Economics Letters 242 (2024) 111886

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

All-pay vs. standard auctions when competing for budget-constrained buyers

Cemil Selcuk

Department of Economics, Cardiff Business School, Cardiff University, United Kingdom

ARTICLE INFO

JEL classification:
D4
D81
D83

Keywords:
All-pay auctions
Directed search
Budget constraints
Competing auctions

ABSTRACT

In a competitive market with budget-constrained buyers, an equilibrium where sellers compete with standard auctions fails to exist if the all-pay format is available. If budgets are not too limited, then all-pay auctions emerge as the preferred selling format.

1. Introduction

Consider a competitive market where transactions are settled with auctions and buyers differ in their budgets. Which auction format – first price, second price, or all-pay – should sellers pick when competing for customers? The choice can be crucial: should most sellers pick one format, a seller can switch to another, draw in more customers, and gain an advantage.

With budget-constrained buyers, all-pay auctions revenue-dominate standard auctions (first and second-price), which provides an edge in a competitive market and leads to two important results. First, an equilibrium in which sellers compete with standard auctions fails to exist if the all-pay format is available. Second, if the budget is not too small, in the unique symmetric equilibrium, sellers compete with all-pay auctions and, despite different budgets, buyers enjoy equal expected payoff. This outcome mirrors that of homogeneous buyers and shows that, unlike standard auctions, the all-pay format can circumvent the budget constraint.

Auctions with budget constraints are studied extensively; see Balseiro et al. (2023) or Kotowski (2020) for a recent review. This body of research, however, overlooks competition: it typically assumes that sellers already have multiple bidders and does not examine how they initially attract them. In our setup, selecting an auction rule affects the number of bidders and the composition of high/low types among them. To account for competition, we rely on the directed search approach in the tradition of Burdett et al. (2001). Price mechanism selection within this literature is studied extensively; see for instance Severinov and Virag (2024) and the review therein.[1] To the best of our knowledge, this note is the first attempt to integrate the concepts of all-pay auctions, budget constraints, and competition simultaneously.

2. Model

The economy consists of a large number of risk-neutral buyers and sellers, with buyer–seller ratio λ. Each seller has one unit of a good and aims to sell it at a price exceeding his reservation price, 0. Similarly, each buyer seeks to purchase one unit and is willing to pay up to his reservation price, 1. A fraction σ of buyers, "low types", have limited budgets and can pay up to $b < 1$, while the remaining buyers, "high types", can pay up to 1. The type of buyer is private information, but the parameters λ, σ, and b are common knowledge. In the first stage of the game, sellers simultaneously choose an auction format m and a reserve price of r^m. The possible formats are first-price, second-price, and all-pay auctions. First and second price auctions are payoff equivalent for buyers and the seller (Che and Gale, 1996; Selcuk, 2017); thus we refer to them as *standard auctions* and let s represent standard auctions and ap represent all-pay auctions.

In the second stage, buyers observe sellers' selections and pick one store to visit. If the customer is alone, then he pays the reserve price. If there are $n \geq 2$ buyers, then bidding ensues. If the trade takes place at price r then the seller realizes payoff r and the buyer realizes $1 - r$. Following the directed search literature, we focus on visiting strategies that are symmetric and anonymous on and off the equilibrium path, which, in a large market implies that the distribution of demand at any

E-mail address: selcukc@cardiff.ac.uk.

[1] A closely related paper is Selcuk (2017), which studies pricing mechanism selection within the context of fixed pricing and standard auctions. In this note, we focus on all-pay auctions and standard auctions.

https://doi.org/10.1016/j.econlet.2024.111886
Received 4 June 2024; Received in revised form 14 July 2024; Accepted 15 July 2024
Available online 17 July 2024

store is Poisson (Burdett et al., 2001). Therefore, the probability that a seller with terms (m, r^m) meets n customers of type $i = h, l$ is given by $z_n(x_i^m)$, where x_i^m is the arrival rate of type i at format m and

$$z_n(x) = e^{-x}x^n/n! \tag{1}$$

As arrivals are independent, the distribution of the *total demand* is also Poisson with $x_h^m + x_l^m$.

