \mathbf{b}^r \quad \mathbf{b}^c + \mathbf{b}^r] \quad (8)$$

in which $\mathbf{b}^L$ and $\mathbf{b}^U$ are the lower bound and upper bound of the interval parameters, respectively. $\mathbf{b}^c$ and $\mathbf{b}^r$ are the center value and radius of the interval parameters, respectively.

Denoting there are N_b sample points chosen in the l -th interval uncertainty b , for a given sample point of interval uncertainty $b_j \in [\mathbf{b}^L \quad \mathbf{b}^U]$, the state space in Eq. (7) is simplified with only random uncertainty $\boldsymbol{\theta}$, thus the response $y(t)$ is the function of $\boldsymbol{\theta}$. For further adjustment of the deterministic displacement constraint in the sequential optimization as shown in Section 4, the PDF of the response at all the moments during $[t_0, t_f]$ should be obtained. Thus the PDEM can be applied to this system according to Ref [55] as follows:

$$\frac{\partial p_{y\Theta}(y, \boldsymbol{\theta}, b_j, t)}{\partial t} + \dot{y}(\boldsymbol{\theta}, b_j, t) \frac{\partial p_{y\Theta}(y, \boldsymbol{\theta}, b_j, t)}{\partial y} = 0 \quad (9)$$

in which $p_{y\Theta}(y, \boldsymbol{\theta}, b_j, t)$ is the joint PDF of $y(t)$ and $\boldsymbol{\Theta}$ under the sample point b_j , and $\dot{y}(\boldsymbol{\theta}, b_j, t)$ is the corresponding velocity. The corresponding initial condition of Eq. (9) is

$$p_{y\Theta}(y, \boldsymbol{\theta}, t)|_{t=0} = \delta(y - y_0) p_{\Theta}(\boldsymbol{\theta}, b_j) \quad (10)$$

In the above equation y_0 is the initial condition of $y(t)$ and $\delta(\bullet)$ is the Dirac

function, which satisfies the following two equations:

$$\delta(y) = \begin{cases} \infty & y = y_0 \\ 0 & y \neq y_0 \end{cases} \quad (11)$$

$$\int_{-\infty}^{+\infty} \delta(y) dy = 1 \quad (12)$$

The selectivity of the Dirac function in Eq. (13) is adopted to calculate Eq. (10):

$$\int_{-\infty}^{+\infty} \delta(y - y_0) p_{\Theta}(\mathbf{\theta}, b_j) dy = p_{\Theta}(y_0, \mathbf{\theta}, b_j) \quad (13)$$

At the same time, the boundary condition is:

$$p_{Y\Theta}(y, \mathbf{\theta}, b_j, t)|_{y \rightarrow \pm\infty} = 0 \quad (14)$$

By solving Eq. (9) with the initial condition in Eq. (10), the joint PDF can be obtained.

Then the PDF of the response $y(t)$ can be calculated by integral:

$$p_Y(y, b_j, t) = \int_{\Omega_0} p_{Y\Theta}(y, \mathbf{\theta}, b_j, t) d\mathbf{\theta} \quad (15)$$

in which Ω_0 is the distribution domain of random uncertainties.

The failure of a dynamic system can be defined as the response exceeding the allowable value, namely the passage occurs. Thus, the probabilistic reliability under sample point b_j is:

$$R_p(b_j, t) = P_r \{y(\tau) \in \Omega_s, t_0 \leq \tau \leq t\} \quad (16)$$

in which Ω_s denotes the safe domain, namely $g(t) \geq 0$ $t_0 \leq t \leq t_f$. Eq (16) means once the dynamic response enters the failure domain at a certain moment, the probability it carries is zero, namely the probability dissipates as the sample $\mathbf{\theta}_i$ shown in Fig. 2. Thus the probabilistic reliability in Eq. (16) is monotonically non-increasing.

Considering the probability dissipation, Eq. (9) is changed into:

$$\frac{\partial p_{Y\Theta}(y, \mathbf{\theta}, b_j, t)}{\partial t} + \dot{y}(\mathbf{\theta}, b_j, t) \frac{\partial p_{Y\Theta}(y, \mathbf{\theta}, b_j, t)}{\partial y} = -H[y(\mathbf{\theta}, b_j, t)] p_{Y\Theta}(y, \mathbf{\theta}, b_j, t) \quad (17)$$

in which $H[\bullet]$ is the Heaviside function, namely

$$H[y(\mathbf{\theta}, b_j, t)] = \begin{cases} 0 & y(\mathbf{\theta}, b_j, t) \in \Omega_s \\ 1 & y(\mathbf{\theta}, b_j, t) \notin \Omega_s \end{cases} \quad (18)$$

The detailed numerical solution procedure for Eq. (9) and Eq. (17) is shown in **Appendix**

A. After solving Eq. (17), then the probabilistic reliability under the sample point b_j can be calculated:

$$R_p(b_j, t) = \int_{\Omega_s} p_Y(y, b_j, t) dy \quad (19)$$

In most situations, the analytical solution of Eq. (17) does not exist and numerical solution is performed as shown in Fig. 2. In this process, the total variation diminishing schemes (abbreviated as TVD) [56] is used to calculate the joint PDF and integrals in Eq. (15) and Eq. (19) are replaced with summations.

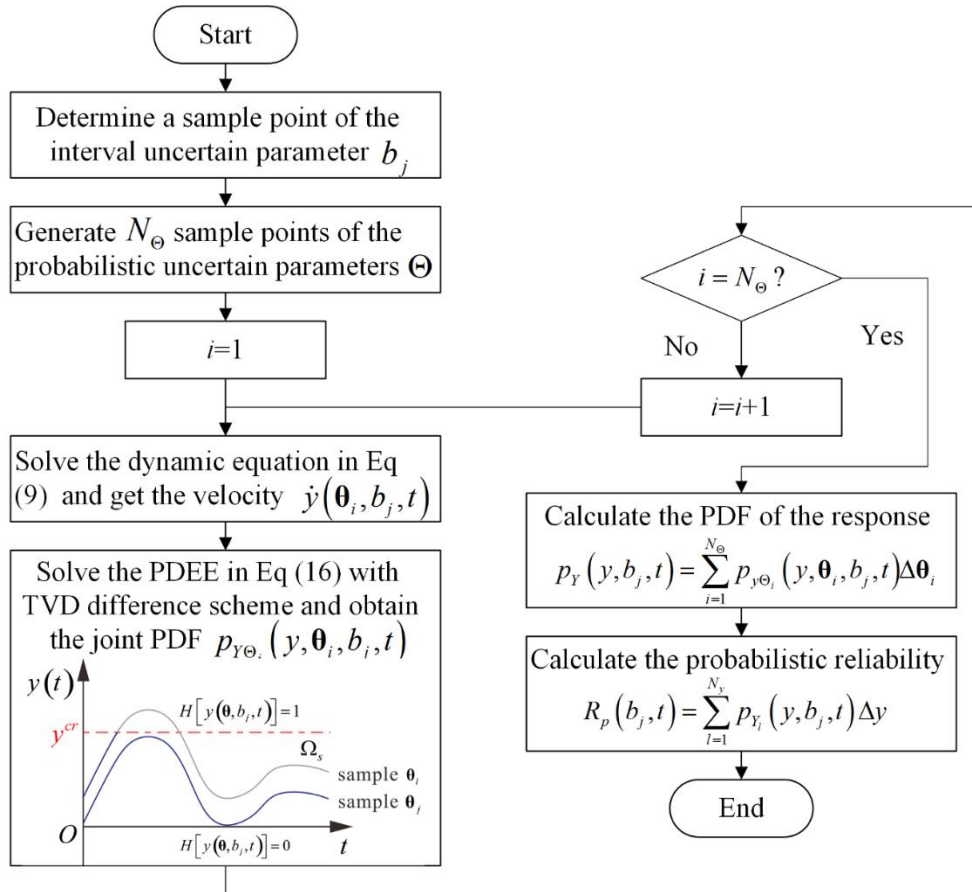


Fig. 2 The whole process of the numerically solving probabilistic reliability based on PDEM

In this way the probabilistic reliability is obtained. In the next part, the interval uncertainties will be introduced into the analysis, and the hybrid reliability will be obtained.

3.2 The hybrid reliability evaluation with the adaptive dichotomy method

Under a certain sample point b_j of the interval uncertainty b , the probabilistic time-dependent reliability can be obtained by the PDEM. The time-dependent reliability at the final analyzed moment t_f is the lowest and concerned. The probabilistic reliability can be simply

expressed as:

$$R_p(b_j, t) \Big|_{t=t_f} = R_p(b_j) \quad (20)$$

When a series of sample points of the interval uncertainty b are chosen, a series of probabilistic reliability corresponding to different b_j will be obtained as shown in Fig. 3. Then the hybrid reliability at the final moment can be defined as the average probabilistic reliability over the interval uncertainties

$$R = \frac{1}{b^U - b^L} \int_{b^L}^{b^U} R_p(b) db \quad (21)$$

The integral part can be regarded as the area between the reliability curve $R_p(b)$ and the interval uncertainty axis b within the interval $[b^L, b^U]$ shown in Fig. 3 (c). Proper sample points within the interval $[b^L, b^U]$ should be chosen to approximately fit the reliability curve and reach a balance between accuracy and efficiency. Thus, the adaptive dichotomy method is used.

As shown in Fig. 3 (a), $b_3 = \frac{b_1 + b_2}{2}$ is the dichotomy of the interval $[b_1, b_2]$ and S is the area of the trapezoid within the interval $[b_1, b_2]$. Similarly the area S_1 within the interval $[b_1, b_3]$, the area S_2 in the interval $[b_3, b_2]$ can be calculated

$$\begin{aligned} S_1 &= 0.5(R_p(b_1) + R_p(b_3))(b_3 - b_1) \\ S_2 &= 0.5(R_p(b_2) + R_p(b_3))(b_2 - b_3) \\ S &= 0.5(R_p(b_1) + R_p(b_2))(b_2 - b_1) \end{aligned} \quad (22)$$

If $R_p(b)$ is linear enough, the following convergence criterion should be met:

$$|S - (S_1 + S_2)| < \varepsilon_s \quad (23)$$

in which ε_s is the convergence criterion of the adaptive dichotomy method. Eq. (23) means that for a uncertain interval $[b_1, b_2]$, if the difference between the original trapezoid area and the sum of two dichotomized trapezoid areas is small enough, the reliability curve within the interval $[b_1, b_2]$ is linear enough. Namely the existing dichotomy b_3 can guarantee enough accuracy, and the adaptive dichotomy method can be terminated. A smaller ε_s means a more accurate fitting of $R_p(b)$ but more computation. For the determine of ε_s , the hybrid

reliability result is the judgement. When decreasing ε_s , the corresponding hybrid reliability R can be obtained. If R does not obviously change with the decrease of ε_s , then the value of ε_s is small enough to guarantee the accuracy of R . In later numerical examples this value is set as $\varepsilon_s = 10^{-3}$.

