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Ordered random walks with heavy tails*

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Abstract

This paper continues our previous work [4] where we have constructed a k-dimensional random walk conditioned to stay in the Weyl chamber of type A. The construction was done under the assumption that the original random walk has $k-1$ moments. In this note we continue the study of killed random walks in the Weyl chamber, and assume that the tail of increments is regularly varying of index $\alpha < k-1$. It appears that the asymptotic behaviour of random walks is different in this case. We determine the asymptotic behaviour of the exit time, and, using this information, construct a conditioned process which lives on a partial compactification of the Weyl chamber.

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1 Main results and discussion

1.1 Introduction

This note is a continuation of our paper [4]. In [4] we constructed a k-dimensional random walk conditioned to stay in the Weyl chamber of type A. The conditional version of the random walk was defined via Doob's h-transform. The form of the corresponding harmonic function has been suggested by Eichelsbacher and König [6]. This construction was performed under the optimal moment conditions and required the existence of $k-1$ moments of the random walk.

The main aim of the present work is to consider the case when that moment condition is not fulfilled. Instead of the existence of $(k-1)$-th moment of the increment, we shall assume that the tail function is regularly varying of index $2 < \alpha < k-1$. This assumption significantly changes the behaviour of the random walk. It turns out that the asymptotic behaviour of the exit time from the Weyl chamber depends not only on

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the number of walks but also on the index α. The typical sample path behaviour for the occurrence of large exit times is different as well. The main reason for that is that the large exit times are caused by one (or several) big jumps of the random walk.

We now introduce some notation. Let $S = (S_1, S_2, \dots, S_k)$ be a k-dimensional random walk with

$$S_j(n) = \sum_{i=1}^{n} X_j(i),$$

where $\{X_j(i)\}_{i,j\ge 1}$ are independent copies of a random variable X. Let W denote the Weyl chamber of type A, i.e.,

$$W = \left\{x \in \mathbb{R}^k : x_1 < x_2 < \dots < x_k\right\}.$$

Let τ_x denote the first exit time of random walk with starting point $x \in W$, that is,

$$\tau_x := \min\{n \ge 1 : x + S(n) \notin W\}.$$

The main purpose of the present paper is to study the asymptotic behaviour of $\mathbf{P}(\tau_x > n)$ and to construct a model for ordered random walks. Recall that in order to define a random walk conditioned to stay in W, one should find a Doob h-transform

$$\mathbf{E}[h(x + S(1)), \tau_x > 1] = h(x) > 0, x \in W.$$

We say that a function which satisfies the latter condition is harmonic. However, it seems that it is not possible to find a harmonic function for the Doob h-transform under present conditions. Therefore, we use a partial compactification of W, which is based on the sample path behaviour of the random walk S on the event $\{\tau_x > n\}$. (Recall that a more formal way consists in applying an h-transform with a harmonic function.) Finally, we prove a functional limit theorem for random walks conditioned to stay in the Weyl chamber up to big, but finite, time.

To simplify our proofs we shall restrict our attention to the case $\alpha \in (k-2, k-1)$. However, it will be clear from the proof, that our method works also for smaller values of α.

1.2 Tail distribution of τ_x

We shall assume that $\mathbf{E}X = 0$. This assumption does not restrict the generality, since τ_x depends only on differences of coordinates of the random walk S. We consider a situation when increments have $k-2$ finite moments, i.e., $\mathbf{E}|X|^{k-2} < \infty$. Under this condition, for $(S_1, S_2, \dots, S_{k-1})$ we can construct a harmonic function by using results of [4]. Denote this function by $V^{(k-1)}$. We introduce also the following functions:

$$v_1(x) = V^{(k-1)}(x_1, x_2, \dots, x_{k-1}) \text{ and } v_2(x) = V^{(k-1)}(x_2, x_3, \dots, x_k), \quad x \in W.$$

It is easy to see that these functions are superharmonic for our original k-dimensional random walk, i.e.,

$$\mathbf{E}\left[v_i(x + S(1)), \tau_x > 1\right] \le v_i(x), \quad i = 1, 2$$

and every inequality is strict at least for one $x \in W$. Thus, the function

$$v(x) := pv_1(x) + qv_2(x)$$

is also superharmonic for all non-negative $p, q \ge 0$ with $p + q = 1$.

To state our first result we introduce a convolution of v with the Green function of random walk in the Weyl chamber:

$$U(x) := \sum_{l=0}^{\infty} \mathbf{E}\left[v(x + S(l)), \tau_x > l\right], \quad x \in W.$$

Theorem 1.1. *Assume that*

$$\mathbf{P}(X > u) \sim \frac{p}{u^\alpha} \text{ and } \mathbf{P}(X < -u) \sim \frac{q}{u^\alpha}, \text{ as } u \to \infty, \tag{1.1}$$

for some $\alpha \in (k-2, k-1)$ and some $k \geq 4$. Then $U(x)$ is a strictly positive superharmonic function, i.e., $\mathbf{E}\left[U(x + S(1)), \tau_x > 1\right] < U(x)$ for all $x \in W$. Moreover,

$$\mathbf{P}(\tau_x > n) \sim \theta U(x) n^{-\alpha/2-(k-1)(k-2)/4} \quad \text{as } n \to \infty, \tag{1.2}$$

where θ is an absolute constant.

There is a very simple strategy behind formula (1.2). For the event $\{\tau_x > n\}$ to occur either the random walk on the top or the random walk on the bottom should jump away, i.e., $X_k(l) \approx \sqrt{n}$ or $X_1(l) \approx -\sqrt{n}$ for some $l \geq 1$. After such a big jump we have a system of $k-1$ random walks with bounded distances between each other and one random walk on the characteristic distance $\sqrt{n}$. This implies that the probability that all k random walks stay in W up to time n is of the same order as the probability that $k-1$ random walk stay in W up to time $n-l$. But it follows from (1.1) that $\mathbf{E}[|X|^{k-2}] < \infty$. So we can apply Theorem 1 from [4], which says that the latter probability is of order $n^{-(k-1)(k-2)/4}$. Since $\mathbf{P}(|X| > \sqrt{n}) \sim n^{-\alpha/2}$, we see that $\mathbf{P}(\tau_x > n)$ is of order $n^{-\alpha/2-(k-1)(k-2)/4}$. This strategy sheds also some light on the structure of the function $U(x)$: the l-th summand in the series corresponds to the case when big jump occurs at time $l+1$.

1.3 Construction of a conditioned random walk

Since U is not harmonic, we can not use the Doob h-transform with this function to define a random walk, conditioned to stay in W for all times. (More precisely, an h-transform with a superharmonic function leads to strict substochastic transition kernel.) An alternative approach via distributional limit does not work as well: using Theorem 1.1 we can define $\hat{P}(x, A)$ for any $x \in W$ and for any bounded $A \subset W$ by the relation

$$\begin{aligned}
\hat{P}(x, A) &= \lim_{n\to\infty} \mathbf{P}(x + S(1) \in A | \tau_x > n) \\
&= \lim_{n\to\infty} \int_A \mathbf{P}(x + S(1) \in dy, \tau_x > 1) \frac{\mathbf{P}(\tau_y > n-1)}{\mathbf{P}(\tau_x > n)} \\
&= \int_A \mathbf{P}(x + S(1) \in dy, \tau_x > 1) \frac{U(y)}{U(x)} \\
&= \frac{\mathbf{E}\left[U(x + S(1)), \tau_x > 1, x + S(1) \in A\right]}{U(x)}.
\end{aligned}$$

Then we can extend $\hat{P}(x, \cdot)$ to a finite measure on the Borel subsets of W. But this measure is not probabilistic, since

$$\hat{P}(x, W) = \frac{\mathbf{E}\left[U(x + S(1)), \tau_x > 1\right]}{U(x)} = \frac{U(x) - v(x)}{U(x)} < 1.$$

We lose the mass because of an “infinite” jump in the first step. Indeed, according to the optimal strategy in Theorem 1.1, one of the random walks should have a jump of order $n^{1/2}$, and we let n go to infinity. This infinite jump is the reason, why a Markov chain, corresponding to the kernel $\hat{P}(x, A)$ has almost sure finite lifetime. Similar effects have been observed already in other models. Bertoin and Doney [1] have proven that a one-dimensional random walk with negative drift and regularly varying tail conditioned to stay positive has finite lifetime. Jacka and Warren [10] have shown that the same effect appears in the Kolmogorov K2 chain.