3. Auctions

Bidding ensues if $n \geq 2$, so consider a store with $n \geq 2$ customers. Low types arrive at rate x_l^m and high types arrive at rate x_h^m. The distribution of the number of low types, therefore, is $binomial(n, \theta)$, where $\theta = x_l^m / \left(x_h^m + x_l^m\right)$ is the probability that a buyer is a low type.

Lemma 1. *Fix $\theta \in (0, 1)$ and $n \geq 2$. With first and second-price auctions buyers earn*

$$u_{h,n}^s = \theta^{n-1}(1-b) \quad \text{and } u_{l,n}^s = \theta^{n-1}(1-b)/n. \tag{2}$$

With all pay auctions, they earn

$$u_{h,n}^{ap} = \max\left\{\theta^{n-1} - b, 0\right\} \quad \text{and} \quad u_{l,n}^{ap} = \max\left\{\theta^{n-1}/n - b, 0\right\}. \tag{3}$$

The seller's expected revenue, on the other hand, is equal to

$$\pi_n^m = 1 - n\theta u_{l,n}^m - n(1-\theta)u_{h,n}^m, \text{ where } m = s, ap. \tag{4}$$

The seller extracts more revenue using all-pay auctions than standard auctions.

Proof. Eq. (2) obtains from Selcuk (2017). For (3) we follow the method in Che and Gale (1996): A buyer with budget w receives a surplus of $u(w)$ in equilibrium in the all-pay auction, where $u(w) = \max_{0 \leq y \leq w} F(y)^{n-1} - y$ is the payoff that accrues if all others bid their entire budget. The cumulative mass function F is given by: (i) $F(w) = 0$ if $w < b$, (ii) $F(w) = \theta$ if $b \leq w < 1$ (iii) $F(w) = 1$ if $w \geq 1$. High types' budget is equal to 1, so their payoff is obtained via

$$u_{h,n}^{ap} := u(1) = \max_{0 \leq y \leq 1} F(y)^{n-1} - y.$$

When $0 \leq y < b$ the maximum value of $F(y)^{n-1} - y$ is 0; when $b \leq y < 1$ its maximum value is $\theta^{n-1} - b$; and finally when $y = 1$ its maximum value is 0. Combining, we have $u_{h,n}^{ap} = \max\left\{\theta^{n-1} - b, 0\right\}$.

Low types' budget is b, so their expected payoff is obtained via

$$u_{l,n}^{ap} := u(b) = \max_{0 \leq y \leq b} F(y)^{n-1} - y.$$

When $0 \leq y < b$ the maximum value of $F(y)^{n-1} - y$ is 0; and when $y = b$ its maximum value is $\theta^{n-1}/n - b$.[2] Thus $u_{l,n}^{ap} = \max\left\{\theta^{n-1}/n - b, 0\right\}$.

The payoffs obtained by buyers and the seller add up to the total surplus, 1, thus

$$\pi_n^m + \sum_{i=0}^{n} \binom{n}{i} \theta^i (1-\theta)^{n-i} \left[i u_{l,n}^m + (n-i) u_{h,n}^m \right] = 1,$$

leading to (4). The last claim obtains as $u_{h,n}^{ap} < u_{h,n}^s$ and $u_{l,n}^{ap} < u_{l,n}^s$, thus $\pi_n^{ap} > \pi_n^s$. ■

Remark 1. Without budget constraints, standard and all-pay auctions are revenue equivalent.

Substituting $b = 1$ into (2) and (3) yields $u_{h,n}^s = u_{l,n}^s = u_{h,n}^{ap} = u_{l,n}^{ap} = 0$, confirming the claim.[3] The Remark implies that our main results indeed stem from the presence of budget-constrained buyers.

If $b < 1$ then all-pay auctions generate more revenue than standard auctions. This happens because, in all-pay auctions, buyers must pay their bid whether they win or lose, leading them to bid lower amounts. These lower bids make budget constraints less restrictive, as bids are less influenced by budget limitations. Consequently, the negative impact of budget constraints on revenue is diminished, allowing the seller to achieve higher revenues.