On the contrary, if Eq. (23) is not met, for the intervals do not meet Eq. (23), the dichotomy should be performed again to these intervals. Taking the interval $[b_1 \ b_3]$ as an example, the corresponding dichotomy point $b_{1,3} = \frac{b_1 + b_3}{2}$ is added as shown in Fig. 3 (b) and the updated areas are

$$\begin{aligned} S_{1,1} &= 0.5(R_p(b_1) + R_p(b_{1,3}))(b_{1,3} - b_1) \\ S_{1,2} &= 0.5(R_p(b_{1,3}) + R_p(b_3))(b_3 - b_{1,3}) \\ S_1 &= 0.5(R_p(b_3) + R_p(b_1))(b_3 - b_1) \end{aligned} \quad (24)$$

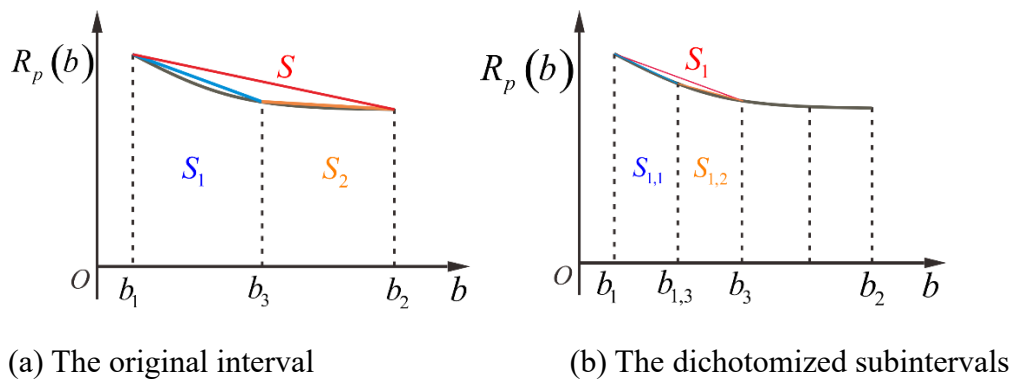
and the convergence criterion is similar with Eq. (23):

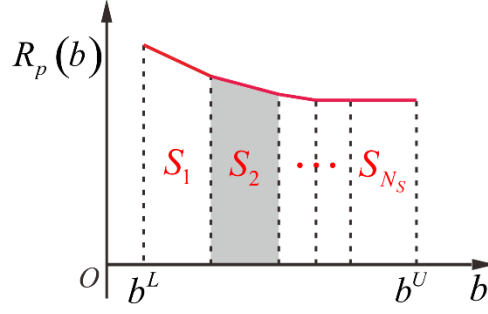
$$|S_1 - (S_{1,1} + S_{1,2})| < \varepsilon_s \quad (25)$$

For the first iteration, $b_1 = \underline{b}$ and $b_2 = \bar{b}$. Then above process is repeated until all the divided intervals meet the convergence criterion. In this way the original integral area in Eq. (21) is approximately divided into several trapezoids. Then the continuous integral in Eq. (21) can be transformed into the sum of several trapezoidal areas, namely

$$R = \frac{1}{b^U - b^L} \sum_{i=1}^{N_s} S_i \quad (26)$$

in which N_s is the total amount of trapezoids obtained by the adaptive dichotomy method, and S_i is the area of the i -th trapezoid.





(c) The calculation of the hybrid reliability

Fig. 3 The diagrams of the adaptive dichotomy method

The whole process is shown in Fig. 4.

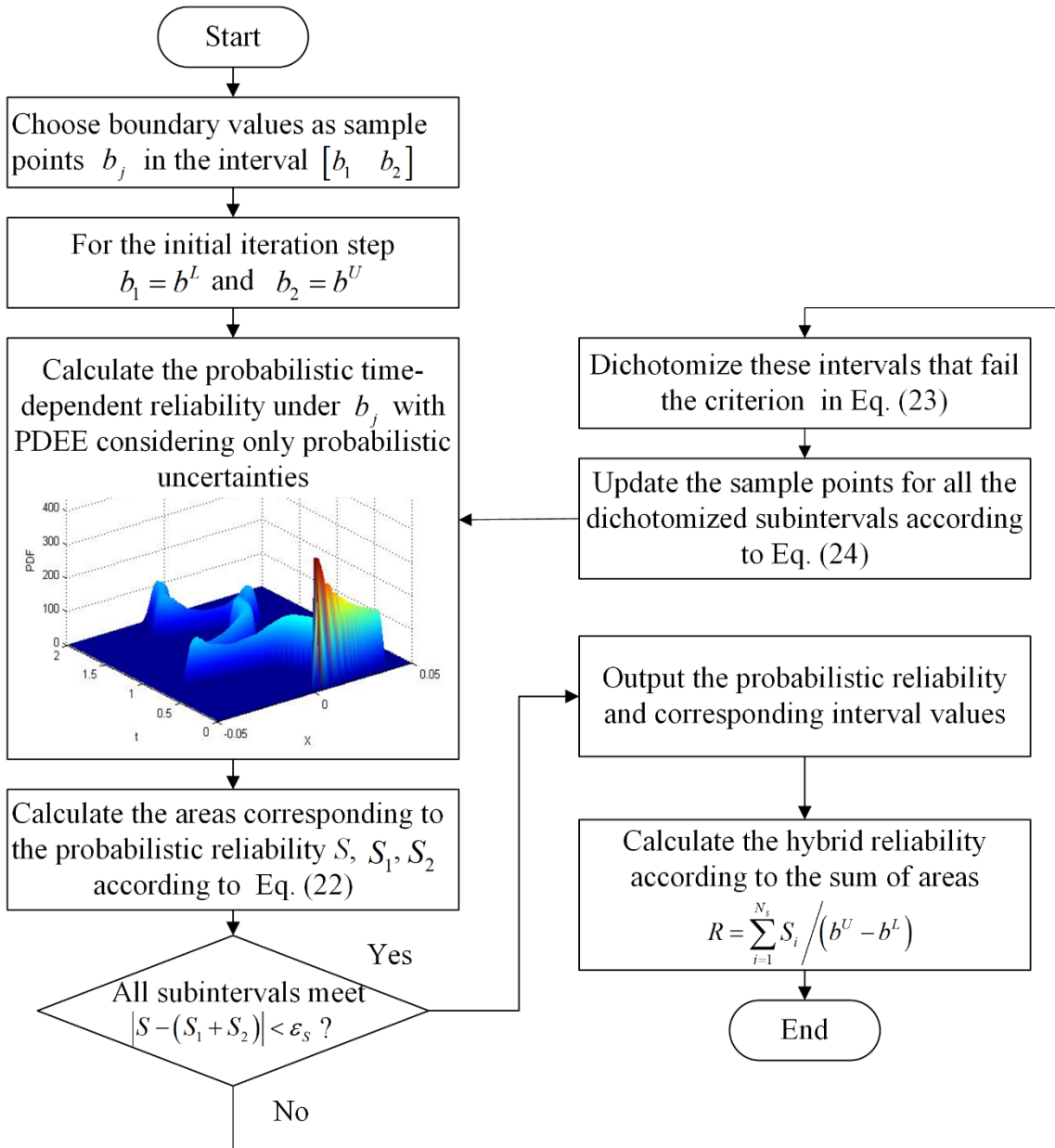


Fig. 4 The flowchart of the adaptive dichotomy method

4 The PID gains hybrid reliable design based on the sequential optimization

4.1 The optimal design based on the double-loop optimization strategy

With the hybrid reliability evaluation, the hybrid uncertainties' influence on the system's safety is reflected. Then the deterministic displacement constraint $g(t) = y^{cr} - \max(|y_{\max}|, |y_{\min}|) \geq 0$ in Eq. (6) is replaced with hybrid reliability constraint $R \geq R^{cr}$, and the reliable design optimization equation is established:

$$\begin{aligned}
 & \text{find } K_p, K_I, K_D \\
 & \min J = \max(|u_{\max}|, |u_{\min}|) \\
 & \text{s.t. } R \geq R^{cr} \\
 & \quad K_{p\min} \leq K_p \leq K_{p\max} \\
 & \quad K_{I\min} \leq K_I \leq K_{I\max} \\
 & \quad K_{D\min} \leq K_D \leq K_{D\max}
 \end{aligned} \tag{27}$$

in which R^{cr} is the allowable value of the reliability. Eq. (27) is a reliable optimization with hybrid uncertainties. It means that the optimal PID gains should make the objective smallest and meet the reliability requirement, namely the hybrid reliability is equal or larger than the allowable value R^{cr} .

The direct strategy to solve Eq. (27) is the double-loop optimization in Fig. 5, in which the optimization of PID gains and reliability evaluation are coupled. The outer loop is the optimization of PID gains. To avoid local optimal solutions, the intelligent optimization algorithms are always adopted in the outer loop such as simulated annealing and particle swarm optimization. Thus, many sample points of PID gains are generated by these optimization algorithms in an iteration step and the corresponding optimization objective and hybrid reliability corresponding to each sample point are obtained by substituting to the inner loop. Then new sample points of PID gains are generated according to these results in the next iteration step until the optimization is converged.

The inner loop is the hybrid reliability evaluation. For all the sample points of PID gains generated from the outer loop, the hybrid reliability evaluation is performed in which PDEM is repeated several times corresponding to the sample points of interval uncertainties chosen by the adaptive dichotomy method. When an iteration of outer loop is performed, the whole

process of the inner loop is repeated for all the generated sample points. The coupled optimization processes result in large calculation amounts. To improve the optimization efficiency, the sequential optimization strategy is utilized.

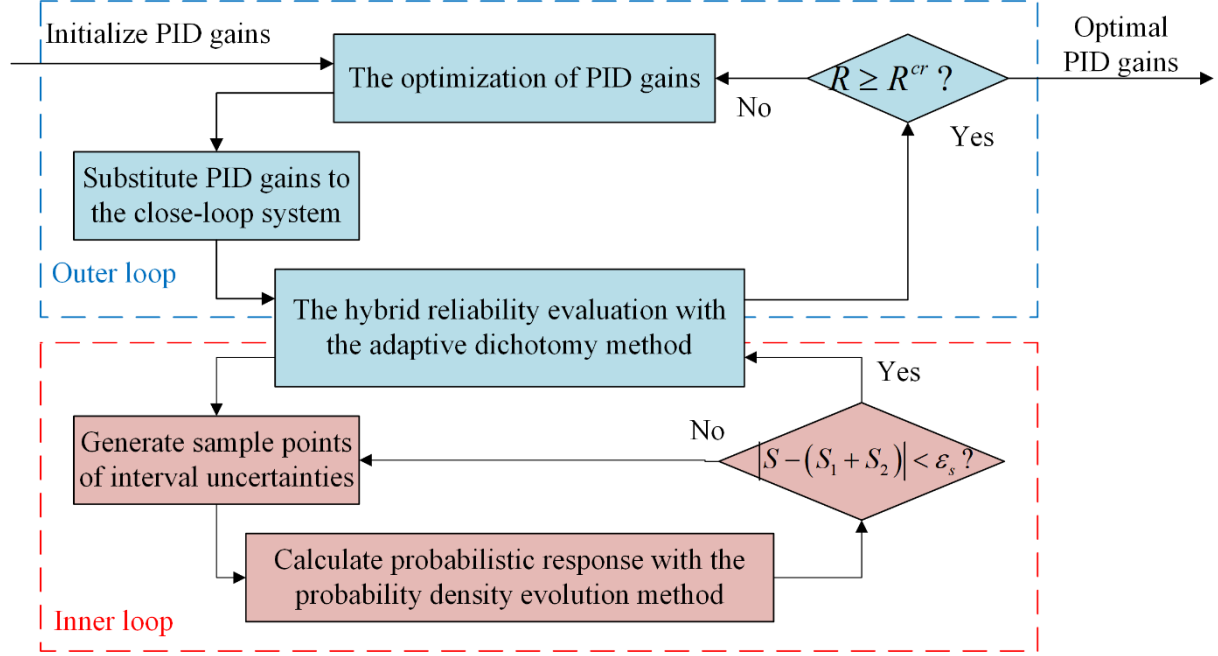


Fig. 5 The diagram of the double-loop optimization

4.2 The optimal design based the sequential optimization strategy

The sequential optimization [57-59] is widely used in aircraft optimization design. For the PID controller reliable design with hybrid uncertainties, the sequential optimization decouples the optimization of PID gains and reliability evaluation. A deterministic optimization with original displacement constraints is performed. Then the deterministic constraint in Eq. (4) is adjusted according to the reliability result in each iteration step and then the updated deterministic optimization is performed. The above process is repeated until the convergence criterion is met.

The adjustment of the deterministic displacement constraint in i -th iteration is

$$y_{i+1}^{cr} = y_i^{cr} + \Delta y_i^{cr} \quad (28)$$

which is a significant part in the sequential optimization. In the i -th iteration step, the offset Δy_i^{cr} is calculated according to the PDF of the response $y(b^c, t)$ obtained by PDEM at a specified moment t_s . The detailed choose strategies of t_s and Δy_i^{cr} will be stated later.

