Detecting and Classifying Algebra Objects in Fusion Categories

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Summary

Algebra objects are categorical structures that appear in a wide range constructions and classification problems throughout category theory and mathematical physics. However, they can be difficult to find and describe explicitly so it is of great importance to develop methods of detecting and classifying them. The aim of this work is to develop such methods in the setting of fusion categories, which can be viewed as a categorical generalisation of a ring. We explore two approaches to this problem.

We shall describe how different types of monoidal functors can be used to preserve algebraic structures, and shall construct a specific functor with the aim to classify algebra objects in the monoidal center of the category of group-graded vector spaces, $\mathcal{Z}(\mathsf{Vect}_G^\omega)$. We shall use an explicit description of this category in terms of Yetter-Drinfeld modules over the group Hopf algebra, and explore which algebraic structures can be preserved using the constructed functor. We shall classify a class of Frobenius algebras in terms of a choice of cohomological data.

We also look at an alternate approach, which works not with an explicit category but with a fusion ring that can be extracted from the data of a fusion category. The representation theory of these rings will be studied using Non-negative Integer Matrix representations (NIM-reps), and we will describe how NIM-reps can be constructed from algebra objects. We shall look at this relationship in detail, providing a method of using NIM-reps to detect potential algebra structures. We will demonstrate how this technique works by classifying the NIM-reps of 3 families of fusion rings, providing a list of potential algebra objects and a platform to develop this technique further.

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Notation

Here we shall fix some notation that shall be used throughout this thesis.

- $\mathbb{k}$ – an algebraically closed field of arbitrary characteristic,
- $1_{\mathbb{k}}$ – multiplicative identity of the field $\mathbb{k}$,
- $\mathsf{dim}_{\mathbb{k}}(U)$ – the dimension of a $\mathbb{k}$-vector space U over its base field,
- $\mathbb{Z}_{+}$– the semi-ring of positive integers with zero,
- G – a finite group,
- e– the group identity,
- $\{e\}$ – the trivial group,
- $\mathsf{Ob}(\mathcal{C})$ – objects of a category $\mathcal{C}$,
- $\mathsf{Hom}_{\mathcal{C}}(X, Y)$ – collection of morphisms between objects $X, Y \in \mathsf{Ob}(\mathcal{C})$.

Chapter 1

Introduction

The focus of this work is to develop methods of detecting and classifying algebra objects, which are a generalisation of associative unital algebras over a field $\Bbbk$ to the more general setting of category theory. These objects are useful in a wide range of mathematical areas, such as abstract algebra and representation theory, and for providing new examples of categories. They also appear in many constructions in topological and conformal fields theories (TFT, CFT respectively). We shall primarily work with fusion categories, a particular type of category endowed with a variety of structures such as a direct sum and tensor product, which can be viewed as an abstraction of a ring.

Following the techniques used to study $\Bbbk$-algebras, we can define modules of an algebra A in some monoidal category $\mathcal{C}$ to be pairs (M, ρ) consisting of an object in $\mathcal{C}$ and an action morphism. By also taking all morphisms in $\mathcal{C}$ that are compatible with these module actions, we can construct the category of A-modules, $\mathsf{Mod}_{\mathcal{C}}(A)$. If $\mathcal{C}$ is braided monoidal, a similar category consisting of local modules, $\mathsf{Mod}_{\mathcal{C}}^{\mathsf{loc}}(A)$, can be constructed. These constructions give us new categories which can possess desirable properties, depending on any additional structure the algebra may have, for example:

Proposition ([EGNO15, Proposition 7.8.30])**.** *Let A be a separable algebra in a fusion category $\mathcal{C}$. Then $\mathsf{Mod}_{\mathcal{C}}(A)$ is a semisimple.*

Theorem ([KO02, Theorem 4.5], [LW23, Theorem 4.12])**.** *If $\mathcal{C}$ is a modular tensor category and A is a rigid Frobenius algebra in $\mathcal{C}$, then the category $\mathsf{Mod}_{\mathcal{C}}^{\mathsf{loc}}(A)$ of local modules over A in $\mathcal{C}$ is also modular.*

As well as being a rich source of new categories, we can also use these constructions to study the representation theory of fusion categories themselves. This is analogous to the ring theoretic setting, with a $\mathcal{C}$-module category being another category equipped with some functorial action on $\mathcal{C}$. It has been shown in [EGNO15, Section 7.10], (see also [Ost03a, Theorem 1], [BW85, Chapter 3]), that if $\mathcal{C}$ is a fusion category, then any $\mathcal{C}$-module category is equivalent to the category $\mathsf{Mod}_{\mathcal{C}}(A)$, where A is some suitably chosen algebra object. This result is explicitly stated in Theorem 5.2.14, and is a major motivation for the study of algebra objects as it gives us the tools to study these categories that are only dependent on objects internal to the category itself.

Some further motivation for finding algebra objects lies in the aforementioned areas of mathematical physics. A TFT, as described in [Ati88], consists of a symmetric monoidal functor from a category of cobordisms to a symmetric monoidal category, such as the category of vector spaces over $\Bbbk$. The category of 2-dimensional oriented cobordisms, written as 2Cob, has objects as 1 dimensional closed oriented manifolds and morphisms as 2-dimensional oriented manifolds whose "in-boundary" is the source object and whose "out-boundary" is the target object. This category is generated by a single object, the circle S^1, with morphisms being generated by six generating cobordisms (see Figure 1.1) that satisfy a number of conditions, where composition is done by concatenation.

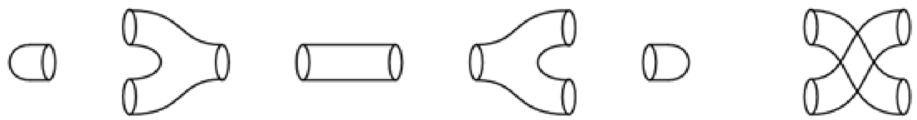


Figure 1.1: Generating cobordisms for 2Cob, read from left to right

If we take this to be the source category of a TFT, then the conditions on the cobordisms give the image of S^1 the structure of a Frobenius algebra. This leads to the following classification:

Theorem ([Koc03])**.** *There is an equivalence between the category of 2D TFTs and the category of commutative Frobenius algebras in* $\mathsf{Vect}_{\Bbbk}$.

Results for algebraic structures and invariants in higher dimensional and extended TFTs are plentiful, a selection of which can be found, for example, in [RT91, Hen96, DR21, DRGG+22, BGR21, Meu23, SP11].

In rational CFT, modular fusion categories appear as categories of representations over the underlying vertex operator algebras [Hua08]. Algebra objects can be used in a variety of constructions in this setting - for example, possible extensions of a VOA are in a one-to-one correspondence with commutative algebras in its category of representations [CKM24, HKL15]. A second construction describes boundary conditions of a rational CFT using modules over Frobenius algebras internal to the representation category associated to certain VOAs [FFRS06, FRS02].

Hence these algebra objects are of great value, but finding them can be difficult in practice. One of the main existing results comes in the setting of the pointed fusion categories Vect_G^ω. They consist of G-graded vector spaces and grading-preserving linear maps, and have a tensor product whose associativity is controlled by a 3-cocycle $\omega \in C^3(G, \Bbbk^\times)$. Given a subgroup $N \subseteq G$ and a 2-cocycle $\kappa \in C^2(N, \Bbbk^\times)$, such that $d\kappa = \omega|_N$, one may construct a twisted group algebra $(\Bbbk[N], \kappa)$. These algebras have the rich structure of a connected special Frobenius algebra, and are all that is needed to describe a whole class of algebras in Vect_G^ω.

Theorem A ([Ost03b][Nat17])**.** Every connected separable algebra in Vect_G^ω is Morita equivalent to some choice of twisted group algebra $(\Bbbk[N], \kappa)$.

Other works classifying algebra objects and module categories include [MMP+23] for group-theoretical fusion categories, [Gal12, MM12] for so-called near-group categories, and recently [Kik23] for connected étale algebras in low-rank modular fusion categories. The work done during this PhD contributes to these works by extending current results and exploring new potential methods of finding algebras. Two papers have been published as a result of this work: [HLRC23] and [HR24]. The structure of this thesis is as follows;

In Chapter 2, we review the basic constructions of category theory that will be used in this thesis, moving from monoidal categories to fusion categories with the category of G-graded vector spaces, Vect_G^ω, being used as a guiding example.

In Chapter 3, we provide an introduction to the main topic of this thesis, algebra objects. This includes special types such as Frobenius and Hopf algebras. We shall construct their categories of modules and describe how we can preserve the algebraic structures through different types of functors. Important examples are discussed, and an explicit description of the monoidal center of the category of G-graded vector spaces, $\mathcal{Z}(\mathsf{Vect}_G^\omega)$, is given using Yetter-Drinfeld modules.

In Chapter 4, we cover the results from [HLRC23], which is joint work with Robert Laugwitz and Ana Ros Camacho. In this work we extend the classification of Theorem A to the monoidal center $\mathcal{Z}(\mathsf{Vect}_G^\omega)$. By lifting this classification, we are able to talk additionally about commutative algebras. The case that $\Bbbk$ has characteristic 0 was covered in [DS17], and our work extends this to a field of arbitrary characteristic.

We first provide a definition of a lifted twisted group algebra, which allows us to view $(\Bbbk[N], \kappa)$ as an algebra object in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$ by adding a Yetter-Drinfeld action, governed by a function $\epsilon : G \times N \to \Bbbk^\times$. This algebra is denoted by $B(N, \kappa, \epsilon)$ and is a commutative Frobenius algebra that provides a partial classification of algebra objects.

Proposition (See Proposition 4.2.6). *Let B be a separable commutative algebra in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$ such that $B_e = \Bbbk$, where $\Bbbk$ is the underlying field. Then B is isomorphic as an algebra in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$ to $B(N, \kappa, \epsilon)$ for $N = \{g \in G \mid B_g \neq 0\}$.*

To improve this classification, it would be beneficial to remove the condition on the trivially-graded component. This is done using a Frobenius monoidal functor, which is a functor functor $F : \mathcal{C} \to \mathcal{D}$ of monoidal categories that satisfies certain compatibility conditions, with respect to the monoidal structures of $\mathcal{C}$ and $\mathcal{D}$. These functors are particularly powerful as they send Frobenius algebras in $\mathcal{C}$ to ones in $\mathcal{D}$.

Theorem (See Propositions 4.3.5 and 4.3.6). *There is a braided separable Frobenius monoidal functor*

$$I : \mathcal{Z}(\mathsf{Vect}_H^{\omega|_H}) \to \mathcal{Z}(\mathsf{Vect}_G^\omega)$$

that is compatible with the braiding structure.

Using this functor, we can transport the twisted group algebras $B(N, \kappa, \epsilon)$ in $\mathcal{Z}(\mathsf{Vect}_H^{\omega|_H})$ to a commutative Frobenius algebra $I(B(N, \kappa, \epsilon)) := A(H, N, \kappa, \epsilon)$ in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$. This allows us to provide a full classification of connected, commutative separable algebras in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$, completely lifting the results of [Ost03b],[Nat17] to the monoidal center.

Theorem (See Theorem 4.4.3). *Let G be a finite group with $\omega \in C^3(G, \Bbbk^\times)$, a subgroup H of G and a tuple (N, κ, ε) defining the twisted group algebra $B(N, \kappa, \epsilon)$ in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$.*

(a) If $|N| \cdot |G : H| \in \Bbbk^\times$, then the algebra $A(H, N, \kappa, \epsilon)$ is a connected commutative special Frobenius algebra.

(b) Every connected separable commutative algebra in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$ is of the form $A(H, N, \kappa, \epsilon)$, for some choice of data H, N, κ, ϵ.

In Chapter 5, we cover results from [HR24], based on work with Ana Ros Camacho. In this project, we explored methods of detecting algebra objects without using an explicit description of the category in question, as was the case with $\mathcal{Z}(\mathsf{Vect}_G^\omega)$. This involves stripping away much of the categorical data of a fusion category until we are are left with the underlying ring.

For a fusion category $\mathcal{C}$, we can construct the Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ using isomorphism classes of simple objects as a basis where the ring addition and multiplication are induced from the categorical structure. These are a special type of ring called fusion rings, and can be studied by considering Non-negative Integer Matrix representations (NIM-reps) over them. They are of interest to us as every semisimple $\mathcal{C}$-module category, and therefore every separable algebra via $\mathsf{Mod}_{\mathcal{C}}(A)$, gives rise to a NIM-rep over $\mathrm{Gr}(\mathcal{C})$.

The converse direction is not true however, as there is no guarantee every NIM-rep can be categorified to a $\mathcal{C}$-module category. This means we need an extra condition to check whether a NIM-rep could come from an algebra in $\mathcal{C}$. We introduce this criterion in Section 5.2, and say that the NIM-rep is admissible if it is satisfied.

In Section 5.3, we compute the NIM-reps for 3 families of fusion rings. First, we consider the group rings $R(G)$, which are the underlying Grothendieck rings of Vect_G^ω. We verify that their NIM-reps are parameterised by conjugacy classes of subgroups $H \subseteq G$ [DFZ90, EK95], and compute that all admissible algebras have the object structure of a group algebra, which directly matches the classification in Theorem A.

The second type of fusion ring we look at are the near-group fusion rings $K(G, \alpha)$, where $\alpha \in \mathbb{Z}_+$. The basis of this ring is the group G plus one additional, non-invertible object X. The multiplication of the group elements is simply given by the group operation, and with the non-invertible object X by

$$Xg = gX = X, \quad X^2 = \sum_{g \in G} g + \alpha X.$$

Categories whose Grothendieck ring is of this form are called near-group categories, and results on when such categorifications occur can be found in [TY98, Ost15, ENO10].

For some NIM-rep over $K(G, \alpha)$, the action over the group part is exactly the same as the group ring case, and is given by a family of subgroups $\{H_i\}_{i \in I}$. We can write the action of X in terms of a matrix $\mathbf{X}$ acting on the basis elements of the NIM-rep, which is subject to the matrix equation

$$\mathbf{XBX} = \alpha \cdot \mathbf{X} + |G| \cdot \mathbf{B}^{-1},$$

where $\mathbf{B} = \mathrm{diag}(\{|G : H_i|\}_{i \in I})$. Solutions to this matrix equation that satisfy the additional NIM-rep conditions are in general difficult to find, as there is no restriction on the order of the matrices involved. The order of the matrices corresponds to the number of subgroups $\{H_i\}$ governing the group part of the action. We provide a classification of NIM-reps when this is either 1 or 2 subgroups.

Proposition (See Proposition 5.3.9)**.** *NIM-reps over $K(G,\alpha)$ consisting of one group orbit are parameterised completely by pairs $(H, x_{1,1})$, containing a subgroup $H \subseteq G$ and a non-negative integer $x_{1,1} \in \mathbb{Z}_2$, such that $\alpha = x_{1,1}|G:H| - \frac{|H|}{x_{1,1}}$, $x_{1,1}$ divides $|H|$, and $(x_{1,1})^2|G:H| \geqslant |H|$.*

Proposition (See Proposition 5.3.10)**.** *All indecomposable NIM-reps over $K(G,\alpha)$ consisting of two group orbits are parameterised completely by tuples $(H_1, H_2, x_{1,1}, x_{2,2})$, containing two subgroups $H_1, H_2 \subseteq G$ and two non-negative integers $x_{1,1}, x_{2,2} \in \mathbb{Z}_2$, such that $\alpha = x_{1,1}|G : H_1| + x_{2,2}|G : H_2|$, $|G|$ divides $|H_1||H_2|$, and $(\frac{|H_1||H_2|}{|G|} + x_{1,1}x_{2,2})$ is a square number.*

For the purpose of detecting admissible algebra structures in near-group categories, we do not need to delve any further.

Proposition (See Proposition 5.3.14 and Corollary 5.3.15)**.** *A connected separable algebra in $K(G,\alpha)$ has one of two following forms;*

- *For a NIM-rep parameterised by $(H, c_{1,1})$, the corresponding algebra is given by $\oplus_{h\in H} b_h \oplus c_{1,1}X$ as an object.*
- *For a NIM-rep parameterised by $(\{e\}, G, 0, \alpha)$, the corresponding algebra object is simply $[1]$, the form of the monoidal unit $\mathbb{1}$ algebra.*

This result extends our understanding of algebra objects in near-group categories, building upon work in [Gal12] and [MM12], but also highlights possible restrictions with this method. There are seemingly no group algebra objects, other than the trivial case, appearing in near-group categories, which is an unexpected result. This is likely to be a result of us removing all morphism data away to get to the fusion ring, which we discuss in Remark 5.3.16.

The third fusion ring we look at comes from a modular tensor category (A_1, l), which can be constructed out of a quantum group of type A_1 at level $l \in \mathbb{Z}_+$, following [NWZ22]. One can take its full subcategory $(A_1, l)_{\frac{1}{2}}$, and we compute that there is only a single NIM-rep over the fusion ring $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$. This recovers results outlined in [EK95, Ost03a] for algebra objects in $(A_1, l)_{\frac{1}{2}}$.

We discuss ways to continue both branches of this research in Chapter 6. Appendix A contains an introduction to the cohomology of groups, which can be used to describe various constructions appearing in Chapter 4. We also include a number of proofs missing throughout this chapter. In Appendix B we collect some basic definitions and results relating to group actions, which are used in Chapter 5.

Chapter 2

Categories

Throughout this thesis, we will be working in the framework of tensor and fusion categories. In this chapter, we shall introduce the necessary background of these areas, assuming only basic prior knowledge of categories. The main references used are [EGNO15, Lan13, TV17]

2.1 Basics to Fusion Categories

To begin, we introduce the conditions required for a category to be *abelian* category, following [Lan13].

Definition 2.1.1. A category $\mathcal{C}$ is called *additive* if

- For any objects X, Y in $\mathsf{Ob}(\mathcal{C})$, the morphism collection $\mathsf{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group, and the composition of morphisms is bilinear with respect to the group operation,
- There is a zero object $0 \in \mathsf{Ob}(\mathcal{C})$,
- For any two objects $X, Y \in \mathsf{Ob}(\mathcal{C})$, there is a distinguished object that is both a direct sum and product, denoted by $X \oplus Y$.

Definition 2.1.2. Let $\mathcal{C}$ be an additive category. It is called *abelian* if

- Every morphism has a kernel and a cokernel,
- Every monomorphism occurs as the kernel of a morphism, and every epimorphism occurs as the cokernel of a morphism.

Definition 2.1.3. Let $\mathcal{C}$ be an abelian category. A non-zero object $X \in \mathsf{Ob}(\mathcal{C})$ is called *simple* if its only subobjects are 0 and X. An object is called *semisimple* if it is isomorphic to a direct sum of simple objects, and $\mathcal{C}$ is said to be semisimple if all objects have this property.

We can further say that an abelian category is $\Bbbk$-*linear* if the morphisms groups are in fact $\Bbbk$-vector spaces, and composition of morphisms is $\Bbbk$-linear.

Definition 2.1.4. A $\Bbbk$-linear abelian category $\mathcal{C}$ is *locally finite* if, for any two objects $X, Y \in \mathsf{Ob}(\mathcal{C})$, $\mathsf{Hom}_{\mathcal{C}}(X, Y)$ is a finite-dimensional $\Bbbk$-vector space and every object has a finite filtration by simple objects. Further, $\mathcal{C}$ is *finite* if, in addition, there are finitely many isomorphism classes of simple objects.

Before we continue expanding our categorical structures, it is useful to define the following notion.

Definition 2.1.5. Let $\mathcal{C}$ be a $\Bbbk$-linear, locally finite abelian category. The *Grothendieck group* $\mathrm{Gr}(\mathcal{C})$ of $\mathcal{C}$ is the free abelian group generated by isomorphism classes X_i of simple objects in $\mathcal{C}$. To each object $X \in \mathsf{Ob}(\mathcal{C})$, we canonically associate the class $[X] \in \mathrm{Gr}(\mathcal{C})$ given by

$$[X] = \sum_i [X : X_i] X_i,$$

where $[X : X_i]$ is the multiplicity of X_i in a Jordan-Hölder series of X.

We now look at how to introduce products into a categorical setting, through the structure of a monoidal category.

Definition 2.1.6. A *monoidal category* $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ is a tuple that consists of the following;

- A category $\mathcal{C}$,
- A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the *tensor product*,
- A distinguished unit object $\mathbb{1} \in \mathsf{Ob}(\mathcal{C})$,
- Three families of natural isomorphisms;
  - Associators $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$,
  - Left unitors $l_X : \mathbb{1} \otimes X \to X$,
  - Right unitors $r_X : X \otimes \mathbb{1} \to X$.

This data must be such that, for all objects $X, Y, Z, W \in \mathsf{Ob}(\mathcal{C})$, the following diagrams (known as the pentagon and triangle axioms respectively) commute;

$$
\begin{array}{ccccc}
 & & ((X \otimes Y) \otimes Z) \otimes W & & \\
 & \swarrow^{\alpha_{X,Y,Z} \otimes \mathsf{Id}_W} & & \searrow^{\alpha_{X \otimes Y, Z, W}} & \\
(X \otimes (Y \otimes Z)) \otimes W & & & & (X \otimes Y) \otimes (Z \otimes W) \\
\downarrow{\scriptstyle \alpha_{X, Y \otimes Z, W}} & & & & \downarrow{\scriptstyle \alpha_{X,Y,Z \otimes W}} \\
X \otimes ((Y \otimes Z) \otimes W) & & \xrightarrow{\mathsf{Id}_X \otimes \alpha_{Y,Z,W}} & & X \otimes (Y \otimes (Z \otimes W))
\end{array}
$$

$$
\begin{array}{ccc}
(X \otimes \mathbb{1}) \otimes Y & \xrightarrow{\alpha_{X,\mathbb{1},Y}} & X \otimes (\mathbb{1} \otimes Y) \\
 & {\scriptstyle r_X \otimes \mathsf{Id}_Y} \searrow \quad \swarrow {\scriptstyle \mathsf{Id}_X \otimes l_Y} & \\
 & X \otimes Y &
\end{array}
$$

We are interested in functors between monoidal categories that respect the tensor products, which we shall now detail.

Definition 2.1.7. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha, l, r)$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}}, \alpha', l', r')$ be monoidal categories. A *lax monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ that is equipped with a pair (μ, η), which consists of

- A family of natural morphisms $\mu_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_{\mathcal{C}} Y)$ in $\mathcal{D}$,
- a morphism $\eta : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ in $\mathcal{D}$,

such that the following diagrams commute;

$$
\begin{array}{ccc}
(F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) & \xrightarrow{\alpha'_{F(X),F(Y),F(Z)}} & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \\
\downarrow{\scriptstyle \mu_{X,Y} \otimes \mathsf{Id}_{F(Z)}} & & \downarrow{\scriptstyle \mathsf{Id}_{F(X)} \otimes \mu_{Y,Z}} \\
F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} F(Z) & & F(X) \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \\
\downarrow{\scriptstyle \mu_{X \otimes_{\mathcal{C}} Y, Z}} & & \downarrow{\scriptstyle \mu_{X, Y \otimes_{\mathcal{C}} Z}} \\
F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow{F(\alpha_{X,Y,Z})} & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z))
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{1}_{\mathcal{D}} \otimes_{\mathcal{D}} F(X) & \xrightarrow{\eta \otimes \mathsf{Id}_{F(X)}} & F(\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} F(X) \\
\downarrow{\scriptstyle l'_X} & & \downarrow{\scriptstyle \mu_{\mathbb{1}_{\mathcal{C}}, X}} \\
F(X) & \xleftarrow{F(l_X)} & F(\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} X)
\end{array}
\qquad
\begin{array}{ccc}
F(X) \otimes_{\mathcal{D}} \mathbb{1}_{\mathcal{D}} & \xrightarrow{\mathsf{Id}_{F(X)} \otimes \eta} & F(X) \otimes_{\mathcal{D}} F(\mathbb{1}_{\mathcal{C}}) \\
\downarrow{\scriptstyle r'_X} & & \downarrow{\scriptstyle \mu_{X, \mathbb{1}_{\mathcal{C}}}} \\
F(X) & \xleftarrow{F(r_X)} & F(X \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}})
\end{array}
$$

We will often refer to the pair (μ, η) as a lax monoidal structure on F.

Definition 2.1.8. We can similarly define an *oplax monoidal* structure (ν, ϵ) on a functor $F : \mathcal{C} \to \mathcal{D}$, a family of natural morphisms $\nu_{X,Y} : F(X \otimes_{\mathcal{C}} Y) \to F(X) \otimes_{\mathcal{D}} F(Y)$, for all objects $X, Y \in \mathsf{Ob}(\mathcal{C})$, and a morphism $\epsilon : F(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ that satisfies compatibility conditions similar to those of the lax monoidal structure, with the arrows reversed.

If the morphisms in a lax monoidal structure (μ, η) are isomorphisms, then we can immediately define an oplax monoidal structure by taking the inverse morphisms (μ^{-1}, η^{-1}). Such a functor is known as a *strong monoidal functor*. However, this is not the only way to have a monoidal functor with both lax and oplax structures.

Definition 2.1.9. A functor $F : \mathcal{C} \to \mathcal{D}$ between two monoidal categories $\mathcal{C}, \mathcal{D}$ is *Frobenius monoidal* if it has both a lax monoidal structure (μ, η) and oplax monoidal structure (ν, ϵ) that are compatible in the sense that the following diagrams commute:

$$\begin{array}{ccc}
F(X) \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) & \xrightarrow{\mathsf{Id}_{F(X)} \otimes \nu_{Y,Z}} & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \\
{\scriptstyle \mu_{X, Y \otimes_{\mathcal{C}} Z}} \downarrow & & \downarrow {\scriptstyle \alpha'^{-1}_{F(X),F(Y),F(Z)}} \\
F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z)) & & (F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) \\
{\scriptstyle F(\alpha^{-1}_{X,Y,Z})} \downarrow & & \downarrow {\scriptstyle \mu_{X,Y} \otimes \mathsf{Id}_{F(Z)}} \\
F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow{\nu_{X \otimes_{\mathcal{C}} Y, Z}} & F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} F(Z)
\end{array}$$

$$\begin{array}{ccc}
F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} F(Z) & \xrightarrow{\nu_{X,Y} \otimes \mathsf{Id}_{F(Z)}} & (F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) \\
{\scriptstyle \mu_{X \otimes_{\mathcal{C}} Y, Z}} \downarrow & & \downarrow {\scriptstyle \alpha'_{F(X),F(Y),F(Z)}} \\
F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \\
{\scriptstyle F(\alpha_{X,Y,Z})} \downarrow & & \downarrow {\scriptstyle \mathsf{Id}_{F(X)} \otimes \mu_{Y,Z}} \\
F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z)) & \xrightarrow{\nu_{X, Y \otimes_{\mathcal{C}} Z}} & F(X) \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z)
\end{array}$$

If a Frobenius monoidal functor satisfies

$$\mu_{X,Y} \circ \nu_{X,Y} = \mathsf{Id}_{F(X,Y)}, \tag{2.1.0.1}$$

for all $X, Y \in \mathsf{Ob}(\mathcal{C})$, we say that it is *separable*.

For some notion of invertibility of objects, we define dual objects.

Definition 2.1.10. Let $\mathcal{C}$ be a monoidal category, and $X \in \mathsf{Ob}(\mathcal{C})$. A *left dual* of X is an object $X^* \in \mathsf{Ob}(\mathcal{C})$ that is equipped with two morphisms in $\mathcal{C}$, $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$ and $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$ such that the compositions

$$X \xrightarrow{l_X^{-1}} \mathbb{1} \otimes X \xrightarrow{\mathrm{coev}_X \otimes \mathsf{Id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathsf{Id}_X \otimes \mathrm{ev}_X} X \otimes \mathbb{1} \xrightarrow{r_X} X \tag{2.1.0.2}$$

$$X^* \xrightarrow{r_{X^*}^{-1}} X^* \otimes \mathbb{1} \xrightarrow{\mathsf{Id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathsf{Id}_X} \mathbb{1} \otimes X^* \xrightarrow{l_{X^*}} X^* \tag{2.1.0.3}$$

are the identity morphisms $\mathsf{Id}_X, \mathsf{Id}_{X^*}$, respectively.

The notion of a *right dual* of X can be defined similarly, consisting of an object ${}^*X \in \mathsf{Ob}(\mathcal{C})$ and morphisms $\widetilde{\mathrm{ev}}_X : X \otimes {}^*X \to \mathbb{1}$, $\widetilde{\mathrm{coev}}_X : \mathbb{1} \to {}^*X \otimes X$, and satisfying similar conditions to Equations 2.1.0.2 and 2.1.0.3.

If a left or right dual exists, then it is unique up to unique isomorphism [EGNO15, Proposition 2.10.5]. An object in $\mathcal{C}$ is said to be *rigid* if it has both left and right duals. If all objects are rigid, then we say that the monoidal category $\mathcal{C}$ is rigid.

Proposition 2.1.11 ([EGNO15] Proposition 2.10.8)**.** *Let $\mathcal{C}$ be a monoidal category, and take $X \in \mathsf{Ob}(\mathcal{C})$ such that X has a left dual X^*. Then, for all $Y, Z \in \mathsf{Ob}(\mathcal{C})$, there are natural isomorphisms*

$$\mathsf{Hom}_{\mathcal{C}}(X^* \otimes Y, Z) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(Y, X \otimes Z),$$
$$\mathsf{Hom}_{\mathcal{C}}(Y \otimes X, Z) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(Y, Z \otimes X^*)$$

Similarly, If X has a right dual ${}^*X$, then there are natural isomorphisms

$$\mathsf{Hom}_{\mathcal{C}}(X \otimes Y, Z) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(Y, {}^*X \otimes Y)$$
$$\mathsf{Hom}_{\mathcal{C}}(Y \otimes {}^*X, Z) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(Y, Z \otimes X)$$

We now have a lot of categorical structures that can be unified.

Definition 2.1.12. Let $\mathcal{C}$ be a locally finite $\mathbb{k}$-linear rigid monoidal abelian category. If the tensor product bifunctor is bilinear on morphisms, and the monoidal unit $\mathbb{1}$ is a simple object, we say that $\mathcal{C}$ is a *tensor category.* If, in addition, $\mathcal{C}$ is finite and semisimple, we say that $\mathcal{C}$ is a *fusion category.*

Example 2.1.13. (Vect)

The category whose objects are finite dimensional $\Bbbk$-vector spaces and morphisms are $\Bbbk$-linear maps is both abelian and monoidal with the standard direct sum $\oplus_{\Bbbk}$ and tensor product $\otimes_{\Bbbk}$ of vector spaces. The associator and unitor morphisms are the obvious maps, with $\Bbbk$ acting as the monoidal unit. Due to the finiteness condition, we can define the duals of any vector space $U \in \mathsf{Ob}(\mathsf{Vect})$ as $U^* = {}^*U = \mathsf{Hom}_{\Bbbk}(U, \Bbbk)$. This category is semisimple as any finite-dimensional vector space is isomorphic to a direct sum of copies of the base field $\Bbbk$, and $\Bbbk$ is clearly simple as a vector space over itself. Hence, Vect is a fusion category.

Before looking at more examples of fusion categories, we introduced an important cohomological structure. From now on, G will denote a finite group.

Definition 2.1.14. A *normalised 3-cocycle* on G is a function $\omega : G \times G \times G \to \Bbbk^\times$ that satisfies

$$\omega(gh, k, l)\omega(g, h, kl) = \omega(g, h, k)\omega(g, hk, l)\omega(h, k, l) \tag{2.1.0.4}$$

$$\omega(g, h, e) = \omega(g, e, h) = \omega(e, g, h) = 1 \tag{2.1.0.5}$$

where e is the group identity, and for all $g, h, k, l \in G$.

For more details on the basics of cohomology and how this formula is derived, see Appendix A.

With this, we can construct an important example of fusion category that arises when we consider G-graded vector spaces.

Example 2.1.15. ($\mathsf{Vect}_G, \mathsf{Vect}_G^\omega$) A vector space U is G-graded if it can be decomposed as a direct sum of vector spaces that is labelled by G, i.e

$$U = \bigoplus_{g \in G} U_g.$$

A morphism of G-graded vector spaces U, V is a $\Bbbk$-linear map $f : U \to V$ that respects the G-grading, in the sense that $f(U_g) \subseteq V_g$. We shall denote the category consisting of these objects and morphisms as Vect_G. This category has a direct sum canonically induced by the direct sum of vector spaces. The simple objects for this category are denoted by $\Bbbk_g$, which is the vector space that is the zero vector space in all components apart from the g-th component, which is the base field. It is easy to see this makes Vect_G semisimple.

A tensor product can be given to this category by defining the g-th component of $U \otimes V$ as

$$(U \otimes V)_g = \bigoplus_{h \in G} U_{gh^{-1}} \otimes V_h, \quad \text{for all } g \in G.$$

If we restrict this to the simple objects, the tensor product gives the relation

$$\mathbb{k}_g \otimes \mathbb{k}_h \cong \mathbb{k}_{gh}.$$

This allows us to identify that a monoidal structure can be defined where $\mathbb{1} = \mathbb{k}_e$ and with $\alpha_{\mathbb{k}_g,\mathbb{k}_h,\mathbb{k}_k} = \mathsf{Id}_{\mathbb{k}_{ghk}} : \mathbb{k}_{ghk} \to \mathbb{k}_{ghk}$, and $l_{\mathbb{k}_g} = r_{\mathbb{k}_g} = \mathsf{Id}_{\mathbb{k}_g} : \mathbb{k}_g \to \mathbb{k}_g$, with the compatibility conditions being trivially satisfied.

