MODULAR PROPERTIES OF $\mathfrak{sl}_2$ TORUS 1-POINT FUNCTIONS

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Contents

1. Introduction

At least as early as 1970, conformal symmetry had been considered in statistical mechanics in the context of critical phenomena, seen as a generalisation of scale invariance to rescaling by a position-dependent parameter [1]. By 1984, a family of two-dimensional quantum field theories enjoying conformal symmetry known as the minimal models had been described, one of the advantages being correlation functions were entirely determined by conformal symmetry [2]. In the context of string theory, the surface which a string sweeps out in space-time – the world-sheet – is the setting for a two-dimensional conformal field theory. Scattering amplitudes of increasing order order in perturbation theory correspond to considering conformal field theory on increasing genus Riemann surfaces. An important theme of conformal field theory is well definedness on these, that is, not only on the Riemann sphere, but on all Riemann surfaces in the sense that n-point correlation functions should be well defined [3, 4].

The modern notion of vertex algebras arose in the context of Monstrous Moonshine in [5] from a study of representations of Kac-Moody algebras, instances of infinite-dimensional Lie algebras. Vertex operator algebras were described in [6] by considering an action by the Virasoro algebra, in the context of moonshine, a phenomenon that unexpectedly suggested a link between the modular j-function and the Monster finite group. (From the high energy physics point of view, the modes of the stress-energy tensor of any two-dimensional conformal field theory generate the Virasoro algebra.) In fact, this link is elucidated by the study of vacuum torus 1-point functions at the vacuum vector which give characters (graded dimensions), as the j-function arises as such a character for a vertex operator algebra dubbed the moonshine module [7, 8]. However, vacuum 1-point functions have played a role in other research avenues, including the classification of rational (bulk) conformal field theories from a given vertex operator algebra [9, 10], number theoretic and combinatorial problems [11–13] and the classification of certain families of vertex operator algebras [14–16].

On the other hand, general torus 1-point functions have received considerably less attention. To the best of our knowledge, in the context of vertex operator algebras, the only case considered in the literature is the family of minimal models [17]. One of the goals of the thesis is to detail another family of examples: the important class of simple affine vertex operator algebras $L(k,0)$ constructed from the Lie algebra $\mathfrak{sl}_2$ at non-negative integral levels k. In the physics literature this is referred to as the SU(2) Wess-Zumino-Witten model whose starting point is an action of an SU(2)-valued field on a world-sheet $\mathbb{R}\times S^1$; a pedagogic introduction to how the theory is set up may be found in [18, Chapter 6]. We note that some relevant, but different work is performed in [19, 20] where generalisations of Jack symmetric polynomials are constructed in the context of affine Lie algebras.

To provide more context for readers unfamiliar with n-point correlation functions in conformal field theory, on higher genus Riemann surfaces these are constructed from those on the sphere by gluing together points to add handles. For each pair of points glued in this way, the number of points in the correlation function decreases by two and the genus of the surface increases by one. The configuration of these points determines the complex structure of the resulting surface with many different configurations giving equivalent complex structures. All configurations giving an equivalent complex structure are famously related by the actions of mapping class groups. Due to conformal invariance being closely related to the existence of complex structure, one may be tempted to conclude that a well defined conformal field theory should not be able to distinguish Riemann surfaces with equivalent complex structures. However, this is only true for bulk or full conformal field theory. For chiral conformal field theory, which is the focus here (specifically its algebraic axiomatisation in the form of vertex operator algebras), one merely has that the mapping class groups act on the spaces of chiral correlation functions, as opposed to this action being trivial. All considerations from here on will be purely chiral and so henceforth correlation function or n-point function will refer to the chiral version. In the special case of the torus with either 0 or 1 points, the mapping class groups are, respectively, $\mathrm{PSL}_2(\mathbb{Z})$ and B_3 (the Braid group on three strands). Recall that B_3 is the universal central extension of $\mathrm{PSL}_2(\mathbb{Z})$. It turns out that the action of B_3 on torus 1-point functions can always be rescaled using multiplier systems to yield an action by its quotient $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Hence the properties of torus correlation functions are commonly presented in terms of Γ. The groups $\mathrm{PSL}_2(\mathbb{Z})$ and Γ are somewhat confusingly both commonly referred to as "the modular group" Γ in the literature and so one speaks of modular invariance of torus 1-point functions.

While [4] gives a compelling motivation for the role of modular invariance, this has only been rigorously established for conformal field theories constructed from rational C_2-cofinite vertex operator algebras. In this setting, torus n-point functions where the n points only take insertions from the vertex operator algebra can be constructed as traces of a product of n copies of the vertex operator algebra action on some module M. We call these n-point functions *vacuum torus n-point functions* because the vertex operator algebra is sometimes also called the *vacuum module* (note that the insertions need not be the vacuum vector of the vertex operator algebra). The special case of $n = 1$ with the insertion being the vacuum vector (this can also be thought of as a 0-point function) is called the character of M. In [21], Zhu proved the modular invariance of such vacuum torus n-point functions and in particular showed that vacuum torus 1-point functions are closed under the action of Γ. The properties of the Γ representations arising from vacuum torus 1-point functions have been heavily studied. Much of this work, for example the congruence property [22] rests on using the theory of tensor categories [23] and Verlinde's formula [24, 25]. An alternative route to studying congruence now exists in the case of characters due to developments in number theory and a proof of the unbounded denominator conjecture [26]. It states that for a modular form $f(\tau) \in \mathbb{Q}[[q^{1/N}]]$ on the upper half plane, for a positive integer N, if it is not modular for a congruence subgroup then it has unbounded denominators in its coefficients. However, again, this does not apply to general torus 1-point functions such as those studied in the thesis. While non-congruence subgroups outnumber congruence subgroups of Γ, the modular forms of the former are more poorly understood, partially attributed to the difficulty of defining suitable Hecke operators [27, Section 2.1, Section 2.3].

The transition from vacuum torus n-point functions to general torus n-point functions requires the replacement of vertex operator algebra actions by intertwining operators. This case, as mentioned above, has so far received far less attention within the literature and is the focus here. For rational C_2-cofinite vertex operator algebras the modular invariance of general torus 1-point functions was shown in [28]. This was generalised to orbifolds in [29] and to torus n-point functions in [30]. However, as aforementioned, neither insights from Verlinde's formula nor the unbounded denominator property apply here.

Part of the thesis will review packaging these 1-point functions into vectors will yield vector-valued modular forms transforming under a representation of the modular group, and we provide results on the congruence properties of these representations (or non-congruence) for dimensions up to three, including a more general statement on the occurrence of non-congruence for $\mathfrak{sl}_2$ and results on spaces of vector-valued modular forms given a representation, for dimension up to four. That general torus 1-point functions of vertex operator algebras may be a source of congruence and non-congruence representations has also been observed for the Virasoro minimal models in [17].

A dual goal of the thesis has been to improve the toolkit available for studying torus 1-point functions and this has entailed employing a categorical approach as well, specifically modular tensor categories that enjoy a rich structure in addition to those of an ordinary tensor category. More specifically, it will be shown that not only do non-congruence representations arise, but that our case is a source of infinite families and in the aforementioned dimensions provide explicit formulae for their q-series. This analytic number theoretic data will then be contrasted with the output of the categorical approach from the modular tensor categories formed by $L(k,0)$-modules.

A natural avenue of study beyond the thesis would be considering the same analysis for affine vertex operator algebras from higher rank Lie algebras, such as simply the rank two case, $\mathfrak{sl}_3$. There are also two ways to further refine the torus 1-point functions studied here. Rather than only tracking the L_0 eigenvalues, one can also include an additional variable tracking the h_0 eigenvalue from the Cartan subalgebra, which leads to considering Jacobi forms and one can continue this for higher rank. The other means of refinement is considering twisted modules, where the notion of twist depends on a choice of (not necessarily proper) subgroup of the automorphism group of the vertex operator algebra. This also rapidly enlarges the dimensions of representations, since even for a finite cyclic subgroup $\langle g \rangle$ of order n, one must include in the vector-valued modular form the torus 1-point functions for modules twisted by all powers of the generator up to g^{n-1}, as modular transformations take us between 1-point functions with different twists.

The thesis is organised as follows. Section 3 to Section 7 provide a pedagogic exposition of the less specialised background material for the thesis, based on the references cited therein. Specifically, Section 3 reviews the basic calculus of formal distributions and Section 4 and Section 5 put it to use in formulating the vertex algebra theory needed. Section 6 covers the basics about modularity and modular forms. Finally, Section 7 introduces the categorical notions required for defining modular tensor categories.

In Section 8 we review vector-valued modular forms and the analytic number theory to study them. Torus 1-point functions are defined using intertwining operators and it is shown how vector-valued modular forms emerge by constructing vectors whose entries are torus 1-point functions.

Section 9 reviews and develops general tools to characterise the space of all torus 1-point functions (as modules over the algebra of holomorphic modular forms and the algebra $\mathcal{R}$ of modular differential operators) obtained by varying the insertion vector over an entire simple vertex operator algebra module. The main results are Proposition 9.1, which gives sufficient conditions for the span of torus 1-point functions obtained from Virasoro descendants of certain vectors to be a cyclic $\mathcal{R}$-module, and Theorem 9.3 which gives sufficient conditions for the span of all torus 1-point functions to be a cyclic $\mathcal{R}$-module.

Section 10 introduces the simple affine vertex operator algebra associate to $\mathfrak{sl}_2$ at non-negative integral levels, which was reviewed in Section 5.2. The main result of the section is the multi-part Theorem 10.2, which collects the most important general results surrounding the analysis of torus 1-point functions for the aforementioned case. These include finding vectors to insert giving non-zero torus 1-point functions, establishing linear independence among a certain set of these functions, obtaining that vectors generated from these functions are weakly holomorphic vector-valued modular forms, and providing necessary and sufficient conditions for when these are holomorphic.

Section 11 studies the representations of the modular group arising in Section 10 and the associated spaces of vector-valued modular forms. We show that such representations in the case of dimension one and two are always congruence, and upon reaching dimension three, that there exists an infinite family of non-congruence representations. For all the aforementioned dimensions we provide explicit distinguished vector-valued modular forms from which all others are generated and the exact levels for which the space of all holomorphic vector-valued modular forms is obtained from torus 1-point functions, that is, when they provide such generators. For dimension four we describe the space of all torus 1-point functions in those cases that a relevant space of holomorphic vector-valued modular forms is a cyclic module over the algebra of modular differential operators. For general dimensions we identify levels of affine $\mathfrak{sl}_2$ for which the representation is non-congruence, if it is irreducible.

Section 12 uses the fact categories of modules over rational C_2-cofinite vertex operator algebras are modular tensor categories. The parallel to torus 1-point functions from the categorical perspective are 3-point coupling spaces and we study the action of the braid group on three strands on these. In the case of vacuum 1-point functions this leads to the familiar S and T matrices of the modular group described in Section 6. These are invariants of modular tensor categories that are, however, known not to be complete invariants [31]. Repeating this procedure for general torus 1-point functions has the potential to yield finer invariants. We derive explicit formulae for this action in terms categorical data (specifically twists and fusing matrices) in Theorem 12.2 and conclude by showing the irreducibility of a representation in the dimension four case, complementing the results of Section 11.

We refer to Section 2 for a list of recurring important notation, to facilitate non-sequential readings of the thesis, or those skipping part of the review of material.

Appendix A includes an explanation of the code used for the example in Section 12.1 and Appendix B outlines the code used to compute the explicit q-series of the vector-valued modular forms in Table 1 and Table 2, being the dimension two and dimension three cases respectively.

We remark that the original theorems of this thesis appear in author's own paper [32], but with substantial additional exposition to provide greater accessibility and more context.

2. Symbols

For the convenience of the reader, a list of the most commonly used symbols appearing in the thesis has been compiled.

$\mathbb{N}$ — set of positive integers, $1, 2, 3, \ldots$

$\mathbb{N}_0$ — set of non-negative integers, $0, 1, 2, 3, \ldots$

$\mathbb{R}$ — set of real numbers

$\mathbb{C}$ — set of complex numbers

$\mathbb{H}$ — complex upper-half plane

q — nome $\mathsf{e}(\tau) = e^{2\pi i\tau}$

G_{2k} — Eisenstein series of weight $2k$

B_ℓ — ℓth Bernoulli number

$\mathcal{M}$ — $\mathbb{C}$-algebra of integral weight holomorphic modular forms

$\eta(\tau)$ — Dedekind eta function

$j(\tau)$ — modular j-function

∂_k — modular derivative in weight k

${}_mF_n$ — generalised hypergeometric function, Gaussian for $m = 2, n = 1$

$\mathrm{wt}(\cdot)$ — conformal weight

$\mathrm{wt}[\cdot]$ — square bracket conformal weight

ω — conformal vector

$L(\lambda)$ — highest weight λ module of finite $\mathfrak{sl}_2$

$L(k, \lambda)$ — affine $\mathfrak{sl}_2$ module of highest weight λ and level k

c — central charge

$\mathcal{R}$ — skew polynomial ring of modular differential operators

Γ — modular group

$\mathcal{M}$ — $\mathbb{C}$-algebra of integral weight holomorphic modular forms

$\mathcal{M}_k$ — subspace of $\mathcal{M}$ of weight k

ν_r — multiplier system of weight r

$\mathcal{M}^!(k, \rho, \nu)$ — vector space of weakly holomorphic vector-valued modular forms of weight k, representation ρ and multiplier system ν

$\mathcal{H}(k,\rho,\nu)$ — vector space of holomorphic vector-valued modular forms of weight k, representation ρ and multiplier system ν

$\mathcal{M}(\rho,\nu)$ — vector space of all weakly holomorphic vector-valued modular forms for representation ρ and multiplier system ν

$\mathcal{H}(\rho,\nu)$ — vector space of all holomorphic vector-valued modular forms for representation ρ and multiplier system ν

$\psi^{\mathcal{Y}}(u,\tau)$ — trace function of u at τ for the intertwiner $\mathcal{Y}$

$Y(\cdot,z)$ — field map or module map in variable z

$Y[\cdot,z]$ — transformed expansion $Y(\cdot,\mathrm{e}^z-1)\mathrm{e}^{z\mathrm{wt}(\cdot)}$

$\mathcal{Y}$ — intertwining operator

$\psi^{\mathcal{Y}}(u,\tau)$ — 1-point function for intertwining operator $\mathcal{Y}$, insertion vector u, as a function of τ

$\Psi(\cdot,\tau)$ — column vector of all 1-point functions given intertwining operators and the fusion rules

$\mathcal{C}_1(\cdot)$ — space of 1-point functions with insertion from a given simple module $\cdot$

$\mathcal{C}_1^u(\cdot)$ — space of 1-point functions with insertion from a given simple module $\cdot$ evaluated at $u\in\cdot$

$\mathcal{V}(\rho_\lambda)$ — space of all evaluations of $\Psi(\cdot,\tau)$ at any insertion vector

$\mathcal{V}^u(\rho_\lambda)$ — subspace of $\mathcal{V}(\rho_\lambda)$ of all evaluations on Virasoro descendants of the insertion vector

3. Formal distributions

Formal distributions and the necessary calculus to work with them are introduced, based on both [33] and [34], as they are instrumental in the majority of calculations to follow in the bulk of the thesis. We elect to work over an arbitrary $\mathbb{C}$-algebra R before specialising to $R = \operatorname{End} V$, linear operators on a vector space V, the case relevant to vertex algebras.

The vector space $R[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$ of formal distributions consists of elements of the form,

$$A(z_1, \dots, z_n) = \sum_{i_1 \in \mathbb{Z}} \cdots \sum_{i_n \in \mathbb{Z}} A_{i_1 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n}, \tag{3.1}$$

where $n \in \mathbb{N}$ and the coefficients $A_{i_1 \cdots i_n} \in R$. Note that if we consider multiplication by another element $B(z_1, \cdots, z_n) \in R[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$, the resultant sum,

$$\sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_n \in \mathbb{Z}} \left(\sum_{i_1 \in \mathbb{Z}} \cdots \sum_{i_n \in \mathbb{Z}} A_{i_1 \cdots i_n} B_{k_1 - i_1, \cdots, k_n - i_n} \right) z_1^{k_1} \cdots z_n^{k_n} \tag{3.2}$$

need not exist as it may have coefficients which do not lie in R since the inner sum is not guaranteed to converge. However (3.2) is an element of $R[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$ if either $A(z_1, \dots, z_n)$ or $B(z_1, \cdots, z_n)$ consist of finitely many terms, i.e. is a Laurent polynomial. This motivates the enumeration of several subspaces in the single variable case, relevant to vertex algebras:

- Formal Laurent polynomials $R[z^{\pm 1}]$:

$$R[z^{\pm 1}] = \left\{ \sum_{n \in \mathbb{Z}} r_n z^n \mid r_n \in R, \quad \text{finitely many } r_n \neq 0 \right\} \tag{3.3}$$

- Formal power series $R[[z]]$:

$$R[[z]] = \left\{ \sum_{n \in \mathbb{Z}} r_n z^n \mid r_n \in R \right\} \tag{3.4}$$

- Truncated formal Laurent series $R((z))$:

$$R((z)) = \left\{ \sum_{n \in \mathbb{Z}} r_n z^n \mid r_n \in R, \quad r_n = 0 \text{ for sufficiently negative } n \right\} \tag{3.5}$$

In this thesis, we adopt the convention as in [34] that the binomial expansion for formal variables z, w for $n \in \mathbb{Z}$ is,

$$(z+w)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^{n-k} w^k, \tag{3.6}$$

i.e. always expanding in non-negative powers of the second variable. When $z, w \in \mathbb{C}$ this is equivalent to expanding for $|z| > |w|$. Note that we handle negative n with the prescription,

$$\binom{n}{k} = \begin{cases} (-1)^k \binom{-n+k-1}{k} & k \geq 0 \\ (-1)^{n-k} \binom{-k-1}{n-k} & k \leq n \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

It will also be useful to introduce the formal distribution,

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n \in \mathbb{C}[[z^{\pm 1}], \tag{3.8}$$

often called the delta distribution, which obeys the following proposition.

Proposition 3.1. *The delta function has the following properties:*

(1) *For* $A(z) \in R[z^{\pm 1}]$,

$$A(z)\delta(z) = A(1)\delta(z). \tag{3.9}$$

(2) *For* $A(z,w) \in (\operatorname{End} V)[[z^{\pm 1}, w^{\pm 1}]]$,

$$A(z,w)\delta\left(\frac{z}{w}\right) = A(z,z)\delta\left(\frac{z}{w}\right) = A(w,w)\delta\left(\frac{z}{w}\right) \tag{3.10}$$

as an expression acting on an element of V*, providing* $\lim_{z \to w} A(z,w)$ *exists.*

Our convention is consistent in the thesis with [34] however it should be noted that other authors define their delta distribution such that $\delta(z-1)$ is required in Equation (3.9), so as to be consistent with the usual delta function. The definition of the delta distribution can be motivated by considering the following two embeddings,

$$\iota_+ : \mathbb{C}(z) \hookrightarrow \mathbb{C}((z)), \quad \iota_- : \mathbb{C}(z) \hookrightarrow \mathbb{C}((z^{-1})) \tag{3.11}$$

where ι_+ is a Laurent expansion in z and ι_- is a Laurent expansion in z^{-1} respectively. We find that,

$$\iota_+\left(\frac{1}{1-z}\right) - \iota_-\left(\frac{1}{1-z}\right) = \sum_{n\geq 0} z^n - \left(-\sum_{n<0} z^n\right) = \delta(z). \tag{3.12}$$

Observe that $\iota_-\left((1-z)^{-1}\right) = (-1+z)^{-1}$ following the binomial expansion convention. When $z, w \in \mathbb{C}$, the delta function is then a difference of expansions in the domains $|z| > |w|$ and $|z| < |w|$ of the same rational function. This is dubbed the one variable expansion of zero.

We define differentiation for $A(z) = \sum_{n\in\mathbb{Z}} A_n z^n \in R[[z^{\pm 1}]]$ as,

$$A'(z) = \frac{\mathrm{d}}{\mathrm{d}z} A(z) = \sum_{n\in\mathbb{Z}} n A_n z^{n-1} \tag{3.13}$$

and partial differentiation analogously for multiple formal variables. We also introduce a formal residue operation $\mathrm{Res}_z : R[[z^{\pm 1}]] \to R$ as,

$$\mathrm{Res}_z A(z) = \text{coefficient of } z^{-1}. \tag{3.14}$$

This allows us to elucidate in what sense $\delta(z)$ or another element of $\mathbb{C}[[z^{\pm 1}]]$ can be called a distribution. Namely for $f(z) \in \mathbb{C}[z^{\pm 1}]$ we define a linear functional $\mathbb{C}[z^{\pm 1}] \to \mathbb{C}$ by,

$$f(z) \mapsto \mathrm{Res}_z f(z)\delta(z) = f(1)\mathrm{Res}_z \delta(z) = f(1). \tag{3.15}$$

4. Vertex algebras

We define both vertex algebras and vertex operator algebras in this section, as well as their modules and a notion of maps between these modules known as intertwining operators which will be instrumental to construct 1-point functions later as the traces of such operators. We draw again from both [33] and [34].

We begin with a suitable definition of vertex algebras for our purposes:

Definition 4.1. A vertex algebra $(V, Y, \mathbf{1})$ consists of a vector space V, vacuum vector $\mathbf{1}$ and a field map $Y(\cdot, z) : V \to (\operatorname{End} V)[[z^{\pm 1}]]$ given by,

$$Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}, \quad v_n \in \operatorname{End} V, \quad v \in V. \tag{4.1}$$

These are subject to the following conditions where $u, v \in V$:

(1) $u_n v = 0$ for a sufficiently large n, which is to say $Y(u,z)v$ has finitely many terms with a negative power of z i.e. $Y(u,z)v \in V((z))$.

(2) $Y(\mathbf{1}, z) = \mathrm{id}_V$, i.e. the field associated to the vacuum vector is the identity operator on V.

(3) $Y(v,z)\mathbf{1} \in V[[z]]$ so that we are able to take the limit $\lim_{z\to 0} Y(v,z)\mathbf{1} = v$.

(4) The following Jacobi identity is satisfied:

$$\begin{aligned} &z_1^{-1}\delta\left(\frac{z_2 - z_3}{z_1}\right) Y(u,z_2)Y(v,z_3) - z_1^{-1}\delta\left(\frac{z_3 - z_2}{-z_1}\right) Y(v,z_3)Y(u,z_2) \\ &= z_3^{-1}\delta\left(\frac{z_2 - z_1}{z_3}\right) Y(Y(u,z_1)v, z_3). \end{aligned} \tag{4.2}$$

Observe that from the quantum field theory perspective, the limit in Definition 4.1(3) is physically taking the initial state at $-\infty$. Furthermore, the vacuum condition implies the map $V \to \operatorname{End}(V)$ given by $v \mapsto v_{-1}$ is injective, thus so is Y. In combination with Definition 4.1(3), this gives a state-operator correspondence between elements of V and fields. It should be noted to correctly evaluate three-variable expressions of the form in Definition 4.1(4) our binomial expansion convention must be followed so that for example,

$$z_1^{-1}\delta\left(\frac{z_2 - z_3}{z_1}\right) = \sum_{n\in\mathbb{Z}} z_1^{-n-1}(z_2 - z_3)^n = \sum_{n\in\mathbb{Z}}\sum_{\ell=0}^{\infty} \binom{n}{\ell} z_1^{-n-1} z_2^{n-\ell}(-z_3)^\ell. \tag{4.3}$$

Furthermore, some authors such as [33] choose to already impose a grading condition on V and define a translation operator $T : V \to V$ satisfying $T\mathbf{1} = 0$ and $[T, Y(v,z)] = \partial_z Y(v,z)$ for $v \in V$. This latter point is redundant as we can define T acting as $Tv = v_{-2}\mathbf{1}$. The Jacobi identity may also be substituted for a locality axiom, namely that there exists an $N \in \mathbb{N}$ for which the commutation relation

$$(z_1 - z_2)^N [Y(u,z_1), Y(v,z_2)] = 0 \tag{4.4}$$

holds as a formal power series in $(\operatorname{End} V)[[z_1^\pm, z_2^\pm]]$ for all $u, v \in V$. This is a looser condition than requiring the commutativity of all fields. The term originates from the fact that imposing the condition physically forbids action at a distance.

Example. If the locality condition is replaced by the stronger condition of commutativity, the definition of a vertex algebra reduces to that of an associative commutative algebra. Given such an algebra R, we can always choose the unit as the vacuum vector $\mathbf{1}$ and define a map $Y : R \to R$ by $Y(A)\mathbf{1} = A$ with only the z^0 term non-vanishing. The product is given by $A \cdot B = Y(A)B$ for $A, B \in R$ along with commutativity $Y(A)Y(B) = Y(B)Y(A)$ which will satisfy the axioms of a vertex algebra.

Definition 4.2. A vertex operator algebra is a $\mathbb{Z}$-graded vector space,

$$V = \bigoplus_{n\in\mathbb{Z}} V_{(n)} \tag{4.5}$$

such that $\dim V_{(n)} < \infty$ and $V_{(n)} = 0$ for a sufficiently negative n, with the structure of a vertex algebra $(V, Y, \mathbf{1})$ and a conformal vector $\omega \in V_{(2)}$ satisfying the following conditions:

- The Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{4.6}$$

  for central charge $c \in \mathbb{C}$ is satisfied by the modes $\{L_n\}_{n\in\mathbb{Z}}$ of ω defined by $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \sum_{n\in\mathbb{Z}} \omega_n z^{-n-1}$ so that $L_n = \omega_{n+1}$.
- For $v \in V_{(n)}$, we have that $L_0 v = nv$ so that the grading on V corresponds to the L_0 eigenvalue decomposition, denoting $n = \mathrm{wt}(v)$ the conformal weight of v.

- The derivative property $Y(L_{-1}v,z) = \partial_z Y(v,z)$ holds.

Note that $\mathrm{wt}(v_n) = \mathrm{wt}(v) - n - 1$ for homogeneous $v \in V$ so that the endomorphism v_n takes us from V_m to $V_{m+\mathrm{wt}(v)-n-1}$ for $m,n \in \mathbb{Z}$. Since the grading corresponds to the conformal weights, ω by definition has conformal weight 2 and the vacuum $\mathbf{1} \in V_{(0)}$. The choice of expanding ω in terms of $z^{-n-\mathrm{wt}(\omega)}$ for the modes L_n is the typical physics convention for expanding the field associated to any state, but in the vertex operator algebra literature is often reserved only for the conformal vector.

The Virasoro algebra Vir arises by considering the central extension of the Lie algebra of continuous derivations of $\mathbb{C}((t))$, namely $\mathrm{Der}\,\mathbb{C}((t)) = \mathbb{C}((t))\partial_t$. That is, it fits into the short exact sequence,

$$0 \to \mathbb{C}\mathbf{c} \to \mathrm{Vir} \to \mathrm{Der}\,\mathbb{C}((t)) \to 0, \tag{4.7}$$

where Vir is generated topologically by $\mathbf{c}$ and $L_n = -t^{n+1}\partial_t$ for $n \in \mathbb{Z}$. (Note that the Virasoro algebra can also be stated in a coordinate-independent manner.)

Example. A natural choice is that of the Virasoro vertex operator algebra. Putting aside the technical construction, its underlying vector space is spanned by elements of the form,

$$L_{r_1} \cdots L_{r_m}\mathbf{1}, \quad r_1 \le \cdots \le r_m \le -2 \tag{4.8}$$

for $m \ge 0$ where we have chosen a Poincaré–Birkhoff–Witt ordering and $\mathbf{1}$ is a suitably chosen vacuum vector. Here we have the relation $L_n\mathbf{1} = 0$ for $n \ge -1$, $\mathbf{c}$ acts as c and we can reorder using the Virasoro commutation relations. L_n has conformal weight $-n$ and the vacuum $\mathbf{1}$ has conformal weight 0. The conformal vector is $\omega = L_{-2}\mathbf{1}$ which has conformal weight 2 as expected.

In what will follow in the main body of the thesis on vector-valued modular forms – in order to state results in greater generality where possible – we require the definition of a C_2-cofinite vertex operator algebra.

Definition 4.3. A vertex operator algebra V is said to be C_n-cofinite if $V/C_n(V)$ is finite-dimensional where $C_n(V) = \{v_{-n}w | v, w \in V\}$.

Then the C_2-cofinite property is the $n = 2$ case. It has been shown [35] that C_2-cofiniteness implies that $V/C_n(V)$ is finite-dimensional for $n > 2$.

4.1. **Modules.** We now turn to defining modules of vertex (operator) algebras, and henceforth for this subsection V will denote a vertex algebra $(V, Y, \mathbf{1})$ with the usual structure.

Definition 4.4. A vector space W is a V-module if it is equipped with the module field map $Y_W(\cdot, z) : V \to (\mathrm{End}\, W)[[z^{\pm}]]$ explicitly given by,

$$v \mapsto Y_W(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \tag{4.9}$$

where $v \in V$, subject to the following conditions for $w \in W$:

- $v_n w = 0$ for n sufficiently large.
- $Y_W(\mathbf{1}, z) = \mathrm{id}_W$.
- It satisfies the analogous Jacobi identity,

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_W(u, z_1) Y_W(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_W(v, z_2) Y_W(u, z_1) \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_W(Y(u, z_0)v, z_2) \end{aligned} \tag{4.10}$$

  where also $u \in V$.

Clearly V is a V-module since Y_W reduced to the field map of V satisfies the axioms. In contrast to the definition of a vertex algebra however, Y_W need not be injective and the analogous state-operator correspondence is not a given.

Definition 4.5. Let V have the additional structure of a vertex operator algebra. W is a V-module if it fulfils Definition 4.4 with the following additional conditions:

- W possesses a grading,

$$W = \bigoplus_{\ell\in\mathbb{C}} W_{(\ell)} \tag{4.11}$$

  where $W_{(\ell)} = \{w \in W | L_0 w = \ell w\}$ for L_0 in the expansion of $Y_W(\omega, z)$ (as opposed to the field map of V) and $\dim W_{(\ell)} < \infty$.
- $W_{(\ell)} = 0$ for $\mathrm{Re}(\ell)$ sufficiently negative.

It should be noted that L_0 arising from $Y_W(\omega,z)$ satisfies the Jacobi identity providing W is a V-module for V as a vertex algebra and furthermore $[L_{-1}, Y_W(v,z)] = \partial_z Y_W(v,z)$ for $v \in V$, so these are not additional requirements.

An important property which we will make use of is a vertex operator algebra being rational, which we define as follows.

Definition 4.6. A vertex operator algebra V is rational if every V-module is completely reducible.

Analogous to Lie algebra representation theory, rationality for a vertex operator algebra implies that there are finitely many non-isomorphic simple V-modules, i.e. modules for which no non-zero proper submodule exists. In the preamble on the necessary category theory background, rationality is also shown to have a categorical definition.

4.2. **Intertwining operators.** One of the requirements to construct 1-point functions later is to first introduce the notion of intertwining operators. A more comprehensive account of intertwining operators and the tensor structures they give rise to can be found in [36].