After a deterministic optimal design, the corresponding PID gains can be obtained. Then

PDEM can be performed under the center value b^c of interval uncertainties and the bounds of the response can be obtained as shown in Fig. 6. It should be stressed that the bounds $\bar{y}(t)$ and $\underline{y}(t)$ are only caused by random uncertainties, which is different with interval parameters such as in Eq. (8). Considering the relationships between $|y_{\max}|$ and $|y_{\min}|$, and the value of R , there are mainly four different cases as shown in Fig. 6, in which t_s is chosen. Then the value of Δy_i^{cr} can be obtained as shown in Fig. 7

Denote the static reliability at the moment t_s as η . According to Eq. (17) and Eq. (19), the $R_p(b_j, t)$ is non-increasing with time and thus $\eta \geq R$. When η decreases R may also decrease if possibility dissipation happens and vice versa. η can influence the change of R and is easier to be calculated because it avoids the time integral of R . Thus R can guide the adjustment of y^{cr} in the sequential optimization.

Case 1: $R < 1$ and $|y_{\max}| > |y_{\min}|$

This means the system has a possibility of failure at the final moment t_f and the failure may happen at the peak. And there exist a moment just before the passage occurs. This is the analyzed moment t_s , namely $\bar{y}(t_s) = y^{cr}$.

As shown in Fig. 7 (a)-(c), according to the PDF $p_y(y, b^c, t_s)$ at the moment t_s , η is calculated as follows:

$$\eta = \begin{cases} \frac{y^{cr} - \underline{y}(t_s)}{\bar{y}(t_s) - \underline{y}(t_s)} & y^{cr} > \bar{y}(t_s) \\ 1 & y^{cr} = \bar{y}(t_s) \\ 1 - \frac{S_{\text{shadow}}}{S_{\text{total}}} & y^{cr} < \bar{y}(t_s) \end{cases} \quad (29)$$

in which S_{total} and S_{shadow} are the areas of the total region and shadow region shown in Fig. 7(b), respectively. These two areas are calculated by the integral of PDF, namely

$$S_{\text{total}} = \int_{\underline{y}(t_s)}^{\bar{y}(t_s)} p_Y(y, b^c, t_s) dy \quad \text{and} \quad S_{\text{shadow}} = \int_{y^{cr}}^{\bar{y}(t_s)} p_Y(y, b^c, t_s) dy$$

The difference between the allowable reliability R^{cr} and final moment R is $\Delta R = R^{cr} - R$. In the i -th iteration step, at the moment t_s , $\eta = 1$ is met in the initial state. Then $\eta_i = 1 + \Delta R$ and the corresponding allowable value y_{cal}^{cr} can be calculated with $p_Y(y, b^c, t)$ according to Eq. (29). In this way Δy_i^{cr} can be obtained:

$$\Delta y_i^{cr} = \bar{y}(t_s) - y_{cal}^{cr} \quad (30)$$

then y_{i+1}^{cr} can be obtained in Eq. (28) and the next iteration can be performed.

Case 2: $R = 1$ and $|y_{\max}| > |y_{\min}|$

This means the system is safe at the final moment t_f and the passage does not happen.

In this situation, the moment at $y_{\max}$ is chosen as t_s . In this situation the calculation of Δy_i^{cr} is:

$$\Delta y_i^{cr} = y^{cr} - y_{\max} + \bar{y}(t_s) - y_{cal}^{cr} \quad (31)$$

The first two terms in Eq (31) mean moving the allowable value to the maximum absolute peak of the response interval. Then the state is transformed into Fig. 7 (a) and the last two terms are obtained as expressed according to Eq. (29).

Case 3: $R < 1$ and $|y_{\max}| < |y_{\min}|$

This means the system has a possibility of failure at the final moment t_f and the failure may happen at the trough. And there exist a moment just before the passage occurs. This is the analyzed moment t_s , namely $\underline{y}(t_s) = y^{cr}$.

Similar with Case 1 as shown in Fig. 7 (d)-(f), according to the PDF $p_y(y, b^c, t_s)$ at the moment t_s , η is calculated as follows:

$$\eta = \begin{cases} \frac{\bar{y}(t_s) + y^{cr}}{\bar{y}(t_s) - \underline{y}(t_s)} & -y^{cr} < \underline{y}(t_s) \\ 1 & -y^{cr} = \underline{y}(t_s) \\ 1 - \frac{S_{\text{shadow}}}{S_{\text{total}}} & -y^{cr} > \underline{y}(t_s) \end{cases} \quad (32)$$

in which S_{total} and S_{shadow} are the areas of the total region and shadow region shown in Fig. 7(e), respectively. These two areas are calculated by the integral of PDF, namely

$$S_{\text{total}} = \int_{\underline{y}(t_s)}^{\bar{y}(t_s)} p_Y(y, b^c, t_s) dy \quad \text{and} \quad S_{\text{shadow}} = \int_{\underline{y}(t_s)}^{y^{cr}} p_Y(y, b^c, t_s) dy$$

At the moment t_s , $\eta = 1$ is also met in the initial state. Then $\eta_i = 1 + \Delta R$ and the corresponding allowable value y_{cal}^{cr} can be calculated with $p_Y(y, b^c, t)$ according to Eq. (32). In this way Δy_i^{cr} can be obtained:

$$\Delta y_i^{cr} = -y_{cal}^{cr} - \underline{y}(t_s) \quad (33)$$

then y_{i+1}^{cr} can be obtained in Eq. (28) and the next iteration can be performed.

Case 4: $R = 1$ and $|y_{\max}| < |y_{\min}|$

This means the system is safe at the final moment t_f and the passage does not happen.

In this situation, the moment at $y_{\min}$ is chosen as t_s . In this situation the calculation of Δy_i^{cr} is:

$$\Delta y_i^{cr} = y_{\min} + y^{cr} - y_{cal}^{cr} - \underline{y}(t_s) \quad (34)$$

The first two terms in Eq (34) mean moving the allowable value to the maximum absolute trough of the response interval. Then the state is transformed into Fig. 7 (d) and the last two terms are obtained as expressed according to Eq. (32).

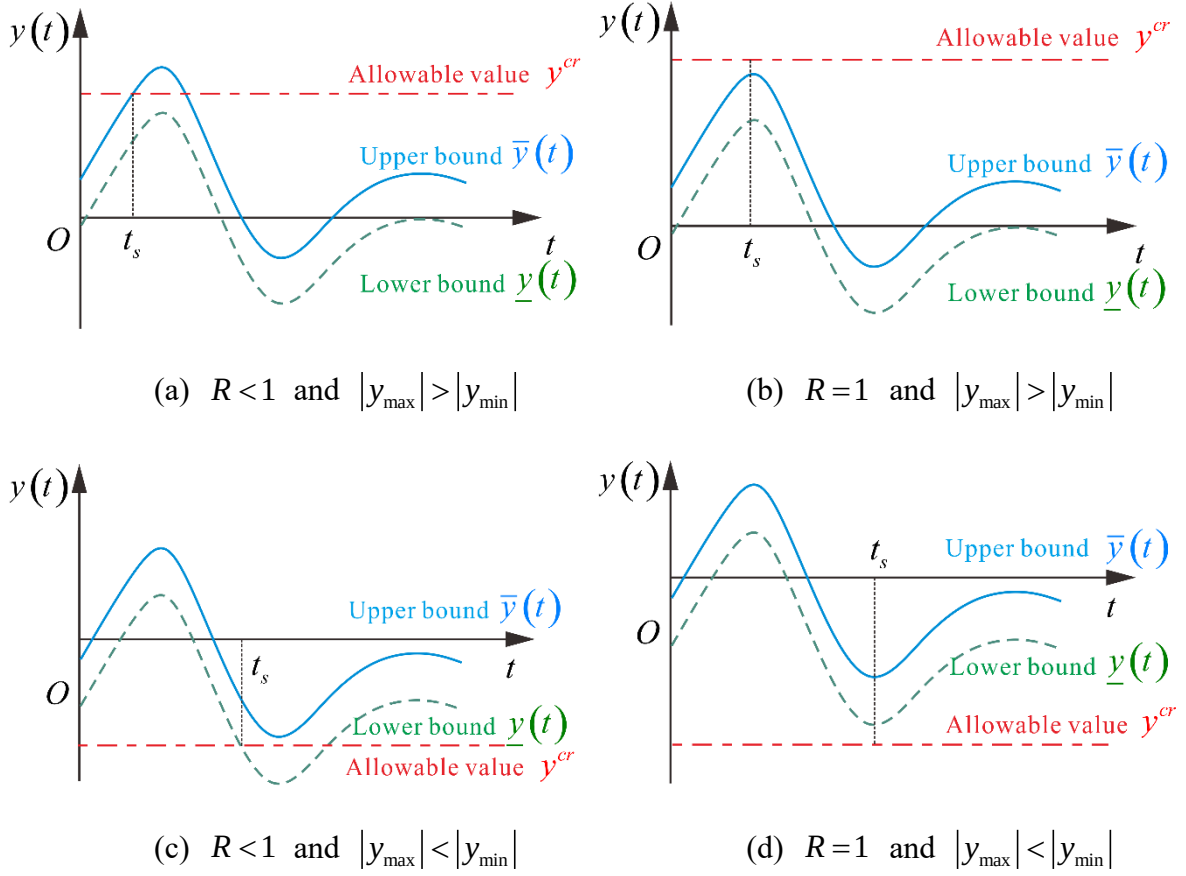


Fig. 6 The different cases of the response bounds under b^c

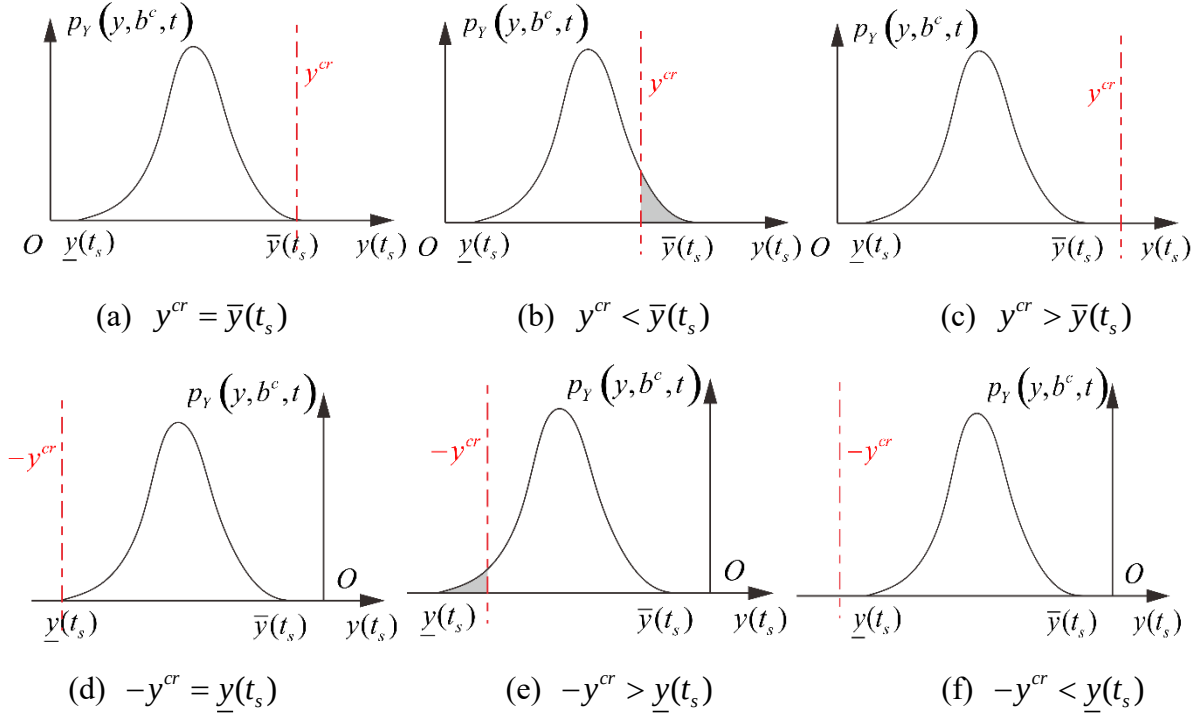


Fig. 7 The different states of updating the deterministic displacement constraint

Generally the sequential optimization process cannot reach the exact value of the required

reliability R^{cr} in Eq (27). In Eq (27) a smaller control force objective J always means a smaller hybrid reliability, thus the convergence judgement of control force objective J can be replaced by the hybrid reliability R , which is easier to be applied to the sequential optimization. The convergence judgement of the above sequential optimization can be set as:

$$0 \leq R - R^{cr} \leq \varepsilon_R \quad (35)$$

in which ε_R is the convergence criterion of the optimization. A large hybrid reliability always corresponds to a larger J , thus the optimization objective J in the sequential optimization can be replaced with minimizing R while meeting the constraint. The final optimization equation in the sequential optimization is:

$$\begin{aligned} & \text{find } K_p, K_I, K_D \\ & \min R \\ & \text{s.t. } 0 \leq R - R^{cr} \leq \varepsilon_R \\ & \quad K_{p\min} \leq K_p \leq K_{p\max} \\ & \quad K_{I\min} \leq K_I \leq K_{I\max} \\ & \quad K_{D\min} \leq K_D \leq K_{D\max} \end{aligned} \quad (36)$$

The whole HRBSO process is stated in Fig. 8.