Using the remaining piece of group data, this category is rigid with dual objects given on simple objects by by $\mathbb{k}_g^* = {}^*\mathbb{k}_g = \mathbb{k}_{g^{-1}}$. Thus Vect_G is a fusion category.

We can make this more general. As the associator is a map $\mathbb{k}_{ghk} \to \mathbb{k}_{ghk}$ it can be some scalar multiple of the identity map, provided the scalar satisfies the pentagon axiom. We can adjust the unitors similarly, provided they satisfy the triangle axiom.

If we take a 3-cocycle $\omega \in H^3(G, \mathbb{k}^\times)$, as described in Equation (2.1.0.4), then we can set the associator to be $\alpha_{\mathbb{k}_g,\mathbb{k}_h,\mathbb{k}_k} = \omega(g,h,k)^{-1}\mathsf{Id}_{\mathbb{k}_{ghk}}$, and unitors $l_{\mathbb{k}_g} = \omega(e,e,g)id_{\mathbb{k}_g}, r_{\mathbb{k}_g} = \omega(g,e,e)^{-1}\mathsf{Id}_{\mathbb{k}_g}$, with the pentagon and triangle axioms being satisfied exactly by the data from Definition 2.1.14. We denote this category by Vect_G^ω. The rest of the categorical structure is the same as Vect_G, so Vect_G^ω is also a fusion category.

Remark 2.1.16. The unitors of Vect_G^ω satisfy $l_U = r_U = \mathsf{Id}_U$, for all $U \in \mathsf{Ob}(\mathsf{Vect}_G^\omega)$, if and only if ω is normalized, that is Equation (2.1.0.5). From now on, we shall always assume that this is the case.

Further, we are using the opposite convention to [EGNO15] which takes $\alpha_{\mathbb{k}_g,\mathbb{k}_h,\mathbb{k}_k} = \omega(g,h,k)\mathsf{Id}_{\mathbb{k}_{ghk}}$. This changes nothing about the categorical structure, but will help with the computations in Chapter 4.

Definition 2.1.17 (Follow up from Definition 2.1.5)**.** Let $\mathcal{C}$ be a tensor category. Then the Grothendieck group $\mathrm{Gr}(\mathcal{C})$ can be given a natural multiplication induced by the tensor product, defined by

$$[X_i] \otimes [X_j] := [X_i \otimes X_j] = \sum_{k \in I} [X_i \otimes X_j : X_k] X_k.$$

This multiplication is associative, and gives $\mathrm{Gr}(\mathcal{C})$ the structure of a ring. We shall call this the *Grothendieck ring* of $\mathcal{C}$.

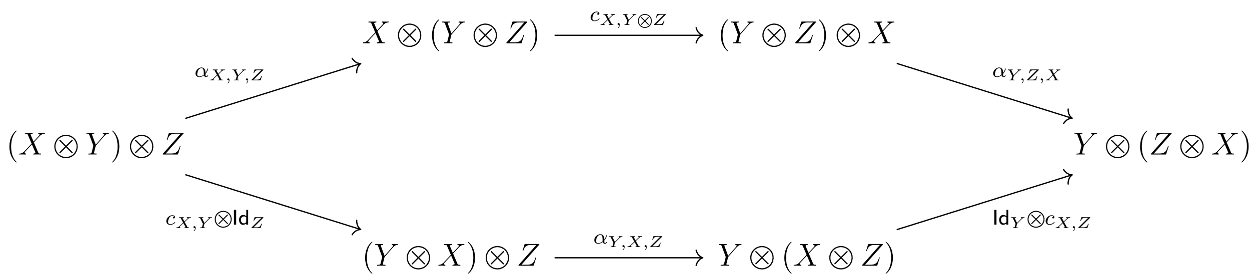
2.2 Braided Categories

In this section we introduce the structure of a braiding, which is a way to add commutativity to the tensor product present in a monoidal category.

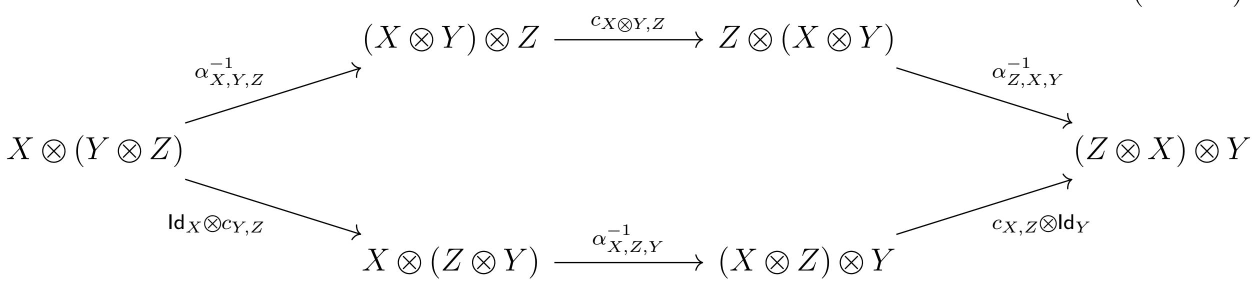
Definition 2.2.1. A *braided monoidal* category is a pair $(\mathcal{C}, c)$ consisting of

- A monoidal category $\mathcal{C}$,
- A family of natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$, called the *braiding*,

such that the following diagrams (the hexagon axioms) commute;



(2.2.0.1)



(2.2.0.2)

Example 2.2.2. The category Vect from Example 2.1.13 has a canonical braided structure given by the flip map,

$$c_{U,V} : U \otimes_{\Bbbk} V \to V \otimes_{\Bbbk} U,$$
$$u \otimes_{\Bbbk} v \mapsto v \otimes_{\Bbbk} u.$$

Example 2.2.3. Consider Vect_G^{ω} from Example 2.1.15. If we restrict a braiding to the simple objects, it becomes a morphism $c_{\Bbbk_g,\Bbbk_h} : \Bbbk_{gh} \to \Bbbk_{hg}$. As the morphism must respect the G-grading, the only way this can exist is if G is abelian.

Assuming that is the case, a braiding must be some scalar multiple the identity morphism,

$$c_{\Bbbk_g,\Bbbk_h} = \psi(g,h)^{-1}\mathsf{Id}_{\Bbbk_{gh}},$$

with the hexagon axioms being equivalent to the identities

$$\omega(h,k,g)\psi(g,hk)\omega(g,h,k) = \psi(g,k)\omega(h,g,k)\psi(h,k),$$
$$\omega(k,g,h)^{-1}\psi(gh,k)\omega(g,h,k)^{-1} = \psi(g,k)\omega(g,k,h)^{-1}\psi(h,k).$$

Viewing $\psi : G \times G \to \Bbbk^\times$ as a function, then the pair (ω, ψ) is known as an *abelian cocycle.*

From this example, we see that not every monoidal category can be endowed with a braiding. Even the example of Vect_G^ω requires us to make the fairly strong restriction to abelian groups. However, there is a way to construct a category with a braiding from any monoidal category.

Definition 2.2.4. (Center Category Construction) Let $\mathcal{C}$ be a monoidal category. The *center category* of $\mathcal{C}$ is the category $\mathcal{Z}(\mathcal{C})$, whose objects are pairs (Z, γ), consisting of an object $Z \in \mathsf{Ob}(\mathcal{C})$ and a family of natural isomorphisms,

$$\gamma_Y : Y \otimes Z \to Z \otimes Y,$$

called the *half-braiding*, that satisfies the single hexagon axiom Equation (2.2.0.2), where c is replaced with γ.

A morphism in $\mathcal{Z}(\mathcal{C})$ from (X, γ) to (Z, γ') is a morphism $f : X \to Z$ in $\mathcal{C}$ such that, for all $Y \in \mathsf{Ob}(\mathcal{C})$, the square

$$\begin{array}{ccc} Y \otimes X & \xrightarrow{\gamma_Y} & X \otimes Y \\ \downarrow{\scriptstyle \mathsf{Id}_Y \otimes f} & & \downarrow{\scriptstyle f \otimes \mathsf{Id}_Y} \\ Y \otimes Z & \xrightarrow{\gamma'_Y} & Z \otimes Y \end{array} \tag{2.2.0.3}$$

commutes.

Remark 2.2.5. This construction is also known as the Drinfeld center or the monoidal center.

Proposition 2.2.6 ([EGNO15, Section 7.13])**.** *The center category $\mathcal{Z}(\mathcal{C})$ is monoidal, with tensor product given by*

$$(X, \gamma) \otimes (Z, \gamma') := (X \otimes Z, \bar{\gamma}),$$

where $\bar{\gamma}$ is defined by the commutative diagram

$$\begin{array}{ccccc} Y \otimes (X \otimes Z) & \xrightarrow{\alpha^{-1}_{Y,X,Z}} & (Y \otimes X) \otimes Z & \xrightarrow{\gamma_Y \otimes \mathsf{Id}_Z} & (X \otimes Y) \otimes Z \\ \downarrow{\scriptstyle \bar{\gamma}_Y} & & & & \downarrow{\scriptstyle \alpha_{X,Y,Z}} \\ (X \otimes Z) \otimes Y & \xleftarrow[\alpha^{-1}_{X,Z,Y}]{} & X \otimes (Z \otimes Y) & \xleftarrow[\mathsf{Id}_X \otimes \gamma'_Y]{} & X \otimes (Y \otimes Z), \end{array}$$

The monoidal unit is $(\mathbb{1}, l^{-1}r)$, with the same associator and unitors as $\mathcal{C}$.

Now that we have monoidality, we can give this category a braiding.

Proposition 2.2.7 ([EGNO15, Proposition 8.5.1])**.** *The center category $\mathcal{Z}(\mathcal{C})$ is a braided monoidal category, with the braiding given by*

$$c_{(X,\gamma),(Z,\gamma')} := \gamma'_X.$$

Example 2.2.8. ($\mathcal{Z}(\mathsf{Vect}^{\omega}_G)$) Objects in this category consist of G-graded vector spaces that are equipped with a collection of half-braiding isomorphisms ,$\gamma_g : \delta_g \otimes V \to V \otimes \delta_g$, one for each simple object in Vect^{ω}_G. If we pick a degree $d \in G$, then using the tensor product from Example 2.1.15 we find that $\delta_g \otimes V_d \cong V_{g^{-1}d}$ and so we can induce linear isomorphisms

$$V_{g^{-1}dg} \tilde{\to} V_d \otimes (\delta_g \otimes \delta_{g^{-1}}) \cong V_d.$$

Thus objects in this category consist of G-graded vector spaces that are equipped with isomorphisms between their G-conjugated components.

We can also extend the notion of a braiding to functors between monoidal categories.

Definition 2.2.9. Given two braided monoidal categories $(\mathcal{C}, c)$ and $(\mathcal{D}, d)$, a *braided lax monoidal functor* is a functor $F : \mathcal{C} \to \mathcal{D}$ with a lax monoidal structure (μ, η), such that the following diagram commutes;

$$\begin{array}{ccc} F(X) \otimes_{\mathcal{D}} F(Y) & \xrightarrow{d_{F(X),F(Y)}} & F(Y) \otimes_{\mathcal{D}} F(X) \\ {\scriptstyle \mu_{X,Y}} \downarrow & & \downarrow {\scriptstyle \mu_{Y,X}} \\ F(X \otimes_{\mathcal{C}} Y) & \xrightarrow{F(c_{X,Y})} & F(Y \otimes_{\mathcal{C}} X) \end{array}$$

Similarly, a *braided oplax monoidal functor* is a functor $F : \mathcal{C} \to \mathcal{D}$ with an oplax structure (ν, ϵ), satisfying a similar compatibility condition.

Remark 2.2.10. If we instead have a strong monoidal functor with a given monoidal structure, then it is braided lax monoidal with respect to this structure if and only if it is braided oplax monoidal with the respect to the oplax structure given by the inverse coherence morphisms. If this is the case, we can simply say that the functor is a *braided monoidal functor*.

Chapter 3

Algebra Objects in Category Theory

In this section, we will introduce the main object of interest to this thesis - algebra objects.

3.1 Basic definitions

Definition 3.1.1. Let $\mathcal{C}$ be a monoidal category. An *algebra object* in $\mathcal{C}$ is a triple (A, m, u) consisting of

- An object $A \in \mathsf{Ob}(C)$,
- A multiplication morphism $m : A \otimes A \to A$ in $\mathcal{C}$,
- A unit morphism $u : \mathbb{1} \to A$ in $\mathcal{C}$,

such that the following diagrams commute;

$$\begin{array}{ccc} (A \otimes A) \otimes A & \xrightarrow{\alpha_{A,A,A}} & A \otimes (A \otimes A) \\ \downarrow{\scriptstyle m \otimes \mathsf{Id}_A} & & \downarrow{\scriptstyle \mathsf{Id}_A \otimes m} \\ A \otimes A & & A \otimes A \\ & \searrow^{m} \quad A \quad {}^{m}\swarrow & \end{array} \tag{3.1.0.1}$$

$$\begin{array}{ccc} \mathbb{1} \otimes A & \xrightarrow{u \otimes \mathsf{Id}_A} & A \otimes A \\ & {\scriptstyle l_A}\searrow \quad \swarrow{\scriptstyle m} & \\ & A & \end{array} \qquad \begin{array}{ccc} A \otimes \mathbb{1} & \xrightarrow{\mathsf{Id}_A \otimes u} & A \otimes A \\ & {\scriptstyle r_A}\searrow \quad \swarrow{\scriptstyle m} & \\ & A & \end{array} \tag{3.1.0.2}$$

These meaning of these axioms are that the algebra needs to be associative and unital, with respect to the category it is living in. Indeed, the algebra objects in the category Vect are exactly the standard associative unital $\Bbbk$-algebras.

Example 3.1.2. There are some basic algebra objects that can be defined for any monoidal category. The monoidal unit $\mathbb{1}$ has the structure of an algebra with $m = l_{\mathbb{1}} = r_{\mathbb{1}}$ and $u = \mathsf{Id}_{\mathbb{1}}$. Further, if X is a rigid object, then $A = X \otimes X^*$ is an algebra with $m = \mathsf{Id}_X \otimes \mathrm{ev}_X \otimes \mathsf{Id}_{X^*}$ and $u = \mathrm{coev}_X$.

We shall now consider an important example of an algebra that can be constructed from a finite group H.

Example 3.1.3 (Group Algebra in Vect)**.** The vector space freely generated by H has the structure of an algebra object in Vect, with multiplication and unit given by

$$m(g \otimes h) = gh, \; u(1_{\Bbbk}) = e,$$

for all $g, h \in H$, with $e \in H$ being the group identity.

Example 3.1.4 (Twisted Group Algebra in Vect_G^ω)**.** Let H be a subgroup of G. Then we can alternately view the group algebra as the object

$$\Bbbk[H] = \bigoplus_{h \in H} \Bbbk_h,$$

in Vect_G^ω. When restricted to simple objects, the multiplication morphism is given by

$$m := \kappa(g,h)^{-1} \mathsf{Id}_{\Bbbk_{gh}} : \Bbbk_{gh} \to \Bbbk_{gh},$$

where $\kappa : H \times H \to \Bbbk^\times$ is a function that must satisfy the following condition as a result of the associativity condition;

$$\omega(g,h,k) = \kappa(gh,k)\kappa(g,h)\kappa(g,hk)^{-1}\kappa(h,k)^{-1}.$$

In terms of cohomology, see Appendix A, this is equivalent to asking that $\kappa \in C^2(H, \Bbbk^\times)$ be a 2-cocycle such that $\mathrm{d}\,\kappa = \omega|_H$. The unit morphism is the canonical embedding of $\Bbbk_e$ into $\Bbbk[H]$. Taking ω, κ to be trivial, we recover the standard group algebra as an object in Vect_G. We shall denote this algebra by $(\Bbbk[H], \kappa)$

We can also define the dual notion, a coalgebra object.

Definition 3.1.5. A *coalgebra object* is a triple (C, Δ, ε) consisting of an object $C \in \mathsf{Ob}(\mathcal{C})$, a comultiplication morphism $\Delta : C \to C \otimes C$ and a counit morphism $\varepsilon : C \to \mathbb{1}$ that satisfy coassociativity and counitality axioms similar to Equations 3.1.0.1 and 3.1.0.2, with the arrows reversed.

Definition 3.1.6. Let $(A, m_A, u_A), (B, m_B, u_B)$ be algebra objects in $\mathcal{C}$. A morphism of algebras is a morphism $f : A \to B$ in $\mathcal{C}$ that satisfies

$$f \circ m_A = m_b \circ (f \otimes f).$$

Where there is no ambiguity, we shall refer to the algebra (A, m, u) simply as A.

Analogously to the classical case, we can study algebra objects through their representation theory, looking at how they act on other objects within the category. This leads to the following definition.

Definition 3.1.7. Let A be an algebra object in a monoidal category $\mathcal{C}$. A *right A-module* in $\mathcal{C}$ is a pair (M, ρ), consisting of

- An object $M \in \mathsf{Ob}(\mathcal{C})$,
- An action morphisms $\rho : M \otimes A \to M$ in $\mathcal{C}$,

such that following diagrams commute;

$$\begin{array}{ccc} (M \otimes A) \otimes A & \xrightarrow{\alpha_{M,A,A}} & M \otimes (A \otimes A) \\ \downarrow{\scriptstyle \rho \otimes \mathsf{Id}_A} & & \downarrow{\scriptstyle \mathsf{Id}_M \otimes m} \\ M \otimes A & & M \otimes A \\ & \searrow^{\rho} \quad M \quad {}^{\rho}\swarrow & \end{array} \tag{3.1.0.3}$$

$$\begin{array}{ccc} M \otimes \mathbb{1} & \xrightarrow{\mathsf{Id}_M \otimes u} & M \otimes A \\ & \searrow^{r_M} \quad M \quad {}^{\rho}\swarrow & \end{array} \tag{3.1.0.4}$$

Remark 3.1.8. Other types of modules can be defined, for example a *left* A-module, which has the algebra acting from the left-hand side instead. Additionally, an A-*bimodule* is an object M that is both a left and right A-module, such that the left and right actions, ρ^l and

ρ^r respectively, commute in the sense that

$$\rho^r \circ (\rho^l \otimes \mathsf{Id}_A) = \rho^l \circ (\mathsf{Id}_A \otimes \rho^r) \circ \alpha_{A \otimes M \otimes A}.$$

In this thesis, we shall predominantly use right A-modules, and shall simply refer to these as A-modules when no distinction is needed.

Example 3.1.9. Given an A-module (M, ρ) and an object $X \in \mathsf{Ob}(\mathcal{C})$, the tensor product $X \otimes M$ can also be given the structure of an A-module, with the action morphism being the composition

$$\rho' : (X \otimes M) \otimes A \xrightarrow{\alpha_{X,M,A}} X \otimes (M \otimes A) \xrightarrow{\mathsf{Id}_X \otimes \rho} X \otimes M.$$

Definition 3.1.10. We can also study the representation theory of a coalgbra C using (right) C-comodules, which are objects $N \in \mathsf{Ob}(\mathcal{C})$ paired with a coaction morphism $\delta : N \to N \otimes A$, satisfying similar diagrams to that of a module, with the arrows reversed.

Definition 3.1.11. A *morphism of right A-modules* (M, ρ_M) and (N, ρ_N) is a morphism $f : M \to N$ in $\mathcal{C}$ that satisfies

$$f \circ \rho_M = \rho_N \circ (f \otimes \mathsf{Id}_A).$$

Morphisms of left A-modules and A-bimodules can be defined similarly.

Example 3.1.12. Every algebra can be viewed as a right (and indeed left) module over itself, with the action morphism given by the algebra multiplication.

Example 3.1.13. The object $A \otimes A$ has the structure of a right A-module, with action morphism given by

$$\rho^r_{A \otimes A} = (\mathsf{Id}_A \otimes m_A)\alpha_{A,A,A}.$$

Similarly, it has a left A-module structure with

$$\rho^l_{A \otimes A} = (m_A \otimes \mathsf{Id}_A)\alpha^{-1}_{A,A,A}.$$

In this sense, $A \otimes A$ can be viewed as a A-bimodule with the bimodule commutativity given by the associativity of A.

3.2 Frobenius Algebras

We shall now look at some types of objects that have both an algebra and coalgebra structure.

Definition 3.2.1. A *Frobenius algebra object* in $\mathcal{C}$ is a tuple $(A, m, u, \Delta, \varepsilon)$, where

- (A, m, u) is an algebra object in $\mathcal{C}$,
- (A, Δ, ε) is a coalgebra object in $\mathcal{C}$,

such that the comultiplication morphism $\Delta : A \to A \otimes A$ is a morphism of A-bimodules, where $A \otimes A$ has the A-bimodule structure described in Example 3.1.13.

The compatability condition between the algebra and coalgebra structure in a Frobenius algebra can be viewed explicitly as asking for the following diagram to commute;

$$\begin{array}{ccccc} (A\otimes A)\otimes A & \xleftarrow{\Delta\otimes\mathsf{Id}_A} & A\otimes A & \xrightarrow{\mathsf{Id}_A\otimes\Delta} & A\otimes(A\otimes A) \\ & & \downarrow{\scriptstyle m} & & \\ \Big\downarrow{\scriptstyle \alpha_{A,A,A}} & & A & & \Big\downarrow{\scriptstyle \alpha^{-1}_{A,A,A}} \\ & & \downarrow{\scriptstyle \Delta} & & \\ A\otimes(A\otimes A) & \xrightarrow{\mathsf{Id}_A\otimes m} & A\otimes A & \xleftarrow{m\otimes\mathsf{Id}_A} & (A\otimes A)\otimes A \end{array} \tag{3.2.0.1}$$

We shall make use of this condition as the main definition of a Frobenius algebra, however there are a variety of equivalent definitions. For example, we can make use of the unit and counit morphisms to construct the following compositions,

$$A \otimes A \xrightarrow{m} A \xrightarrow{\varepsilon} \mathbb{1}, \quad \mathbb{1} \xrightarrow{u} A \xrightarrow{\Delta} A \otimes A.$$

We can check that these form a non-degenerate pairing, i.e they satisfy the rigidity conditions from Equations 2.1.0.2 and 2.1.0.3 which in particular means that $A \cong A^* \cong {}^*A$, so A is self-dual in $\mathcal{C}$.

Conversely, if we have an algebra A in $\mathcal{C}$ that is self-dual with an associative non-degenerate pairing $\mathrm{ev}_A : A \otimes A \to \mathbb{1}$ in the sense that $\mathrm{ev}_A(m \otimes \mathsf{Id}_A) = \mathrm{ev}_A(\mathsf{Id}_A \otimes m)\alpha_{A,A,A}$, then we can define a coalgebra structure through

$$\Delta : A \cong A \otimes \mathbb{1} \xrightarrow{\mathsf{Id}_A \otimes \mathrm{coev}_A} A \otimes (A \otimes A) \xrightarrow{\alpha^{-1}_{A,A,A}} (A \otimes A) \otimes A \xrightarrow{m \otimes \mathsf{Id}_A} A \otimes A \tag{3.2.0.2}$$

$$\varepsilon : A \cong A \otimes \mathbb{1} \xrightarrow{\mathsf{Id}_A \otimes u} A \otimes A \xrightarrow{\mathrm{ev}_A} \mathbb{1} \tag{3.2.0.3}$$

A detailed proof of this can be found in [CD20], and gives us the following lemma.

Lemma 3.2.2. *Let A be an algebra in $\mathcal{C}$. It is Frobenius if and only if it is equipped with an associative non-degenerate pairing $A \otimes A \to \mathbb{1}$.*

Example 3.2.3. The twisted group algebra $(\Bbbk[H], \kappa)$ in Vect_G^ω can be endowed with a coalgebra structure, with

$$\Delta(\Bbbk_h) = \bigoplus_{k \in H} \kappa(k, k^{-1}h) \Bbbk_k \otimes \Bbbk_{k^{-1}h} \ , \quad \varepsilon(\Bbbk_h) = \delta_{h,e}$$

for all $h \in H$. The comultiplication condition is satisfied as $d\kappa = \omega|_H$.

In addition, this coalgebra structure gives $(\Bbbk[H], \kappa)$ the structure of a Frobenius algebra. To check that the left-hand square in Equation (3.2.0.1) commutes, we require the following compositions to be equal;

$$\kappa(g^{-1}h, k)\omega(g, g^{-1}h, k)^{-1}\kappa(g, g^{-1}h)^{-1} = \kappa(g, g^{-1}hk)^{-1}\kappa(h, k),$$

for all $g, h, k \in H$, which is exactly the condition that $d\kappa = \omega|_H$. The second condition follows analogously.

There are a variety of adjectives which we can attach to the algebra objects seen so far. These allow us to impose different categorical structures on the modules of such algebras, but we shall explore this idea in the next section. For now, we define the following properties;

Definition 3.2.4.

- Let A be an algebra object in a $\Bbbk$-linear abelian monoidal category $\mathcal{C}$.
  - A is *indecomposable* if it is not isomorphic to a direct sum of non-trivial algebras in $\mathcal{C}$.
  - A is *connected* if $\mathsf{dim}_\Bbbk \mathsf{Hom}_\mathcal{C}(\mathbb{1}, A) = 1$
  - A is *separable* if there exists a morphism $\Delta' \colon A \to A \otimes A$ that is a morphism of A-bimodules, and satisfies $m\Delta' = \mathsf{Id}_A$.
- Suppose $\mathcal{C}$ is braided. Then an algebra A in $\mathcal{C}$ is *commutative* if $mc_{A,A} = m$.
- A Frobenius algebra $(A, m, u, \Delta, \varepsilon)$ in $\mathcal{C}$ is *special* if $m\Delta = \beta_A \mathsf{Id}_A$ and $\varepsilon u = \beta_\mathbb{1}\ \mathsf{Id}_\mathbb{1}$ for non-zero $\beta_A, \beta_\mathbb{1} \in \Bbbk^\times$.

Remark 3.2.5.

- If an algebra is connected, then it is also indecomposable as there is only a single choice of unit morphism, up to a scalar.
- There are alternate, equivalent definitions of separability, such as asking A to admit a morphism $\theta : A \to \mathbb{1}$ where $\theta \circ u = \mathsf{Id}_\mathbb{1}$ and $\theta \circ m$ is non-degenerate.

- We can also notice that the condition for an algebra to be separable is exactly asking for $\Delta' : A \to A \otimes A$ to be a morphism of A-bimodules, as in the definition of a Frobenius algebra. In fact, if an algebra is special Frobenius, we can always construct a separable algebra with a coproduct that has been scaled as $\tilde{\Delta} = \beta_A^{-1}\Delta$.

Proposition 3.2.6 ([MMP⁺23], Proposition 5.9)**.** *The twisted group algebra* $(\Bbbk[H], \kappa)$ *as described in Example 3.2.3 is a connected Frobenius algebra in* Vect_G^ω*. When* $|H| \in \Bbbk^\times$*, this algebra is special Frobenius.*

Proof. This algebra is connected in Vect_G^ω, as morphisms must respect the G-grading and so any morphism $u : \mathbb{1} \to \Bbbk[H]$ is simply a map from $\Bbbk_e \to \Bbbk_e$. This morphism space is clearly one-dimensional.

For the special condition, we can directly calculate that

$$m\Delta = |H|\mathsf{Id}_A, \quad \varepsilon u = \mathsf{Id}_{\mathbb{1}}.$$

□

We will now look at how we can preserve the algebraic structures discussed so far using the various types of monoidal functors defined in Definitions 2.1.7 - 2.1.9, and Definition 2.2.9.

Proposition 3.2.7. *Let* $\mathcal{C}$,$\mathcal{D}$ *be* $\Bbbk$*-linear abelian monoidal categories.*

a) Suppose $F : \mathcal{C} \to \mathcal{D}$ *is a lax/oplax/Frobenius monoidal functor, and* A *an algebra/coalgebra/Frobenius algebra in* $\mathcal{C}$*. Then* $F(A)$ *is an algebra/coalgebra/Frobenius algebra in* $\mathcal{D}$*.*

b) Suppose $F : \mathcal{C} \to \mathcal{D}$ *is a separable Frobenius monoidal functor such that* $\epsilon \circ \eta \neq 0$*, and the monoidal unit* $\mathbb{1}_{\mathcal{D}}$ *in* $\mathcal{D}$ *is simple. If* A *is a special Frobenius algebra in* $\mathcal{C}$ *then* $F(A)$ *is a special Frobenius algebra in* $\mathcal{D}$*.*

c) Suppose $\mathcal{C}, \mathcal{D}$ *are further assumed to be braided, and* $F : \mathcal{C} \to \mathcal{D}$ *a braided lax monoidal functor. If* A *is a commutative algebra in* $\mathcal{C}$*, then* $F(A)$ *is commutative in* $\mathcal{D}$*.*

Proof. Part a) Using the lax monoidal structure (μ, η) on $F : \mathcal{C} \to \mathcal{D}$, the object $F(A)$ can be given multiplication and unit morphisms by the following compositions respectively;

$$m_{F(A)} : F(A) \otimes_{\mathcal{D}} F(A) \xrightarrow{\mu_{A,A}} F(A \otimes_{\mathcal{C}} A) \xrightarrow{F(m_A)} F(A),$$
$$u_{F(A)} : \mathbb{1}_{\mathcal{D}} \xrightarrow{\eta} F(\mathbb{1}_{\mathcal{C}}) \xrightarrow{F(u_A)} F(A).$$

The algebra compatibility conditions from Definition 3.1.1 can then be easily verified using the lax monoidal compatibility conditions in Definition 2.1.7.

The statement for coalgebras is proven analagously. If we have a coalgebra C and an oplax monoidal structure (ν, ϵ) on $F : \mathcal{C} \to \mathcal{D}$, then $F(C)$ has comultiplication and counit given by

$$\Delta_{F(C)} : F(C) \xrightarrow{F(\Delta_C)} F(C \otimes_{\mathcal{C}} C) \xrightarrow{\nu_{C,C}} F(C) \otimes_{\mathcal{D}} F(C),$$
$$\varepsilon_{F(C)} : F(C) \xrightarrow{F(\varepsilon_C)} F(\mathbb{1}_{\mathcal{C}}) \xrightarrow{\epsilon} \mathbb{1}_{\mathcal{D}}.$$

where the defining conditions of a oplax monoidal functor make this into a coalgebra. The Frobenius monoidal functor conditions can be easily verified to coincide with the Frobenius algebra condition for the algebra and coalgebra structures given above.

Part b) For the second statement, as $\mathbb{1}_{\mathcal{D}}$ is simple, we have that $\epsilon \circ \eta = \beta' \mathsf{Id}_{\mathbb{1}_{\mathcal{D}}}$ for some non-zero $\beta' \in \Bbbk^{\times}$. The conditions for $F(A)$ to be a special Frobenius algebra can then be calculated directly;

$$m_{F(A)} \circ \Delta_{F(A)} = F(m_A) \circ \mu_{A,A} \circ \nu_{A,A} \circ F(\Delta_A) = F(m_A \circ \Delta_A) = \beta_A F(\mathsf{Id}_A) = \beta_A \mathsf{Id}_{F(A)},$$
$$\varepsilon_{F(A)} \circ u_{F(A)} = \epsilon \circ F(\varepsilon_A \circ u_A) \circ \eta = \beta_{\mathbb{1}_{\mathcal{C}}} \epsilon \circ \eta = (\beta_{\mathbb{1}_{\mathcal{C}}} \beta') \mathsf{Id}_{\mathbb{1}},$$

and so $F(A)$ is special Frobenius.

Part c) Using the multiplication of $F(A)$ given above, the condition that this is a commutative algebra from Definition 3.2.4 is equivalent to asking that the perimeter of the following diagram commutes;

$$\begin{array}{ccccc} F(A) \otimes_{\mathcal{D}} F(A) & \xrightarrow{\mu_{A,A}} & F(A \otimes_{\mathcal{C}} A) & \xrightarrow{F(m_A)} & F(A) \\ \downarrow{\scriptstyle d_{A,A}} & & \downarrow{\scriptstyle F(c_{A,A})} & \nearrow{\scriptstyle F(m_A)} & \\ F(A) \otimes_{\mathcal{D}} F(A) & \xrightarrow{\mu_{A,A}} & F(A \otimes_{\mathcal{C}} A) & & \end{array}$$

This is satisfied as the square on the left commutes by the braided lax monoidal condition from Definition 2.2.9, and the triangle commutes by functorality and by A being commutative. □

3.3 Categories of Modules

We shall now use algebra objects and their modules to construct new categories. This is one reason why we study algebras in category theory as they are a great source of other examples of categories. For this section, we assume that all categories are abelian and $\Bbbk$-linear.

Given an algebra object in a monoidal category $\mathcal{C}$, we define the category $\mathsf{Mod}_{\mathcal{C}}(A)$ to consist of all A-modules and A-module morphisms in $\mathcal{C}$. This allows us to view the representation theory of an algebra object as a categorical construction itself.

Remark 3.3.1. Here we are taking right A-modules, but the categories of left A-module and A-bimodules can be constructed in an analogous way.