Having in mind this picture with "infinite" jumps, we can construct a conditioned random walk, which lives on the following set

$$\hat{W} := W \cup W_1 \cup W_2,$$

where

$$W_1 = \{(x_1, x_2, \ldots, x_{k-1}, \infty),\, x_1 < x_2 < \ldots < x_{k-1}\}$$
$$W_2 = \{(-\infty, x_2, x_3, \ldots, x_k),\, x_2 < x_3 < \ldots < x_k\}.$$

We define the transition probability by the following relations:

1. If $x \in W$ and $A \subset W$, then
$$\hat{P}(x, A) = \frac{\mathbf{E}\left[U(x + S(1)), \tau_x > 1, x + S(1) \in A\right]}{U(x)}.$$
2. If $x \in W$ and $A = A' \times \{\infty\} \subset W_1$, then
$$\hat{P}(x, A) = \frac{p\mathbf{E}\left[v_1(x + S(1)), \tau_x^{(1)} > 1, x + S^{(1)}(1) \in A'\right]}{U(x)}.$$
3. If $x \in W$ and $A = \{-\infty\} \times A' \subset W_2$, then
$$\hat{P}(x, A) = \frac{q\mathbf{E}\left[v_2(x + S(1)), \tau_x^{(2)} > 1, x + S^{(2)}(1) \in A'\right]}{U(x)}.$$
4. If $x \in W_1$ and $A = A' \times \{\infty\} \subset W_1$, then
$$\hat{P}(x, A) = \frac{\mathbf{E}\left[v_1(x + S(1)), \tau_x^{(1)} > 1, x + S^{(1)}(1) \in A'\right]}{v_1(x)}.$$
5. If $x \in W_2$ and $A = \{-\infty\} \times A' \subset W_2$, then
$$\hat{P}(x, A) = \frac{\mathbf{E}\left[v_2(x + S(1)), \tau_x^{(2)} > 1, x + S^{(2)}(1) \in A'\right]}{v_2(x)}.$$

Here

$$\tau_x^{(i)} := \min\{n \ge 1 : \, x^{(i)} + S^{(i)} \notin W^{(i)}\}, \; i = 1, 2,$$
$$x^{(1)} := (x_1, x_2, \ldots, x_{k-1}), \quad x^{(2)} := (x_2, x_3, \ldots, x_k),$$
$$S^{(1)} := (S_1, S_2, \ldots, S_{k-1}), \quad S^{(2)} := (S_2, S_3, \ldots, S_k)$$

and $W^{(1)} = W^{(2)} = \{x \in \mathbb{R}^{k-1} : x_1 < x_2 < \ldots < x_{k-1}\}$.

The asymptotic behaviour of the corresponding Markov chain, say $\{\hat{S}(n), n \ge 0\}$, can be described as follows. One of the random walks jumps away at time m with probability $\mathbf{E}\left[v(x + S(m-1)), \tau_x > m - 1\right]/U(x)$. Then we restart our process, which has from now on one frozen coordinate, either $-\infty$ or ∞, and $k - 1$ ordered random walks. But for $k - 1$ random walks we can apply Theorem 3 of [4]. As a result we have that the limit of $\left\{\hat{S}([nt])/\sqrt{n}, t \in [r_n/n, 1]\right\}$ converges weakly to a process $\{X(t), t \in (0, 1]\}$, where r_n is such that $r_n \to \infty$. (We need this additional restriction because of jumps at bounded times.) The limit can be constructed as follows: Let $D(t)$ denote here the $(k-1)$-dimensional Dyson Brownian motion starting from zero. With some probability $p(x)$ we add to $D(t)$ one coordinate with constant value ∞, and with probability $q(x) = 1 - p(x)$ we add the coordinate with value $-\infty$.

We have constructed a model of ordered random walks on an enlarged state space by formalising an intuitive picture of big jumps. But it remains unclear whether one can find a harmonic function for the substochastic kernel $\mathbf{P}(x + S(1) \in dy, \tau_x > 1)$. If such a function exists, then one can construct a model of ordered random walks on the original Weyl chamber. We conjecture that there are no harmonic functions for ordered random walks with heavy tails. The examples from [1, 10], which we have mentioned above, support this conjecture.

The most standard way to describe the set of harmonic functions consists in the study of the corresponding Martin boundary. We found only a few results on Martin boundary for killed random walks. Doney [5] found sufficient and necessary conditions for existence of harmonic functions in one-dimensional case. The proof relies on the Wiener-Hopf factorisation, which seems to work in the one-dimensional case only. In a series of papers [7, 8, 9] by Ignatiouk-Robert, and by Ignatiouk-Robert and Loree Martin boundaries for killed random walks with non-zero drift in a half-space and in a quadrant have been studied. In all these papers the Cramer condition has been imposed. Next-neighbour random walks with zero mean in the Weyl chamber have been studied by Raschel [13, 14]. In our situation all the increments are heavy-tailed. This means that one needs another method for finding the Martin boundary.

1.4 Conditional limit theorem for S

In this paragraph we turn our attention to the behaviour of $\{S([nt])/\sqrt{n}, t \le 1\}$ conditioned on $\{\tau_x > n\}$. Since one of the random walks should have a jump of order $\sqrt{n}$ on the event $\{\tau_x > n\}$, this conditioning will not lead to an infinite jump, as it happens in the case of conditioning on $\{\tau_x = \infty\}$.

We define

$$X^{(n)}(t) := \frac{x + S([nt] \wedge r_n)}{\sqrt{n}}, \quad t \in [0, 1].$$

Here $r_n \to \infty$ and $r_n = o(n)$. (Again, we need to go away from zero, because of a big jump occurring at the very beginning.) In order to state our limit theorem we have to introduce a limiting process, say X. Denote

$$p(x) := \frac{p \sum_{l=0}^{\infty} \mathbf{E}\left[v_1(x + S(l)), \tau_x > l\right]}{U(x)}, \ q(x) := \frac{q \sum_{l=0}^{\infty} \mathbf{E}\left[v_2(x + S(l)), \tau_x > l\right]}{U(x)}$$

and

$$\psi(r) := \lim_{a \to 0} \frac{\mathbf{P}\left(B_1(t) < a + B_2(t) < \ldots < (k-2)a + B_{k-1}(t) < r + B_k(t), \ t \le 1\right)}{\mathbf{P}\left(B_1(t) < a + B_2(t) < \ldots < (k-2)a + B_{k-1}(t), \ t \le 1\right)}.$$

$B(t)$ denote here a k-dimensional Brownian motion. The distribution of the starting point, $X(0)$, is given by

$$\mu_x(dy) = q(x) f(-y_1) dy_1 \prod_{i=2}^{k} \delta_0(dy_i) + p(x) f(y_k) dy_k \prod_{i=1}^{k-1} \delta_0(dy_i),$$

where $f(x) = \theta^{-1} \psi(x) x^{-\alpha-1} 1_{\mathbb{R}_+}(x)$.

Further, given $X(0) = y$, we define

$$\mathcal{L}\left(X\right) = \lim_{a \to 0} \mathcal{L}\left(y(a) + B(t), t \in [0, 1] \middle| \tau_{y(a)}^{bm} > 1\right),$$

where $y(a) = y + a(0, 1, 2, \ldots, k-1)$ and $\tau_y^{bm} = \min\{t : y + B(t) \notin W\}$.

Theorem 1.2. *Under the conditions of Theorem 1.1,*

$$\{X^{(n)}|\tau_x > n\} \Rightarrow X$$

in the Skorohod topology on $C[0,1]$.

It is worth mentioning that the limiting process is not invariant with respect to the starting position of the random walk. More precisely, the distribution of $X(0)$ depends on x through $p(x)$ and $q(x)$. Clearly this happens because of one large jump i the beginning. An analogous result can be proven also for random walks with $\mathbf{E}|X|^{k-1} < \infty$, but the limiting process will start always at zero.

1.5 Some remarks on the general case.

Although the informal picture behind Theorems 1.1 and 1.2 is quite simple, the proofs are very technical. In the case of smaller values of α, i.e. $\alpha < k-2$, one has to overcome even more technical difficulties, which are of the combinatorial nature. However, it is clear that our approach works in the case $\alpha < k-2$ as well. In this paragraph we describe the behaviour of ordered random walks for such values of α.

First, in order to stay in W at least up to time n, the random walk S should have $k_\alpha := k - [\alpha+1]$ big jumps. Then it may happen that at least two jumps go in the same direction (upwards or downwards). The values of all these jumps should be ordered. As a result one gets the following relation:

$$\mathbf{P}(\tau_x > n) \sim \ U(x) n^{-\alpha k_\alpha/2} n^{-(k-k_\alpha)(k-k_\alpha-1)/4}$$

with some superharmonic function U. Second, to construct ordered random walks we need to add all vectors with k_α infinite coordinates. Finally, in Theorem 1.2 one has to change the distribution of X_0 only: The limiting process will start from a random point with k_α non-zero coordinates.

Unfortunately, the case of integer values of α remains unsolved. If, for example, $\alpha = k-1$, then, the jumps of order $\sqrt{n}$ do not contribute to $\mathbf{P}(\tau_x > n)$. Therefore, we can not use the method proposed in the present work.

We next turn to the case $\alpha < 2$. We believe that $k-2$ walks should jump away. But in this situation the typical size of a 'big' jump is $n^{1/\alpha}$. Thus, the probability of occurrence of $k-2$ such jumps is $(n^{1/\alpha})^{-\alpha(k-2)} = n^{2-k}$.

Noting further that, without any assumptions on the distribution of X, two random walks do not change their order with probability $n^{-1/2}$. As a result we have the following conjecture: If (1.1) holds with some $\alpha \in (0,2)$, then there exists a positive function $U(x)$ such that

$$\mathbf{P}(\tau_x > n) \sim U(x) n^{-k+3/2}.$$

Our next remark concerns other Weyl chambers. König and Schmid [11] have shown that the approach proposed in [4] works also in Weyl chambers of types C and D. It is easy to see that, using the method from the present paper, one can prove analogues of Theorems 1.1 and 1.2 for chambers of types C and D. Moreover, since big negative jumps lead to exit from these two regions, the corresponding optimal strategies are even simpler then in the chamber of type A.

We conclude this subsection by commenting on our assumption (1.1). First, it is clear that we could assume that the tail of $|X|$ is regularly varying with index α. Second, it is possible to consider slightly more general heavy-tailed distributions in order to derive asymptotics for $\mathbf{P}(\tau_x > n)$. The minimal assumptions we need are

(a) $\sup_{u>0} \mathbf{P}(X > cu)/\mathbf{P}(X > u) < \infty$ for any fixed $c \in (0,1)$,

(b) $\mathbf{P}(X > u - \sqrt{u}) \sim \mathbf{P}(X > u)$ as $u \to \infty$.

