# Semiconcavity, viscosity solutions and the square distance in Carnot groups.

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**Abstract** We give an overview on semiconcavity, starting from the standard notion, up to more recent generalizations in a different geometrical context, such as Carnot groups; focusing in particular on the viscosity characterization by bounds for second derivatives. We then apply these theories to show some recent results obtained by the author, in collaboration with Qing Liu and Ye Zhang. In particular we show that the square Carnot-Carathéodory distance is h-semiconcave in step 2 Carnot groups.

**Key words:** Carnot groups, Heisenberg group, convexity, semiconcavity, Sub-Riemannian geometries, Hömander condition, Hamilton-Jacobi equations.
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## 1 The Euclidean Case

In this section we briefly recall the standard notions of concavity and semiconcavity, used in the Euclidean space. We also give a quick overview on viscosity solutions, focusing in particular on the viscosity characterization for concave and semiconcave functions by bounds for the second derivatives. The theory is usually stated for convex and semiconvex functions, but since the focus later is the semiconcavity of the square distance, we state all definitions and properties for concave and semiconcave functions. We recall that a function $u$ is concave (respectively semiconcave) if and only if $-u$ is convex (respectively semiconvex).

***Definition*** Given $u:\mathbb{R}^n\to\mathbb{R}$, we say that $u$ is concave if and only if

$$u\big(\lambda\, p+(1-\lambda)\,q\big)\geq \lambda\, u(p)+(1-\lambda)\,u(q) \quad \forall\,\lambda\in[0,1] \text{ and } \forall\, p,q\in\mathbb{R}^n.$$

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Recall $u$ is convex if and only if $-u$ is concave. □

***Example*** Some very well-known examples are all affine functions, which are both convex and concave on $\mathbb{R}^n$, the modulus $|x|$ is convex, thus $-|x|$ is concave; similarly $u(x) = x^2$ is convex while $u(x) = -x^2$ is concave. □

It is also easy to show that, whenever $u \in C^2$, then $u$ is concave if and only if $D^2u \leq 0$, where this means that the $n \times n$ (symmetric) matrix of second derivatives $D^2u$ is semi-negative definite (i.e. all the eigenvalues are non-positive).

A very important generalization of the notion of concavity is the definition of semiconcavity. We next recall some equivalent notions and refer to [**?**] for a beautiful overview on this theory. Note that also in this case, $u$ semiconvex if and only if $-u$ semiconcave.

***Definition (Semiconcavity)*** Given $u : \mathbb{R}^n \to \mathbb{R}$, the following statements are equivalent: $\exists C > 0$ such that

1. $p \mapsto u(p) - \frac{C}{2}|p|^2$ is concave.
2. $\forall \lambda \in [0,1]$, $p, q \in \mathbb{R}^n$, $\lambda\, u(p) + (1-\lambda)\, u(q) - u\big(\lambda\, p + (1-\lambda)\, q\big) \leq C|p-q|^2$.
3. $\forall\, p, v \in \mathbb{R}^n$, $u(p+v) + u(p-v) - 2u(p) \leq C|v|^2$ (mid-point semiconcavity).

In this case the function $u$ is called semiconcave. □

If in addition we assume $u \in C^2$, then $u$ is semiconcave if and only if $\exists C > 0$ such that $D^2u \leq C\, Id_{n\times n}$. Another key property of semiconcave (or semiconvex) functions is that they are locally Lipschitz continuous, which also implies that the gradient is (uniformly) locally bounded.

***Example***

1. Concave functions are also semiconcave.
2. $C^2$ functions are both semiconvex and semiconcave in every bounded set.
3. The Euclidean square distance is both semiconcave and semiconvex in $\mathbb{R}^n$ since the Hessian is $2Id$, while the distance from the origin is only semiconvex (since it is convex), but it is not semiconcave at the origin.
4. The CC distance from the origin is locally semiconcave outside the origin [**?**, **?**] in ideal Carnot groups (e.g. in the Heisenberg group).