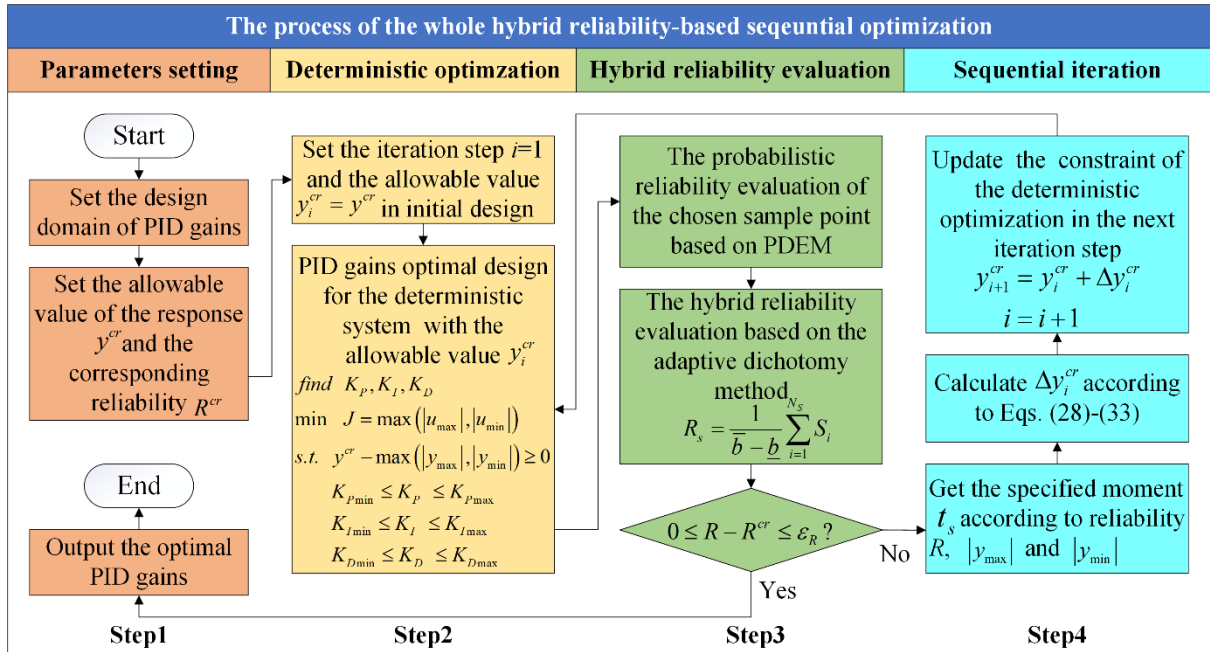


Fig. 8 The process of the whole HRBSO

In this way an optimal PID controller considering both energy-consuming and safety with hybrid random and interval uncertainties can be obtained.

4.3 The improvements of HRBSO compared with existing frameworks

(a) The types of uncertainties and reliability evaluation

In our former work such as Ref [60, 61], only interval uncertainties are taken into consideration and then interval propagation methods such as subinterval methods in Eq. (37) or collocation methods in Eq. (38) are used to calculate the uncertain response related with interval uncertainties in Fig. 9. Then the time-dependent reliability is calculated based on the first-passage theory as shown in Eq. (39).

$$\mathbf{X}^I(t) = \bigcup_{i=1,2,\dots,s^I} \mathbf{X}_i^I(t) = \left[\min_{i=1,2,\dots,s^I} \mathbf{X}_i(t), \max_{i=1,2,\dots,s^I} \overline{\mathbf{X}}_i(t) \right] \quad (37)$$

$$P_r(\xi_{\max}^j)_{\max}^j = \max(P_r(\xi_{\max}^j), P_r(-1), P_r(1)) \quad (38)$$

$$P_r(\xi_{\min}^j)_{\min}^j = \min(P_r(\xi_{\min}^j), P_r(-1), P_r(1))$$

$$R_s = 1 - \max \left[\text{Pos}(t_0) + \sum_{k=1}^{N-1} \frac{S_{\text{interference}}^k}{S_{\text{total}}^k} \right] \quad (39)$$

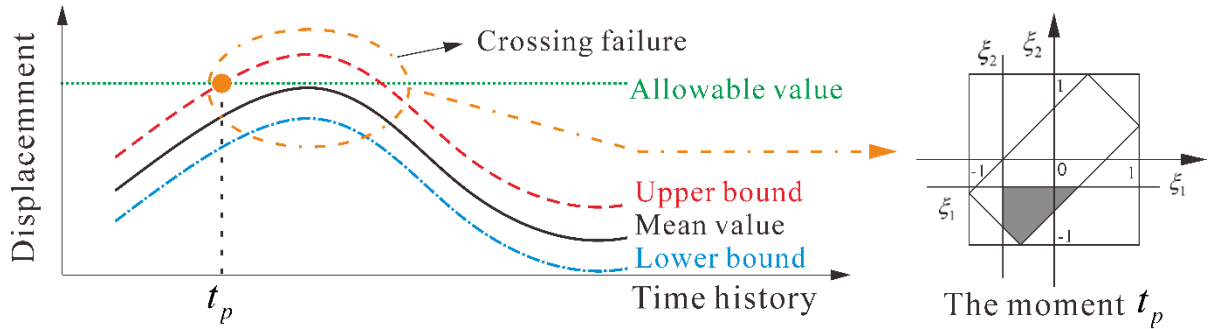


Fig. 9 Schematic diagram of feasible and interference regions in standard interval

In this work the random uncertainties are also taken into consideration, namely the usage is wider in complex practical engineering. The uncertain under a certain point of interval uncertainty is calculated by PDEM and then the hybrid reliability is calculated according to Eq. (26). Different uncertainties correspond to different uncertain response calculation and different reliability evaluation methods.

(b) The perform of sequential optimization

In the Ref [1] the sequential optimization is also adopted. However different reliability evaluation methods lead to different adjustments of the allowable value in the deterministic designs. In the Ref [1] the reliability is obtained from the extreme state function (uncertain response) as shown in Fig. 10 (a), thus the adjustment is performed directly according to the

final hybrid reliability as shown in Fig. 10 (b) and Eq. (40)

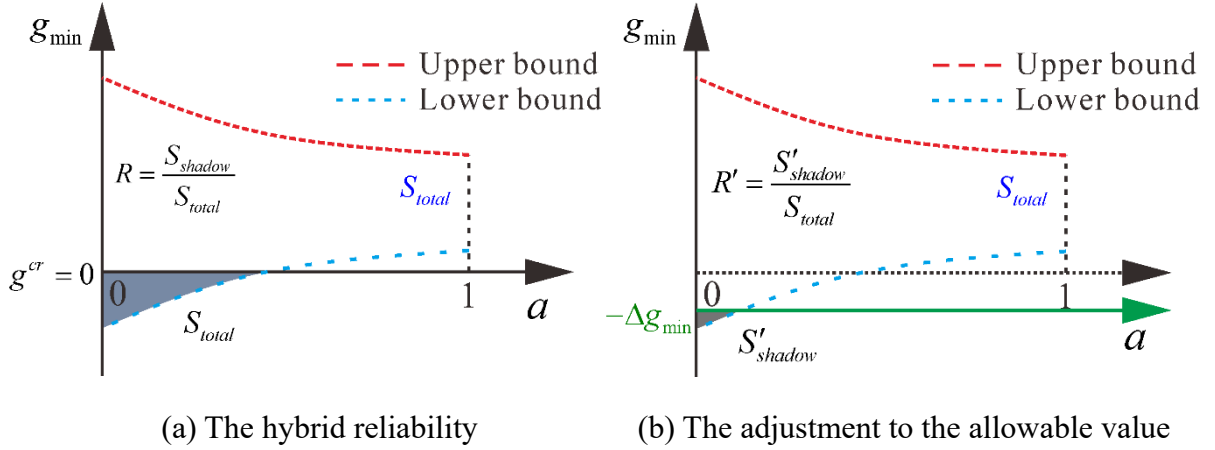


Fig. 10 The sequential optimization with fuzzy and interval uncertainties

$$\begin{aligned}
 & \text{find } \Delta g_{\min} \\
 & \min \Delta R = R - R^{cr} \\
 & \text{s.t. } 0 \leq R - R^{cr} \leq \varepsilon_R
 \end{aligned} \tag{40}$$

However, the hybrid reliability in this paper is obtained from the average of possible reliability. It is hard to perform adjustment directly to the uncertain probabilistic response because of TVD difference scheme. Thus a static reliability η in Eq. (29) or Eq. (32) is defined. In this way the adjustment is decided according to η , whose change can indirectly reflect the change of hybrid reliability.

5 Numerical examples

5.1 Mass-spring-damper system

A 5DOFs mass-spring-damper system is shown in Fig. 11. There is only one translational degree of freedom ux in the horizontal direction. All the five blocks have the same mass, damping and stiffness. The mass m , the damping c and the stiffness k are uncertain as shown in Table 1. A horizontal impulse disturbance load $f(t)$ with the amplitude of 10000 N is applied to the block 5. The horizontal displacement response of block 5 is $x(t)$. One PID controller is used and the control force $u(t)$ is applied at block 4. The maximum value of the deterministic open-loop response (corresponding to $m = \mu(m) = 1\text{kg}$, $c = c^c = 5\text{N}\cdot\text{s/m}$ and

$k = \mu(k) = 1100 \text{ N/m}$ is $\max(x^c) = 0.604 \text{ mm}$.

The controller design requirement is that the absolute value of the displacement response $|x(t)|$ does not exceed the allowable value $x^{cr} = 0.03 \text{ m}$ after $t^{cr} = 1.25 \text{ s}$ with the reliability allowable value $R^{cr} = 0.95$. The convergence criterion of the HRBSO in Eq. (35) is $0 \leq R - 0.95 \leq 0.02$.

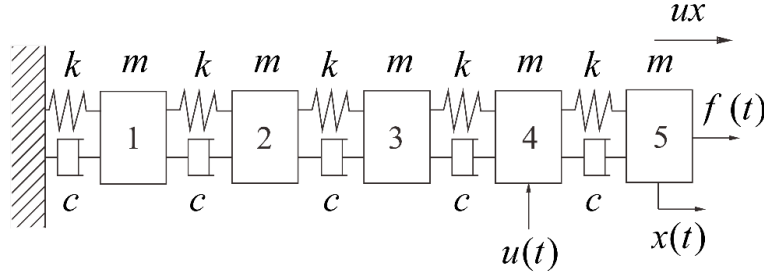


Fig. 11 Schematic diagram of the 5DOFs mass-spring-damper system

Table 1 The details of uncertainties in the mass-spring-damper system

Uncertainty type		Detailed distribution
m	Uniform distribution random uncertainty	$m \sim U(0.95, 1.05) \text{ kg}$
c	Interval uncertainty	$c \in [4.75 \ 5.25] \text{ N} \cdot \text{s/m}$
k	Normal distribution random uncertainty	$k \sim N(1000, 33.33^2) \text{ N/m}$

(a) The accuracy of the adaptive dichotomy method

The design ranges and optimal PID gains obtained by Eq (36) are listed in Table 1. The whole optimization process is performed in MATLAB. The initial gains are obtained from the deterministic design with the initial allowable value of displacement $x^{cr} = 0.03 \text{ m}$ and the detailed results are shown in Fig. 12. In Fig. 12 (a), only obviously improved optimization objectives in each iteration are shown. x^{cr} and R in each iteration step are respectively the allowable value in deterministic design and the reliability under the corresponding optimized PID gains. Fig. 12 (b) shows the PDF of the response under the center value of the interval uncertainty. the whole optimization process is performed in MATLAB, and the initial gains mean the initial point in the first deterministic design of HRBSO automatically set by PSO

function. To show the accuracy of the adaptive dichotomy method, in which only 5 sample points are required to fit the reliability curve, 50 sample points are uniformly selected in the uncertain interval as the comparison of the probabilistic reliability result under the same optimal PID gains and the corresponding probabilistic reliability is shown in Fig. 12 (c). It should be stressed that more sample points can obtain a more accurate reliability curve, the choose of 50 sample points is just empirical judgement because it is much more than 5 sample points in the adaptive dichotomy method.