Without assumption (b) the asymptotics will be different. This happens due to the fact that without (b) the resulting asymptotics will be influenced by the fluctuations corresponding to the Central Limit Theorem, see [2] for one-dimensional case. Assumption (a) is essential as well. The reasons for that are the following: in the heavy-tailed setting the asymptotics of the exit time is closely connected with large deviations probabilities $\mathbf{P}(S_n > x)$ of heavy-tailed random walks. In one-dimensional case this was demonstrated in [2] using the Wiener-Hopf factorization. In turn, to find asymptotics for the large deviations probabilities in the heavy-tailed setting one should make some assumptions about the tail of the distribution. If $\mathbf{E}\xi^t = \infty$ for some t, then one usually assumes (a) or regular variation, see for example [3]. Without any assumptions asymptotics might simply be different.

However (a) and (b) are not a big generalization as for the regular varying distributions

$$\mathbf{P}(X > cu)/\mathbf{P}(X > u) \to c^{-\alpha} \text{ as } u \to \infty.$$

2 Finiteness of the superharmonic function

Proposition 2.1. *Under the assumptions of Theorem 1.1,*

$$U(x) = \sum_{l=0}^{\infty} \mathbf{E}\left[v(x + S(l)), \tau_x > l\right] < \infty.$$

We first introduce some notation. For every $\varepsilon > 0$ denote

$$W_{n,\varepsilon} := \left\{x \in \mathbb{R}^k : |x_j - x_i| > n^{1/2-\varepsilon} \text{ for all } 1 \le i < j \le k\right\}$$

and let

$$\nu(n) := \min\{j \ge 1 : x + S(j) \in W_{n,\varepsilon}\}$$

be the first time the random walk enters this region.

Proof. Fix $\delta > 0$. Let $\eta^{\pm}$ be the times of first 'big' jumps upwards and downwards, i.e.,

$$\eta^+ = \min\left\{l \ge 1 : X_k(l) > n^{(1-\delta)/2}\right\} \text{ and } \eta^- = \min\left\{l \ge 1 : X_1(l) < -n^{(1-\delta)/2}\right\}.$$

Let $\eta = \min\{\eta^+, \eta^-\}$ be the first big jump.

First we note that

$$\begin{aligned}\mathbf{E}\left[v(x + S(n)), \tau_x > n\right] = \mathbf{E}&\left[v(x + S(n)), \tau_x > n, \nu(n) \le n^{1-\varepsilon}\right] \\ &+ \mathbf{E}\left[v(x + S(n)), \tau_x > n, \nu(n) > n^{1-\varepsilon}\right]. \end{aligned} \tag{2.1}$$

To estimate the second term we apply Proposition 4 of [4] to obtain

$$c_* \Delta_1^{(i)}(x) \le v_i(x) \le c^* \Delta_1^{(i)}(x), \quad i = 1, 2,$$

where

$$\Delta_1^{(1)}(x) := \prod_{1 \le i < j \le k-1} (1 + |x_j - x_i|) \quad \text{and} \quad \Delta_1^{(2)}(x) := \prod_{2 \le i < j \le k} (1 + |x_j - x_i|).$$

Then, according to Lemma 8 in [4],

$$\begin{aligned}
&\mathbf{E}\left[v(x+S(n)),\tau_x>n,\nu(n)>n^{1-\varepsilon}\right]\\
&\le p\mathbf{E}\left[v_1(x+S(n)),\tau_x^{(1)}>n,\nu(n)>n^{1-\varepsilon}\right]\\
&\qquad+q\mathbf{E}\left[v_2(x+S(n)),\tau_x^{(2)}>n,\nu(n)>n^{1-\varepsilon}\right]\\
&\le C\left(\Delta_1^{(1)}(x)+\Delta_1^{(2)}(x)\right)\exp\{-Cn^{\varepsilon}\}\\
&\le Cv(x)\exp\{-Cn^{\varepsilon}\}. \qquad (2.2)
\end{aligned}$$

This gives us an estimate for the second term of (2.1).

The rest of the proof is devoted to estimation of the first summand in (2.1). We split this term in three parts: with big jump upwards, big jump downwards and no big jumps,

$$\begin{aligned}
\mathbf{E}&\left[v(x+S(n)),\tau_x>n,\nu(n)\le n^{1-\varepsilon}\right]\\
&\le \mathbf{E}\left[v(x+S(n)),\tau_x>n,\eta^+\le\nu(n)\le n^{1-\varepsilon}\right]\\
&\quad+\mathbf{E}\left[v(x+S(n)),\tau_x>n,\eta^-\le\nu(n)\le n^{1-\varepsilon},\eta^+>\nu(n)\right]\\
&\quad+\mathbf{E}\left[v(x+S(n)),\tau_x>n,\nu(n)\le n^{1-\varepsilon},\eta>\nu(n)\right]\\
&=: E_{up}+E_{down}+E_{no}.
\end{aligned}$$

We construct estimates for each of terms separately and then combine them. We apply the resulting estimate recursively several times and prove the claim.

Big jump upwards: Using the Markov property, we get

$$\begin{aligned}
E_{up}=\sum_{l=1}^{n^{1-\varepsilon}}\int_W &\mathbf{P}(x+S(l)\in dy,\tau_x>l,\eta^+=l,\nu(n)\ge l)\\
&\times\mathbf{E}[v_1(y+S(n-l)),\tau_y>n-l,\nu(n)\le n^{1-\varepsilon}-l].
\end{aligned}$$

Since v_1 is harmonic for the system of $k-1$ random walks,

$$\mathbf{E}[v_1(y+S(n)),\tau_y>n]\le\mathbf{E}[v_1(y+S(n)),\tau_y^{(1)}>n]=v_1(y)$$

for all $n\ge 1$. Therefore,

$$\mathbf{E}[v_1(y+S(n-l)),\tau_y>n-l]\le v_1(y)$$

and, consequently,

$$E_{up}\le\sum_{l=1}^{n^{1-\varepsilon}}\mathbf{E}\left[v_1(x+S(l)),\tau_x>l,\eta^+=l\right].$$

Using the Markov property once again, we have

$$\begin{aligned}
&\mathbf{E}\left[v_1(x+S(l)),\tau_x>l,\eta^+=l\right]\\
&=\int_W\mathbf{P}(x+S(l-1)\in dy,\tau_x>l-1,\eta^+>l-1)\\
&\qquad\times\mathbf{E}\left[v_1(y+X(1)),\tau_y>1,X_k(1)>n^{(1-\delta)/2}\right].
\end{aligned}$$

The random variable X_k is independent of $X_1,\ldots,X_{k-1}$,

$$\begin{aligned}
&\mathbf{E}\left[v_1(y+X(1)),\tau_y>1,X_k(1)>n^{(1-\delta)/2}\right]\\
&\le\mathbf{E}\left[v_1(y+X(1)),\tau_y^{(1)}>1,X_k(1)>n^{(1-\delta)/2}\right]\\
&=\mathbf{E}\left[v_1(y+X(1)),\tau_y^{(1)}>1\right]\mathbf{P}\left(X_k(1)>n^{(1-\delta)/2}\right).
\end{aligned}$$

Hence,

$$\mathbf{E}\left[v_1(x+S(l)),\tau_x>l,\eta^+=l\right]\le pn^{-\alpha(1-\delta)/2}\mathbf{E}\left[v_1(x+S(l-1)),\tau_x>l-1\right].$$

Summing up over $l\le n^{1-\varepsilon}$, we obtain

$$E_{up}\le pn^{-\alpha(1-\delta)/2}\sum_{l=1}^{n^{1-\varepsilon}}\mathbf{E}\left[v_1(x+S(l-1)),\tau_x>l-1\right]. \tag{2.3}$$

Big jump downwards: We now turn our attention to the case when all jumps of the random walk on the top are bounded by $n^{(1-\delta)/2}$. First of all we note that according to one of Fuk-Nagaev inequalities, see Corollary 1.11 in [12],

$$\mathbf{P}\left(\max_{j\le n^{1-\varepsilon}}\left[S_k(j)1\{\eta^+>j\}\right]>n^{1/2-r(\delta)}\right)\le\exp\{-Cn^{\delta^2/2}\}, \tag{2.4}$$

where $r(\delta)=\delta/2-\delta^2/2$. This yields

$$\begin{aligned}
&\mathbf{E}\left[v_1(x+S(n)),\tau_x>n,\max_{j\le\nu(n)}S_k(j)>n^{1/2-r(\delta)},\nu(n)\le n^{1-\varepsilon},\eta^+>\nu(n)\right]\\
&\quad\le\mathbf{E}\left[v_1(x+S(n)),\tau_x^{(1)}>n,\max_{j\le n^{1-\varepsilon}}\left[S_k(j)1\{\eta^+>j\}\right]>n^{1/2-r(\delta)}\right]\\
&\quad=\mathbf{E}\left[v_1(x+S(n)),\tau_x^{(1)}>n\right]\mathbf{P}\left(\max_{j\le n^{1-\varepsilon}}\left[S_k(j)1\{\eta^+>j\}\right]>n^{1/2-r(\delta)}\right)\\
&\quad\le v_1(x)\exp\{-Cn^{\delta^2/2}\}.
\end{aligned} \tag{2.5}$$