Proposition 3.3.2 ([EGNO15, Section 7.8])**.** *Let A be an algebra object in a $\Bbbk$-linear finite abelian category $\mathcal{C}$. Then $\mathsf{Mod}_{\mathcal{C}}(A)$ is also a $\Bbbk$-linear finite abelian category .*

If we wished to make $\mathsf{Mod}_{\mathcal{C}}(A)$ a fusion category we would need it to be semisimple and monoidal, amongst other things. For this to be possible, we need to make some additional assumptions on both our algebra object and the category it lives in.

Proposition 3.3.3. *Let A be a connected algebra in a monoidal category $\mathcal{C}$. Then A is simple when viewed as an object in $\mathsf{Mod}_{\mathcal{C}}(A)$.*

Proof. Take objects $X \in \mathsf{Ob}(\mathcal{C})$, and$M \in \mathsf{Ob}(\mathsf{Mod}_{\mathcal{C}}(A))$. By [EGNO15, Lemma 7.8.12], there is a natural isomorphism

$$\mathsf{Hom}_{\mathsf{Mod}_{\mathcal{C}}(A)}(X \otimes A, M) \cong \mathsf{Hom}_{\mathcal{C}}(X, M), \tag{3.3.0.1}$$

where $X \otimes A$ is the A-module from Example 3.1.9, and on the right-hand side M is viewed as an object of $\mathcal{C}$. If we specify that $X = \mathbb{1}$ and $M = A$ viewed as a module over itself, then this isomorphism becomes

$$\mathsf{Hom}_{\mathsf{Mod}_{\mathcal{C}}(A)}(A, A) \cong \mathsf{Hom}_{\mathcal{C}}(\mathbb{1}, A) \cong \Bbbk,$$

as A is connected, and so A is simple in $\mathsf{Mod}_{\mathcal{C}}(A)$. ☐

Suppose that $\mathcal{C}$ is an abelian braided monoidal category. Then any right A-module (M, ρ^r) can be given the structure of a left A-module, with the left A-action given by the composition.

$$\rho^l : A \otimes M \xrightarrow{c_{A,M}} M \otimes A \xrightarrow{\rho^r} M$$

Given two right A-modules $(M, \rho^r_M), (N, \rho^r_N)$, we define their *tensor product over A* as the coequaliser of the diagram

$$M \otimes A \otimes N \underset{\mathsf{Id}_M \otimes \rho^l_N}{\overset{\rho^r_M \otimes \mathsf{Id}_N}{\rightrightarrows}} M \otimes N \longrightarrow M \otimes_A N, \tag{3.3.0.2}$$

Proposition 3.3.4 ([Par95, Proposition 1.4], [KO02, Theorem 1.5])**.** *Suppose A is a commutative algebra in a braided monoidal category $\mathcal{C}$. Then $\mathsf{Mod}_{\mathcal{C}}(A)$ is a monoidal category with monoidal unit A.*

The last important remaining condition is that of semisimplicity. This requires us to restrict the underlying category even further.

Proposition 3.3.5 ([EGNO15, Proposition 7.8.30])**.** *Let A be a separable algebra in a fusion category $\mathcal{C}$. Then $\mathsf{Mod}_{\mathcal{C}}(A)$ is semisimple.*

Thus we have seen that commutative connected separable (or any combination of the three adjectives) algebras provide a wealth of structure on their categories of modules. It is these types of algebras which we shall focus on.

However, there is one structure currently missing - a braiding. This can be introduced by looking at a special type of module.

Definition 3.3.6. Let A be an algebra in a braided monoidal category $\mathcal{C}$. An A-module (M, ρ) is said to be *local* if the module action is invariant under a double braiding, that is

$$\rho = \rho \circ c_{A,M} \circ c_{M,A}. \tag{3.3.0.3}$$

We shall write $\mathsf{Mod}^{\mathsf{loc}}_{\mathcal{C}}(A)$ for the category of local modules. This can be viewed as a full monoidal subcategory of $\mathsf{Mod}_{\mathcal{C}}(A)$.

Proposition 3.3.7 ([Par95, Theorem 2.5])**.** *(Pareigis) Let A be a commutative algebra in a braided monoidal category $\mathcal{C}$. Then the braiding of $\mathcal{C}$ descends directly on to the category $\mathsf{Mod}^{\mathsf{loc}}_{\mathcal{C}}(A)$, making it braided monoidal.*

Local modules are also very useful when working in the context of modular categories. It has been proven that when A is a particular type of commutative algebra in a modular tensor category $\mathcal{C}$, the category $\mathsf{Mod}^{\mathsf{loc}}_{\mathcal{C}}(A)$ is also modular [KO02, LW23]. This has applications in rational CFT. By [HKL15, Theorem 3.2], extensions of a VOA correspond to commutative algebras in its category of modules. This goes further, as it is also shown that the representation category of these extensions is equivalent to the category of local modules for the corresponding algebra [HKL15, Theorem 3.4]. Further properties of categories arising from local modules are discussed in [FRS04],[FFRS06].

3.4 Hopf Algebras

Let $\mathcal{C}$ be a braided monoidal category. We can then give the object $A \otimes A$ the structure of an algebra, with multiplication and unit morphisms

$$m_{A\otimes A} = (m_A \otimes m_A) \circ (\mathsf{Id}_A \otimes c_{A,A} \otimes \mathsf{Id}_A), \quad u_{A\otimes A} = u_A \otimes u_A,$$

where we have omitted the associator and unitor maps.

This allows us to define a second structure that combines an algebra and coalgebra structure, by slightly modifying the definition of a Frobenius algebra.

Definition 3.4.1. A *bialgebra* in a braided monoidal category $\mathcal{C}$ is a tuple $(A, m, u, \Delta, \varepsilon)$ such that

- (A, m, u) is an algebra object in $\mathcal{C}$,
- (A, Δ, ε) is a coalgebra object in $\mathcal{C}$, such that $\Delta : A \to A \otimes A$ is a morphism of algebras in $\mathcal{C}$.

A special type of bialgebra is the structure of a Hopf algebra.

Definition 3.4.2 (Hopf Algebras)**.** A *Hopf algebra* in Vect is a bialgebra $(\mathcal{H}, m, u, \Delta, \varepsilon)$ that is equipped with a map $S : \mathcal{H} \to \mathcal{H}$, called the antipode, such that the diagram

$$\begin{array}{ccccc} \mathcal{H}\otimes\mathcal{H} & \xleftarrow{\Delta} & \mathcal{H} & \xrightarrow{\Delta} & \mathcal{H}\otimes\mathcal{H} \\ \downarrow{\scriptstyle \mathsf{Id}_{\mathcal{H}}\otimes S} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle S\otimes \mathsf{Id}_{\mathcal{H}}} \\ & & \mathbb{1} & & \\ & & \downarrow{\scriptstyle u} & & \\ \mathcal{H}\otimes\mathcal{H} & \xrightarrow{m} & \mathcal{H} & \xleftarrow{m} & \mathcal{H}\otimes\mathcal{H} \end{array}$$

commutes.

Notation 3.4.3. To simplify the equations involved when working with Hopf algebras in Vect, we can use a notation convention known as Sweedler notation;

$$m(x \otimes y) = xy,\ \Delta(x) = \sum_i x_1^{(i)} \otimes x_2^{(i)} = x_1 \otimes x_2,\ u(1_{\Bbbk}) = 1_{\mathcal{H}}.$$

Using this, the antipode condition can be rewritten as

$$x_1 S(x_2) = \varepsilon(x) \cdot 1_{\mathcal{H}} = S(x_1) x_2$$

An interesting question is when do the notions of Frobenius and Hopf algebras coincide.

Definition 3.4.4. A *left integral* of $\mathcal{H}$ is a non-zero morphism $\varsigma : \mathbb{1} \to \mathcal{H}$ in $\mathcal{C}$ such that

$$m \circ (\mathsf{Id}_{\mathcal{H}} \otimes \varsigma) = \varsigma \circ \varepsilon.$$

A *right cointegral* of $\mathcal{H}$ is a non-zero morphism $\chi : \mathcal{H} \to \mathbb{1}$ in $\mathcal{C}$ such that

$$(\chi \otimes \mathsf{Id}_{\mathcal{H}}) \circ \Delta = u \circ \chi,$$

Proposition 3.4.5 ([CD20, Lemmas 3.8,3.9])**.** *Let $\mathcal{H}$ be a Hopf algebra in $\mathcal{C}$ that has a left integral and a right cointegral. Then $\mathcal{H}$ can be given the structure of a Frobenius algebra.*

Sketch of proof: Using this data, we can construct an associative non-degenerate pairing associated to $\mathcal{H}$ is given by the compositions

$$f := \mathcal{H} \otimes \mathcal{H} \xrightarrow{m} \mathcal{H} \xrightarrow{\chi} \mathbb{1}, \quad g := \mathbb{1} \xrightarrow{\varsigma} \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{S \otimes \mathsf{Id}_{\mathcal{H}}} \mathcal{H} \otimes \mathcal{H}.$$

Then, by Lemma 3.2.2, this gives $\mathcal{H}$ the structure of a Frobenius algebra. □

In general, the coproduct of the Frobenius algebra given by this pairing will not be the same as the Hopf algebra structure. This can be seen by considering the group algebra in Vect.

Example 3.4.6. The group algebra $\Bbbk[G]$ from Example 3.1.3 can be given the structure of a Hopf algebra, with

$$\Delta(g) = g \otimes g\ ,\ \varepsilon(g) = 1\ ,\ S(g) = g^{-1}.$$

It also admits a left integral $\varsigma(1_{\Bbbk}) = \sum_{g \in G} g$ and right cointegral $\chi(g) = \delta_{g,e}$, and so by the previous result we should be able to recover a Frobenius algebra structure on H. If we use the non-degenerate pairing from Proposition 3.4.5 to form the Frobenius coproduct and counit from Equations 3.2.0.2 and 3.2.0.3, we find that

$$\Delta_{\mathrm{Frob}}(h) = \sum_{g \in G} hg^{-1} \otimes g, \quad \varepsilon_{\mathrm{Frob}}(h) = \delta_{h,e}.$$

Up to some relabelling of indices, this is exactly the Frobenius structure from Example 3.2.3, and is clearly different to the Hopf coalgebra structure.

If we restrict to the category of vector spaces, any finite-dimensional Hopf algebra satisfies these conditions.

Theorem 3.4.7. [LS69] *Every finite-dimensional Hopf* $\Bbbk$*-algebra is a Frobenius* $\Bbbk$*-algebra.*

3.5 Yetter-Drinfeld Modules

For a general Hopf algebra, its category of modules is braided if and only if the Hopf algebra is quasi-triangular [Kas95, Sch20]. This is a fairly restrictive condition and, similarly to when we introduced the center category, can be circumvented by constructing a new category. This is done by defining a particular type of module that incorporates all of the Hopf algebra structure morphisms.

Definition 3.5.1. Let $\mathcal{H}$ be a Hopf algebra in a braided monoidal category $\mathcal{C}$. A *(left-left) Yetter-Drinfeld module* is a triple such that

- (U, ρ) is a left $\mathcal{H}$-module,
- (U, δ) is a left $\mathcal{H}$-comodule,

such that

$$\begin{array}{ccccc}
\mathcal{H} \otimes U & \xrightarrow{\quad \rho \quad} & U & \xrightarrow{\quad \delta \quad} & \mathcal{H} \otimes U \\
{\scriptstyle \Delta^2 \otimes \delta} \big\downarrow & & & & \big\uparrow {\scriptstyle m^2 \otimes \rho} \\
\mathcal{H}^{\otimes 4} \otimes U & \xrightarrow{\mathsf{Id}_{\mathcal{H}} \otimes \tilde{c}_{\mathcal{H},\mathcal{H},\mathcal{H}} \otimes \mathsf{Id}_U} & \mathcal{H}^{\otimes 4} \otimes U & \xrightarrow{\mathsf{Id}_{\mathcal{H} \otimes \mathcal{H}} \otimes S \otimes \mathsf{Id}_{\mathcal{H} \otimes U}} & \mathcal{H}^{\otimes 4} \otimes U
\end{array}$$

commutes, where $\Delta^2 \otimes \delta = (\mathsf{Id}_{\mathcal{H}} \otimes \Delta \otimes \mathsf{Id}_{\mathcal{H} \otimes U})(\Delta \otimes \delta)$, $m^2 \otimes \rho = (m \otimes \rho)(\mathsf{Id}_{\mathcal{H}} \otimes m \otimes \mathsf{Id}_{\mathcal{H} \otimes U})$, and $\tilde{c}_{\mathcal{H},\mathcal{H},\mathcal{H}} = (\mathsf{Id}_{\mathcal{H}} \otimes c_{\mathcal{H},\mathcal{H}})(c_{\mathcal{H},\mathcal{H}} \otimes \mathsf{Id}_{\mathcal{H}})(\mathsf{Id}_{\mathcal{H}} \otimes c_{\mathcal{H},\mathcal{H}})$ and the associators have been suppressed.

A morphism of Yetter-Drinfeld modules U, V over $\mathcal{H}$ is a morphism $f : U \to V$ in $\mathcal{C}$ that is both a morphism of $\mathcal{H}$-modules and $\mathcal{H}$-comodules.

Remark 3.5.2. When working in Vect, we can write the left action and coaction maps in Sweedler notation as

$$\rho_U(h \otimes v) = h \cdot v, \ \delta(v) = v_{(-1)} \otimes v_{(0)}.$$

The Yetter-Drinfeld condition can then be written as

$$\delta(h \cdot v) = h_1 v_{(-1)} S(h_2) \otimes h_{(3)} \cdot v_{(0)} \tag{3.5.0.1}$$

We shall denote the category of (left-left) Yetter-Drinfeld modules by ${}^{\mathcal{H}}_{\mathcal{H}}\mathsf{YD}$. We can similarly define (left-right),(right-left) and (right-right) Yetter-Drinfeld modules over $\mathcal{H}$, where the first component describes the type of $\mathcal{H}$-module and the second the type of $\mathcal{H}$-comodule. The resulting categories are denoted by ${}^{\mathcal{H}}\mathsf{YD}_{\mathcal{H}}, {}_{\mathcal{H}}\mathsf{YD}^{\mathcal{H}}, \mathsf{YD}^{\mathcal{H}}_{\mathcal{H}}$ respectively. However, these four categories are all equivalent, see [BCPO19], so we can pick whichever description is most convenient. For the rest of this thesis, we shall work with the (left-left) description, so shall just refer to these as Yetter-Drinfeld modules.

The category of Yetter-Drinfeld modules inherits the properties of $\Bbbk$-linear abelian finite directly from $\mathcal{C}$, analagously to the standard module case. Also similarly to $\mathsf{Mod}_{\mathcal{C}}(A)$, as we are necesarily in a braided category, we can construct a tensor product of Yetter-Drinfeld modules. However, we can also extend this to a braided structure, with the coalgebra structure taking the place of the local structure.

Proposition 3.5.3 ([HS20, Theorem 3.4.13][Sch20, Propositions 4.4.6,4.4.7])**.** *Let $\mathcal{H}$ be a Hopf algebra in a braided monoidal category $\mathcal{C}$. Then*

- *The category ${}^{\mathcal{H}}_{\mathcal{H}}\mathsf{YD}$ of Yetter-Drinfeld modules is monoidal, with the underlying tensor product of $\mathcal{C}$ on objects, and the $\mathcal{H}$-action and coaction of $(U \otimes V)$ given by the following compositions:*

$$\rho_{U\otimes V} : \mathcal{H} \otimes U \otimes V \xrightarrow{\Delta \otimes \mathsf{Id}_{U\otimes V}} \mathcal{H} \otimes \mathcal{H} \otimes U \otimes V \xrightarrow{\mathsf{Id}_{\mathcal{H}} \otimes c_{\mathcal{H},U} \otimes \mathsf{Id}_V} \mathcal{H} \otimes U \otimes \mathcal{H} \otimes V \xrightarrow{\rho_U \otimes \rho_V} U \otimes V,$$
$$\delta_{U\otimes V} : U \otimes V \xrightarrow{\delta_U \otimes \delta_V} \mathcal{H} \otimes U \otimes \mathcal{H} \otimes V \xrightarrow{\mathsf{Id}_{\mathcal{H}} \otimes c_{U,\mathcal{H}} \otimes V} \mathcal{H} \otimes \mathcal{H} \otimes U \otimes V \xrightarrow{m \otimes \mathsf{Id}_{U\otimes V}} \mathcal{H} \otimes U \otimes V.$$

The unit object is the same as the underlying category, $\mathbb{1}$, with the unitors acting as its H-action and coaction morphisms. The rest of the structure is inherited directly from $\mathcal{C}$.

- *The category ${}^{\mathcal{H}}_{\mathcal{H}}\mathsf{YD}$ can be given a braided structure, with*

$$c'_{U,V} : U \otimes V \xrightarrow{\delta_U \otimes \mathsf{Id}_V} \mathcal{H} \otimes U \otimes V \xrightarrow{\mathsf{Id}_{\mathcal{H}} \otimes c_{U,V}} \mathcal{H} \otimes V \otimes U \xrightarrow{\rho_V \otimes \mathsf{Id}_U} V \otimes U \tag{3.5.0.2}$$

Example 3.5.4 (${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}$)**.** Let G be a finite group, and recall the group Hopf algebra $\Bbbk[G]$ from Example 3.4.6. Suppose we have a Yetter-Drinfeld module U over $\Bbbk[G]$. The $\Bbbk G$ - coaction allows us to define a G-grading on U, see [Sch20, Example 2.2.8], so we can write $U = \bigoplus_{g \in G} U_g$, with $\delta(u_d) = d \otimes u_d$, for a homogenous element $u_d \in U_d$.

The Yetter-Drinfeld condition now becomes equivalent to

$$\delta(h \cdot u_d) = hdh^{-1} \otimes h \cdot u_d,$$

which states that the action of h on a component $u_d \in U_d$ will take it to an element in the space graded by the conjugate of d by h , i.e $h \cdot d \in U_{hdh^{-1}}$.

The tensor product of two two Yetter-Drinfeld modules simply becomes the tensor product of G-graded vector spaces from Example 2.1.15, with the $\Bbbk[G]$-action being given by

$$\rho_{U \otimes V}(h \otimes (u_d \otimes v_f)) = h \cdot u_d \otimes h \cdot v_f.$$

for $u_d \in U_d, v_f \in V_f$. The braiding is given by

$$c_{U,V}(u_d \otimes v_f) = d \cdot v_f \otimes u_d.$$

Corollary 3.5.5. *The category ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}$ can be given the structure of a braided fusion category.*

Proof. Follows from combining Proposition 3.5.3 and the preceding discussion with [Sch20, Example 4.5.2 and Proposition 4.5.5]. □

It is apparent that there are a number of similarities between this category and the center category of $\mathcal{Z}(\mathsf{Vect}_G)$. Both categories can be described in terms of G-graded vector spaces with the same monoidal structure, and the Yetter-Drinfeld condition appears to induce a morphism from $V_d \to V_{gdg^{-1}}$, which is exactly the structure required in the monoidal center, see Example 2.2.8. These similarities are due to the following equivalence.

Proposition 3.5.6. *The categories ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}$ and $\mathcal{Z}(\mathsf{Vect}_G)$ are equivalent as braided monoidal categories.*

Proof. First, we define a functor

$$F : {}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD} \to \mathcal{Z}(\mathsf{Vect}_G) \tag{3.5.0.3}$$
$$(U, \rho) \mapsto (U, \gamma),$$
$$f : (U, \rho) \to (V, \rho') \mapsto f : (U, \gamma) \to (V, \gamma')$$

where the G-graded vector space U is mapped to itself, and its half braiding is given by

$$\gamma_X : X \otimes U \to U \otimes X,\ \gamma_X(x_d \otimes u) = d \cdot u \otimes x_d,$$

for all $X \in \mathsf{Ob}(\mathcal{Z}(\mathsf{Vect}_G))$, $x_d \in X_d$. Morphisms are sent to themselves as maps of the unchanged underlying vector spaces.

For this to be a morphism of G-graded vector spaces, we require that $\gamma_X(x_d \otimes u_f)$ has degree df, or equivalently that $dfd^{-1} = \deg(d \cdot u_f)$. This implies that $d \cdot u_f \in U_{dfd^{-1}}$ which is simply the Yetter-Drinfeld condition, as discussed in Example 3.5.4. That the half-braiding satisfies Equation (2.2.0.2) follows immediately as ρ is a $\Bbbk[G]$-action.

The functor F sends a morphism in $\mathsf{Hom}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}}(U, V)$ is sent to the same map on the underlying vector space in $\mathsf{Hom}_{\mathcal{Z}(\mathsf{Vect}_G)}(U, V)$. For a morphism $f : (U, \gamma) \to (V, \gamma')$ to be a morphism in $\mathcal{Z}(\mathsf{Vect}_G)$, we must satisfy Equation (2.2.0.3), which is equivalent in our case to

$$f(d \cdot v) \otimes x_d = d \cdot f(v) \otimes x_d,$$

which clearly holds if and only if f is a morphism of Yetter-Drinfeld modules. Hence we have a well-defined functor that is full and faithful.

For essential surjectivity, consider a generic object $(V, \gamma) \in \mathsf{Ob}(\mathcal{Z}(\mathsf{Vect}_G))$. If we define the map

$$\rho : \Bbbk[G] \otimes V \xrightarrow{\gamma_{\Bbbk[G]}} V \otimes \Bbbk[G] \xrightarrow{\varepsilon} V \otimes \mathbb{1} \cong V,$$

then this is a $\Bbbk[G]$-action that satisfies the Yetter-Drinfeld condition, similarly to earlier, because γ is a morphism in $\mathcal{Z}(\mathsf{Vect}_G)$. Using the counit of $\Bbbk[G]$, we recover that the image of ρ under F is exactly the initial half-braiding.

The functor F is clearly monoidal as both categories have the same underlying monoidal structure (tensor product, unit object, associator and unitors), with the lax and oplax structures both being given by identity morphisms. The braided monoidal condition from Definition 2.2.9 is similarly immediately satisfied, as the braiding $c_{F(U,\rho),F(V,\rho')}(u_d \otimes v_f) =$

$\gamma_U(u_d \otimes v_f) = d \cdot v_f \otimes u_d$, which is exactly the braiding in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}$. □

Remark 3.5.7. For any general Hopf algebra $\mathcal{H}$, the is an equivalence of tensor categories between ${}^{\mathcal{H}}_{\mathcal{H}}\mathsf{YD}$ and the center of the category of representations of $\mathcal{H}$, denoted by $\mathcal{Z}(\mathsf{Rep}(\mathcal{H}))$ (See [EGNO15, Proposition 7.15.3], [Maj00, Example 9.1.8]). As $\mathsf{Rep}(\Bbbk[G]) \cong \mathsf{Vect}_G$, Proposition 3.5.6 is an extension of this result to include compatibility with the braiding.

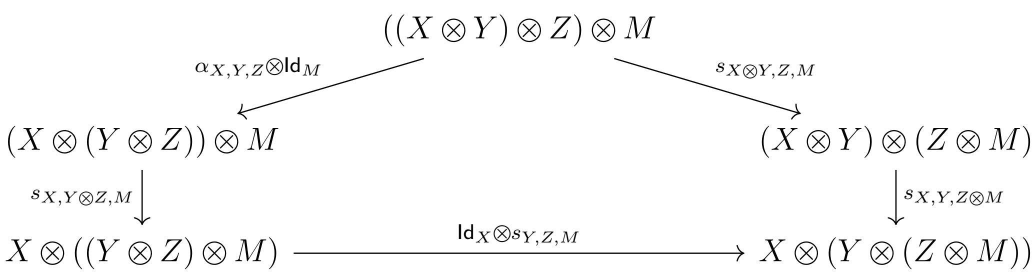
3.6 Module Categories

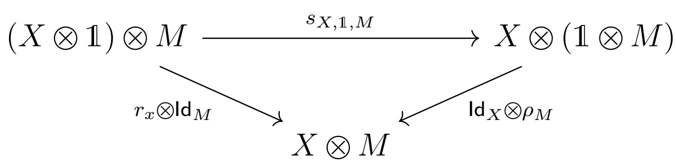
As well as using modules to study algebras, there is a way to study categories themselves in a similar manner.

Definition 3.6.1. Let $\mathcal{C}$ be a monoidal category. A *left module category* over $\mathcal{C}$ is a tuple $(\mathcal{M}, \otimes, s, \lambda)$, consisting of

- A category $\mathcal{M}$,
- A bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, called the *module action*,
- Two families of natural isomorphisms
  - Module associators $s_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$,
  - Left module unitor $\lambda_M : \mathbb{1} \otimes M \to \mathbb{1}$,

  such that the following diagrams commute;





for any objects $X, Y, Z \in \mathsf{Ob}(\mathcal{C}), M \in \mathsf{Ob}(\mathcal{M})$

Definition 3.6.2. A $\mathcal{C}$-module category $\mathcal{M}$ is said to be *indecomposable* if it cannot be written as direct sum of non-zero $\mathcal{C}$-module categories.

Similarly to the case of algebras, any monoidal category is a left module category over itself, with the original tensor product, associator and left unitor filling the roles of the module category structure.

Example 3.6.3. Let A be an algebra object in $\mathcal{C}$. The category $\mathsf{Mod}_{\mathcal{C}}(A)$ has a natural structure of a $\mathcal{C}$-module category, with the module product given by

$$\begin{aligned} \otimes : \mathcal{C} \times \mathsf{Mod}_{\mathcal{C}}(A) &\to \mathsf{Mod}_{\mathcal{C}}(A) \\ X \times (M, \rho) &\mapsto (X \otimes M, \rho'), \end{aligned}$$

where $(X \otimes M, \rho')$ is the A-module described in Example 3.1.9. The module associator and left module unitor are given by their monoidal counterparts. This is indecomposable as a $\mathcal{C}$-module category when the algebra A is indecomposable in $\mathcal{C}$.

Definition 3.6.4. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{C}$-module categories, with module associators s and s' respectively. A *$\mathcal{C}$-module functor* is a functor $F : \mathcal{M} \to \mathcal{N}$ that is equipped with a family of natural isomorphisms

$$\Lambda_{X,M} : F(X \otimes M) \to X \otimes F(M),$$

such that the following diagrams commute;

$$\begin{array}{ccccc} & & F((X \otimes Y) \otimes M) & & \\ & \swarrow^{F(s_{X,Y,M})} & & \searrow^{\Lambda_{X \otimes Y, M}} & \\ F(X \otimes (Y \otimes M)) & & & & (X \otimes Y) \otimes F(M) \\ \downarrow{\scriptstyle \Lambda_{X, Y \otimes M}} & & & & \downarrow{\scriptstyle s'_{X,Y,F(M)}} \\ X \otimes F(Y \otimes M) & & \xrightarrow{\mathsf{Id}_X \otimes \Lambda_{X,M}} & & X \otimes (Y \otimes F(M)). \end{array}$$

$$\begin{array}{ccc} F(\mathbb{1} \otimes M) & \xrightarrow{\Lambda_{\mathbb{1},M}} & \mathbb{1} \otimes F(M) \\ & {\scriptstyle F(\lambda_M)} \searrow \quad \swarrow {\scriptstyle \lambda_{F(M)}} & \\ & F(M) & \end{array}$$

for any objects $X, Y \in \mathsf{Ob}(\mathcal{C}), M \in \mathsf{Ob}(\mathcal{M})$.

Definition 3.6.5. Two algebras A, B in $\mathcal{C}$ are *Morita equivalent* if and only if $\mathsf{Mod}_{\mathcal{C}}(A)$ and $\mathsf{Mod}_{\mathcal{C}}(B)$ are equivalent as $\mathcal{C}$-module categories.

Generally, all of the information that is required to describe how a particular algebra object acts is encoded in its category of modules, so classifying algebra objects up to Morita equivalence is sufficient for classification results.

We shall finish this section by looking at some equivalences on the morphism spaces of module categories.

Proposition 3.6.6 ([EGNO15, Proposition 7.1.6])**.** *Let $\mathcal{C}$ be a rigid monoidal category, and let $\mathcal{M}$ be a $\mathcal{C}$-module category. Then there is a natural isomorphism*

$$\mathsf{Hom}_{\mathcal{M}}(X^* \otimes M, N) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{M}}(M, X \otimes N).$$

The next two propositions come from [EGNO15, Section 7.9]

Proposition 3.6.7. *Let $\mathcal{C}$ be a finite tensor category, and $\mathcal{M}$ be a $\mathcal{C}$-module category. Take objects $X \in \mathsf{Ob}(\mathcal{C}), M, N \in \mathsf{Ob}(\mathcal{M})$. Then there exists an object $\underline{\mathsf{Hom}}(M, N) \in \mathsf{Ob}(\mathcal{C})$ and a natural isomorphism*

$$\mathsf{Hom}_{\mathcal{M}}(X \otimes M, N) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(X, \underline{\mathsf{Hom}}(M, N)).$$

The object $\underline{\mathsf{Hom}}(M, N)$ is called the *internal Hom* from M to N. As well as allowing us to describe the morphisms of a module category in terms of the category acting on it, it also gives us a family of algebra objects.

Proposition 3.6.8. *The object $\underline{\mathsf{Hom}}(M, M)$ has the structure of an algebra object in $\mathcal{C}$ for all choices of $M \in \mathsf{Ob}(\mathcal{M})$.*

Example 3.6.9. If we take $\mathcal{M} = \mathsf{Mod}_{\mathcal{C}}(A)$ for some algebra A in $\mathcal{C}$, then the isomorphism in Equation (3.3.0.1) defines the internal hom for this category to be $\underline{\mathsf{Hom}}(A, M) = M$. In particular, the internal hom algebra $\underline{\mathsf{Hom}}(A, A) = A$ is simply the original algebra.

Chapter 4

Algebras in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$

We shall now begin looking at how we can find algebra objects in fusion categories. In this chapter, we shall tackle the problem using direct, hands-on methods by classifying algebra objects in a given category.

Recall the twisted group algebra $(\Bbbk[H], \kappa)$ in Vect_G^ω from Example 3.1.4. As this algebra is connected and separable, its category of modules $\mathcal{M}(H, \kappa) := \mathsf{Mod}_{\mathsf{Vect}_G^\omega}((\Bbbk[H], \kappa))$ is an indecomposable semisimple module category over Vect_G^ω by Proposition 3.3.5. These particular module categories acts as equivalence class representatives for all such Vect_G^ω-module categories.

Theorem 4.0.1 ([Ost03b], [Nat17]).

- *Every indecomposable semisimple Vect_G^ω-module category is equivalent as a Vect_G^ω-module category to one of the form $\mathcal{M}(H, \kappa)$, where $\omega|_H$ is trivial .*
- *We have that $\mathcal{M}(H, \kappa)$ and $\mathcal{M}(H', \kappa')$ are equivalent as Vect_G^ω-module categories if and only if H and H' are conjugate as subgroups of G, and the 2-cocycle $\kappa'^{-1}\kappa^x\gamma_x|_{H'\times H'}$ is trivial in $H^2(H', \Bbbk^\times)$, where $\kappa^x(h_1, h_2) = \kappa(xh_1x^{-1}, xh_2x^{-1})$, and $\gamma_x(h_1, h_2) := \gamma(x)(h_1, h_2)$, where γ is some map that shall be introduced later (see Equation (4.1.0.3)).*

Corollary 4.0.2. *Every connected separable algebra in Vect_G^ω is Morita equivalent to a twisted group algebra $(\Bbbk[H], \kappa)$.*

This collection of results are extremely powerful as they allows us to classify all connected separable algebras in Vect_G^ω in terms of objects that admit a Frobenius algebra structure. Moreover, we can describe when such objects are equivalent in terms of a cohomological condition.

Introduced in [ENO05], a *group-theoretical fusion category* is a category of the form $\mathsf{Bimod}_{\mathsf{Vect}_G^\omega}((\Bbbk[H], \kappa))$, for some choice of twisted group algebra where $\omega|_H$ is trivial. They are a generalisation of Vect_G^ω, as taking the twisted group algebra to be the monoidal unit algebra $\mathbb{1}$, viewed as $(\Bbbk[\{e\}], 1)$, returns the original category. As such, we can ask whether the classification of Theorem 4.0.1 can be extended to this more general setting in any way. This question has been answered in [MMP$^+$23]. There, a Frobenius monoidal functor $\Phi : \mathcal{C} \to \mathsf{Bimod}_{\mathcal{C}}(A)$ was constructed for any special Frobenius algebra A in a monoidal category $\mathcal{C}$. Applying this to the case of twisted group algebras in Vect_G^ω, we get a family of Frobenius algebras $\Phi((\Bbbk[H], \kappa))$ in the group-theoretical fusion categories, called *twisted Hecke algebras*. It is then shown that these new algebras are indecomposable and separable, and act as Morita equivalence class representatives for all such algebras in $\mathsf{Bimod}_{\mathcal{C}}(A)$.

There are other ways to extend Vect_G^ω, and we have already come across one. The monoidal center $\mathcal{Z}(\mathsf{Vect}_G^\omega)$ contains Vect_G^ω as a subcategory, so we can similarly ask how we can extend Theorem 4.0.1 to this case. This was done in the case that $\Bbbk$ has characteristic 0 in [DS17]. In this chapter, we shall discuss work first appearing in [HLRC23] that extends this result to the case that $\Bbbk$ has arbitrary characteristic.