While convexity and concavity are key geometrical properties for sets and functions, semiconvexity and semiconcavity turn out to be key regularity properties, for example a function that is both semiconvex and semiconcave is $C^{1,1}$, see [**?**].

The bounds on their first and second derivatives have also many applications in the theory of PDEs.

In the next session we characterise concavity and semiconcavity by looking at the second derivatives, also for function not necessarily $C^2$, interpreting the sign/bounds for the second derivatives in the viscosity sense.

## 2 Viscosity solutions and semiconcavity

The theory of viscosity solutions has been formally introduced by M.G. Crandall and P.-L. Lions in 1981 [**?**, **?**] and developed in the following years with the contributions of many other authors. In this section we recall the main definitions and apply it to a characterization for concave and semiconcave functions.

***Definition*** We say that a continuous function $u : \Omega \to \mathbb{R}$ is a

1. viscosity subsolution at some point $x_0 \in \Omega$, if and only if, for any test function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at $x_0$, then
$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$
2. viscosity supersolution at some point $x_0 \in \Omega$, if and only if, for any test function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at $x_0$, then
$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0.$$
3. viscosity solution at a point $x_0 \in \Omega$ if $u$ is both a viscosity subsolution and a viscosity supersolution.

***Example*** If we consider the eikonal equation $|u'(x)| = 1$ on $(-1,1)$, with vanishing boundary conditions $u(-1) = 0 = u(1)$, by simply applying Rolle's Theorem, one can show that there are no classical solutions, in fact any function $C^1((-1,1)) \cap C([-1,1])$ solving the vanishing boundary condition, have at least one point where $u' = 0$. By simply applying the previous definition, it is easy to check that the function $u(x) = -|x| + 1$ is the (only) viscosity solution of the eikonal equation, satisfying the given vanishing boundary conditions. □

Viscosity solutions are a very good notion of generalised solutions for a large class of partial differential equations. In order to show that, we need to verify that they are consistent with classical solutions, which means the following:

1. Whenever $u \in C^2$ is viscosity solution, then $u$ is also classical solution. That is very easy to check since we can take $u = \varphi$ for both the viscosity supersolution and the viscosity subsolution properties, which trivially implies
$$F(x_0, u(x_0), Du(x_0), D^2u(x_0)) = 0, \quad \forall x_0 \in \Omega.$$
2. To show that classical solutions are indeed also viscosity solutions is more tricky and one needs to assume the following additional assumption on the structure of the PDE:
$$F(x,z,p,M) \leq F(x,z,p,N), \quad \forall M \geq N, \tag{1}$$
where $M$ and $N$ are symmetric $n \times n$-matrices, therefore $M \geq N$ needs to be interpreted e.g. as all eigenvalues for the matrix $M - N$ are non-negative.
Whenever the PDE satisfies the previous property of monotonicity w.r.t. the second derivatives, the PDE is called *degenerate elliptic*. Assuming this additional

property we can show that classical solutions are also viscosity solutions, in fact, given $\varphi \in C^2(\Omega)$ such that $u-\varphi \in C^2(\Omega)$ has a local maximum at $x_0$, then

$$\begin{cases} D(u-\varphi)(x_0)=0, \\ D^2(u-\varphi)(x_0)\leq 0. \end{cases}$$

Hence $u$ classical solution, together with property (**??**), give

$$F(x_0,u(x_0),Du(x_0),D^2u(x_0))=0 \quad \Rightarrow \; F(x_0,u(x_0),D\varphi(x_0),D^2\varphi(x_0))\leq 0,$$

i.e. the viscosity subsolution property. Similarly one can check the viscosity supersolution property.