While our main focus is on comparing standard and all-pay auctions, a natural question arises: What is the revenue-maximizing mechanism above, i.e. when a seller faces $n \geq 2$ buyers, some of whom are budget-constrained? In the Appendix, we examine this question in detail and show that if $b \geq \theta^{n-1}$, the all-pay auction format is indeed a revenue-maximizing mechanism. This means that by employing all-pay auctions, the seller can extract the maximum possible surplus from buyers, further justifying the exploration of the all-pay format in a model of competition.

4. Competition

A type $i = h, l$ buyer's expected utility from visiting a store with the format $m = s, ap$ is given by

$$U_i^m(r^m, x_h^m, x_l^m) = z_0\left(x_h^m + x_l^m\right)(1 - r^m) + \sum_{n=1}^{\infty} z_n\left(x_h^m + x_l^m\right) u_{i,n+1}^m. \tag{5}$$

With probability z_0 the buyer is alone and pays the reserve price r^m. With probability z_n he finds n other buyers, so in total there are $n + 1$ buyers and the expected utility is $u_{i,n+1}^m$.

Now consider a store competing with rule m. The expected profit is given by

$$\begin{aligned} \Pi^m(r^m, x_h^m, x_l^m) &= z_1\left(x_h^m + x_l^m\right) r^m + \sum_{n=2}^{\infty} z_n\left(x_h^m + x_l^m\right)\pi_n^m \\ &= 1 - z_0\left(x_h^m + x_l^m\right) - x_h^m U_h^m - x_l^m U_l^m. \end{aligned} \tag{6}$$

With probability z_1 the store gets a single customer and charges the reserve r^m. With probability z_n the store gets $n \geq 2$ customers and the corresponding payoff is π_n^m. The second line obtains after substituting (4) for π_n^m and using the definitions of U_h^m and U_l^m from (5).

Following the directed search literature, let $\overline{U}_i$ denote the maximum expected utility ("market utility") a type $i = h, l$ customer can obtain in the market. For now we treat $\overline{U}_i$ as given, subsequently it will be determined endogenously. Consider a seller who advertises (m, r^m) and suppose that buyers respond with arrival rates $x_h^m \geq 0$ and $x_l^m \geq 0$. These rates satisfy

$$x_i^m > 0 \text{ if } U_i^m\left(r^m, x_h^m, x_l^m\right) = \overline{U}_i \text{ and 0 otherwise.} \tag{7}$$

The tuple $\left(r^m, x_h^m, x_l^m\right)$ must generate an expected utility of $\overline{U}_h$ for high types, else they will stay away, and $\overline{U}_l$ for low types, else they will stay away. Note that $U_i^m \leq \overline{U}_i$ by definition. Each seller chooses m and r^m to maximize Π^m but realizes that x_h^m and x_l^m are determined via (7).[4] Furthermore, if the seller attracts low types, then r^m must satisfy the budget constraint, i.e.

$$x_l^m > 0 \Rightarrow r^m \leq b. \tag{8}$$

To close the model, we need a feasibility condition ensuring that the weighted sum of expected queue lengths across stores equals the aggregate buyer–seller ratios for both types. Letting α denote the fraction of sellers opting for all-pay auctions, we have

$$\alpha x_h^{ap} + (1-\alpha)x_h^s = (1-\sigma)\lambda \text{ and } \alpha x_l^{ap} + (1-\alpha)x_l^s = \sigma\lambda. \tag{9}$$

[2] The term θ^{n-1}/n arises because a tie occurs at b when all other buyers are low types, which occurs with probability θ^{n-1}. For tiebreaking we need $1/n$. Selcuk (2024) – an earlier, working paper version of this paper – offers a detailed characterization of the all-pay auction equilibrium and confirms the same equilibrium payoffs as in (3).

[3] In general, with risk-neutral bidders, independent private values, and no budget limitations, standard auctions, and all-pay auctions are revenue equivalent, e.g. see Klemperer (1999).