Next we need to analyse the case when the top random walk is always less than $n^{1/2}$. Hence,

$$\begin{aligned}
&E_{down}\le v_1(x)\exp\{-Cn^{\delta^2/2}\}+\\
&\sum_{l=1}^{n^{1-\varepsilon}}\mathbf{E}\left[v_1(x+S(n)),\tau_x>n,\eta^-=l\le\nu(n)\le n^{1-\varepsilon},\max_{j\le\nu(n)}S_k(j)<n^{1/2},\eta^+>\nu(n)\right]\\
&=v_1(x)\exp\{-Cn^{\delta^2/2}\}+\sum_l E_{down,l}.
\end{aligned}$$

Clearly in the definition of $E_{down,l}$ the big jump occurs at time l. Also note that we have excluded the possibility that the top random walk goes up without a big jump. Applying the Markov property again,

$$\begin{aligned}
E_{down,l}&=\int_W\mathbf{P}\left(x+S(l)\in dy,\tau_x>l,\eta^-=l\right)\\
&\quad\times\mathbf{E}\left[v_1(y+S(n-l)),\tau_y>n-l,\nu(n)<n^{1-\varepsilon}-l,\max_{j\le\nu(n)}S_k(j)<n^{1/2}\right]\\
&=:\int_W\mathbf{P}\left(x+S(l)\in dy,\tau_x>l,\eta^-=l\right)E_{after,l}(y).
\end{aligned}$$

Using the Markov property for the multiplier,

$$\begin{aligned}
E_{after,l}(y)=\sum_{r=1}^{n^{1-\varepsilon}-l}\int_{W_{n,\varepsilon}}&\mathbf{P}\left(y+S(r)\in dz,\tau_y>r=\nu(n),\max_{j\le r}S_k(j)<n^{1/2}\right)\\
&\times\mathbf{E}\left[v_1(z+S(n-l-r)),\tau_z>n-l-r\right].
\end{aligned}$$

It follows from the martingale property of v_1 that

$$\begin{aligned}&\mathbf{E}\left[v_1(z+S(n-l-r)),\tau_z>n-l-r\right]\\&\qquad\le\mathbf{E}\left[v_1(z+S(n-l-r)),\tau_z^{(1)}>n-l-r\right]=v_1(z).\end{aligned}$$

Consequently,

$$E_{after,l}(y)\le\mathbf{E}\left[v_1(y+S(\nu(n))),\tau_y>\nu(n),\max_{j\le\nu(n)}S_k(j)<n^{1/2}\right]. \qquad (2.6)$$

It follows from Proposition 4 of [4] that, uniformly in $z\in W_{n,\varepsilon}$,

$$\begin{aligned}v_1(z)&\le C\prod_{1\le i<j\le k-1}(z_j-z_i)\\&\le C\frac{(z_k-z_1)^{k-2}}{\prod_{2\le l\le k-1}(z_k-z_l)}\prod_{2\le i<j\le k}(z_j-z_i)\\&\le C(z_k-z_1)^{k-2}n^{-(1/2-\varepsilon)(k-2)}v_2(z).\end{aligned}$$

Therefore, since $S(\nu(n))\in W_{n,\varepsilon}$ and $S_k(\nu(n))<n^{1/2}$ it follows from the latter inequality and (2.6) that

$$\begin{aligned}&E_{after,l}(y)\\&\le Cn^{-(1/2-\varepsilon)(k-2)}\\&\quad\times\mathbf{E}\left[(n^{1/2}-S_1(\nu(n)))^{k-2}v_2(y+S(\nu(n))),\tau_y>\nu(n),\nu(n)\le n^{1-\varepsilon}-l\right]\\&\le Cn^{-(1/2-\varepsilon)(k-2)}\\&\quad\times\mathbf{E}\left[\left(n^{1/2}-y_1-M_1(n^{1-\varepsilon})\right)^{k-2}v_2(y+S(\nu(n))),\tau_y^{(2)}>\nu(n),\nu(n)\le n^{1-\varepsilon}-l\right],\end{aligned}$$

where $M_1(n):=\min_{k\le n}S_1(k)$.
Using now the fact that the sequence $v_2(y+S(n))1\{\tau_y^{(2)}>n\}$ is a martingale, we get

$$\begin{aligned}&E_{after,l}(y)\\&\quad\le Cn^{-(1/2-\varepsilon)(k-2)}\mathbf{E}\left[\left(n^{1/2}-y_1-M_1(n^{1-\varepsilon})\right)^{k-2}v_2(y+S(n)),\tau_y^{(2)}>n\right]\\&\quad=Cn^{-(1/2-\varepsilon)(k-2)}\mathbf{E}\left[\left(n^{1/2}-y_1-M_1(n^{1-\varepsilon})\right)^{k-2}\right]\mathbf{E}\left[v_2(y+S(n)),\tau_y^{(2)}>n\right]\\&\quad=Cn^{-(1/2-\varepsilon)(k-2)}\mathbf{E}\left[\left(n^{1/2}-y_1-M_1(n^{1-\varepsilon})\right)^{k-2}\right]v_2(y).\end{aligned}$$

According to the Rosenthal inequality, see [15],

$$\mathbf{E}|M_1(n)|^{k-2}\le Cn^{(k-2)/2}.$$

Combining this with the Doob inequality, we get

$$\mathbf{E}|M_1(n)|^{k-2}\le\left(\frac{k-2}{k-3}\right)^{k-2}\mathbf{E}|M_1(n)|^{k-2}\le Cn^{(k-2)/2}.$$

Therefore,

$$E_{after,l}(y)\le Cn^{-(1/2-\varepsilon)(k-2)}v_2(y)\left(|y_1|^{k-2}+n^{(k-2)/2}\right).$$

Using this bound, we get

$$
\begin{aligned}
&E_{down,l}\\
&\le Cn^{-(1/2-\varepsilon)(k-2)}\int_W \mathbf{P}\left(x+S(l)\in dy,\tau_x>l,\eta^-=l\right)v_2(y)\left(|y_1|^{k-2}+n^{(k-2)/2}\right)\\
&=Cn^{-(1/2-\varepsilon)(k-2)}\mathbf{E}\left[\left(|x+S_1(l)|^{k-2}+n^{(k-2)/2}\right)v_2(x+S(l)),\tau_x>l,\eta^-=l\right]
\end{aligned}
$$

We split the latter expectation in two parts. First on the event $\{S_1(l-1)\ge -n^{1/2}\}$ we have

$$
\begin{aligned}
&\mathbf{E}\left[|x+S_1(l)|^{k-2}v_2(x+S(l)),\eta^-=l,\tau_x>l,S_1(l-1)\ge -n^{1/2}\right]\\
&\quad\le C\mathbf{E}\left[\left((-X_1(l))^{k-2}+n^{(k-2)/2}\right)v_2(x+S(l)),\tau_x>l,\eta^-=l\right]\\
&\quad\le C\mathbf{E}\left[v_2(x+S(l-1)),\tau_x>l-1\right]\\
&\qquad\qquad\times\mathbf{E}\left[\left((-X_1(l))^{k-2}+n^{(k-2)/2}\right),X_1(l)<-n^{(1-\delta)/2}\right]\\
&\quad\le Cn^{(k-2)/2-\alpha(1-\delta)/2}\mathbf{E}\left[v_2(x+S(l-1)),\tau_x>l-1\right]. \qquad (2.7)
\end{aligned}
$$

Second the probability of event $\{S_1(l-1)<-n^{1/2}\}$ is negligible due to the Fuk-Nagaev inequality,

$$
\mathbf{P}\left(S_1(l-1)<-z,\eta^->l-1\right)\le\exp\left\{-Cz/n^{(1-\delta)/2}\right\},\quad z>n^{1/2}.
$$

Therefore, in view of the martingale property of $v_2(y+S_n)1\{\tau_y^{(2)}>n\}$,

$$
\begin{aligned}
&\mathbf{E}\left[|x+S_1(l-1)|^{k-2}v_2(x+S(l)),\eta^-=l,\tau_x>l,S_1(l-1)<-n^{1/2}\right]\\
&\quad\le v_2(x)\mathbf{E}\left[|x+S_1(l-1)|^{k-2}\eta^->l-1,S_1(l-1)<-n^{1/2}\right]\\
&\quad\le v_2(x)\exp\left\{-Cn^{-\delta/2}\right\}.
\end{aligned}
$$

Combining the latter estimate with (2.7) and using a bound

$$
|x+S_1(l)|^{k-2}\le 2^{k-3}\left(|x+S_1(l-1)|^{k-2}+(-X_1(l))^{k-2}\right),
$$

we get

$$
\begin{aligned}
&\mathbf{E}\left[|x+S_1(l-1)|^{k-2}v_2(x+S(l)),\eta^-=l,\tau_x>l\right]\\
&\le v_2(x)\exp\left\{-Cn^{-\delta/2}\right\}+Cn^{(k-2)/2-\alpha(1-\delta)/2}\mathbf{E}\left[v_2(x+S(l-1)),\tau_x>l-1\right]. \qquad (2.8)
\end{aligned}
$$

From (2.7) and (2.8) we conclude

$$
E_{down,l}\le v_2(x)\exp\left\{-Cn^{-\delta/2}\right\}+Cn^{-\alpha/2+\delta_1}\mathbf{E}\left[v_2(x+S(l-1)),\tau_x>l-1\right],
$$

where $\delta_1=\varepsilon(k-2)+\alpha\delta/2$. Summing up over l and taking into account (2.5), we obtain

$$
\begin{aligned}
E_{down}&\le v_2(x)\exp\left\{-Cn^{-\delta/2}\right\}\\
&\quad+Cn^{-\alpha/2+\delta_1}\sum_{l=1}^{n^{1-\varepsilon}}\mathbf{E}\left[v_2(x+S(l-1)),\tau_x>l-1\right]. \qquad (2.9)
\end{aligned}
$$