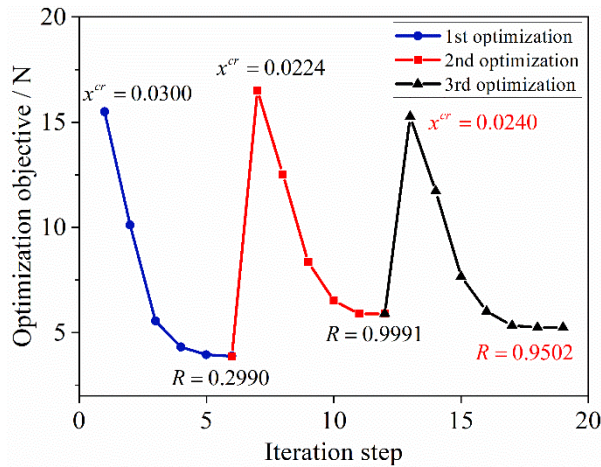
The relative errors at these 50 sample points are shown in Fig. 12 (d), in which

$$e_{relative} = \frac{p_a - p_{50}}{p_{50}} \times 100\% \quad (41)$$

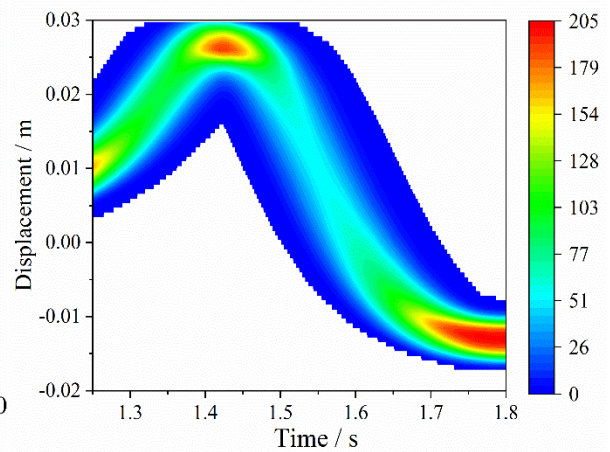
where p_{50} is the probabilistic reliability calculated by the uniformly distributed 50 points and p_a is the probabilistic reliability calculated by the adaptive dichotomy method.

Table 2 The design ranges and optimal PID gains of the mass-spring-damper system

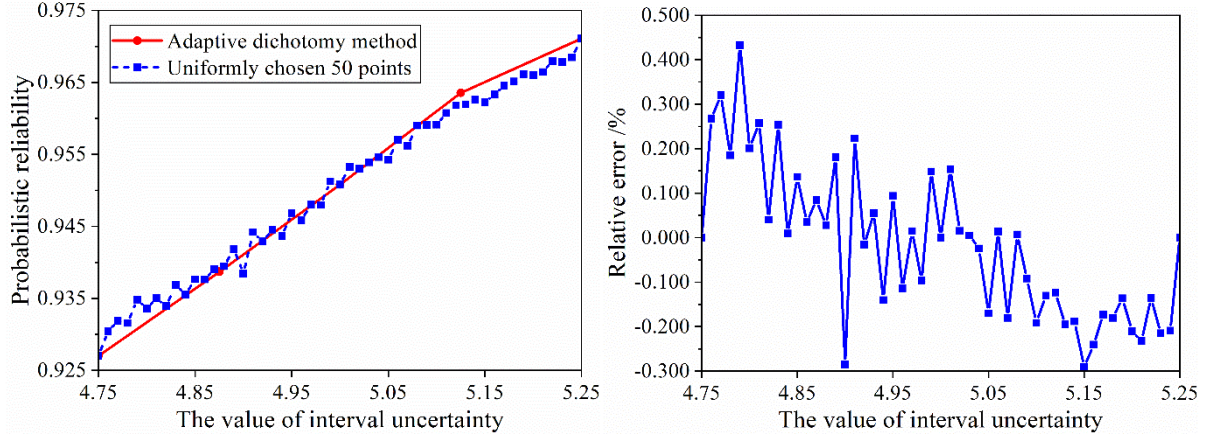
	K_p	K_I	K_D	J / N
Upper bounds of the design domain	100	20	20	--
Lower bounds of the design domain	0	0	0	--
The initial gains	59.7512	0	1.6904	3.8523
The optimal gains	75.3642	0.0110	2.7425	5.2627



(a) The iteration history



(b) The response under center value



(c) The probabilistic reliability of each sample

(d) The relative errors

Fig. 12 The detailed results of the mass-spring-damper system with the optimal PID gains

From Table 2 it is obvious that the objective J under optimal PID gains is larger than the initial PID gains, which shows the conflict between the safety and energy-consuming, thus the reliable design is essential. From Fig. 12 (c) and (d) the reliability curves obtained from the two methods are similar and the relative errors are smaller than 0.5%. Moreover, the hybrid reliability calculated according to the 50 points is 0.9503, which also shows the accuracy of the adaptive dichotomy method.

(b) The efficiency of the adaptive dichotomy method

To show the efficiency of the sequential strategy, the traditional double-loop optimization strategy is adopted to carry out the same optimization solution. The detailed results are shown in Fig. 13 and detailed comparison between the two methods are shown in Table 3.

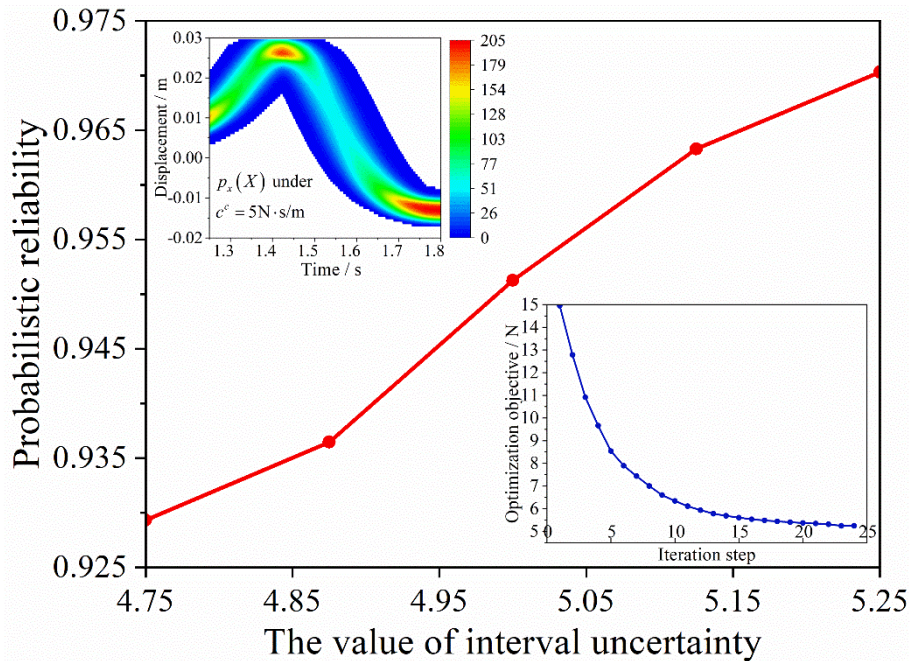


Fig. 13 The uncertain response, hybrid reliability results and iterations obtained from the

double-loop optimization strategy

Table 3 The detailed results obtained from the two methods

	The sequential optimization	The double-loop optimization
K_p	75.3642	75.3193
K_I	0.0110	0.3683
K_D	2.7425	2.7520
J / N	5.2627	5.2450
Total calculation time (s)	12204	333860
Call number of the PDEM	25	847
Call number of the hybrid reliability evaluation	3	121

From Fig. 12, Fig. 13 and Table 3 the optimal PID gains and detailed results obtained from the sequential optimization strategy and the double-loop optimization strategy are fairly similar. The sequential optimization strategy is much more efficiency than traditional double-loop strategy. Both optimization strategy spends most time on the hybrid reliability evaluation, in which repeated PDEM is time-consuming as stated in Section 4.1. The sequential optimization strategy calls much less hybrid reliability evaluation than the double-loop strategy, thus it can save time and obtain the fairly similar results.

(c) The accuracy of the PDEM

To show the accuracy of PDEM, a simple equation in the Example 1 of Ref [56] is analyzed and the detail is shown in **Appendix B**, in which the results have easily-obtained analytic solutions and shows the accuracy of PDEM.

For this numerical example, the analytic solution is hard to be calculated, thus the MCS is adopted to evaluate the accuracy of the PDEM. $N_{MCS} = 10000$ samples are selected according to the probabilistic uncertainties in the MCS to calculate the PDF of the response under the center value of the interval uncertainty. Denote the step of displacement in the TVD difference scheme as Δy , then the $P_y(Y)$ at a certain displacement Y at the moment t can be calculated:

$$P_y(Y, t) = \frac{1}{N_{MCS} \Delta y} \sum_{j=1}^{n_y} I(y_j, t) \quad (42)$$

in which y_j is the response result that meets $Y - \Delta y / 2 \leq y_j < Y + \Delta y / 2$ at the moment t and n_y is the corresponding amount of samples. $I(y_j, t)$ obtains

$$I(y_j, t) = \begin{cases} 1 & |y(\tau)| \leq y^{cr}, \forall t^{cr} \leq \tau \leq t \\ 0 & \text{otherwise} \end{cases} \quad (43)$$

Eq. (43) means if the response $y(\tau)$ exceeds the allowable value y^{cr} before the moment t after the required moment t^{cr} , the possibility at this moment is dissipated. The PDF results at $t = 1.4s$, $t = 1.6s$ and $t = 1.8s$ are displayed in Fig. 14. It can be seen that at a certain moment, the PDF obtained by PDEM is similar with the results obtained by MCS, and the accuracy of PDEM is proved.