[4] Here, following the convention in the directed search literature, e.g. Eeckhout and Kircher (2010), we focus on pure strategies, i.e. sellers do not randomize when picking a mechanism or selecting a reserve price.

Definition 1. A directed search equilibrium is an outcome where sellers pick (m, r^m) to maximize profits; arrival rates (x_h^m, x_l^m) satisfy buyers' indifference (7); the demand distribution is given by (1); and finally, the budget and feasibility constraints (8) and (9) are satisfied.

Lemma 2. *Fix a reserve price r and arrival rates x_h and x_l. The all-pay format leads to higher profits, i.e. $\Pi^{ap} > \Pi^s$.*

Recall that $u_{i,n}^{ap} < u_{i,n}^s$ for all $i = h, l$ and $n \geq 2$. Since r and z_n are controlled for, equation (5) implies $U_i^{ap} < U_i^s$. Substituting these into (6) reveals that $\Pi^{ap} > \Pi^s$. This echoes Lemma 1. There, n was known, so the claim was based on ex-post payoffs $u_{h,n}$, $u_{l,n}$ and π_n. Here n is uncertain and the claim is about the ex-ante payoffs U_h, U_l and Π.

Proposition 1. *If the all-pay format is available then an equilibrium where all sellers adopt standard auctions fails to exist.*

Proof. Conjecture an equilibrium in which sellers adopt standard auctions. Selcuk (2017) characterizes this outcome in detail, and shows that each seller sets the same reserve price r^s and receives $\sigma\lambda$ low type and $(1-\sigma)\lambda$ high type customers. Along this outcome sellers earn

$$\Pi^s(r^s, \lambda, \sigma) = 1 - z_0(\lambda) - \lambda(1-\sigma)U_h^s(r^s, \lambda, \sigma) - \sigma\lambda U_l^s(r^s, \lambda, \sigma).$$

Crucially, $U_h^s > U_l^s$, i.e. high types obtain a higher payoff than low types. Now, consider a seller who switches to all-pay auctions. We will show that this seller can earn more while still offering buyers U_h^s and U_l^s. After the switch, there are three parameters: the reserve price r, the total demand, x, and the composition of demand, θ. The fact that they are different from (r^s, λ, σ) makes the comparison difficult. To solve this, fix the total demand λ and note that per Lemma 2

$$U_h^{ap}(r, \lambda, \theta) < U_h^s(r^s, \lambda, \sigma) \text{ and } U_l^{ap}(r, \lambda, \theta) < U_l^s(r^s, \lambda, \sigma) \text{ when } r = r^s \text{ and } \theta = \sigma.$$

Moreover, both U_h^{ap} and U_l^{ap} fall in r and rise in θ. The first claim is immediate from (5) whereas the second follows from the fact that $u_{h,n}^{ap}$ and $u_{l,n}^{ap}$ both rise in θ. Thus, there exists some $\hat{\theta} > \sigma$ and $\hat{r} < r^s$ satisfying $U_h^{ap}(\hat{r}, \lambda, \hat{\theta}) = U_h^s(r^s, \lambda, \sigma)$ and $U_l^{ap}(\hat{r}, \lambda, \hat{\theta}) = U_l^s(r^s, \lambda, \sigma)$, i.e. if the store sets $\hat{r}$ then it still gets λ buyers, but with a high/low composition $\hat{\theta}$ instead of σ. The store earns

$$\Pi^{ap}(\hat{r}, \lambda, \hat{\theta}) = 1 - z_0(\lambda) - \lambda(1-\hat{\theta})\underbrace{U_h^{ap}(\hat{r}, \lambda, \hat{\theta})}_{=U_h^s(r^s,\lambda,\sigma)} - \hat{\theta}\lambda\underbrace{U_l^{ap}(\hat{r}, \lambda, \hat{\theta})}_{=U_l^s(r^s,\lambda,\sigma)}U_l^{ap}(\hat{r}, \lambda, \hat{\theta}).$$