To do this, we need a good working description of Vect_G^ω. We shall use a modified version of the equivalence in Proposition 3.5.6, using *twisted Yetter-Drinfeld modules* to ensure we can work with non-trivial choices of ω. Then, in the spirit of the group-theoretical case, we shall construct a Frobenius monoidal functor that allows us to use the Frobenius algebra structure of the twisted group algebras in an attempt to extend the classification of indecomposable separable algebras.

4.1 Category of twisted Yetter-Drinfeld Modules

To generalise the equivalence from Proposition 3.5.6, we define a twisted Yetter-Drinfeld module over the Hopf algebra $\Bbbk[G]$,following [Maj98, Proposition 3.2], to be a vector space U that has a G-grading (i.e. a $\Bbbk[G]$-coaction), and a $\Bbbk[G]$-action that satisfies the twisted action condition

$$h \cdot (k \cdot u_d) = \tau(h,k)(d) hk \cdot u_d. \tag{4.1.0.1}$$

for all $h, k \in G$, $u_d \in U_d$, where the coefficient $\tau(h,k)(d)$ is given by

$$\tau(h,k)(d) := \frac{\omega(h,k,d)\omega(hkd(hk)^{-1},h,k)}{\omega(h,kdk^{-1},k)}. \tag{4.1.0.2}$$

This construction follows Morphisms of these objects are the same as those in the non-twisted case, and thus we denote the category of twisted Yetter-Drinfeld modules over $\Bbbk[G]$ by ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}$.

Notation 4.1.1. For brevity, from now on we shall simply write YD instead of Yetter-Drinfeld.

This category is finite $\Bbbk$-linear abelian in the same way as ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}$, and much of the other structure making ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}$ a fusion category, from Corollary 3.5.5, can be used for the twisted version. The only additional condition is that the twisted action condition is satisfied.

Lemma 4.1.2. *The category of twisted YD modules can be given the structure of a monoidal category by using the same monoidal structure of* $\mathsf{Vect}^{\omega}_{G}$*, and defining the twisted* $\Bbbk[G]$*-action on the tensor product* $U \otimes V$ *to be*

$$h \cdot (u_d \otimes v_f) = \gamma(h)(d,f)(h \cdot u_d \otimes h \cdot v_f),$$

for $h \in G$ *and* $u_d \in U_d, v_f \in V_f$*, and where*

$$\gamma(h)(d,f) := \frac{\omega(h,d,f)\omega(hdh^{-1},hfh^{-1},h)}{\omega(hdh^{-1},h,f)}. \tag{4.1.0.3}$$

.

Proof. On the underlying vector spaces, the tensor product is the same as the untwisted case, being the tensor product of G-graded vector spaces. The additional structure required is the defined twisted $\Bbbk[G]$-action, which we need to show satisfies Equation (4.1.0.1). This means we must verify that

$$h \cdot (k \cdot (v_d \otimes u_f)) = \tau(h,k)(df)\, hk \cdot (u_d \otimes v_f).$$

Expanding out both sides, we find that

$$\begin{aligned} h \cdot (k \cdot (u_d \otimes v_f)) &= \gamma(k)(d,f) h \cdot (k \cdot u_d \otimes k \cdot v_f) \\ &= \gamma(k)(d,f)\gamma(h)(kdk^{-1},kfk^{-1})(h \cdot (k \cdot u_d) \otimes h \cdot (k \cdot v_f)) \\ &= \gamma(k)(d,f)\gamma(h)(kdk^{-1},kfk^{-1})\tau(h,k)(d)\tau(h,k)(f)(hk \cdot u_d \otimes hk \cdot v_f), \\ \tau(h,k)(df)hk \cdot (u_d \otimes v_f) &= \tau(h,k)(df)\gamma(hk)(d,f)(hk \cdot u_d \otimes hk \cdot v_f). \end{aligned}$$

Thus, we require that the following equality holds;

$$\gamma(k)(d,f)\gamma(h)\left(kdk^{-1},kfk^{-1}\right)\tau(h,k)(d)\tau(h,k)(f) = \tau(h,k)(df)\gamma(hk)(d,f). \tag{4.1.0.4}$$

Showing this requires us to use the definitions of τ and γ, and to make extensive use of the 3-cocycle condition from Equation (2.1.0.4). This holds, and the full calculation is given in Proposition A.2.1.

We now have a well-defined tensor product in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$. As for the rest of the monoidal structure, we take the monoidal unit, associator and unitors as those of Vect_G^ω. The only extra condition we need to verify is that the associator and unitors are morphisms of twisted YD modules.

As we have assumed that ω is normalised, by Remark 2.1.16, the unitors are simply identity morphisms and are clearly compatible with the twisted $\Bbbk[G]$-action. For the associator given by $\alpha_{\Bbbk_g,\Bbbk_h,\Bbbk_k} = \omega(g,h,k)^{-1}\mathsf{Id}_{\Bbbk_{ghk}}$, we need to check that

$$\alpha\left(h \cdot ([\Bbbk_g \otimes \Bbbk_{g'}] \otimes \Bbbk_{g''})\right) = h \cdot \left(\alpha\left([\Bbbk_g \otimes \Bbbk_{g'}] \otimes \Bbbk_{g''}\right)\right),$$

for all $h, g, g', g'' \in G$. This is equivalent to the equality

$$\omega^{-1}(hgh^{-1}, hg'h^{-1}, hg''h^{-1})\gamma(h)(gg', g'')\gamma(h)(g, g') = \gamma(h)(g, g'g'')\gamma(h)(g', g'')\omega^{-1}(g, g', g''). \tag{4.1.0.5}$$

which is shown to hold in Proposition A.2.2. Thus ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$ is a monoidal category. □

Lemma 4.1.3. *For two twisted YD modules $U, V \in \mathsf{Ob}({}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega)$, there is a braiding given by*

$$\begin{aligned} c_{U,V} : U \otimes V &\to V \otimes U \\ u_d \otimes v_f &\mapsto d \cdot v_f \otimes u_d, \end{aligned}$$

giving ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$ the structure of a braided monoidal category.

Proof. Once again, this is the same braiding structure as the untwisted case. However, due to the non-trivial associator, we have to verify that the hexagon axioms in Equations 2.2.0.1 and 2.2.0.2 still hold. They are equivalent to the identities

$$\begin{aligned} \omega(dfd^{-1}, d, h)^{-1} &= \omega(dfd^{-1}, dhd^{-1}, d)^{-1}\gamma(d)(f,h)\omega(d,f,h)^{-1}, \\ \omega(d, fhf^{-1}, f) &= \omega(d,f,h)\omega(dfhf^{-1}d^{-1}, d, f)\tau(d,f)(h)^{-1}, \end{aligned}$$

respectively. These hold by directly substituting in the definitions of τ and γ from Equation (4.1.0.2) and Equation (4.1.0.3).

Now we need to check that $c_{U,V}$ is a morphisms of twisted YD-modules, meaning

$$c_{U,V}(g \cdot (u_d \otimes v_f)) = g \cdot (c_{U,V}(u_d \otimes v_f)),$$

which is equivalent to the identity

$$\gamma(g)(d, f)\tau(gdg^{-1}, g)(f) = \gamma(g)(dfd^{-1}, d)\tau(g, d)(f), \tag{4.1.0.6}$$

This is proven in Proposition A.2.3. □

Similarly to Proposition 3.5.3 and Corollary 3.5.5, the remaining structure for ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}$ to be a fusion category is inherited directly from the underlying category. Hence ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}$ is a braided fusion category.

Proposition 4.1.4. *The categories* ${}^{\Bbbk G}_{\Bbbk G}\mathsf{YD}^{\omega}$ *and* $\mathcal{Z}(\mathsf{Vect}^{\omega}_G)$ *are equivalent as braided monoidal categories.*

Proof. This proof is analagous to the untwisted case from Proposition 3.5.6. We use the same functor from Equation (3.5.0.3), and the only additional condition that we need to check is that the proposed half-braiding still satisfies the half-braiding condition Equation (2.2.0.2) with non-trivial associator.

This condition is now equivalent to

$$\tau(h, k)(d)\omega(h, kdk^{-1}, k) = \omega(h, k, d)\omega(hkdk^{-1}h^{-1}, h, k),$$

which is exactly the definition of τ from Equation (4.1.0.2). All other conditions remain unchanged, thus the proof is concluded. □

4.2 Twisted group algebras

Now that we have an explicit description of $\mathcal{Z}(\mathsf{Vect}^{\omega}_G) \cong {}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}$, we need to come up with a candidate family of algebras that will serve as equivalence class representatives, similarly to the twisted group algebras for Vect^{ω}_G. We will do this by attempting to lift the twisted group algebras to the monoidal center.

We begin by introducing a collection of cohomological data.

Notation 4.2.1. Let $N \lhd G$ be a normal subgroup of G and $\kappa\colon N \times N \to \Bbbk^\times$ be a map satisfying

$$\omega(n,m,k) = \kappa(n,m)\kappa(m,k)^{-1}\kappa(nm,k)\kappa(n,mk)^{-1}, \qquad \kappa(n,1) = \kappa(1,n) = 1, \tag{4.2.0.1}$$

for all $n, m, k \in N$. In addition, let

$$\epsilon\colon G \times N \to \Bbbk^\times, \quad (g,n) \mapsto \epsilon_g(n)$$

be a map satisfying, for all $g, k \in G$ and $n, m \in N$, that

$$\tau(g,k)(n) = \frac{\epsilon_g(knk^{-1})\epsilon_k(n)}{\epsilon_{gk}(n)}, \tag{4.2.0.2}$$

$$\gamma(g)(n,m) = \frac{\epsilon_g(nm)}{\epsilon_g(n)\epsilon_g(m)} \cdot \frac{\kappa(gng^{-1}, gmg^{-1})}{\kappa(n,m)}, \tag{4.2.0.3}$$

$$\kappa(nmn^{-1}, n) = \epsilon_n(m)\kappa(n,m). \tag{4.2.0.4}$$

If we combine Equation (4.2.0.2) and Equation (4.2.0.3) with the fact that κ is normalised, we find that ϵ is also normalised in the sense that

$$\epsilon_h(1) = 1, \qquad \text{and} \qquad \epsilon_1(n) = 1.$$

Remark 4.2.2. Recall that the twisted group algebra (Example 3.1.4) in Vect_G^ω required the data of a 2-cocycle whose differential was equal to a particular 3-cocycle. The above data mirrors this setup.

We can view the maps $\epsilon(g,n) := \epsilon_g(n)$ and κ as a normalized element $\epsilon \oplus \kappa$ in the truncated total complex $\widetilde{F}^2_{\mathrm{Tot}}(G, N, \Bbbk^\times)$, where G acts on N by conjugation. For details of this construction, see Appendix A.3. Further, the element defined by $T(\omega) = \omega \oplus \gamma \oplus \tau$ is a 3-cocycle in $\widetilde{F}^3_{\mathrm{Tot}}(G, G, \kappa^\times)$, shown in Example A.3.2.

Then, by calculating the differential $\mathrm{d}^2_{\mathrm{Tot}}(\epsilon \oplus \kappa)$, see Example A.3.3, we find that Equations (4.2.0.1)–(4.2.0.3) are equivalent to

$$\mathrm{d}^2_{\mathrm{Tot}}(\epsilon \oplus \kappa) = T(\omega)|_{(G,N)},$$

where the 3-cocycle on the right-hand side has been restricted to $\widetilde{F}^3_{\mathrm{Tot}}(G, N, \Bbbk^\times)$.

This data now allows us to lift the twisted group algebra $(\Bbbk[G], \kappa)$ to the monoidal center, by adding a compatible $\Bbbk[G]$-action in the following way.

Proposition 4.2.3 (Algebras $B(N, \kappa, \epsilon)$)**.** *Assume we have a tuple $(G, N, \omega, \kappa, \epsilon)$ as described in Notation 4.2.1. Consider the $\Bbbk$-vector space $B(N, \kappa, \epsilon)$ with $\Bbbk$-basis $\{e_n \mid n \in N\}$, and define*

(i) $g \cdot e_n = \epsilon_g(n) e_{gng^{-1}}$, *for* $g \in G$;

(ii) $\delta(e_n) = n \otimes e_n$, *that is,* e_n *is homogeneous of degree* $n \in G$;

(iii) *multiplication* m_B *given by* $m_B(e_n \otimes e_m) = \kappa(n, m)^{-1} e_{nm}$ *for all* $n, m \in N$;

(iv) *unit* $u(1_\Bbbk) = e_e$.

Then $B(N, \kappa, \epsilon)$ is an algebra object in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$. Further, this algebra is connected and commutative.

Proof. First, we check that B is a twisted YD module. There is a G-grading, so we have a $\Bbbk[G]$-coaction. For the proposed action to be a twisted $\Bbbk[G]$-action, we need

$$g \cdot (k \cdot e_n) = \epsilon_k(n) g \cdot e_{knk^{-1}} = \epsilon_g(knk^{-1}) \epsilon_k(n) e_{gknk^{-1}g^{-1}},$$
$$\tau(g, k)(n) gk \cdot e_n = \tau(g, k)(n) \epsilon_{gk}(n) e_{gknk^{-1}g^{-1}},$$

to be equal, and so this is equivalent to

$$\epsilon_g(knk^{-1}) \epsilon_k(n) = \tau(g, k)(n) \epsilon_{gk}(n),$$

which is exactly Equation (4.2.0.2).

The Yetter-Drinfeld condition Equation (3.5.0.1) is satisfied as

$$\delta(g \cdot e_n) = \epsilon_g(n) \delta(e_{gng^{-1}}) = \epsilon_g(n) gng^{-1} \otimes e_{gng^{-1}} = gng^{-1} \otimes g.e_n.$$

So B is indeed an object in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$.

The proposed multiplication and unit morphisms of B are the same as those for the twisted group algebra in Vect_G^ω, from Example 3.1.4, so we already know that they are associative and unital by Equation (4.2.0.1). Hence we just need to verify that they are morphisms in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$.

For the multiplication to be a morphism of YD modules, we need

$$g \cdot m_B(e_n \otimes e_m) = m_B(g \cdot (e_n \otimes e_m)).$$

Computing these, we get that

$$\begin{aligned} g \cdot m_B(e_n \otimes e_m) =& \kappa(n,m)^{-1} g \cdot e_{nm} = \kappa(n,m)^{-1}\epsilon_g(nm) e_{gnmg^{-1}}, \\ m_B(g \cdot (e_n \otimes e_m)) &= \gamma(g)(n,m) m_B(g \cdot e_n \otimes g \cdot e_m) \\ &= \gamma(g)(n,m)\epsilon_g(n)\epsilon_g(m) m_B(e_{gng^{-1}} \otimes e_{gmg^{-1}}) \\ &= \gamma(g)(n,m)\epsilon_g(n)\epsilon_g(m)\kappa(gng^{-1}, gmg^{-1})^{-1}) e_{gnmg^{-1}}, \end{aligned}$$

and so they are equal by Equation (4.2.0.3). Hence B is an algebra object in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$.

For the adjectives, this algebra is clearly connected as the trivially-graded component is necessarily 1-dimensional, and so as morphisms in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$ respect the G-grading, the morphism space $\mathsf{Hom}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}(\mathbb{1}, B)$ is also 1-dimensional.

To see commutativity, we compute that

$$m_B c_{B,B}(e_n \otimes e_m) = m_B(n.e_m \otimes e_n) = \epsilon_n(m) m_B(e_{nmn^{-1} \otimes e_n}) = \kappa^{-1}(nmn^{-1}, n)\epsilon_n(m)(e_{nm}),$$

and thus the condition $m_B \circ c_{B,B} = m_B$ is exactly satisfied by Equation (4.2.0.4). □

By adding a Yetter-Drinfeld structure, we have successfully lifted the twisted group algebras from Example 3.1.4 to the monoidal center, which are dependent on a choice of subgroup (normal this time) and a 2-cocycle with specified differential (in the total complex). Additionally, by making a further imposition, we have incorporated the braided structure of this case to make these algebras commutative.

To complete this generalisation, we would like to also bring forward the Frobenius structure of the twisted group algebras.

Proposition 4.2.4. *The algebras $B = B(N, \kappa, \epsilon)$ are Frobenius algebras in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$ with coalgebra structure given by*

$$\Delta_B(e_n) = \sum_{m \in N} \kappa(m, m^{-1}n)\; e_m \otimes e_{m^{-1}n}, \qquad \varepsilon_B(e_n) = \delta_{n,e}, \quad \textit{for all } n \in N.$$

Additionally, if $|N| \in \Bbbk^\times$, then B is a special Frobenius algebra.

Proof. This coalgebra structure is the same as for the twisted group algebra in Vect^ω_G from

Example 3.2.3, so we simply need to check that these comultiplication and counit morphisms are morphisms of twisted YD modules.

We compute that

$$\begin{aligned}
g \cdot \Delta_B(e_n) &= \sum_{m \in N} \kappa(m, m^{-1}n) g \cdot (e_m \otimes e_{m^{-1}n}) \\
&= \sum_{m \in N} \kappa(m, m^{-1}n) \gamma(g)(m, m^{-1}n)\ g \cdot e_m \otimes g \cdot e_{m^{-1}n} \\
&= \sum_{m \in N} \kappa(m, m^{-1}n) \gamma(g)(m, m^{-1}n) \epsilon_g(m) \epsilon_g(m^{-1}n)\ e_{gmg^{-1}} \otimes e_{gm^{-1}ng^{-1}}, \\
\Delta_B(g \cdot e_n) &= \epsilon_g(n) \Delta_B(e_{gng^{-1}}) = \sum_{m \in N} \epsilon_g(n) \kappa(m, m^{-1}gng^{-1}) e_m \otimes e_{m^{-1}gng^{-1}}.
\end{aligned}$$

As N is a normal subgroup, in the second sum we can make the index substitution of $m \mapsto gmg^{-1}$. This allows us to compare the two expressions, which are equal if and only if

$$\gamma(g)(m, m^{-1}n) \kappa(m, m^{-1}n) \epsilon_g(m) \epsilon_g(m^{-1}n) = \epsilon_g(n) \kappa(gmg^{-1}, gm^{-1}ng^{-1}),$$

which holds by Equation (4.2.0.3)

If we calculate the compositions from the definition of special Frobenius, we get that

$$m_B \Delta_B(e_n) = \sum_{k \in n} e_n = |N| e_n, \quad \varepsilon_B u_B(1) = 1.$$

Thus we see that B is special Frobenius if and only if $|N| \in \Bbbk^\times$. □

Remark 4.2.5. Recall by Remark 3.2.5 that any special Frobenius algebra gives rise to a separable algebra by scaling the coproduct. In the case that $|N| \in \Bbbk^\times$, this is done by instead equipping $B(N, \kappa, \epsilon)$ with the coproduct $\Delta'_B = |N|^{-1} \Delta_B$. This is exactly the coproduct that appears on the twisted group algebra in [MMP+23].

As a result, we have now successfully lifted the twisted group algebras $(\Bbbk[G], \kappa)$ in Vect_G^ω to algebras $B(N, \kappa, \epsilon)$ in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$, preserving the connected separable Frobenius algebra structures. Additionally, we have gained commutativity to utilise the additional structure gained from moving to the monoidal center. We now wish to follow Theorem 4.0.1 and use these algebras to classify some collection of algebra objects in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$.

Proposition 4.2.6. *Let B be a separable commutative algebra in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$ such that $B_e = \Bbbk$. Then B is isomorphic as an algebra in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$ to $B(N, \kappa, \epsilon)$ for $N = \{g \in G \mid B_g \neq 0\}$.*

Proof. Using the alternative definition of separability from Remark 3.2.5, we have that the morphism

$$B \otimes B \xrightarrow{m} B \to \mathbb{1} \cong \Bbbk_e$$

is non-degenerate in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$. Hence, to respect the grading, this restricts to a non-degenerate morphism

$$B_g \otimes B_g^{-1} \to \mathbb{1},$$

hence any element $b \in B_g$, where B_g is non-zero, is a unit. In particular, this means that $ab \in B_{gh}$ is non-zero if and only if $a \neq 0 \in B_g, b \neq 0 \in B_h$. Thus, the following subset

$$N = \mathrm{Supp}(G) := \{g \in G | B_g \neq 0\}$$

is a subgroup of G. In fact, by the twisted YD condition we have that $g \cdot b \in B_{gng^{-1}}$, for all $g \in G$, $b \in B_n$, we get that N is a normal subgroup of G.

If we then consider elements $a, c \in B_g$, $b \in B_{g^{-1}}$ such that a, c, ab, bc are non-zero, we have by associativity that $(ab)c = \omega(g, g^{-1}, g)a(bc)$, so a, c are proportional. This means that $\mathsf{dim}_\Bbbk B_g \leqslant 1$ for all $g \in G$, and we can choose a $\Bbbk$-basis $\{e_n\}_{n \in N}$ for B.

Also as as $\mathsf{dim}_\Bbbk B_g \leqslant 1$, the multiplication and $\Bbbk[G]$-actions of B are determined by scalars $\kappa(n, m), \epsilon_g(n) \in \Bbbk^\times$ respectively, satisfying

$$\begin{aligned} e_n e_m &= \kappa(n, m)^{-1} e_{nm}, \qquad \forall n, m \in N. \\ g \cdot e_n &= \epsilon_g(n) e_{gng^{-1}}, \qquad \forall g \in G, n \in N. \end{aligned}$$

This gives us the triple (N, κ, ϵ), and it follows from B being an algebra in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$ that the conditions in Notation 4.2.1 are satisfied (as seen in the proof of Proposition 4.2.3). □

So now we have a family of algebras in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$ that are represented by the algebras $B(N, \kappa, \epsilon)$. However, we currently have the fairly strong condition that the trivially-graded component of these algebras is trivial. We shall now explore how we can extend the collection of algebras in this classification by means of a Frobenius monoidal functor.

4.3 Constructing a Frobenius Monoidal Functor

To further utilise the lifted twisted group algebras $B(N, \kappa, \epsilon)$, we will construct a functor that allows us to transport the Frobenius algebra structures.

Let $H \subseteq G$ be a subgroup of G. Then $\omega|_H \in C^3(H, \Bbbk^\times)$. We shall make a slight abuse of notation and write ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega := {}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^{\omega|_H}$.

Consider a functor

$$I : {}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega \to {}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega.$$

On objects, for any $U \in \mathsf{Ob}({}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega)$, the underlying target vector space is given by

$$U \mapsto I(U) := \Bbbk G \otimes U,$$

and we impose upon it the relation

$$gh \otimes u_d = \tau(g,h)(d)^{-1} g \otimes h \cdot u_d, \tag{4.3.0.1}$$

for $g \in G,\ h, d \in H$.

On morphisms, for $f \in \mathsf{Hom}_{{}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega}(U, V)$, the functor acts as $f \mapsto I(f) := \mathsf{Id}_{\Bbbk[G]} \otimes f$. For this to be a well-defined functor, we need to endow $I(U)$ with the structure of a twisted YD module. This is done with a $\Bbbk[G]$-action given by

$$\begin{aligned} \rho_{I(U)} &: \Bbbk[G] \otimes I(U) \to I(U) \\ & g \otimes (k \otimes u_d) \mapsto g \rhd (k \otimes u_d) := \tau(g,k)(d) gk \otimes u_d, \end{aligned} \tag{4.3.0.2}$$

and $\Bbbk[G]$-coaction given by

$$\begin{aligned} \delta_{I(U)} &: I(U) \to \Bbbk[G] \otimes I(U) \\ & (g \otimes v_d) \mapsto gdg^{-1} \otimes (g \otimes v_d). \end{aligned} \tag{4.3.0.3}$$

Lemma 4.3.1. *$I\,(U)$ has the structure of a twisted YD module over $\Bbbk[G]$.*

Proof. First, we have to check that Equation (4.3.0.2) is a valid $\Bbbk[G]$-coaction. For compatibility with the comultiplication of $\Bbbk[G]$, we require the following two equations to be equal,

$$\begin{aligned} (\Delta \otimes \mathsf{Id})\,\delta\,(g \otimes u_d) &= (\Delta \otimes \mathsf{Id})\left(gdg^{-1} \otimes g \otimes u_d\right) = gdg^{-1} \otimes gdg^{-1} \otimes g \otimes u_d, \\ (\mathsf{Id} \otimes \delta)\,\delta\,(g \otimes u_d) &= gdg^{-1} \otimes (g \otimes u_d) = gdg^{-1} \otimes gdg^{-1} \otimes g \otimes u_d. \end{aligned}$$

which is immediate. Compatibility with the counit is also immediate.

For the proposed map to be a twisted action, we need Equation (4.1.0.1) to hold, which for the proposed $\Bbbk[G]$-action is equivalent to

$$\tau(h,k)(d)\tau(g,hk)(d) = \tau(g,h)(kdk^{-1})\tau(gh,k)(d).$$

This holds and is proven in Proposition A.2.4. Unit compatibility is immediate as ω is normalised.

The final thing to check is the YD compatibility condition from Definition 3.5.1. For this, we require that for any $g, k \in G$, $d \in H$, the following two compositions are equal;

$$\begin{aligned} g \otimes (k \otimes u_d) \mapsto g \otimes g{\otimes}g \otimes kdk^{-1} \otimes (k \otimes u_d) \mapsto g \otimes kdk^{-1} \otimes g^{-1} \otimes g \otimes (k \otimes u_d) \mapsto \\ \mapsto gkdk^{-1}g^{-1} \otimes g \rhd (k \otimes u_d) = \tau(g,k)(d) gkdk^{-1}g^{-1} \otimes (gk \otimes u_d), \\ g \otimes (k \otimes u_d) \mapsto g \rhd (k \otimes u_d) = \tau(g,k)(d)(gk \otimes u_d) \mapsto \tau(g,k)(d) gkdk^{-1}g^{-1} \otimes (gk \otimes u_d), \end{aligned}$$

which is clearly the case. □

As discussed earlier, we wish to use this functor to transport our found Frobenius algebras between categories. As such, we need to check that the functor I is compatible with the monoidal structures involved. We begin by looking at a lax structure for I.

For this, we require natural morphisms $I(U) \otimes I(V) \to I(U \otimes V)$ hence we should consider vectors of the form $(g \otimes u_d) \otimes (k \otimes v_f)$. If $gH = kH$, then we can use the relation in Equation (4.3.0.1) to rewrite this vector as

$$(g \otimes u_d) \otimes (k \otimes v_f) = (g \otimes u_d) \otimes (g(g^{-1}k) \otimes v_f) = \tau(g, g^{-1}k)(f)^{-1} (g \otimes u_d) \otimes (g \otimes (g^{-1}k) \cdot v_f).$$

As such, in this case it is sufficient to consider vectors of the form $(g \otimes u_d) \otimes (g \otimes v_f)$.

From this observation, we make the choice to only consider vectors of this form, by defining the natural transformation $\mu_{U,V} \colon I(U) \otimes I(V) \to I(U \otimes V)$ as sending a vector $(g \otimes u_d) \otimes (k \otimes v_f)$ to zero unless $g^{-1}k \in H$. If $gH = kH$, then

$$\mu_{U,V}((g \otimes u_d) \otimes (g \otimes v_f)) = \gamma(g)(d,f)^{-1} g \otimes (u_d \otimes v_f). \tag{4.3.0.4}$$

For the unit morphism of the lax monoidal structure, we define a morphism

$$\eta \colon \mathbb{1} \to I(\mathbb{1}), \qquad 1_{\Bbbk} \mapsto \sum_i g_i \otimes 1_k, \tag{4.3.0.5}$$

where $\{g_i\}_{i \in I}$ is a set of representatives for the left cosets of H in G, i.e. $G = \coprod_i g_i H$.

Remark 4.3.2. We can combine these steps to alternatively write this natural transformation μ as

$$\mu_{U,V}\colon I(U)\otimes I(V) \longrightarrow I(U\otimes V),$$
$$(g\otimes u_d)\otimes(k\otimes v_f) \mapsto \begin{cases}\zeta(g,d,k,f)g\otimes(u_d\otimes g^{-1}k\cdot v_f), & \text{if } gH = kH,\\ 0, & \text{else,}\end{cases}$$

where

$$\zeta(g,d,k,f) = \Big(\tau(g,g^{-1}k)(f)\gamma(g)(d,g^{-1}kfk^{-1}g)\Big)^{-1}.$$

Lemma 4.3.3. *The pair (μ,η) as defined in Equations 4.3.0.4 and 4.3.0.5 equips the functor I with a lax monoidal structure.*

Proof. Firstly, the natural morphism $\mu_{U,V}$ is a morphism of YD modules as the condition

$$\mu_{U,V}(k\rhd((g\otimes u_d)\otimes(g\otimes v_f))) = k\rhd(\mu_{U,V}((g\otimes u_d)\otimes(g\otimes v_f)))$$

is equivalent to the equality

$$\tau(k,g)(df)\gamma(g)(d,f)^{-1} = \gamma(kg)(df)^{-1}\tau(k,g)(d)\tau(k,g)(f)\gamma(k)(gdg^{-1},gfg^{-1}),$$

which is exactly Equation (4.1.0.4). For the unit morphism, this is immediate as the unit object $\mathbb{1}$ has trivial YD action.

Per our discussion prior to Remark 4.3.2, it suffices to check that the lax monoidal condition in Definition 2.1.7 holds on vectors of the form $((g\otimes u_d)\otimes(g\otimes v_f))\otimes(g\otimes w_h)$. If we compute the relevant composition of maps on these vectors, we get that

$$\begin{aligned}&I(\alpha_{U,V,W})\mu_{U\otimes V,W}(\mu_{U,V}\otimes \mathsf{Id}_{I(W)})(((g\otimes u_d)\otimes(g\otimes v_f))\otimes(g\otimes w_h))\\ &= \gamma(g)(d,f)^{-1}\gamma(g)(df,h)^{-1}\omega(d,f,h)^{-1}g\otimes(u_d\otimes(v_f\otimes w_h)),\\ &\mu_{U,V\otimes W}(\mathsf{Id}_{I(U)}\otimes\mu_{V,W}\alpha_{I(U),I(V),I(W)}(((g\otimes u_d)\otimes(g\otimes v_f))\otimes(g\otimes w_h))\\ &= \omega(gdg^{-1},gfg^{-1},ghg^{-1})\gamma(g)(f,h)^{-1}\gamma(g)(d,fh)^{-1}g\otimes(u_d\otimes(v_f\otimes w_h)).\end{aligned}$$

These equations are equal due to Equation (4.1.0.5).

The unitality conditions hold immediately as $\gamma(g)(d,e) = 1 = \gamma(g)(e,d)$. □

We shall now consider an op-lax monoidal structure on I. Let us define the natural transformation;

$$
\begin{aligned}
\nu_{U,V} \colon I\left(U \otimes V\right) &\to I\left(V\right) \otimes I\left(V\right), \\
g \otimes \left(u_d \otimes v_f\right) &\mapsto \gamma\left(g\right)\left(d, f\right)\left(g \otimes u_d\right) \otimes \left(g \otimes v_f\right),
\end{aligned} \tag{4.3.0.6}
$$

for any $d, f \in H$, and $V, U \in \mathsf{Ob}\left(\mathcal{Z}(\mathsf{Vect}_H^\omega)\right)$. We also define the morphism

$$
\epsilon : I(\mathbb{1}) \to \mathbb{1}, \qquad \epsilon(g \otimes 1_{\Bbbk}) = 1_{\Bbbk}. \tag{4.3.0.7}
$$

Lemma 4.3.4. *The pair (ν, ϵ) defined by Equations 4.3.0.6 and 4.3.0.7 equips the functor I with an op-lax monoidal structure.*

Proof. As always, we need to check that the defined maps are morphisms of YD modules. For the morphism ϵ, this follows from ω being normalised and $\mathbb{1}$ having trivial YD action, as

$$
\epsilon(k \rhd (g \otimes 1_{\Bbbk})) = \tau(k, g)(e)\epsilon(kg \otimes 1_{\Bbbk}) = 1_{\Bbbk} = k \rhd 1_{\Bbbk} = k \rhd \epsilon(g \otimes 1_{\Bbbk}).
$$

For the natural transformation ν, we calculate that

$$
\nu_{U,V}(k \rhd (g \otimes (u_d \otimes v_f))) = k \rhd (\nu_{U,V}(g \otimes (u_d \otimes v_f)))
$$

is equivalent to the identity

$$
\tau(k, g)(df)\gamma(kg)(d, f) = \tau(k, g)(d)\tau(k, g)(f)\gamma(g)(d, f)\gamma(k)(gdg^{-1}, gfg^{-1})
$$

This is the same condition as in Equation (4.1.0.4), so $\nu_{U,V}$ is a morphism of YD modules.

The op-lax monoidal condition can be verified in exactly the same way as the proof of the lax monoidal structure, and we also get that this is equivalent to both Equation (4.1.0.5) being true and $\gamma(g)(d, e) = 1 = \gamma(g)(e, d)$. □

So, by Proposition 3.2.7, we now have a functor that preserves algebra and coalgebra structures. The only thing left to verify is that Frobenius algebra structures are preserved under I, which will enable us to transport the twisted group algebras $B(N, \kappa, \varepsilon)$.