No big jumps: It remains to consider the case with no big jumps before the stopping time $\nu(n)$. If all the jumps are bounded by $n^{(1-\delta)/2}$, then, as it was shown in the proof of Lemma 16 of [4],

$$\mathbf{E}\left[v_1(x+S(n)),\tau_x>n,|S(\nu(n))|>n^{1/2-\delta/4},\eta\ge\nu(n),\nu(n)\le n^{1-\varepsilon}\right] \le C\exp\left\{-Cn^{-\delta/4}\right\}. \quad (2.10)$$

If the random walk starts from $y\in W_{n,\varepsilon}$ with $|y|\le n^{1/2-\delta/4}$, then one can use the standard KMT-coupling to show that

$$\begin{aligned}\mathbf{E}\left[v_1(y+S(n)),\tau_y>n\right] &\sim \mathbf{E}\left[\Delta^{(1)}(y+S(n)),\tau_y>n\right]\\ &\sim \mathbf{E}\left[\Delta^{(1)}(y+B(n)),\tau_y^{bm}>n\right]\\ &\sim \frac{\Delta^{(1)}(y)}{n^{(k-1)/2}}\mathbf{E}\left[\Delta^{(1)}(B(1))|\tau^{bm}_{y/\sqrt{n}}>1\right], \end{aligned}\quad (2.11)$$

where

$$\Delta^{(1)}(x)=\prod_{1\le i<j\le(k-1)}(x_j-x_i).$$

Moreover, if γ is sufficiently small, $\gamma<\delta/8$, then, using the same arguments,

$$\begin{aligned}\mathbf{E}\left[v_1(y+S(n^{1-\gamma})),\tau_y>n^{1-\gamma}\right] &\sim \mathbf{E}\left[\Delta^{(1)}(y+S(n^{1-\gamma})),\tau_y>n^{1-\gamma}\right]\\ &\sim \mathbf{E}\left[\Delta^{(1)}(y+B(n^{1-\gamma})),\tau_y^{bm}>n^{1-\gamma}\right]\\ &\sim \frac{\Delta^{(1)}(y)}{n^{(1-\gamma)(k-1)/2}}\mathbf{E}\left[\Delta^{(1)}(B(1))|\tau^{bm}_{y/n^{(1-\gamma)/2}}>1\right]. \end{aligned}\quad (2.12)$$

Since $y/n^{(1-\gamma)/2}\to 0$, we have

$$\mathbf{E}\left[\Delta^{(1)}(B(1))|\tau^{bm}_{y/\sqrt{n}}>1\right]\sim\mathbf{E}\left[\Delta^{(1)}(B(1))|\tau^{bm}_{y/n^{(1-\gamma)/2}}>1\right].$$

From this relation, estimates (2.11) and (2.12), and the strong Markov property we infer that

$$\begin{aligned}&\mathbf{E}\left[v_1(x+S(n)),\tau_x>n,|S(\nu(n))|\le n^{1/2-\delta/4},\eta\ge\nu(n),\nu(n)\le n^{1-\varepsilon}\right]\\ &\sim n^{-\gamma(k-1)/2}\\ &\times\mathbf{E}\left[v_1(x+S(n^{1-\gamma})),\tau_x>n^{1-\gamma},|S(\nu(n))|\le n^{1/2-\delta/4},\eta\ge\nu(n),\nu(n)\le n^{1-\varepsilon}\right].\end{aligned}$$

Hence

$$\begin{aligned}&\mathbf{E}\left[v_1(x+S(n)),\tau_x>n,|S(\nu(n))|\le n^{1/2-\delta/4},\eta\ge\nu(n),\nu(n)\le n^{1-\varepsilon}\right]\\ &\le Cn^{-\gamma(k-1)/2}\mathbf{E}\left[v_1(x+S(n^{1-\gamma})),\tau_x>n^{1-\gamma}\right].\end{aligned}\quad (2.13)$$

Final recursion: Putting (2.2), (2.3), (2.9)–(2.13) together, we obtain

$$\begin{aligned}\mathbf{E}\left[v_1(x+S(n)),\tau_x>n\right] &\le Cn^{-\alpha/2+\delta_1}\sum_{l=1}^{n^{1-\varepsilon}}\mathbf{E}\left[v(x+S(l-1)),\tau_x>l-1\right]\\ &+Cn^{-\gamma(k-1)/2}\mathbf{E}\left[v_1(x+S(n^{1-\gamma})),\tau_x>n^{1-\gamma}\right]+C\exp\{-Cn^{\delta/4}\}.\end{aligned}$$

Because of the symmetry, an analogous bound holds for $\mathbf{E}\left[v_2(x+S(n)),\tau_x>n\right]$. Consequently,

$$\mathbf{E}\left[v(x+S(n)),\tau_x>n\right]\le Cn^{-\alpha/2+\delta_1}\sum_{l=1}^{n^{1-\varepsilon}}\mathbf{E}\left[v(x+S(l-1)),\tau_x>l-1\right]$$
$$+Cn^{-\gamma(k-1)/2}\mathbf{E}\left[v(x+S(n^{1-\gamma})),\tau_x>n^{1-\gamma}\right]+C\exp\{-Cn^{\delta/4}\}.$$

Iterating this bound N times and recalling that $\alpha<k-1$, we obtain

$$\mathbf{E}\left[v(x+S(n)),\tau_x>n\right]\le Cn^{-\alpha/2+\delta_1}\sum_{l=1}^{n}\mathbf{E}\left[v(x+S(l-1)),\tau_x>l-1\right]$$
$$+Cn^{-(1-(1-\gamma)^N)(k-1)/2}v(x)+C\exp\{-Cn^{(1-\gamma)^{N-1}\delta/4}\}.$$

If N is such that $(1-(1-\gamma)^N)(k-1)/>\alpha$, then

$$\mathbf{E}\left[v(x+S(n)),\tau_x>n\right]$$
$$\le Cn^{-\alpha/2+\delta_1}\sum_{l=1}^{n}\mathbf{E}\left[v(x+S(l-1)),\tau_x>l-1\right]+C(x)n^{-\alpha/2}. \tag{2.14}$$

We know that $\mathbf{E}\left[v(x+S(l-1)),\tau_x>l-1\right]\le v(x)$. Entering with this into (2.14), we get

$$\mathbf{E}\left[v(x+S(n)),\tau_x>n\right]\le C(x)n^{1-\alpha/2+\delta_1}. \tag{2.15}$$

If $\alpha>4$, then making δ_1 sufficiently small, we see that $\mathbf{E}\left[v(x+S(n)),\tau_x>n\right]$ is summable. If $\alpha\le 4$, then applying (2.15) to every expectation on the right hand side of (2.14), we get

$$\mathbf{E}\left[v(x+S(n)),\tau_x>n\right]\le C(x)n^{2-\alpha+2\delta_1}.$$

We are done if $\alpha>3$. If it is not the case, then we enter with the new bound into (2.14), and so on. The N-th iteration will give the bound of order $n^{N(1-\alpha/2+\delta_1)}$. If $N(1-\alpha/2+\delta_1)<-1$, then we have the desired summability. □

3 Proof of Theorem 1.1

We start by estimating the tail of τ_x for paths without big jumps.

Lemma 3.1. *Let $A_r(n)$ denote the event $\{X_1(l)\ge -rn^{1/2}, X_k(l)\le rn^{1/2}$ for all $l\le n\}$. Then*

$$\mathbf{P}\left(\tau_x>n,A_r(n)\right)\le Cr^{k-1-\alpha}n^{-\alpha/2-(k-1)(k-2)/4}.$$

Proof. First, if $\eta>\nu(n)$, then, repeating coupling arguments from the proof of Proposition 2.1, we obtain

$$\mathbf{P}\left(\tau_x>n,A_r(n),\eta>\nu(n),\nu(n)\le n^{1-\varepsilon}\right)$$
$$\sim n^{-\gamma k(k-1)/4}\mathbf{P}\left(\tau_x>n^{1-\gamma},A_r(n^{1-\gamma}),\eta>\nu(n),\nu(n)\le n^{1-\varepsilon}\right)$$
$$\le n^{-\gamma k(k-1)/4}\mathbf{P}\left(\tau_x>n^{1-\gamma},A_r(n^{1-\gamma})\right). \tag{3.1}$$

We next assume that the random walk on the bottom jumps before $\nu(n)$ but the random walk on the top does not jump, i.e. $\eta^-\le\nu(n)$ and $\eta^+>\nu(n)$. It follows from (2.4) that

$$\mathbf{P}\left(\tau_x>n,A_r(n),\eta^-\le\nu(n)<\eta^+,\nu(n)\le n^{1-\varepsilon}\right)$$
$$\le\mathbf{P}\left(\tau_x>n,A_r(n),\eta^-\le\nu(n)\le n^{1-\varepsilon},S_k(\nu(n))\le n^{1/2-r(\delta)}\right)+\exp\{-Cn^{\delta^2/2}\}.$$

Applying now estimate (33) from [4], we get

$$\mathbf{P}\left(\tau_x > n, A_r(n), \eta^- \le \nu(n) \le n^{1-\varepsilon}, S_k(\nu(n)) \le n^{1/2-r(\delta)}\right) \le \frac{C}{n^{k(k-1)/4}}$$
$$\times \mathbf{E}\left[\Delta(x+S(\nu(n))), \tau_x > \nu(n), A_r(\nu(n)), \eta^- \le \nu(n) \le n^{1-\varepsilon}, S_k(\nu(n)) \le n^{1/2-r(\delta)}\right].$$