Definition 4.7. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and let (U_i, Y_{U_i}) for $i = 1,2,3$ be V-modules. Consider the linear map,

$$\begin{aligned} \mathcal{Y} :& U_1 \otimes U_2 \to U_3\{z\}[\log(z)] \\ & u_1 \otimes u_2 \mapsto \mathcal{Y}(u_1,z)u_2 = \sum_{\substack{s\in\mathbb{C}\\ t\geq 0}} (u_1)_{s,t} u_2 z^{-s-1} \log(z)^t \end{aligned} \tag{4.12}$$

for integral t where $\{z\}$ denotes the unbounded power series with arbitrary complex exponents and $\log(z)$ is a formal variable distinct from z which is defined by the property $\frac{\mathrm{d}}{\mathrm{d}z}\log(z) = z^{-1}$, with $u_1 \in U_1$ and $u_2 \in U_2$ respectively. Note that the modes $(u_1)_{s,t}$ are maps $U_2 \to U_3$ to comply with the definition. We denote such a map $\mathcal{Y}$ an intertwining operator of type $\binom{U_3}{U_1, U_2}$ if it satisfies the following conditions:

- For fixed $u_1 \in U_1$, $u_2 \in U_2$ and $s \in \mathbb{C}, t \geq 0$,

$$(u_1)_{s+\ell,t} u_2 = 0 \tag{4.13}$$

for sufficiently large $\ell \in \mathbb{Z}$.

- For any $u_1 \in U_1$ and $u_2 \in U_2$,

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Y}(u_1,z)u_2 = \mathcal{Y}(L_{-1}u_1,z)u_2. \tag{4.14}$$

- For any $v \in V, u_1 \in U_1, u_2 \in U_2$,

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_{U_3}(v,z_1)\mathcal{Y}(u_1,z_2)u_2 = z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\mathcal{Y}(u_1,z_2)Y_{U_2}(v,z_1)u_2 \\ + z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{Y}(Y_{U_1}(v,z_0)u_1,z_2)u_2, \end{aligned} \tag{4.15}$$

where the delta function is as introduced in Section 3. It should be observed that Equation (4.13) is an analogous truncation condition to the field map for vertex algebras and likewise Equation (4.15) is a Jacobi identity. In terms of notation, we use $\binom{U_3}{U_1,U_2}$ to denote both the type of an intertwining operator and the space of intertwining operators of such type. In the case of C_2-cofinite rational vertex operator algebras, the $\log(z)$ term disappears from the intertwining operators (see [37] for more details). We are interested in the case when $U_2 = U_3$ so that given two V-modules U and W we will be utilising intertwining operators from $\binom{W}{U,W}$ which will allow us to define a trace operation.

Definition 4.8. For an L_0 eigenvector $u \in U_{\mathrm{wt}(u)}$, we define the $\mathcal{Y}$ zero mode $o^{\mathcal{Y}}(u)$ of u to be the coefficient of $z^{-\mathrm{wt}(u)}$ in the expansion of $\mathcal{Y}(u,z)$ where $\mathcal{Y}$ is an intertwining operator of type $\binom{W}{U,W}$, where W is the appropriate module depending on context.

This zero mode is aptly named since $\mathrm{wt}(u_{\mathrm{wt}(u)-1}) = 0$ to ensure it preserves L_0 eigenvalues in the sense that $o^{\mathcal{Y}}(u)(W_m) \subset W_m$ for all L_0 eigenvalues $m \in \mathbb{C}$.

Example. In order to show explicit intertwining operators, we turn to the affine Heisenberg vertex operator algebra at level 1; we draw from the exposition in [38] but refer the reader to [39] for a treatment in more modern notation.

Whilst any real finite-dimensional vector space $\mathfrak{h}$ suffices, we immediately specialise to $\mathfrak{h} = \mathbb{R}$ equipped with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. With $\mathfrak{h}$ viewed as a real abelian Lie algebra, set $\hat{\mathfrak{h}} = \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}\mathbf{1}$ where $\mathbf{1}$ commutes with all elements. For $\alpha \in \mathfrak{h}_{\mathbb{C}} = \mathbb{C}$ and $\alpha_n = \alpha \otimes t^n$, we have the relations,

$$[\alpha_n, \beta_m] = n\langle \alpha, \beta \rangle \delta_{n+m,0} \mathbf{1} \tag{4.16}$$

for $n, m \in \mathbb{Z}$ and $\alpha_n, \beta_m \in \hat{\mathfrak{h}}$. We now decompose $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0 \oplus \hat{\mathfrak{h}}_-$ where $\hat{\mathfrak{h}}_0 = \mathfrak{h}_{\mathbb{C}} \otimes 1 \oplus \mathbb{C}\mathbf{1}$ and $\hat{\mathfrak{h}}_\pm = \{\alpha_n : \alpha \in \mathfrak{h}_{\mathbb{C}}, \pm n > 0\}$. In order to construct the space of states for the vertex operator algebra, we will require so-called Fock spaces created as induced modules,

$$\mathcal{F}_\lambda = \mathrm{Ind}_{\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0}^{\hat{\mathfrak{h}}} \mathbb{C}|\lambda\rangle \tag{4.17}$$

for $\lambda \in \mathfrak{h}_{\mathbb{C}}$. Here $\hat{\mathfrak{h}}_+$ annihilates the highest weight vector $|\lambda\rangle$ so that $\hat{\mathfrak{h}}_+|\lambda\rangle = 0$ and $\alpha_0|\lambda\rangle = \langle \alpha, \lambda \rangle |\lambda\rangle$. Furthermore, $\mathbf{1}|\lambda\rangle = |\lambda\rangle$ and $\hat{\mathfrak{h}}_-$ acts freely so that the Fock spaces are comprised of words in $\hat{\mathfrak{h}}_-$ subject to the relations. $\mathcal{F}_0$ is chosen as the vertex operator algebra with field map given by,

$$Y(\alpha_{-1}|0\rangle, z) = \alpha(z) = \sum_{n=0}^{\infty} \alpha_n z^{-n-1}. \tag{4.18}$$

The Fock spaces $\mathcal{F}_\lambda$ for $\lambda \in \mathfrak{h}_{\mathbb{C}}$ are modules for $\mathcal{F}_0$ as a vertex operator algebra, with analogous module maps as the field map in Equation (4.18). For $\mu, \nu, \rho \in \mathfrak{h}_{\mathbb{C}}$, the fusion rules imply that,

$$\dim \binom{\mathcal{F}_\rho}{\mathcal{F}_\mu \, \mathcal{F}_\nu} = \begin{cases} 1 & \rho = \mu + \nu \\ 0 & \rho \neq \mu + \nu \end{cases} \tag{4.19}$$

so that there is a single non-zero intertwining operator (up to rescaling) only when $\rho = \mu + \nu$. Consider the group algebra $\mathbb{C}[\mathfrak{h}_{\mathbb{C}}]$ as an abelian group under addition with basis elements e^α for all $\alpha \in \mathfrak{h}_{\mathbb{C}}$. To each e^α we associate a map $s^\alpha : \mathcal{F}_\lambda \to \mathcal{F}_{\lambda+\alpha}$ given by $w|\lambda\rangle \mapsto w|\lambda + \alpha\rangle$ for $w \in U(\hat{\mathfrak{h}}_-)$ where $U(\cdot)$ denotes the universal enveloping algebra. An intertwining operator of type $\binom{\mathcal{F}_{\mu+\nu}}{\mathcal{F}_\mu \, \mathcal{F}_\nu}$ written as a linear map $I_{\mu,\nu} : \mathcal{F}_\mu \otimes \mathcal{F}_\nu \to \mathcal{F}_{\mu+\nu}[[z^{\pm 1}]]z^{\langle \mu, \nu \rangle}$ for $\mu, \nu \in \mathfrak{h}_{\mathbb{C}}$ is given by,

$$I_{\mu,\nu}(p|\mu\rangle, z)q|\nu\rangle = z^{\langle \mu, \nu \rangle} s^\mu E^-(\mu, z) Y(p|0\rangle, z) E^+(\mu, z) q|\nu\rangle \tag{4.20}$$

for $p, q \in U(\hat{\mathfrak{h}}_-)$, where,

$$E^{\pm}(\alpha, z) = \exp\left(\mp \sum_{n=1}^{\infty} \frac{\alpha_{\pm n}}{n} z^{\mp n} \right) \tag{4.21}$$

understood as the expanded series. Observe that $E^{\pm}(\alpha, z)$ has strictly positive or negative modes respectively and we have not had to introduce the caveat of normal ordering, as in Equation (4.20) we have manually chosen annihilating operators to be rightmost. This example is a rare instance where the intertwining operators can be given simply and explicitly, contrary to the case of $\mathfrak{sl}_2$ in the bulk of the thesis to follow.

5. Vertex operator algebras associated to affine Lie algebras

The application of the theory of modularity of vector-valued modular forms to the affine $\mathfrak{sl}_2$ vertex operator algebra necessitates introducing the formalism by which one constructs a vertex operator algebra given an affine Lie algebra. We will also include how the corresponding vertex operator algebra modules arise from the modules of the affine Lie algebra, before specialising to $\mathfrak{sl}_2$ and present its fusion rules which will be relevant to constructing vector-valued modular forms from 1-point functions later.

Let $\mathfrak{g}$ be a Lie algebra adorned with a symmetric, bilinear invariant form which we take without loss of generality to be the Killing form, $\kappa(\cdot,\cdot)$. The associated affine Lie algebra $\hat{\mathfrak{g}}$ has underlying vector space,

$$\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t^{\pm 1}]\oplus\mathbb{C}\mathbf{k}, \tag{5.1}$$

where $\mathbf{k}$ is a non-zero central element and the commutation relations are,

$$[a\otimes t^m, b\otimes t^n] = [a,b]\otimes t^{m+n} + m\kappa(a,b)\delta_{m+n,0}\mathbf{k} \tag{5.2}$$

where $a,b\in\mathfrak{g}$ and $m,n\in\mathbb{Z}$. Note that the underlying vector space of $\hat{\mathfrak{g}}$ possesses a $\mathbb{Z}$-grading,

$$\hat{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \hat{\mathfrak{g}}_{(n)} \tag{5.3}$$

where $\hat{\mathfrak{g}}_{(0)} = \mathfrak{g}\oplus\mathbb{C}\mathbf{k}$ and for $n\neq 0$, $\hat{\mathfrak{g}}_{(n)} = \mathfrak{g}\otimes t^{-n}$. It will be useful to enumerate the following subalgebras:

- All non-zero powers of t, i.e.

$$\hat{\mathfrak{g}}_{(\pm)} = \bigoplus_{n>0} \hat{\mathfrak{g}}_{(\pm n)}. \tag{5.4}$$

- All positive powers of t and the finite part, i.e.

$$\hat{\mathfrak{g}}_{(\leq 0)} = \bigoplus_{n\leq 0} \hat{\mathfrak{g}}_{(n)} = \hat{\mathfrak{g}}_{(-)}\oplus\mathfrak{g}\oplus\mathbb{C}\mathbf{k}. \tag{5.5}$$

Note that $\hat{\mathfrak{g}}_{(\leq 0)}$ denotes elements of positive or zero degree in t but when the grading is matched with $L(0)$ eigenvalues later, these elements will have negative or zero $L(0)$ weight. Define $\mathbb{C}_k$ as a $\hat{\mathfrak{g}}_{(\leq 0)}$-module wherein $\hat{\mathfrak{g}}_{(-)}\oplus\mathfrak{g}$ acts trivially and $\mathbf{k}$ acts as multiplication by $k \in \mathbb{C}$ for now. Consider the induced module,

$$V_{\hat{\mathfrak{g}}}(k,0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{(\leq 0)})} \mathbb{C}_k. \tag{5.6}$$

Take $\mathbf{1} = 1 \in \mathbb{C} \subset V_{\hat{\mathfrak{g}}}(k,0)$ as what will be the vacuum element, and define,

$$a(z) = \sum_{n\in\mathbb{Z}} (a \otimes t^n) z^{-n-1} \in \hat{\mathfrak{g}}[[z^{\pm}]] \tag{5.7}$$

where $a \in \mathfrak{g}$ and $a(n) = a \otimes t^n$ for $n \in \mathbb{Z}$. Then,

$$V_{\hat{\mathfrak{g}}}(k,0) = \bigoplus_{n\geq 0} V_{\hat{\mathfrak{g}}}(k,0)_{(n)} \tag{5.8}$$

where $V_{\hat{\mathfrak{g}}}(k,0)_{(n)}$ is spanned by elements of the form,

$$a^{(1)}(-m_1)\cdots a^{(r)}(-m_r)\mathbf{1}, \tag{5.9}$$

where $r \geq 0$, $a^{(i)} \in \mathfrak{g}$ and $m_i \geq 1$ for $i = 1,\dots,r$ and $n = m_1 + \cdots + m_r$. Here $a^{(r)}(-m_r)$ refers to the $-m_r$ mode as opposed to the field. Note that we have the relation $a(n)\mathbf{1} = 0$ for $a \in \mathfrak{g}$ providing $n \geq 0$ and such expressions may be manipulated using the aforementioned commutation relations. We now have the ingredients to state the following theorem:

Theorem 5.1. *Let $k \in \mathbb{C}$. There exists a unique vertex algebra structure $(V_{\hat{\mathfrak{g}}}(k,0), Y, \mathbf{1})$ on $V_{\hat{\mathfrak{g}}}(k,0)$ where $\mathbf{1} = 1 \in \mathbb{C} \subset V_{\hat{\mathfrak{g}}}(k,0)$ is the vacuum vector and the field map is given by,*

$$Y(a^{(1)}(n_1)\cdots a^{(r)}(n_r)\mathbf{1}, z) = a^{(1)}(z)_{n_1} \cdots a^{(r)}(z)_{n_r} \mathrm{id}_{V_{\hat{\mathfrak{g}}}(k,0)} \tag{5.10}$$

where $r \geq 0$ and $a^{(i)} \in \mathfrak{g}, n_i \in \mathbb{Z}$ for $i = 1,\dots,r$.

We say that $(V_{\hat{\mathfrak{g}}}(k,0), Y, \mathbf{1})$ has level k and it remains to be shown when it also possesses the additional structure of a vertex operator algebra. Henceforth we assume $\mathfrak{g}$ is finite-dimensional and that the Killing form is non-degenerate, which is the case for all semi-simple Lie algebras. Let $d = \dim \mathfrak{g}$ and choose any basis $\{e^{(1)},\dots,e^{(d)}\}$ of $\mathfrak{g}$ with a dual $\{e_{(1)},\dots,e_{(d)}\}$ with respect to the Killing form such that $\kappa(e^{(i)}, e_{(j)}) = \delta^i_j$

for $i, j = 1, \ldots, d$. Define the Casimir element,

$$\Omega = \sum_{i=1}^{d} e^{(i)} e_{(i)} \in U(\mathfrak{g}) \tag{5.11}$$

that acts on $\mathfrak{g}$ as multiplication by $2h$ where $h \in \mathbb{C}$. When we normalise such that $\kappa(\alpha, \alpha) = 2$ where $\alpha \in \mathfrak{h}^*$ is any long root of $\mathfrak{g}$ in the dual of the Cartan subalgebra $\mathfrak{h}$, then h is the dual Coxeter number. Set,

$$\omega = \frac{1}{2(k+h)} \sum_{i=1}^{d} e^{(i)}(-1) e^{(i)}(-1)\mathbf{1} \in V_{\hat{\mathfrak{g}}}(k,0)_{(2)} \tag{5.12}$$

which will be our conformal vector, from which we construct,

$$L(z) = \sum_{n \in \mathbb{Z}} \omega_n z^{-n-1} = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \tag{5.13}$$

We are now in a position to specify the vertex operator algebra structure on $V_{\hat{\mathfrak{g}}}(k,0)$.

Theorem 5.2. *Let $\mathfrak{g}$ be as assumed and $k \neq -h$. Then $V_{\hat{\mathfrak{g}}}(k,0)$ is a vertex operator algebra with $c = dk/(k+h)$ with Equation (5.12) as conformal vector and the $\mathbb{Z}$-grading on $V_{\hat{\mathfrak{g}}}(k,0)$ corresponding to a grading by conformal weight, i.e. L_0 eigenvalues.*

We will not concern ourselves with $k = -h$ since in the affine $\mathfrak{sl}_2$ case to follow, we will only consider integral positive level. However, see for example [40] for the critical level case $k = -h$. We also have the following useful commutation relation for how the Virasoro generators act on elements of $V_{\hat{\mathfrak{g}}}(k,0)$, derived from Equation (5.12) and Equation (5.2),

$$[L_m, a_n] = -n a_{m+n} \tag{5.14}$$

for $a \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.

5.1. **Modules of affine vertex operator algebras.** We now come to determining the modules of the vertex operator algebra associated to $\hat{\mathfrak{g}}$ which we refer to as the affine $\hat{\mathfrak{g}}$ vertex operator algebra. Define for a finite-dimensional irreducible $\mathfrak{g}$-module U the $\hat{\mathfrak{g}}$-module $L_{\hat{\mathfrak{g}}}(k,U)$ by,

$$L_{\hat{\mathfrak{g}}}(k,U) = \mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(U)/N. \tag{5.15}$$

Here N is the sum of all proper $\hat{\mathfrak{g}}$-submodules of $\mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(U)$, where $\mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(U)$ is the induced module,

$$\mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(U) = U(\hat{\mathfrak{g}}) \otimes_{\hat{\mathfrak{g}}_{(\leq 0)}} U \tag{5.16}$$

analogous to constructing $V_{\hat{\mathfrak{g}}}(k,0)$. As a vector space, $\mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(U) = U(\hat{\mathfrak{g}}_{(+)}) \otimes_{\mathbb{C}} U$ where $\hat{\mathfrak{g}}_{(-)}$ acts trivially on U and $\mathbf{k}$ as k.

Proposition 5.3. *For any finite-dimensional irreducible module U, the irreducible $\hat{\mathfrak{g}}$-module $L_{\hat{\mathfrak{g}}}(k,U)$ is an irreducible module for $V_{\hat{\mathfrak{g}}}(k,0)$ as a vertex operator algebra and up to equivalence, all $V_{\hat{\mathfrak{g}}}(k,0)$-modules are of such form.*

Note that $V_{\hat{\mathfrak{g}}}(k,0)$ is not necessarily a simple vertex operator algebra, and we now concern ourselves exclusively with the simple vertex operator algebra $L_{\hat{\mathfrak{g}}}(k,0)$, corresponding to $V_{\hat{\mathfrak{g}}}(k,0)$ quotiented by N. This is the $U = \mathbb{C}$ case. Now let $L(\lambda)$ be the highest weight irreducible $\mathfrak{g}$-module whose highest weight is $\lambda \in \mathfrak{h}^*$. We now enumerate all the irreducible modules for $L_{\hat{\mathfrak{g}}}(k,0)$ as a vertex operator algebra.

Theorem 5.4. *Let the level k be non-negative and integral. Every irreducible module for $L_{\hat{\mathfrak{g}}}(k,0)$ as a vertex operator algebra is equivalent to $L_{\hat{\mathfrak{g}}}(k,L(\lambda))$ for some dominant integral weight $\lambda \in \mathfrak{h}^*$ and $\lambda(\theta) \leq k$ where θ is the highest root of $\mathfrak{g}$.*

Recall that a weight $\lambda \in \mathfrak{h}^*$ is said to be dominant if $\kappa(\lambda,\xi) \geq 0$ for every positive root $\xi \in \mathfrak{h}^*$ (or for any choice of symmetric bilinear invariant form).

To elaborate on the effect of quotienting by N, consider the root space decomposition,

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha \tag{5.17}$$

where Δ is the root system of $\mathfrak{g}$ and $\mathfrak{g}_\alpha = \{a \in \mathfrak{g} | [h,a] = \alpha(h)a, h \in \mathfrak{h}\}$. Choose $e_\theta \in \mathfrak{g}_\theta$ and $f_\theta \in \mathfrak{g}_{-\theta}$ such that $\kappa(e_\theta, f_\theta) = 1$. Then the relation in $L_{\hat{\mathfrak{g}}}(k,L(\lambda))$ imposed by quotienting by N is,

$$e_\theta(-1)^{k+1} v = 0 \tag{5.18}$$

where v is the highest weight vector of $L(\lambda)$.

5.2. **Affine $\mathfrak{sl}_2$.** Having introduced the formalism of affine vertex operator algebras, we turn to setting up the specific case of $\mathfrak{g} = \mathfrak{sl}_2$ which will be the focus of the thesis.

Consistent with Theorem 5.4, the level $k \geq 0$ and we denote the vertex operator algebra as $L(k,0)$. The central charge is,

$$c = \frac{3k}{k+2}. \tag{5.19}$$

A Chevalley basis is given by $\{e,h,f\}$ which satisfies the standard commutation relations,

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f, \tag{5.20}$$

where h spans a choice of Cartan subalgebra $\mathfrak{h} \cong \mathbb{C}h$. We identify the the root lattice with the even integers, $Q \cong 2\mathbb{Z}$ and the weight lattice with integers, $P \cong \mathbb{Z}$. Recall that we have chosen the normalisation $\kappa(h,h) = 2$ with the only other non-vanishing pairing being $\kappa(e,f) = 1$. The representation theory of $L(k,0)$ is semi-simple following Theorem 5.4 and we denote a complete set of representatives of simple modules by $L(k,\mu)$ for $0 \leq \mu \leq k$ with μ being the finite $\mathfrak{sl}_2$ weight. The conformal weight (or L_0 eigenvalue) of the highest weight vector is,

$$h_\mu = \frac{\mu(\mu+2)}{4(k+2)}. \tag{5.21}$$

6. Modularity

In this section we provide the background in modular forms required, utilising [41], in order for the introduction of vector-valued modular forms later in the thesis to be comprehensible. Here and throughout, τ will always lie in the upper-half plane denoted by $\mathbb{H} = \{z \in \mathbb{C} | \text{im}(\tau) > 0\}$.

6.1. Modular group.

The modular group $\Gamma = \text{SL}_2(\mathbb{Z})$, that is, 2×2 matrices over the integers with unit determinant,

$$\text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z}, ad-bc = 1 \right\}. \tag{6.1}$$

It has generators on the upper-half plane $\mathsf{S}: \tau \mapsto -\frac{1}{\tau}$ and $\mathsf{T}: \tau \mapsto \tau + 1$ in terms of the action on τ but may also be represented as matrices by,

$$\mathsf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{6.2}$$

where the action on $\mathbb{H}$ is given by,

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{6.3}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. The modular group admits the presentation,

$$\Gamma = \left\langle \mathsf{S}, \mathsf{T} | \mathsf{S}^4 = 1, (\mathsf{ST})^3 = \mathsf{S}^2 \right\rangle. \tag{6.4}$$

To later discuss congruence and non-congruence representations of the modular group, we introduce the principal congruence subgroups:

Definition 6.1. For integral $N \geq 1$, the principal congruence subgroup of level N in the modular group Γ denoted $\Gamma(N)$ is,

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}. \tag{6.5}$$

6.2. **Modular forms.** Rather than beginning with the definition of a modular form, we consider a weaker notion first. Further generalisations will be introduced when we come to the modular properties of traces of intertwining operators.

Definition 6.2. A weakly modular form of weight $k \in \mathbb{Z}$ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the transformation property,

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \tag{6.6}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Note it is sufficient to check this transformation property for S and T only since these generate the modular group. The case $k = 0$ corresponds to invariance under Γ. The following example is such a case and will be used in later calculations.

Example. The j-invariant or j-function $j(\tau)$ is a weight $k = 0$ modular function characterised uniquely by having a simple pole at infinity and the values,

$$j(i) = 1728, \quad j\left(e^{2\pi i/3}\right) = 0. \tag{6.7}$$

An implication of the transformation property under T is that $f(\tau + 1) = f(\tau)$ so it is periodic and has a Fourier expansion. To f we can construct a function $g : \mathcal{D} \to \mathbb{C}$ where $\mathcal{D}$ is the punctured unit disk and g is such that $f(\tau) = g(e^{2\pi i\tau})$. Then rather than thinking of a Fourier expansion, one can think of g as having the Laurent expansion,

$$g(q) = \sum_{n\in\mathbb{Z}} a_n q^n \tag{6.8}$$

for $q \in \mathcal{D}$ where $q = e^{2\pi i\tau}$ and coefficients $a_n \in \mathbb{C}$. q is often referred to as the nome in the modular forms literature and will be defined as such henceforth.

Example. The j-function has the following q-expansion,

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots \tag{6.9}$$

Note that the expansion does not contain terms of lower order than q^{-1} due to the simple pole at infinity.

Definition 6.3. A modular form of weight $k \in \mathbb{Z}$ is a weakly modular form f of weight k such that f is holomorphic on $\mathbb{H}$ and f is holomorphic at ∞.

The point of considering the Laurent expansion is that one defines holomorphicity at ∞ as being that this series extends to $q = 0$ at the origin of the punctured disk, which corresponds to $\mathrm{im}(\tau) \to \infty$. This is equivalent to f being bounded as $\mathrm{im}(\tau) \to \infty$. To describe spaces of modular forms, we introduce the following series:

Definition 6.4. For $k \in \mathbb{N}$, the Eisenstein series of weight $2k$ is given by,

$$G_{2k}(\tau) = -\frac{B_{2k}}{(2k)!} + \frac{2}{(2k-1)!}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n} \tag{6.10}$$

where B_ℓ denotes the ℓth Bernoulli number given by the generating series,

$$\sum_{\ell=0}^{\infty} B_\ell \frac{x^\ell}{\ell!} = \frac{x}{e^x - 1}. \tag{6.11}$$

For $k \geq 2$, $G_{2k}(\tau)$ is a holomorphic modular form of weight $2k$. For later convenience, let $\mathcal{M}$ denote the $\mathbb{C}$-algebra of integral weight holomorphic modular forms and $\mathcal{M}_k$ the subspace of $\mathcal{M}$ of weight k holomorphic modular forms. We have that $\mathcal{M} = \mathbb{C}[G_4, G_6]$. Furthermore, the j-function can be defined in terms of the Eisenstein series. In our conventions,

$$j(\tau) = \frac{34560 G_4(\tau)^3}{20 G_4(\tau)^3 - 49 G_6(\tau)^2}. \tag{6.12}$$

Finally, we introduce the Dedekind eta function,

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{6.13}$$

This has the following S and T transformation properties,

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau), \quad \eta(\tau+1) = e^{\pi i/12}\eta(\tau). \tag{6.14}$$

It is therefore not a modular form, strictly speaking based on the definitions introduced. However, as we will see in Section 8, there is a notion of it transforming as a modular form of real weight, rather than integral.

7. Modular tensor categories

We assume a familiarity with the rudiments of category theory, however we refer the reader otherwise to [42]. Our exposition draws from [43] and serves as a primer for modular tensor categories. These will be used in our analysis of modular actions from categorical data which will recast several results through the lens of categories rather than employing analytic number theory on modular forms. However, we will have to build up the definition of a modular tensor category starting with the notion of a monoidal category.

Definition 7.1. A monoidal category $(\mathscr{C}, \otimes, a, \mathbf{1}, \iota)$ also called a tensor category consists of a category $\mathscr{C}$, a tensor product bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, a natural isomorphism,

$$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W) \tag{7.1}$$

for $U, V, W \in \mathscr{C}$ called the associativity isomorphism, a unit object $\mathbf{1} \in \mathscr{C}$ and $\iota : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ also an isomorphism. These must satisfy the following two conditions:

- The pentagon axiom,

$$\begin{array}{ccccc} & & ((U \otimes V) \otimes W) \otimes Z & & \\ & \swarrow^{a_{U,V,W} \otimes \mathrm{id}_Z} & & \searrow^{a_{U \otimes V, W, Z}} & \\ (U \otimes (V \otimes W)) \otimes Z & & & & (U \otimes V) \otimes (W \otimes Z) \\ \downarrow{\scriptstyle a_{U, V \otimes W, Z}} & & & & \downarrow{\scriptstyle a_{U,V,W \otimes Z}} \\ U \otimes ((V \otimes W) \otimes Z) & & \xrightarrow{\mathrm{id}_U \otimes a_{V,W,Z}} & & U \otimes (V \otimes (W \otimes Z)) \end{array}$$

  where $U, V, W, Z \in \mathscr{C}$.
- Left $U \mapsto \mathbf{1} \otimes U$ and right $U \mapsto U \otimes \mathbf{1}$ multiplication respectively by $\mathbf{1}$ as functors are autoequivalences, known as the unit axiom, where $U \in \mathscr{C}$.

Example. The category of R-modules over a commutative ring R is a monoidal category. Here the tensor product $\otimes_R$ over R is the choice of bifunctor and R itself is chosen as $\mathbf{1}$, viewed as an R-module. Choosing $R = \mathbb{F}$ as a field $\mathbb{F}$ amounts to studying the category of $\mathbb{F}$-vector spaces $\mathsf{Vec}_{\mathbb{F}}$ over $\mathbb{F}$ or choosing $R = \mathbb{Z}$ gives the category of abelian groups. In the case of $\mathsf{Vec}_{\mathbb{F}}$ the bifunctor is then the usual tensor product over a field and from the unit axiom, it is clear $\mathbf{1}$ is the field $\mathbb{F}$ in question.

To this definition of a monoidal category we will append other properties to arrive at a modular tensor category. We now follow with a definition that essentially describes a category as being akin to that of say, abelian groups. Recall that $\hom_{\mathscr{C}}(\cdot,\cdot)$ denotes the hom functor in the category $\mathscr{C}$.

Definition 7.2. A category $\mathscr{C}$ is an $\mathbb{F}$-linear abelian category for a field $\mathbb{F}$ if the following conditions are satisfied:

- Additive:
  - (a) $\hom_{\mathscr{C}}(U,V)$ has the structure of an abelian group for $U,V \in \mathscr{C}$.
  - (b) There exists a zero object $0 \in \mathscr{C}$ such that $\hom_{\mathscr{C}}(0,0) = 0$.
  - (c) $\mathscr{C}$ admits finite coproducts, which is to say, there exists a direct sum bifunctor $\oplus : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$.
- Abelian:
  - (a) Kernels and cokernels exist for all morphisms.
  - (b) For every morphism $f : U \to V$ in $\mathscr{C}$, there exists an isomorphism,

$$\tilde{f} : \mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f)). \tag{7.2}$$

  - (c) Every monomorphism is a kernel and every epimorphism is a cokernel.
- $\mathbb{F}$-linear:
  - (a) $\hom_{\mathscr{C}}(U,V)$ has the structure of a vector space over $\mathbb{F}$ for $U,V \in \mathscr{C}$.

Note here that the cokernel of a morphism $f : U \to V$ is an object W along with a morphism $g : V \to W$ such that the diagram,

$$\begin{array}{ccc} & & V \\ & \overset{f}{\nearrow} & \big\downarrow g \\ U & \xrightarrow[0_{UW}]{} & W \end{array} \tag{7.3}$$

commutes, where 0_{UW} is the zero morphism. We further require this be universal, in the sense that for another pair (g', W'), there exists a unique morphism $u : W \to W'$ such that $g' = u \circ g$. For example, for the case of modules, the cokernel of a morphism is the quotient of its target by the image of the morphism.

Example. $\mathsf{Vec}_{\mathbb{F}}$ is an abelian category. The zero object is simply the zero-dimensional vector space over $\mathbb{F}$ and the direct sum of vector spaces serves as the coproduct. The other properties are consequences of linear algebra and as the hom spaces are vector spaces over $\mathbb{F}$ it is also $\mathbb{F}$-linear.