The all-pay store provides buyers with the same payoffs as other stores ($U_l^{ap} = U_l^s$ and $U_h^{ap} = U_h^s$). However, it attracts a higher proportion of low types and a lower proportion of high types ($\hat{\theta} > \sigma$). Given that $U_h^s > U_l^s$, it follows that $\Pi^{ap} > \Pi^s$, making the deviation profitable. ■

Proposition 2. *If $b \geq \sigma$ then in the unique symmetric equilibrium sellers compete with all-pay auctions with a reserve $r^* = 0$. Each seller, on average, receives $\sigma\lambda$ low type and $(1-\sigma)\lambda$ high type buyers. Buyers earn $U_h^{ap} = U_l^{ap} = z_0(\lambda)$, while sellers earn $\Pi^{ap} = 1 - z_0(\lambda) - z_1(\lambda)$.*

Proof. Consider an outcome where sellers adopt the all-pay format, i.e. $\alpha = 1$. Substituting $\alpha = 1$ into (9) implies that each seller receives $\sigma\lambda$ low types and $(1-\sigma)\lambda$ high type customers. The condition $b > \sigma$ ensures that $b > \sigma^{n-1}$ for all $n \geq 2$; thus per Lemma 1 $u_{h,n}^{ap} = u_{l,n}^{ap} = 0$. Substituting these into (5) yields $U_h^{ap} = U_l^{ap} = z_0(\lambda)(1-r)$. A seller solves

$$\max_\lambda \Pi^{ap} = \max_\lambda 1 - z_0(\lambda) - \sigma\lambda U_l^{ap} - (1-\sigma)\lambda U_h^{ap}$$
$$\text{s.t. } U_l^{ap} = \overline{U}_l \text{ and } U_h^{ap} = \overline{U}_h.$$

The fact $U_h^{ap} = U_l^{ap}$ implies $\overline{U}_l = \overline{U}_h = \overline{U}$. Thus, the problem becomes $\max_\lambda 1 - z_0(\lambda) - \lambda\overline{U}$. The objective function is concave and the first-order condition yields $z_0(\lambda) = \overline{U}$. It follows that $z_0(\lambda)(1-r) = z_0(\lambda)$ implying that $r^* = 0$. Each seller, thus, earns $\Pi^{ap} = 1 - z_0(\lambda) - z_1(\lambda)$.

We now show that one cannot profitably deviate to a standard format. At all-pay stores we have $U_h^{ap} = U_l^{ap} = \overline{U}$, whereas at the deviating store we have $U_h^s > U_l^s$, i.e. with standard auctions low types obtain a strictly lower expected utility than high types. So, there are three scenarios:

- The store attracts low types only. This requires $U_l^s = \overline{U} > U_h^s$, but contradicts $U_h^s > U_l^s$.
- The store attracts both types. This requires $U_l^s = \overline{U}$ and $U_h^s = \overline{U}$, but contradicts $U_h^s > U_l^s$.
- The store attracts high types only, i.e. $U_h^s = \overline{U} > U_l^s$, which is feasible, so we focus on this.

The seller solves $\max_x \Pi^s = 1 - z_0(x) - xU_h^s$ s.t. $U_h^s = \overline{U}$. As above, the first order condition implies $z_0(x) = \overline{U}$. The store attracts high types only, i.e. $\theta = 0$, so per Lemma 1

$$u_{h,n}^s = 0 \text{ for all } n \geq 2 \implies U_h^s = z_0(x)(1-r) \implies r = 0.$$

The equalities $U_h^s = \overline{U} = z_0(\lambda)$ imply that $x = \lambda$, thus $\Pi^s = 1 - z_0(\lambda) - z_1(\lambda)$, i.e. the seller earns as much as Π^{ap}, which provides no incentive to deviate; hence the all-pay equilibrium remains. Uniqueness of the symmetric equilibrium follows from Proposition 1. ■

The payoffs here are the same as in a model with homogeneous buyers.[5] The all-pay format ensures that buyers with different budgets receive the same payoff, effectively circumventing the budget constraint. This holds true as long as the budget is not too low or the proportion of low types is not too high ($b \geq \sigma$). In contrast, standard auction formats trigger the budget constraint as soon as b drops below 1 (Selcuk, 2017).