On $W_{n,\varepsilon}$ holds

$$\Delta(x+S(\nu(n))) \le (x_k - x_1 + S_k(\nu(n)) - S_1(\nu(n)))^{k-1} v_1(x+S(\nu(n))).$$

Consequently,

$$\mathbf{E}\left[\Delta(x+S(\nu(n))), \tau_x > \nu(n), A_r(\nu(n)), \eta^- \le \nu(n) \le n^{1-\varepsilon}, S_k(\nu(n)) \le n^{1/2-r(\delta)}\right]$$
$$\le \mathbf{E}\left[\left(n^{1/2-r(\delta)} - S_1(\nu(n))\right)^{k-1} v_2(x+S(\nu(n))), \tau_x > \nu(n), A_r(\nu(n)), \eta^- \le \nu(n) \le n^{1-\varepsilon}\right].$$

Repeating arguments from the proof of Proposition 2.1, we get

$$\mathbf{E}\left[\Delta(x+S(\nu(n))), \tau_x > \nu(n), A_r(\nu(n)), \eta^- \le \nu(n) \le n^{1-\varepsilon}, S_k(\nu(n)) \le n^{1/2-r(\delta)}\right]$$
$$\le \sum_{l=1}^{n^{1-\varepsilon}} \mathbf{E}\left[v_2(x+S(l-1)), \tau_x > l-1\right] \mathbf{E}\left[X^{k-1}, n^{(1-\delta)/2} \le X \le an^{1/2}\right]$$
$$\le r^{k-1-\alpha} n^{(k-1-\alpha)/2} \sum_{l=1}^{\infty} \mathbf{E}\left[v_2(x+S(l-1)), \tau_x > l-1\right]$$

As a result we have

$$\mathbf{P}\left(\tau_x > n, A_r(n), \eta^- \le \nu(n) < \eta^+, \nu(n) \le n^{1-\varepsilon}\right)$$
$$\le \frac{Cr^{k-1-\alpha}}{n^{\alpha/2+(k-1)(k-2)/4}} \sum_{l=1}^{\infty} \mathbf{E}\left[v_2(x+S(l-1)), \tau_x > l-1\right]. \tag{3.2}$$

Analogously,

$$\mathbf{P}\left(\tau_x > n, A_r(n), \eta^+ \le \nu(n) < \eta^-, \nu(n) \le n^{1-\varepsilon}\right)$$
$$\le \frac{Cr^{k-1-\alpha}}{n^{\alpha/2+(k-1)(k-2)/4}} \sum_{l=1}^{\infty} \mathbf{E}\left[v_1(x+S(l-1)), \tau_x > l-1\right]. \tag{3.3}$$

Therefore, it remains to consider the case when $\eta^+ \le \nu(n)$ and $\eta^- \le \nu(n)$. Because of the symmetry we may assume that $\eta^+ \le \eta^-$. Then

$$\mathbf{P}\left(\tau_x > n, \eta^+ \le \eta^- \le \nu(n) \le n^{1-\varepsilon}\right)$$
$$\le \sum_{l=1}^{n^{1-\varepsilon}} \mathbf{P}\left(\tau_x > n, \eta^+ = \eta^- = l\right) + \sum_{l=1}^{n^{1-\varepsilon}} \sum_{j=l+1}^{n^{1-\varepsilon}} \mathbf{P}\left(\tau_x > n, \eta^+ = l, \eta^- = j\right).$$

First we note

$$\mathbf{P}\left(\tau_x > n, \eta^+ = \eta^- = l\right) \le C \int_W \mathbf{P}(x+S(l-1) \in dy, \tau_x > l-1) n^{-\alpha+\delta} \frac{\tilde{v}(y)}{n^{(k-2)(k-3)/4}},$$

where

$$\tilde{v}(x) = V^{(k-2)}(x_2, x_3, \ldots, x_{k-1}).$$

Using now the bound

$$\mathbf{E}\left[\tilde{v}(x+S(l)),\tau_x^{(1)}>l\right]\le Cv_1(x)l^{-(k-2)/2},\quad l\ge 1,\tag{3.4}$$

we obtain

$$\begin{aligned}\sum_{l=1}^{n^{1-\varepsilon}}\mathbf{P}\left(\tau_x>n,\eta^+=\eta^-=l\right)&\le\frac{C}{n^{\alpha-\delta+(k-2)(k-3)/4}}\sum_{l=0}^{n}\mathbf{E}\left[\tilde{v}(x+S(l)),\tau_x^{(1)}>l\right]\\&\le\frac{Cv_1(x)}{n^{\alpha-\delta+(k-2)(k-3)/4}}\sum_{l=1}^{n}l^{-(k-2)/2}\\&\le Cv_1(x)\frac{\log n}{n^{\alpha-\delta+(k-2)(k-3)/4}}\\&=o\left(n^{-\alpha/2-(k-1)(k-2)/4}\right).\end{aligned}$$

Furthermore, applying (3.4) once again, we get

$$\begin{aligned}&\mathbf{P}\left(\tau_x>n,\eta^+=l,\eta^-=j\right)\\&\quad\le C\int_W\mathbf{P}(x+S(j-1)\in dy,\tau_x>j-1,\eta^+=l)n^{-\alpha/2+\delta/2}\frac{\tilde{v}(y)}{n^{(k-2)(k-3)/4}}\\&\quad\le C\int_W\mathbf{P}(x+S(l-1)\in dy,\tau_x>l-1)n^{-\alpha+\delta}\frac{v_1(y)}{n^{(k-2)(k-3)/4}}\frac{1}{(j-l)^{(k-2)/2}}.\end{aligned}$$

This implies that

$$\begin{aligned}&\sum_{l=1}^{n^{1-\varepsilon}}\sum_{j=l+1}^{n^{1-\varepsilon}}\mathbf{P}\left(\tau_x>n,\eta^+=l,\eta^-=j\right)\\&\quad\le\frac{C\log n}{n^{\alpha-\delta+(k-2)(k-3)/4}}\sum_{l=0}^{\infty}\mathbf{E}\left[v_1(x+S(l)),\tau_x>l\right]\\&\quad=o\left(n^{-\alpha/2-(k-1)(k-2)/4}\right).\end{aligned}$$

As a result we have the bound

$$\mathbf{P}\left(\tau_x>n,\eta^+\le\eta^-\le\nu(n)\le n^{1-\varepsilon}\right)=o\left(n^{-\alpha/2-(k-1)(k-2)/4}\right).\tag{3.5}$$

Combining (3.1) – (3.5), we arrive at the inequality

$$\mathbf{P}\left(\tau_x>n,A_r(n)\right)\le n^{-\gamma k(k-1)/4}\mathbf{P}\left(\tau_x>n^{1-\gamma},A_r(n^{1-\gamma})\right)+\frac{Cr^{k-1-\alpha}}{n^{\alpha/2+(k-1)(k-2)/4}}.$$

Iterating N times we get

$$\begin{aligned}\mathbf{P}\left(\tau_x>n,A_r(n)\right)\le n^{-(1-(1-\gamma)^N)k(k-1)/4}&\mathbf{P}\left(\tau_x>n^{(1-\gamma)^N},A_{n^{(1-\gamma)^N}}(a)\right)\\&+\frac{Cr^{k-1-\alpha}}{n^{\alpha/2+(k-1)(k-2)/4}}.\end{aligned}$$

Choosing N sufficiently large, we arrive at the desired inequality. □

Lemma 3.2. *If S is as in Theorem 1 of [4], then there exists a constant C such that*

$$\mathbf{P}(\tau_x>n)\le\frac{CV(x)}{n^{k(k-1)/4}},\quad x\in W.$$

Proof. It follows from Proposition 4 of [4] that $V(x) \sim \Delta(x)$ uniformly in $x \in W_{n,\varepsilon}$. This and inequality (33) from [4] imply that

$$\mathbf{P}(\tau_x > n, \nu(n) \le n^{1-\varepsilon}) \le \frac{C}{n^{k(k-1)/4}} \mathbf{E}\left[V(x + S(\nu(n))), \tau_x > \nu(n), \nu(n) \le n^{1-\varepsilon}\right].$$

Recalling that the sequence $V(x + S(n))1\{\tau_x > n\}$ is a martingale, we conclude that

$$\begin{aligned}
&\mathbf{E}\left[V(x + S(\nu(n))), \tau_x > \nu(n), \nu(n) \le n^{1-\varepsilon}\right] \\
&\quad \le \mathbf{E}\left[V(x + S(\nu(n) \wedge n^{1-\varepsilon})), \tau_x > \nu(n) \wedge n^{1-\varepsilon}\right] = V(x).
\end{aligned}$$

To complete the proof it remains to note that

$$\mathbf{P}(\nu(n) > n^{1-\varepsilon}) \le e^{-Cn^{\varepsilon}}$$

and that $\inf_{x \in W} V(x) > 0$. □

Lemma 3.3. *If $x_k = r\sqrt{n}$ and $x_1, \ldots, x_{k-1}$ are fixed, then*

$$\mathbf{P}(\tau_x > n) \sim \psi(r) \frac{v_1(x)}{n^{(k-1)(k-2)/4}}. \tag{3.6}$$

Moreover,

$$\psi(a) \le Cr^{k-1}, \quad r > 0. \tag{3.7}$$

Proof. It is clear that

$$\mathbf{P}(\tau_x > n) = \mathbf{P}(\tau_x > n, \nu(n) \le n^{1-\varepsilon}) + O\left(e^{-Cn^{\varepsilon}}\right).$$