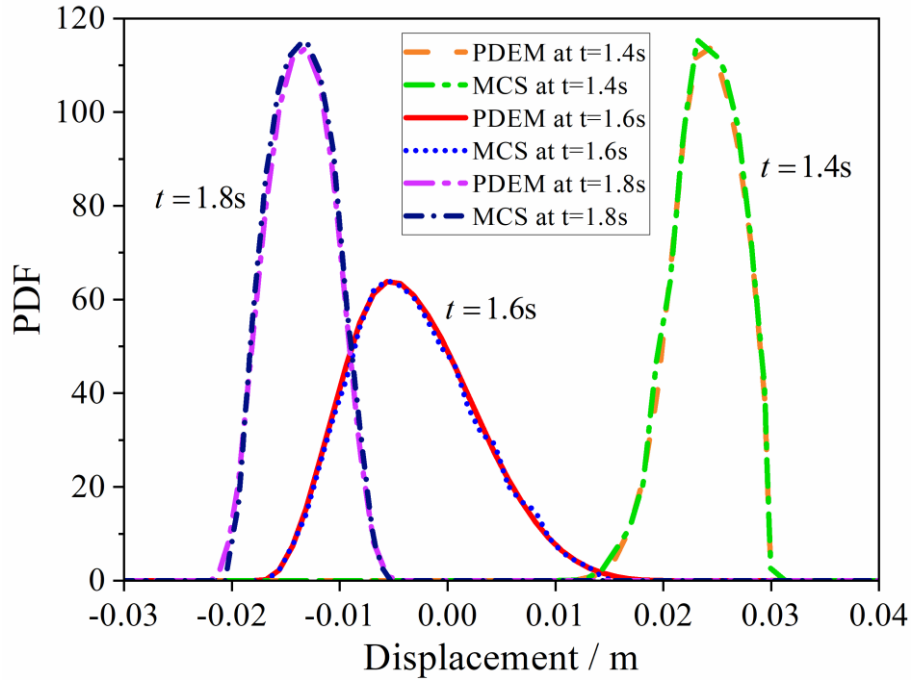


Fig. 14 The PDF results at three moments obtained by PDEM and MCS

5.2 A stiffened plate fixed on four sides

A stiffened plate is shown in Fig. 15, in which 6 DOFs of four sides of the plate are constrained. The corresponding finite element model is shown in Fig. 16, and the whole model is established in ANSYS with the elements SHELL181. The density ρ is interval uncertainty $\rho^I = [7410 \ 8190] \text{kg/m}^3$ and the elastic modules E is random form

$E \sim N(190, 12.667^2)$ GPa as shown in Fig. 16. The damping matrix is the Rayleigh damping $\mathbf{P} = p_\alpha \mathbf{M} + p_\beta \mathbf{K}$, in which $p_\alpha = 0.05$ and $p_\beta = 0.001$. Two step disturbance loads with the amplitude $f_A = 10$ N are applied on the symmetric locations of the plate, which are shown as the red arrows. The sensor is placed at the square orange point and the actuator is placed at the blue circular point. The maximum value of the deterministic open-loop response (corresponding to $\rho = \rho^c = 7800 \text{ kg/m}^3$ and $E = \mu(E) = 190 \text{ GPa}$) is $\max(z^c) = 1.609 \text{ mm}$.

The controller design requirement is that the absolute value of the displacement response $|z(t)|$ does not exceed the allowable value $z^{cr} = 0.6 \text{ mm}$ after $t^{cr} = 0.2 \text{ s}$ with the reliability allowable value R^{cr} , in which the detailed value of R^{cr} will be analyzed later. The convergence criterion of the HRBSO in Eq. (35) is $0 \leq R - R^{cr} \leq 0.02$. The HRBSO with different R^{cr} is performed and detailed results with optimal PID gains are shown in Table 4 and Fig. 17. The design ranges and optimal PID gains with different R^{cr} are listed in Table 4. The whole optimization process is performed in MATLAB, and the initial gains mean the initial point in the first deterministic design of HRBSO automatically set by PSO function. The detailed results are shown in Fig. 17. In each subgraph of Fig. 17, the top-left corner is the PDF of close-loop response under center value of density and the bottom-right corner is the iteration of HRBSO.

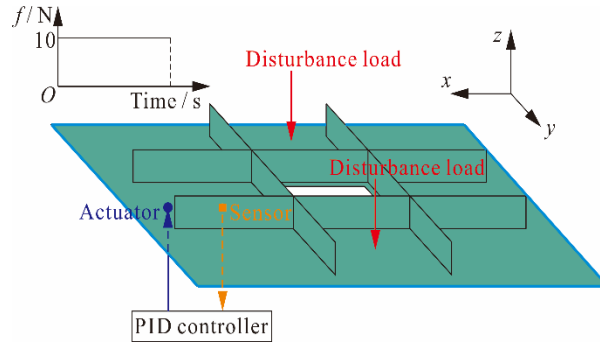


Fig. 15 The stiffened plate fixed on four sides

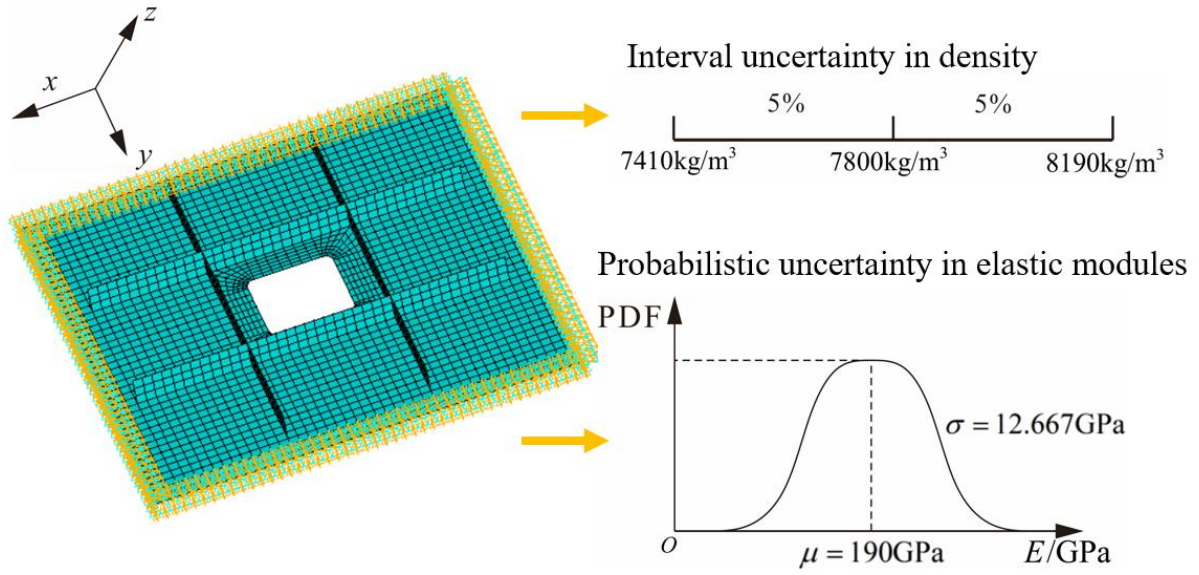
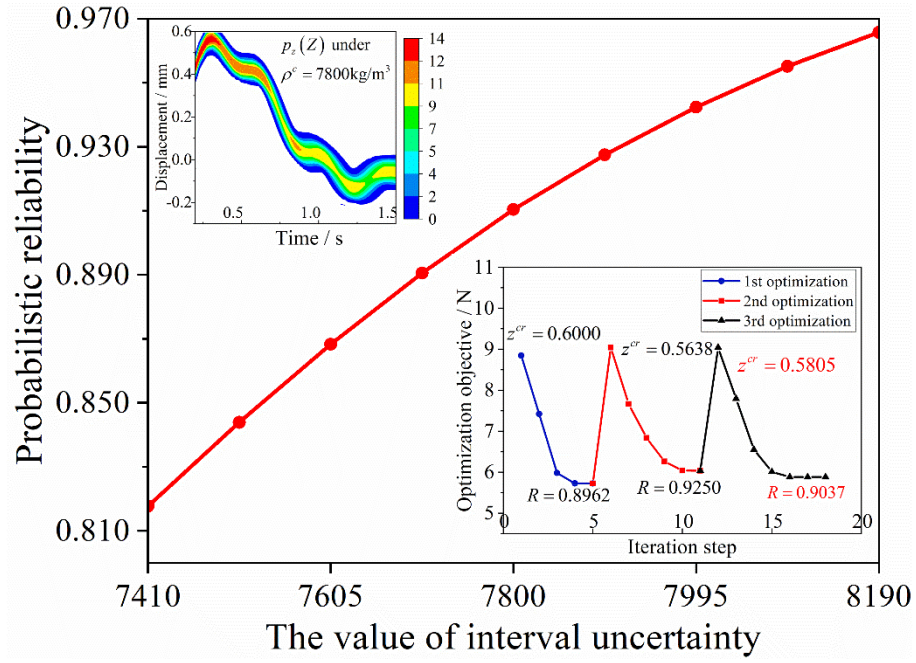


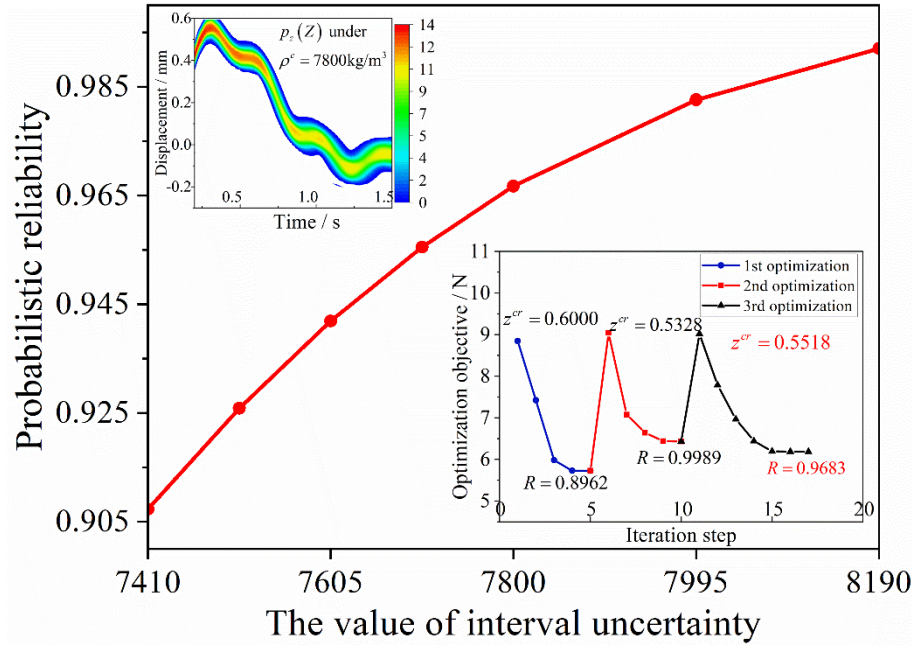
Fig. 16 The interval density and random elastic modules uncertainties of the plate

Table 4 The design ranges and optimal PID gains of the stiffened plate

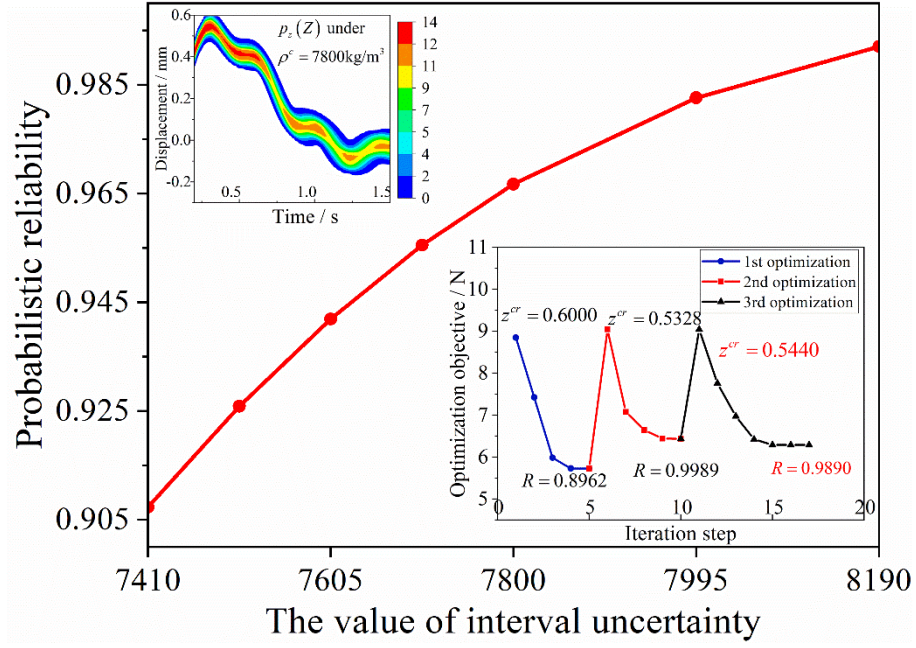
	K_p	K_I	K_D	J / N
Upper bounds of the design domain	10	5	5	--
Lower bounds of the design domain	4	0	0	--
The initial gains	4	0	1.092	4.7502
The optimal gains with $R^{cr} = 0.9$	4.2235	0	1.3205	4.9345
The optimal gains with $R^{cr} = 0.95$	4.2085	0	1.3883	5.2215
The optimal gains with $R^{cr} = 0.97$	4.0021	0	1.4918	5.2892



(a) $R^{cr} = 0.9$



(b) $R^{cr} = 0.95$



(c) $R^{cr} = 0.97$

Fig. 17 The uncertain response, hybrid reliability results and iterations with different R^{cr}

From all the iterations in Fig. 17, a higher hybrid reliability result is based on the deterministic design with smaller z^{cr} . The reliability in an iteration step smaller than R^{cr} will cause a smaller allowable value z^{cr} , namely a safer constraint in the next iteration step and vice versa. These proves the rationality of adjusting the allowable value in the deterministic design. From Table 4 it is obvious than a larger allowable value of reliability R^{cr} corresponds to a larger objective J . These results also show that achieving a higher requirement of system safety means consuming more energy. Thus, the set of R^{cr} in HRBSO should balance the safety and energy-consuming of the system. This also demonstrate the rationality of replacing minimizing J with minimizing R in Eq. (36).