Proposition 4.3.5. *The functor $I \colon {}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega \to {}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$ is a Frobenius monoidal functor.*

Proof. The claim that I is a Frobenius monoidal functor follows from checking the diagrams in Definition 2.1.9. For the first condition, we note that both compositions use the lax monoidal structure μ and so it is clear that we can simply consider vectors of the form $(g\otimes u_d)\otimes(g\otimes(v_f\otimes w_h))$, as otherwise we will end up with 0. We compute that this condition is equivalent to the identity

$$\omega(gdg^{-1}, gfg^{-1}, ghg^{-1})\gamma(g)(f,h)\gamma(g)(d,f)^{-1} = \gamma(g)(d,fh)^{-1}\gamma(g)(df,h)\omega(d,f,h)$$

which is simply Equation (4.1.0.5) rearranged. The second diagram holds similarly. $\square$

Hence, I preserves Frobenius algebras, which in particular preserves the lifted twisted group algebras $B(N,\kappa,\epsilon) \in \mathsf{Ob}(\mathcal{Z}(\mathsf{Vect}_G^\omega))$. These algebras have other properties, such as commutativity and special Frobenius, which we would also like to preserve.

Proposition 4.3.6. *The functor I is a braided separable Frobenius monoidal functor, and $\epsilon \circ \eta \neq 0$ if and only if $|G:H| \in \Bbbk^\times$.*

Proof. The condition in Equation (2.1.0.1) is satisfied as

$$\begin{aligned}\mu_{U,V}\nu_{U,V}(g\otimes(u_d\otimes v_f)) =& \gamma(g)(d,f)\mu_{U,V}((g\otimes u_d)\otimes(g\otimes v_f)) \\ &= \gamma(g)(d,f)\gamma(g)(d,f)^{-1}(g\otimes(u_d\otimes v_f)) = (g\otimes(u_d\otimes v_f)),\end{aligned}$$

so I is a separable Frobenius monoidal functor. The condition on the unit and counit morphisms are also easily verified, as

$$\epsilon\circ\eta(1_\Bbbk) = \epsilon\left(\sum_i g_i\otimes 1_\Bbbk\right) = |G:H|\cdot 1_\Bbbk.$$

To show that it is compatible with the braiding, we need to verify I is both braided lax monoidal and braided oplax monoidal.

First, we check that the lax monoidal structure (μ,η) is compatible with the braiding as in Definition 2.2.9.

By our earlier discussion, $I(c_{U,V})\mu_{U,V}$ will be zero on all vectors that are not some linear combination of vectors of the form $(g\otimes u_d)\otimes(g\otimes v_f)$. To see that this is also true for $\mu_{V,U}c_{I(U),I(V)}$, we compute

$$\mu_{V,U}c_{I(U),I(V)}((g\otimes u_d)\otimes(k\otimes v_f)) = \tau(gdg^{-1},k)(f)\mu_{V,U}((gdg^{-1}k\otimes v_f))\otimes(g\otimes u_d))$$

Now this term is non-zero only when $dg^{-1}k^{-1} \in H$, which is equivalent to requiring $g^{-1}k \in H$, as $d \in H$. Hence we can restrict to vectors of the proposed form. When we do so, we compute that

$$\begin{aligned}\mu_{V,U}c_{I(U),I(V)}((g\otimes u_d)\otimes(g\otimes v_f)) &= \mu_{V,U}((gdg^{-1}\rhd(g\otimes v_f))\otimes(g\otimes u_d))\\ &= \tau(gdg^{-1},g)(f)\mu_{V,U}((gd\otimes v_f)\otimes(g\otimes u_d))\\ &= \tau(gdg^{-1},g)(f)\tau(g,d)(f)^{-1}\mu_{V,U}((g\otimes d\cdot v_f)\otimes(g\otimes u_d))\\ &= \tau(gdg^{-1},g)(f)\tau(g,d)(f)^{-1}\gamma(g)(dfd^{-1},d)^{-1}g\otimes(d\cdot v_f\otimes u_d),\\ I(c_{U,V})\mu_{U,V}((g\otimes u_d)\otimes(g\otimes v_f)) &= \gamma(g)(d,f)^{-1}I(c_{U,V})(g\otimes(u_d\otimes v_f))\\ &= \gamma(g)(d,f)^{-1}g\otimes(d\cdot v_f\otimes u_d).\end{aligned}$$

By expanding out both coefficients using the definitions of τ and γ from Equations 4.1.0.2 and 4.1.0.3, we see that these are equal.

The braided oplax monoidal condition follows similarly. The result of this computation gives

$$\gamma(g)(dfd^{-1},d)=\tau(g,d)(f)^{-1}\tau(gdg^{-1},g)(f)\gamma(g)(d,f),$$

which is the same identity, just rearranged.

□

As a result of this, by Proposition 3.2.7, the functor I preserves a whole variety of algebraic structures.

Corollary 4.3.7.
- *If A is an algebra/coalgebra/Frobenius algebra in ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$, then $I(A)$ is an algebra/coalgebra/Frobenius algebra in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$.*
- *If A is a commutative algebra in ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$, then $I(A)$ is commutative in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$.*
- *If A is a special Frobenius algebra in ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$ and $|G:H|\in\Bbbk^\times$, then$I(A)$ is special Frobenius in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$.*

So we have now achieved our goal of constructing a functor that will enable us to preserve the structures on the twisted group algebras $B(N,\kappa,\epsilon)$ in ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$. Before we apply the functor I to this case, let's first consider how the trivial algebra object $\mathbb{1}$, which is a commutative Frobenius algebra, behaves under I.

Example 4.3.8. Let us define $A_H := I(\mathbb{1})$ to be the image of the monoidal unit of ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$ under the functor I. Explicitly, A_H is spanned as a $\Bbbk$-vector space by $\{g\otimes 1_\Bbbk | g\in G\}$, with the relation Equation (4.3.0.1) becoming simply that $g \otimes 1_{\Bbbk} = k \otimes 1_{\Bbbk}$ if and only if $g^{-1}k \in H$. From now on, we shall denote these basis vectors as $\upsilon_{gH} := g \otimes 1_{\Bbbk}$.

As $\mathbb{1}$ has trivial YD module structure, using Equation (4.3.0.2) and Equation (4.3.0.3), we get that A_H is a YD module with

$$k \cdot \upsilon_{gH} = \delta_{kgH}, \qquad \delta(\upsilon_{gH}) = 1 \otimes \upsilon_{gH}.$$

So A_H is completely contained in its trivially-graded component.

Using Proposition 3.2.7, A_H is an algebra in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}$ with multiplication and unit given by

$$m_{A_H}(\upsilon_{gH} \otimes \upsilon_{kH}) = \begin{cases} \upsilon_{gH}, & \text{if } g^{-1}k \in H, \\ 0, & \text{otherwise,} \end{cases} \qquad u_{A_H}(1_{\Bbbk}) = \sum_i \delta_{g_i H} =: 1_{A_H},$$

where $\{g_i\}$ is a set of H-coset representatives.

Similarly, A_H is a coalgebra with comultiplication and counit

$$\Delta_{A_H}(\upsilon_{gH}) = \upsilon_{gH} \otimes \upsilon_{gH}, \qquad \varepsilon_{A_H}(\upsilon_{gH}) = 1_{\Bbbk}.$$

These structures make A_H a commutative Frobenius algebra in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}$.

Lemma 4.3.9. *Assume $|G : H| \in \Bbbk^{\times}$. Then the algebra A_H is a connected special Frobenius algebra in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}$.*

Proof. The algebra is connected as A_H is concentrated in G-degree e, it is a G-module and $\mathsf{Hom}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}}(\mathbb{1}, A_H) \subseteq (A_H)^G$. The latter space of G-invariant elements in A_H is one-dimensional since A_H is given by functions on a transitive G-set.

The special Frobenius condition follows immediately from Corollary 4.3.7. We can also see this by computing explicitly that

$$m_{A_H}\Delta_{A_H}(\upsilon_{gH}) = \upsilon_{gH}\upsilon_{gH} = \upsilon_{gH}, \qquad \varepsilon_{A_H}(1_{A_H}) = |G : H|.$$

□

As well as being a nice example of how the functor I works, the algebra A_H also allows us to impose more structure on the image of this functor.

Lemma 4.3.10. *For any object V in $\mathsf{Ob}({}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^{\omega})$, $I(V)$ is a local A_H-module.*

Proof. We propose that the right action $a_{I(U)} : I(U) \otimes A_H \to I(U)$ is given by

$$(g \otimes u_d) \cdot v_{kH} = \begin{cases} g \otimes u_d, & \text{if } k^{-1}g \in H, \\ 0, & \text{otherwise.} \end{cases}$$

For this action to have compatibility with the multiplication of A_H, we check that the two compositions

$$a_{I(U)}(a_{I(U)} \otimes \mathsf{Id}_{A_H})(((g \otimes u_d) \otimes v_{kH}) \otimes v_{fH}) = \begin{cases} g \otimes u_d & \text{if } g^{-1}k, g^{-1}f \in H \\ 0 & \text{else} \end{cases},$$

$$a_{I(U)}(\mathsf{Id}_{I(U)} \otimes m_{A_H})\alpha_{I(U),A_H,A_H}(((g \otimes u_d) \otimes v_{kH}) \otimes v_{fH}) = \begin{cases} g \otimes u_d & \text{if } k^{-1}f, g^{-1}k \in H \\ 0 & \text{else} \end{cases},$$

are equal, which is clear as H is a group.

The unitality condition is easily satisfied;

$$(g \otimes u_d) \cdot 1_{A_H} = \sum_i (g \otimes u_d) \cdot v_{g_i H} = g \otimes u_d.$$

So $I(U)$ is a right A_H-module. To check the locality condition from Equation (3.3.0.3), we compute that

$$\begin{aligned} a_{I(U)}c_{A_H,I(U)}c_{I(U),A_H}((g \otimes u_d) \otimes v_{kH}) &= a_{I(U)}c_{A_H,I(U)}(v_{gdg^{-1}kH} \otimes (g \otimes u_d)) \\ &= a_{I(U)}(g \otimes u_d) \otimes v_{gdg^{-1}kH} = \begin{cases} g \otimes u_d, & \text{if } k^{-1}gd^{-1} \in H, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

As $d \in H$, this is just the result of applying the right action $a_{I(V)}$ only. □

From this, we can actually restrict I to a functor

$$I : {}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega \to \mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}(A_H).$$

As A_H is a commutative algebra in a braided monoidal category, its category of local modules is braided monoidal by Proposition 3.3.7. We now check whether this induced functor is compatible with these monoidal and braiding structures.

By Proposition 3.3.4, the tensor product of $X, Y \in \mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}(A_H)$ is given by the relative tensor product $X \otimes_{A_H} Y$, with the left action induced by braiding in the following way;

$$a^l_X := a^r_X c_{A_H,X}.$$

If we then compute $I(U) \otimes_{A_H} I(V)$, we find that the vector $(g \otimes u_d) \otimes (k \otimes v_f)$ is zero if and only if $gH \neq kH$, and so as vector space we have that

$$I(U) \otimes_{A_H} I(V) = (I(U) \otimes I(V))/S, \text{ where } S = \mathrm{span}_{\Bbbk}\{(g \otimes u_d) \otimes (k \otimes v_f) | g^{-1}k \notin H\}.$$

For this to be a valid quotient in $\mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}}(A_H)$, we need to verify that S is a subobject of $I(U) \otimes I(V)$.

Lemma 4.3.11. *The subspace $S = span_{\Bbbk}\{(g \otimes u_d) \otimes (k \otimes v_f) | g^{-1}k \notin H\,, u_d \in U_d, v_f \in V_f\}$ is a subobject of $I(U) \otimes I(V)$ in $\mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}}(A_H)$.*

Proof. Firstly, S has a G-grading as

$$\delta((g \otimes u_d) \otimes (l \otimes v_f)) = gdg^{-1}lfl^{-1} \otimes \big((g \otimes u_d) \otimes (l \otimes v_f)\big)$$

gives a G-homogeneous spanning set. Secondly, S is closed under the twisted $\Bbbk G$-action as

$$h \triangleright ((g \otimes u_d) \otimes (k \otimes v_f)) = \gamma(h)(gdg^{-1}, kfk^{-1})\tau(h,g)(d)\tau(h,k)(f)(hg \otimes u_d) \otimes (hk \otimes v_f)$$

is in S because $(hk)^{-1}(hg) \notin H$ if and only if $k^{-1}g \notin H$.

Finally, S is closed under the right action of A_H since

$$((g \otimes u_d) \otimes (k \otimes v_f)) \cdot v_{hH} = \begin{cases} (g \otimes u_d) \otimes (k \otimes v_f), & \text{if } hH = kH, \\ 0, & \text{else,} \end{cases}$$

is clearly in S. Hence S is a subobject of $I(U) \otimes I(V)$. □

Corollary 4.3.12. *As objects in $\mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}}(A_H)$, $I(U) \otimes_{A_H} I(V) \cong I(U) \otimes I(V)/S$.*

Proposition 4.3.13. *The functor I induces a separable Frobenius monoidal functor I : ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^{\omega} \to \mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}}(A_H)$.*

Proof. Due to our description of $I(U) \otimes_{A_H} I(V)$ we can naturally extend the op-lax monoidal structure (ν, ϵ) to

$$\bar{\nu}_{U,V} \colon I(U \otimes V) \xrightarrow{\nu_{U,V}} I(U) \otimes I(V) \longrightarrow I(U) \otimes_{A_H} I(V),$$

$$g \otimes (u_d \otimes v_f) \mapsto \gamma(g)(d,f)(g \otimes u_d) \otimes (g \otimes v_f),$$

on the induced functor.

The lax-monoidal structure is induced similarly. If we use the form of $\mu_{U,V}$ given in Remark 4.3.2, we see that S is in the kernel of $\mu_{U,V}$ and so we can induce a quotient morphism

$$\begin{aligned} \bar{\mu}_{U,V} \colon I(U) \otimes_{A_H} I(V) &\to I(U \otimes V) \\ (g \otimes u_d) \otimes (k \otimes v_f) &\mapsto \zeta(g,d,k,f) g \otimes (u_d \otimes g^{-1}k \cdot v_f) \end{aligned}$$

in $\mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}(A_H)$. All of the conditions for this to be a separable Frobenius monoidal functor are satisfied directly.

□

Proposition 4.3.14. *The monoidal functor $I : {}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega \to \mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}(A_H)$ is braided lax monoidal.*

Proof. To see that I is a braided lax monoidal functor, consider the diagram

$$\begin{array}{ccccc} I(U) \otimes I(V) & \longrightarrow & I(U) \otimes_{A_H} I(V) & \xrightarrow{\bar{\nu}_{U,V}} & I(U \otimes V) \\ \downarrow{\scriptstyle c_{I(U),I(V)}} & & \downarrow{\scriptstyle c'_{U,V}} & & \downarrow{\scriptstyle I(c_{U,V})} \\ I(V) \otimes I(U) & \longrightarrow & I(V) \otimes_{A_H} I(U) & \xrightarrow{\bar{\nu}_{V,U}} & I(V \otimes U), \end{array}$$

where the unlabeled morphisms are the coequalisers defined in Equation (3.3.0.2). The left-most square commutes by definition of the braiding in $\mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}(A_H)$, and the perimeter commutes by naturality. Hence the right-most square commutes, which is exactly the condition for the functor I to be braided lax monoidal.

□

The following result was proved in [DS17, Theorem 3.7], and here we give a proof utilising the functor I we have developed.

Theorem 4.3.15. *The functor I defines an equivalence of braided monoidal categories between ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$ and $\mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}(A_H)$.*

Proof. Recall that on morphisms, the functor I is given by the map

$$\mathsf{Hom}_{{}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega}(U,V) \to \mathsf{Hom}_{\mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}(A_H)}(\Bbbk G \otimes U, \Bbbk G \otimes V)$$

$$f \mapsto I(f) = \mathsf{Id}_{\Bbbk G} \otimes f$$

This map is injective, which can be seen by the restriction of morphisms to the subspace $e \otimes V$ of $I(V)$, meaning $I(f) = I(g)$ only when then morphisms agree on the V component, which is just $f = g$. Hence I is faithful.

To prove that I is full, suppose $q\colon \Bbbk G \otimes_{A_H} U \to \Bbbk G \otimes_{A_H} V$ is a morphism in $\mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}}(A_H)$. Then, for all $g, k \in G$, $u \in U$, we have

$$q(g \otimes u) \cdot \upsilon_{kH} = q((g \otimes u) \cdot \upsilon_{kH}) = \begin{cases} q(g \otimes u), & \text{if } k^{-1}g \in H, \\ 0, & \text{otherwise.} \end{cases}$$

We choose a set of coset representatives $\{g_i\}$ for H in G such that $g_1 = e$. For a local module $L \in \mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}}(A_H)$, the action of the idempotent elements υ_{g_iH} define a family of idempotent endomorphisms

$$\iota_g\colon L \to L, \quad l \mapsto l \cdot \upsilon_{gH} \qquad \iota_g \in \mathsf{End}_{\mathcal{Z}(\mathsf{Vect}_G^{\omega})}(L). \tag{4.3.0.8}$$

If we then set $L^i = \mathrm{Im}(\iota_{g_i})$, we get a direct sum decomposition $L = \bigoplus_i L^i$.

If we apply this to the module $I(V)$, it is clear that $I(V)^i = g_i \otimes V$ and consists of all elements that are non-zero under the action of υ_{g_iH}. If we consider $q(g_i \otimes u)$, then by the above

$$q(g_i \otimes u) \cdot \upsilon_{g_jH} = \begin{cases} q(g_i \otimes u) & \text{if } i = j \\ 0 & \text{else.} \end{cases} \tag{4.3.0.9}$$

So $q(g_i \otimes u)$ is contained in the subspace $g_i \otimes V$ of $I(V)$. Hence, there exists a vector $v_i \in V$ such that $q(g_i \otimes u) = g_i \otimes v_i$. But, as q is a morphism in $\mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}}(A_H)$, it commutes with the twisted $\Bbbk G$-action, which gives us that

$$g_i \otimes v_i = q(g_i \otimes u) = q(g_i \cdot (e \otimes u)) = g_i \cdot q(e \otimes u) = g_i \otimes v_1. \tag{4.3.0.10}$$

This clearly implies that $v_i = v_1$ for all i, which allows us to define a $\Bbbk$-linear map

$$\bar{q} : U \to V, \quad u \mapsto v_1, \tag{4.3.0.11}$$

which satisfies $I(\bar{q}) = q$, as

$$q(g \otimes u) = g \otimes \bar{q}(u), \tag{4.3.0.12}$$

and which can be easily checked that $\bar{q}$ is a morphism of YD modules over $\Bbbk H$.

The last thing to check is essential surjectivity. For this, consider a general local module $L \in \mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}(A_H)$, with idempotent decomposition as described above.

We now observe that $l \in L^i$ if and only if $g_j g_i^{-1} \cdot l \in L^j$. In particular, $l \in L^1$ if and only if $g_i \cdot l \in L^i$. Further, restricting the twisted action of L makes L^1 into a submodule. The assumption that L is a local module implies that if $l_d \in L^1$ has degree $|l| = d$, then $l = l \cdot \upsilon_{dH}$. We can write $d = g_i h$, with $h \in H$, and find that $l \in L^i$. However, as the subspaces L^i intersect trivially it follows that $d \in H$. Thus, L^1 correspond to an object in ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$.

Using the twisted right G-action on L, we define a map

$$\pi \colon \Bbbk G \otimes L_1 \longrightarrow L, \qquad g \otimes l \mapsto g \cdot l.$$

The map π is surjective. Indeed, L_1 is given by all elements of the form $l \cdot \upsilon_H$, with $l \in L$. Now,

$$g_i \left(\frac{(g_i^{-1} l)}{\gamma(g_i)(g_i, |l|)} \cdot \upsilon_H \right) = \frac{\tau(g_i)(g_i|l|g_i^{-1}, 1)}{\gamma(g_i)(g_i, |l|)} (g_i(g_i^{-1} l)) \cdot (g_i \upsilon_H)) = \frac{\gamma(g_i)(g_i, |l|)}{\gamma(g_i)(g_i, |l|)} l \cdot \upsilon_{g_i H} = l \cdot \upsilon_{g_i H}.$$

Thus, $l \cdot \upsilon_{g_i H}$ is in the image of π and hence, for any $l \in L$, $l = \sum_i l \cdot \upsilon_{g_i H} \in \mathrm{Im}(\pi)$. It follows that

$$\pi(gh \otimes v_d) = (gh)l = \tau(g,h)(d)^{-1} g(hl) = \tau(g,h)(d)^{-1} g \otimes h v_d.$$

So by Equation (4.3.0.1), π descends to a quotient map $\overline{\pi} \colon I(L^1) \to L$ which is still surjective. The right twisted action by $g \in G$ gives an isomorphism of vector spaces and hence $\mathsf{dim}(L^i) = \mathsf{dim}(L^1)$ for all i. This shows that

$$\mathsf{dim} L = |G : H| \mathsf{dim}_\Bbbk L^1 = \mathsf{dim}_\Bbbk I(L^1).$$

Thus, $\overline{\pi}$ is injective and hence gives an isomorphism $I(L^1) \cong L$. □

4.4 Frobenius algebras in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$

Now that we have completed our construction and exploration of the monoidal functor $I : {}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega \to \mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}$, it is time to return to the twisted group algebras $B(N, \kappa, \epsilon)$.

Definition 4.4.1 ($A(H,N,\kappa,\epsilon)$)**.** We define $A := A(H, N, \kappa, \epsilon)$ to be the commutative Frobenius algebra $I(B)$, for the algebra $B = B(N, \kappa, \epsilon)$ from Proposition 4.2.3 in ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$.

While we now by Corollary 4.3.7 that this is a commutative Frobenius algebra, we can explicitly describe the structure.

We have that $B(N,\kappa,\epsilon)$ has a basis given by $\{e_n | n \in N\}$, so under the functor I these take on the form $a_{g,n} := g \otimes e_n$, for some $g \in G$. These elements clearly form a basis for the underlying space of A, but we need to account for the relations in Equation (4.3.0.1);

$$gh \otimes e_n = \tau(g,h)(n)^{-1} g \otimes h \cdot e_n = \tau(g,h)(n)^{-1}\epsilon_h(n) g \otimes e_{hnh^{-1}},$$

so A is subject to the relations

$$a_{gh,n} = \tau(g,h)(n)^{-1}\epsilon_h(n) a_{g,hnh^{-1}}, \forall g \in G, h \in H, n \in N.$$

By Equations 4.3.0.2 and 4.3.0.3, we get that the $\Bbbk[G]$-coaction of these elements is $\delta(a_{g,n}) = gng^{-1} \otimes a_{g,n}$, and the $\Bbbk[G]$- action is given by

$$k \cdot a_{g,n} = \tau(k,g)(n) a_{kg,n}.$$

This completes the description of $A(H,N,\kappa,\epsilon)$ as a twisted YD-module in $\mathsf{Mod}^{\mathsf{loc}}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}}(A_H)$. For the algebra structure, we need to make use of Proposition 3.2.7 where the multiplication of A is the composition

$$m_A : A \otimes A = I(B) \otimes I(B) \xrightarrow{\mu_{B,B}} I(B \otimes B) \xrightarrow{I(m_B)} I(B) = A,$$

which on our basis elements is given by

$$(g \otimes e_n)(k \otimes e_m) = \begin{cases} \gamma(g)(n,m)^{-1}\kappa(n,m)^{-1}(g \otimes (e_{nm})) & \text{if } g^{-1}k \in H \\ 0 & \text{else.} \end{cases}$$

The unit $u_A : \mathbb{1} \xrightarrow{eta} I(\mathbb{1}) \xrightarrow{I(u_B)} I(B) = A$ results in $1_A := u_A(1_\Bbbk) = \sum_{i \in I} a_{g_i,1}$.

The coalgebra structure is derived similarly, this time using the oplax monoidal structure of I. Once we do this and put it together, we get the following lemma.

Lemma 4.4.2. *The explicit structure of $A = A(H,N,\kappa,\epsilon)$ as a Frobenius algebra in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^{\omega}$ is as follows;*

(a) A is the $\Bbbk$*-vector space spanned by* $\{a_{g,n} \mid g \in G, n \in N\}$*, subject to the relations*

$$a_{gh,n} = \tau(g,h)(n)^{-1}\epsilon_h(n)a_{g,hnh^{-1}}, \ \forall h \in H.$$

(b) The twisted YD module structure is given by

(i) left $\Bbbk G$*-coaction given by* $\delta(a_{g,n}) = gng^{-1} \otimes a_{g,n}$*;*

(ii) twisted G-action given by $k \cdot a_{g,n} = \tau(k,g)(n)a_{kg,n}$*, for* $k \in G$*.*

(c) The Frobenius algebras structure is given by the

(iii) multiplication m_A *given by*

$$a_{g,n}a_{g,m} = \gamma(g)(n,m)^{-1}\kappa(n,m)^{-1}a_{g,nm},$$

for $g \in G$ *and* $n, m \in N$*, and* $a_{g,n}a_{k,m} = 0$ *if* $gH \neq kH$*;*

(iv) unit u_A *given by* $1_A = \sum_{i \in I} a_{g_i,1}$*;*

(v) coproduct Δ_A *given by*

$$\Delta_A(a_{g,n}) = \sum_{m \in N} \gamma(g)(m, m^{-1}n)\kappa(m, m^{-1}n) \ a_{g,m} \otimes a_{g,m^{-1}n},$$

(vi) counit ε_A *given by* $\varepsilon_A(a_{g,n}) = \delta_{n,1}$*.*

Now, we have an explicit description of a transfer of the twisted group algebras under the functor I. We can now use the algebras $A(H, N, \kappa, \epsilon)$ to provide a version of Theorem 4.0.1 that extends to the monoidal center $\mathcal{Z}(\mathsf{Vect}_G^\omega) \cong {}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$.

Theorem 4.4.3. *Let G be a finite group with* $\omega \in C^3(G, \Bbbk^\times)$*, H a subgroup of G and a tuple* (N, κ, ε) *as in Notation 4.2.1.*

(a) If $|N| \cdot |G : H| \in \Bbbk^\times$*, then the algebra* $A = A(H, N, \kappa, \epsilon)$ *from Definition 4.4.1 is a connected commutative special Frobenius algebra in* $\mathcal{Z}(\mathsf{Vect}_G^\omega)$*.*

(b) Every connected separable commutative algebra in $\mathcal{Z}(\mathsf{Vect}_G^\omega)$ *is of the form* $A(H, N, \kappa, \epsilon)$*, for some choice of data* H, N, γ, ϵ*.*

Proof. Part (a) We already have that $A = A(H, N, \kappa, \epsilon)$ is a commutative Frobenius algebra by combining Propositions 4.2.3 and 4.2.4 with Corollary 4.3.7.

A is connected since $\mathsf{Hom}_{{}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega}(\mathbb{1}, A) = (A_1)^G$, the space of G-invariant elements in the $\Bbbk G$-module A_1. This space is 1-dimensional as $A_1 = \Bbbk(G/H)$ and G acts by left translation.

By Proposition 4.2.4, the algebra $B(N,\kappa,\epsilon)$ is special Frobenius when $|N| \in \Bbbk^\times$. This is preserved by I if and only if $|G:H| \in \Bbbk^\times$ by Corollary 4.3.7. Thus we are left with the required condition.

Part (b) Assume that we have a connected separable commutative algebra $\mathcal{A}$ in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$. If we consider the subobject of trivial degree $\mathcal{A}_e$, the restriction of the multiplication on $\mathcal{A}$ to $\mathcal{A}_e$ makes it an algebra. Restriction of the $\Bbbk[G]$-action makes $\mathcal{A}_e$ a $\Bbbk[G]$-module, with untwisted action. Thus we can view $\mathcal{A}_e$ as an algebra object in $\mathsf{Mod}_{\mathsf{Vect}}(\Bbbk[G])$. The conditions of connected, separable commutative restrict to $\mathcal{A}_e$. Then, by [KO02, Theorem 2.2], $\mathcal{A}_e \cong \Bbbk(G/H) = A_H$ for some $H \subseteq G$.

Now, if we restrict the multiplication of $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ in the right component only, we get that $\mathcal{A}$ is a right local $\mathcal{A}_e = A_H$- module. Thus, by the equivalence in Theorem 4.3.15, $\mathcal{A} \cong I(B)$ for a connected separable commutative algebra B in ${}^{\Bbbk[G]}_{\Bbbk[G]}\mathsf{YD}^\omega$. Now, $\mathsf{dim}_\Bbbk B_e = 1$ as

$$\mathsf{dim}_\Bbbk \mathcal{A}_e \geqslant (\mathsf{dim}_\Bbbk B_e)(\mathsf{dim}_\Bbbk A_H) = (\mathsf{dim}_\Bbbk B_e)(\mathsf{dim}_\Bbbk \mathcal{A}_e),$$

and so B is isomorphic, as an algebra in ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$ to an algebra of the form $B(N,\kappa,\epsilon)$ by Proposition 4.2.6. □

What remains is to consider when two of the representative algebras are isomorphic to each other.

Proposition 4.4.4. *Fix H and $\omega \in C^3(H,\Bbbk^\times)$ and let (N,κ,ϵ) and (N',κ',ϵ') be tuples satisfying the conditions of Notation 4.2.1. Then $B = B(N,\kappa,\epsilon)$ and $B' = B(N',\kappa',\epsilon')$ are isomorphic as algebras in ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$ if and only if $N = N'$ and $\epsilon'\epsilon^{-1} \oplus \kappa'\kappa^{-1}$ is zero in $\widetilde{H}^2_{\mathrm{Tot}}(H,N,\Bbbk^\times)$.*

Proof. Suppose we have an isomorphism of algebras in ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^\omega$, given by $\phi : B \to B'$. Thus $\mathsf{dim} B = \mathsf{dim} B'$ and hence $|N| = |N'|$. Additionally, ϕ must be a morphism of twisted YD modules over $\Bbbk[H]$ and so is an isomophism of G-graded vector spaces. This implies that $N = N'$ and $\phi(e_n) = \sigma(n)e_n$, for some scalar $\sigma(n) \in \Bbbk^\times$.

We also have, as ϕ is a morphism of algebras, that

$$\phi(e_n e_m) = \phi(\kappa(n,m)^{-1}e_{nm}) = \kappa(n,m)^{-1}\sigma(nm)e_{nm} = \sigma(n)\sigma(m)\kappa'(n,m)^{-1}e_{nm}.$$

Thus, ϕ is a morphism of algebras if and only if

$$\frac{\kappa'(n,m)}{\kappa(n,m)} = \frac{\sigma(n)\sigma(m)}{\sigma(nm)},$$

which is that $\kappa'\kappa^{-1} = \partial^{0,1}(\sigma)$.

Further, ϕ is a morphism of twisted YD modules. Thus,

$$\phi(h \cdot e_n) = \epsilon_h(n)\sigma(hnh^{-1})e_{hnh^{-1}} = \epsilon'_h(n)\sigma(n)e_{hnh^{-1}} = h \cdot \phi(e_n).$$

Thus, ϕ is a morphism of twisted YD modules if and only if

$$\frac{\epsilon'_h(n)}{\epsilon_h(n)} = \frac{\sigma(hnh^{-1})}{\sigma(n)}.$$

This condition gives that $\epsilon^{-1}\epsilon' = \mathrm{d}^{0,1}(\sigma)$. Combining these results, and using the differential of the total complex, we see that

$$(\epsilon' \oplus \kappa') \cdot (\epsilon \oplus \kappa)^{-1} = \frac{\epsilon'}{\epsilon} \oplus \frac{\kappa'}{\kappa}$$

equals $\mathrm{d}^1_{\mathrm{Tot}}(\sigma)$ and hence is zero in $\widetilde{H}^2_{\mathrm{Tot}}(H, N, \Bbbk^\times)$ as claimed. □

Remark 4.4.5. With this we have succeeded in extending Theorem 4.0.1 to the monoidal center in a very natural way, showing that a version of the twisted group algebras act as representatives for all commutative connected separable algebras in $\mathcal{Z}(\mathsf{Vect}^\omega_G)$. However, this is a slightly more restrictive result than Theorem 4.0.1, as we have only proven the conditions for the twisted group algebras to be isomorphic as opposed to Morita equivalent.

Let us consider how the equivalence condition of Theorem 4.0.1 might look for twisted group algebra by considering algebras $B(N, \kappa, \epsilon)$ and $B(N', \kappa', \epsilon')$. Firstly, the condition that N and N' are conjugate as subgroups of H is equivalent to the condition that $N = N'$, as they are normal subgroups of H. Secondly, both algebras have a single piece of cohomological data in the complex $\widetilde{F}^2_{\mathrm{Tot}}(H, N, \Bbbk^\times)$. Thus, we can straightforwardly write the condition on these cocycles to be that

$$(\epsilon' \oplus \kappa')^x (\epsilon \oplus \kappa)^{-1} \bar{\gamma_x}$$

is trivial in $\widetilde{H}^2_{\mathrm{Tot}}(H, N, \Bbbk^\times)$, where x is some group element, possibly in $H \ltimes N$, and where $\bar{\gamma_x}$ is some element in this complex whose second component is the γ_x defined in Equation (4.1.0.3).

This would preserve all the conditions in Theorem 4.0.1, and is consistent with the results found in Proposition 4.4.4. Indeed, if we were to take the conjugating element x as the identity element and we assume that $\bar{\gamma_x}$ is trivial in this case, similarly to the original γ_x, then we recover the required conditions of Proposition 4.4.4. This provides compelling evidence that this result should hold and motivates the following conjecture.