***Example***

1. First order PDEs (as for example the eikonal equation or in general Hamilton-Jacobi equations) are always degenerate elliptic.
2. Linear elliptic and parabolic equations (e.g. Poisson equations or heat equations) are degenerate elliptic.
3. Hyperbolic equations (such as the wave equation) are not degenerate elliptic.

Viscosity solutions have usually very good existence, uniqueness and stability properties. For detailed theorems and suitable assumptions we refer to [**?**, **?**, **?**, **?**, **?**, **?**, **?**] which give a complete overview on the theory from different point of views.

We can now use the theory of viscosity solutions to chartacterize concave and semiconcave functions.

***Theorem ([?])*** Let $u$ lower semicontinuous on $\Omega \subset \mathbb{R}^n$ open, then $u$ is concave, if and only if, $-D^2u(x)\geq 0$, in the viscosity sense, in $\Omega$. This means that $D^2\phi(x_0)\geq 0$ for all smooth $\phi \in C^2\left(\Omega\right)$ such that $u-\varphi$ has a local minimum at some point $x_0 \in \Omega$ (viscosity supersolution property). □

The previous theorem can be applied to semiconcave functions: this means that, given $u$ lower semicontinuous on $\Omega \subset \mathbb{R}^n$ open, then $u$ is semiconcave, if and only if,

$$-D^2u(x)\geq -C\,Id_{n\times n}, \quad \text{in the viscosity sense, in } \Omega.$$

## 3 Carnot groups.

In this session we are going to give a brief overview on Carnot groups, recalling in particular the main definitions and properties. In particular we focus on the gauge (also called homogeneous) distance and the Carnot-Carathéodory (shortly CC) distance.

***Definition*** A Carnot group $\left(\mathbb{G},\cdot\right)$ is a simply connected nilpotent Lie group, whose Lie algebra g has a finite stratification, i.e.

$$g = \bigoplus_{j=1}^{r} g_j, \quad \text{i.e.} \begin{cases} [g_1, g_j] = g_{j+1}, \ \forall j = 1, \dots, r-1, \\ [g_1, g_r] = \{0\}. \end{cases}$$

If $g_r \neq \{0\}$, then $r$ is the step of the Carnot group $\mathbb{G}$. □

***Example*** The $n$-dimensional Heisenberg group $\mathbb{H}^n$ is a step 2 Carnot group defined on $\mathbb{R}^{2n+1}$ by the following non-abelian operation:

$$p \cdot q = \left( \bar{x}_1 + \bar{y}_1, \bar{x}_2 + \bar{y}_2, x_3 + y_3 + \frac{\langle \bar{x}_1, \bar{y}_2 \rangle - \langle \bar{x}_2, \bar{y}_1 \rangle}{2} \right),$$

where $p = (\bar{x}_1, \bar{x}_2, x_3), q = (\bar{y}_1, \bar{y}_2, y_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, when $+$ denotes the Euclidean sum and and $\langle \ , \ \rangle$ the standard inner product in $\mathbb{R}^n$.

For all $\lambda > 0$, we consider the automorphism $\Delta_\lambda : g \to g$ defined by $\Delta_\lambda(X) = \lambda^i X$, for all $X \in g_i$ and $i = 1, \dots, r$. By using the finite stratification of g, we can deduce:

$$\Delta_\lambda(X) = \lambda X_1 + \cdots + \lambda^r X_r, \quad \text{for } X = X_1 + \cdots + X_r \text{ and } X_i \in g_i.$$

On a Carnot group $\mathbb{G}$, the exponential map is a diffeomorphism from g to $\mathbb{G}$ (see [**?**] for a definition and properties). Therefore we can define the dilations on the group $\mathbb{G}$ as the smooth map $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ given by $\delta_\lambda = exp \circ \Delta_\lambda \circ exp^{-1}$, where by $exp$ we indicate the exponential map. Note that the dilations give a family of anisotropic scalings whenever $r \geq 2$.