5. Conclusion

In our model buyers have identical valuations, which somewhat restricts the scope of our findings. Pai and Vohra (2014) study the optimal auction design involving financially constrained buyers with differing valuations and conclude that a modified all-pay auction rule is the most effective approach. This suggests that in a competitive market with differing valuations, all-pay auctions should still maintain their edge over standard auctions.

With widening income inequality, budget limitations are now more critical than ever. Although the all-pay format may not be as widely utilized as the standard auction formats in business practices, our results suggest that its potential to address budget issues should not be underestimated.

Data availability

No data was used for the research described in the article.

Appendix

Consider a store with $n \geq 2$ buyers. Recall that with probability θ a buyer is a low type and with probability $1-\theta$ he is a high type. What is the revenue-maximizing mechanism in this circumstance?

In the spirit of Pai and Vohra (2014), a mechanism consists of a tuple (w, p_h, p_l), where $w \in [0,1]$ is a weight and p_h and p_l are payments. Buyers report their types truthfully to the seller. In response, the seller

[5] In a setup with homogeneous buyers (Eeckhout and Kircher, 2010) show that there exists a continuum of equilibria in which sellers compete with "payoff complete" mechanisms. In any such equilibrium, the expected demand at a store is λ; sellers earn $1 - z_0(\lambda) - z_1(\lambda)$ no matter which rule they compete with whereas buyers earn $z_0(\lambda)$ no matter which seller's rule they join in. We show that all-pay auctions achieve the same even with budget-constrained buyers.

assigns a weight of w to each high type and $1-w$ to each low type and requests payments of p_h from high types and p_l from low types. The weights determine the probability of each type receiving the good. Given this arrangement, the expected payoff of a high-type buyer is equal to

$$u_h(w,p_h)=\sum_{i=0}^{n-1}\binom{n-1}{i}\theta^i(1-\theta)^{n-1-i}\frac{w}{w(n-i)+(1-w)i}-p_h.$$

The buyer is a high type and he faces $n-1$ other buyers whose types are unknown to him. The binomial terms represent the probability of encountering i low types and $n-1-i$ high types. Including himself, there are $n-i$ high types, each with weight w, and i low types each with weight $1-w$. The fraction, therefore, is his chance of acquiring the item. Similarly, the expected payoff of a low type is given by

$$u_l(w,p_l)=\sum_{i=0}^{n-1}\binom{n-1}{i}\theta^i(1-\theta)^{n-1-i}\frac{1-w}{w(n-i-1)+(1-w)(i+1)}-p_l.$$

On expected terms, there are $n\theta$ low types and $n(1-\theta)$ high types in the store, so we have

$$\pi+n(1-\theta)u_h+n\theta u_l=1 \quad\Rightarrow\quad \pi=1-n(1-\theta)u_h-n\theta u_l.$$

The first equation follows because the payoffs obtained by buyers and the seller add up to the total surplus, 1. To maximize π the seller must extract as much surplus from buyers as possible (ideally $u_h=u_l=0$) by selecting the payments p_h, p_l and the weight w. In doing so, he faces two constraints

BC: $p_l\le b$ and IC: $u_h=u_l$.

The first is the budget constraint, whereas the second is the incentive constraint, ensuring that buyers report their types truthfully. Indeed, if $u_h<u_l$ then high types will pretend to be low types and if $u_l<u_h$ then low types will pretend to be high types; so we must have $u_h=u_l$.[6]

The seller can easily extract all the surplus from high types, as he faces no constraint in setting p_h. Indeed, u_h rises in w and falls in p_h, so even when $w=1$, there exists some $p_h<1$ that ensures $u_h(1,p_h)=0$. In contrast, low types can only pay up to b, so the most a seller can extract from them is $p_l=b$. Noting that u_l falls in w and in p_l, and that $p_l\le b$, the seller optimally sets $w=1$, i.e. he makes sure that low types can acquire the item only if no high type is present. Setting $w=1$ shrinks u_l, and therefore, avoids the budget constraint as much as possible. In doing so, the seller does not need to worry about u_h, because even when $w=1$ there exists some $p_h<1$ satisfying $u_h=0$. Substituting $w=1$ into u_h and u_l we have[7]

$$u_h(1,p_h)=\frac{1-\theta^n}{n(1-\theta)}-p_h \quad\text{and}\quad u_l(1,p_l)=\frac{\theta^{n-1}}{n}-p_l.$$