Conjecture 4.4.6. *The algebras $B(N, \kappa, \epsilon)$ and $B(N', \kappa', \epsilon')$ in ${}^{\Bbbk[H]}_{\Bbbk[H]}\mathsf{YD}^{\omega}$ are Morita equivalent if and only if $N = N'$ and $(\epsilon' \oplus \kappa')^x(\epsilon \oplus \kappa)^{-1}\bar{\gamma_x}$ is trivial in $\widetilde{H}^2_{\mathrm{Tot}}(H, N, \Bbbk^\times)$.*

This conjecture should be provable using a similar approach to the non-center case found in [Nat17], but finding a suitable form for the remaining part of $\bar{\gamma_x}$ has proven to be tricky. I hope to complete this result in future works, to fully complete the extension of Theorem 4.0.1 to the braided case.

Chapter 5

Detecting Algebra Objects via NIM-reps

We have so far seen a method of classifying algebra objects when we have a lot of structure in the relevant fusion category. In particular, we used the Yetter-Drinfeld module description of $\mathcal{Z}(\mathsf{Vect}_G^\omega)$ to utilise many of the nice properties of the group Hopf algebra. However, this relies on a great deal of work already done and it can be difficult to get such a nice description for more complicated categories. Hence, it is beneficial to explore alternative ways of detecting algebras. In this section, we shall explore how this can be done for fusion categories using the construction of Non-negative Integer Matrix representations (NIM-reps) .

5.1 Fusion Rings

We start this section by introducing the basic definitions needed for this construction.

Definition 5.1.1. A $\mathbb{Z}_+$*-ring* is a pair (R, B) consisting of

- A ring R that is free as a $\mathbb{Z}$-module,
- A fixed basis $B = \{b_i\}_{i\in I}$, where I is an indexing set, such that for all $b_i, b_j \in B$, we have that

  $$b_i b_j = \sum_{k\in I} c_{ij}^k b_k$$

  where $c_{ij}^k \in \mathbb{Z}_+$,
- The ring identity 1_R is a non-negative linear combination of the basis elements. If the identity is in the basis, we say that (R, B) is *unital*.

Notation 5.1.2. In general, we shall refer to a $\mathbb{Z}_+$ ring as (R, B), but whenever clear from context we may refer to it simply as R.

Looking at this definition, we can intuitively see a connection to a semisimple tensor category, which in fact gives us our first example.

Example 5.1.3. Let $\mathcal{C}$ be a semisimple locally finite $\Bbbk$-linear abelian monoidal category with biexact tensor product (i.e a tensor category where we do not assume the existence of duals, and where $\mathbb{1}$ is not assumed to be simple). Then the Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ from Definition 2.1.17 is a $\mathbb{Z}_+$-ring, with the basis given by isomorphism classes of simple objects and the structure coefficients being the Jordan-Hölder multiplicities.

If we assume that the unit object $\mathbb{1}$ is simple, then $\mathrm{Gr}(\mathcal{C})$ is a unital $\mathbb{Z}_+$-ring.

Our goal is to work with fusion categories, so we need some extra conditions on our $\mathbb{Z}_+$-ring to capture the remaining structure.

Let $I_0 \subset I$ be the subset of the index set containing all basis element $b_i, i \in I_0$, that appear in the decomposition of the ring identity 1_R. Then we can define a group homomorphism $\vartheta : R \to \mathbb{Z}$ by:

$$\vartheta(b_i) = \begin{cases} 1 & i \in I_0, \\ 0 & i \notin I_0. \end{cases}$$

If we have a unital $\mathbb{Z}_+$-ring, and take $b_1 = 1_R$, then $\vartheta(b_i) = \delta_{1,i}$.

Definition 5.1.4. A $\mathbb{Z}_+$-ring (R, B) is called a *based ring* if there exists an involution $()^* \colon I \to I$, $i \mapsto i^*$ of the indexing set I such that the induced map

$$a = \sum_{i \in I} a_i b_i \mapsto a^* = \sum_{i \in I} a_i b_{i^*}, \tag{5.1.0.1}$$

where $a_i \in \mathbb{Z}$, is an anti-involution of the ring R and

$$\vartheta\left(b_i b_j\right) = \begin{cases} 1 & i = j^*, \\ 0 & i \neq j^*. \end{cases} \tag{5.1.0.2}$$

Proposition 5.1.5 (Proposition 3.1.6, [EGNO15])**.** *In any based ring, the number $c_{i,j}^{k^*}$ is invariant under cyclic permutations of i,j,k.*

Additionally, in the unital case, it is easy to see that $\vartheta(b_i b_j) = c_{i,j}^1$, and so if we view the involution as some notion of duality, then the condition in Equation (5.1.0.2) says that the only way for the unit 1_R to map into a product of basis elements $b_i b_j$ is if they are dual to each other. This is similar to the condition that a category is rigid, and further motivates the following definition;

Definition 5.1.6. A *fusion ring* is a unital, based ring of finite rank.

For a fusion ring, we can restructure the conditions on the involution into a clearer form.

Proposition 5.1.7. *(Rigidity property) A fusion ring (R, B) can be equipped with a symmetric bilinear form $(-,-) : R \times R \to \mathbb{Z}$ that satisfies the condition $(b_i b_j, b_k) = (b_j, b_{i^*} b_k)$.*

Proof. Let $(-,-)$ be the symmetric bilinear form defined by the condition

$$(b_i, b_j) = \delta_{i,j}.$$

Then we can rewrite Equation (5.1.0.2) as $\vartheta(b_i b_j) = (b_i, b_{j^*}) = (1, b_i b_j)$. Then, by associativity, we have that $(1, (b_i b_j) b_k) = (1, b_i(b_j b_k))$ and so

$$((b_i b_j)^*, b_k) = (b_{j^*} b_{i^*}, b_k) = (b_{i^*}, b_j b_k)$$

where the first equality uses that the induced map 5.1.0.1 is an anti-involution. We then get the result by relabelling the indices. $\square$

As alluded to by the naming, we can now link fusion rings to fusion categories.

Example 5.1.8 (Follow up from Example 5.1.3)**.** Let $\mathcal{C}$ be a fusion category. Then the unital $\mathbb{Z}_+$-ring $\mathrm{Gr}(\mathcal{C})$ is a fusion ring. It has finite rank as $\mathcal{C}$ is finite, and the involution on the basis is induced by taking duals,

$$[X]^* = [X^*] \tag{5.1.0.3}$$

for all $X \in \mathsf{Ob}(\mathcal{C})$. The symmetric bilinear form corresponds to

$$([X], [Y]) = \dim_{\Bbbk} \left(\mathrm{Hom}_{\mathcal{C}} (X, Y) \right), \tag{5.1.0.4}$$

by Schur's Lemma, with the rigidity property of Proposition 5.1.7 following from Proposition 2.1.11.

The classification of fusion rings is a substantial open problem, and is currently only explicitly known for small rank, and even then with a range of further conditions imposed. For details, see [Ost15], [LPR22],[VS23]. However, there are (at least) two families of fusion rings that can be constructed from a group.

Example 5.1.9 (*Group rings*)**.** Let $\mathbb{Z}G$ be the integer span of G, which is a ring with multiplication given by the group operation. Then the *group fusion ring* $R(G) := (\mathbb{Z}G, G)$ is a fusion ring with involution $g^* = g^{-1}$.

Example 5.1.10 ((Near-Group rings)[Ost15, Sie03])**.** As well as G, we fix an integer $\alpha \in \mathbb{Z}_+$. The *near-group fusion ring* is the fusion ring constructed by taking the integer span of the set $G \cup \{X\}$ where the multiplication of the group elements is the group operation, and with the element X is:

$$Xg = gX = X, \quad X^2 = \sum_{g \in G} g + \alpha X.$$

The group elements have involution given by the group inverse, and the element X is self-dual, i.e $X^* = X$. This is a fusion ring and shall be denoted by by $K(G, \alpha)$. A fusion category whose Grothendieck ring is a near-group fusion ring is called a near-group fusion category

Remark 5.1.11. To be able to use fusion rings to study fusion categories, we need to be aware of problems that can occur with their categorification. If we consider the group ring $R(G)$, then it can be seen that this is exactly the Grothendieck ring of the category Vect_G^ω from Example 2.1.15. However, the Grothendieck ring doesn't see the choice of associator, only that one exists, and so $R(G)$ is the Grothendieck ring for all possible choice of ω. This loss of categorical data restricts the ability of using fusion rings to see categorical structures such as algebra objects in their entirety.

In the other direction, there are fusion rings that admit no categorification to a fusion category. The near-group fusion ring $K(\{e\}, \alpha)$ built from the trivial group gives an infinite family of fusion rings, however it has been shown in [Ost02] that a categorification only exists when $\alpha = 0$ or 1. Thus any structures detected on these fusion rings for $\alpha \geqslant 2$ don't actually correspond to a categorical structure. It is important to keep in mind that anything we may find with this approach has to be checked further for categorification obstructions. For near-group fusion rings, the particular case that $\alpha = 0$ is called a *Tambara-Yamagami fusion ring*, and it is known that these fusion rings admit a categorification if and only if G is abelian [TY98]. Results for more general cases can be found in [Ost15, Appendix A], for example.

Example 5.1.12 (*Ising fusion ring*, $K(\mathbb{Z}_2, 0)$)**.** Let $B = \{1, X, Y\}$, and R be the integer span $\mathbb{Z}B$ with addition defined linearly and multiplication given by the fusion rules

$$X^2 = 1 + Y, \quad Y^2 = 1 \quad XY = YX = X.$$

$(\mathbb{Z}B, B)$ is a fusion ring with the self-dual involution $X^* = X, Y^* = Y$.

The Ising fusion ring is of particular note as it is the only Tambara-fusion ring that can be given the structure of a modular tensor category, which follows from [Tho12, Theorem IV.5.2] .

5.2 Non-negative Integer Matrix Representations

In this section, we introduce the main tool that we will use to detect algebra objects from fusion rings, namely NIM-reps, following [BPPZ00, BPRZ21, Gan02, Gan05, Gan06]. First, we need the notion of a module over a $\mathbb{Z}_+$-ring.

Definition 5.2.1. Let (R, B) be a $\mathbb{Z}_+$-ring. A $\mathbb{Z}_+$*-module* is a pair (T, M) consisting of

- An R-module T, with action given by $\rhd\colon R \times T \to T$,
- A fixed $\mathbb{Z}$-basis $M = \{m_l\}_{l \in L}$, where L is an indexing set, such that, for any $m_l \in M$, $b_i \in B$, we have that
$$b_i \rhd m_l = \sum_{k \in L} a_{i,l}^k m_k,$$
where $a_{i,l}^k \in \mathbb{Z}_+$.

For this to be compatible with the additional structure of a fusion ring, we need to have a condition that works with the involution structure.

Definition 5.2.2. Let (R, B) be a fusion ring. A *non-negative integer matrix representation* (NIM-rep) of (R, B) is a $\mathbb{Z}_+$-module (T, M) that satisfies the following condition;

- *(Rigidity condition):* Let (T, M) have the symmetric bilinear form $(-,-) : T \times T \to \mathbb{Z}$ defined by
$$(m_l, m_k) = \delta_{l,k}$$
for any $l, k \in L$. Then we must have, for any $i \in I, l, k \in L$
$$(b_i \rhd m_l, m_k) = (m_l, b_{i^*} \rhd m_k). \tag{5.2.0.1}$$

We note here that this is extremely similar to Proposition 5.1.7. However, in this definition rigidity is a condition and not a property.

Remark 5.2.3. For a NIM-rep, the symmetric bilinear form of (t, m_l), for $t \in T, m_l \in M$, counts the multiplicity of m_l in the basis decomposition of t. This then gives us that $(t, t) = 1$ if and only if $t \in M$.

Notation 5.2.4. Contrary to the case of a fusion ring, when working with a NIM-rep we may refer to a NIM-rep simply as M instead of (T, M).

Example 5.2.5 (Follow-up from Definition 5.1.8)**.** If we take a semi-simple module category $\mathcal{M}$ over $\mathcal{C}$, then the Grothendieck group $\mathrm{Gr}(\mathcal{M})$ is a NIM-representation of $\mathrm{Gr}(\mathcal{C})$, with the isomorphism classes of simple objects acting as the basis. The NIM-rep action of $\mathrm{Gr}(\mathcal{M})$ on $\mathrm{Gr}(\mathcal{C})$ is induced from the module category action of $\mathcal{M}$ on $\mathcal{C}$, and the rigidity condition follows immediately by Proposition 3.6.6.

We shall later calculate the NIM-reps for three types of fusion categories, but a first example of one is immediate as a fusion ring can always be taken as a NIM-rep of itself, with the module action being the ring multiplication. In this case, the rigidity property found in proposition 5.1.7 satisfies the required rigidity condition in Equation (5.2.0.1).

Remark 5.2.6. Suppose we have a NIM-rep (T, M) over a fusion ring (R, B) that satisfies $b_i \rhd m_j = 0_R$ for some $b_i \in B, m_j \in M$. The rigidity condition then imposes that $m_j = 0_R \in M$. However, the only way 0_R appears in the NIM-rep basis is if $\{0_R\} = M$, (i.e this NIM-rep is the zero NIM-rep). We shall remove this NIM-rep from future considerations.

Definition 5.2.7 ([BD12])**.** Let $(T, M), (T', M')$ be two NIM-reps over a fusion ring (R, B). A *NIM-rep morphism* is a function $\psi\colon M \to M'$ inducing a $\mathbb{Z}$-linear map between the modules.

If ψ is a bijection, and the induced map is an isomorphism of R-modules, then we say that the NIM-reps are *isomorphic*.

We can also define the notion of a direct sum between NIM-reps, in the obvious way; The direct sum of two NIM-reps $(T, M), (T', M')$ over a fusion ring (R, B) is the R-module $T \oplus T'$ with distinguished basis $M \oplus M'$. Now that we have this idea, we can talk about indecomposable NIM-reps.

Definition 5.2.8 ([EGNO15] Section 3.4, [Ost03a] Lemma 2.1.)**.** A NIM-rep is *indecomposable* if it is not isomorphic to a non-trivial direct sum of NIM-reps.

Remark 5.2.9. There is also the notion of an *irreducible NIM-rep*, meaning one which has no non-trivial sub-NIM-reps. However, this is equivalent to the notion of indecomposability.

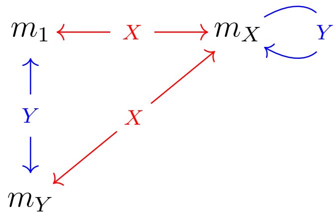
When working with NIM-reps, we can visually express their structure in the form of a multi-digraph. For basic definitions in graph theory, see [GYZ13].

Definition 5.2.10. Let M be a NIM-rep over (R, B). The corresponding *NIM-graph* is constructed in the following way:

- Put a labelled node for every NIM-basis element $m_l \in M$,
- A directed arrow, labelled by $b_i \in B$, with source m_l and target m_k for every copy of m_k in $b_i \rhd m_l$.

In particular, every node in a NIM-graph will have a self-loop labelled by the ring identity 1_R, which we omit for simplicity.

Example 5.2.11. If we consider the Ising fusion ring from Example 5.1.12 as a NIM-rep over itself, the corresponding NIM-graph is given by;



As well as being a convenient way to express the structure of a NIM-rep, we can also extract properties of the NIM-rep directly from the NIM-graph.

Lemma 5.2.12. *A NIM-rep is indecomposable if and only if the corresponding NIM-graph is connected.*

To begin relating algebra objects in $\mathcal{C}$ to NIM-reps, we first note the following result;

Lemma 5.2.13. *Any indecomposable separable algebra A in a fusion category $\mathcal{C}$ gives rise to an indecomposable NIM-rep over* $\mathrm{Gr}(\mathcal{C})$.

Proof. By Proposition 3.3.5 and Example 3.6.3, the $\mathcal{C}$-module category $\mathsf{Mod}_{\mathcal{C}}(A)$ is indecomposable and semisimple. This then gives an a indecomposable NIM-rep over $\mathrm{Gr}(\mathcal{C})$ as outlined in Example 5.2.5. □

To explore this correspondence further, we state the following theorem using results from [EGNO15, Section 7.10]:

Theorem 5.2.14. *Let $\mathcal{C}$ be a fusion category, $\mathcal{M}$ an indecomposable semisimple $\mathcal{C}$-module category, and $N \in \mathsf{Ob}(\mathcal{M})$ such that $[N]$ generates $\mathrm{Gr}(\mathcal{M})$ as a based $\mathbb{Z}_+$-module over $\mathrm{Gr}(\mathcal{C})$. Then there is an equivalence $\mathcal{M} \simeq \mathsf{Mod}_{\mathcal{C}}(A)$ of $\mathcal{C}$-module categories, where $A = \underline{\mathrm{Hom}}(N, N)$.*

Every category of modules $\mathsf{Mod}_{\mathcal{C}}(A)$ coming from a indecomposable separable algebra satisfies the conditions of this theorem, with $N = A$, as $\underline{\mathrm{Hom}}(A, A) = A$ from Example 3.6.9. Thus, if were able to find all such $\mathcal{C}$-module categories, we would have the category of modules for all indecomposable separable algebras, which would be one step closer to a Morita equivalence classification. This is a difficult classification problem, but we will focus on attempting it by using NIM-reps. To do this, let's translate the conditions in this theorem into the language of NIM-reps;

Lemma 5.2.15. *Suppose we are in the setup of Theorem 5.2.14. Then the element* $[N]$ *of the NIM-rep* $\mathrm{Gr}(\mathcal{M})$ *satisfies the condition that, for all basis elements* $[N_i]$*, there exists a basis element* $[X_i]$ *in* $\mathrm{Gr}(\mathcal{C})$ *such that* $[X_i] \rhd [N] = [N_i]$.

Proof. As $[N]$ generates $\mathrm{Gr}(\mathcal{M})$ as a $\mathbb{Z}_+$-module, we have that for all $[M] \in \mathrm{Gr}(\mathcal{M})$, there exists some $[X] \in \mathrm{Gr}(\mathcal{C})$ such that $[X] \rhd [N] = [M]$. If we restrict to the basis given by simple objects $[N_i]$, this becomes

$$[X] \rhd [N] = \sum_j [X : X_j][X_j] \rhd [N] = [N_i],$$

where we have also expanded out $[X]$ in terms of basis elements. Then, using the symmetric bilinear form, we see that this is only possible when there is only one j which gives exactly $[X_j] \rhd [N] = [N_i]$. ☐

If the module category is of the form $\mathsf{Mod}_{\mathcal{C}}(A)$, then by taking A to be connected we get that A is simple on $\mathsf{Mod}_{\mathcal{C}}(A)$ by Proposition 3.3.3 . Thus the distinguished element $[A]$ in $\mathrm{Gr}(\mathsf{Mod}_{\mathcal{C}}(A))$ will be simple, leading to the following definition.

Definition 5.2.16. We shall call a NIM-rep (T, M) over the fusion ring (R, B) *admissible* if there exists an element $m_0 \in M$ such that, for every other element $m_i \in M$, there exists an element $b_j \in B$ that satisfies $b_j \rhd m_0 = m_i$.

Remark 5.2.17. We now have a condition to search for, as a NIM-rep coming from a connected separable algebra will always be admissible. It is only a necessary condition, however, as there is no guarantee that an admissible NIM-rep can be realised as $\mathrm{Gr}(\mathsf{Mod}_{\mathcal{C}}(A))$ for some algebra A. Thus, we can use NIM-reps as a means of detecting separable algebras, but not yet to completely classify them.

In the case that we have an admissible NIM-rep, we are able to recover a portion of the data that makes up the corresponding algebra object.

Proposition 5.2.18. *Let (T, M) be an admissible NIM-rep over the fusion ring* $\mathrm{Gr}(\mathcal{C})$*. If it is the underlying NIM-rep of an indecomposable semisimple $\mathcal{C}$-module category, as in the setup of Theorem 5.2.14, then the decomposition of the algebra $A = \underline{\mathrm{Hom}}(N, N)$ is given by $\bigoplus_{i \in I} a_i b_i$, where a_i is the number of self-loops of m_0 labelled by b_i in the NIM-graph of (T, M).*

Proof. Using the isomorphism from [EGNO15, Equation 7.21] applied to the algebra A, we have that

$$\mathsf{Hom}_{\mathcal{C}}(X, A) \cong \mathsf{Hom}_{\mathcal{M}}(X \rhd N, N).$$

By Schur's Lemma, if we restrict X to the simple objects of C, then X appears in the decomposition of A if and only if N is in the decomposition of $X \rhd N$. But by restricting to the NIM-reps picture, and the identification of m_0 with N, we see that this occurs exactly when X labels a self-loop on m_0. This gives the result. □

As a consequence, by classifying all admissible NIM-reps over $\mathrm{Gr}(\mathcal{C})$, we can produce a list of all choices for the object structure of an indecomposable separable algebra in $\mathcal{C}$. These can then be checked individually for possible multiplication and unit morphisms. This has to be done separately, as the loss of categorical data that occurs when moving to the fusion ring and NIM-reps picture means we cannot detect these structures from the corresponding NIM-reps. This is in addition to checking for any other categorification obstructions as discussed in Remark 5.1.11.

Example 5.2.19. To demonstrate how this method works, consider the Ising fusion ring as a NIM-rep over itself. By looking at the NIM-graph from Example 5.2.11, we see that this is admissible by setting $m_0 = m_1$, as $X \rhd m_1 = m_X$ and $Y \rhd m_1 = m_Y$.

The distinguished basis object m_0 only has a self-loop labeled by the ring identity 1. As discussed in Remark 5.1.11, this fusion ring admits a categorification, and so the algebra object that corresponds to this NIM-rep is $A = \mathbb{1}$, the monoidal unit.

5.3 Computing NIM-Reps and admissible algebras

We will now compute the NIM-reps for three families of fusion categories, and use them to provide a list of all possible connected separable algebra object structures within them.

5.3.1 Group Fusion Rings

To begin, we will consider the group fusion rings $R(G)$ from Example 5.1.9, which as previously mentioned are the underlying fusion rings of Vect_G^ω. These results have been computed previously in [DFZ90, EK95], but we present them here to demonstrate our method of finding potential algebra object structures. This section shall make use of notions and results to do with group actions, which can be found in Appendix B.

Proposition 5.3.1. *Let (T, M) be a NIM-rep over the fusion ring $R(G)$. Then the NIM-rep action restricts to a group action on M.*

Proof. This can be seen as

$$(g \rhd m_l, g \rhd m_l) = (m_l, g^{-1} \rhd (g \rhd m_l)) = (m_l, m_l) = 1$$

and so $g \rhd m_l$ is in the basis of M. Thus the NIM-action restricts to a group action $G \times M \to M$. □

Proposition 5.3.2. *A NIM-rep M over $R(G)$ is indecomposable if and only if the induced group action is transitive.*

Proof. If a NIM-rep M over $R(G)$ is not indecomposable, then its corresponding group action will always be partitioned into G-orbits by restricting the action to the NIM-reps that sum to give M. Thus the group action is transitive only if the NIM-rep is indecomposable.

Conversely, if the group action is not transitive, then we can write it as the sum of some finite combination of G-actions $\rhd_i \colon G \times M_i \to M_i$. It is easy to see that each M_i is a $\mathbb{Z}$-basis for another NIM-rep over $R(G)$. Hence the NIM-rep is indecomposable only if the group action is transitive. □

We can use this to explicitly descibe the structure of NIM-reps of $R(G)$. Using properties of transitive group orbits, see Proposition B.0.4, the basis elements of an indecomposable NIM-rep M will be parameterised by the left cosets of some subgroup $H \subseteq G$. Let $\{g_i\}_i$ be a set of coset representatives, then the induced G-action will be given by, for some $k \in G$,

$$k \rhd m_{g_i} = m_{g_j}, \text{ where } kg_i \in g_j H.$$

We shall denote this NIM-rep by $M(H)$.

Classifying NIM-reps over $R(G)$ now comes down to classifying transitive G-actions, which allows us to state the following result.

Proposition 5.3.3. *Two NIM-reps $M(H), M(K)$ over $R(G)$ are isomorphic if and only if H, K are conjugate subgroups of G.*

Proof. This follows immediately by combining Proposition 5.3.2 and Proposition B.0.5. $\square$

We now have all possible NIM-reps over the group ring $R(G)$, so we can establish which are admissible. It turns out in this fairly simple case that there are no obstructions to admissibility.

Proposition 5.3.4 (Admissible NIM-reps for group rings)**.** *All indecomposable NIM-reps over a group fusion rings $R(G)$ are admissible.*

Proof. Let $M(H)$ be a NIM-rep over $R(G)$, and let $\{g_i\}_i$ be a set of left coset representatives, such that $g_0 = e$, parameterising the NIM-basis elements of $M(H)$. If we set $m_0 = m_{g_0} = m_e$, then the condition in Definition 5.2.16 is satisfied as $g_i \rhd m_e = m_{g_i}$. In fact, we could choose any other basis element m_{g_i} to be m_0, as $g_j g_i^{-1} \rhd m_{g_i} = m_{g_j}$. Hence $M(H)$ is admissible. $\square$

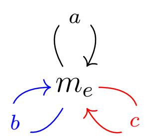
Remark 5.3.5. The algebra object associated to an admissible NIM-rep $M(H)$ over $R(G)$ is given by the elements that act trivially in $M(H)$, which are exactly the elements in H. Thus, any connected separable algebra in Vect_G^ω is of the form $\bigoplus_{h\in H} b_h$. This agrees with the classification of algebras in Vect_G^ω in [Nat17, Ost03b].

Example 5.3.6 (NIM-reps of $R(\mathbb{Z}_2 \times \mathbb{Z}_2)$)**.** The Klein-four group has presentation

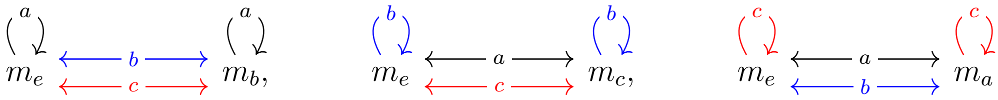
$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{a, b, c | a^2 = b^2 = c^2 = abc = e\}$$

There are 3 isomorphism classes of subgroups in $\mathbb{Z}_2 \times \mathbb{Z}_2$;

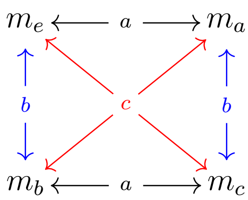
- $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a subgroup of itself – Then $M(\mathbb{Z}_2 \times \mathbb{Z}_2)$ has a single basis element corresponding to the group identity e. The NIM-rep graph is given by



- $\mathbb{Z}_2$ – There are 3 conjugacy classes of subgroups in this case; $H_1 = \{e, a\}, H_2 = \{e, b\}, H_3 = \{e, c\}$. The NIM-reps $M(H_1), M(H_2), M(H_3)$ have two basis elements parameterised by coset representatives $\{e, b\}, \{e, c\}, \{e, a\}$ respectively.



- The trivial subgroup $H = \{e\}$ – The basis elements of $M(H)$ are simply parameterised by elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

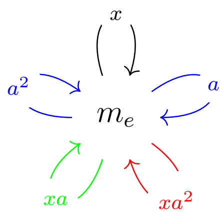


Example 5.3.7 (NIM-reps of $R(D_3)$)**.** Consider the dihedral group D_3, with presentation

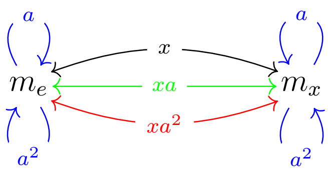
$$D_3 = \{x, a \mid x^2 = a^3 = e,\ xa = a^{-1}x\}$$

There are four conjugacy classes of subgroups of D_3, giving four isomorphism classes of NIM-reps;

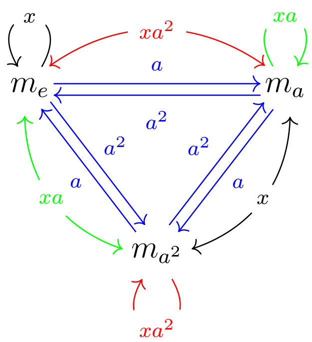
- D_3 as a subgroup of itself. This NIM-graph simply consists of a single basis element, with each group ring element acting trivially.



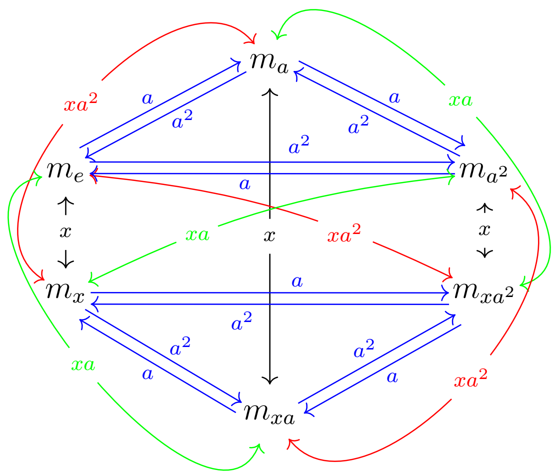
- The isomorphism class of $\mathbb{Z}_3$, given by $H = \{e, a, a^2\}$.



- The isomorphism class of $\mathbb{Z}_2$ is given by 3 conjugate subgroups, $H = \{e, x\}, \{e, xa\}, \{e, xa^2\}$. This gives one NIM-graph, up to isomorphism of NIM-reps. We shall label our graph using the subgroup $H = \{e, x\}$;



- The trivial subgroup $H = \{e\}$. The basis elements are parameterised by the elements of D_3;



5.3.2 Near-Group Fusion Rings

We shall now apply the same techniques to the near-group fusion rings $K(G, \alpha)$ from Example 5.1.10.

As the fusion ring $K(G, \alpha)$ is the union of a group G and a non-invertible element X, the action of this fusion ring on a NIM-rep M can be split into the action of a group G on M, which corresponds to a group action by the previous section, and the action of X on M.

Unlike the group ring case, the group action coming from $K(G, \alpha)$ on M is not necessarily transitive. This can be seen by considering the example of the Ising fusion ring as a NIM-rep over itself.

If we consider the Ising NIM-graph, see Example 5.2.11, then we see that there are two group orbits involved; The NIM-basis elements $\{m_e, m_Y\}$ with trivial stabiliser group, and $\{m_X\}$ with $\mathbb{Z}_2$ as the stabiliser group. It is clear that the reason we have two group orbits in the same indecomposable NIM-rep is the action of the non-invertible element X, which connects them.

In general, the NIM-rep basis M can be partitioned into G-orbits which are connected through X. This partition is given by

$$M = \bigcup_{i=1}^{p} \{m_l^i\}_{1 \leqslant l \leqslant |G:H_i|}, \tag{5.3.2.1}$$

where the i-label counts the p distinct G-orbits, each governed by a stabiliser subgroup $\{H_i\}$. This suffices to describe the action of G on M, so we now need to focus only on the action of X. Additionally, we know that a NIM-rep is indecomposable if and only if its NIM-graph is connected, which only happens if every orbit is connected to at least one other by X. If there are p orbits, where $p \geqslant 3$, this implies that there are at least $p-2$ orbits that are connected to two or more other group orbits.

Next, lets consider how X acts on elements in the same group orbit.

Proposition 5.3.8. *Let M be an indecomposable NIM-rep over $K(G,\alpha)$, and fix a group orbit label i in the partition of M. Then $X \rhd m_{l_1}^i = X \rhd m_{l_2}^i$ for all $1 \leqslant l_1, l_2 \leqslant |G : H_i|$.*

Proof. As $m_{l_1}^i$ and $m_{l_2}^i$ are in the same G-orbit, we can write

$$m_{l_1}^i = g \rhd m_{l_2}^i,$$

for some $g \in G$. Then we compute that

$$X \rhd m_{l_1}^i = X \rhd (g \rhd m_{l_2}^i) = (Xg) \rhd m_{l_2}^i = X \rhd m_{l_2}^i,$$

using the near group fusion rules from Example 5.1.10. □

So we see that the action of X is constant within group orbits. As such, we can simplify our notation, and define

$$x_{i,j} := (X \rhd m_l^i, m_k^j) = (X \rhd m_k^j, m_l^i) = x_{j,i}$$

which is constant for all l, k, by Proposition 5.3.8, and the symmetry comes from X being self-dual.

Using this alongside the partition of M, we can now express the action of X as

$$X \rhd m_l^i = \sum_{j=1}^{p} x_{i,j} \sum_{k=1}^{|G:H_j|} m_k^j. \tag{5.3.2.2}$$

By acting on both sides with X, we compute that

$$\begin{aligned} X^2 \rhd m_l^i &= \sum_{j=1}^{p} x_{i,j} \sum_{k=1}^{|G:H_j|} X \rhd m_k^j \\ \implies \sum_{g\in G} g \rhd m_l^i + \alpha X \rhd m_l^i &= \sum_{j=1}^{p} x_{i,j} |G:H_j| X \rhd m_k^j \\ \implies |H_i| \sum_{k=1}^{|G:H_i|} m_k^i + \alpha X \rhd m_l^i &= \sum_{j=1}^{p} x_{i,j} |G:H_j| X \rhd m_l^j. \end{aligned} \tag{5.3.2.3}$$

Using the symmetric bilinear form of M, we can count the multiplicities on group orbit by applying it to Equation (5.3.2.3).