Definition 7.3. A category $\mathscr{C}$ which is an $\mathbb{F}$-linear abelian category is finite if it is equivalent to the category of finite-dimensional R-modules for a finite-dimensional $\mathbb{F}$-algebra R.

Example. The category of finite-dimensional vector spaces over $\mathbb{F}$ is a finite category, and is a subcategory of $\mathsf{Vec}_{\mathbb{F}}$.

With these definitions in place, we now turn to defining the notions of a braiding, rigidity and ribbon structure, which are applied to monoidal categories.

Definition 7.4. A braided structure on a monoidal category $\mathscr{C}$ consists of a natural isomorphism $c_{U,V} : U \otimes V \xrightarrow{\sim} V \otimes U$ such that the following diagrams are commutative for $U, V, W \in \mathscr{C}$:

- $$\begin{array}{ccccccc} & & U\otimes(V\otimes W) & \xrightarrow{c_{U,V\otimes W}} & (V\otimes W)\otimes U & & \\ & \overset{a_{U,V,W}}{\nearrow} & & & & \overset{a_{V,W,U}}{\searrow} & \\ (U\otimes V)\otimes W & & & & & & V\otimes(W\otimes U) \\ & \underset{c_{U,V}\otimes \mathrm{id}_W}{\searrow} & & & & \underset{\mathrm{id}_V\otimes c_{U,W}}{\nearrow} & \\ & & (V\otimes U)\otimes W & \xrightarrow[a_{V,U,W}]{} & V\otimes(U\otimes W) & & \end{array}$$

- $$\begin{array}{ccccccc} & & (U\otimes V)\otimes W & \xrightarrow{c_{U\otimes V,W}} & W\otimes(U\otimes V) & & \\ & \overset{a^{-1}_{U,V,W}}{\nearrow} & & & & \overset{a^{-1}_{W,u,V}}{\searrow} & \\ U\otimes(V\otimes W) & & & & & & (W\otimes U)\otimes V \\ & \underset{\mathrm{id}_U\otimes c_{V,W}}{\searrow} & & & & \underset{c_{U,W}\otimes \mathrm{id}_V}{\nearrow} & \\ & & U\otimes(W\otimes V) & \xrightarrow[a^{-1}_{U,W,V}]{} & (U\otimes W)\otimes V & & \end{array}$$

Example. Continuing with $\mathsf{Vec}_{\mathbb{F}}$, an obvious choice of braiding is $c_{U,V} : u \otimes_{\mathbb{F}} v \mapsto v \otimes_{\mathbb{F}} u$, where $u \in U, v \in V$ and $U, V \in \mathsf{Vec}_{\mathbb{F}}$. In fact, this braiding satisfies $c_{V,U} \circ c_{U,V} =$

$\mathrm{id}_{U\otimes V}$ which also makes it a symmetric monoidal category. Any symmetric monoidal category is a braided monoidal category.

For $U \in \mathscr{C}$ we denote its left dual by U^* if there exists an evaluation $\mathrm{ev}_U : U^* \otimes U \to \mathbf{1}$ and coevaluation $\mathrm{coev}_U : \mathbf{1} \to U \otimes U^*$ morphism respectively, for which the compositions,

$$\begin{aligned} &U \xrightarrow{\mathrm{coev}_U\otimes \mathrm{id}_U} (U\otimes U^*)\otimes U \xrightarrow{a_{U,U^*,U}} U\otimes(U^*\otimes U) \xrightarrow{\mathrm{id}_U\otimes \mathrm{ev}_U} U, \\ &U^* \xrightarrow{\mathrm{id}_U\otimes \mathrm{coev}_U} U^*\otimes(U\otimes U^*) \xrightarrow{a^{-1}_{U^*,U,U^*}} (U^*\otimes U)\otimes U^* \xrightarrow{\mathrm{ev}_U\otimes \mathrm{id}_{U^*}} U^*, \end{aligned} \tag{7.4}$$

are identity morphisms. Similarly a right dual ${}^*U$ is defined with morphisms $\mathrm{ev}'_U : U\otimes {}^*U \to \mathbf{1}$ and $\mathrm{coev}'_U : \mathbf{1} \to U\otimes {}^*U$ for which the compositions,

$$\begin{aligned} &U \xrightarrow{\mathrm{id}_U\otimes \mathrm{coev}'_U} U\otimes({}^*U\otimes U) \xrightarrow{a^{-1}_{U,{}^*U,U}} (U\otimes {}^*U)\otimes U \xrightarrow{\mathrm{ev}'_U\otimes \mathrm{id}_U} U, \\ &{}^*U \xrightarrow{\mathrm{coev}'_U\otimes \mathrm{id}_{{}^*U}} ({}^*U\otimes U)\otimes {}^*U \xrightarrow{a_{{}^*U,U,{}^*U}} {}^*U\otimes(U\otimes {}^*U) \xrightarrow{\mathrm{id}_{{}^*U}\otimes \mathrm{ev}'_U} {}^*U, \end{aligned} \tag{7.5}$$

are identity morphisms.

Definition 7.5. A monoidal category $\mathscr{C}$ is said to be rigid if all objects have left and right duals.

Example. In the category of finite-dimensional vector spaces over $\mathbb{F}$, we can define the dual V^* of a vector space V over $\mathbb{F}$ as being the linear functionals $\hom(V,\mathbb{F})$. The evaluation is then $\mathrm{ev}_V : v^*\otimes_{\mathbb{F}} v \mapsto \langle v^*, v\rangle$, the pairing of $v^* \in V^*$ with $v \in V$. Choosing a basis $\{e_i\}$ of V and dual basis $\{f_i\}$ of V^* for $i = 0,\dots,\dim V-1$, we can then define the coevaluation as

$$\mathrm{coev}_V : \gamma \mapsto \gamma \sum_{i=0}^{\dim V-1} e_i \otimes_{\mathbb{F}} f_i \tag{7.6}$$

for the scalar $\gamma \in \mathbb{F}$. As the category of finite-dimensional vector spaces over $\mathbb{F}$ is rigid, we need only specify one evaluation and coevaluation respectively.

Definition 7.6. A braided rigid monoidal category $\mathscr{C}$ is ribbon if it possesses a ribbon structure, i.e. a twist $\theta \in \mathrm{Aut}(\mathrm{id}_{\mathscr{C}})$ for which,

$$\theta_{U\otimes V} = (\theta_U \otimes \theta_V)\circ c_{V,U}\circ c_{U,V} \tag{7.7}$$

and $(\theta_U)^* = \theta_{U^*}$ for all $U, V \in \mathscr{C}$.

Example. For the category of finite-dimensional vector spaces over $\mathbb{F}$ and the braiding as introduced before, the twist must satisfy $\theta_{U\otimes V} = \theta_U \otimes \theta_V$ since $c_{V,U} \circ c_{U,V} = \mathrm{id}_{U\otimes V}$. The twist is trivially $\theta_V = \mathrm{id}_V$.

For further intuition for a ribbon structure, we refer to [43, Remark 8.10.4] which makes a comparison to quadratic forms. We now have all the necessary properties to define a modular tensor category. Assume that $\mathbb{F}$ is algebraically closed and characteristic 0.

Definition 7.7. A modular tensor category is a finite (including abelian, $\mathbb{F}$-linear), semi-simple, ribbon (including braided, rigid) monoidal category whose S-matrix is non-degenerate, where the S-matrix is defined by,

$$S = (s_{U,V})_{U,V\in\mathcal{O}(\mathscr{C})}, \quad s_{U,V} = \mathrm{tr}(c_{V,U} \circ c_{U,V}) \tag{7.8}$$

where $\mathcal{O}(\mathscr{C})$ is the set of isomorphism classes of simple objects of $\mathscr{C}$.

8. Modular properties of traces of intertwining operators

In this section we introduce the notion of vector-valued modular forms and the analytic number theory to study them. Torus 1-point functions are defined using intertwining operators and vector-valued forms are shown to emerge when arranging vectors whose entries are torus 1-point functions.

8.1. Vector-valued modular forms.

For convenience, we use the notation that for a variable x we denote $\mathbf{e}(x) = \mathrm{e}^{2\pi i x}$. In particular, recall that $q = \mathbf{e}(\tau)$ recalling τ lies in the complex upper-half plane so that q lies in the interior of the complex unit disk, as in Section 6. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we set,

$$j_k(\gamma;\tau) = (c\tau + d)^k \tag{8.1}$$

where $k \in \mathbb{R}$, not to be conflated with the j-function defined in Equation (6.7). In order to accommodate a real-valued modular weight, we must also introduce multiplier systems.

Definition 8.1. A function $\nu\colon \mathrm{SL}_2(\mathbb{Z}) \to \{r \in \mathbb{C} \mid |r| = 1\}$ is called a *multiplier system for* Γ *of weight* $k \in \mathbb{R}$ if for any $A, B \in \mathrm{SL}_2(\mathbb{Z})$ it satisfies

$$\nu(AB) j_k(AB;\tau) = \nu(A)\nu(B) j_k(A;B\tau) j_k(B;\tau). \tag{8.2}$$

The *cusp parameter* of ν is the unique $m \in \mathbb{R}$ such that $0 \le m < 12$ and $\nu(T) = \mathbf{e}(m/12)$.

Note that the relation (8.2) implies that multiplier systems are uniquely characterised by their values on the generators S and T. The purpose of multiplier systems is to redefine projective representations of $\mathrm{SL}_2(\mathbb{Z})$ (specifically representations of the braid group on three strands $\mathsf{B}_3 = \langle \mathsf{S}, \mathsf{T} \mid (\mathsf{ST})^3 = \mathsf{S}^2 \rangle$, where S^4 acts as a phase) so that they are no longer projective, as we shall see shortly. A more computational viewpoint is that introducing real weights spoils the cocycle condition of j_k which must be restored with the multiplier system.

Observe that if ν is a multiplier system of weight k, then it is also one of weight $k + n$ for any $n \in \mathbb{Z}$. The remainder of this subsection does not depend on the choice of multiplier system one wishes to consider. Nevertheless, for use later we note that for each $r \in \mathbb{R}$ there exists a multiplier system ν_r of weight r satisfying

$$\nu_r(\mathsf{T}) = \mathbf{e}\left(\frac{r}{12}\right), \qquad \nu_r(\mathsf{S}) = \mathbf{e}\left(\frac{-r}{4}\right), \qquad \nu_r(\mathsf{ST}) = \mathbf{e}\left(\frac{-r}{6}\right) \tag{8.3}$$

(see, for example, [44, Proposition 2.3.2]). Recall that the eta function η introduced in Equation (6.13) was not an integral weight modular form, but the multiplier system ν_r is precisely that which makes η^{2r} transform as a modular form of weight r or equivalently thinking of η as a modular form of weight $\frac{1}{2}$. Note that r is the cusp parameter for ν_r if and only if $0 \le r < 12$.

Multiplier systems allow us to define an action of Γ on tuples of functions (called vectors) on $\mathbb{H}$ via the following:

Definition 8.2. Let $\rho\colon \Gamma \to \mathsf{GL}(d, \mathbb{C})$ be a d-dimensional representation of Γ and consider holomorphic functions $f_1, \dots, f_d\colon \mathbb{H} \to \mathbb{C}$ arranged into a vector

$$F = (f_1, \dots, f_d)^t, \tag{8.4}$$

where $\mathbf{x}^t$ denotes the transpose of a vector $\mathbf{x}$.

(1) The vector F is a d-dimensional *weakly holomorphic vector-valued modular form of weight* $k \in \mathbb{R}$ on Γ for the representation ρ and a multiplier system ν, if the following hold.
   (i) Each f_j is meromorphic at the cusp $\mathrm{i}\infty$.
   (ii) For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$F|_k^\nu \gamma = \rho(\gamma) F, \tag{8.5}$$

   where we define $|_k^\nu \gamma$ on each f_j by

$$\left(f_j|_k^\nu \gamma\right)(\tau) = \nu(\gamma)^{-1} j_k(\gamma; \tau)^{-1} f_j(\gamma\tau) \tag{8.6}$$

   and extend the definition of $|_k^\nu$ component wise to F.

(2) The vector F is a *holomorphic vector-valued modular form* if it is a weakly holomorphic vector-valued modular form for which each f_j is holomorphic at the cusp $i\infty$.

The notion of holomorphicity at the cusp at infinity is as explained in Section 6. Here we see that if the multiplier system were omitted from the action (8.6), then the $\mathsf{S}^4 = 1$ relation of Γ would not necessarily hold. This would therefore define a projective action, or alternatively, an action of B_3.

For a fixed representation $\rho\colon \Gamma \to \mathsf{GL}(d,\mathbb{C})$ and multiplier system ν of weight $k \in \mathbb{R}$, the corresponding vector spaces of weakly holomorphic and holomorphic vector-valued modular forms of weight k for representation ρ and multiplier system ν are denoted $\mathcal{M}^!(k,\rho,\nu)$ and $\mathcal{H}(k,\rho,\nu)$, respectively. As noted above, multiplier systems only determine weights up to shifts by integers and these weights are always in the same integer coset as the cusp parameter m of ν. We therefore denote by $\mathcal{M}^!(\rho,\nu) = \bigoplus_{n\in\mathbb{Z}} \mathcal{M}^!(m+n,\rho,\nu)$ and $\mathcal{H}(\rho,\nu) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}(m+n,\rho,\nu)$, respectively, the spaces of all weakly holomorphic and holomorphic vector-valued modular forms for the pair (ρ,ν). Further, $\mathcal{H}(\rho,\nu)$ always admits a minimal weight $p_0 \in \mathbb{R}$ such that $\mathcal{H}(\rho,\nu) = \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}(p_0+n,\rho,\nu)$ and $\mathcal{H}(p_0-\ell,\rho,\nu) = 0$ for all $\ell \in \mathbb{N}$.

We will always assume that $\rho(\mathsf{T})$ is diagonal with

$$\rho(\mathsf{T}) = \mathrm{diag}(\mathbf{e}(r_1),\dots,\mathbf{e}(r_d)), \tag{8.7}$$

for real numbers $r_1,\dots,r_d$, in particular, $\rho(T)$ is a unitary matrix. This assumption can always be made for vertex operator algebras with a semisimple representation theory (which is the case we will specialise to shortly) as the intertwining operators can then be chosen without loss of generality to take values in simple modules. A further simplifying assumption that is common in the number theory literature is that $\rho(\mathsf{S}^2)$ is a scalar matrix. Since S^2 generates the centre of Γ and has finite order (in the standard realisation of Γ as integral 2×2 matrices with unit determinant, we have $\mathsf{S}^2 = -1$), the vector space underlying the representation ρ always admits a direct sum decomposition with $\rho(\mathsf{S}^2)$ acting as a scalar on each summand. This assumption therefore primarily simplifies the presentation of certain theorems and hence is not necessary. In the context of vertex operator algebras, S^2 carries the additional interpretation of being the *charge conjugation involution* (the functor which assigns a module to its dual), so $\rho(\mathsf{S}^2)$ cannot be diagonal in a basis of intertwining operators which only take values in simple modules, if there are modules which are not self dual. For an account of the role of these assumptions in number theory one can consult, for example, [45] or [46]. In the latter, it is also assumed that $0 \leq r_1, \ldots, r_d < 1$, but this will not be required here. With these assumptions, if $F \in \mathcal{M}^!(k, \rho, \nu)$ we can replace Definition 8.2.1.i with the condition that each f_j has a Fourier expansion of the form

$$f_j(\tau) = q^{\lambda_j} \sum_{n=0}^{\infty} a_n q^n, \tag{8.8}$$

for some real numbers λ_j (see, for example, [45] for more discussion). As described in [46], a holomorphic vector-valued modular form F requires each f_j to have an expansion (8.8) where

$$0 \leq \lambda_j \equiv r_j + \frac{m}{12} \pmod{\mathbb{Z}}, \tag{8.9}$$

and m is the cusp parameter of the multiplier system ν.

Definition 8.3. Let $\rho \colon \Gamma \to \mathsf{GL}(d, \mathbb{C})$ be a representation such that $\rho(T)$ is diagonal and unitary as in (8.7) and ν a multiplier system. A set of non-negative real numbers $\{\lambda_1, \ldots, \lambda_d\}$ satisfying (8.9) is called an *admissible set* for (ρ, ν). The *minimal admissible set* for (ρ, ν) is the unique admissible set which additionally satisfies $\lambda_j < 1$ for $1 \leq j \leq d$.

As pointed out in [46], and which follows from (8.8) and (8.9) above, for a minimal admissible set $\{\lambda_1, \ldots, \lambda_d\}$ we have that every non-zero $F \in \mathcal{H}(\rho, \nu)$ has the form

$$\begin{pmatrix} q^{\lambda_1 + n_1} \sum_{n=0}^{\infty} a_1(n) q^n \\ \vdots \\ q^{\lambda_d + n_d} \sum_{n=0}^{\infty} a_d(n) q^n \end{pmatrix}, \tag{8.10}$$

where for each $j = 1, \ldots, d$ we have $a_j(0) \neq 0$ and n_j are non-negative integers. Note that $a_j(n)$ refers to the nth coefficient for the jth entry in the column vector.

To provide conditions for a lower bound on the minimal weight p_0, among other things, we require the modular derivative in weight $k \in \mathbb{R}$, which is defined as

$$\partial = \partial_k = \frac{1}{2\pi i} \frac{d}{d\tau} + k G_2(\tau), \tag{8.11}$$

on weight k (vector-valued) modular forms and is then extended linearly. The modular derivative increments the weight of (vector-valued) modular forms by two. In particular, the homogeneous subspaces of the ring of integral weight holomorphic modular forms are related by $\partial_k \mathcal{M}_k \subset \mathcal{M}_{k+2}$. For $n \in \mathbb{N}$ and $\phi \in \mathcal{M}_k$ we let $\partial^n \phi$ denote the composition of operators $\partial_{k+2n} \circ \partial_{k+2(n-1)} \circ \cdots \circ \partial_{k+2} \circ \partial_k \phi$. This allows us to consider an order $n \in \mathbb{N}$ monic modular differential equation in weight $k \in \mathbb{R}$, which is an ordinary differential equation of the form

$$\left(\partial_k^n + \sum_{j=0}^{n-2} \phi_{2(n-j)} \partial_k^j \right) f = 0 \tag{8.12}$$

in the disk $|q| < 1$, where $\phi_j \in \mathcal{M}_j$ for each j. For more details about monic modular differential equations see, for example, [46, 47].

The modular derivative can be adjoined to the algebra of integral weight modular forms $\mathcal{M}$ to form

$$\mathcal{R} = \left\{ \phi_0 + \phi_1 \partial + \cdots + \phi_n \partial^n \mid \phi_i \in \mathcal{M}, n \geq 0 \right\}, \tag{8.13}$$

the skew polynomial ring of modular differential operators, where addition is defined component wise, and multiplication is characterised by $\partial \cdot \phi = \phi \partial + \partial_k \phi$ for $\phi \in \mathcal{M}_k$. In fact, for any $F \in \mathcal{M}^!(k, \rho, \nu)$, defining ∂F to be ∂ applied component wise, we find $\partial F \in \mathcal{M}^!(k+2, \rho, \nu)$. Similarly, for $\phi \in \mathcal{M}_k$ we let ϕF denote the vector F with ϕ multiplied component wise. Thus both $\mathcal{M}^!(\rho, \nu)$ and $\mathcal{H}(\rho, \nu)$ are left $\mathcal{R}$-modules with $\mathcal{H}(\rho, \nu)$ as an $\mathcal{R}$-submodule.

We conclude this section with convenient criteria for determining when the components of a holomorphic vector-valued modular form span the solution space of a monic modular differential equation and for determining the minimal weight p_0 of the space $\mathcal{H}(\rho, \nu)$ of holomorphic vector-valued modular forms.

Theorem 8.4 (Mason [47], Marks [46, Theorem 2.8]). *Let $\rho \colon \Gamma \to \mathsf{GL}(d, \mathbb{C})$ be a representation such that $\rho(T)$ is diagonal and unitary as in (8.7), ν a multiplier system, and $\{\lambda_1, \ldots, \lambda_d\} \subset \mathbb{R}$ the minimal admissible set for (ρ, ν). Consider a holomorphic vector-valued modular form $F \in \mathcal{H}(k, \rho, \nu)$ which must therefore have an expansion of the form (8.10) such that the leading exponent of the jth component is $\lambda_j + n_j$,*

$n_j \in \mathbb{N}_0$. *If the components of F are linearly independent over $\mathbb{C}$, then the weight p is bounded below by the inequality*

$$p \geq \frac{12\left(\sum_j \left(\lambda_j + n_j\right)\right)}{d} + 1 - d, \tag{8.14}$$

and equality holds if and only if the components of F span the solution space of a monic modular differential equation. In particular, the minimal weight p_0 of $\mathcal{H}(\rho, \nu) = \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}(p_0 + n, \rho, \nu)$ satisfies

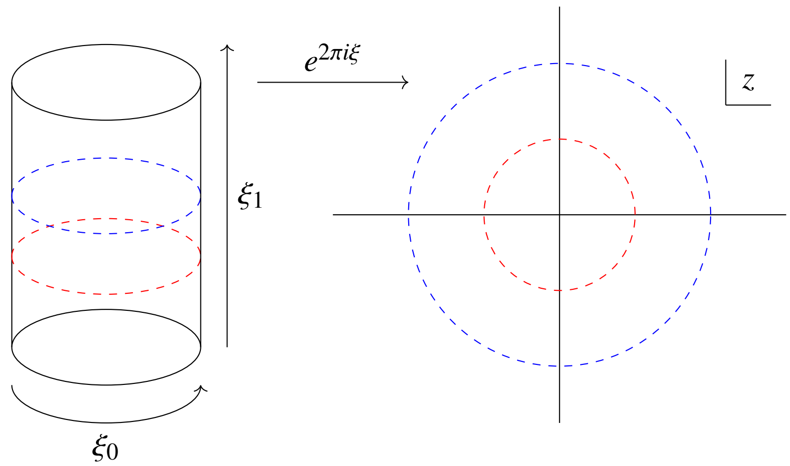
$$p_0 \geq \frac{12\left(\sum_j \left(\lambda_j\right)\right)}{d} + 1 - d. \tag{8.15}$$

8.2. **Modularity of torus 1-point functions.** The purpose of this section is to introduce torus 1-point functions which can be constructed in a number of ways. Here we will define them using suitable traces of intertwining operators as aforementioned. We mostly follow the conventions of [33], see [36] for an exhaustive account of intertwining operators and the tensor structures that arise from them. As a reminder of our notation for vertex operator algebras, let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra, where V denotes the underlying vector space, Y the field map, $\mathbf{1}$ the vacuum vector, and ω the conformal vector. The central charge of the Virasoro algebra generated by the field expansion of $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ will be denoted $\mathbf{c}$. Further, let (U, Y_U) be a V-module, with U the underlying vector space and Y_U the field map (or action) representing the vertex operator algebra V. We will always assume that modules are graded by generalised L_0 eigenvalues, that is,

$$U = \bigoplus_{n \in \mathbb{C}} U_n, \qquad U_n = \{u \in U \mid \exists m \in \mathbb{N}, (L_0 - n)^m u = 0\}. \tag{8.16}$$

For a homogeneous element $u \in U_n$, we denote the *conformal weight* n of u by $\mathrm{wt}(u) = n$ as in Definition 4.2. We will soon specialise to rational C_2-cofinite vertex operator algebras. Among other helpful properties, such vertex operator algebras only admit modules for which L_0 acts semisimply with finite-dimensional eigenspaces, and all eigenvalues are rational, bounded below, and discrete.

FIGURE 1. Schematic illustration of the homeomorphism between the cylinder and annulus, with increasing ξ_1 corresponding to moving radially outward in the annulus.



We now come to the definition of a torus 1-point function, utilising the intertwining operators introduced in Section 4.2. To motivate this definition, the usual means of reaching the torus in conformal field theory on the plane is to go through the cylinder first. Reaching the cylinder from the plane is accomplished by compactifying one coordinate, so that one is left with a cylinder of infinite length in the uncompactified direction. If $\xi = \xi_0 + i\xi_1$ is the complex coordinate on the cylinder, ξ_0 runs around in the compactified direction and ξ_1 is its length. It is homeomorphic to the annulus with complex coordinate z where we choose a mapping, say $z = e^{2\pi i\xi}$. Then slices of constant ξ_1 correspond to concentric circles, as illustrated in Figure 1. The torus is then obtained by making the identification $z \sim qz$ where here $q \in \mathbb{C}$ is non-zero, which we can visualise as wrapping the cylinder to form a torus. This geometric process is equivalent to taking a q-trace [21, Section 4.2], which gives us the following definition.

Definition 8.5. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra, (U, Y_U), (W, Y_W) be V-modules, and $\mathcal{Y}$ be an intertwining operator of type $\binom{W}{U,\ W}$. The *torus 1-point function associated to* $\mathcal{Y}$ is the trace

$$\psi^{\mathcal{Y}}(u,\tau) = \mathrm{tr}_W\, o^{\mathcal{Y}}(u) q^{L(0)-\frac{\mathsf{c}}{24}}, \qquad u \in U. \tag{8.17}$$

Recall $o^{\mathcal{Y}}(u) = u_{\mathrm{wt}(u)-1}$ is the zero mode with respect to the intertwining operator $\mathcal{Y}$. If the intertwining operator $\mathcal{Y}$ is clear from context, we will omit $\mathcal{Y}$ as a label for zero modes and torus 1-point functions.

Beyond the standard expansions of fields, we will also need to consider transformed expansions

$$Y[a,z] = Y(a, \mathrm{e}^z - 1)\mathrm{e}^{z\,\mathrm{wt}(a)} = \sum_{n\in\mathbb{Z}} a_{[n]} z^{-n-1}, \qquad a \in V_{\mathrm{wt}(a)}, \tag{8.18}$$

and extended linearly, which implies the formula

$$a_{[n]} = \mathrm{Res}_z \left(Y(a,z)(\log(1+z))^n (1+z)^{\mathrm{wt}(a)-1} \right). \tag{8.19}$$

This is motivated by the mapping between the plane and cylinder mentioned earlier. For example, for $a \in V_{\mathrm{wt}(a)}$ we have

$$\begin{aligned} a_{[0]} &= \mathrm{Res}_z \left(Y(a,z)(1+z)^{\mathrm{wt}(a)-1} \right) \\ &= \mathrm{Res}_z \left(\sum_{n\in\mathbb{Z}} \sum_{j=0}^{\infty} \binom{\mathrm{wt}(a)-1}{j} a_n z^{-n-1+j} \right) \\ &= \sum_{j=0}^{\infty} \binom{\mathrm{wt}(a)-1}{j} a_j \end{aligned} \tag{8.20}$$

so that if $a \in V_1$ we obtain $a_{[0]} = a_0$ where we have used the binomial convention Equation (3.6). In fact, the map (8.18) gives V another structure of a vertex operator algebra of central charge $\mathbf{c}$ with the same vacuum vector and conformal element $\tilde{\omega} = \omega - \frac{\mathbf{c}}{24}\mathbf{1}$ (see [21, Section 4] for details). Similar to above, defining $L_{[n]}$ via $Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} L_{[n]} z^{-n-2}$ gives us a *square bracket grading*

$$U = \bigoplus_{n\in\mathbb{C}} U_{[n]}, \qquad U_{[n]} = \{u \in U \mid \exists m \in \mathbb{N}, (L_{[0]} - n)^m u = 0\}. \tag{8.21}$$

If $u \in U_{[n]}$ we write $\mathrm{wt}[u] = n$. In the case where the vertex operator algebra V is rational and C_2-cofinite, we again have that $L_{[0]}$ acts semisimply with finite-dimensional eigenspaces, and that all eigenvalues are rational, bounded below, and discrete. Additionally, if we assume that U is simple then the minimal L_0 and $L_{[0]}$ eigenvalues will be equal and denoted h_U, which is a rational number [48], and recall is called the *conformal weight of the module U*. Further, for $n \in \mathbb{N}_0$ we have $U_{[h_U+n]} \subset \bigoplus_{m=0}^{n} U_{h_U+m}$ and $U_{h_U+n} \subset \bigoplus_{m=0}^{n} U_{[h_U+m]}$.

For later use we prepare some helpful identities involving torus 1-point functions and square bracket expansions.

Proposition 8.6 (Zhu [21], Miyamoto [28, Propsition 3.1 and 3.3])**.** *Let V be a vertex operator algebra, U, W be V-modules, and $\mathcal{Y}$ an intertwining operator of type $\binom{W}{U,\ W}$. Then for any $a \in V$ and $u \in U$ we have*

$$\psi^{\mathcal{Y}}(a_{[0]}u, \tau) = 0 \tag{8.22}$$

and

$$\psi^{\mathcal{Y}}(a_{[-1]}u, \tau) = \mathrm{tr}|_W o(a)o(u)q^{L_0-\frac{\mathbf{c}}{24}} + \sum_{\ell=1}^{\infty} G_{2\ell}(\tau)\psi^{\mathcal{Y}}(a_{[2\ell-1]}u, \tau). \tag{8.23}$$

The identity (8.22) can be specialised and refined as follows.

Proposition 8.7. *Let V be a vertex operator algebra and U a V-module with a decomposition into generalised $L_{[0]}$ eigenspaces as in (8.21). Suppose that for any $x, y \in V_1$ the binary operation $[x, y] = \mathrm{Res}_z\, Y(x, z)y = x_0 y$ furnishes V_1 with the structure of a finite-dimensional reductive Lie algebra and thus each homogeneous space $U_{[m]}$ is a module over V_1. Let $U_{[m]} = U_{[m]}^{triv} \oplus U_{[m]}^{non\text{-}triv}$ be the unique decomposition into the maximal trivial submodule $U_{[m]}^{triv}$ and its complement $U_{[m]}^{non\text{-}triv}$ containing all non-trivial simple summands. Then for any $u \in U_{[m]}^{non\text{-}triv}$, any V-module W, and any intertwining operator $\mathcal{Y}$ of type $\binom{W}{U,W}$, we have*

$$\psi^{\mathcal{Y}}(u, \tau) = 0. \tag{8.24}$$

Proof. Recall that for $a \in V_1$ the square bracket and non-square bracket zero modes coincide, that is, $a_{[0]} = a_0$. Since by assumption V_1 is a reductive Lie algebra, we can decompose $U_{[m]}$ into a direct sum of irreducible V_1-modules and group them into trivial and non-trivial ones. In particular

$$U_{[m]}^{\text{triv}} = \{u \in U_{[m]} \mid \forall a \in V_1, a_0 u = 0\}, \qquad U_{[m]}^{\text{non-triv}} = \{a_0 u \mid a \in V_1, u \in U_{[m]}\}. \quad (8.25)$$

Therefore, if u lies in a simple non-trivial submodule of $U_{[m]}$, then it also lies in the image of some $a \in V_1$ and thus by Proposition 8.6 the result follows. ∎

This result will be utilised later in determining the appropriate insertion vectors taken from modules of the $\mathfrak{sl}_2$ vertex operator algebra to yield non-zero vector-valued modular forms.

Note that a sufficient condition for V_1 being a Lie algebra under the bracket given above is $\dim(V_0) = 1$ and $\dim(V_n) = 0$ for $n < 0$. If in addition to being rational V is C_2-cofinite and $L_1 V_1 = 0$, then V_1 is reductive [49], which is the case for the examples we shall consider later.