5.3 An aircraft wing

An aircraft wing composed of composite materials is shown in Fig. 18, which contains two spars and five ribs. The ends of two spars are constrained with 6 DOFs. A z-direction impulse disturbance load with the amplitude of 1000N is applied to the wing. The finite element model is established in ANSYS with SHELL181 elements. The densities ρ are interval uncertain and elastic modules E are random uncertain as shown in Table 5, in which $\rho^c = \rho_0$ and $\mu(E) = E_0$. Three different combinations of uncertainties are analyzed and the optimal

PID gains are shown in Table 6. The sensor is placed at the square orange point and the actuator is placed at the blue circular point. The maximum value of the deterministic open-loop response (corresponding to $\rho = \rho^c$ and $E = \mu(E)$) is $\max(z^c) = 3.492\text{mm}$. The convergence criterion of the HRBSO in Eq. (35) is $0 \leq R - R^{cr} \leq 0.02$. The controller design requirement is that the absolute value of the displacement response $|z(t)|$ does not exceed the allowable value $z^{cr} = 2\text{ mm}$ after $t^{cr} = 0.005\text{s}$ with the reliability allowable value R^{cr} . The optimal PID gains are shown in Table 6 and Fig. 19.

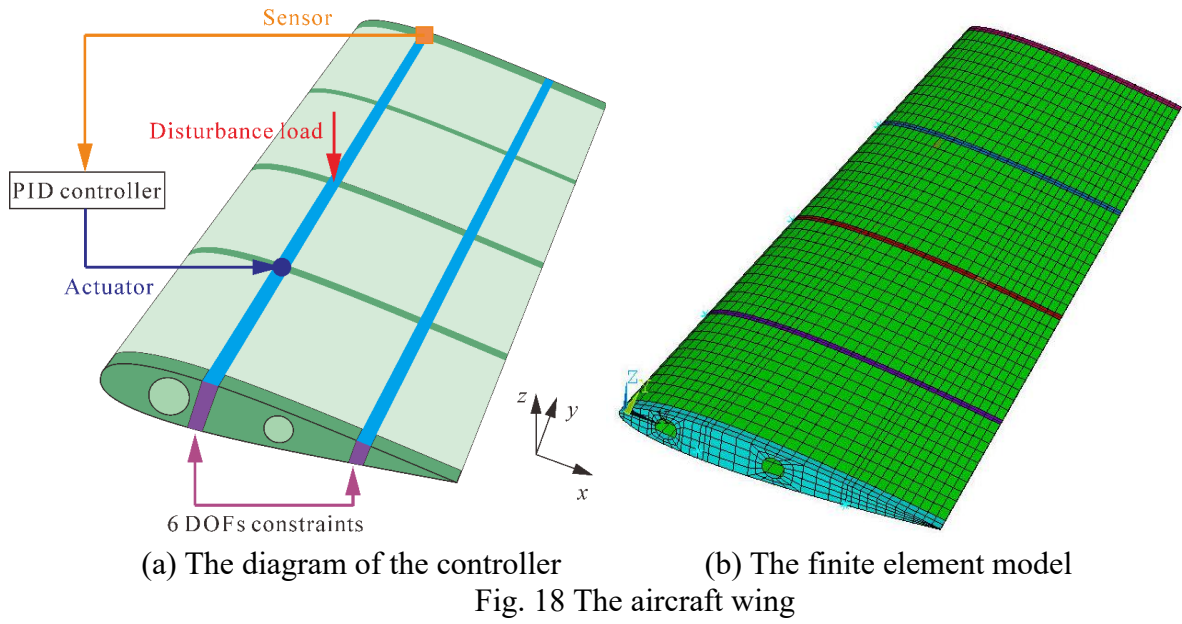
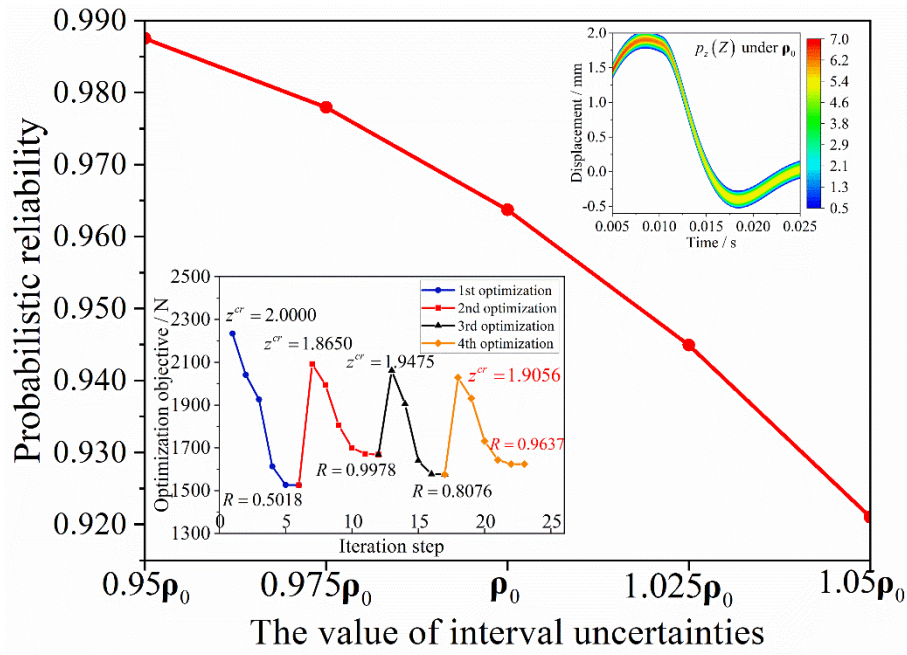


Table 5 The details of uncertainties in the aircraft wing

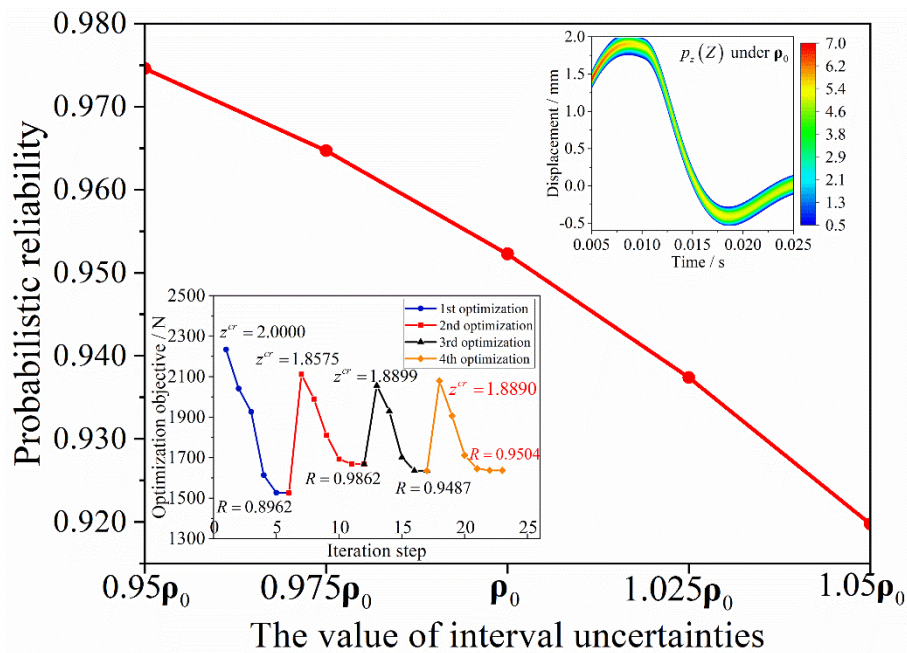
		Uncertainty type	Detailed distribution
Case 1	ρ	Interval uncertainties	$\rho \in [0.95\rho_0 \quad 1.05\rho_0]$
	E	Normal distribution random uncertainties	$E \sim N(E_0, 0.0167^2 E_0^2)$
Case 2	ρ	Interval uncertainties	$\rho \in [0.95\rho_0 \quad 1.05\rho_0]$
	E	Normal distribution random uncertainties	$E \sim N(E_0, 0.0333^2 E_0^2)$
Case 3	ρ	Interval uncertainties	$\rho \in [0.90\rho_0 \quad 1.10\rho_0]$
	E	Normal distribution random uncertainties	$E \sim N(E_0, 0.0167^2 E_0^2)$

Table 6 The optimal PID gains of three cases

	K_p	K_I	K_D	J/N
Upper bounds of the design domain	1000	10	10	--
Lower bounds of the design domain	500	0	0	--
Case 1	544.017	0	2.7527	1619.8
Case 2	550.6608	0	2.8196	1636.6
Case 3	540.7724	0.9016	2.7976	1624.5



(a) Case 1



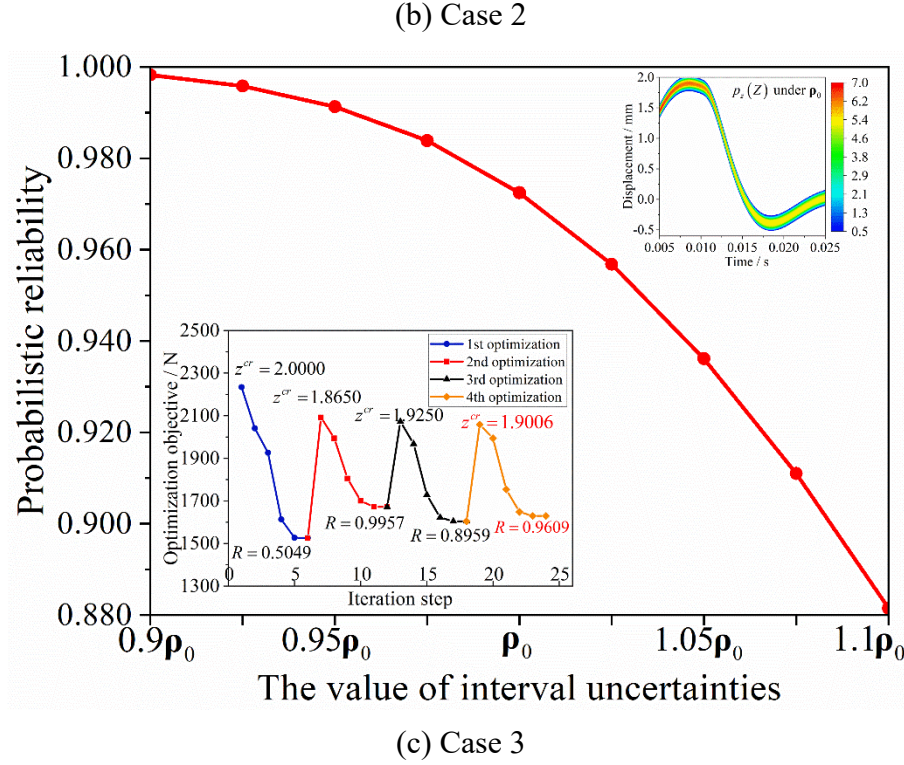


Fig. 19 The uncertain response, hybrid reliability results and iterations in three cases

From Table 6 and Fig. 19, Case 1 consume less energy than the other two cases. In Case 1 the radiuses of interval uncertainties and standard deviations of random uncertainties are smaller, which means under the same PID gains Case 1 has a smaller failure possibility than other cases. The uncertainties are quantified with interval or random forms before HRBSO, thus more accurate quantification of uncertainties can save more energy in HRBSO.