Using again the stratification of g and the exponential map, we can introduce a gauge (or homogeneous) norm as

$$\|x\|_G := \left( \sum_{i=1}^{r} |\bar{x}_i|^{\frac{2r!}{i}} \right)^{\frac{1}{2r!}}, \quad x = (\bar{x}_1, \dots, \bar{x}_r) \in \mathbb{G}.$$

The gauge norm rescales with the dilations: $\|\delta_\lambda(x)\|_G = \lambda \|x\|_G$. Note that $\|x\|_G^{2r!}$ is a polynomial, so it is always smooth (in the standard sense). As already remarked smooth functions are trivially both semiconvex and semiconcave in every bounded set. Given $p, q \in \mathbb{G}$, the gauge distance is $d_G(p,q) = \left\| q^{-1} \cdot p \right\|_G$.

To introduce the CC distance on Carnot groups we need to consider the sub-Riemannian manifold structure of canonical left-invariant vector fields defined on Carnot groups. We refer to [**?**, **?**] for details. Here we just recall that we can identify the horizontal distribution $\mathscr{D} \subset T\mathbb{G}$ generated by $g_1$ by using a suitable family of left invariant vector fields $X_1, \dots, X_m$ where $m = \dim g_1$.

The key notion necessary to introduce the CC distance is the one of admissible (horizontal) curves: a horizontal curve $\gamma$ is every absolutely continuous curve defined on the group, such that $\dot{\gamma}(t) \in \mathscr{D}_{\gamma(t)}$, a.e. $t$, that means $\dot{\gamma}(t) = \alpha_1(t) X_1(\gamma(t)) + \cdots + \alpha_m(t) X_m(\gamma(t))$, for $\alpha_1, \dots, \alpha_m$ suitable measurable functions and $X_1, \dots, X_m$

generating $\mathscr{D}$. On $\mathscr{D}$ we can define a Riemannian metric $g(\cdot,\cdot)$, so for horizontal curves, the length can be introduced as $l(\gamma)=\int_0^1\sqrt{g(\dot\gamma(t),\dot\gamma(t))}\,dt$.

***Definition*** Given $p,q\in\mathbb{G}$, the Carnot-Carathéodory (CC) distance is

$$d(p,q)=\inf\left\{l(\gamma)\,|\,\gamma:[0,1]\to\mathbb{G},\text{ horizontal }\gamma(0)=p,\gamma(1)=q\right\}.$$

□

## 4 h-semiconcavity and the square CC distance.

In this section we recall how to adapt the previous definitions and properties of concave and semiconcave functions to Carnot groups, and in particular the viscosity characterization by the intrinsic (horizontal) second derivatives. The horizontal convexity (shortly h-convexity) was first introduced in [**?**, **?**, **?**], and later generalised to the case of general Hörmader vector fields in [**?**, **?**]. We refer also to [**?**] for an overview on the subject. As done for the Euclidean counterpart, in this section, we write all definitions and properties directly for the h-concave, respectively h-semiconcave, case.

***Definition*** Given $\Omega\subset\mathbb{G}$ open and $u$ lower semicontinuous, $u$ is h-concave if and only if

$$u(p\cdot h)+u(p\cdot h^{-1})-2u(p)\le 0,$$

$\forall\,p\in\Omega, h\in\mathscr{H}_0=\mathscr{D}_0$ such that $[p\cdot h^{-1},p\cdot h]:=\{p\cdot\tau h\,|\,\tau\in[-1,1]\}\subset\Omega$. □

In step 2 Carnot groups, (Euclidean) concavity implies h-concavity.

Recall that, given $\mathscr{X}=\{X_1,\dots,X_m\}$ a generic family of (left-invariant) vector fields, the intrinsic gradient is given by

$$D_{\mathscr{X}}u=(X_1u,\dots,X_mu)^t=\sigma Du,$$

where $\sigma=[X_1,\dots,X_m]^t$ is the $m\times n$ matrix associated to the family $\mathscr{X}$.