For the remainder of this thesis we assume that V is a vertex operator algebra for which the conformal weights are bounded below by 0, the conformal weight 0 space V_0 is 1-dimensional, V is C_2-cofinite, the contragredient or graded dual V^* satisfies $V^* \cong V$, and V is rational, i.e., the category of admissible modules, $\mathsf{Rep}(V)$, is semisimple. The C_2-cofiniteness of V implies that $\mathsf{Rep}(V)$ admits only a finite number of inequivalent simple V-modules $V = W^1, \ldots, W^{d_V}$ for some $d_V \in \mathbb{N}$ [21] (see also [50]). In this notation, for each $\mu \in \{1, \ldots, d_V\}$, we let h_μ denote the conformal weight of W^μ, where $h_1 = 0$. Further, C_2-cofiniteness also implies that the central charge and conformal weights of all modules are rational [51, Corollaries 5.10 and 5.11] (see also [48, Theorem 1.1]). Note that the assumptions $\dim(V_0) = 1$, $V \cong V^*$, and that the conformal weights are bounded below by 0 are not required for the space of torus 1-point functions to be closed under the action of the modular group [21] (see also [48]). They are necessary to prove that $\mathsf{Rep}(V)$ is rigid and, additionally, a modular tensor category [25].

Given a simple V-module W^λ we introduce the vector space of intertwining operators

$$\mathcal{I}_\lambda = \bigoplus_\mu \binom{W^\mu}{W^\lambda \; W^\mu}, \qquad (8.26)$$

recalling that $\binom{W^\mu}{W^\lambda\ W^\mu}$ simultaneously denotes the space of intertwining operators of such type, so that $\mathcal{I}_\lambda$ consists of all intertwining operators for the modules that W^λ may act on. Let $N^\mu_{\lambda,\mu} = \dim\binom{W^\mu}{W^\lambda\ W^\mu}$ then,

$$\dim \mathcal{J}_\lambda = \sum_\mu N^\mu_{\lambda,\mu}. \tag{8.27}$$

When W^λ is the vertex operator algebra V, the intertwining operators specialise to the module field maps Y_{W^μ} giving the action of V on W^μ and these are a distinguished basis for $\binom{W^\mu}{W^\lambda\ W^\mu}$ in which case $N^\mu_{\lambda,\mu} = 1$ for all μ.

The space of 1-point functions, $\mathcal{C}_1(W^\lambda)$, with insertion from a simple module W^λ admits a number of characterisations in various level of generality, however, for rational C_2-cofinite vertex operator algebras this space can always be realised as the span

$$\mathcal{C}_1\left(W^\lambda\right) = \{\psi^{\mathcal{Y}}(-,\tau) \mid \mathcal{Y} \in \mathcal{I}_\lambda\}, \tag{8.28}$$

see [29, Theorem 5.1] for details, which we use as the definition here, for simplicity. Further, we define the space of torus 1-point functions evaluated at $u \in W^\lambda$ to be

$$\mathcal{C}^u_1\left(W^\lambda\right) = \{\psi^{\mathcal{Y}}(u,\tau) \mid \mathcal{Y} \in \mathcal{I}_\lambda\}. \tag{8.29}$$

Bounds on dimensions are then given by

$$\dim\left(\mathcal{C}^u_1\left(W^\lambda\right)\right) \leq \dim\left(\mathcal{C}_1\left(W^\lambda\right)\right) \leq \dim(\mathcal{I}_\lambda) = \sum_{\mu=1}^{d_V} N^\mu_{\lambda,\mu}. \tag{8.30}$$

Theorem 8.8 (Miyamoto [28, Theorem 5.1], Yamauchi [29, Theorem 5.1], Huang [30, Theorem 7.3])**.** *Let V be a rational C_2-cofinite vertex operator algebra. Then for any simple module W^λ and any homogeneous vector $u \in W^\lambda_{[\mathrm{wt}[u]]}$, every torus 1-point function $\psi(u,\tau) \in \mathcal{C}^u_1(W^\lambda)$ evaluated at u is a holomorphic function on $\mathbb{H}$. For $\gamma \in \Gamma$ and $u \in W^\lambda_{[\mathrm{wt}[u]]}$,*

$$\psi^{\mathcal{Y}}(u,\tau)|_\gamma = \nu_{\mathrm{wt}[u]}(\gamma)^{-1} j_{\mathrm{wt}[u]}(\gamma;\tau)^{-1} \psi^{\mathcal{Y}}(u,\gamma\tau) \in \mathcal{C}^u_1\left(W^\lambda\right) \tag{8.31}$$

defines an action of Γ on $\mathcal{C}^u_1(W^\lambda)$ which lifts to an action on $\mathcal{C}_1(W^\lambda)$.

Recall that the weight of a multiplier system can be freely shifted by integers and that for u as in the theorem above $\mathrm{wt}[u] - h_\lambda \in \mathbb{Z}$. We can therefore define the right action of Γ on $\mathcal{C}_1^u(W^\lambda)$ to be

$$\left(\psi^{\mathcal{Y}}|^\lambda_{\mathrm{wt}[u]}\gamma\right)(u,\tau) = \nu_{h_\lambda}(\gamma)^{-1} j_{\mathrm{wt}[u]}(\gamma;\tau)^{-1}\psi^{\mathcal{Y}}(u,\gamma\tau), \tag{8.32}$$

thus making the multiplier system independent of the choice of vector $u \in W^\lambda$.

To transition from considering right Γ actions on $\mathcal{C}_1^u(W^\lambda)$ to vector-valued modular forms, we need to choose elements in $\mathcal{C}_1^u(W^\lambda)$ to form the components of a vector, which we shall now do. Let $\mathcal{B}^\mu_\lambda$ be a basis for the space of intertwining operators of type $\binom{W^\mu}{W^\lambda\ W^\mu}$ which is the empty set if λ,μ are such that $N^\mu_{\lambda,\mu} = 0$ as the corresponding space of intertwining operators will vanish. Let

$$\Xi_\lambda = \bigcup_\mu \mathcal{B}^\mu_\lambda \tag{8.33}$$

be the union of all these bases for fixed λ, thus forming a basis of $\mathcal{I}_\lambda$. The space of intertwining operators $\mathcal{I}_\lambda$ and the spaces of torus 1-point functions $\mathcal{C}_1(W^\lambda)$ and $\mathcal{C}_1^u(W^\lambda)$ are related by the linear maps

$$\begin{aligned} \mathrm{tr}^\lambda \colon \mathcal{I}_\lambda &\to \mathcal{C}_1\left(W^\lambda\right), & \mathrm{ev}_u \colon \mathcal{C}_1\left(W^\lambda\right) &\to \mathcal{C}_1^u\left(W^\lambda\right), \\ \mathcal{Y} &\mapsto \mathrm{tr}\, o^{\mathcal{Y}}(-) q^{L(0)-\frac{\mathsf{c}}{24}} & f(-,\tau) &\mapsto f(u,\tau) \end{aligned} \tag{8.34}$$

that is, the first map is the taking of traces and the second is evaluation at the vector $u \in W^\lambda$. These maps are surjective by construction, hence their composition is too, yet they need not be injective. In particular, certain choices of $u \in W^\lambda$ can lead to large kernels. For example, if V is the simple affine vertex operator algebra constructed from $\mathfrak{sl}_3$ at level 3, with W^λ chosen to be V, and $u = \mathbf{1}$, then we have $|\Xi_\lambda| = 10$ while $\dim(\mathcal{C}_1^{\mathbf{1}}(V)) = 6$. That is, there are 10 simple modules up to equivalence, yet the span of characters is only 6-dimensional. Indeed, Proposition 8.7 shows that there can exist non-zero $u \in W^\lambda$ for which ev_u is the zero map.

Setting $\delta(\lambda) = \dim \mathcal{C}_1(W^\lambda)$, let $\Delta_\lambda \subset \mathrm{tr}^\lambda(\Xi_\lambda) = \{\psi^1, \dots \psi^{\delta(\lambda)}\}$ be a linearly independent subset of the image of the basis Ξ_λ and hence a basis of $\mathcal{C}_1(W^\lambda)$, and define the vector $\Psi_\lambda = (\psi^1, \dots, \psi^{\delta(\lambda)})^t$. Then Theorem 8.8 can be restated as follows.

Theorem 8.9. *For any $u \in W^{\lambda}_{[\mathrm{wt}[u]]}$ the space $\mathcal{C}_1^u(W^\lambda)$ carries a* T*-unitarisable representation $\rho_\lambda\colon \Gamma \to \mathrm{GL}(\delta(\lambda),\mathbb{C})$ such that $\Psi_\lambda(u,\tau)$ is a $\delta(\lambda)$-dimensional weakly holomorphic vector-valued modular form of weight* $\mathrm{wt}[u]$*, representation ρ_λ, and multiplier system ν_{h_λ}. That is, $\Psi_\lambda(u,\tau) \in \mathcal{M}^!(\mathrm{wt}[u],\rho_\lambda,\nu_{h_\lambda})$. For each component ψ^j of Ψ_λ let μ_j be the corresponding module label, that is, ψ^j is the trace of an intertwining operator of type $\binom{W^{\mu_j}}{W^\lambda\, W^{\mu_j}}$. Then additionally*

$$\rho_\lambda(T) = \mathrm{diag}\left\{\mathbf{e}(r_1),\ldots,\mathbf{e}\left(r_{\delta(\lambda)}\right)\right\}, \tag{8.35}$$

where

$$r_j = h_{\mu_j} - \frac{\mathbf{c}}{24} - \frac{h_\lambda}{12}, \qquad 1 \le j \le \delta(\lambda). \tag{8.36}$$

Moreover, if $h_{\mu_j} - \mathbf{c}/24 \ge 0$ for all $1 \le j \le \delta(\lambda)$, then $\Psi_\lambda(u,\tau) \in \mathcal{H}(\mathrm{wt}[u],\rho_\lambda,\nu_{h_\lambda})$.

Proof. By construction every basis element $\psi^j \in \Delta_\lambda$ is the image of an intertwining operator that takes values in a simple module. The exponents of q in the series expansion of ψ^j will therefore only differ by integers and hence the matrix for T will be diagonal and all diagonal entries will be complex numbers of modulus 1. That $\Psi_\lambda(u,\tau)$ is a $|\Xi_\lambda|$-dimensional weakly holomorphic vector-valued modular form of weight $\mathrm{wt}[u]$, representation ρ_λ, and multiplier system ν_{h_λ} follows from Theorem 8.8 (for additional details, see [44, Proposition 2.5.2]). Taking $\gamma = \mathsf{T}$ in (8.5) gives

$$\begin{aligned}\rho_\lambda(\mathsf{T})\Psi_\lambda(u,\tau) &= \nu_{h_\lambda}(\mathsf{T})^{-1} j_{\mathrm{wt}[u]}(\mathsf{T};\tau)^{-1}\Psi_\lambda(u,\tau+1)\\ &= \mathbf{e}\left(-\frac{h_\lambda}{12} - \frac{\mathbf{c}}{24}\right)\mathrm{diag}\left\{\mathbf{e}\left(h_{\mu_1}\right),\ldots,\mathbf{e}\left(h_{\mu_{\delta(\lambda)}}\right)\right\}\Psi_\lambda(u,\tau),\end{aligned} \tag{8.37}$$

and thus (8.35) and (8.36). Meanwhile, $\Psi_\lambda(u,\tau)$ is holomorphic at $i\infty$ if and only if each component function ψ^j is holomorphic at $i\infty$, and this is true if and only if $h_{\mu_j} - \mathbf{c}/24 \ge 0$ for all $1 \le j \le \delta(\lambda)$. ∎

We stress that choices were made to construct the vector $\Psi_\lambda(u,\tau) \in \mathcal{M}^!(\rho_\lambda,\nu_{h_\lambda})$ from $\mathcal{C}_1(W^\lambda)$. For example, for any $U \in \mathrm{GL}_d(\mathbb{C})$ the components of the vector $U\Psi_\lambda(u,\tau)$ will also form a basis of $\mathcal{C}_1(W^\lambda)$ and hence give rise to an equivalent representation ρ_U related to the previously constructed representation via $\rho_U(\gamma) = U\rho_\lambda(\gamma)U^{-1}$ for all $\gamma \in \Gamma$. Furthermore, $U\Psi_\lambda(u,\tau) \in \mathcal{M}^!(\mathrm{wt}[u], \rho_U, \nu_{h_\lambda})$. The association of $\mathcal{C}_1(W^\lambda)$ to vector-valued modular forms is therefore only determined up to a choice of basis.

A natural question to ask, is if there is a discrepancy between the dimension of the space of intertwining operators $\mathcal{I}_\lambda$ and that of the unevaluated torus 1-point functions $\mathcal{C}_1(W^\lambda)$, or equivalently, if the trace map tr^λ in (8.34) has a non-trivial kernel. In [21, Theorem 5.3.1] it was shown that the kernel is trivial in the special case $W^\lambda = V$. However, it is currently not known, if this is true or false for general W^λ. A sufficient condition for the kernel being trivial is the existence of a vector $u \in W^\lambda$ such that the image of the basis Ξ_λ under the composition $\mathrm{ev}_u \circ \mathrm{tr}^\lambda$ is linearly independent, as in this case the inequalities (8.30) are all equalities. Such vectors will also be shown to exist below (not assuming $W^\lambda = V$) in the example of the simple affine vertex operator algebra constructed from $\mathfrak{sl}_2$ at any non-negative integer level.

9. General results on spaces of 1-point functions

In this section we develop the tools needed to characterise the space of all torus 1-point functions (as modules over the algebra of holomorphic modular forms and the algebra $\mathcal{R}$ of modular differential operators) obtained by varying the insertion vector $u \in W^\lambda$ over an entire simple vertex operator algebra module. Specifically, let

$$\mathcal{V}(\rho_\lambda)_n = \mathrm{span}_{\mathbb{C}}\left\{\Psi_\lambda(u,\tau) \mid u \in W^\lambda_{[h_\lambda+n]}\right\} \tag{9.1}$$

for $n \in \mathbb{N}_0$ and

$$\mathcal{V}(\rho_\lambda) = \bigoplus_{n=0}^{\infty} \mathcal{V}(\rho_\lambda)_n. \tag{9.2}$$

That is, $\mathcal{V}(\rho_\lambda)$ is the space of all evaluations of the $\Psi(-,\tau)$ at any $u \in W^\lambda$. Additionally, for $u \in W^\lambda$ define

$$\mathrm{Vir}(u) = \mathrm{span}_{\mathbb{C}}\left\{L_{[-n_1]}\cdots L_{[-n_\ell]}u \mid n_1,\dots,n_\ell \in \mathbb{N}_0, \ell \in \mathbb{N}_0\right\} \subset W^\lambda, \tag{9.3}$$

which is the space of Virasoro descendants of u. Note that $\mathrm{Vir}(u)$ is a Virasoro module, that is, closed under the action of the Virasoro algebra, if and only if u is a singular vector (an $L_{[0]}$-eigenvector that is annihilated by all positive Virasoro modes). For $u \in W^\lambda$, set

$$\mathcal{V}^u(\rho_\lambda) = \mathrm{span}_{\mathbb{C}}\{\Psi_\lambda(w,\tau) \mid w \in \mathrm{Vir}(u)\} \subset \mathcal{V}(\rho_\lambda), \qquad \mathcal{V}^u(\rho_\lambda)_n = \mathcal{V}^u(\rho_\lambda) \cap \mathcal{V}(\rho_\lambda)_n. \tag{9.4}$$

That is, $\mathcal{V}^u(\rho_\lambda)$ is the subspace of $\mathcal{V}(\rho_\lambda)$ consisting of all evaluations of $\Psi_\lambda(-,\tau)$ on Virasoro descendants of u. Recall the ring of modular differential operators given in (8.13).

The next result gives a condition on the insertion vector which implies evaluating the 1-point function on all Virasoro descendants gives you a space lying within or equal to $\mathcal{R}\Psi^\lambda(u,\tau)$, namely that u should attain the minimal conformal weight for a non-zero 1-point function.

Proposition 9.1. *Let $u \in W^\lambda_{[\mathrm{wt}[u]]}$ be a homogeneous vector whose conformal weight is such that the condition $\Psi_\lambda(w,\tau) = 0$ is satisfied for all homogeneous vectors $w \in W^\lambda_{[\mathrm{wt}[w]]}$ with $\mathrm{wt}[w] < \mathrm{wt}[u]$. Then $\mathcal{V}^u(\rho_\lambda) \subseteq \mathcal{R}\Psi_\lambda(u,\tau)$. Furthermore, if $-\mathrm{wt}[u] \notin \mathbb{N}_0$*

then $\mathcal{V}^u(\rho_\lambda) = \mathcal{R}\Psi_\lambda(u,\tau)$*, that is,* $\mathcal{V}^u(\rho_\lambda)$ *is cyclic as an* $\mathcal{R}$*-module and is generated by* $\Psi_\lambda(u,\tau)$*.*

Proof. This proof is a generalisation of [52, Proposition 2(b)].

We first prove $\mathcal{V}^u(\rho_\lambda) \subseteq \mathcal{R}\Psi_\lambda(u,\tau)$. Consider $\Psi_\lambda(w,\tau) \in \mathcal{V}^u(\rho_\lambda)$ and note $w \in W^\lambda_{[h_\lambda+N]}$ for some $N \in \mathbb{N}_0$. Set $k = \mathrm{wt}[u] - h_\lambda$. We will prove by induction on N that $\Psi_\lambda(w,\tau) \in \mathcal{R}\Psi_\lambda(u,\tau)$. If $N < k$ then $\Psi_\lambda(w,\tau) = 0$ by assumption. Note that if $N = k$, then w is a scalar multiple of u since $w \in \mathrm{Vir}(u)$ and so $\Psi_\lambda(w,\tau) \in \mathcal{R}\Psi_\lambda(u,\tau)$ for $N \le k$.

Suppose the result holds for an arbitrary $N \in \mathbb{N}_0$ and consider the case $w \in W^\lambda_{[h_\lambda+N+1]}$. Since $w \in \mathrm{Vir}(u)$ and w is not a scalar multiple of u, we may assume without loss of generality that $w = L_{[-n_1]}L_{[-n_2]}\cdots L_{[-n_t]}u$, where n_j equals 1 or 2 for all $1 \le j \le t$ since $L_{[-1]}$ and $L_{[-2]}$ generate $L_{[-n]}$ for all $n > 0$. In the case $n_1 = 1$, we have by (8.22) (recall $\omega_{[n+1]} = L_{[n]}$) that $\Psi_\lambda(w,\tau) = 0$. If $n_1 = 2$, then setting $x = L_{[-n_2]}\cdots L_{[-n_t]}u$ and using (8.23) we have

$$\Psi_\lambda(w,\tau) = \partial\Psi_\lambda(x,\tau) + \sum_{j=2}^{\infty} G_{2j}(\tau)\Psi_\lambda(L_{[2j-2]}x,\tau). \tag{9.5}$$

Since $\mathrm{wt}[x]$ and $\mathrm{wt}[L_{[2j-2]}x]$ are both strictly less than $\mathrm{wt}[w] = h_\lambda + N + 1$, our induction hypothesis implies $\Psi_\lambda(w,\tau) \in \mathcal{R}\Psi_\lambda(u,\tau)$.

We turn to showing that $\mathcal{R}\Psi_\lambda(u,\tau) \subseteq \mathcal{V}^u(\rho_\lambda)$ if $-\mathrm{wt}[u] \notin \mathbb{N}_0$. Recall every element in $\mathcal{R}\Psi_\lambda(u,\tau)$ is of the form

$$\left(\phi_0 + \phi_1\partial + \cdots + \phi_t\partial^t\right)\Psi_\lambda(u,\tau), \tag{9.6}$$

for $\phi_\ell \in \mathcal{M}$ with $0 \le \ell \le t$ and $t \in \mathbb{N}_0$. We also have that ϕ_ℓ is a linear combination of terms $G_4(\tau)^i G_6(\tau)^j$ for some $i,j \in \mathbb{N}_0$. Thus, by linearity, we need only show that $\mathcal{V}^u(\rho_\lambda)$ is closed under taking modular derivatives ∂, and multiplication by $G_4(\tau)$ and $G_6(\tau)$ if $-\mathrm{wt}[u] \notin \mathbb{N}_0$. For $z \in \mathrm{Vir}(u)$ and $r \in \mathbb{N}_0$, set $x_r(z) = L_{[-2]}L^{2r}_{[-1]}z$. Then for $r \ge 1$,

$$\Psi_\lambda(x_r(z),\tau) = \alpha G_{2r+2}(\tau)\Psi_\lambda(z,\tau) + \sum_{\ell=r+2}^{\infty} G_{2\ell}(\tau)\Psi_\lambda\left(x_{\ell,r,z},\tau\right) \tag{9.7}$$

(see the proof of [52, Proposition 2(b)]), where α is a complex number that is 0 if and only if $\mathrm{wt}[z] = 0$, and $x_{\ell,r,z} \in \mathrm{Vir}(u)$ satisfies $\mathrm{wt}[x_{\ell,r,z}] < \mathrm{wt}[z]$ for $\ell \ge r+2$.

We first claim that for any $\ell \geq 2$ and $\Psi_\lambda(w,\tau) \in \mathcal{V}^u(\rho_\lambda)$ we have $G_{2\ell}(\tau)\Psi_\lambda(w,\tau) \in \mathcal{V}^u(\rho_\lambda)$. We prove this by induction on wt$[w]$, and more specifically, by induction on the non-negative integer n such that wt$[w] = $ wt$[u]+n$. In the case $n=0$, we have w is a scalar multiple of u and (9.7) gives

$$\Psi_\lambda(x_r(w),\tau) = \alpha G_{2r+2}(\tau)\Psi_\lambda(w,\tau) + \sum_{\ell=r+2}^{\infty} G_{2\ell}(\tau)\Psi_\lambda\left(x_{\ell,r,w},\tau\right) = \alpha G_{2r+2}(\tau)\Psi_\lambda(w,\tau) \tag{9.8}$$

since wt$[x_{\ell,r,w}] <$ wt$[u]$ so that $\Psi_\lambda(x_{\ell,r,z},\tau) = 0$ for $\ell \geq r+2$ by assumption. Here α is non-zero by our assumption $-$wt$[u] \notin \mathbb{N}_0$. Thus $G_{2r+2}(\tau)\Psi_\lambda(w,\tau) \in \mathcal{V}^u(\rho_\lambda)$ for $r \geq 1$ and the base case is established. Assume $G_{2\ell}(\tau)\Psi_\lambda(w,\tau) \in \mathcal{V}^u(\rho_\lambda)$ for any $\ell \geq 2$ when wt$[w] = $ wt$[u]+m$ and $0 \leq m \leq n$. Consider the $n+1$ case, i.e., wt$[w] = $ wt$[u]+n+1$. Then (9.7) becomes

$$\Psi_\lambda(x_r(w),\tau) = \alpha G_{2r+2}(\tau)\Psi_\lambda(w,\tau) + \sum_{\ell=r+2}^{\infty} G_{2\ell}(\tau)\Psi_\lambda\left(x_{\ell,r,w},\tau\right) \tag{9.9}$$

where wt$[x_{\ell,r,w}] <$ wt$[w] = $ wt$[u]+n+1$ for $\ell \geq r+2$. Again, α is non-zero since wt$[w] >$ wt$[u]$ and $-$wt$[u] \notin \mathbb{N}_0$, so wt$[w] \neq 0$. As such, we have by our induction hypothesis that $G_{2\ell}(\tau)\Psi_\lambda(x_{\ell,r,w},\tau) \in \mathcal{V}^u(\rho_\lambda)$ for $\ell \geq r+2$. It follows that $G_{2r+2}(\tau)\Psi_\lambda(w,\tau) \in \mathcal{V}^u(\rho_\lambda)$ for $r \geq 1$.

Next, we establish that $\partial\Psi_\lambda(w,\tau) \in \mathcal{V}^u(\rho_\lambda)$ for any $w \in$ Vir(u). Indeed, (8.23) gives

$$\Psi_\lambda(x_0(w),\tau) = \partial\Psi_\lambda(w,\tau) + \sum_{\ell=2}^{\infty} G_{2\ell}(\tau)\Psi_\lambda\left(x_{\ell,0,w},\tau\right) \tag{9.10}$$

where wt$[x_{\ell,0,w}] <$ wt$[w]$. Since $x_{\ell,0,w} \in$ Vir(u) and $G_{2\ell}(\tau)\Psi_\lambda(x_{\ell,0,w},\tau) \in \mathcal{V}^u(\rho_\lambda)$ for $\ell \geq 2$ we have $\partial\Psi_\lambda(w,\tau) \in \mathcal{V}^u(\rho_\lambda)$. Thus $\mathcal{V}^u(\rho_\lambda)$ is closed under the action of $\mathcal{R}$ and hence $\mathcal{V}^u(\rho_\lambda) = \mathcal{R}\Psi_\lambda(w,\tau)$. ∎

Since the above proposition gives the decomposition of $\mathcal{V}^u(\rho_\lambda)$ as an $\mathcal{R}$-module it is natural to ask how it decomposes as an $\mathcal{M}$-module. To this end we require the following proposition.

Proposition 9.2 (Marks-Mason [53, Theorem 1], Marks [46, Lemma 2.7])**.** *Let $F \in \mathcal{H}(k,\rho,v)$ be a d-dimensional holomorphic vector-valued modular form of weight*

k for a representation and multiplier system (ρ,ν), whose components form a fundamental set of solutions for a monic modular differential equation. Then the set $\{F,\partial F,\dots\partial^{d-1}F\}$ is an $\mathcal{M}$-basis for $\mathcal{R}F$. Further, if $c(d,n)\in\mathbb{N}$ are the coefficients of the series expansion

$$\frac{1-t^{2d}}{\left(1-t^2\right)\left(1-t^4\right)\left(1-t^6\right)}=\sum_{n\geq 0}c(d,n)t^n, \tag{9.11}$$

then the weight $k+n$ homogeneous subspace $(\mathcal{R}F)_n=\{(\phi_0+\phi_1\partial+\phi_2\partial^2+\cdots)F\mid\phi_j\in\mathcal{M}_{n-2j},j\geq 0\}$ satisfies

$$\dim((\mathcal{R}F)_n)=c(d,n). \tag{9.12}$$

Proof. In [53, Theorem 1] it is shown that $\mathcal{H}(k,\rho,\nu)$ is free and rank d over $\mathcal{M}$. Further, in [46, Lemma 2.7] it is shown that $\{F,\partial F,\dots,\partial^{d-1}F\}$ is linearly independent over $\mathcal{M}$. By assumption, the components of F form a fundamental set of solutions for a monic modular differential equation of degree d, hence $\partial^d F$ lies in the $\mathcal{M}$-span of $\{F,\partial F,\dots,\partial^{d-1}F\}$. Thus $\{F,\partial F,\dots,\partial^{d-1}F\}$ is an $\mathcal{M}$-basis for $\mathcal{R}F$. The series (9.11) is the Hilbert-Poincaré series for $\mathcal{R}F$, see [46, Equation 2.9]. ∎

To provide further motivation for the following key theorem, we note it will be instrumental in Section 11 in order to describe $\mathcal{V}^u(\cdot)$ and $\mathcal{V}(\cdot)$ for the appropriate representation in question, namely by giving sufficient conditions for when they are cyclic $\mathcal{R}$-modules. In combination with other results this will allow for a closed-form of the 1-point functions to be written in some dimensions.

Theorem 9.3. *Let $u\in W^\lambda_{[\mathrm{wt}[u]]}$ be a homogeneous vector whose conformal weight satisfies the condition $\Psi_\lambda(w,\tau)=0$ for all homogeneous vectors $w\in W^\lambda_{[\mathrm{wt}[w]]}$ with* $\mathrm{wt}[w]<\mathrm{wt}[u]$. *Let $\{\mu_1,\dots,\mu_{\delta(\lambda)}\}$ (recall $\delta(\lambda)=\dim(\mathcal{C}_1(W^\lambda))$) be the leading exponents of $\Psi_\lambda(u,\tau)$ and let $\mu_{\min},\mu_{\max}$ be the least and greatest leading exponent, respectively. Suppose further that the following four conditions hold.*

(1) *The subspace $N=\{w\in W^\lambda_{[\mathrm{wt}[u]]}\mid\Psi_\lambda(w,\tau)=0\}$ has codimension 1 in $W^\lambda_{[\mathrm{wt}[u]]}$.*

(2) *The exponents $\{\mu_1,\dots,\mu_{\delta(\lambda)}\}$ saturate the inequality (8.14), that is, it is an equality.*

(3) *The exponents* $\{\mu_1,\ldots,\mu_{\delta(\lambda)}\}$ *of* $\Psi_\lambda(u,\tau)$ *are minimal among all vector-valued modular forms in* $\mathcal{V}(\rho_\lambda)$*, that is, for any* $F\in\mathcal{V}(\rho_\lambda)$ *the leading exponent of the jth component will be at least* μ_j.

(4) *The space* $\mathcal{H}(\rho_\lambda,\nu_{h_\lambda-12\mu_{min}})$ *is cyclic over* $\mathcal{R}$.

Then

$$\mathcal{V}^u(\rho_\lambda)\subset\mathcal{V}(\rho_\lambda)\subset\eta^{24\mu_{\min}}\mathcal{H}\left(\rho_\lambda,\nu_{h_\lambda-12\mu_{\min}}\right) \tag{9.13}$$

and $\{\mu_1-\mu_{\min},\ldots,\mu_{\delta(\lambda)}-\mu_{\min}\}$ *is an admissible set for* $(\rho_\lambda,\nu_{h_\lambda-12\mu_{\min}})$*. Moreover, let* $\{\lambda_1,\ldots,\lambda_{\delta(\lambda)}\}$ *be the minimal admissible set for* $(\rho_\lambda,\nu_{h_\lambda-12\mu_{\min}})$ *and define*

$$M=\sum_{i=1}^{\delta(\lambda)}(\mu_i-\mu_{\min}-\lambda_i). \tag{9.14}$$

Then

$$c(\delta(\lambda),n)=\dim\left(\mathcal{V}^u(\rho_\lambda)_n\right)\le\dim(\mathcal{V}(\rho_\lambda)_n)\le c\left(\delta(\lambda),n+12\frac{M}{\delta(\lambda)}\right), \tag{9.15}$$

where $c(\delta(\lambda),n)$ *are the graded dimensions of* (9.12)*. If additionally*

(5) $\mu_{\max}-\mu_{\min}<1$,

then this is a necessary and sufficient condition for $\{\mu_1-\mu_{\min},\ldots,\mu_{\delta(\lambda)}-\mu_{\min}\}$ *to be the minimal admissible set for* $(\rho_\lambda,\nu_{h_\lambda-12\mu_{\min}})$*. Finally, if in addition to the five conditions above*

(6) $-\mathrm{wt}[u]\notin\mathbb{N}_0$,

then

$$\mathcal{V}^u(\rho_\lambda)=\mathcal{V}(\rho_\lambda)=\eta^{24\mu_{\min}}\mathcal{H}\left(\rho_\lambda,\nu_{h_\lambda-12\mu_{\min}}\right) \tag{9.16}$$

and

$$\dim(\mathcal{V}(\rho_\lambda)_n)=c(\delta(\lambda),n). \tag{9.17}$$

Proof. By Proposition 9.1, $\mathcal{V}^u(\rho_\lambda)\subset\mathcal{R}\Psi_\lambda(u,\tau)$ and $\mathcal{V}^u(\rho_\lambda)=\mathcal{R}\Psi_\lambda(u,\tau)$ if $-\mathrm{wt}[u]\notin\mathbb{N}_0$. Since the exponents of $\Psi_\lambda(u,\tau)$ saturate the inequality (8.14), so do the exponents of $\eta^{-24\mu_{\min}}\Psi_\lambda(u,\tau)$ and the exponents of each component function are all non-negative, hence $\eta^{-24\mu_{\min}}\Psi_\lambda(u,\tau)$ is holomorphic. Since η commutes with modular derivatives, we further have that $\mathcal{R}\eta^{-24\mu_{\min}}\Psi_\lambda(u,\tau) = \eta^{-24\mu_{\min}}\mathcal{R}\Psi_\lambda(u,\tau) = \eta^{-24\mu_{\min}}\mathcal{V}^u(\rho_\lambda)$. Additionally, the leading exponents of $\Psi_\lambda(u,\tau)$ are minimal among all vector-valued modular forms in $\mathcal{V}(\rho_\lambda)$, we therefore have $\eta^{-24\mu_{\min}}\mathcal{V}(\rho_\lambda) \subset \mathcal{H}(\rho_\lambda, \nu_{h_\lambda - 12\mu_{\min}})$ and (9.13) follows. The bounds on the graded dimension then follow from the fact that $\mathcal{V}^u(\rho_\lambda)$ and $\mathcal{H}(\rho_\lambda, \nu_{h_\lambda-12\mu_{\min}})$ are both cyclic $\mathcal{R}$-modules and the weight of the cyclic generator of $\mathcal{H}(\rho_\lambda, \nu_{h_\lambda-12\mu_{\min}})$ differs from the weight of $\Psi_\lambda(u,\tau)$ by $12\frac{M}{\delta(\lambda)}$.