5.4 Solution process of numerical examples

All the three numerical examples have the same process of solving RHBSO as shown in Fig. 8. Then take the solution of the mass-spring-damper system in Section 5.1 as an example. The detailed solution process is as follows:

After parameters setting corresponding to Step 1 of Fig. 8, the deterministic design corresponding to Step 2 of Fig. 8 is performed according to Eq. (6) and an example of the iteration is shown in Fig. 12 (a), in which different iteration lines corresponds to different allowable value in Eq. (6). These different allowable values are obtained from the iterations of sequential optimization. Then the hybrid reliability is calculated under the obtained PID gains according to Eq. (26) corresponding to Step 3 of Fig. 8. In the hybrid reliability evaluation, the uncertain response under a sample of uncertain response is calculated by PDEM as shown

in Eq. (17) and an example of the PDF of uncertain response is shown in Fig. 12 (b). The choosing sample points of interval uncertainty is based on the adaptive dichotomy method in Eq. (25) and an example of the reliability curve is shown in Fig. 12 (c). Next the allowable value in Eq. (6) is updated according to Eq. (28) corresponding to Step 4 of Fig. 8. If the convergence judgement in Eq. (35) is not met then the deterministic design with updated allowable value will performed. Other numerical examples obey the similar process.

6 Conclusion

Vibration active control is a significant problem in practical engineering, in which the hybrid uncertainties' effect cannot be ignored. A novel study on controller design based on hybrid reliability sequential optimization strategy considering interval and random uncertainties is proposed in this paper. Firstly, probability density evolution method is used to calculate the uncertain response considering only random uncertainties, in which the obtained PDF is used in later sequential optimization. Then the hybrid reliability is defined as the average probabilistic reliability within the interval uncertainty, and the adaptive dichotomy method is utilized to fit the reliability curve with enough accuracy and efficiency. Next the hybrid reliability is introduced to the optimization as a constraint, thus the PID gains reliable design considering random and interval hybrid uncertainties can be performed to balance the energy-consuming and safety. Finally, the sequential optimization strategy is used to perform the optimization with high efficiency, in which the deterministic design and hybrid reliability evaluation are decoupled and the deterministic constraint is adjusted according to the reliability result.

Three numerical examples are presented to demonstrate the accuracy and practicability of the proposed framework, and the following findings are obtained: (1) In the first example, the accuracy and the efficiency of the adaptive dichotomy method in the HRBSO framework are demonstrated. (2) In the second example, the results show the conflict between safety and energy-consuming, which shows the rationality of replacing the optimization objective with minimizing the hybrid reliability. (3) The third example shows that the proposed HRBSO can deal with different situations with different uncertainties, and more accurate quantification of uncertainties can save more energy. The hybrid reliable design considering random and interval

uncertainties is just beginning, more kinds of uncertainties will be taken into consideration in further work.

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Appendix A

The specific numerical solution of Eq. (9) with TVD scheme is stated in Ref. [56], only core formulas are shown there. The difference of solving Eq. (17) compared with Eq. (9) is introducing the absorption boundary condition. Denoting the differential steps of time and response as Δt and Δy , respectively and the joint PDF $p_{Y\Theta}(j\Delta y, \theta, b_l, m\Delta t)$ is simply expressed as $p_{Y\Theta}(j, \theta, m)$, in which b_l of interval uncertainties keeps unchanged. The major steps are shown as follows:

Step1: Select discretized samples in the domain Ω_θ . Denote the selected points as θ_i , in which $i=1, 2, \dots, N$. N is the amount of selected samples. In this paper θ_i are selected uniformly and the step is $\Delta\theta$.

Step2: Establish the discretized initial condition according to Eq. (10):

$$p_{Y\Theta}(y, \theta_i, t)|_{t=0} = \begin{cases} \frac{1}{\Delta y} p_\Theta(\theta_i, b_l), & y \in [y - \Delta y, y + \Delta y] \\ 0, & \text{otherwise} \end{cases} \quad (\text{A.1})$$

Step3: Solving Eq. (2) to obtain the velocity $\dot{y}(\theta_i, b_l, t)$ results corresponding to all the samples θ_i

Step4: Solving Eq. (9) or Eq. (17) with TVD difference scheme:

$$\begin{aligned}
 p_{Y\Theta}(j, \theta_i, m+1) = & p_{Y\Theta}(j, \theta_i, m) - r_L \left[\frac{1}{2} (g_m + |g_m|) (p_{Y\Theta}(j, \theta_i, m) - p_{Y\Theta}(j-1, \theta_i, m)) \right. \\
 & + \frac{1}{2} (g_m - |g_m|) (p_{Y\Theta}(j+1, \theta_i, m) - p_{Y\Theta}(j, \theta_i, m)) \Big] \\
 & - \frac{1}{2} (1 - |r_L g_m|) |r_L g_m| [\psi(r_{j+0.5}^+, r_{j+0.5}^-) (p_{Y\Theta}(j+1, \theta_i, m) - p_{Y\Theta}(j, \theta_i, m)) \\
 & - \psi(r_{j-0.5}^+, r_{j-0.5}^-) (p_{Y\Theta}(j, \theta_i, m) - p_{Y\Theta}(j-1, \theta_i, m))]
 \end{aligned}
 \tag{A.2}$$

in which $r_L = \frac{\Delta t}{\Delta x}$ and:

$$r_{j+0.5}^+ = \frac{p_{Y\Theta}(j+2, \theta_i, m) - p_{Y\Theta}(j+1, \theta_i, m)}{p_{Y\Theta}(j+1, \theta_i, m) - p_{Y\Theta}(j, \theta_i, m)} \tag{A.3}$$

$$r_{j+0.5}^- = \frac{p_{Y\Theta}(j, \theta_i, m) - p_{Y\Theta}(j-1, \theta_i, m)}{p_{Y\Theta}(j+1, \theta_i, m) - p_{Y\Theta}(j, \theta_i, m)} \tag{A.4}$$

$$r_{j-0.5}^+ = \frac{p_{Y\Theta}(j+1, \theta_i, m) - p_{Y\Theta}(j, \theta_i, m)}{p_{Y\Theta}(j, \theta_i, m) - p_{Y\Theta}(j-1, \theta_i, m)} \tag{A.5}$$

$$r_{j-0.5}^- = \frac{p_{Y\Theta}(j-1, \theta_i, m) - p_{Y\Theta}(j-2, \theta_i, m)}{p_{Y\Theta}(j, \theta_i, m) - p_{Y\Theta}(j-1, \theta_i, m)} \tag{A.6}$$

$$g_m = \frac{1}{2} \left[\dot{y}(\theta_i, b_l, m\Delta t) + \dot{y}(\theta_i, b_l, (m-1)\Delta t) \right] \tag{A.7}$$

Detailed meanings of other functions and variables in Eq. (A.2) can refer to Ref. [56]. The Courant-Friedrichs-Lewy condition for Eq. (9) and Eq. (17) is:

$$|r_L g_m| \leq 1 \tag{A.8}$$

For Eq. (17) the absorption boundary condition is additionally introduced in the difference as:

$$p_{Y\Theta}(j, \theta_i, m) = 0, x \notin \Omega_s \tag{A.9}$$

Step5: Taking numerical integration to get the numerical PDF

$$p_Y(j, m) = \Delta \theta \sum_{i=1}^N p_{Y\Theta}(j, \theta_i, m) \tag{A.10}$$

Appendix B

To show the accuracy of PDEM, the following equation in the Example 1 of Ref [56] is analyzed:

$$\ddot{X} + \omega^2 X = 0 \quad (B.1)$$

ω is the only probabilistic uncertainty uniformly distributed in $\left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]$ and the initial conditions are:

$$X(t)|_{t=0} = 0.1, \quad \dot{X}(t)|_{t=0} = 0 \quad (B.2)$$

The PDEM used in this article is also performed to this example. The PDF of uncertain response is obtained as shown in Fig. 20 (a). The detailed PDF at $t=0.9s$, $t=1.0s$ and $t=1.1s$ is shown in Fig. 20 (b), which are basically the same with the results in Ref [56]. To further show the accuracy of the obtained results, the expectation and the mean square results are shown in Fig. 20 (c) and (d). The analytical solutions are shown in Eq. (A7) and Eq. (A8) of Ref [56], respectively.

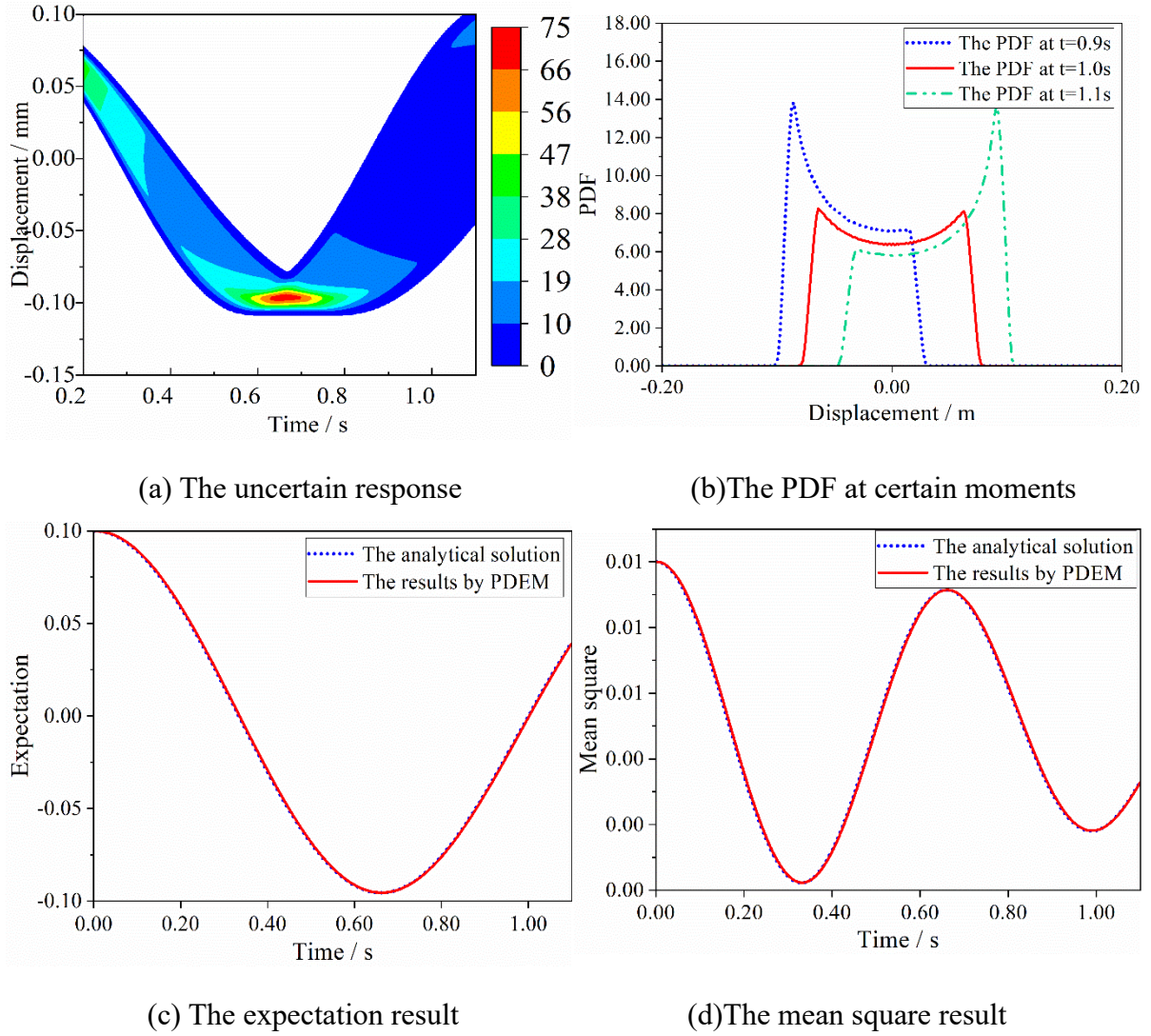


Fig. 20 The detailed results of the equation example

With this equation example, the accuracy of the adopted PDEM is proved.