- If we apply the form for the fixed orbit i, $(-, m_l^i)$, we get

$$|H_i| + \alpha x_{i,i} = \sum_{j=1}^{p} x_{i,j}^2 |G:H_j|, \tag{5.3.2.4}$$

- If there is more than one orbit, we can apply the form for some orbit $q \neq i$, $(-, m_l^q)$ resulting in

$$\alpha x_{i,q} = \sum_{j=1}^{p} x_{i,j} x_{j,q} |G:H_j|. \tag{5.3.2.5}$$

The way of finding NIM-reps over a near-group fusion ring is thus to find sets $\{H_i\}$, $\{x_{i,j}\}$ that solve these equations. This problem can become quite large as the number of group orbits increases, so it would be beneficial to find a way neater way to display these equations.

Let us define the following matrices;

$$\mathbf{X} := (x_{i,j})_{i,j}, \ \mathbf{B} = \operatorname{diag}(|G:H_i|_i).$$

The matrix $\mathbf{X}$ encodes all the data required to describe the action of X. Converting

Equation (5.3.2.3) to matrix form, we get the matrix equation

$$\mathbf{XBX} = \alpha \cdot \mathbf{X} + |G| \cdot \mathbf{B}^{-1}. \tag{5.3.2.6}$$

which fully encodes the action of X on M.

As the matrix $\mathbf{B}$ is invertible, we obtain a quadratic matrix equation in the variable $\mathbf{XB}$;

$$(\mathbf{XB})^2 = \alpha \cdot (\mathbf{XB}) + |G| \cdot \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. The coefficient matrices are diagonal and so commute with each other, meaning we can solve this equation using the quadratic equation, resulting in

$$\mathbf{XB} = \frac{1}{2}\alpha \cdot \mathbf{I} \pm \sqrt{(\frac{\alpha^2}{4} + |G|) \cdot \mathbf{I}} = \frac{1}{2}\alpha \cdot \mathbf{I} \pm \sqrt{(\frac{\alpha^2}{4} + |G|)} \cdot \mathbf{Y}, \tag{5.3.2.7}$$

where $\mathbf{Y}$ is a square root of the identity matrix $\mathbf{I}$.

Some of the conditions of a NIM-rep can reduce the problem. Firstly, all of the elements in $\mathbf{XB}$ are non-negative, and so all of the non-diagonal elements of $\mathbf{Y}$ must have the same sign, such that it is the same as the choice made for the $\pm$ in front of the square root. If $\mathbf{Y}$ is a square root of the identity matrix, then so is $-\mathbf{Y}$, so for this reason we can always take the positive sign along with $\mathbf{Y}$. Further, the elements of $\mathbf{XB}$ are always integers and so the non-diagonal elements of $\mathbf{Y}$ must be divisible by $(\frac{\alpha^2}{4} + |G|)^{-1/2}$.

To begin, we consider the case that the NIM-rep M contains a single group orbit.

Proposition 5.3.9. *NIM-reps over $K(G, \alpha)$ consisting of one group orbit are parameterised by pairs $(H, x_{1,1})$, consisting of*

- *A subgroup $H \subseteq G$,*
- *A non-negative integer $x_{1,1} \in \mathbb{Z}_2$,*

such that $\alpha = x_{1,1}|G : H| - \frac{|H|}{x_{1,1}}$, $x_{1,1}$ divides $|H|$, and $(x_{1,1})^2|G : H| \geqslant |H|$.

Proof. As there is only a single group orbit in M, we need a single stabiliser subgroup $H \subseteq G$. The action of X is given by a non-negative integer $x_{1,1} \in \mathbb{Z}_+$. In fact, we can take $x_{1,1}$ to be strictly positive, as otherwise we have the situation that $X \rhd m_l^1 = 0$, which is the zero NIM-rep that we have discounted in Remark 5.2.6

The matrices in Equation (5.3.2.7) are one-dimensional and so the only positive square root of $\mathbf{I} = 1$ is clearly $\mathbf{Y} = 1$. If we input our data into Equation (5.3.2.7), we get the equation

$$x_{1,1}|G:H| = \frac{1}{2}\alpha + \sqrt{\frac{\alpha^2}{4} + |G|}$$

If we rearrange this and then square both sides, we get

$$\frac{\alpha^2}{4} + |G| = (x_{1,1})^2|G:H|^2 - \alpha x_{1,1}|G:H| + \frac{\alpha^2}{4}.$$

Solving this equation for α gives us the condition that

$$\alpha = x_{1,1}|G:H| - \frac{|H|}{x_{1,1}},$$

and the remaining conditions are a result of α needing to be a non-negative integer. □

The two orbit case results in a 2-by-2 matrix equation, and there is also a complete classification of square roots of the identity matrix in this case.

Proposition 5.3.10. *All indecomposable NIM-reps over $K(G,\alpha)$ consisting of two group orbits are parameterised by tuples $(H_1, H_2, x_{1,1}, x_{2,2})$, consisting of*

- *Two subgroups $H_1, H_2 \subseteq G$,*
- *Two non-negative integers $x_{1,1}, x_{2,2} \in \mathbb{Z}_2$.*

such that $\alpha = x_{1,1}|G:H_1| + x_{2,2}|G:H_2|$, $|G|$ divides $|H_1||H_2|$, and $(\frac{|H_1||H_2|}{|G|} + x_{1,1}x_{2,2})$ is a square number.

Proof. We take $H_1, H_2 \subseteq G$ as the subgroups that are stabilisers of the two group orbits in M. The 2-by-2 matrix $\mathbf{X}$ is symmetric, so we require three non-negative integers $x_{1,1}, x_{1,2}, x_{2,2} \in \mathbb{Z}_+$ to describe the action of X.

Square roots of the 2-by-2 identity matrix come in two forms;

- Diagonal matrices whose diagonal elements are from the set $\{-1, 1\}$,
- Matrices of the form $\mathbf{Y} = \begin{pmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & -y_{1,1} \end{pmatrix}$, where $y_{1,1}^2 + y_{1,2}y_{2,1} = 1$ and $y_{1,2}, y_{2,1} \neq 0$.

If we were to pick a square root of the first form, we would immediately get that $x_{1,2} = 0$ as all the matrices on the right-hand side of Equation (5.3.2.7) are diagonal. But this contradicts the indecomposability of the NIM-rep, as there is no longer any way to travel between the two group orbits. Thus we can only have a square root of the second form. By inputting this into Equation (5.3.2.7), we get the following system of equations:

$$\begin{aligned} x_{1,1}|G:H_1| &= \frac{1}{2}\alpha + \sqrt{\frac{\alpha^2}{4} + |G|y_{1,1}}, \\ x_{2,2}|G:H_2| &= \frac{1}{2}\alpha - \sqrt{\frac{\alpha^2}{4} + |G|y_{1,1}}, \\ x_{1,2}|G:H_2| &= \sqrt{\frac{\alpha^2}{4} + |G|y_{1,2}}, \\ x_{1,2}|G:H_1| &= \sqrt{\frac{\alpha^2}{4} + |G|y_{2,1}}. \end{aligned}$$

By adding the first two equations together, we get that $\alpha = x_{1,1}|G:H_1| + x_{2,2}|G:H_2|$. By multiplying the first two equations together and the last two equations together, we obtain

$$\begin{aligned} x_{1,1}x_{2,2}|G:H_1||G:H_2| &= \frac{\alpha^2}{4} - \left(\frac{\alpha^2}{4} + |G|\right)(y_{1,1})^2, \\ (x_{1,2})^2|G:H_1||G:H_2| &= \left(\frac{\alpha^2}{4} + |G|\right)y_{1,2}y_{2,1}. \end{aligned}$$

Using the defining relation of $\mathbf{Y}$, we can combine these two equations together to get that

$$\begin{aligned} ((x_{1,2})^2 - x_{1,1}x_{2,2})|G:H_1||G:H_2| =& \left(\frac{\alpha^2}{4} + |G|\right)(y_{1,1}^2 + y_{1,1}y_{2,2}) - \frac{\alpha^2}{4} = |G| \\ \implies (x_{1,2})^2 =& \frac{|H_1||H_2|}{|G|} + x_{1,1}x_{2,2}. \end{aligned}$$

We see that $x_{1,2}$ can be given in terms of the rest of the input data, with the remaining conditions coming from $x_{1,2}$ being a positive integer.

The explicit structure of the action of X on a two orbit indecomposable NIM-rep is given by

$$\begin{aligned} X \triangleright m_i^1 &= x_{1,1}\sum_{k=1}^{|G:H_1|} m_k^1 + \sqrt{\frac{|H_1||H_2|}{|G|} + x_{1,1}x_{2,2}}\sum_{k=1}^{|G:H_2|} m_k^2, \\ X \triangleright m_i^2 &= \sqrt{\frac{|H_1||H_2|}{|G|} + x_{1,1}x_{2,2}}\sum_{k=1}^{|G:H_1|} m_k^1 + x_{2,2}\sum_{k=1}^{|G:H_2|} m_k^2. \end{aligned}$$

□

To proceed further in this way, we would need an explicit classification of square roots of the identity matrix of order 3 or more, which to date is an open problem. To illustrate that this is not possible, consider the following example.

A Hadamard matrix of order n is an n-by-n matrix containing only -1 and $+1$ that satisfies $H_n H_n = nI_n$ and has rows that are mutually orthogonal. The matrix defined by $Y := \frac{1}{\sqrt{n}} H_n$ is thus a square root of the identity matrix. It is still unknown whether there exists a Hadamard matrix of order $4k$ for every positive integer k, [Pal33], [MS22].

For our purposes, a Hadamard matrix doesn't provide a NIM-rep unless $n = 1$, as H_n contains exactly $n(n-1)/2$ elements that are -1. This contradicts the fact that all non-diagonal elements of Y, of which there are $n(n-1)$, must be non-negative for it to provide a solution to Equation (5.3.2.7). The only value that this construction is valid for is $n = 1$, which is just the known square root $+1$, which we have already considered. However, it is sufficient to demonstrate that we can't provide such a complete classification further on.

We can still construct examples of NIM-reps containing 3 or more G-orbits.

Example 5.3.11. (NIM-rep over $K(\mathbb{Z}_q, q-1), q$ prime) Consider the p-by-p matrix that has $-\frac{1}{2}\alpha$ as its diagonal elements, and 1 for all off-diagonal elements;

$$Z_p = \begin{pmatrix} -\frac{1}{2}\alpha & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & -\frac{1}{2}\alpha \end{pmatrix}$$

If we square this matrix, we get that

$$(Z_p^2)_{i,i} = \frac{\alpha^2}{4} + p - 1, \quad (Z_p^2)_{i,j} = p - 2 - \alpha, \ i \neq j.$$

Thus, for Z_p^2 to be a square root of $(\frac{\alpha^2}{4} + |G|) \cdot I_p$ we require that

$$\alpha = p - 2, \text{ making } (Z_p^2)_{i,j} = 0,$$

and then

$$|G| = p - 1, \text{ making } (Z_p^2)_{i,i} = \frac{\alpha^2}{4} + |G|.$$

So Z_p has the potential to provide a NIM-rep over $K(G, p-2)$, where $|G| = p - 1$ for some

$p \in \mathbb{N}$. Whether a valid solution to Equation (5.3.2.7) exists depends on the choice of group and p, but we consider the special case that $p = q + 1$, where q is a prime number.

In this case, we must have that $G = \mathbb{Z}_q$, so we are seeking to check whether $Z_{(q+1)}$ provides a NIM-rep over $K(\mathbb{Z}_q, q-1)$. This NIM-rep consists of $q+1$ group orbits, so we need $q+1$ subgroups $H_i \subseteq \mathbb{Z}_q$.

Inputting Z_p into Equation (5.3.2.7), we get that

$$CB = \frac{1}{2}\alpha + Z_p = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix}.$$

We must have that $x_{i,i} = 0$ for all $1 \leqslant i \leqslant q+1$. Further, for all $1 \leqslant i, j \leqslant q+1, i \neq j$, we have that

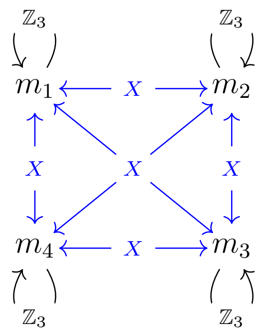
$$x_{i,j}|G : H_i| = 1.$$

As the only subgroups of $\mathbb{Z}_q$ are the trivial subgroup and the $\mathbb{Z}_q$ itself, the only choice that leads to an integer value of $x_{i,j}$ is if $H_i \cong \mathbb{Z}_q$ for all i and $x_{i,j} = 1$. Hence, we have a NIM-rep over $K(\mathbb{Z}_q, q-1)$ that consists of $q+1$ basis elements, all which are acted on by $\mathbb{Z}_q$ trivially.

The action of X is given by

$$X \rhd m^i = \sum_{\substack{j=1 \\ j \neq i}}^{q+1} m^j,$$

so each group orbit is connected to all orbits but itself by a single copy of X. For example, when $q = 3$ the NIM-graph of this NIM-rep over $K(\mathbb{Z}_3, 2)$ is given by,



For other examples, it comes down to finding proposed square roots and computing them directly. However, it turns out that for the case of the Tambara-Yamagami fusion rings, (the case $\alpha = 0$), we already have everything needed to completely classify their NIM-reps.

Corollary 5.3.12. *All indecomposable NIM-reps over the Tambara-Yamagami fusion ring $K(G, 0)$ consist of at most two group orbits.*

Proof. As $\alpha = 0$, the matrix $\mathbf{XBX}$ in Equation (5.3.2.6) must be diagonal. Element-wise, this means that

$$(\mathbf{XBX})_{i,j} = \sum_{k=1}^{p} x_{i,k} x_{k,j} |G : H_i| = 0,$$

for all $i \neq j$. For this to hold, we must have that every two distinct group orbits are not connected to a common third group orbit. But by the indecomposability of the NIM-rep, this is not possible when there are 3 or more orbits. □

So our classification of NIM-reps containing 1 or 2 group orbits is sufficient to construct all NIM-reps over a Tambara-Yamagami fusion ring.

Example 5.3.13 (NIM-reps over the Ising fusion ring, $K(\mathbb{Z}_2, 0)$)**.** Recall the Ising fusion ring Example 5.1.12, which can be viewed as the Tambara-Yamagami fusion ring $K(\mathbb{Z}_2, 0)$.

- NIM-reps with 1 G-orbit.

  By Proposition 5.3.9, a pair $(H, x_{1,1})$ gives a 1-orbit NIM-rep if $x_{1,1}|\mathbb{Z}_2 : H| = \frac{|H|}{x_{1,1}}$, as $\alpha = 0$. There are two choices of subgroup in $\mathbb{Z}_2$:

  - The trivial subgroup $H = \{e\}$; Here we must have $2x_{1,1} = \frac{1}{x_{1,1}}$, which has no integer solutions.
  - The group $\mathbb{Z}_2$; This must satisfy $x_{1,1} = \frac{2}{x_{1,1}}$, which again has no integer solutions.

  From this, we conclude that there are no 1-orbit NIM-reps of $K(\mathbb{Z}_2, 0)$.

- NIM-reps with 2 G-orbits.

  By Proposition 5.3.10, a tuple $(H_1, H_2, x_{1,1}, x_{2,2})$ gives a NIM-rep if $\alpha = 0 = x_{1,1}|G : H_1| + x_{2,2}|G : H_2|$, which occurs only when $x_{1,1} = x_{2,2} = 0$.

  As a consequence, we have that $(x_{1,2})^2 = \frac{|H_1||H_2|}{|G|}$. There are three possible pairs of subgroups for which $|G|$ divides $|H_1||H_2|$;

  - $(H_1 = \mathbb{Z}_2, H_2 = \mathbb{Z}_2)$; Here, we get that $(x_{1,2})^2 = 2$ which is not a square number, hence doesn't give a valid solution.

– $(H_1 = \{e\}, H_2 = \mathbb{Z}_2)$; We get that $(x_{1,2})^2 = 1$ which is valid. Hence $(\{e\}, \mathbb{Z}_2, 0, 0)$ gives a NIM-rep over $K(\mathbb{Z}_2, 0)$. We note that the pair $(H_1 = \mathbb{Z}_2, H_2 = \{e\})$ gives an isomorphic NIM-rep.

Thus there is only a single NIM-rep, up to isomorphism, over the Ising fusion ring, parameterised by $(\{e\}, \mathbb{Z}_2, 0, 0)$. This is exactly the Ising fusion ring as a NIM-rep over itself.

Beyond this, the classification of NIM-reps over near-group fusion rings is currently a procedural, case-by-case problem. However, if we turn our attention back to finding admissible algebras we have all the tools required.

Proposition 5.3.14. *An indecomposable NIM-rep over $K(G, \alpha)$ is admissible only if it consists of either one group orbit, parameterised by $(H, x_{1,1})$, or two group orbits, parameterised by $(\{e\}, G, 0, x_{2,2})$.*

Proof. Suppose we have a 1-orbit NIM-rep parameterised by $(H, x_{1,1})$. Then it is admissible in an analagous way as the group ring case, Lemma 5.2.15, as we can take the NIM-basis element as m_e, which is connected to all other basis elements by the action of G.

Now suppose that we have a NIM-rep consisting of 2 or more group orbits. For this to be admissible, we must have a distinguished basis element m_0 that connects to all other basis elements. Let us set H_1 as the stabiliser group of the orbit containing m_0. Consider a basis element $n \in M$ that lies in a different group orbit to m_0, with stabiliser group H_2. We can only reach it by the action of X, with $X \rhd m_0 = n$. This gives us that $x_{1,1} = 0, x_{1,2} = 1$.

If we then act with X again, we get that

$$X \rhd n = X^2 \rhd m_0 = |H_1| \sum_{i \in G \cdot m_0} m_i + \alpha X \rhd m_0 = |H_1| \sum_{i \in G \cdot m_0} m_i + \alpha n,$$

where the sum runs over elements in the same G-orbit as m_0. In particular, we see that n is only connected to one other group orbit, meaning that M contains two group orbits by indecomposability.

Further, we can read off that $x_{2,2} = \alpha$, and as $X \rhd n = gX \rhd n$ for all $g \in G$, we must have that n is the only element in its G-orbit. Hence the stabiliser group is given by $H_2 = G$. Along with $x_{1,2} = 1$, this implies that $H_1 = \{e\}$.Hence, a two orbit NIM-rep is admissible only if it is of the form $(\{e\}, G, 0, \alpha)$.

The converse direction for both cases is immediate, by construction. □

Corollary 5.3.15. *From this, we get two forms of algebra objects in $K(G, \alpha)$ coming from admissible NIM-reps;*

- *For a NIM-rep parameterised by $(H, x_{1,1})$, the potential algebra structure is given as an object by $\oplus_{h \in H} b_h \oplus x_{1,1} X$.*

- *For a NIM-rep parameterised by $(\{e\}, G, 0, \alpha)$, the potential algebra object is simply* $[1]$*, the form of the monoidal unit $\mathbb{1}$ algebra.*

Remark 5.3.16. While this process gives gives us the structure of indecomposable semisimple module categories, we don't currently have a complete picture of what happens with the algebras.

For example, we can view $\mathsf{Vect}_{\mathbb{Z}_2}$ as a fusion subcategory of the Ising category $C(\mathbb{Z}_2, 0)$, meaning we are able to view the group algebras $\Bbbk[\{e\}], \Bbbk[\mathbb{Z}_2]$ as algebra objects in $C(\mathbb{Z}_2, 0)$. Both are connected separable in $\mathsf{Vect}_{\mathbb{Z}_2}$, yet by Example 5.3.13 we only have a single NIM-rep over the fusion ring $K(\mathbb{Z}_2, 0)$. So in the transfer to the near-group case something occurs which our method cannot see, almost certainly due to the fact we can make no statements about the morphisms involved. We can observe by Corollary 5.3.15 that in all cases of near-group categories we lose the group algebra structures. This is something which will form the basis of further refining this method.

Example 5.3.17 (NIM-reps of $K(\mathbb{Z}_2 \times \mathbb{Z}_2, \alpha)$)**.** As discussed, every choice of α has one admissible 2-orbit NIM-rep that corresponds to the trivial algebra object. Hence we just need to classify the 1-orbit NIM-reps. There are three isomorphism classes of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

- The trivial subgroup $H = \{e\}$; As $x_{1,1}$ needs to divide $|H|$, we have that $x_{1,1} = 1$. Then $\alpha = 3$, meaning we have a NIM-rep over $K(G, 3)$.

- The subgroup $H = \mathbb{Z}_2$; We can have either $x_{1,1} = 1$, meaning $\alpha = 0$, or $x_{1,1} = 2$ which gives $\alpha = 3$. We note that we will still get 3 different NIM-reps, depending on our choice of conjugacy class of $\mathbb{Z}_2$.

- The group $\mathbb{Z}_2 \times \mathbb{Z}_2$; For $x_{1,1} = 1$, we have $\alpha = -3$ which is invalid. The other choices of $x_{1,1} = 2$ and $x_{1,1} = 4$ give $\alpha = 0$ and $\alpha = 3$ respectively.

  Thus, the only non-trivial algebras in $K(\mathbb{Z}_2 \times \mathbb{Z}_2, \alpha)$ are given by NIM-reps

  - $(\mathbb{Z}_2, 1)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, 2)$ over $K(\mathbb{Z}_2 \times \mathbb{Z}_2, 0)$,
  - $(\{e\}, 1)$, $(\mathbb{Z}_2, 2)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, 4)$ over $K(\mathbb{Z}_2 \times \mathbb{Z}_2, 3)$

5.3.3 $(A_1, l)_{\frac{1}{2}}$ Fusion Rings

We will now consider a modular tensor category, which can be constructed out of a quantum group of type A_1 at level $l \in \mathbb{Z}_+$, following [NWZ22]. In this section, we shall compute the NIM-reps over its corresponding fusion ring.

Definition 5.3.18. The Grothendieck ring $\mathrm{Gr}((A_1, l))$ has basis $\{V_i\}_{i\in[0,l]}$, with the fusion coefficients of $V_iV_j = \sum_{k=0}^{l} c_{i,j}^k V_k$ given by:

$$c_{i,j}^k = \begin{cases} 1, & \text{if } |i-j| \leqslant k \leqslant \min(i+j, 2k-i-j) \text{ and } k \equiv i+j \bmod 2, \\ 0, & \text{else.} \end{cases} \tag{5.3.3.1}$$

This is a commutative ring.

We can take a full modular subcategory $(A_1, l)_{\mathrm{pt}}$ by taking the simple objects $\mathrm{Irr}((A_1, l)_{\mathrm{pt}}) = \{V_0, V_l\}$, [NWZ22]. The fusion rules of this subcategory are simply $V_l^2 = V_0$, which are $\mathbb{Z}_2$. Hence the Grothendieck ring $\mathrm{Gr}(\mathrm{Irr}((A_1, l)_{\mathrm{pt}})) = R(\mathbb{Z}_2)$ is a group ring, which we have studied previously.

When l is a positive odd integer, we can define a second modular subcategory $(A_1, l)_{\frac{1}{2}}$ by taking the full subcategory with simple objects

$$\mathrm{Irr}((A_1, l)_{\frac{1}{2}}) = \left\{ V_{2i} \mid 0 \leqslant i \leqslant \frac{l-1}{2} \right\}.$$

Lemma 5.3.19 ([NWZ22] Section 4)**.** *Let l be a positive odd integer. Then there is an equivalence of modular tensor categories $(A_1, l) \simeq (A_1, l)_{\frac{1}{2}} \boxtimes (A_1, l)_{\mathrm{pt}}$*

Now, for any NIM-rep over $\mathrm{Gr}((A_1, l))$, we naturally get a NIM-rep over both $R(\mathbb{Z}_2)$ and $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$ by restricting to the each subcategory along the natural embedding. The NIM-reps over the fusion ring $\mathrm{Gr}((A_1, l))$ were classified in [EK95] and are in one-to-one correspondence with simply laced Dynkin diagrams with Coxeter number $h = l + 2$, and in the case that l is an odd integer, we have the NIM-rep as $\mathrm{Gr}((A_1, l))$ viewed as a NIM-rep over itself only. Restriction to $R(\mathbb{Z}_2)$ and $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$ gives the self NIM-rep also. However, we know by Proposition 5.3.1 there are two NIM-reps over $R(\mathbb{Z}_2)$, so there is not a lifting to $\mathrm{Gr}((A_1, l))$ for every NIM-rep on its subcategories. This leads to the natural question of whether there are any NIM-reps of $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$ that don't come from the larger category.

When $l = 1$, we are left with the trivial ring which can be viewed as the group ring $R(\{e\})$. Hence we shall assume $l \geqslant 3$.

Definition 5.3.20. For an object $V_i \in \mathrm{Gr}((A_1, l))$, we define the *length* of V_i to be

$$\mathrm{length}(V_i) := \sum_{k=0}^{l} c_{i,i}^{k}.$$

Remark 5.3.21. The length outputs the number of objects in the decomposition of V_i^2.In the fusion ring $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$, it is simple to check using the fusion rules that $V_i \neq V_j$ if and only if $\mathrm{length}(V_i) \neq \mathrm{length}(V_j)$.

By arranging objects in sequence of increasing length, we obtain the sequence

$$V_0, V_{l-1}, V_2, V_{l-3}, ..., V_{(l\pm1)/2},$$

where the sign of the end object depends on whether $l = \mp 1 \bmod 4$.

Remark 5.3.22. If we have objects $V_i, V_j \in \mathrm{Gr}((A_1, l)_{\frac{1}{2}})$ such that $\mathrm{length}(V_i) > \mathrm{length}(V_j)$, then it is easily seen $(V_i^2 \rhd m_p, m_p) \geqslant (V_j^2 \rhd m_p, m_p)$. As V_i is self-dual, If $V_i \rhd m_p \in M$, then

$$1 = (V_i \rhd m_p, V_i \rhd m_p) = (V_i^2 \rhd m_p, m_p) \geqslant (V_j^2 \rhd m_p, m_p) = 1,$$

implying that $V_j \rhd m_p$ is in M also.

Now, if we assume that M is admissible and has a basis of order d, we immediately get that the objects that satisfy $V_j \rhd m_0 \in M$ are exactly those of $\mathrm{length}(V_j) \leqslant d$.

Unlike the case of group an near-group fusion categories, the ring $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$ has no invertible elements. As such, we can make the following statement.

Lemma 5.3.23. *Let M be an admissible NIM-rep over $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$, and $V_i \neq V_j \in \mathrm{Gr}((A_1, l)_{\frac{1}{2}})$ such that $V_i \rhd m_0, V_j \rhd m_0 \in M$. Then $V_i \rhd m_0 \neq V_j \rhd m_0$.*

Proof. If we assume that $V_i \rhd m_0 = V_j \rhd m_0$, then as the fusion ring is commutative, we have that

$$V_i^2 \rhd m_0 = V_i V_j \rhd m_0 = V_j V_i \rhd m_0 = V_j^2 \rhd m_0.$$

But by Remark 5.3.21, as V_i and V_j are distinct, we must have that $\mathrm{length}(V_i) \neq \mathrm{length}(V_j)$, meaning one of them contains more objects in their decomposition than the other. Then, the only way that $V_i^2 \rhd m_0 = V_j^2 \rhd m_0$ is if some V_k in the decomposition of V_i^2 or V_j^2 (whichever has larger length), satisfies $V_k \rhd m_0 = 0$. But this then leads to the zero NIM-rep, which we have discounted. □

As we have assumed $l \geqslant 3$, every fusion ring we are considering contains the object V_{l-1}, so we shall focus on exploring how this acts on a NIM-rep.

Proposition 5.3.24. *In any NIM-rep M over the fusion ring* $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$, $(V_{l-1} \rhd m_p, m_q) \leqslant 1$ *for all* $m_p, m_q \in M$.

Proof. Let us assume that $(V_{l-1} \rhd m_p, m_q) = a^q_{l-1,p} \geqslant 2$. Then we can use the fusion rules in Equation (5.3.3.1) to obtain

$$m_p + V_2 \rhd m_p = V_{l-1}^2 \rhd m_p = a^q_{l-1,p} V_{l-1} \rhd m_q + \sum_{\substack{k \in L \\ k \neq q}} a^k_{l-1,p} V_{l-1} \rhd m_k$$

Applying the form $(-, m_p)$, and using the rigidity condition of the NIM-rep, we find that

$$(V_2 \rhd m_p, m_p) \geqslant a^q_{l-1,p}(V_{l-1} \rhd m_q, m_p) - 1 \geqslant 3 \tag{5.3.3.2}$$

The fusion rules in Equation (5.3.3.1) give that when $l \geqslant 3$,

$$V_{2j}V_2 = V_{2j-2} + V_{2j} + V_{2j+2}, \text{ when } 1 \leqslant j \leqslant \frac{l-3}{2},$$

and

$$V_2 V_{l-1} = V_{l-3} + V_{l-1}$$

We set $h_{i,p} := \sum_{k \in L} a^k_{i,p}$, which counts the number of NIM-basis elements in the decomposition of $V_i \rhd m_p$.

Applying the fusion rules to $V_{2j}V_2 \rhd m_p = \sum_{k \in L} a^k_{2,p} V_{2j} \rhd m_k$, we obtain

$$V_{2j-2} \rhd m_p + V_{2j+2} \rhd m_p = (a^p_{2,p} - 1)V_{2j} \rhd m_p + \sum_{\substack{k \in L \\ k \neq p}} a^k_{2,p} V_{2j} \rhd m_k, \quad 1 \leqslant j \leqslant \frac{l-3}{2}$$
$$V_{l-3} \rhd m_p = (a^p_{2,p} - 1)V_{l-1} \rhd m_p + \sum_{\substack{k \in L \\ k \neq p}} a^k_{2,p} V_{l-1} \rhd m_k$$

By noting that $h_{i,p} > 1$ for all choices of i, p, and $a^p_{2,p} \geqslant 3$ by Equation (5.3.3.2), when we count the NIM-basis elements on each side we obtain the following inequalities;

$$h_{2j-2,p} + h_{2j+2,p} \geqslant 2h_{2j,p} + h_{2,p} - 3, \quad 1 \leqslant j \leqslant \frac{l-3}{2},$$
$$h_{l-3,p} \geqslant h_{l-1,p} + h_{2,p} - 3$$

By taking the inequality for each $1 \leqslant j \leqslant \frac{l-1}{2}$ and summing them together, we obtain

$$
\begin{aligned}
1 + h_{2,p} + 2h_{4,p} + ... + 2h_{l-3,p} + h_{l-1,p} \geqslant 2h_{2,p} + 2h_{4,p} + ... + 2h_{l-1,p} + \frac{l-1}{2}(h_{2,p} - 3) \\
\implies \frac{3l-1}{2} \geqslant \frac{l+1}{2}h_{2,p} + h_{l-1,p} \geqslant \frac{l+1}{2}h_{2,p} + 2 \\
\implies \frac{3l-5}{l+1} \geqslant h_{2,p}
\end{aligned}
$$

where the last inequality in the second line follows due to our initial assumption. However, the last inequality cannot be satisfied as the fraction on the left-hand side is strictly less than 3 for all values of l, which contradicts $(V_2 \rhd m_p, m_p) = a^p_{2,p} \geqslant 3$. Hence we have a contradiction, and so, $(V_{l-1} * m_p, m_q) \leqslant 1$, for all $m_p, m_q \in M$. □

Lemma 5.3.25. *In any NIM-rep M over* $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$, $(V^2_{l-1} \rhd m_p, m_p) \leqslant 3$ *for all* $m_p, m_q \in M$.

Proof. If we assume that $(V^2_{l-1} \rhd m_p, m_p) > 4$, then by Proposition 5.3.24, we have that $h_{l-1,p} = (V^2_{l-1} \rhd m_p, m_p) > 4$. By the fusion rules in Equation (5.3.3.1), we have that $(V_2 \rhd m_p, m_p) \geqslant 3$. We are in a very similar setup to the proof of Proposition 5.3.24, which if we follow through results in the inequality

$$\frac{3l-9}{l+1} \geqslant h_{2,p}.$$

This fraction is also strictly less than 3, so we obtain the desired contradiction. □

Proposition 5.3.26. *Let M be an admissible NIM-rep over* $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$. *Then there exists no $m_k \in M$ such that $(V^2_{l-1} \rhd m_k, m_k) = 3$.*

Proof. Assume there exists some $m_k \in M$ such that $(V^2_{k-1} \rhd m_k, m_k) = 3$. By Lemma 5.3.25, we can write

$$V_{k-1} \rhd m_k = m_x + m_y + m_z,$$

where $m_x, m_y, m_z \in M$ are distinct NIM-basis elements. The fusion rule of V^2_{l-1} gives us that $(V_2 \rhd m_k, m_k) = 2$.