If $\mu_{\max} - \mu_{\min} < 1$, then $\{\mu_1 - \mu_{\min}, \dots, \mu_{\delta(\lambda)} - \mu_{\min}\}$ is a minimal admissible set for $(\rho_\lambda, \nu_{h_\lambda-12\mu_{\min}})$. Thus $\eta^{-24\mu_{\min}}\Psi_\lambda(u,\tau)$ has the same weight as the cyclic generator of $\mathcal{H}(\rho_\lambda, \nu_{h_\lambda-12\mu_{\min}})$, which lies in a 1-dimensional weight space, that is, $\eta^{-24\mu_{\min}}\Psi_\lambda(u,\tau)$ is a non-zero scalar multiple of the cyclic generator. Hence $\eta^{-24\mu_{\min}}\mathcal{V}^u(\rho_\lambda) = \mathcal{H}(\rho_\lambda, \nu_{h_\lambda-12\mu_{\min}})$, which implies (9.16) and also the dimension formula (9.17). ■

Proposition 9.1 and Theorem 9.3 generalise Lemma 2.1, Theorem 3.5, and Corollary 3.3 of [17]. Additionally, we note here that the necessary condition on the values of wt$[u]$ is absent in the statement of Lemma 2.1 in [17], and thus also Theorem 3.5 and Corollary 3.3 in loc. cit. Indeed, in the notation of that paper, these results and the relevant discussion should all include the assumption that $-h_{m,n} \notin \mathbb{N}_0$. Fortunately, in the application of these results to the Virasoro minimal models in [17, Section 3], the analysis of small dimensions automatically excludes $-h_{m,n} \in \mathbb{N}_0$ with one exception. This exception is in the 1-dimensional setting where the trivial case of $(m,n) = (1,1)$ is included in the second statement of Theorem 3.7 when it should not be. However, this corresponds to the Virasoro minimal model at central charge $c = 0$ (i.e., the trivial vertex operator algebra isomorphic to $\mathbb{C}$) acting on itself. In this case (up to rescaling) there is only one torus 1-point function and it is constant.

10. 1-POINT FUNCTIONS OF AFFINE $\mathfrak{sl}_2$

We will now be applying the theory and tools of the preceding sections to the case of the affine vertex operator algebra $L(k,0)$ associated to $\mathfrak{sl}_2$ at non-negative integral level k. The particulars of this construction were covered in Section 5.2. Recall that $L(k,\mu)$ for $0 \le \mu \le k$ is the set of representatives of simple modules where μ is the h_0 eigenvalue of the highest weight vector.

In order to obtain non-vanishing trace functions $\psi^{\mathcal{Y}}(u,\tau)$, we need to find non-vanishing spaces of intertwining operators of type $\binom{L(k,\mu)}{L(k,\lambda)\ L(k,\mu)}$, and a suitable basis Ξ_λ as defined in (8.33). Note that as intertwining operator spaces for triples of simple $L(k,0)$-modules are always at most 1-dimensional, we will identify the basis vectors in Ξ_λ with the index μ appearing in $\binom{L(k,\mu)}{L(k,\lambda)\ L(k,\mu)}$. Further, let ψ^μ denote the trace of the basis intertwining operator corresponding to the label μ over the module $L(k,\mu)$, as in (8.17).

Proposition 10.1. *Let* $0 \le \lambda \le k$*, then*

$$\Xi_\lambda = \begin{cases} \left\{\mu \,\middle|\, \frac{\lambda}{2} \le \mu \le k - \frac{\lambda}{2}\right\} & \lambda \text{ even},\\ \emptyset & \lambda \text{ odd}. \end{cases} \tag{10.1}$$

In particular, if λ *is even then* $|\Xi_\lambda| = k - \lambda + 1$.

Proof. This is an immediate consequence of the $L(k,0)$ fusion rules given by $L(k,\lambda) \boxtimes L(k,\mu) = \bigoplus_\nu N^\nu_{\lambda\mu} L(k,\nu)$, where

$$N^\nu_{\lambda\mu} = \begin{cases} 1 & \text{if } |\lambda-\mu| \le \nu \le \min\{\lambda+\mu, 2k-\lambda-\mu\} \text{ and } \lambda+\mu+\nu \equiv 0 \pmod 2,\\ 0 & \text{otherwise.} \end{cases} \tag{10.2}$$

■

The fusion rules (10.2) were originally presented in the physics literature in [54,55] and predate vertex operator algebras. They were later proved in [56, Theorem 1] and [57, Corollary 3.2.1].

We next record many properties of the traces ψ^μ, with the remainder of the section dedicated to proving these properties. Throughout the section for any $0 \le \lambda \le k$ we denote the highest weight vector of $L(k,\lambda)$ by $|\lambda\rangle$. The following theorem combines many results and requires some preparation, but we present it first to contextualise the steps needed for its proof.

Theorem 10.2. *Let $0 \le \lambda \le k$ with λ even.*

(1) *Let $0 \le n \le \frac{\lambda}{2} - 1$, then $\psi^\mu(v,\tau) = 0$ for all $v \in L(k,\lambda)_{[h_\lambda+n]}$ and $\mu \in \Xi_\lambda$, where $L(k,\lambda)_{[h_\lambda+n]}$ denotes a homogeneous space with respect to the square bracket grading (8.21).*

(2) *For any $\mu \in \Xi_\lambda$, the subspace $N = \{v \in L(k,\lambda)_{[h_\lambda+\frac{\lambda}{2}]} \mid \psi^\mu(v,\tau) = 0\}$ has codimension 1 in $L(k,\lambda)_{[h_\lambda+\frac{\lambda}{2}]}$. Hence there is a unique vector $u \in L(k,\lambda)_{[h_\lambda+\frac{\lambda}{2}]}$, up to rescaling or addition by elements in N, such that $\psi^\mu(u,\tau) \neq 0$. The vector u can be chosen to be $f_{[-1]}^{\frac{\lambda}{2}}|\lambda\rangle$ or, more generally, as any element*

$$u \in f_{-1}^{\frac{\lambda}{2}}|\lambda\rangle + \bigoplus_{n=0}^{\frac{\lambda}{2}-1} L(k,\lambda)_{h_\lambda+n}, \tag{10.3}$$

where $L(k,\lambda)_{h_\lambda+n}$ denotes a homogeneous space with respect to the standard conformal grading (8.16).

(3) *For any $\mu \in \Xi_\lambda$ and $u \in L(k,\lambda)$ as in (10.3), the leading exponent of $\psi^\mu(u,\tau)$ is*

$$h_\mu - \frac{\mathbf{c}}{24} = \frac{2\mu^2 + 4\mu - k}{8(k+2)}. \tag{10.4}$$

These leading exponents saturate the inequality (8.14), that is, it is an equality.

(4) *For $u \in L(k,\lambda)$ as in (10.3), the intertwining operator basis Ξ_λ can be normalised such that all coefficients of the series expansion of $\psi^\mu(u,\tau)$ are rational for each $\mu \in \Xi_\lambda$.*

(5) *Let $u \in L(k,\lambda)$ be as in (10.3). The set $\{\psi^\mu(u,\tau) \mid \mu \in \Xi_\lambda\}$ is linearly independent in $\mathcal{C}_1^u(L(k,\lambda))$, the space of torus 1-point functions evaluated at u, and the set $\{\psi^\mu(-,\tau) \mid \mu \in \Xi_\lambda\}$ is linearly independent in $\mathcal{C}_1(L(k,\lambda))$, the space of (un-evaluated) torus 1-point functions. The dimension of both of these spaces is $k-\lambda+1$, the cardinality of Ξ_λ. Thus the vector $\Psi_\lambda(u,\tau) = (\psi^\mu(u,\tau) \mid \mu \in \Xi_\lambda)^t = (\psi^{\frac{\lambda}{2}}(u,\tau), \dots, \psi^{k-\frac{\lambda}{2}}(u,\tau))^t$ is a $|\Xi_\lambda| = k-\lambda+1$-dimensional weakly holomorphic vector-valued modular form of weight $h_\lambda + \frac{\lambda}{2}$, representation ρ_λ, and multiplier*

system ν_{h_λ}*. Moreover,*

$$\rho_\lambda(\mathsf{T}) = \mathrm{diag}\left\{\mathbf{e}\left(r_{\frac{\lambda}{2}}\right), \dots, \mathbf{e}\left(r_{k-\frac{\lambda}{2}}\right)\right\}, \tag{10.5}$$

where

$$r_\mu = h_\mu - \frac{\mathbf{c}}{24} - \frac{h_\lambda}{12}, \qquad \mu \in \Xi_\lambda. \tag{10.6}$$

(6) *For arbitrary* $w \in L(k,\lambda)_{[\mathrm{wt}[w]]}$ *we have that* $\Psi_\lambda(w,\tau)$ *is a holomorphic vector-valued modular form if* $\lambda \geq -2+\sqrt{2k+4}$*. Additionally, for* $u \in L(k,\lambda)$ *as in* (10.3)*, the following are equivalent.*
   (i) $\Psi_\lambda(u,\tau) \in \mathcal{H}(\rho_\lambda, \nu_{h_\lambda})$.
   (ii) $\lambda \geq -2+\sqrt{2k+4}$.
   (iii) $k \geq d-2+\sqrt{2d-1}$ *or* $k \leq d-2-\sqrt{2d-1}$*, where* $d = |\Xi_\lambda|$ *is the dimension of* $\Psi_\lambda(u,\tau)$.

First note that the conformal weight 1 space of $L(k,0)$ is isomorphic to $\mathfrak{sl}_2$ and hence all conformal weight spaces of the simple modules $L(k,\lambda)$ completely reduce into finite direct sums of finite-dimensional simple $\mathfrak{sl}_2$ modules. Thus, by Proposition 8.7, for each $\mu \in \Xi_\lambda$ we have $\psi^\mu(-,\tau)$ vanishes when restricted to a conformal weight space of $L(k,\lambda)$ that does not contain the trivial $\mathfrak{sl}_2$ module. Therefore, we need to find the conformal weight space in $L(k,\lambda)$ at which the trivial module first appears.

Lemma 10.3. *Let* $0 \leq \lambda \leq k$*,* λ *even, and* $n \in \mathbb{Z}$*.*

(1) *For* $\lambda \geq 2$ *and* $n < \frac{\lambda}{2}$*, the multiplicity of the trivial* $\mathfrak{sl}_2$ *module in the conformal weight spaces* $L(k,\lambda)_{h_\lambda+n}$ *is* 0.
(2) *For* $\lambda \geq 0$*, the multiplicity of the trivial* $\mathfrak{sl}_2$ *module in the conformal weight spaces* $L(k,\lambda)_{h_\lambda+\frac{\lambda}{2}}$ *is* 1.

Proof. If $n<0$, then $L(k,\lambda)_{h_\lambda+n} = \{0\}$. Meanwhile, for $n \geq 0$ and $\alpha \in \mathbb{Z}$ we define

$$L(k,\lambda)_{h_\lambda+n:\,\alpha} = \left\{w \in L(k,\lambda)_{h_\lambda+n} \mid h_0 w = \alpha w\right\}. \tag{10.7}$$

We compute the multiplicity of the trivial $\mathfrak{sl}_2$ module by considering character formulae obtained from the Bernstein-Gelfand-Gelfand (BGG) resolution of $L(k,\lambda)$ in terms of Verma modules $V(k,\sigma)$ (where σ again denotes the $\mathfrak{sl}_2$ weight, that is, the h_0 eigenvalue, of the generating highest weight vector)

$$\begin{aligned}\cdots \to V(k,\lambda_i)\oplus V(k,\lambda_{-i}) \to \cdots \to V(k,\lambda_2)\oplus V(k,\lambda_{-2}) \to \\ \to V(k,\lambda_1)\oplus V(k,\lambda_{-1}) \to V(k,\lambda) \to L(k,\lambda) \to 0,\end{aligned} \tag{10.8}$$

where for $j \in \mathbb{Z}$,

$$\lambda_{2j} = \lambda + 2j(k+2), \quad \text{and} \quad \lambda_{2j-1} = -\lambda - 2 + 2j(k+2). \tag{10.9}$$

The above resolution is given in [58, Section 4] using results from [59]. Recall that the character of a Verma module is given by

$$\mathrm{ch}[V(k,\mu)] = \mathrm{tr}_{V(k,\mu)} z^{h_0} q^{L_0 - \frac{\mathbf{c}}{24}} = \frac{z^{\mu} q^{h_\mu - \frac{\mathbf{c}}{24}}}{\prod_{m\geq 1}(1-z^2q^m)(1-q^m)(1-z^{-2}q^{m-1})}. \tag{10.10}$$

The Verma character formulae, in turn, yield the character formula for simple modules via the BGG resolution above. We have

$$\begin{aligned} q^{\frac{\mathbf{c}}{24}-h_\lambda}\mathrm{ch}[L(k,\lambda)] &= q^{\frac{\mathbf{c}}{24}-h_\lambda}\mathrm{tr}_{L(k,\mu)} z^{h_0} q^{L_0-\frac{\mathbf{c}}{24}} \qquad (10.11)\\ &= q^{\frac{\mathbf{c}}{24}-h_\lambda}\mathrm{ch}[V(k,\lambda)] + q^{\frac{\mathbf{c}}{24}-h_\lambda}\sum_{i\geq 1}(-1)^i(\mathrm{ch}[V(k,\lambda_i)] + \mathrm{ch}[V(k,\lambda_{-i})])\\ &= q^{\frac{\mathbf{c}}{24}-h_\lambda}\sum_{i\geq 0}(-1)^i(\mathrm{ch}[V(k,\lambda_i)] - \mathrm{ch}[V(k,\lambda_{-i-1})])\\ &= \sum_{i\geq 0}(-1)^i q^{h_{\lambda_i}-h_\lambda}\frac{\sum_{n=0}^{\lambda_i} z^{\lambda_i-2n}}{\prod_{m\geq 1}(1-z^2q^m)(1-q^m)(1-z^{-2}q^m)}, \qquad (10.12)\end{aligned}$$

where multiplication by the factor $q^{\frac{\mathbf{c}}{24}-h_\lambda}$ shifts the exponents of the above power series such that the coefficient of $z^m q^n$ is the dimension of $L(k,\lambda)_{h_\lambda+n:\,m}$. The last equality in the above character formula uses that $\lambda_{-m} = -2-\lambda_{m-1}$, $m \in \mathbb{Z}$, and hence the conformal weights corresponding to these $\mathfrak{sl}_2$ weights satisfy $h_{\lambda_{-m}} = h_{\lambda_{m-1}}$. The expansion of (10.12) up to degree $\frac{\lambda}{2}$ in q will allow us to conclude the lemma.

Note that all even weight simple $\mathfrak{sl}_2$ modules have a 1-dimensional weight 0 space. Further the weight 2 space vanishes for the trivial module, while it is 1-dimensional for all other even weight simple $\mathfrak{sl}_2$ modules. Therefore the difference $\dim(L(k,\lambda)_{h_\lambda+m:0}) - \dim(L(k,\lambda)_{h_\lambda+m:2})$ is the multiplicity of the trivial module in $L(k,\lambda)_{h_\lambda+m}$. This difference is also equal to the difference of the coefficients of z^0q^m

and z^2q^m in the character formula (10.12) above. Further, note that $h_{\lambda_m} - h_{\lambda_0}$ increases monotonically in $m > 0$, and in particular,

$$h_{\lambda_1} - h_\lambda = k - \lambda + 1, \quad h_{\lambda_2} - h_\lambda = k + \lambda + 3 > \frac{\lambda}{2}. \tag{10.13}$$

Thus if we wish to expand $q^{\frac{c}{24} - h_\lambda}\mathrm{ch}[L(k,\lambda)]$ up to degree $q^{\frac{\lambda}{2}}$ it is sufficient to only consider the summands coming from $i = 0, 1$ in the character formula (10.12). To simplify formulae, we introduce the notation $(q)_i = \prod_{m=1}^{i}(1-q^m), i \geq 0$ and record the q-series identity [60, Equation 9.16]

$$\frac{1}{\prod_{m\geq 1}(1-z^2q^m)(1-z^{-2}q^m)} = \sum_{n\in\mathbb{Z}} z^{2n} \sum_{i\geq 0} \frac{q^{2i+|n|}}{(q)_i(q)_{i+|n|}}. \tag{10.14}$$

This identity is a consequence of the identity

$$\frac{1}{\prod_{m\geq 1}(1-z^2q^m)} = \sum_{j\geq 0}\frac{q^j}{(q)_j}z^{2j}, \tag{10.15}$$

in [61, Equation 2.2.5]. Thus

$$\begin{aligned} q^{\frac{c}{24}-h_\lambda}\mathrm{ch}[L(k,\lambda)] &= \frac{\sum_{n=0}^{\lambda} z^{\lambda-2n} - q^{k+1-\lambda}\sum_{n=0}^{2(k+1)-\lambda} z^{2(k+1)-\lambda-2n}}{\prod_{m\geq 1}(1-z^2q^m)(1-q^m)(1-z^{-2}q^m)} + \mathcal{O}(q^{\lambda+k+3}) \\ &= \left[\sum_{n\in\mathbb{Z}} z^{2n}\sum_{i\geq 0}\frac{q^{2i+|n|}}{(q)_i(q)_{i+|n|}(q)_\infty}\right]\left[\sum_{m=0}^{\lambda} z^{\lambda-2m} - q^{k-\lambda+1}\sum_{m=0}^{2(k+1)-\lambda} z^{2(k+1)-\lambda-2m}\right] + \mathcal{O}(q^{\lambda+k+3}). \end{aligned} \tag{10.16}$$

Collecting the z^0 terms gives

$$\sum_{m=0}^{\lambda}\sum_{i\geq 0}\frac{q^{2i+|\frac{\lambda}{2}-m|}}{(q)_i(q)_{i+|\frac{\lambda}{2}-m|}(q)_\infty} - q^{k-\lambda+1}\sum_{m=0}^{2(k+1)-\lambda}\sum_{i\geq 0}\frac{q^{2i+|k+1-\frac{\lambda}{2}-m|}}{(q)_i(q)_{i+|k+1-\frac{\lambda}{2}-m|}(q)_\infty} + \mathcal{O}(q^{\lambda+k+3}), \tag{10.17}$$

while collecting the z^2 terms gives

$$\sum_{m=0}^{\lambda}\sum_{i\geq 0}\frac{q^{2i+|\frac{\lambda}{2}-m-1|}}{(q)_i(q)_{i+|\frac{\lambda}{2}-2m-2|}(q)_\infty} - q^{k-\lambda+1}\sum_{m=0}^{2(k+1)-\lambda}\sum_{i\geq 0}\frac{q^{2i+|k-\frac{\lambda}{2}-m|}}{(q)_i(q)_{i+|k-\frac{\lambda}{2}-m|}(q)_\infty} + \mathcal{O}(q^{\lambda+k+3}). \tag{10.18}$$

The difference of the z^0 and z^2 terms is therefore

$$\sum_{m\geq 0}\left(\dim\left(L(k,\lambda)_{h_\lambda+m:0}\right) - \dim\left(L(k,\lambda)_{h_\lambda+m:2}\right)\right)q^m$$

$$= \sum_{i\geq 0}\left[\frac{q^{2i+\frac{\lambda}{2}}}{(q)_i(q)_{i+\frac{\lambda}{2}}(q)_\infty} - \frac{q^{2i+\frac{\lambda}{2}+1}}{(q)_i(q)_{i+\frac{\lambda}{2}+1}(q)_\infty}\right] \tag{10.19}$$

$$-q^{k-\lambda+1}\sum_{i\geq 0}\left[\frac{q^{2i+k+1-\frac{\lambda}{2}}}{(q)_i(q)_{i+k+1-\frac{\lambda}{2}}(q)_\infty} - \frac{q^{i+k+2-\frac{\lambda}{2}}}{(q)_i(q)_{i+k+2-\frac{\lambda}{2}}(q)_\infty}\right] + \mathcal{O}(q^{\lambda+k+3})$$

$$= q^{\frac{\lambda}{2}} + \mathcal{O}(q^{\frac{\lambda}{2}+1}), \tag{10.20}$$

where we have used that only the first term of the first summand at $i = 0$ contributes to $q^{\frac{\lambda}{2}}$. Thus, both parts of the lemma follow. ∎

Lemma 10.4. *Let $0 \leq \lambda \leq k$, λ even, and $u \in L(k,\lambda)$ be as in (10.3).*

(1) *For all $\mu \in \Xi_\lambda$, the trace $\psi^\mu(u,\tau)$ is non-vanishing and the leading exponent is*

$$h_\mu - \frac{\mathbf{c}}{24} = \frac{2\mu^2+4\mu-k}{8(k+2)}. \tag{10.21}$$

These leading exponents saturate the inequality (8.14), that is, it is an equality.

(2) *The intertwining operator underlying the trace $\psi^\mu(u,\tau)$ can be normalised such that all coefficients of the series expansion are rational.*

Proof. Recall that we denote by $o(u)$ the coefficient of $z^{-\operatorname{wt}(u)}$ in the series expansion of u inserted into the intertwining operator underlying the trace ψ^μ. Therefore,

$$\psi^\mu(u,\tau) = \operatorname{tr}_{L(k,\mu)} o(u) q^{L_0-\frac{\mathbf{c}}{24}} = q^{h_\mu-\frac{\mathbf{c}}{24}}\sum_{n=0}^{\infty} q^n \operatorname{tr}_{L(k,\mu)_{h_\mu+n}} o(u). \tag{10.22}$$

We compute the coefficient of the leading term corresponding to $n = 0$ and show that it is non-zero, which in turn will imply that formula (10.21) gives the leading exponent. Note that $L(k,\mu)_{h_\mu}$ is a module over the finite-dimensional Lie algebra $\mathfrak{sl}_2$ by restriction, and it is isomorphic to the simple highest weight module $L(\mu)$ of highest weight μ. We choose the basis $\{v_i = f_0^i|\mu\rangle\}_{i=0}^{\mu}$ of $L(k,\mu)_{h_\mu} \cong L(\mu)$ and the corresponding dual basis $\{\phi_i\}_{i=0}^{\mu} \subset L(\mu)^* \cong L(\mu)$. Let $\langle\ ,\ \rangle$ denote the standard pairing between $L(\mu)$ and its dual space so that the standard left action of $\mathfrak{sl}_2$ on the dual space is characterised by $\langle x_0\psi, w\rangle = -\langle \psi, x_0 w\rangle$, $x \in \mathfrak{sl}_2$, $\psi \in L(\mu)^*$, $w \in L(\mu)$. With these conventions we have

$$\phi_i = (-1)^i\frac{(\mu-i)!}{i!\mu!}e_0^i\phi_0. \tag{10.23}$$

Then

$$\mathrm{tr}_{L(k,\mu)_{h_\mu}} o\left(f_{-1}^{\frac{\lambda}{2}}|\lambda\rangle\right) = \sum_{i=0}^{\mu}\left\langle \phi_i, o\left(f_{-1}^{\frac{\lambda}{2}}|\lambda\rangle\right) v_i\right\rangle. \tag{10.24}$$

To further evaluate this expression we recall the Jacobi identity (4.15). In that identity we set $v = f_{-1}\mathbf{1}$, $U_1 = L(k,\lambda)$, $U_2 = U_3 = L(k,\mu)$, multiply both sides by z_0^{-1}, and take the residue in z_0 and z_1 to obtain the identity

$$\mathcal{Y}(f_{-1}u_1, z_2)u_2 = \sum_{s\geq 0} z_2^s f_{-s-1}\mathcal{Y}(u_1,z_2)u_2 + z_2^{-s-1}\mathcal{Y}(u_1,z_2) f_s u_2. \tag{10.25}$$

Specialising further to $u_2 = v_i$ and noting that for $s \geq 1$ we have $f_s v_i = 0$ and $\phi_i(f_{-s}w) = 0$ for any $w \in L(k,\lambda)$, we obtain

$$\begin{aligned}\mathrm{tr}_{L(k,\mu)_{h_\mu}} o\left(f_{-1}^{\frac{\lambda}{2}}|\lambda\rangle\right) &= \sum_{i=0}^{\mu}\left\langle \phi_i, o(|\lambda\rangle) f_0^{\frac{\lambda}{2}} v_i\right\rangle \\ &= \sum_{i=0}^{\mu-\frac{\lambda}{2}}\left\langle \phi_i, o(|\lambda\rangle) f_0^{\frac{\lambda}{2}} v_i\right\rangle \\ &= (-1)^i\frac{(\mu-i)!}{i!\mu!}\sum_{i=0}^{\mu-\frac{\lambda}{2}}\left\langle e_0^i\phi_0, o(|\lambda\rangle) f_0^{\frac{\lambda}{2}} v_i\right\rangle.\end{aligned} \tag{10.26}$$

Here, the second equality is due to $f_0^{\frac{\lambda}{2}} v_i$ vanishing for $i > \mu - \frac{\lambda}{2}$, while the third equality follows from the formula above for the basis and its dual. Evaluating the action on the dual space and using the identity $[e_0, o(|\lambda\rangle)] = o(e_0|\lambda\rangle) = 0$ (which follows similarly to (10.25) by taking appropriate residues of the Jacobi identity (4.15)) we obtain

$$\mathrm{tr}_{L(k,\mu)_{h_\mu}} o\left(f_{-1}^{\frac{\lambda}{2}}|\lambda\rangle\right) = \sum_{i=0}^{\mu-\frac{\lambda}{2}}\frac{(\mu-i)!}{i!\mu!}\left\langle \phi_0, o(|\lambda\rangle) e_0^i f_0^{\frac{\lambda}{2}+i} v_0\right\rangle. \tag{10.27}$$

Observe that

$$e_0^i f_0^{i+\frac{\lambda}{2}}|\mu\rangle = \frac{(\frac{\lambda}{2}+i)!}{(\frac{\lambda}{2})!}\frac{(\mu-\frac{\lambda}{2})!}{(\mu-\frac{\lambda}{2}-i)!} f_0^{\frac{\lambda}{2}}|\mu\rangle, \tag{10.28}$$

which combined with the Jacobi identity $[f_0, o(w)] = o(f_0 w)$ to move $f_0^{\frac{\lambda}{2}}$ back into the intertwining operator zero mode yields

$$\mathrm{tr}_{L(k,\mu)_{h_\mu}} o\left(f_{-1}^{\frac{\lambda}{2}}|\lambda\rangle\right) = \left\langle \phi_0, o\left(f_0^{\frac{\lambda}{2}}|\lambda\rangle\right) v_0\right\rangle (-1)^{\frac{\lambda}{2}} \sum_{i=0}^{\mu-\frac{\lambda}{2}}\frac{(\mu-i)!}{i!\mu!}\frac{\left(\frac{\lambda}{2}+i\right)!\left(\mu-\frac{\lambda}{2}\right)!}{\left(\frac{\lambda}{2}\right)!\left(\mu-\frac{\lambda}{2}-i\right)!}. \tag{10.29}$$

Since $\mu \geq \frac{\lambda}{2}$ and $i \leq \mu - \frac{\lambda}{2}$, the sum is strictly positive and rational. Thus, $\mathrm{tr}_{L(k,\mu)_{h_\mu}} o(f_{-1}^{\frac{\lambda}{2}}|\lambda\rangle)$ is non-vanishing if and only if $\langle \phi_0, o(f_0^{\frac{\lambda}{2}}|\lambda\rangle)v_0\rangle$ is, which in turn must be non-zero because the intertwining operator is.

We turn to showing that the leading exponents provide an equality in (8.14). Set $\mu_n = \frac{\lambda}{2} + n$ for $n = 0, 1, \ldots, k-\lambda$. Then $\Xi_\lambda = \{\mu_0, \mu_1, \ldots, \mu_{k-\lambda}\}$. Recalling $|\Xi_\lambda| = k - \lambda + 1$, we have

$$\frac{\sum_{n=0}^{k-\lambda}\left(h_{\mu_n} - \frac{\mathbf{c}}{24}\right)}{k-\lambda+1} = \frac{4k(k+2) - 2\lambda(k+1) + \lambda^2}{48(k+2)}, \tag{10.30}$$

and thus

$$\frac{12\left(\sum_{n=0}^{k-\lambda} h_{\mu_n} - \frac{\mathbf{c}}{24}\right)}{k-\lambda+1} + \lambda - k = \frac{\lambda(\lambda+2k+6)}{4(k+2)}. \tag{10.31}$$

Finally, we note that the above is equal to

$$h_\lambda + \frac{\lambda}{2} = \frac{\lambda(\lambda+2k+6)}{4(k+2)} \tag{10.32}$$

and hence the equality in (8.14) is obtained.

Next we show that the intertwining operator can be normalised such that the trace $\psi^\mu(u,\tau)$ has rational coefficients. First note that since the level k is integral, the commutation relations of the affine generators e_n, h_n, f_n all have integral structure constants. Further, the two generating singular vectors of the maximal proper submodule of the Verma module $V(k,\mu)$ can be normalised to have integral expansions in the standard Poincaré-Birkhoff-Witt (PBW) basis. Therefore, a basis of the simple quotient $L(k,\mu)$ can be chosen such that its representatives in $V(k,\mu)$ expand in the standard PBW basis with rational coefficients. See [62] for a description of such bases.