The intrinsic Hessian is given by the $m\times m$-matrix

$$D^2_{\mathscr{X}}u=\big(X_i(X_ju)\big)_{i,j=1}^m.$$

Since in general $D^2_{\mathscr{X}}u$ may not be symmetric, we consider the symmetrized matrix

$$\big(D^2_{\mathscr{X}}u\big)^*=\left(\frac{X_i(X_ju)+X_j(X_iu)}{2}\right)_{i,j=1}^m=\sigma D^2u\,\sigma^t+\left(\frac{\nabla_{X_j}X_i+\nabla_{X_i}X_j}{2}\cdot Du\right)_{i,j=1}^m.$$

Using the horizontal derivatives, we can generalise the viscosity characterization for concave functions to the case of h-concave functions. The following theorem has been proved in Carnot groups in [?, ?, ?] and later generalized to the case of Hörmander vector fields in [?].

***Theorem*** Let $u \in C(\mathbb{R}^n)$, then $u$ is h-concave, if and only if, $-\left(D^2_{\mathscr{X}} u\right)^* \geq 0$, in the viscosity sense, in $\mathbb{R}^n$. □

The previous definition and result can be generalised to the smiconcave case.

***Definition*** $u : \mathbb{G} \to \mathbb{R}$ is h-semiconcave if $\exists\, C > 0$ such that $\forall p \in \mathbb{G}$ and $\forall h \in \mathscr{H}_0$, $u(p \cdot h) + u(p \cdot h^{-1}) - 2u(p) \leq C|h|^2$. Note that $h \in \mathscr{H}_0 \Leftrightarrow h = (h_1, \dots h_m, 0, \dots, 0)$ with $m = dim(\mathrm{g}_1)$, that implies $\|h\|_G = |h|$ (Euclidean norm). □

By using the previous theorem, it is trivial to show that, given $u : \mathbb{G} \to \mathbb{R}$ lower semicontinuous, $u$ is h-semiconcave if and only if $-\left(D^2_{\mathscr{X}} u\right)^* \geq -C\, Id_{m\times m}$ in the viscosity sense.

By using the viscosity characterization, in [?] we could prove that the square CC distance from the origin is h-semiconcave in the whole space. First we want to recall that in [?, ?] it was proved that in ideal group the CC distance from the origin is locally Euclidean semiconcave outside the origin. Since ideal groups are a subfamily of step 2 Carnot groups, this result implies local $h$-semiconcavity outside the origin. While the dilations can be used to extend the horizontal (not the Euclidean) property at infinity, the result can be proved to be false at the origin (see [?] for more details). Nevertheless the result can be used to prove the following theorem in ideal group. For non ideal step 2 groups, one can instead use the recently developed notion of C-nearly horizontal semiconcavity and the related results in [?].

***Theorem ([?])*** In all step 2 groups $\mathbb{G}$, $d^2(p,0)$ is $h$-semiconcave in $\mathbb{G}$. □

Note that the result is optimal in the sense that we can show that in the Engel group (simplest example step 3 Carnot group) no powers of the CC distance are h-semiconcave near the origin. Moreover, differently from the Euclidean case, one can show that in the Heisenberg group the square CC distance from the origin is not h-semiconvex.

In [?] the previous theorem, together with the stability of viscosity supersolution w.r.t. infimum, have also been used to prove h-semiconcavity properties for the viscosity solutions of a large class of degenerate Hamilton-Jacobi equations, by using the metric Hopf-Lax formula developed in [?, ?].

## References


1. Alvarez O., Lasry, J.-M., Lions, P.-L.: Convex viscosity solutions and state constraints. Journal de Mathématiques Pures et Appliquées **76** (3), 265–288 (1997).
2. Badreddine Z., Rifford, L.: Measure contraction properties for two-step analytic sub-Riemannian structures and Lipschitz Carnot groups. Ann. Inst. Fourier **70** (6), 2303–2330 (2020).
3. Bardi, M., Capuzzo Dolcetta, I.: Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations. Birkhäuser, Basel (1997).