- If $b\ge\theta^{n-1}/n$ then there exists payments $p_h<1$ and $p_l\le b$ that ensure $u_h=u_l=0$. These payments extract all the surplus from buyers (thus maximize π) and they satisfy BC and IC.
- If, however, $b<\theta^{n-1}/n$ then the seller sets $p_l=b$. Here u_l is bounded below by $\theta^{n-1}/n-b>0$. The seller can still pick a high enough p_h to ensure $u_h=0$ but this violates IC, as high types would pretend to be low types. Therefore p_h has to be adjusted to ensure that $u_h=u_l=\theta^{n-1}/n-b$, satisfying IC.

To sum up, buyers' payoffs associated with revenue maximization satisfy

$$u_h^*=u_l^*=\max\left\{\theta^{n-1}/n-b,0\right\}.$$

Implementation. Recall that with all-pay auctions, we have

$$u_{h,n}^{ap}=\max\left\{\theta^{n-1}-b,0\right\} \quad\text{and}\quad u_{l,n}^{ap}=\max\left\{\theta^{n-1}/n-b,0\right\}.$$

- Clearly $u_{l,n}^{ap}=u_l^*$, thus low types' payoffs can be achieved via all-pay auctions. As for high types, if $b\ge\theta^{n-1}$ then $u_{h,n}^{ap}=u_h^*=0$; so their payoffs, too, can be achieved through all-pay auctions. Consequently, if $b\ge\theta^{n-1}$, then the all-pay rule is a revenue-maximizing mechanism.
- If, however, $b<\theta^{n-1}$ then $u_{h,n}^{ap}>u_h^*$. In this region, the all-pay rule leaves too much surplus to high types; so it is not revenue maximizing.

In characterizing the all-pay equilibrium in the paper (Proposition 2) we require $b\ge\sigma$. Noting that in symmetric equilibrium $\theta=\sigma$, this is a sufficient condition for $b\ge\theta^{n-1}$. Thus, in the region of interest ($b\ge\sigma$) the all-pay format is indeed a revenue-maximizing mechanism.

References

Balseiro, S., Kroer, C., Kumar, R., 2023. Contextual standard auctions with budgets: Revenue equivalence and efficiency guarantees. Manage. Sci. 69 (11).
Burdett, K., Shi, S., Wright, R., 2001. Pricing and matching with frictions. J. Polit. Econ. 109 (5).
Che, Y.K., Gale, I., 1996. Expected revenue of all-pay auctions and first-price sealed-bid auctions with budget constraints. Econom. Lett. 50 (3).
Eeckhout, J., Kircher, P., 2010. Sorting versus screening: Search frictions and competing mechanisms. J. Econom. Theory 145 (4).
Klemperer, P., 1999. Auction theory: A guide to the literature. J. Econ. Surv. 13 (3).
Kotowski, M.H., 2020. First price auctions with budget constraints. Theor. Econ. 15.
Pai, M.M., Vohra, R., 2014. Optimal auctions with financially constrained buyers. J. Econom. Theory 150.
Selcuk, C., 2017. Auctions vs. fixed pricing: Competing for budget constrained buyers. Games Econom. Behav. 103.
Selcuk, C., 2024. Competition for budget-constrained buyers: Exploring all-pay auctions. arXiv:2404.08762.
Severinov, S., Virag, G., 2024. Who wants to be an auctioneer? J. Econom. Theory 217.

[6] If $p_h>b$, then low types cannot afford to pretend to be high types, so, in this case, we only need $u_h\ge u_l$.

[7] When calculating u_l, all terms vanish except when $i=n-1$. This scenario corresponds to the outcome where all other buyers are also low types, resulting in the low-type buyer winning with a probability of $1/n$.