Finally, note that when expressing dual basis vectors in simple finite $\mathfrak{sl}_2$-modules in terms of the dual of the highest vector (as in (10.23)) all normalisation factors are again rational. Thus every computation of $\mathrm{tr}_{L(k,\mu)_{h_\mu+m}} o(f_{-1}^{\frac{\lambda}{2}}|\lambda\rangle)$ will reduce to $\langle \phi_0, o(f_0^{\frac{\lambda}{2}}|\lambda\rangle)v_0\rangle$ multiplied by a sum of products of rational numbers, hence to ensure that the coefficients of $\psi^\mu(u,\tau)$ are rational it is necessary and sufficient to normalise the intertwining operator such that $\langle \phi_0, o(f_0^{\frac{\lambda}{2}}|\lambda\rangle)v_0\rangle$ is rational, which can always be done. ■

We now have all results needed to prove Theorem 10.2.

Proof of Theorem 10.2. Recall that for all non-negative integers m we have $L(k,\lambda)_{[h_\lambda+m]} \subset \bigoplus_{n=0}^{m} L(k,\lambda)_{h_\lambda+n}$, hence Part 1 follows from Lemma 10.3.1.

To conclude Part 2 note that Lemma 10.3.2 bounds the codimension of the subspace N above by 1, while Lemma 10.4.1 bounds it below by 1, hence the codimension is 1.

Part 3 is given in Lemma 10.4.1, while Part 4 is Lemma 10.4.2.

Finally, for Part 5, we show linear independence of the traces ψ^μ evaluated at u. Recall that the leading exponents are

$$h_\mu - \frac{\mathbf{c}}{24} = \frac{2\mu^2+4\mu-k}{8(k+2)}, \qquad \frac{\lambda}{2} \le \mu \le k - \frac{\lambda}{2}. \tag{10.33}$$

Observe that the numerator is quadratic in μ with a minimum at $\mu = -1$ which is below the range of μ hence all exponents are distinct. Thus, the set $\{\psi^\mu(u,\tau) \mid \mu \in \Xi_\lambda\}$ is linearly independent and thus so is $\{\psi^\mu(-,\tau) \mid \mu \in \Xi_\lambda\}$.

Finally, we turn to Part 6. By Theorem 8.9 we know $\Psi_\lambda(w,\tau)$ is a weakly holomorphic vector-valued modular form. It remains to show that if $\lambda \ge -2+\sqrt{2k+4}$, then all exponents of each component of $\Psi_\lambda(w,\tau)$ are non-negative. By Proposition 10.1, the smallest possible leading exponent in the q-expansions among all components is $h_\mu - \mathbf{c}/24$ for $\mu = \lambda/2$. All other exponents are larger since $h_{\mu_1} - \mathbf{c}/24 \ge h_{\mu_2} - \mathbf{c}/24$ if $\mu_1 \ge \mu_2$. Thus, we are assured all exponents will be non-negative if $h_{\lambda/2} - \mathbf{c}/24 \ge 0$. By (10.4) this is equivalent to

$$\frac{\lambda^2+4\lambda-2k}{16(k+2)} \ge 0, \tag{10.34}$$

which in turn amounts to $\lambda^2+4\lambda-2k \ge 0$. This establishes the holomorphicity of $\Psi_\lambda(w,\tau)$, as desired.

As discussed above, the smallest exponent occurring in any q-expansion in the components of $\Psi_\lambda(u,\tau)$ is $h_{\lambda/2} - \mathbf{c}/24$. The same argument as above now gives the equivalence between Parts 6.i and 6.ii. Meanwhile, we recall that 10.2.5 gives $d = k-\lambda+1$, or $\lambda = k-d+1$. Plugging this into the inequality $\lambda^2+4\lambda-2k \ge 0$ we obtained above and solving for k gives the equivalence between Parts 6.i and 6.iii. ∎

11. Analysing representations and spaces of torus 1-point functions

In this section an analysis will be undertaken of the vector-valued modular forms obtained from the 1-point functions of the simple affine vertex operator algebra associated to $\mathfrak{sl}_2$ at non-negative integral levels. In particular, one of the properties investigated will be congruence or non-congruence of the representations of the modular group. We review this property below:

Definition 11.1. A representation $f : \Gamma \to \mathrm{GL}_n(\mathbb{C})$ of the modular group for $n \in \mathbb{N}$ is said to be non-congruence if the kernel $\ker f$ does not contain any principal congruence subgroups, and is said to be congruence otherwise.

Recall that the principal congruence subgroups are reviewed in Definition 6.1. We determine congruence or non-congruence for large families of examples, and also whether the representations associated to these forms have finite or infinite image. Additionally, we give complete descriptions of the spaces of vector-valued modular forms of dimension at most three.

Recall the ring of integral weight modular forms $\mathcal{M} = \mathbb{C}[G_4, G_6]$ and the skew polynomial ring of modular differential operators $\mathcal{R}$. Throughout this section let $0 \le \lambda \le k$, with λ even, and $\Psi_\lambda(u,\tau)$ denote the vector-valued modular form defined in Theorem 10.2.5, with $u \in L(k,\lambda)$ as in (10.3). Consider the cyclic $\mathcal{R}$-submodule $\mathcal{R}\Psi_\lambda(u,\tau)$ of $\mathcal{M}^!(\rho_\lambda, \nu_{h_\lambda})$ generated by $\Psi_\lambda(u,\tau)$. In this section we will consider how the $\mathcal{R}$-modules $\mathcal{R}\Psi_\lambda(u,\tau)$, $\mathcal{V}^u(\rho_\lambda)$, $\mathcal{V}(\rho_\lambda)$, $\mathcal{H}(\rho_\lambda, \nu_{h_\lambda})$ and $\mathcal{M}^!(\rho_\lambda, \nu_{h_\lambda})$ are interrelated.

Proposition 11.2. *Let $2 \le \lambda \le k$, λ even, and $u \in L(k,\lambda)$ be as in (10.3). Then $\mathcal{V}^u(\rho_\lambda) = \mathcal{R}\Psi_\lambda(u,\tau)$.*

Proof. Suppose $w \in L(k,\lambda)_{[h_\lambda+\ell]}$ for some $\ell \in \mathbb{Z}$. If $\ell < \lambda/2$ we have $\Psi_\lambda(w,\tau) = 0$ by Theorem 10.2.1. The result now follows from Proposition 9.1 after noting that $\mathrm{wt}[u] > 0$. ∎

Next, we prepare a sufficient condition for concluding the irreducibility of a representation of Γ.

Lemma 11.3. *Let $d \in \mathbb{N}$ and let $\rho\colon \Gamma \to \mathsf{GL}(d,\mathbb{C})$ be a d-dimensional representation of Γ such that $\rho(T)$ is diagonalisable with eigenvalues $\{\lambda_1,\dots,\lambda_d\}$. If every non-empty proper subproduct of* $\det(\rho(T)) = \prod_{i=1}^{d} \lambda_i$ *is not a 12th root of unity, then ρ is irreducible.*

Proof. Recall that taking the determinant $\det(\rho)$ of ρ yields an element in the group of characters $\mathrm{Hom}(\Gamma,\mathbb{C}^\times)$ and that this group is cyclic of order 12 (one choice of cyclic generator assigns $\mathsf{T} \mapsto \mathbf{e}\left(\frac{1}{12}\right)$ $\mathsf{S} \mapsto \mathbf{e}\left(\frac{3}{4}\right)$). In particular, every element in $\mathrm{Hom}(\Gamma,\mathbb{C}^\times)$ maps T to some 12th root of unity. Note further that any invariant subspace of the representation ρ admits a basis of $\rho(\mathsf{T})$ eigenvectors so taking the determinant of the representation restricted to this subspace will map T to a product of $\rho(\mathsf{T})$-eigenvalues with as many factors as the dimension of the subspace and this product would need to be a 12th root of unity. So if no non-empty product of $\rho(\mathsf{T})$-eigenvalues is a 12th root of unity, then ρ admits no non-trivial invariant subspace. ∎

11.1. **Dimension one.** We begin by considering 1-dimensional vector-valued modular forms.

Theorem 11.4. *Let $0 \le \lambda \le k$, λ even, $u \in L(k,\lambda)$ as in (10.3), and ρ_λ be the representation associated to $\Psi_\lambda(u,\tau)$. The dimension of the vector-valued modular form $\Psi_\lambda(u,\tau)$ is 1 if and only if $\lambda = k$ and hence the level k is even. Moreover, in this case the following hold.*

(1) *The representation ρ_k is irreducible and congruence.*
(2) *The representation ρ_k satisfies $\rho_k(\mathsf{S}) = \mathbf{e}\left(\frac{k}{8}\right)$ and $\rho_k(\mathsf{T}) = \mathbf{e}\left(\frac{k}{24}\right)$. In particular, ρ_k is trivial if and only if k is a multiple of 24.*
(3) *We have the inclusion $\mathcal{V}(\rho_k) \subset \mathcal{H}(\rho_k,\nu_{h_k})$ for all even $k \ge 0$. The inclusion is an equality if $2 \le k \le 14$ and it is proper if $k = 0$ or $k \ge 16$.*
(4) *There exists a normalisation of the intertwining operator underlying $\Psi_k(u,\tau)$ such that $\Psi_k(u,\tau) = \eta^{\frac{3k}{2}}$ for all even $k \ge 0$. Further, for $k \ge 2$ we have the identity of $\mathcal{R}$-modules*

$$\mathcal{V}^u(\rho_k) = \mathcal{V}(\rho_k) = \mathcal{R}\eta^{\frac{3k}{2}}. \tag{11.1}$$

As $\mathcal{M}$-modules each of the above is free of rank 1 with basis $\{\eta^{\frac{3k}{2}}\}$. For $n \in \mathbb{N}_0$,

$$\dim(\mathcal{V}(\rho_k))_n = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 2 \\ \lfloor \frac{n}{12} \rfloor & \text{if } n \equiv 2 \pmod{12} \\ \lfloor \frac{n}{12} \rfloor + 1 & \text{otherwise,} \end{cases} \tag{11.2}$$

where $\lfloor x \rfloor$ denotes the floor function of a real number x.

Proof. Theorem 10.2.5 gives that ρ_k is a 1-dimensional representation for even k. Further, ρ_k is irreducible as it is 1-dimensional and we note any irreducible 1-dimensional representation must also be congruence (see, for example, [17, Section 3.1] for more details). This gives Part 1.

For Part 2, note that the representation ρ_k is trivial if and only $\rho_k(\mathsf{T}) = 1$. By Theorem 10.2.5, $\rho_k(\mathsf{T}) = \mathbf{e}\left(\frac{k}{24}\right)$, thus the representation is trivial if and only if k is a multiple of 24.

We consider Parts 3 and 4 together. In the case $k = 0$, it is well known that $L(0,0) \cong \mathbb{C}$, and it follows that $\mathcal{V}(\rho_k) = \mathbb{C}$, which is strictly contained in $\mathcal{H}(\rho_0, \nu_{h_0})$. Therefore, we consider the case $k > 0$, and thus $\mathrm{wt}[u] > 0$ by Theorem 10.2. Note for dimensions three or less spaces of holomorphic vector-valued modular forms are always cyclic $\mathcal{R}$-modules, and hence all six conditions in Theorem 9.3 are satisfied, which implies

$$\mathcal{V}^u(\rho_k) = \mathcal{V}(\rho_k) = \eta^{\frac{3k}{2}} \mathcal{H}\left(\rho_k, \nu_{-\frac{k}{2}}\right) \tag{11.3}$$

and the dimension formula (11.2) follows using Equation (9.17). Further, the cyclic generator of $\mathcal{H}(\rho_k, \nu_{-\frac{k}{2}})$ has weight 0 and thus can be chosen to be 1, which gives the formula $\Psi_k(u, \tau) = \eta^{\frac{3k}{2}}$. ∎

11.2. **Dimension two.** We turn to describing the 2-dimensional setting and prepare some notation. Let $j(\tau)$ be Klein's j-invariant (normalised so that the leading term is q^{-1}) and $J(\tau) = j(\tau)/1728$. Additionally, for $a, b, c \in \mathbb{C}$, c not a negative integer, and a variable z, let ${}_2F_1(a, b; c; z)$ denote the Gaussian hypergeometric function, which is given by

$${}_2F_1(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)^n (b)^n}{(c)^n} \frac{z^n}{n!} \tag{11.4}$$

with $(x)^n$ being the (rising) factorial for $x \in \mathbb{C}$ given by $(x)^n = x(x+1)\cdots(x+n-1)$ for $n \in \mathbb{N}$. Finally, set

$$\Phi = \begin{pmatrix} J^{\frac{1}{24}} {}_2F_1\left(\frac{-1}{24}, \frac{7}{24}; \frac{3}{4}; J^{-1}\right) \\ J^{-\frac{5}{24}} {}_2F_1\left(\frac{5}{24}, \frac{13}{24}; \frac{5}{4}; J^{-1}\right) \end{pmatrix}, \tag{11.5}$$

where $J^{-1} = 1/J$.

Theorem 11.5. *Let $0 \le \lambda \le k$, λ even, $u \in L(k,\lambda)$ as in (10.3), and let ρ_λ be the representation associated to $\Psi_\lambda(u,\tau)$. The dimension of the vector-valued modular form $\Psi_\lambda(u,\tau)$ is 2 if and only if $\lambda = k-1$ and hence the level k is odd.*

(1) *The representation ρ_{k-1} is irreducible. Moreover, among all indecomposable representations ρ' of Γ satisfying $\rho'(T) = \mathrm{diag}(\mathbf{e}\left(\frac{k-2}{24}\right), \mathbf{e}\left(\frac{k+4}{24}\right))$, ρ_{k-1} is the unique (up to isomorphism) one that is irreducible.*

(2) *The representation ρ_{k-1} is congruence with congruence level $N = 8$ for $k \equiv 2 \pmod 3$ and $N = 24$ otherwise.*

(3) *We have an inclusion $\mathcal{V}(\rho_{k-1}) \subset \mathcal{H}(\rho_{k-1}, \nu_{h_{k-1}})$ for all odd $k \ge 3$. This inclusion is an equality if $3 \le k \le 13$ and proper if $k \ge 15$.*

(4) *There exists a normalisation of the intertwining operator underlying $\Psi_{k-1}(u,\tau)$ such that*

$$\Psi_{k-1}(u,\tau) = \eta^{\frac{3k^2+2k-5}{2(k+2)}} \Phi \tag{11.6}$$

for all odd $k \ge 1$. Further, for $k \ge 3$ we have the identity of $\mathcal{R}$ modules

$$\mathcal{V}^u(\rho_{k-1}) = \mathcal{V}(\rho_{k-1}) = \mathcal{R}\eta^{\frac{3k^2+2k-5}{2(k+2)}} \Phi. \tag{11.7}$$

As $\mathcal{M}$-modules each of the above is free of rank 2 with basis $\{\Psi_{k-1}(u,\tau), \partial\Psi_{k-1}(u,\tau)\} = \{\Psi(u,\tau), \Psi_{k-1}(L_{[-2]}u,\tau)\}$. For $n \in \mathbb{N}_0$,

$$\dim\left(\mathcal{V}(\rho_{k-1})_n\right) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 2 \\ \lfloor \frac{n}{6} \rfloor + 1 & \text{otherwise.} \end{cases} \tag{11.8}$$

Proof. That ρ_{k-1} is a 2-dimensional representation if and only if $\lambda = k-1$ is even follows directly from Theorem 10.2.5. To show Part 1 we use the criterion in

Lemma 11.3 and note by Theorem 10.2.5 that

$$\rho_{k-1}(\mathsf{T}) = \operatorname{diag}\left(\mathbf{e}\left(\frac{k-2}{24}\right), \mathbf{e}\left(\frac{k+4}{24}\right)\right). \tag{11.9}$$

Thus, by Lemma 11.3, ρ_{k-1} is irreducible if neither of the $\rho_{k-1}(\mathsf{T})$-eigenvalues are a 12th root of unity. This is clearly the case, since k is odd. The fact that ρ_{k-1} is the unique irreducible representation among indecomposable representations with T given by the formula (11.9) is due to [63, Theorem 3.1].

Part 2 follows from [63, Theorem 3.7], where are all 2-dimensional irreducible finite image representations are classified. They turn out to all be congruence representations. The congruence levels are recorded in the tables following that theorem, where each representation is characterised by the fractions (or rather the smallest non-negative representative of their integer coset) that appear in the formula (11.9) for $\rho_{k-1}(\mathsf{T})$. The congruence level is then always the order of $\rho_{k-1}(\mathsf{T})$, that is, 8 if $k \equiv 2 \pmod 3$ and 24 otherwise. A simple calculation reveals that for all odd k each $\rho_{k-1}(\mathsf{T})$ corresponds to a case in [63, Table 3].

We consider Parts 3 and 4 together. By Theorem Theorem 10.2.3 the leading exponents of $\Psi_{k-1}(u,\tau)$ are

$$\mu_{\min} = h_{\frac{k-1}{2}} - \frac{\mathbf{c}}{24} = \frac{k^2-3}{16(k+2)}, \ \mu_{\max} = h_{\frac{k+1}{2}} - \frac{\mathbf{c}}{24} = \frac{k^2+4k+5}{16(k+2)} = \frac{k^2-3}{16(k+2)} + \frac{1}{4}. \tag{11.10}$$

Clearly these exponents are non-negative if and only if $k \geq 3$, which proves the inclusion in Part 3. Since spaces of holomorphic vector-valued modular forms of dimension three or less are always cyclic $\mathcal{R}$ modules, and since the two exponents above differ by $\frac{1}{4}$, all six assumptions of Theorem 9.3 apply if $k \geq 3$. Hence

$$\mathcal{V}^u(\rho_{k-1}) = \mathcal{V}(\rho_{k-1}) = \eta^{3\frac{k^2-3}{2(k+2)}} \mathcal{H}\left(\rho_{k-1}, \nu_{\frac{2-k}{2}}\right) \tag{11.11}$$

for $k \geq 3$. For $k = 1$, by Proposition 9.1, we can still assert $\mathcal{V}^u(\rho_{k-1}) \subset \eta^{3\frac{k^2-3}{2(k+2)}} \mathcal{H}(\rho_{k-1}, \nu_{\frac{2-k}{2}})$. Further, for all odd $k \geq 1$ we have that $\eta^{-3\frac{k^2-3}{2(k+2)}} \Psi(u,\tau)$ is a cyclic generator for $\mathcal{H}(\rho_{k-1}, \nu_{\frac{2-k}{2}})$, since it has the right weight and spans a 1-dimensional weight space. By construction, a cyclic generator of $\mathcal{H}(\rho_{k-1}, \nu_{\frac{2-k}{2}})$ has leading exponents $\{0, \frac{1}{4}\} = \{\lambda_1, \lambda_2\}$ (these exponents are a minimal admissible set) and the weight of this cyclic generator is $\frac{1}{2}$, in particular it satisfies the equality in (8.15) and so its components form a fundamental system of solutions to a monic modular differential equation. This allows us to use [46, Theorem 3.1] and its proof (which additionally requires $\rho(\mathsf{S}^2)$ be a scalar matrix, but this is automatic due to ρ_{k-1} being irreducible; see the discussion below (8.7)). This gives the existence of a non-zero vector-valued modular form F of weight $p_0 = 6(\lambda_1 + \lambda_2) - 1 = \frac{1}{2}$ such that $\mathcal{H}(\rho_{k-1}, \nu_{h_{k-1}}) = \mathcal{R}F$ (as $\mathcal{R}$-modules) and $\mathcal{H}(\rho_{k-1}, \nu_{h_{k-1}}) = \mathcal{M}F \oplus \mathcal{M}\partial F$ (as $\mathcal{M}$-modules). Moreover, the component functions of F form a fundamental system of solutions of a second order monic modular differential equation of the form

$$\left(\partial_{p_0}^2 + \phi\right) f = 0, \tag{11.12}$$

where $\phi \in \mathcal{M}_4$. Note that since $\mathcal{M}_4 = \mathrm{span}_{\mathbb{C}}\{G_4\}$, up to a scalar p_1 we have that (11.12) can be rewritten as

$$\left(\partial_{p_0}^2 - p_1 G_4\right) f = 0. \tag{11.13}$$

This equation is characterised by its indicial roots being λ_1 and λ_2 (these are related to p_1 via $p_1 = 180(\lambda_1 - \lambda_2)^2 - 5)$ which are the exponents of the first and second component functions of F, respectively, as detailed in the proof of [46, Theorem 3.1] (cf. (8.8) and (8.9)). Meanwhile, [64, Proposition 2.2] (see also [65, Section 4.1]) gives that the functions

$$\begin{aligned} f_1 &= \eta^{2p_0} J^{-\frac{6(\lambda_1-\lambda_2)+1}{12}} {}_2F_1\left(\frac{6(\lambda_1-\lambda_2)+1}{12}, \frac{6(\lambda_1-\lambda_2)+5}{12}; \lambda_1 - \lambda_2 + 1; J^{-1}\right) \\ f_2 &= \eta^{2p_0} J^{-\frac{6(\lambda_2-\lambda_1)+1}{12}} {}_2F_1\left(\frac{6(\lambda_2-\lambda_1)+1}{12}, \frac{6(\lambda_2-\lambda_1)+5}{12}; \lambda_2 - \lambda_1 + 1; J^{-1}\right) \end{aligned} \tag{11.14}$$

form a fundamental set of solutions for (11.13). We make some notes pertaining to our use of [64, 65]. First, loc. cit. assumes integral weights, however, a careful examination of the proof shows that it also holds for real weights. Second, there is a difference of normalisations of Eisenstein series between the G_{2k} in this paper and the E_{2k} of [64, 65] given by $E_{2\ell} = -\frac{(2\ell)!}{B_{2\ell}} G_{2\ell}$ for $\ell \in \mathbb{N}$. Since f_1 and f_2 form a fundamental set of solutions for (11.13), up to a matrix $A \in \mathrm{GL}(2, \mathbb{C})$, we have $A(f_1, f_2)^t = F$. That is, $(f_1, f_2)^t$ is a vector-valued modular form of weight p_0, but

Level k	Cyclic generator $\Psi_{k-1}(u,\tau)$
3	$q^{3/40}\left(1+\frac{1}{5}q-\frac{117}{25}q^2-\frac{84}{125}q^3+\frac{3659}{625}q^4+\cdots\right)$ $q^{13/40}\left(1-\frac{9}{5}q-\frac{2}{25}q^2-\frac{39}{125}q^3-\frac{126}{625}q^4+\cdots\right)$
5	$q^{11/56}\left(1-\frac{19}{7}q-\frac{264}{49}q^2+\frac{6061}{343}q^3+\frac{22963}{2401}q^4+\cdots\right)$ $q^{25/56}\left(1-\frac{33}{7}q+\frac{247}{49}q^2+\frac{1672}{343}q^3-\frac{18183}{2401}q^4+\cdots\right)$
7	$q^{23/72}\left(1-\frac{17}{3}q+\frac{23}{9}q^2+\frac{3128}{81}q^3-\frac{13429}{243}q^4+\cdots\right)$ $q^{41/72}\left(1-\frac{23}{3}q+\frac{170}{9}q^2-\frac{391}{81}q^3-\frac{10948}{243}q^4+\cdots\right)$
9	$q^{39/88}\left(1-\frac{95}{11}q+\frac{2340}{121}q^2+\frac{48165}{1331}q^3-\frac{2895523}{14641}q^4+\cdots\right)$ $q^{61/88}\left(1-\frac{117}{11}q+\frac{5035}{121}q^2-\frac{74100}{1331}q^3-\frac{1011465}{14641}q^4+\cdots\right)$
11	$q^{59/104}\left(1-\frac{151}{13}q+\frac{7611}{169}q^2-\frac{35636}{2197}q^3-\frac{9959957}{28561}q^4+\cdots\right)$ $q^{85/104}\left(1-\frac{177}{13}q+\frac{12382}{169}q^2-\frac{383087}{2197}q^3+\frac{1229442}{28561}q^4+\cdots\right)$
13	$q^{83/120}\left(1-\frac{73}{5}q+\frac{1992}{25}q^2-\frac{18177}{125}q^3-\frac{224261}{625}q^4+\cdots\right)$ $q^{113/120}\left(1-\frac{83}{5}q+\frac{2847}{25}q^2-\frac{48472}{125}q^3+\frac{309009}{625}q^4+\cdots\right)$

TABLE 1. The first five terms of the q-series expansions for $\Psi_{k-1}(u,\tau)$ for all levels k at which $\Psi_{k-1}(u,\tau)$ generates $\mathcal{H}(\rho_{k-1},v_{h_{k-1}})$. In each case the series have been normalised so that the leading coefficient is 1. This can always be achieved by an appropriate choice of normalisation of the intertwining operators in $\Psi_{k-1}(u,\tau)$.

with representation $A\rho_{k-1}A^{-1}$. However, the leading exponents of f_1 and f_2 are λ_1 and λ_2, respectively (we note that this disagrees with [64, Remark 2.3], where there is a minor typographical error listing the exponents in reverse order). Thus, it must be that $A = \mathrm{diag}(\alpha,\beta)$ for some $\alpha,\beta\in\mathbb{C}^\times$ and we have $A\rho_{k-1}A^{-1} = \rho_{k-1}$. In particular, $F_{\alpha,\beta} = (\alpha f_1,\beta f_2)^t$. It remains to show that $(\alpha f_1,\beta f_2)^t$ is equal to the right-hand side of (11.5). This follows immediately from (11.14) by specialising $p_0 = \frac{1}{2}$, $\lambda_1 = 0$, $\lambda_2 = \frac{1}{4}$. The dimension formula (11.8) then follows from the evaluation of (9.17) [46, Corollary 3.2]. Finally the inclusion of Part 3 is an equality if and only the leading exponents of $\Psi(u,\tau)$ lie in the interval $[0,1)$ which happens if and only of $3\le k\le 13$. ∎

See Table 1 for explicit expansions of $\Psi_{k-1}(u,\tau)$ as q-series for the first few values of the level k.

11.3. **Dimension three.** Here we consider the 3-dimensional case. Recall, ${}_3F_2$, the generalised hypergeometric function, which for $a,b,c,d,e\in\mathbb{C}$, d,e not negative integers, and a variable z, is given by

$$ {}_3F_2(a,b,c;d,e;z) = 1 + \sum_{n=1}^{\infty} \frac{(a)^n (b)^n (c)^n}{(d)^n (e)^n} \frac{z^n}{n!}. \tag{11.15} $$

We set

$$ \Phi_k = \begin{pmatrix} J^{\frac{(k+1)}{12(k+2)}} {}_3F_2\left(-\frac{(k+1)}{12(k+2)}, \frac{11k+14}{24(k+2)}, \frac{19k+30}{24(k+2)}; \frac{3k+7}{4(k+2)}, \frac{1}{2}; J^{-1}\right) \\ J^{-\frac{k+1}{6(k+2)}} {}_3F_2\left(\frac{k+1}{6(k+2)}, \frac{3k+5}{6(k+2)}, \frac{5k+9}{6(k+2)}; \frac{5k+9}{4(k+2)}, \frac{5}{8}; J^{-1}\right) \\ J^{-\frac{5k+11}{12(k+2)}} {}_3F_2\left(\frac{5k+11}{12(k+2)}, \frac{9k+19}{12(k+2)}, \frac{13k+27}{12(k+2)}; \frac{3}{2}, \frac{5k+11}{4(k+2)}; J^{-1}\right) \end{pmatrix}, \tag{11.16} $$

where $J = j/1728$ is the same renormalisation of Klein's j-invariant as in the previous section.

Theorem 11.6. *Let $0 \le \lambda \le k$, λ even, $u \in L(k,\lambda)$ as in as in (10.3), ρ_λ the representation associated to $\Psi_\lambda(u,\tau)$. The dimension of the vector-valued modular form $\Psi_\lambda(u,\tau)$ is 3 if and only if $\lambda = k-2$ and hence the level k is even.*

(1) *The representation ρ_{k-2} is irreducible.*
(2) *The representation ρ_{k-2} has finite image. Additionally, the order of $\rho_{k-2}(\mathsf{T})$ is $12(k+2)$ if $k \equiv 4 \pmod 6$, and is $4(k+2)$ otherwise.*
(3) *If the order of $\rho_{k-2}(\mathsf{T})$ does not divide $25{,}401{,}600 = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^2$, then the representation is non-congruence, in particular, this gives an infinite family of non-congruence representations and a finite bound on the number of congruence representations.*
(4) *We have an inclusion $\mathcal{V}(\rho_{k-2}) \subset \mathcal{H}(\rho_{k-2}, \nu_{h_{k-2}})$ for all even $k \ge 4$. This inclusion is an equality if $4 \le k \le 10$ and it is proper if $k \ge 12$.*
(5) *There exists a normalisation of the intertwining operators underlying $\Psi_{k-2}(u,\tau)$ such that*

$$ \Psi_{k-2}(u,\tau) = \eta^{\frac{3k^2-2k-8}{2(k+2)}} \Phi_k \tag{11.17} $$

for all even $k \ge 2$. Further, for $k \ge 4$ we have the identity of $\mathcal{R}$-modules

$$ \mathcal{V}^u(\rho_{k-2}) = \mathcal{V}(\rho_{k-2}) = \mathcal{R}\eta^{\frac{3k^2-2k-8}{2(k+2)}} \Phi_k. \tag{11.18} $$

As $\mathcal{M}$-modules each of the above is free of rank 3 with basis

$$\begin{aligned}&\left\{\Psi_{k-2}(u,\tau),\partial\Psi_{k-2}(u,\tau),\partial^2\Psi_{k-2}(u,\tau)\right\} = \\ &\left\{\Psi_{k-2}(u,\tau),\Psi_{k-2}\left(L_{[-2]}u,\tau\right),\Psi_{k-2}\left(L_{[-2]}^2u+\delta L_{[-4]}u,\tau\right)\right\},\end{aligned} \tag{11.19}$$

where $\delta = \frac{2(16+k-6k^2)}{3(k-2)(4+3k)}$. *For* $n \in \mathbb{N}_0$,

$$\dim\left(\mathcal{V}(\rho_{k-2})_n\right) = \begin{cases} 0, & \text{if } n \equiv 1 \pmod 2, \\ \lfloor \frac{n}{4} \rfloor + 1, & \text{otherwise.} \end{cases} \tag{11.20}$$

Proof. Note that Theorem 10.2.5 gives that ρ_λ is a 3-dimensional representation if and only if $\lambda = k-2$ is even.

To establish Part 1 we use the irreducibility criterion in Lemma 11.3. Note that Theorem 10.2.5 yields the formula

$$\rho_{k-2}(\mathsf{T}) = \operatorname{diag}\left(\mathbf{e}\left(\frac{k(k-2)-6}{24(k+2)}\right),\mathbf{e}\left(\frac{k(k+4)}{24(k+2)}\right),\mathbf{e}\left(\frac{k(k+10)+18}{24(k+2)}\right)\right). \tag{11.21}$$

The fractions in the formula for $\rho_{k-2}(\mathsf{T})$ above multiplied by 12 are, respectively,

$$\begin{aligned}\frac{k}{2}-2+\frac{1}{k+2} &\equiv \frac{1}{k+2} \pmod 1,\\ \frac{k}{2}+\frac{k}{k+2} &\equiv \frac{k}{k+2} \pmod 1,\\ \frac{k}{2}+4+\frac{1}{k+2} &\equiv \frac{1}{k+2} \pmod 1.\end{aligned} \tag{11.22}$$

These are never integral since the numerators on the right-hand sides of the above identities are always strictly less than the denominators. Proper subproducts of two eigenvalues correspond to sums of two of the above fractions. The numerator of such sums cannot exceed $k+1$, so they are never integral for any $k \in 2\mathbb{N}$, concluding no twelfth root of unity arises as a product of one or two $\rho_{k-2}(\mathsf{T})$ eigenvalues. Thus ρ_{k-2} is irreducible.