4. Bardi, M., Dragoni, F.: Convexity and semiconvexity along vector fields. Calculus of Variations and Partial Differential Equations **42** (3-4) , 405–427 (2011).
5. Bardi, M., Dragoni, F.: Subdifferential and properties of convex functions with respect to vector fields. Journal of Convex Analysis **21** (3), 785–810 (2014).
6. Barles, G.: Solutions de Viscosité des Equations de Hamilton-Jacobi. Springer-Verlag, Berlin (1994).
7. Bonfiglioli, A., Lanconelli, E., Uguzzoni, F.: Stratified Lie Groups and Potential Theory for their Sub-Laplacians Springer-Verlag, Berlin (2007).
8. Caffarelli, L., Cabré, X.: Fully Nonlinear Elliptic Equations. American Mathematical Society, Providence (1995).
9. Cannarsa P., Rifford, L.: Semiconcavity results for optimal control problems admitting no singular minimizing controls. Ann. Inst. H. Poincaré C Anal. Non Linéaire, **25** (4), 773–802 (2008).
10. Cannarsa, P., Sinestrari, C.: Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control. Birkhäuser, Basel (2004).
11. Crandall, M.G., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bullet AMS **27** (1), 1–67 (1992).
12. Crandall, M.G., Lions, P.L.: Condition d'unicité pour les solutions generalisées des équations de Hamilton-Jacobi du premier ordre. Acad. Sci. Paris Sér. I Math **292**, 183–186. (1981).
13. Crandall, M.G., Lions, P.L.: Viscosity solutions of Hamiltonian-Jacobi equations. Trans. Am. Math. Soc. **277**, 1–42 (1983).
14. Danielli, D., Garofalo, N., Nhieu, D.-M.: Notions of convexity in Carnot groups. Comm. Anal. Geom. **11** (2), 263–341 (2003).
15. Dragoni, F.: Metric Hopf-Lax formula with semicontinuous data. Discrete and Continuous Dynamical Systems **17** (4), 713–729 (2007).
16. Dragoni, F.: Introduction to viscosity solutions for nonlinear PDEs (2012). Available online.
17. Dragoni, F., Filali, D.: Starshaped and convex sets in Carnot groups and in the geometries of vector fields. Journal of Convex Analysis **4** (24), 1349–1372 (2019).
18. Dragoni, F., Liu, Q., Zhang Y.: Horizontal semiconcavity for the square of Carnot-Carathéodory distance on step 2 Carnot groups and applications to Hamilton-Jacobi equations. Preprint (2024).
19. Evans, L.C.: Partial Differential Equations. American Mathematical Society, Providence (2010).
20. Figalli, A., Rifford L.: Mass transportation on sub-Riemannian manifolds. Geom. Funct. Anal. **20** (1),124–159 (2010).
21. Giga, Y.: Surface Evolution Equations A Level Set Approach. Birkhäuser, Basel (2002).
22. Juutinen, P., Lu, G., Manfredi, J., Stroffolini, B.: Convex functions on Carnot groups. Rev. Mat. Iberoamericana **23** (1), 191–200 (2007).
23. Lu, G., Manfredi, J., Stroffolini, B.: Convex functions in Heisenberg. Group. Calculus of Variations **19**, 1—22 (2004).
24. Manfredi, J., Stroffolini, B.: A version of the Hopf–Lax formula in the Heisenberg group. Communications in Partial Differential Equations **27**, 1139–1159 (2002).
25. Montgomery, R.: A Tour of Subriemannian Geometries, Their Geodesics and Application. American Mathematical Society, Providence (2002).
26. Wang, C.: Viscosity convex functions on Carnot Groups. Proc. Amer. Math. Soc. **133**, 1247–1253 (2005).