Furthermore,

$$\begin{aligned}
&\mathbf{P}\left(\tau_x > n, \nu(n) \le n^{1-\varepsilon}, |S_k(\nu(n))| \ge \theta_n \sqrt{n}\right) \\
&\le \mathbf{P}\left(\max_{j \le n^{1-\varepsilon}} |S_k(j)| \ge \theta_n \sqrt{n}\right) \mathbf{P}(\tau_x^{(1)} > n) \\
&= o\left(n^{-(k-1)(k-2)/4}\right)
\end{aligned}$$

and, in view of Lemma 16 from [4],

$$\mathbf{P}(\tau_x^{(1)} > n, |S(\nu(n))| > \sqrt{n}, \nu(n) \le n^{1-\varepsilon}) = o\left(n^{-(k-1)(k-2)/4}\right).$$

As a result we have

$$\mathbf{P}(\tau_x > n) = \mathbf{P}(\tau_x > n, |S(\nu(n))| \le \theta_n \sqrt{n}, \nu(n) \le n^{1-\varepsilon}) + o\left(n^{-(k-1)(k-2)/4}\right). \tag{3.8}$$

Applying inequality (33) from [4], we obtain the bound

$$\begin{aligned}
&\mathbf{P}\left(\tau_x > n, \nu(n) \le n^{1-\varepsilon}, |S(\nu(n))| \le \theta_n \sqrt{n}\right) \\
&\le \frac{C}{n^{k(k-1)/4}} \mathbf{E}\left[\Delta(x + S(\nu(n))), \tau_x > \nu(n), |S(\nu(n))| \le \theta_n \sqrt{n}\right] \\
&\le \frac{Cr^{k-1}}{n^{(k-1)(k-2)/4}} \mathbf{E}\left[\Delta^{(1)}(x + S(\nu(n))), \tau_x^{(1)} > \nu(n)\right]
\end{aligned}$$

Noting that the expectation on the right converges to $v_1(x)$ and taking into account (3.8), we obtain finally

$$n^{(k-1)(k-2)/4} \mathbf{P}(\tau_x > n) \le Cr^{k-1} v_1(x). \tag{3.9}$$

Using coupling one can show that, uniformly in $x = (x_1, x_2, \dots, x_k) \in W_{n,\varepsilon}$ with $|x_j| \le \theta_n\sqrt{n}$ and $|x_k - r\sqrt{n}| \le \theta_n\sqrt{n}$, holds

$$\mathbf{P}(\tau_x > n) \sim \mathbf{P}(\tau_x^{bm} > n) \sim \frac{\Delta^{(1)}(x)}{n^{(k-1)(k-2)/4}}\psi(r).$$

Consequently,

$$\begin{aligned}
&\mathbf{P}\left(\tau_x > n, \nu(n) \le n^{1-\varepsilon}, |S(\nu(n))| \le \theta_n\sqrt{n}\right)\\
\sim\quad & \frac{\psi(r)}{n^{(k-1)(k-2)/4}}\mathbf{E}\left[\Delta^{(1)}(x + S(\nu(n)))\tau_x > \nu(n), \nu(n) \le n^{1-\varepsilon}, |S(\nu(n))| \le \theta_n\sqrt{n}\right]\\
\sim\quad & \frac{\psi(r)}{n^{(k-1)(k-2)/4}}v_1(x),
\end{aligned}$$

where in the last step we have used Lemmas 15 and 16 from [4]. Combining this relation with (3.8), we get (3.6), and (3.7) follows from (3.9). □

Proof of Theorem 1.1. Denote

$$T^+ = \min\{j \ge 1 : X_k(j) \ge rn^{1/2}\}, \quad T^- = \min\{j \ge 1 : X_1(j) \le -rn^{1/2}\}$$

and

$$T = \min\{T^+, T^-\}.$$

We first derive an upper bound for $\mathbf{P}(\tau_x > n)$. Our starting point will be the following inequality

$$\mathbf{P}(\tau_x > n) \le \sum_{l=1}^{n/2}\mathbf{P}(\tau_x > n, T = l) + \mathbf{P}(\tau_x > n/2, T > n/2). \tag{3.10}$$

According to Lemma 3.1,

$$\mathbf{P}(\tau_x > n/2, T > n/2) \le \frac{Cr^{k-1-\alpha}}{n^{\alpha/2+(k-1)(k-2)/4}}. \tag{3.11}$$

Applying Lemma 3.2 to $(S_1, S_2, \dots, S_{k-1})$, we conclude that, for every $l \le n/2$, holds

$$\begin{aligned}
\mathbf{P}(\tau_x > n, T^+ = l) &\le \int_W \mathbf{P}(x + S(l) \in dy, \tau_x > l, T^+ = l)\mathbf{P}(\tau_y^{(1)} > n/2)\\
&\le \frac{C}{n^{(k-1)(k-2)/4}}\mathbf{E}\left[v_1(x + S(l)), \tau_x > l, T^+ = l\right]\\
&\le \frac{C}{n^{(k-1)(k-2)/4}}\frac{p}{(rn^{1/2})^\alpha}\mathbf{E}\left[v_1(x + S(l-1)), \tau_x > l-1\right].
\end{aligned}$$

And an analogous inequality holds for $\mathbf{P}(\tau_x > n, T^- = l)$. As a result we have

$$\sum_{l=N}^{n/2}\mathbf{P}(\tau_x > n, T^+ = l) \le \frac{Cr^{-\alpha}}{n^{\alpha/2+(k-1)(k-2)/4}}\sum_{l=N}^{\infty}\mathbf{E}\left[v(x + S(l-1)), \tau_x > l-1\right]. \tag{3.12}$$

For every fixed l we have

$$\begin{aligned}
\mathbf{P}(\tau_x > n, T^+ = l) &= \int_W \mathbf{P}(x + S(l) \in dy, \tau_x > l, T^+ = l)\mathbf{P}(\tau_y > n - l)\\
&\sim n^{-(k-1)(k-2)/4}\mathbf{E}\left[v_1(x + S(l))\psi\left(\frac{X_k(l)}{\sqrt{n}}\right), \tau_x > l, T^+ = l\right]\\
&\sim n^{-(k-1)(k-2)/4}\mathbf{E}\left[v_1(x + S(l))\psi\left(\frac{X_k(l)}{\sqrt{n}}\right), \tau_x > l, T^+ = l\right]\\
&\sim n^{-(k-1)(k-2)/4}\mathbf{E}\left[v_1(x + S(l-1)), \tau_x > l-1\right]\mathbf{E}\left[\psi\left(\frac{X_k(l)}{\sqrt{n}}\right), T^+ = l\right].
\end{aligned}$$

Noting that

$$\mathbf{E}\left[\psi\left(\frac{X_k(l)}{\sqrt{n}}\right), T^+ = l\right] \sim pn^{-\alpha/2}\int_a^\infty \psi(y)\alpha y^{-\alpha-1}dy =: \theta(r),$$

we obtain

$$\mathbf{P}(\tau_x > n, T^+ = l) \sim p\theta(r)n^{-\alpha/2-(k-1)(k-2)/4}\mathbf{E}\left[v_1(x+S(l-1))\tau_x > l-1\right].$$

In the same way one can get

$$\mathbf{P}(\tau_x > n, T^- = l) \sim q\theta(r)n^{-\alpha/2-(k-1)(k-2)/4}\mathbf{E}\left[v_2(x+S(l-1))\tau_x > l-1\right].$$

Therefore,

$$\sum_{l=1}^{N-1}\mathbf{P}(\tau_x > n, T = l) \sim \theta(r)n^{-\alpha/2-(k-1)(k-2)/4}\sum_{l=1}^{N-1}\mathbf{E}\left[v(x+S(l-1))\tau_x > l-1\right]. \quad (3.13)$$

Combining (3.11) — (3.13) and noting that (3.7) yields $\theta(r) \le \theta(0) < \infty$, we see that

$$\limsup_{n\to\infty} n^{\alpha/2+(k-1)(k-2)/4}\mathbf{P}(\tau_x > n) \le \theta(0)\sum_{l=1}^{\infty}\mathbf{E}\left[v(x+S(l-1))\tau_x > l-1\right] + Cr^{-\alpha}\sum_{l=N}^{\infty}\mathbf{E}\left[v(x+S(l-1)), \tau_x > l-1\right] + Cr^{k-1-\alpha}.$$

Letting here first $N \to \infty$ and then $r \to 0$, we get

$$\limsup_{n\to\infty} n^{\alpha/2+(k-1)(k-2)/4}\mathbf{P}(\tau_x > n) \le \theta(0)U(x). \quad (3.14)$$

To obtain a corresponding lower bound we note that, for every $N \ge 1$,

$$\mathbf{P}(\tau_x > n) \ge \sum_{l=1}^{N-1}\mathbf{P}(\tau_x > n, T = l).$$

Using now (3.13), we have

$$\liminf_{n\to\infty} n^{\alpha/2+(k-1)(k-2)/4}\mathbf{P}(\tau_x > n) \ge \theta(r)\sum_{l=1}^{N-1}\mathbf{E}\left[v(x+S(l-1))\tau_x > l-1\right].$$

Since N can be chosen arbitrary large

$$\liminf_{n\to\infty} n^{\alpha/2+(k-1)(k-2)/4}\mathbf{P}(\tau_x > n) \ge \theta(r)U(x).$$

Finally, it follows from (3.7) that $\theta(r) = \theta(0) + O(a^{k-1-\alpha})$. Hence,

$$\liminf_{n\to\infty} n^{\alpha/2+(k-1)(k-2)/4}\mathbf{P}(\tau_x > n) \ge \theta(0)U(x).$$

From this inequality and (3.14) we conclude that (1.2) holds with $\theta = \theta(0)$. □

4 Proof of Theorem 1.2

We have to show that

$$\mathbf{E}[f(X^{(n)})|\tau_x > n] \to \mathbf{E}[f(X)] \tag{4.1}$$

for every bounded and continuous $f : C[0,1] \to \mathbb{R}$.