Moving to Part 2, we show that ρ_{k-2} has finite image by using the criterion [66, Proposition 5.1], which states that a 3-dimensional irreducible representation with diagonalisable $\rho(\mathsf{T})$ has finite image if there exist two eigenvalues of $\rho(\mathsf{T})$ whose ratio is -1. To this end, observe that

$$\frac{k(k+10)+18}{24(k+2)} - \frac{k(k-2)-6}{24(k+2)} = \frac{1}{2}. \tag{11.23}$$

Hence ρ_{k-2} has finite image. To determine the order of $\rho_{k-2}(T)$, note that it is the least common multiple of the denominators of the reductions of the fractions in the exponents in (11.21). Since k is even, we take $k = 2\ell$ for $\ell \geq 0$ and consider all three exponents

$$\begin{aligned} \frac{k(k-2)-6}{24(k+2)} &= \frac{2\ell(\ell-1)-3}{24(\ell+1)}, \\ \frac{k(k+4)}{24(k+2)} &= \frac{\ell(\ell+2)}{12(\ell+1)}, \\ \frac{k(k+10)+18}{24(k+2)} &= \frac{2\ell(\ell+5)+9}{24(\ell+1)}. \end{aligned} \tag{11.24}$$

Next we compute the greatest common divisor of the numerator and denominator of the first and third fractions above to reduce them. Denote $a = 2\ell(\ell-1)-3 = 2(\ell-2)(\ell+1)+1$, $b = 2\ell(\ell+5)+9 = 2(\ell+1)(\ell+4)+1$, and $c = 24(\ell+1)$ so that the first and third fractions are equal to $\frac{a}{c}$ and $\frac{b}{c}$, respectively. Note that a, b are odd while c is even, so $\gcd(a,c)$ and $\gcd(b,c)$ will both be odd. Further, if a prime $p \geq 5$ divides c, then it must divide $(\ell+1)$, but then $a \equiv 1 \pmod{p} \equiv b$. So p does not divide $\gcd(a,c)$ or $\gcd(b,c)$. Both of these greatest common divisors are therefore a power of 3. We have that 3 divides a if and only if $\ell \equiv 0,1 \pmod{3}$ and the same is also true for b. Further, if $\ell \equiv 0,1 \pmod{3}$, then $\ell+1 \equiv 1,2 \pmod{3}$, and so 9 does not divide c. Thus,

$$\gcd(a,c) = \gcd(b,c) = \begin{cases} 1 & \ell \equiv 2 \pmod{3} \\ 3 & \ell \equiv 0,1 \pmod{3}. \end{cases} \tag{11.25}$$

Therefore, after reduction, the denominators of the first and third fractions in (11.24) are $24(\ell+1) = 12(k+2)$ if $k \equiv 4 \pmod{6}$, and $8(\ell+1) = 4(k+2)$ if $k \equiv 0,2 \pmod{6}$. Next we see that the reduced denominator of the middle fraction divides $12(\ell+1)$ if $\ell \equiv 2 \pmod{3}$, and that it divides $4(\ell+1)$ if $\ell \equiv 0,1 \pmod{3}$. It follows that the least common multiple of the reduced divisors in (11.24), the order of $\rho_{k-2}(\mathsf{T})$, is as claimed.

Part 3 follows from [66, Corollary 3.5], which states that ρ_{k-2} is non-congruence if there exists a prime dividing $\frac{N}{(N,2^8\cdot 3^4\cdot 5^2\cdot 7^2)}$. This quotient reduces to 1 if and only if N divides $2^8\cdot 3^4\cdot 5^2\cdot 7^2$. Thus, there exists a prime p dividing $\frac{N}{(N,2^8\cdot 3^4\cdot 5^2\cdot 7^2)}$ if and only if N does not divide $2^8\cdot 3^4\cdot 5^2\cdot 7^2$.

We consider Parts 4 and 5 together. By Theorem 10.2.3 the leading exponents of $\Psi_{k-2}(u,\tau)$ are

$$\begin{aligned}\mu_{\min} &= h_{\frac{k-2}{2}} - \frac{\mathbf{c}}{24} = \frac{k^2-2k-4}{16(k+2)},\\ h_{\frac{k}{2}} - \frac{\mathbf{c}}{24} = \frac{k}{16} &= \mu_{\min} + \frac{k+1}{4(k+2)},\\ \mu_{\max} &= h_{\frac{k+2}{2}} - \frac{\mathbf{c}}{24} = \frac{k^2+6k+12}{16(k+2)} = \mu_{\min} + \frac{1}{2}.\end{aligned} \tag{11.26}$$

Clearly these exponents are non-negative if and only if $k \geq 4$, which proves the inclusion in Part 4. Spaces of holomorphic vector-valued modular forms of dimension three or less are always cyclic over $\mathcal{R}$, and the maximal and minimal exponents above differ by $\frac{1}{2}$. Therefore, all six assumptions of Theorem 9.3 apply if $k \geq 4$. Hence

$$\mathcal{V}^u(\rho_{k-2}) = \mathcal{V}(\rho_{k-2}) = \eta^{3\frac{k^2-2k-4}{2(k+2)}}\mathcal{H}\left(\rho_{k-2}, \nu_{\frac{3+k-k^2}{2(k+2)}}\right) \tag{11.27}$$

for $k \geq 4$. For $k=2$, by Proposition 9.1, we can assert $\mathcal{V}^u(\rho_{k-1}) \subset \eta^{3\frac{k^2-2k-4}{2(k+2)}}\mathcal{H}(\rho_{k-1}, \nu_{\frac{2-k}{2}})$, however, for all even $k \geq 2$ we still have that $\eta^{-3\frac{k^2-2k-4}{2(k+2)}}\Psi(u,\tau)$ is a cyclic generator for $\mathcal{H}(\rho_{k-2}, \nu_{\frac{3+k-k^2}{2(k+2)}})$, since it has the right weight and spans a 1-dimensional weight space. By construction, the leading exponents of the cyclic generator of $\mathcal{H}(\rho_{k-2}, \nu_{\frac{3+k-k^2}{2(k+2)}})$ are $\{0, \frac{k+1}{4(k+2)}, \frac{1}{2}\}$ (they form a minimal admissible set) and the weight of this cyclic generator is $\frac{k+1}{k+2}$. This implies the components of this vector-valued modular form form a set of fundamental solutions for a third order monic modular linear differential equation of the form

$$\left(\partial_{p_0}^3 + p_1 G_4 \partial_{p_0} + p_2 G_6\right) f = 0, \tag{11.28}$$

see [65, Equation 15] and the surrounding text. While [65] works in the context of integral weight vector-valued modular forms, a careful analysis of their construction of solutions [65, Equation 16] to (11.28) shows that it is valid for real weight as well. In terms of the leading exponents $\{\lambda_1, \lambda_2, \lambda_3\}$ a set of fundamental solutions is given by

$$f_1 = \eta^{2p_0} J^{-\frac{4\lambda_1-2\lambda_2-2\lambda_3+1}{6}} {}_3F_2\left(\tfrac{4\lambda_1-2\lambda_2-2\lambda_3+1}{6}, \tfrac{4\lambda_1-2\lambda_2-2\lambda_3+3}{6}, \tfrac{4\lambda_1-2\lambda_2-2\lambda_3+5}{6}; \lambda_1-\lambda_2+1, \lambda_1-\lambda_3+1; J^{-1}\right),$$
$$f_2 = \eta^{2p_0} J^{-\frac{4\lambda_2-2\lambda_1-2\lambda_3+1}{6}} {}_3F_2\left(\tfrac{4\lambda_2-2\lambda_1-2\lambda_3+1}{6}, \tfrac{4\lambda_2-2\lambda_1-2\lambda_3+3}{6}, \tfrac{4\lambda_2-2\lambda_1-2\lambda_3+5}{6}; \lambda_2-\lambda_1+1, \lambda_2-\lambda_3+1; J^{-1}\right),$$
$$f_3 = \eta^{2p_0} J^{-\frac{4\lambda_3-2\lambda_1-2\lambda_2+1}{6}} {}_3F_2\left(\frac{4\lambda_3-2\lambda_1-2\lambda_2+1}{6}, \tfrac{4\lambda_3-2\lambda_1-2\lambda_2+3}{6}, \tfrac{4\lambda_3-2\lambda_1-2\lambda_2+5}{6}; \lambda_3-\lambda_1+1, \lambda_3-\lambda_2+1; J^{-1}\right),$$

with $p_0 = 4(\lambda_1+\lambda_2+\lambda_3)-2$. Specialising to $\{\lambda_1,\lambda_2,\lambda_3\} = \{0, \frac{k+1}{4(k+2)}, \frac{1}{2}\}$ gives

$$\begin{aligned}
f_1 &= \eta^{\frac{2k}{k+2}} J^{\frac{(k+1)}{12(k+2)}} {}_3F_2\left(-\frac{(k+1)}{12(k+2)}, \frac{11k+14}{24(k+2)}, \frac{19k+30}{24(k+2)}; \frac{3k+7}{4(k+2)}, \frac{1}{2}; J^{-1}\right),\\
f_2 &= \eta^{\frac{2k}{k+2}} J^{-\frac{k+1}{6(k+2)}} {}_3F_2\left(\frac{k+1}{6(k+2)}, \frac{3k+5}{6(k+2)}, \frac{5k+9}{6(k+2)}; \frac{5k+9}{4(k+2)}, \frac{5}{8}; J^{-1}\right),\\
f_3 &= \eta^{\frac{2k}{k+2}} J^{-\frac{5k+11}{12(k+2)}} {}_3F_2\left(\frac{5k+11}{12(k+2)}, \frac{9k+19}{12(k+2)}, \frac{13k+27}{12(k+2)}; \frac{3}{2}, \frac{5k+11}{4(k+2)}; J^{-1}\right).
\end{aligned} \tag{11.29}$$

The components of the cyclic generator are therefore linear combinations of the above fundamental solutions. More specifically, since the leading exponents of the respective components are $\{0, \frac{k+1}{4(k+2)}, \frac{1}{2}\}$, the cyclic generator must be of the form $(\alpha f_1, \beta f_2, \gamma f_3)^t$, $\alpha,\beta,\gamma \in \mathbb{C}^\times$. Hence

$$\Psi(u,\tau) = \eta^{3\frac{k^2-2k-4}{2(k+2)}} \begin{pmatrix} \alpha f_1 \\ \beta f_2 \\ \gamma f_3 \end{pmatrix} \tag{11.30}$$

and the underlying intertwining operators can be normalised such that $\alpha = \beta = \gamma = 1$, so (11.17) and (11.18) follow. That the left-hand side of (11.19) is an $\mathcal{M}$-basis follows from Theorem 9.3. So all that remains is to relate powers of the modular derivative the action of Virasoro generators. By the $a = \tilde{\omega}$ case of (8.23) we have $\Psi_{k-2}(L_{[-2]}u,\tau) = \partial\Psi_{k-2}(u,\tau)$. Furthermore, using [67, Theorem 5.10], we find that the components ψ^μ, $\frac{k-2}{2} \le \mu \le \frac{k+2}{2}$ of $\Psi(w,\tau)$, $w \in L(k,k-2)$ satisfy

$$\begin{aligned}
\psi^\mu(\tilde{\omega}_{[-3]}u,\tau) &= \psi^\mu(L_{[-4]}u,\tau)\\
&= \sum_{m=1}^{\infty} \binom{m+2}{m} G_{m+3}(\tau)\psi^\mu(L_{[m-1]}u,\tau)\\
&= 3\mathrm{wt}[u] G_4(\tau)\psi^\mu(u,\tau).
\end{aligned} \tag{11.31}$$

We also have

$$\begin{aligned}
\psi^{\mu}\left(\tilde{\omega}_{[-1]}^{2}u,\tau\right) &= \mathrm{tr}\big|_{L(k,\mu)}\left(L_0-\frac{\mathbf{c}}{24}\right)o(\tilde{\omega}_{[-1]}u)q^{L_0-\frac{\mathbf{c}}{24}}+\sum_{m=1}^{\infty}(-1)^{m+1}G_{m+1}(\tau)\psi^{\mu}(L_{[m-1]}L_{[-2]}u,\tau)\\
&= \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{tr}\big|_{L(k,\mu)}o(\tilde{\omega}_{[-1]}u)q^{L_0-\frac{\mathbf{c}}{24}}+(\mathrm{wt}[u]+2)G_2(\tau)\psi^{\mu}(\tilde{\omega}_{[-1]}u,\tau)\\
&+\sum_{m=2}^{\infty}(-1)^{m+1}G_{m+1}(\tau)\psi^{\mu}(L_{[m-1]}L_{[-2]}u,\tau)\\
&= \partial^2\psi^{\mu}(u,\tau)+\sum_{m=2}^{\infty}(-1)^{2m}G_{2m}(\tau)\psi^{\mu}(L_{[2m-2]}L_{[-2]}u,\tau)\\
&= \partial^2\psi^{\mu}(u,\tau)+G_4(\tau)\psi^{\mu}(L_{[2]}L_{[-2]}u,\tau)\\
&= \partial^2\psi^{\mu}(u,\tau)+\left(4\mathrm{wt}[u]+\frac{\mathbf{c}}{2}\right)G_4(\tau)\psi^{\mu}(u,\tau).
\end{aligned}\tag{11.32}$$

Combining (11.31) and (11.32) we find $\partial^2\Psi_{k-2}(u,\tau)=\Psi_{k-2}((L_{[-2]}^2+\delta L_{[-4]})u,\tau)$ with $\delta=-(4\,\mathrm{wt}[u]+\mathbf{c}/24)/(3\,\mathrm{wt}[u])$. Plugging in the formula above for $\mathrm{wt}[u]$ and $\mathbf{c}$ gives the stated formula for δ. ■

Level k	$\Psi_{k-2}(u,\tau)$
4	$q^{1/24}\left(1-\frac{6991}{171}q-\frac{1462930981}{198531}q^2-\frac{11520966474250}{5360337}q^3-\frac{467661528323716250}{627159429}q^4+\cdots\right)$
	$q^{1/4}\left(1+\frac{134}{9}q+\frac{167509}{81}q^2+\frac{24672291010}{45927}q^3+\frac{2054193740460070}{11986947}q^4\cdots\right)$
	$q^{13/24}\left(1-\frac{31}{27}q+\frac{473}{1215}q^2-\frac{27056}{32805}q^3-\frac{1533931}{2657205}q^4+\cdots\right)$
6	$q^{5/32}\left(1-\frac{1041}{20}q-\frac{28822341}{3040}q^2-\frac{34699584029}{12160}q^3-\frac{2170275413391777}{2140160}q^4+\cdots\right)$
	$q^{3/8}\left(1+\frac{74}{5}q+\frac{317943}{130}q^2+\frac{8423595}{13}q^3+\frac{21692516271}{104}q^4+\cdots\right)$
	$q^{21/32}\left(1-\frac{31}{8}q+\frac{423}{128}q^2+\frac{14247}{7168}q^3-\frac{485683}{229376}q^4+\cdots\right)$
8	$q^{11/40}\left(1-\frac{46803}{775}q-\frac{14944931541}{1375625}q^2-\frac{574656427747084}{171953125}q^3-\frac{782261040133149248781}{649123046875}q^4+\cdots\right)$
	$q^{1/2}\left(1+\frac{1704}{125}q+\frac{108483138}{40625}q^2+\frac{5094872662288}{7109375}q^3+\frac{5893213005533601}{25390625}q^4+\cdots\right)$
	$q^{31/40}\left(1-\frac{503}{75}q+\frac{44149}{3125}q^2-\frac{206842}{78125}q^3-\frac{420276376}{17578125}q^4+\cdots\right)$
10	$q^{19/48}\left(1-\frac{44717}{666}q-\frac{14421863479}{1222776}q^2-\frac{243672512766437}{66029904}q^3-\frac{680387381704666662661}{50617747584}q^4+\cdots\right)$
	$q^{5/8}\left(1+\frac{107}{9}q+\frac{2963152}{1053}q^2+\frac{64959522367}{85293}q^3+\frac{5516615806491181}{22261473}q^4+\cdots\right)$
	$q^{43/48}\left(1-\frac{259}{27}q+\frac{8110}{243}q^2-\frac{251140}{6561}q^3-\frac{25036652}{531441}q^4+\cdots\right)$

TABLE 2. The first five terms of the q-series expansions for $\Psi_{k-2}(u,\tau)$ for all levels k at which $\Psi_{k-2}(u,\tau)$ generates $\mathcal{H}(\rho_{k-2},\nu_{h_{k-2}})$. In each case the series have been normalised so that the leading coefficient is 1. This can always be achieved by an appropriate choice of normalisation of the intertwining operators underlying $\Psi_{k-2}(u,\tau)$.

See Table 2 for explicit expansions of $\Psi_{k-2}(u,\tau)$ for the first few values of the level k.

11.4. **Select higher dimensions.** We conclude this section by providing some results concerning non-congruence representations in some higher dimensions. Specifically, we provide a criterion for non-congruence applicable to a particular case of higher dimension and results on certain spaces of vector-valued modular forms in dimension four.

Theorem 11.7. *Let the level be $k = p^t - 2$, where $p > 3$ is prime and t is a positive integer. For $2 \le \lambda \le k$ with λ even, if the representation ρ_λ is irreducible, then it is non-congruence if $t = 1$ or if $t > 1$ and $\lambda + 1 > p^{t-2}$.*

Proof. By Theorem 10.2.5,

$$\rho_\lambda(\mathsf{T}) = \mathrm{diag}\{\mathbf{e}(r_0), \ldots, \mathbf{e}(r_{k-\lambda})\} \tag{11.33}$$

where

$$r_j = \frac{6j^2 + 6j(\lambda+2) + \lambda(\lambda+5) - 3k}{24(k+2)}. \tag{11.34}$$

Evaluating at $k = p^t - 2$, the r_j become

$$r_j = \frac{6j^2 + 6j(\lambda+2) + \lambda(\lambda+5) - 3p^t + 6}{2^3 \cdot 3p^t}. \tag{11.35}$$

Since $p > 3$, the numerator in (11.35) is odd and hence indivisible by 2. Furthermore, the numerator is divisible by 3 if and only if 3 divides $\lambda(\lambda+5)$, which is the case if and only if $\lambda \equiv 0, 1 \pmod 3$. The order of the $\rho_\lambda(\mathsf{T})$ is given by the least common multiple of the reduced denominators. To ensure this includes the factor of p^t, it suffices that p does not divide the numerator for all $j = 0, \ldots, k - \lambda = p^t - 2 - \lambda$. As j increases, the jth numerator is incremented by $18 + 12j + 6\lambda$ which must be divisible by p for all numerators to be divisible by p. This is only the case if p divides 12, i.e., $p = 2, 3$ which we have excluded. Thus the order of $\rho_\lambda(\mathsf{T})$ is $N = 2^3 \cdot 3 \cdot p^t$ if $\lambda \equiv 2 \pmod 3$ and $N = 2^3 \cdot p^t$ otherwise.

To ascertain non-congruence based on the level and dimension of a representation, we follow the argument in [17, Section 3.4]. Namely, for a d-dimensional congruence representation $\rho \colon \Gamma \to \mathsf{GL}(d, \mathbb{C})$ of level N, its image is isomorphic to a quotient of $\mathsf{SL}(2, \mathbb{Z}_N)$. If $N = \prod_i p_i^{s_i}$ is the factorisation of N into distinct primes p_i, then $\mathsf{SL}(2, \mathbb{Z}_\mathsf{N}) \cong \prod_i \mathsf{SL}(2, \mathbb{Z}_{p_i^{s_i}})$ and hence any irreducible representation of $\mathsf{SL}(2, \mathbb{Z}_N)$ can be constructed by tensoring irreducible representations of the $\mathsf{SL}(2, \mathbb{Z}_{p_i^{s_i}})$ factors.

In [68, 69], all irreducible representations of $\mathsf{SL}(2,\mathbb{Z}_{p^t})$, for p prime and $t \in \mathbb{N}$, were classified and their dimensions were determined. In particular, tables summarising the classification are given in [69, Section 9] (a summary of the minimal dimensions of non-trivial representations in English is given in [17, Theorem 3.14]). Specifically the minimal dimensions of a representations of level 2^3 or 3 are 2 and 1, respectively. While for representations of level p^t the minimal dimension is $\frac{1}{2}(p-1)$, if $t = 1$ and $\frac{1}{2}(p^t - t^{t-2})$, if $t \geq 2$. Thus, the minimal dimension among representations of level N may be found as a product of the minimal dimensions of the representations of level $p_i^{s_i}$. Note that requiring each tensor factor to have respective level $p_i^{s_i}$ precludes any of the tensor factors from being trivial. Thus for $t = 1$, we get that the minimal dimension is $p-1$ for both $N = 2^3 \cdot 3 \cdot p^t$ and $N = 2^3 \cdot p^t$. Comparing this to Theorem 10.2.5 at level $k = p^t - 2$ we see that the dimension formula becomes $p - 1 - \lambda$ which is less than $p-1$ if $\lambda \geq 2$. Similarly, if $t \geq 2$ the minimal congruence dimension is $p^t - p^{t-2}$. Hence we have non-congruence, if $p^t - p^{t-2} > p^t - 1 - \lambda$, or equivalently, $\lambda + 1 > p^{t-2}$. ■

Proposition 11.8. *Let $0 \leq \lambda \leq k$, λ even, $u \in L(k,\lambda)$ as in (10.3), and let ρ_λ be the representation associated to $\Psi_\lambda(u,\tau)$. The dimension of the vector-valued modular form $\Psi_\lambda(u,\tau)$ is 4 if and only if $\lambda = k-3$ and hence the level k is odd. If $\mathcal{H}(\rho_{k-3}, \nu_{\frac{-k^2+4k+6}{2(k+2)}})$ is cyclic, then*

$$\mathcal{V}^u(\rho_\lambda) = \mathcal{V}(\rho_\lambda) = \eta^{\frac{3}{2}\frac{k^2-4k-3}{k+2}} \mathcal{H}\left(\rho_{k-3}, \nu_{\frac{-k^2+4k+6}{2(k+2)}}\right). \tag{11.36}$$

Proof. Theorem 10.2.5 gives that ρ_λ is a 4-dimensional representation if and only if $\lambda = k-3$ is even. Next note that as $h_{\frac{k+3}{2}} - h_{\frac{k-3}{2}} = \frac{3}{4}$ all conditions in Theorem 9.3 other than condition 4 (cyclicity over $\mathcal{R}$) obviously hold. Thus the proposition follows for those levels where $\mathcal{H}(\rho_{k-3}, \nu_{\frac{-k^2+4k+6}{2(k+2)}})$ is cyclic. ■

Note that in the case of general non-negative integral levels k and even weight $0 \leq \lambda \leq k$, where one obtains vector-valued modular forms of dimension $d = k - \lambda + 1$, we have $\mu_{\max} - \mu_{\min} = (d-1)/4$. Thus, the fifth condition of Theorem 9.3 holds only for those levels and weights chosen so that $d \leq 4$ or equivalently $k - \lambda \leq 3$. For $d \geq 5$

we therefore have that the inclusion

$$\eta^{-\frac{3}{2}\frac{(k-d+1)(k-d+5)-2k}{k+2}}\mathcal{V}^u(\rho_\lambda) \subset \mathcal{H}\left(\rho_{k-d+1}, \nu_{\frac{(k-d+1)(k-d+3)}{16(k+2)}}\right) \tag{11.37}$$

is proper.

12. Modular actions from categorical data

So far we have studied the properties of traces of intertwining operators directly, that is, using results from analytic number theory on modular forms. However, since categories of modules over rational vertex operator algebras are modular tensor categories, and additionally the categorical and number theoretic notions of modularity coincide [25], we can repeat the above analysis using categorical data. Let $\mathscr{C}$ be a modular tensor category, that is, a monoidal category with many additional structures and properties reviewed in Section 7. To compute the action of the modular group, we will need the graphical calculus (also known as string diagram calculus, see [70, Section 2.3] for an introduction). To convert this abstract action of the modular group into actual matrices we will need to make explicit choices of bases (see [71, Section 2] for an introduction to working in such bases and some helpful identities), just as we needed to choose bases of intertwining operators in Section 8 to obtain vector-valued modular forms. Let $\mathcal{I}$ be a complete set of representatives of simple isomorphism classes of objects in $\mathscr{C}$, with $0 \in \mathcal{I}$ denoting the tensor unit (that is, the vertex operator algebra itself, if $\mathscr{C}$ is a category of vertex operator algebra modules). The rigid dual of a simple object $i \in \mathcal{I}$ will be denoted i^*. For every triple $i, j, k \in \mathcal{I}$, consider the vector space $\hom_{\mathscr{C}}(i \otimes j, k)$ (called a 3-point coupling space) and pick a basis $\{\lambda^{\alpha}_{(i,j)k}\}_{\alpha=0}^{\dim(\hom_{\mathscr{C}}(i\otimes j,k))-1}$ and denote its dual basis by $\{\Upsilon^{\alpha}_{(i,j)k}\}_{\alpha} \subset \hom_{\mathscr{C}}(k, i\otimes j)$, where the evaluation of dual vector on vectors is given by

$$\lambda^{\alpha}_{(i,j)k} \circ \Upsilon^{\beta}_{(i,j)k} = \delta_{\alpha,\beta}\,\mathrm{id}_k \in \hom_{\mathscr{C}}(k,k) = \mathbb{C}\,\mathrm{id}_k, \tag{12.1}$$

where $\delta_{\alpha,\beta}$ is the Kronecker δ. The 3-point coupling space $\hom_{\mathscr{C}}(i \otimes j, k)$ is the categorical counterpart to the space of intertwining operators of type $\binom{k}{i\ j}$ and picking a basis of 3-point couplings is equivalent to picking a basis of intertwining operators. Therefore, a natural categorical question is to ask how to characterise the subcategory of objects which correspond to intertwining operators that map an object to itself (and hence admit a trace as in (8.17)). Recall the adjoint category $\mathscr{C}_{\mathrm{ad}}$ is defined to be the smallest full subcategory of $\mathscr{C}$ containing all objects $i \otimes i^*$, $i \in \mathscr{C}$ and all of their subquotients. Note that this category is closed under taking duals. Another characterisation of $\mathscr{C}_{\mathrm{ad}}$ is as the centraliser of the subcategory of invertible objects [43, Section 4.14].

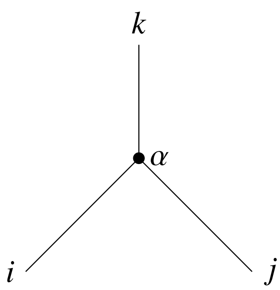
Lemma 12.1. *Let $\mathscr{C}$ be a modular tensor category. A simple object $p \in \mathscr{C}$ admits a non-vanishing* $\hom_{\mathscr{C}}(p \otimes i, i)$ *for some $i \in \mathscr{C}$ if and only if p is in the adjoint subcategory $\mathscr{C}_{\mathrm{ad}}$.*

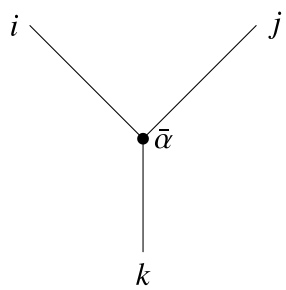
Proof. Recall that Hom spaces in a modular tensor category satisfy natural isomorphisms

$$\hom_{\mathscr{C}}(p \otimes i, i) \cong \hom_{\mathscr{C}}\left(i^* \otimes i, p^*\right). \tag{12.2}$$

Thus the Hom spaces on the right-hand side of the above identification are non-vanishing if and only p^* lies in the adjoint subcategory, which is the case if and only if p does. ∎

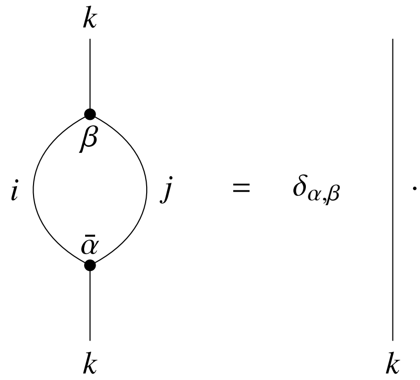
We continue fixing conventions. Note that if $i = 0$ or $j = 0$ then $\dim(\hom_{\mathscr{C}}(0 \otimes j, k)) = \delta_{j,k}$ and $\dim(\hom_{\mathscr{C}}(i \otimes 0, k)) = \delta_{i,k}$. The non-vanishing 3-point coupling spaces $\hom_{\mathscr{C}}(0 \otimes j, j)$ and $\hom_{\mathscr{C}}(j \otimes 0, j)$ are spanned by the left and right unitors, respectively, and so we choose these as our basis elements, that is, $\lambda^0_{(i,0)i} = \ell_i$ and $\lambda^0_{(0,j)j} = r_j$. In our conventions for the graphical calculus we will always read diagrams from bottom to top (also called the optimistic direction). The 3-point couplings and their duals are thus displayed as

$\lambda^{\alpha}_{(i,j)k} =$ 

, $\Upsilon^{\alpha}_{(i,j)k} =$ 

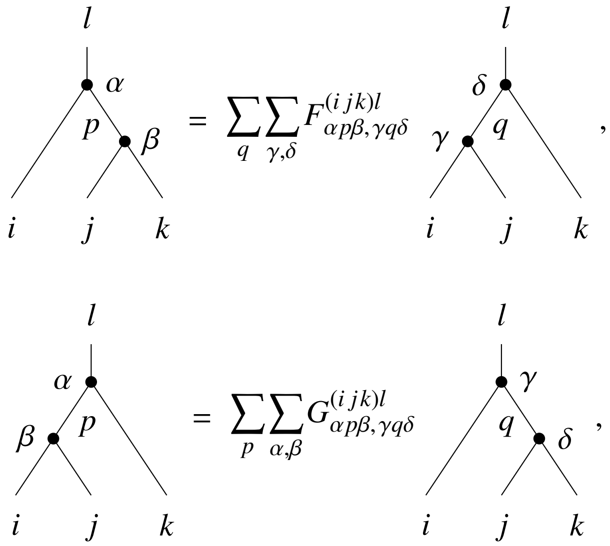
(12.3)

and duality property (12.1) is then expressed as



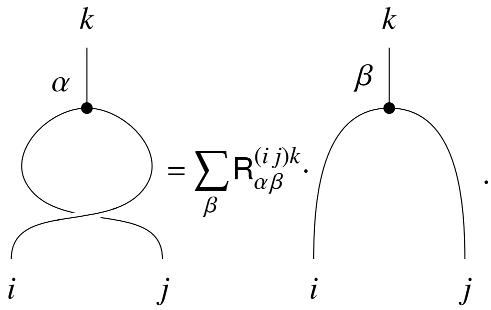
(12.4)

With these choices of bases of 3-point couplings the associator structure morphisms (and their inverses) of $\mathscr{C}$ can be expressed as the matrices



(12.5)

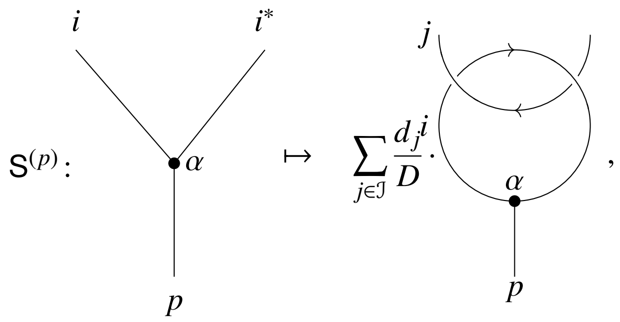
while the braiding isomorphisms are expressed as the matrix

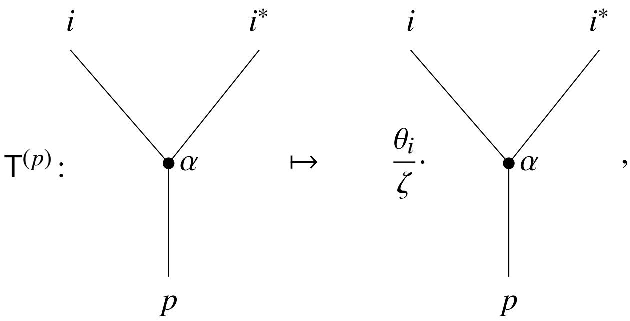


(12.6)

Theorem 12.2 (Bakalov-Kirillov [70, Theorem 3.1.17])**.** *Let $p \in \mathfrak{I}$ and consider the vector spaces $W_{p,i} = \hom_{\mathscr{C}}(p, i \otimes i^*)$, $i \in \mathfrak{I}$, and their direct sum $W_p = \bigoplus_{i\in\mathfrak{I}} W_{p,i}$.*

Define linear maps $\mathsf{S}^p, \mathsf{T}^p : W_p \to W_p$ *via the diagrams*





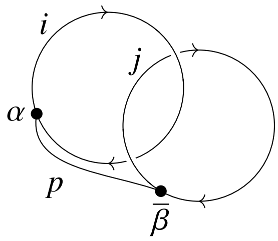
(12.7)

where d_i *is the quantum dimension of* $i \in \mathfrak{I}$, $D = \sum_i d_i^2$, *and* $\zeta = \left(\frac{\sum_i \theta_i d_i}{\sum_i \theta_i^{-1} d_i}\right)^{\frac{1}{6}}$. *Then* $\mathsf{S}^{(p)}, \mathsf{T}^{(p)}$ *satisfy the relations* $\left(\mathsf{S}^{(p)}\mathsf{T}^{(p)}\right)^3 = \left(\mathsf{S}^{(p)}\right)^2$ *and* $\left(\mathsf{S}^{(p)}\right)^4 = \theta_p^{-1}$. *That is,* $\mathsf{S}^{(p)}, \mathsf{T}^{(p)}$ *satisfy the defining relations of the braid group* B_3 *on three strands (the modular group of the torus with one marked point) with the additional relation* $\left(\mathsf{S}^{(p)}\right)^4 = \theta_p^{-1}$ *being the Dehn twist about the marked point.*

The above theorem is a specialisation of [70, Theorem 3.1.17 and 5.5.1], where Theorem 3.1.17 gives the action of the modular group on duals of 3-point coupling spaces and Theorem 5.5.1 gives the action on marked tori (which are the geometric interpretation of traces of intertwining operators). Note that in order to be closer to the conventions of vertex operator algebra literature, we have rescaled the definition of $\mathsf{T}^{(p)}$ by a factor of ζ relative to the conventions of [70]. Note further that the action given in Theorem 12.2 and above is an action of B_3. To deprojectify and obtain an action of Γ one needs to include a multiplier system, which we shall do a posteriori in the $\mathfrak{sl}(2)$ example below. If $\mathscr{C}$ is a category of modules over a rational vertex operator algebra (one satisfying all of the assumptions in the paragraph preceding (8.26)), then the numbers appearing in the theorem above can be expressed in terms of vertex operator algebra data as $D = \mathsf{S}_{0,0}^{-1}$, $d_i = \frac{\mathsf{S}_{i,0}}{\mathsf{S}_{0,0}}$, where $\mathsf{S}_{i,j}$ is the modular S-matrix of characters, and $\theta_p = \mathbf{e}(h_p)$, where h_p is the conformal weight of the simple module p and $\zeta = \mathbf{e}(\frac{\mathbf{c}}{24})$, where $\mathbf{c}$ is the central charge of the vertex operator algebra.