As the NIM-rep is admissible and m_x, m_y, m_y are distinct NIM-basis elements, the cardinality of the NIM-basis M is at least 3, so by Remark 5.3.22 we know that $V_2 \rhd m_0, V_{l-1} \rhd m_0 \in M$. We also know that there exists a $V_j \in (A_1, l)_{\frac{1}{2}}$ such that $V_j \rhd m_0 = m_k$. Using the fusion rules, we find that

$$V_{l-1} \rhd m_k = V_{l-1}V_j \rhd m_0 = V_{l-1-j} \rhd m_0 + V_{l+1-j} \rhd m_0.$$

Acting with V_{l-1} again on both sides and then using the form $(-, m_k)$, we have

$$\begin{aligned} m_k + V_2 \rhd m_k &= V_{k-1}^2 \rhd m_0 = V_j \rhd m_0 + 2V_{j+2} \rhd m_0 + V_{j+4} \rhd m_0 \\ \implies V_2 \rhd m_k &= 2V_{j+2} \rhd m_0 + V_{j+4} \rhd m_0. \end{aligned} \tag{5.3.3.3}$$

A second way to calculate $V_2 \rhd m_k$ is as follows:

$$V_2 \rhd m_k = V_2V_j \rhd m_0 = V_{j-2} \rhd m_0 + V_j \rhd m_0 + V_{j+2} \rhd m_0. \tag{5.3.3.4}$$

By applying the form $(-, m_k)$ to both Equation (5.3.3.3) and Equation (5.3.3.4), we have that

$$2 = 2(V_{j+2} \rhd m_0, m_k) + (V_{j+4} \rhd m_0, m_k), \tag{5.3.3.5}$$

$$1 = (V_{j-2} \rhd m_0, m_k) + (V_{j+2} \rhd m_0, m_k). \tag{5.3.3.6}$$

By the fusion rules Equation (5.3.3.1) and Remark 5.3.22, it is clear that we can only satisfy Equation (5.3.3.5) when $j \leqslant \frac{l\pm1}{2}$, where the sign is determined by $l \equiv \mp 1 \bmod 4$. But then we immediately have that $(V_{j-2} \rhd m_k, m_1) = 0$ by Remark 5.3.22, so the above equations give us that

$$(V_{j+2} \rhd m_0, m_k) = 1, \text{ and } (V_{j+4} \rhd m_0, m_k) = 0. \tag{5.3.3.7}$$

If we calculate the fusion rules, we see that the only term that appears in the expansion of $(V_{j+2}V_j \rhd m_0, m_0)$ and not in that of $(V_{j+4}V_j \rhd m_0, m_0)$ is $(V_2 \rhd m_0, m_0)$. But as $V_2 \rhd m_0 \in M$ and V_2 is non-invertible, this term must always be 0. Hence we can never satisfy Equation (5.3.3.7), giving a contradiction. Thus we must have $(V_{l-1}^2 \rhd m_0, m_0) \leqslant 2$ for all $m_k \in M$. □

Proposition 5.3.27. *Up to isomorphism, there is only one admissible NIM-rep over* $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$.

Proof. Suppose we have an admissible NIM-rep M over $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$ where M has cardinality d. Then there exists a $V_j \in \mathrm{Gr}((A_1, l)_{\frac{1}{2}})$ where $\mathrm{length}(V_j) = d$ and $V_j \rhd m_0 \in M$. Acting with V_{l-1} we get that

$$V_{l-1}V_j \rhd m_0 = V_{l-1-j} \rhd m_0 + V_{l+1-j} \rhd m_0.$$

Using the fusion rules from Equation (5.3.3.1) and Definition 5.3.20, if $j < \frac{l\pm1}{2}$, (where the sign depends on $k \equiv \mp 1 \bmod 4$), then $\text{length}(V_{l+1-j}) = d+1$ and so $V_{l-1-j} \rhd m_0 \notin M$, by Remark 5.3.22. Similarly, if $j > \frac{l\pm1}{2}$, then $V_{l+1-j} \rhd m_0 \notin M$. In both cases, this leads to

$$(V_{l-1}^2 V_j \rhd m_0, V_j \rhd m_0) \geqslant (V_{l-1-j} \rhd m_0, V_{l-1-j} \rhd m_0) + (V_{l+1-j} \rhd m_0, V_{l+1-j} \rhd m_0) = 3.$$

This contradicts $V_j \rhd m_0 \in M$ by Proposition 5.3.26, and so we must have that $j = \frac{l\pm1}{2}$. All other objects V_i have length less than V_j, and so every $m_k \in M$ is of the form $V_i \rhd m_0 \in M$. This gives that the full NIM-rep structure is simply generated by the fusion rules. □

Thus the NIM-rep that comes from restricting down from $\mathrm{Gr}((A_1, l))$ is indeed the only NIM-rep over $\mathrm{Gr}((A_1, l)_{\frac{1}{2}})$. This corresponds to the monoidal unit $\mathbb{1}$.

Chapter 6

Future Directions

We shall now briefly highlight ways in which the work presented in this thesis may be extended.

Regarding the results in Chapter 4, we used a Frobenius monoidal functor to classify Frobenius algebras in the category of (twisted) Yetter-Drinfeld modules over the group Hopf algebra. However, we can define ${}^{\mathcal{H}}_{\mathcal{H}}\mathsf{YD}$ for any Hopf algebra $\mathcal{H}$, so a natural next step is to see how much of our process we can generalise. First, we would need some sort of braided monoidal functor between two categories of YD modules, taking the place of the functor I from Section 4.3.

Proposition 6.0.1. [FLP24, Corollary H] *Let $\varphi : \mathcal{K} \to \mathcal{H}$ be a morphism of Hopf algebras in* Vect*. Then we induce a braided lax monoidal functor*

$$Ind : {}^{\mathcal{H}}_{\mathcal{H}}\mathsf{YD} \to {}^{\mathcal{K}}_{\mathcal{K}}\mathsf{YD}.$$

Further, if $\mathcal{H}$ is finitely generated projective as a left $\mathcal{K}$-module, we get a braided lax monoidal functor

$$CoInd : {}^{\mathcal{K}}_{\mathcal{K}}\mathsf{YD} \to {}^{\mathcal{H}}_{\mathcal{H}}\mathsf{YD}.$$

These functors give a place to start, but it is still yet to be seen whether they admit a Frobenius monoidal structure. Even if not, we still aim to explore what sort of algebraic properties could be preserved and potentially used as classification representatives, akin to the twisted group algebras that we used previously.

Regarding the results from Chapter 5, if we recall the discussion from Remark 5.3.16, we do not currently recover a full picture of the potential algebra structures purely from the NIM-reps. To add to this, if we consider the classification in Theorem 4.0.1 of connected separable algebras in Vect_G^ω , there are occasions when two group algebras $(\Bbbk[H], \kappa), (\Bbbk[H], \kappa')$ formed from the same subgroup but differing 2-cocycles result in non-equivalent module categories. This is a distinction which cannot be made using NIM-reps, once again because we do not see the morphisms at this level. So to advance this work we need to find a way to introduce some more of the categorical data to our construction. As we have both morphisms of fusion rings and morphisms of the NIM-reps they are acting on, this could potentially be done using a bicategory or double category as defined in [Ben98].

Secondly, we can still continue to classify NIM-reps for other families of fusion rings to build up our catalogue of examples which we can test with. Possible options include perfect tensor categories, which appear in the classification of low-rank modular tensor categories [CGP24], such as categories of modules over certain quantum groups [EP21, NWZ22] . This will be useful in building more examples of the method, but as all elements in the corresponding fusion rings will be non-invertible (as opposed to the single non-invertible element for the near-group case) it is likely we will be left with a family of matrix equations to solve, which may prove difficult to find exact solutions to.

Overall, the search for algebra objects in fusion categories is an extremely active area that has lots of potential for future advancements that this preliminary work can contribute to.

Appendix A

Cohomology

Here, we collect the basic definitions and constructions from group cohomology used throughout this thesis, following [Ben98, Section 3.4], [Bro94, Chapter III], [Wei94]. We note that these references typically use the convention of left modules, whereas we use right modules.

A.1 Cohomology of groups

Let R be a ring.

Definition A.1.1. A *chain complex* $M_\bullet$ is a sequence of R-modules, connected together by R-module homomorphisms,

$$\dots \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} M_{n-2} \xrightarrow{d_{n-2}} \dots, \qquad \text{(A.1.0.1)}$$

such that the composition $d_{n-1} \circ d_n = 0$, for all $n \in \mathbb{Z}$. The maps d_n are called differentials. We denote by $C_n(M_\bullet) := \mathrm{Ker}(d_n)$ the space of *n-cycles*, and by $B_n(M_\bullet) := \mathrm{Im}(d_{n+1})$ the space of *n-boundaries.* The *n-th homology group* is the quotient

$$H_n(M_\bullet) := C_n(M_\bullet)/B_n(M_\bullet). \qquad \text{(A.1.0.2)}$$

In a dual manner, we can define a *cochain complex* $M^\bullet$ as a sequence of R-modules and module homomorphisms,

$$\dots \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \xrightarrow{d^{n+1}} M^{n+2} \xrightarrow{d^{n+2}} \dots, \qquad \text{(A.1.0.3)}$$

where the composition of differentials $d^n \circ d^{n-1} = 0$. The spaces of *n-cocycles* $C^n(M^\bullet)$

and *n-coboundaries* $B^n(M^\bullet)$ are defined similarly, with the *n-th cohomolgy group* being the quotient

$$H^n(M^\bullet) := C^n(M^\bullet)/B^n(M^\bullet). \tag{A.1.0.4}$$

Given a group G, we can study its homological properties by constructing the *standard complex* (also known as the bar resolution). If we take $\mathbb{Z}G^n \otimes_{\mathbb{Z}} \mathbb{Z}G$ which is a right $\mathbb{Z}G$-module via right multiplication, and freely generated by n-tuples $(g_1, ..., g_n)$, then we can form the complex

$$\ldots \mathbb{Z}G^n \otimes_{\mathbb{Z}} \mathbb{Z}G \xrightarrow{\partial_n} \mathbb{Z}G^{n-1} \otimes_{\mathbb{Z}} \mathbb{Z}G \xrightarrow{\partial_{n-1}} \ldots\ldots \mathbb{Z}G^1 \otimes_{\mathbb{Z}} \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G.$$

The differentials are the $\mathbb{Z}G$-module homomorphism defined by

$$\partial_n(g_1, \ldots, g_n) = (-1)^n(g_2, \ldots, g_n) + \sum_{i=1}^{n-1}(-1)^{n-i}(g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + (g_1, \ldots, g_{n-1})g_n.$$

We can also use this standard complex to study the cohomology of groups. Given a right $\mathbb{Z}G$-module A, we construct the cocomplex $F^\bullet(G, A)$, where the modules of this cocomplex are the abelian groups of $\mathbb{Z}$-module of maps $G^n \to A$, $F^n(G, A) := \mathsf{Fun}(G^n, A)$,

$$A \xrightarrow{\mathrm{d}^0} F^1(G, A) \xrightarrow{\mathrm{d}^1} F^2(G, A) \ldots\ldots F^{n-1}(G, A) \xrightarrow{\mathrm{d}^{n-1}} F^n(G, A) \ldots.$$

To get the differentials, we can use the the isomorphism

$$\mathsf{Hom}_{\mathbb{Z}G}(\mathbb{Z}G^n \otimes_{\mathbb{Z}} \mathbb{Z}G, A) \cong \mathsf{Fun}(G^n, A) = F^n(G, A),$$

to define d^n as composition with ∂_{n+1}, under this isomorphism.

Explicitly, the differential d^n is given on a map $f\colon G^n \to A$ by

$$\mathrm{d}^n f(g_0, \ldots, g_n) = (-1)^{n+1} f(g_1, \ldots, g_n) + \sum_{i=0}^{n-1}(-1)^{n-i} f(g_0, \ldots, g_i g_{i+1}, \ldots, g_n) + f(g_0, \ldots, g_{n-1}) \cdot g_n, \tag{A.1.0.5}$$

where $\cdot$ denotes the action of G on A. We shall call this the *standard cocomplex*. In practice, we often use the G-module $A = \Bbbk^\times$, with trivial G-action. In this case, we use multiplicative notation.

Example A.1.2. Consider the cocomplex $F^\bullet(G, \Bbbk^\times)$. The differential of a a function

$\omega \in F^3(G, \Bbbk^\times)$ is given by

$$\mathrm{d}^n \omega(g_0, g_1, g_2, g_3) = \omega(g_1, g_2, g_3)\omega(g_0g_1, g_2, g_3)^{-1}\omega(g_0, g_1g_2, g_3)\omega(g_0, g_1, g_2g_3)^{-1}\omega(g_0, g_1, g_2), \tag{A.1.0.6}$$

where we have used the multiplicative notation. If we take ω to be a 3-cocycle, that is that its differential is 0, then we recover the 3-cocycle condition given in Equation (2.1.0.4).

A.2 Cocycle Calculations

Here we collect the computations of various cohomological identites used throughougt Chapter 4. These relate to a normalised 3-cocycle ω (defined in Equation (2.1.0.4)), τ and γ (Equations 4.1.0.2 and 4.1.0.3 respectively).

Proposition A.2.1. *The maps $\tau(h,k)(d)$ and $\gamma(h)(d,f)$ are related by the following identity:*

$$\gamma\left(k\right)\left(d,f\right)\gamma\left(h\right)\left(kdk^{-1},kfk^{-1}\right)\tau\left(h,k\right)\left(d\right)\tau\left(h,k\right)\left(f\right) = \tau\left(h,k\right)\left(df\right)\gamma\left(hk\right)\left(d,f\right).$$

Proof. To prove this holds, we shall make extensive use of the 3-cocycle condition, Equation (2.1.0.4). We first use this with the following intputs to get a series of equalities;

- $g_1 = h, g_2 = k, g_3 = d, g_4 = f$,

$$\omega(h,k,d)\omega(k,d,f) = \frac{\omega(hk,d,f)\omega(h,k,df)}{\omega(h,kd,f)}$$

- $g_1 = h, g_2 = kdk^{-1}, g_3 = kfk^{-1}, g_4 = k$,

$$\omega(h,kdk^{-1},kfk^{-1})\omega(kdk^{-1},kfk^{-1},k) = \frac{\omega(hkdk^{-1},kfk^{-1},k)\omega(h,kdk^{-1},kf)}{\omega(h,kdfk^{-1},k)}$$

- $g_1 = hkdk^{-1}h^{-1}, g_2 = h, g_3 = k, g_4 = f$,

$$\omega(hkdk^{-1}h^{-1},h,k)\omega(h,k,f) = \frac{\omega(hkdk^{-1},k,f)\omega(hkdk^{-1}h^{-1},h,kf))}{\omega(hkdk^{-1}h^{-1},hk,f)}$$

- $g_1 = hkdk^{-1}h^{-1}, g_2 = hkfk^{-1}h^{-1}, g_3 = h, g_4 = k$,

$$\omega(hkdk^{-1}h^{-1},hkfk^{-1}h^{-1},h)\omega(hkfk^{-1}h^{-1},h,k) = \frac{\omega(hkdfk^{-1}h^{-1},h,k)\omega(hkdk^{-1}h^{-1},hkfk^{-1}h^{-1},hk)}{\omega(hkdk^{-1}h^{-1},hkfk^{-1},k)}$$

If we then expand out the left hand side of Equation (4.1.0.4) , and substitute in the above equalities, we are left with

$$\frac{\omega(hkdk^{-1},kfk^{-1},k)\omega(hkdk^{-1}h^{-1},h,kf)}{\omega(hkdk^{-1}h^{-1},h,kfk^{-1})\omega(hkdk^{-1}h^{-1},hkfk^{-1},k)\omega(h,kfk^{-1},k)}\times$$
$$\times\frac{\omega(h,kdk^{-1},kf)\omega(hkdk^{-1},k,f)}{\omega(kdk^{-1},k,f)\omega(h,kd,f)\omega(h,kdk^{-1},k)}\tau(h,k)(df)\gamma(hk)(d,f)$$

But the remaining coefficients vanish by using the 3-cocycle equation with inputs;

– $g_1 = hkdk^{-1}h^{-1}, g_2 = h, g_3 = kfk^{-1}, g_4 = k,$

$$\frac{\omega(hkdk^{-1},kfk^{-1},k)\omega(hkdk^{-1}h^{-1},h,kf)}{\omega(hkdk^{-1}h^{-1},h,kfk^{-1})\omega(hkdk^{-1}h^{-1},hkfk^{-1},k)\omega(h,kfk^{-1},k)} = 1 \quad \text{(A.2.0.1)}$$

– $g_1 = h, g_2 = kdk^{-1}, g_3 = k, g_4 = f,$

$$\frac{\omega(h,kdk^{-1},kf)\omega(hkdk^{-1},k,f)}{\omega(kdk^{-1},k,f)\omega(h,kd,f)\omega(h,kdk^{-1},k)} = 1 \quad \text{(A.2.0.2)}$$

So the equality holds. □

Proposition A.2.2. *The map $\gamma(h)(g,g')$ is related to the 3-cocycle $\omega(g,g',g'')$ by the following identity:*

$$\omega^{-1}(hgh^{-1},hg'h^{-1},hg''h^{-1})\gamma(h)(gg',g'')\gamma(h)(g,g') = \gamma(h)(g,g'g'')\gamma(h)(g',g'')\omega^{-1}(g,g',g'').$$

Proof. We use the 3-cocycle condition in Equation (2.1.0.4) with the following inputs;

– $g_1 = h, g_2 = g, g_3 = g', g_4 = g'',$
– $g_1 = hgh^{-1}, g_2 = hg'h^{-1}, g_3 = hg''h^{-1}, g_4 = h,$
– $g_1 = hgh^{-1}, g_2 = h, g_3 = g', g_4 = g'',$
– $g_1 = hgh^{-1}, g_2 = hg'h^{-1}, g_3 = h, g_4 = g'',$ □

Proposition A.2.3. *The maps $tau(g,d)(f)$ and $\gamma(g)(d,f)$ are also related by the following identity:*

$$\gamma(g)(d,f)\tau(gdg^{-1},g)(f) = \gamma(g)(dfd^{-1},d)\tau(g,d)(f),$$

Proof. We use the 3-cocycle condition with the following inputs;

– $g_1 = gdfd^{-1}g^{-1}, g_2 = g, g_3 = dg^{-1}, g_4 = g,$
– $g_1 = g, g_2 = dfd^{-1}, g_3 = dg^{-1}, g_4 = g,$
–$g_1 = g, g_2 = dg^{-1}, g_3 = gfg^{-1}, g_4 = g,$
– $g_1 = g, g_2 = dg^{-1}, g_3 = g, g_4 = f,$ □

Proposition A.2.4. *The map $\tau(h,k)(d)$ satisfies the following identity:*

$$\tau(h,k)(d)\tau(g,hk)(d) = \tau(g,h)(kdk^{-1})\tau(gh,k)(d).$$

Proof. We use the 3-cocycle condition with the following inputs;

- $g_1 = g, g_2 = h, g_3 = k, g_4 = d,$
- $g_1 = g, g_2 = h, g_3 = kdk^{-1}, g_4 = k,$
- $g_1 = g, g_2 = hkdk^{-1}h^{-1}, g_3 = h, g_4 = k,$
- $g_1 = ghkdk^{-1}h^{-1}g^{-1}, g_2 = g, g_3 = h, g_4 = k,$ □

A.3 Cohomology of crossed product of groups

Following [DS17, Appendix A], we describe how the cohomology of a crossed product of groups can be described using a double complex.

Definition A.3.1. Let H, G be groups, along with together with a left action of H on G by group automorphisms, $H \mapsto \mathrm{Aut}(G), h \mapsto (g \mapsto {}^h g)$. Then the *crossed product*, $G \rtimes H$, is the set $G \times H$ with group multiplication given by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1{}^{h_1}g_2, h_1h_2).$$

Let A be a right $\mathbb{Z}G$-module. Then $F^n(G,A) = \mathsf{Fun}(G^n, A)$ becomes a right H-module when equipped with the action

$$(f \cdot h)(g_1, \ldots, g_n) = f({}^h g_1, \ldots, {}^h g_n).$$

This allows us to define a standard cocomplex $F^\bullet(H, F^n(G,A))$ for each $n \in \mathbb{Z}$.

If we define

$$F^{n,m}(H,G,A) = \mathsf{Fun}(H^n, \mathsf{Fun}(G^m, A)) = F^n(H, F^m(G,A)),$$

then we have two families of module homomorphisms

$$\mathrm{d}^{n,m} \colon F^{n,m}(H,G,A) \to F^{n+1,m}(H,G,A), \quad \partial^{n,m} \colon F^{n,m}(H,G,A) \to F^{n,m+1}(H,G,A),$$

where the first is given directly by the standard comomplex $F^\bullet(H, F^n(G,A))$ and the second is induced by the standard cocomplex $F^\bullet(G,A)$ in the sense that

$$(\partial^{n,m}(f))(h_1,\dots,h_n) = \mathrm{d}^n(f(h_1,\dots,h_n)).$$

The differentials commute, $\mathrm{d}^{n,m+1}\,\partial^{n,m} = \partial^{n+1,m}\,\mathrm{d}^{n,m}$, and so we are left with the system

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial^{n,m-1}} & & \downarrow{\scriptstyle \partial^{n+1,m-1}} & & \downarrow{\scriptstyle \partial^{n+2,m-1}} & & \\
\cdots & \xrightarrow{\mathrm{d}^{n-1,m}} & F^{n,m}(H,G,A) & \xrightarrow{\mathrm{d}^{n,m}} & F^{n+1,m}(H,G,A) & \xrightarrow{\mathrm{d}^{n+1,m}} & F^{n+2,m}(H,G,A) & \xrightarrow{\mathrm{d}^{n+2,m}} & \cdots \\
 & & \downarrow{\scriptstyle \partial^{n,m}} & & \downarrow{\scriptstyle \partial^{n+1,m}} & & \downarrow{\scriptstyle \partial^{n+2,m}} & & \\
\cdots & \xrightarrow{\mathrm{d}^{n-1,m+1}} & F^{n,m+1}(H,G,A) & \xrightarrow{\mathrm{d}^{n,m+1}} & F^{n+1,m+1}(H,G,A) & \xrightarrow{\mathrm{d}^{n+1,m+1}} & F^{n+2,m+1}(H,G,A) & \xrightarrow{\mathrm{d}^{n+2,m+1}} & \cdots \\
 & & \downarrow{\scriptstyle \partial^{n,m+1}} & & \downarrow{\scriptstyle \partial^{n+1,m+1}} & & \downarrow{\scriptstyle \partial^{n+2,m+1}} & & \\
\cdots & \xrightarrow{\mathrm{d}^{n-1,m+2}} & F^{n,m+2}(H,G,A) & \xrightarrow{\mathrm{d}^{n,m+2}} & F^{n+1,m+2}(H,G,A) & \xrightarrow{\mathrm{d}^{n+1,m+2}} & F^{n+2,m+2}(H,G,A) & \xrightarrow{\mathrm{d}^{n+2,m+2}} & \cdots \\
 & & \downarrow{\scriptstyle \partial^{n,m+2}} & & \downarrow{\scriptstyle \partial^{n+1,m+2}} & & \downarrow{\scriptstyle \partial^{n+2,m+2}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where all rows and columns are complexes, and all squares commute. This is an example of a double complex, [Wei94].

To view this system as a single complex, one can consider the associated *truncated double complex* $\widetilde{F}^\bullet_{\mathrm{Tot}}(H,G,A)$, with

$$\widetilde{F}^n_{\mathrm{Tot}}(H,G,A) = \bigoplus_{i=0}^{n-1} F^{i,n-i}(H,G,A),$$
$$\mathrm{d}^n_{\mathrm{Tot}}(f) := \mathrm{d}^{i,n-i}(f) + (-1)^i\partial^{i,n-1}(f), \qquad \text{for } f \in F^{i,n-i}(H,G,A) \text{ with } i < n.$$

We can view an element $f \in \widetilde{F}^{i,n-i}(H,G,A) \subseteq F^n_{\mathrm{Tot}}(H,G,A)$ as a function $f\colon H^i \times G^{n-i} \to A$.

This complex allows us to interpret several of the functions appearing in Chapter 4 as cohomological elements.

Example A.3.2. Let the group G act on itself via conjugation, while G acts on $\Bbbk^\times$ trivially, using multiplicative notation. Then we can form the cocomplex $\widetilde{F}^\bullet_{\mathrm{Tot}}(G,G,\Bbbk^\times)$. If we recall the functions $\tau : (G\times G)\times G \to \Bbbk^\times$, $\gamma : G \times (G\times G) \to \Bbbk^\times$ from Equations 4.1.0.2 and 4.1.0.3, we can view the triple

$$T(\omega) = \omega \oplus \gamma \oplus \tau \in F^{0,3} \oplus F^{1,2} \oplus F^{2,1} = \widetilde{F}^3_{\mathrm{Tot}}(G,G,\Bbbk^\times),$$

with $\gamma(h_1, g_1, g_2) := \gamma(h_1)(g_1, g_2)$ and $\tau(h_1, h_2, g_1) := \tau(h_1, h_2)(g_1)$, as a cochain in the truncated double cocomplex. If we calculate the differential of this element, using multiplicative notation and collecting terms in the same spaces, we get that

$$\mathrm{d}^3_{\mathrm{Tot}}(T(\omega)) = \partial^{0,3}(\omega) \oplus \frac{\mathrm{d}^{0,3}(\omega)}{\partial^{1,2}(\gamma)} \oplus \mathrm{d}^{1,2}(\gamma)\partial^{2,1}(\tau) \oplus \mathrm{d}^{2,1}(\tau). \tag{A.3.0.1}$$

Asking that $T(\omega)$ be a 3-cocycle is thus equivalent to the following conditions;

$$\begin{aligned}
\mathrm{d}^{1,2}(\gamma)\partial^{2,1}(\tau) &= 1 = \frac{\gamma(h_2)(g_1, g_2)\,\gamma(h_1)\left(h_2 g_1 h_2^{-1}, h_2 g_2 h_2^{-1}\right)\tau(h_1, h_2)(g_1)\,\tau(h_1, h_2)(g_2)}{\tau(h_1, h_2)(g_1 g_2)\,\gamma(h_1 h_2)(g_1, g_2)},\\
\partial^{0,3}(\omega) &= 1 = \frac{\omega(g_1 g_2, g_3, g_4)\,\omega(g_1, g_2, g_3 g_4)}{\omega(g_1, g_2, g_3)\,\omega(g_1, g_2 g_3, g_4)\,\omega(g_2, g_3, g_4)},\\
\frac{\mathrm{d}^{0,3}(\omega)}{\partial^{1,2}(\gamma)} &= 1 = \frac{\gamma(h)(g_1 g_2, g_3)\gamma(h)(g_1, g_2)\omega(g_1, g_2, g_3)}{\omega(hg_1h^{-1}, hg_2h^{-1}, hg_2h^{-1})\gamma(h)(g_1, g_2g_3)\gamma(h)(g_2, g_3)},\\
\mathrm{d}^{2,1}(\tau) &= 1 = \frac{\tau(h_2, h_3)(g)\tau(h_1, h_2h_3)(g)}{\tau(h_1h_2, h_3)(g)\tau(h_1, h_2)(h_3 g h_3^{-1})}.
\end{aligned}$$

These are, respectively, that ω is a 3-cocycle, as described by Equation (2.1.0.4), and that Propositions A.2.1 A.2.2, and A.2.4 hold.

Example A.3.3. Let $N \lhd H$ be a normal subgroup and let H act on N by conjugation. The 2-boundaries in the complex $\widetilde{F}^{\bullet}_{\mathrm{Tot}}(H, N, \Bbbk^\times)$ can be parameterised by pairs

$$\epsilon \oplus \kappa \in F^{1,1} \oplus F^{0,2} = F^2_{\mathrm{Tot}}(H, N, \Bbbk^\times).$$

The differential of this element is given by

$$\mathrm{d}^2_{\mathrm{Tot}}(\epsilon \oplus \kappa) = \mathrm{d}^{1,1}(\epsilon) \oplus \frac{\mathrm{d}^{0,2}(\kappa)}{\partial^{1,1}(\epsilon)} \oplus \partial^{0,2}(\kappa).$$

Explicitly, the formulas for each component are given by

$$\begin{aligned}
\mathrm{d}^{1,1}\,\epsilon(h_1, h_2, n_1) &= \frac{\epsilon(h_1, h_2 n_1 h_2^{-1})\epsilon(h_2, n_1)}{\epsilon(h_1 h_2, n_1)},\\
\partial^{1,1}\epsilon(h_1, n_1, n_2) &= \frac{\epsilon(h_1, n_1)\epsilon(h_1, n_2)}{\epsilon(h_1, n_1 n_2)}, \qquad \mathrm{d}^{0,2}\,\kappa(h_1, n_1, n_2) = \frac{\kappa(h_1 n_1 h_1^{-1}, h_1 n_2 h_1^{-1})}{\kappa(n_1, n_2)},\\
\partial^{0,2}\kappa(n_1, n_2, n_3) &= \frac{\kappa(n_1, n_1)\kappa(n_1 n_2, n_3)}{\kappa(n_1, n_2 n_3)\kappa(n_2, n_3)}.
\end{aligned}$$

Appendix B

Group actions

Here we collect some standard definitions and well-known results on group actions, following [Cam99], that are used in Chapter 5.

Definition B.0.1. Let G be a group and S a set.

A *G-action on S* is a binary operation $*\colon G \times S \to S$ such that, for all $s \in S$,

$$e * s = s, \text{ and } (g \cdot h) * s = g * (h * s),$$

where $e \in G$ is the group identity element.We shall call a set endowed with such an action a *G-set.*

Example B.0.2. (Set of left cosets) Let G be a group, and H some subgroup. We can form the set of left cosets $G/H = \{gH | g \in G\}$, which has a G-action defined by $k * gH = kgH$, where $k, g \in G$.

So in this way, the action of a group can be thought of as a particular choice of the symmetries of a set.

Definition B.0.3. Let S, T be two sets with G-actions $*_S$ and $*_T$ respectively. They are *isomorphic as G-sets* if there exists a bijection $f : S \to T$ such that

$$g *_T f(s) = f(g *_S s) \quad \text{for all } s \in S.$$

For a given element s in a G-set S, we can define a distinguished subset of S; The *orbit of G through S* is the subset defined by

$$\mathrm{Orb}(s) = \{g * s | g \in G\},$$

and contains all elements that can be reached from s using the group action. If there exists an $s \in S$ such that $\mathrm{Orb}(s) = S$, then we say that the G-action is *transitive*.

If we have two elements $s, t \in S$, then the subsets $\mathrm{Orb}(s)$ and $\mathrm{Orb}(t)$ are either equal or disjoint. Moreover, we can form an equivalence relation by setting $s \sim t$ if and only if $\mathrm{Orb}(s) = \mathrm{Orb}(t)$. This allows us to partition the set S into a collection of transitive G-sets,

$$S = \bigsqcup_{s \in S/\sim} \mathrm{Orb}(s). \tag{B.0.0.1}$$

So to understand the structure of a G-set, we can focus on the G-action orbits.

To do this, for a given $s \in S$, we can look at the the *stabiliser of* s, which is defined as

$$\mathrm{Stab}(s) = \{g \in G | g * s = s\}.$$

This is a subgroup of G, and contains the elements of g that act trivially on s. This subgroup allows us to relate the G-sets of $\mathrm{Orb}(s)$ and the left coset G-set $G/\mathrm{Stab(s)}$ from Example B.0.2.

Proposition B.0.4. *Let S be a G-set, and take $s \in S$. There is an isomorphism of G-sets between $Orb(s)$ and the G-set of left cosets $G/Stab(s)$.*

Proof. For $x \in \mathrm{Orb}(s)$, there is some element $h \in G$ such that $h * s = x$. The map $f(x) = \{g \in G | g * s = x = h * s\} = h^{-1}\mathrm{Stab}(s)$ is a bijection between Orb(s) and $G/\mathrm{Stab}(s)$. This is an isomorphism of G-sets as

$$k * f(x) = k * h^{-1}\mathrm{Stab}(s) = kh^{-1}\mathrm{Stab}(s) = f(kh^{-1} * s) = f(k * x).$$

□

If we combine this result with Equation (B.0.0.1), then we can study any G-set S using the left-coset G-sets $G/\mathrm{Stab}(s)$. As $\mathrm{Stab}(s)$ is simply some subgroup of G, we have a complete understanding of G-sets by consider the left-coset G-set from Example B.0.2. This is understanding is completed using the following result.

Proposition B.0.5. *Two left coset G-sets G/H,G/K are isomorphic as G-sets if and only if H and K are conjugate subgroups of G.*

Proof. First, suppose that $K = g^{-1}Hg$ for some $g \in G$. Then the map $f : G/H \to G/K$ defined by $f(kH) = kgK$ is an isomorphism of G-sets, with inverse map given by $f^{-1}(lK) = lg^{-1}H$.

Conversely, suppose we have an isomorphism of G-sets $f : G/H \to G/K$. In particular, this gives $f(H) = gK$ for some $g \in G$. Then, for some $h \in H$, we have that

$$gK = f(H) = f(hH) = h * f(H) = hgK. \tag{B.0.0.2}$$

Thus we have that $g^{-1}Hg \subseteq K$.

If we now consider some $k \in K$, we have that

$$f(g^{-1}H) = K = kK = k * f(g^{-1}H) = f(kg^{-1}H). \tag{B.0.0.3}$$

As f is a bijection, this gives that $g^{-1}KG \subseteq H$, and so H and K are conjugate subgroups. $\square$

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