We first note that it suffices to prove that

$$\begin{aligned}\lim_{r\to 0}\lim_{n\to\infty} & n^{\alpha/2+(k-1)(k-2)/4}\mathbf{E}[f(X^{(n)}), T = l, X_k(l) > rn^{1/2}, \tau_x > n] \\ &= p\mathbf{E}[v_1(x+S(l-1)), \tau_x > l-1]\mathbf{E}[f(X), X_k(0) > 0]. \end{aligned} \tag{4.2}$$

for every fixed l. Indeed, in view of the symmetry,

$$\begin{aligned}\lim_{r\to 0}\lim_{n\to\infty} & n^{\alpha/2+(k-1)(k-2)/4}\mathbf{E}[f(X^{(n)}), T = l, X_1(l) < -rn^{1/2}, \tau_x > n] \\ &= q\mathbf{E}[v_2(x+S(l-1)), \tau_x > l-1]\mathbf{E}[f(X), X_1(0) < 0]. \end{aligned} \tag{4.3}$$

Then, combining (4.2) and (4.3), we get

$$\begin{aligned}\lim_{r\to 0}\lim_{n\to\infty} & \mathbf{E}[f(X^{(n)}), T \le N|\tau_x > n] \\ &\sim \frac{\sum_{l=0}^{N-1}\mathbf{E}[v(x+S(l-1)), \tau_x > l-1]}{U(x)}\mathbf{E}[f(X)]\end{aligned}$$

Using Proposition 2.1 and Lemma 3.1, we get (4.1).

In order to prove (4.2) we assume first that our random walk starts from x with $|x_1| < A, \dots, |x_{k-1}| < A$ and $x_k > rn^{1/2}$. It is easy to see that

$$\begin{aligned}&\mathbf{P}\left(\tau_x > n, \nu(n) \le n^{1-\varepsilon}, |S_k(\nu(n))| > n^{1/2-\varepsilon/4}\right) \\ &\le \mathbf{P}\left(\max_{j\le n^{1-\varepsilon}}|S_k(\nu(n))| > n^{1/2-\varepsilon/4}\right)\mathbf{P}(\tau_x^{(1)} > n) = o\left(\frac{1}{n^{(k-1)(k-2)/4}}\right).\end{aligned}$$

Furthermore, it follows from Lemma 16 of [4] that

$$\mathbf{P}\left(\tau_x^{(1)} > n, \nu(n) \le n^{1-\varepsilon}, |S(\nu(n))| > \theta_n n^{1/2}\right) = o\left(\frac{1}{n^{(k-1)(k-2)/4}}\right).$$

As a result we have the following representation

$$\begin{aligned}\mathbf{E}\left[f(X^{(n)}), \tau_x > n\right] = \mathbf{E}&\left[f(X^{(n)}), \tau_x > n, |S(\nu(n))| \le \theta_n n^{1/2}, \nu(n) \le n^{1-\varepsilon}\right] \\ &+ o\left(\frac{1}{n^{(k-1)(k-2)/4}}\right).\end{aligned}$$

Further,

$$\begin{aligned}&\mathbf{E}\left[f(X^{(n)}), \tau_x > n, |S(\nu(n))| \le \theta_n n^{1/2}, \nu(n) \le n^{1-\varepsilon}\right] \\ &= \sum_{l=1}^{n^{1-\varepsilon}}\int_W \mathbf{P}\left(x+S(l)\in dy, \tau_x > l, \nu(n) = l, |S(l)| \le \theta_n n^{1/2}\right) \\ &\qquad\qquad \times \mathbf{E}\left[f_{l,y}(X^{(n)}), \tau_y > n-l\right],\end{aligned}$$

where

$$f_{l,y}(u) = f\left(y1_{\{t\le l/n\}} + u(t)1_{\{t>l/n\}}\right), \quad u \in C[0,1].$$

Using coupling, we obtain

$$\mathbf{E}\left[f_{l,y}(X^{(n)}),\tau_y>n-l\right]\sim\mathbf{E}[f(X)|X_k(0)=x_k/\sqrt{n}]\mathbf{P}(\tau_y^{bm}>n)$$
$$\sim\mathbf{E}[f(X)|X_k(0)=x_k/\sqrt{n}]\frac{\Delta^{(1)}(y)\psi(x_k/\sqrt{n})}{n^{(k-1)(k-2)/4}}.$$

This implies that

$$\begin{aligned}&\mathbf{E}\left[f(X^{(n)}),\tau_x>n,|S(\nu(n))|\le\theta_n n^{1/2},\nu(n)\le n^{1-\varepsilon}\right]\\&\quad\sim\frac{\mathbf{E}[f(X)|X_k(0)=x_k/\sqrt{n}]\psi(x_k/\sqrt{n})}{n^{(k-1)(k-2)/4}}\\&\qquad\times\mathbf{E}\left[\Delta^{(1)}(x+S(\nu(n))),\tau_x>\nu(n),\nu(n)\le n^{1-\varepsilon},|S(\nu(n))|\le\theta_n n^{1/2}\right]\\&\quad\sim v_1(x)\frac{\mathbf{E}[f(X)|X_k(0)=x_k/\sqrt{n}]\psi(x_k/\sqrt{n})}{n^{(k-1)(k-2)/4}},\end{aligned}$$

where in the last step we used Lemmas 15 and 16 from [4]. Therefore,

$$\begin{aligned}&\mathbf{E}[f(X^{(n)}),T=l,X_k(l)>rn^{1/2},\tau_x>n]\\&\quad\sim\int_W\mathbf{P}(x+S(l)\in dy,\tau_x>l,T=l,X_k(l)>rn^{1/2})\\&\qquad\qquad\times\frac{v_1(y)}{n^{(k-1)(k-2)/4}}\mathbf{E}[f(X)|X_k(0)=y_k/\sqrt{n}]\psi(y_k/\sqrt{n})\\&\quad\sim\frac{p\mathbf{E}[v_1(x+S(l-1),\tau_x>l-1]}{n^{\alpha/2+(k-1)(k-2)/4}}\int_a^\infty\mathbf{E}[f(X)|X_k(0)=z]\psi(z)\alpha z^{-\alpha-1}dz.\end{aligned}$$

Since

$$\lim_{r\to0}\int_r^\infty\mathbf{E}[f(X)|X_k(0)=z]\psi(z)\alpha z^{-\alpha-1}dz=\mathbf{E}[f(X),X_k(0)>0],$$

the previous relation implies (4.1). Thus, the proof is finished.

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References

[1] Bertoin, J. and Doney, R.A. On conditioning a random walk to stay nonnegative. *Ann. Probab.*,**22**, (1994), 2152–2167. MR-1331218

[2] Denisov, D. and Shneer, V. (2007). Asymptotics for first-passage times of Lévy processes and random walks. ArXiv Preprint, arXiv:0712.0728.

[3] Denisov, D., Dieker, T. and Shneer, V. Large deviations for random walks under subexponentiality: the big-jump domain. *Ann. Probab.*, **36** (5), (2008) 1946-1991. MR-2440928

[4] Denisov, D. and Wachtel, V. Conditional limit theorems for ordered random walks. *Electron. J. Probab.*, **(**15), (2010),292–322. MR-2609589

[5] Doney, R.A. The Martin boundary and ratio limit theorems for killed random walks. *J. London Math. Soc. (2)*, **58**, (1998), 761–768. MR-1678162

[6] Eichelsbacher, P. and König, W. Ordered random walks. *Electron. J. Probab.*, **13**, (2008), 1307–1336. MR-2430709

[7] Ignatiouk-Robert, I. Martin boundary of a killed random walk on a half-space. *J. Theoret. Probab.*, **21**, (2008), 35–68. MR-2384472

[8] Ignatiouk-Robert, I. and Loree, C. Martin boundary of a killed random walk on a quadrant. *Ann. Probab.*, **38**, (2010), 1106–1142. MR-2674995

[9] Ignatiouk-Robert, I. Martin boundary of a killed random walk on $\mathbb{Z}_+^d$. ArXiv Preprint arXiv:0909.3921.

[10] Jacka, S. and Warren, J. Examples of convergence and non-convergence of Markov chains conditioned not to die. *Electron. J. Probab.*, **7**, (2002), 1–22. MR-1887621

[11] König, W. and Schmid, P. Random walks conditioned to stay in Weyl chambers of type C and D. *Elect. Comm. Probab.*, **15**, (2010), 286–296. MR-2670195

[12] Nagaev, S.V. Large deviations of sums of independent random variables. *Ann. Probab.*, **7**, (1979), 745–789. MR-0542129

[13] Raschel, K. Green functions and Martin compactification for killed random walks related to SU(3). *Elect. Comm. Probab.*, **15**, (2010), 176–190. MR-2651549

[14] Raschel, K. Green functions for killed random walk in the Weyl chamber of Sp(4). *Ann. Inst. H. Poincaré Probab. Statist.*, **47**, (2011), 1001–1019.

[15] Rosenthal, H.P. On the subspaces of L^p ($p > 2$) spanned by sequences of independent random variables. *Israel J. Math.*,**8**, (1970), 273–303. MR-0271721