Theorem 12.3. *Let $\mathscr{C}$ be a modular tensor category with a set of representatives of simple isomorphism classes $\mathfrak{I}$, twist, braiding and fusing matrices given in a choice of basis of 3-point couplings, as described above. Let $p \in \mathfrak{I}$. Then the pull back of $\mathsf{S}^{(p)}, \mathsf{T}^{(p)}$ to spaces of 3-point couplings via evaluation and co-evaluation, expanded in the basis $\{\lambda^i_{(p,i)\alpha}\}$ is given by*

$$\mathsf{S}^{(p)}_{i\alpha,j\beta} = D^{-1}\cdot$$



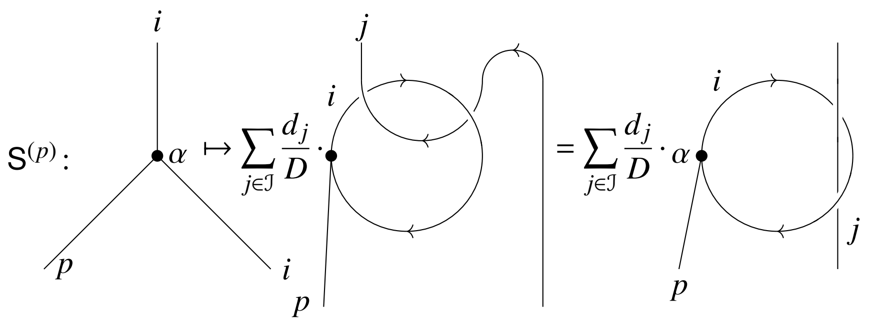
$$= \frac{d_i d_j}{D}\sum_{r\in i^*\otimes j}\sum_{\gamma\delta,\varepsilon}\frac{\theta_r}{\theta_i\theta_j}\mathsf{G}^{(ii^*j)j}_{0;\delta r\gamma}\mathsf{F}^{(ii^*j)j}_{\varepsilon r\gamma;0}\mathsf{G}^{(pir)j}_{\delta i\alpha;\beta j\varepsilon}, \tag{12.8}$$

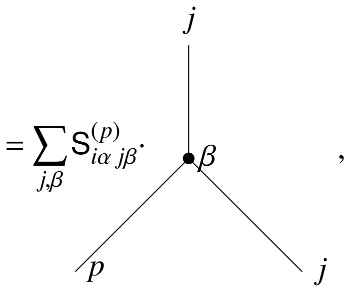
where $\mathsf{S}_{i,j}$ denotes entries of the matrix for the standard S-transformation of characters, γ runs over the basis of $\hom_{\mathscr{C}}(i^*\otimes j, r)$, and ε, δ both run over the basis of $\hom_{\mathscr{C}}(i\otimes r, j)$. The modular T-matrix is given by

$$\mathsf{T}^{(p)}_{i\alpha,j\beta} = \delta_{i,j}\delta_{\alpha,\beta}\frac{\theta_i}{\zeta}. \tag{12.9}$$

The above S and T matrices are the categorical counterpart to the analytic number theoretic ones discussed previously. Note however, that the multiplier system ν_{h_p} has not yet been included in these formulae. A similar diagrammatic formula for $\mathsf{S}^{(p)}_{i\alpha,j\beta}$ already appeared in [4], but with some additional assumptions on dimensions of 3-point coupling spaces, which we do away with here.

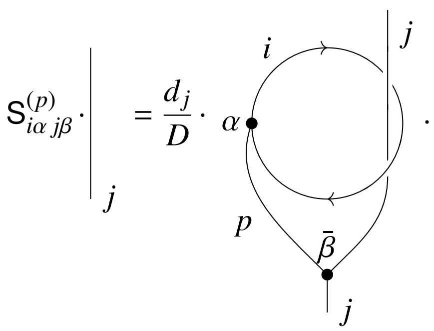
Proof. The formula for $\mathsf{T}^{(p)}$ follows immediately from Theorem 12.2, so we focus on the formula for $\mathsf{S}^{(p)}$. The transferal of $\mathsf{S}^{(p)}$ as it is given in Theorem 12.2 to 3-point couplings via evaluation and co-evaluation is





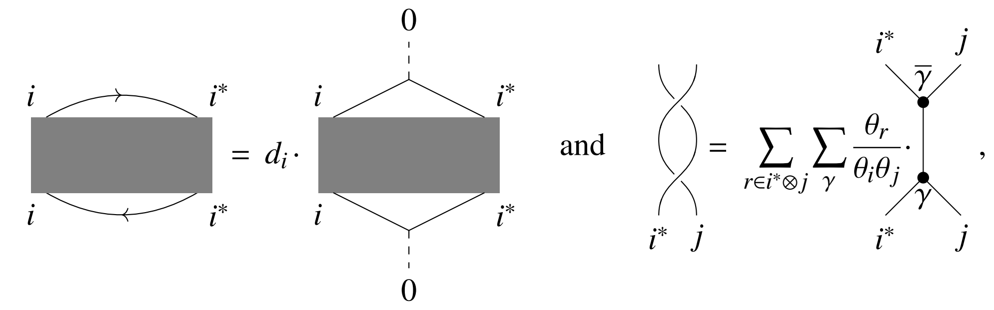
(12.10)

where the first identity uses the straightening axiom of evaluation and co-evaluation to yield a morphism in $\hom_{\mathscr{C}}(p \otimes j, j)$ and the second identity is the expansion of this morphism in our chosen basis, which defines the coefficients $\mathsf{S}^{(p)}_{i\alpha\, j\beta}$. To extract the coefficient in front of each basis vector we pair with the dual basis by attaching $\Upsilon^{\beta}_{(p,j)j}$ to the diagram from below, which will yield a morphism in $\hom_{\mathscr{C}}(j,j) = \mathbb{C}\,\mathrm{id}_j$, proportional to the identity, that is,



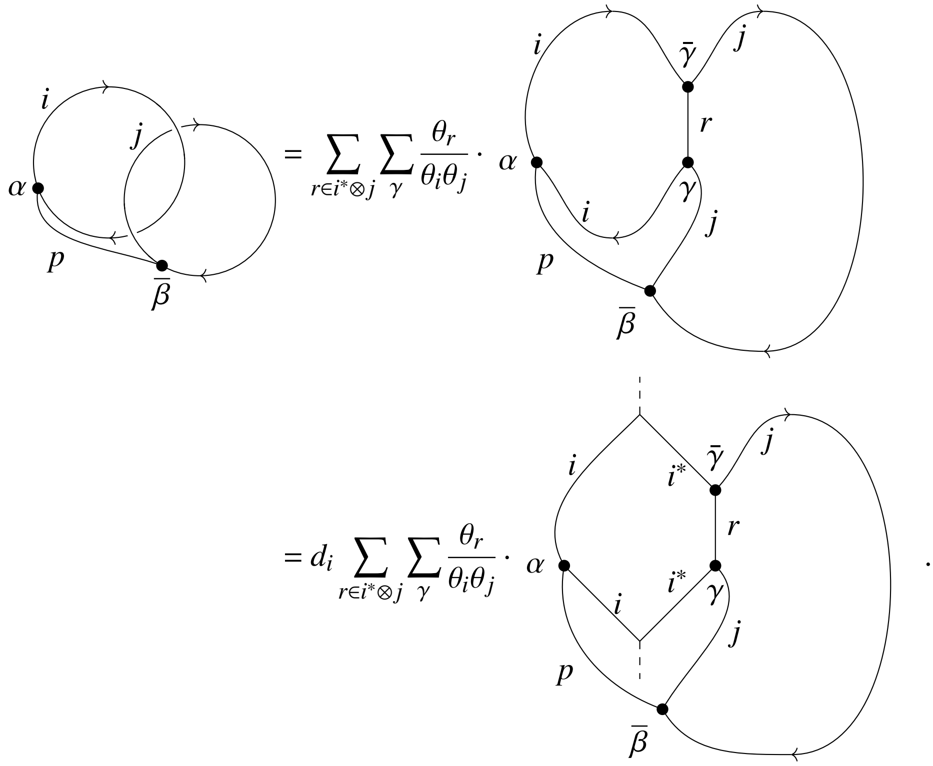
(12.11)

We can then take the trace over j to get a morphism in $\hom_{\mathscr{C}}(0,0) = \mathbb{C}\,\mathrm{id}_0$, that is, we connect the j-strands at the top and bottom of the diagram using evaluation and co-evaluation. The left-hand side of the identity is then just a circle labelled by j and this evaluates to d_j. The right-hand side then becomes string diagram in (12.8). To evaluate the string diagram we need the well known identities



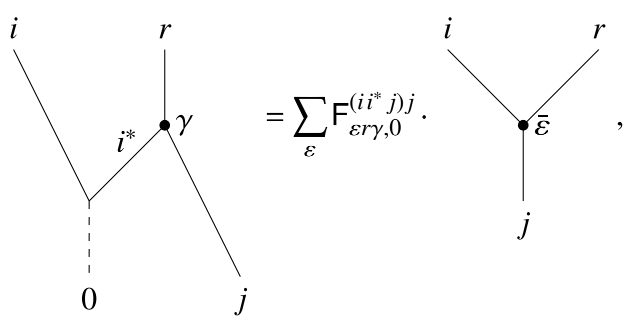
(12.12)

where the grey boxes in the left identity can contain any diagram and where we have also used that $\theta_{i^*} = \theta_i$. Then

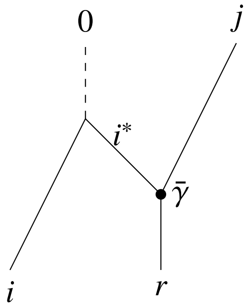


(12.13)

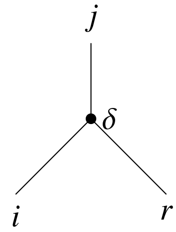
Applying the well known identities



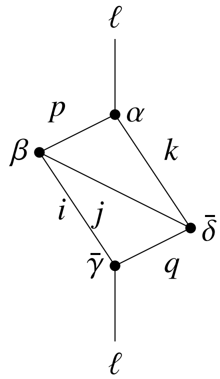
(12.14)



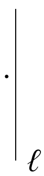
$$= \sum_{\delta} \mathsf{G}^{(i i^* j) j}_{0,\delta r \gamma} \cdot \qquad (12.15)$$



and



$$= \mathsf{G}^{(i j k)\ell}_{\alpha p \beta, \gamma q \delta} \cdot \qquad (12.16)$$



to the last diagram in (12.13) and again using the fact that the circle labelled by j evaluates to d_j yields the formula in (12.8). ■

12.1. **The $\mathfrak{sl}_2$ example.** Explicit formulae for the Moore-Seiberg data of the modular tensor category for affine $\mathfrak{sl}_2$ at level $k \in \mathbb{N}_0$ are known and we reproduce them here to compute some examples. We take the label set of simple modules to be their highest weights $\mathcal{I} = \{0, \ldots, k\}$ with respective conformal weights $h_n = \frac{n(n+2)}{4(k+2)}$, $n \in \mathcal{I}$ and central charge $\mathbf{c} = \frac{3k}{k+2}$. Since for $\mathfrak{sl}_2$ the 3-point coupling spaces are always at most 1-dimensional, no labels are needed for basis vectors (the Greek indices above). The character S-matrix entries are given by

$$\mathsf{S}_{i,j} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi(i+1)(j+1)}{k+2}\right), \quad D = \mathsf{S}_{0,0}^{-1}, \quad d_i = \frac{\mathsf{S}_{i,0}}{\mathsf{S}_{0,0}}. \qquad (12.17)$$

The twist, ζ, braiding and fusing matrices are, respectively, given by

$$\theta_r = \mathbf{e}(h_r), \quad \zeta = \mathbf{e}\left(\frac{\mathbf{c}}{24}\right),$$
$$\mathsf{R}^{(r s)t} = (-1)^{r+s-t} \mathbf{e}\left(\frac{h_r + h_s - h_t}{2}\right), \quad \mathsf{F}^{(r s t)u}_{pq} = \begin{Bmatrix} t/2 & s/2 & p/2 \\ r/2 & u/2 & q/2 \end{Bmatrix}, \qquad (12.18)$$

where

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{a+b-c-d-2e} \sqrt{[2e+1][2f+1]}\Delta(a,b,e)\Delta(a,c,f)\Delta(c,e,d)\Delta(d,b,f)$$

$$\times \sum_{z=\min\{a+b+c+d,a+d+e+f,b+c+e+f\}}^{\max\{a+b+e,a+c+f,b+d+f,c+d+e\}} (-1)^z[z+1]!\big([z-a-b-e]!\left[z-a-c-f\right]!\left[z-b-d-f\right]![z-d-c-e]!$$

$$[a+b+c+d-z]!\left[a+d+e+f-z\right]!\left[b+c+e+f-z\right]!\big)^{-1} \tag{12.19}$$

are quantum group $6j$-symbols and

$$\Delta(a,b,c) = \sqrt{\frac{[-a+b+c]![a-b+c]![a+b-c]!}{[a+b+c+1]!}},$$

$$[n] = \frac{\sin\left(\frac{\pi n}{k+2}\right)}{\sin\left(\frac{\pi}{k+2}\right)}, \qquad [n]! = \prod_{m=1}^{n}[m], \qquad [0]! = 1. \tag{12.20}$$

The above formulae can be found in [72, 73]. Note that the G can be computed from the above data using the identity

$$\mathsf{G}_{pq}^{(ijk)\ell} = \frac{\mathsf{R}^{(jk)q}\mathsf{R}^{(iq)\ell}}{\mathsf{R}^{(ij)p}\mathsf{R}^{(pk)\ell}}\mathsf{F}_{pq}^{(kji)\ell}. \tag{12.21}$$

Consider now the example when the label p of the acting object is equal to k. Then we are in the 1-dimensional case with S and T given by

$$\mathsf{S}^{(k)} = \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\left(h_k+\frac{k}{2}\right)} = \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\frac{3k}{8}}, \qquad \mathsf{T}^{(k)} = \mathrm{e}^{\frac{\pi}{4}\mathrm{i}\frac{k}{2}}. \tag{12.22}$$

In particular, $\mathsf{S}^{(k)}$ and $\mathsf{T}^{(k)}$ are equal to the evaluation of the multiplier system $\nu_{h_k+\frac{k}{2}}$, so the categorical data appear to detect that the natural vector u to use for torus 1-point functions has conformal weight $h_k+\frac{k}{2}$, as we have seen in the sections above. If we divide the above formulae by the ν_{h_k} multiplier system then we recover the formulae in Theorem 11.4.

The above formulae for S and T can also be used to show that Theorem 11.7 admits examples of non-congruent representations of dimension greater than three.

Proposition 12.4. *Choose $p=7$, $t=1$ in Theorem 11.7 and hence $k=5$. Then for $\lambda=2$, the representation ρ_2 is 4-dimensional, irreducible and non-congruence.*

Proof. All of the conditions of Theorem 11.7 except for irreducibility hold by construction. So we only need to show irreducibility. Note that the existence or absence of a non-trivial invariant subspace does not depend on whether a multiplier system is included in the formulae for S or T. Therefore, we work directly with the formulae in Theorem 12.3. Note in particular that all of the eigenvalues of

$$\mathsf{T}^{(2)} = \mathrm{diag}\left(\mathbf{e}\left(\frac{1}{56}\right), \mathbf{e}\left(\frac{11}{56}\right), \mathbf{e}\left(\frac{25}{56}\right), \mathbf{e}\left(\frac{43}{56}\right)\right) \tag{12.23}$$

are distinct. Therefore a non-trivial invariant subspace would need to admit a basis B that is a proper non-empty subset of $\mathsf{T}^{(2)}$-eigenvectors. The columns in $\mathsf{S}^{(2)}$ corresponding to these basis vectors would hence need to contain entries that are 0 in those rows which correspond to $\mathsf{T}^{(2)}$-eigenvectors not in B. However,

$$\mathsf{S}^{(2)} \approx \begin{pmatrix} -0.16-0.33\mathfrak{i} & -0.26-0.55\mathfrak{i} & -0.26-0.55\mathfrak{i} & -0.16-0.33\mathfrak{i} \\ -0.26-0.55\mathfrak{i} & -0.16-0.33\mathfrak{i} & 0.16+0.33\mathfrak{i} & 0.26+0.55\mathfrak{i} \\ -0.26-0.55\mathfrak{i} & 0.16+0.33\mathfrak{i} & 0.16+0.33\mathfrak{i} & -0.26-0.55\mathfrak{i} \\ -0.16-0.33\mathfrak{i} & 0.26+0.55\mathfrak{i} & -0.26-0.55\mathfrak{i} & 0.16+0.33\mathfrak{i} \end{pmatrix} \tag{12.24}$$

has no entries that are 0 and hence the representation must be irreducible. Here we have chosen to give a numerical approximation of $\mathsf{S}^{(2)}$ to two significant digits for simplicity, as the exact expression in terms of radicals is impractically large to present. ∎

Note that $\nu_{h_2}(\mathsf{T}) = \mathbf{e}(\frac{1}{42})$ and so if we divide the diagonal entries in (12.23) by ν_{h_2} and take the product of the first two diagonal entries, we obtain $\mathbf{e}(\frac{1}{6})$, which is a 12th root of unity. Hence Lemma 11.3 does not apply and we were only able to conclude irreducibility because of the categorical formulae.

Appendix A. Modular matrices from categorical data

We document the Sage code used to extract the modular S and T matrices from categorical data, as in Section 12. Sage documentation may be found here [74].

We begin by setting up the conformal weight based on the $\mathfrak{sl}_2$ weight n and level k as well as the central charge:

```
weight(n,k)=n*(n+2)/(4*k+8)
c(k)=k*3/(k+2)
```

The quantum numbers denoted $[n]$ in Equation (12.20) are specified, being dependent also on the level,

```
qnum(n,k)=sin(pi*n/(k+2))/sin(pi/(k+2))
```

as well as the vacuum S-matrix and the R-matrices in Equation (12.18) respectively:

```
Smat(i,j,k)=sqrt(2/(k+2))*sin(pi*(i+1)*(j+1)/(k+2))
Rmat(r,s,t,k)=(-1)^((r+s-t)/2)*exp(-I*pi*(weight(t,k)-weight(r,k)
                                    -weight(s,k)))
```

We define a function that computes a list of simples in the fusion product of two simples, labelled by their integer indices `i,j`:

```
def fuse(i,j,k):
  return range(abs(i-j),min(i+j,2*k-i-j)+1,2)
```

The values of $\Delta(a,b,c)$ in Equation (12.20) necessitate a helper function `h(a,b,c)` and the $6j$-symbols `sixj(a,b,c,d,e,f,k)` as in Equation (12.19) are also defined, dependent on a quantum factorial `qfac(n,k)`:

```
def qfac(n,k):
  return prod([qnum(i,k) for i in range(1,n+1)])
def h(a,b,c,k):
  return sqrt(qfac(-a+b+c,k)*qfac(a-b+c,k)*qfac(a+b-c,k)/qfac(a+b
                                    +c+1,k))
def sixj(a,b,c,d,e,f,k):
```

```
    return ((-1)^(a+b-c-d-2*e))*sqrt(qnum(2*e+1,k)*qnum(2*f+1,k))*h
                              (a,b,e,k)*h(a,c,f,k)*h(c,e,d,k)*h
                              (d,b,f,k)*sum([(qfac(z+1,k)*(-1)^
                              z)/(qfac(z-a-b-e,k)*qfac(z-a-c-f,
                              k)*qfac(z-b-d-f,k)*qfac(z-d-c-e,k
                              )*qfac(a+b+c+d-z,k)*qfac(a+d+e+f-
                              z,k)*qfac(b+c+e+f-z,k)) for z in
                              range(max(a+b+e, a+c+f, b+d+f, c+
                              d+e), min(a+b+c+d, a+d+e+f, b+c+e
                              +f) + 1)])
```

Finally, we require the F-matrix in Equation (12.18) and as well as its inverse `G(r,s,t,u,p,q,k)` in Equation (12.21):

```
def F(r,s,t,u,p,q,k):
    return sixj(t/2,s/2,u/2,r/2,p/2,q/2,k)
def G(r,s,t,u,p,q,k):
    return Rmat(s,t,q,k)*Rmat(r,q,u,k)*F(t,s,r,u,p,q,k)/(Rmat(r,s,p
                              ,k)*Rmat(p,t,u,k))
```

We are now in a position to define the S-matrix entries which accepts as input the indices of the simples and the acting module p:

```
def S(i,j,p,k):
    return Smat(0,0,k)*sum([qnum(i+1,k)*qnum(j+1,k)*exp(-2*pi*I*(
                              weight(i,k)+weight(j,k)-weight(r,
                              k)))*F(i,i,j,j,r,0,k)*G(i,i,j,j,0
                              ,r,k)*G(p,i,r,j,i,j,k) for r in
                              fuse(i,j,k)])
```

To compute the S and T-matrices, we require setting a choice of `level` and acting module `acting_label`, as well as a helper function `admlabels(p,k)` giving the modules on which p acts:

```
def admlabels(p,k):
    return [i for i in range(k+1) if i in fuse(p,i,k)]
```

Computing the matrices follows:

```
Smat=Matrix([[simplify(S(i,j,acting_label,level)) for j in
                                    admlabels(acting_label,level)]
                                    for i in admlabels(acting_label,
                                    level)])
Tmat=diagonal_matrix([exp(2*pi*I*(weight(i,level)-c(level)/24))
                                    for i in admlabels(acting_label,
                                    level)])
```

APPENDIX B. COMPUTING q-SERIES

We document the Sage code used to compute the explicit q-series of the vector-valued modular forms in Section 11.2 and Section 11.3, that is, the dimension two and three cases respectively, when the vector-valued modular form obtained generates the entire space for the representation in question. Sage documentation may be found here [74].

We begin by defining a helper function `modone(ratnum)` calculating the smallest positive representation mod 1, `fuse(i,j,k)` which returns a list of simples in the fusion product of two simples `i,j` at level `k`, and `act_on_labels(p,k)` which returns the modules on which `p` acts:

```
def modone(ratnum):
  return (ratnum.numerator() % ratnum.denominator())/ratnum.
                                          denominator()
def fuse(i,j,k):
  return range(abs(i-j),min(i+j,2*k-i-j)+1,2)
def act_on_labels(p,k):
  return [i for i in range(k+1) if i in fuse(p,i,k)]
```

We define the central charge `c(k)`, the L_0 weight `h(m,k)` of the $\mathfrak{sl}_2$ module as a function of $\mathfrak{sl}_2$ weight `m`, the cusp parameter `cuspparam(p,k)` and define `q` as a symbolic variable for expansions:

```
c(k) = 3*k/(k+2)
h(m, k) = m*(m+2)/(4*(k+2))
def cuspparam(p,k):
  return 12*modone(h(p,k)/12)
var('q')
```

These details can be found in Section 5.2 and Section 8. We now compute the value of the j-function up to 11 terms in the q-expansion, as well as $j^{1/24}$, $j^{-5/24}$ and J^{-1} using Taylor series about $q = 0$ which will be needed, according to Equation (11.5). (Higher order terms can be computed if needed to go beyond what is provided in the thesis.)

```
jqexp = j_invariant_qexp(11).truncate(11)
j124 = (jqexp^(1/24)).taylor(q,0,20)/(1728^(1/24))
j524 = (jqexp^(-5/24)).taylor(q,0,10)/(1728^(-5/24))
jinv = 1728*(jqexp^(-1)).taylor(q,0,10)
```

Now the acting label has to be chosen from a set `level` from which `actlbl` is computed according to Theorem 11.5. Then we can find the pair `adms` of leading powers of q and k_0.

```
actlbl = level-1
adms = [modone(QQ((h(q, level) - c(level)/24))) for q in
                                act_on_labels(actlbl,level)]
k0 = 6*sum(adms) - 1
```

We now compute an expansion of η^{2k_0} and the two relevant hypergeometric functions:

```
etak0 = ((q^(1/24)*qexp_eta(ZZ[['q']], 20).polynomial())^(2*k0)).
                                taylor(q,0,10)
hypergeom1 = hypergeometric([-1/24, 7/24],[3/4],x).series(x,10).
                                truncate().substitute(x==jinv)
hypergeom2 = hypergeometric([5/24, 13/24],[5/4],x).series(x,10).
                                truncate().substitute(x==jinv)
```

The two components of the vector-valued modular form are then computed,

```
modularform1 = expand((etak0*j124*hypergeom1).full_simplify())
modularform2 = expand((etak0*j524*hypergeom2))
```

and the q-expansion can be displayed without the leading q-factors with fractional exponents:

```
mf1lead = ((q^(-adms[0])*modularform1).expand()).coefficient(q,0)
expand(q^(-adms[0])*modularform1/mf1lead)
mf2lead = ((q^(-adms[1])*modularform2).expand()).coefficient(q,0)
expand(q^(-adms[1])*modularform2/mf2lead)
```

A similar code suffices for the dimension three case. Instead now the acting label must be according to Theorem 11.6:

```
actlbl = level-2
```

Now we require $j^{\frac{k+1)}{12(k+2)}}, j^{-\frac{k+1}{6(k+2)}}, j^{-\frac{5k+11}{12(k+2)}}$ and j^{-1}, based on `jexp` as before:

```
jk1exp = (level+1)/(12*(level+2))
jk1 = (jqexp^(jk1exp)).taylor(q,0,20)/(1728^(jk1exp))
jk2exp = -(level+1)/(6*(level+2))
jk2 = (jqexp^(jk2exp)).taylor(q,0,20)/(1728^(jk2exp))
jk3exp = -(5*level+11)/(12*(level+2))
jk3 = (jqexp^(jk3exp)).taylor(q,0,20)/(1728^(jk3exp))
```

Now the triple of leading exponents is given by the following function, as well the new value for k_0,

```
adms = [modone(QQ((h(q, level) - c(level)/24))) for q in
                                act_on_labels(actlbl,level)]
k0 = 4*sum(adms) - 2
```

Lastly, we require the new hypergeometric functions:

```
hypergeom1 = hypergeometric([-(level+1)/(12*(level+2)), (11*level
                                +14)/(24*(level+2)),(19*level+30)
                                /(24*(level+2))],[(3*level+7)/(4*
                                (level+2)),1/2],x).series(x,10).
                                truncate().substitute(x==jinv)
hypergeom2 = hypergeometric([(level+1)/(6*(level+2)), (3*level+5)
                                /(6*(level+2)),(5*level+9)/(6*(
                                level+2))],[(5*level+9)/(4*(level
                                +2)),5/8],x).series(x,10).
                                truncate().substitute(x==jinv)
hypergeom3 = hypergeometric([(5*level+11)/(12*(level+2)), (9*
                                level+19)/(12*(level+2)),(13*
                                level+27)/(12*(level+2))],[3/2,(5
                                *level+11)/(4*(level+2))],x).
                                series(x,10).truncate().
                                substitute(x==jinv)
```

The components of the vector-valued modular form are then readily computed:

```
modularform1 = expand((etak0*jk1*hypergeom1).full_simplify())
modularform2 = expand((etak0*jk2*hypergeom2).full_simplify())
modularform3 = expand((etak0*jk3*hypergeom3).full_simplify())
```

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