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Sparsity and proximity transference in integer programming

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Abstract

We obtain new transference bounds that connect the additive integrality gap and sparsity of solutions for integer linear programs. Specifically, we consider the integer programs $\min\{\boldsymbol{c} \cdot \boldsymbol{x} : \boldsymbol{x} \in P \cap \mathbb{Z}^n\}$, where $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is a polyhedron in the standard form determined by an integer $m \times n$ matrix $\boldsymbol{A}$ and an integer vector $\boldsymbol{b}$. The main result of the paper gives an upper bound for the integrality gap that drops exponentially in the size of the support of the optimal solutions corresponding to the vertices of the integer hull of P. Additionally, we obtain a new proximity estimate for the ℓ_2-distance from a vertex of P to its nearest integer point in P. We also strengthen previously known bounds for the integer Carathéodory rank, a key sparsity characteristic which estimates the minimum size of the support of an integer point in P in terms of the matrix $\boldsymbol{A}$. The proofs make use of the results from the geometry of numbers and convex geometry.

Keywords Integrality gap · Proximity · Sparsity

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1 Introduction and main results

The proximity and sparsity of solutions to integer programs are well-established directions of research in the theory of mathematical programming.

Proximity-type results study approximations of the solutions to integer programs by the solutions of linear programming relaxations. This is a traditional research topic with the first contributions dated back at least to Gomory [19, 20]. The most influential results in this area also include the proximity bounds by Cook et al. [15] and by Eisenbrand and Weismantel [18]. For more recent contributions, we refer the reader to Celaya et al. [13], Lee et al. [21, 22], and Paat et al. [24].

Sparsity-type results study the size of support of solutions to integer programs. This area of research takes its origin from the integer Carathéodory theorems of Cook et al. [14] and Sebő [26] and later major contributions by Eisenbrand and Shmonin [17]. In recent years, this topic has been studied in numerous papers, including Abdi et al. [1], Aliev et al. [2, 4, 5], Berndt et al. [7], Dubey and Liu [16] and Oertel et al. [23].

Recent works of Lee et al. [21] and Aliev et al. [3] show that the proximity and sparsity areas are highly interconnected. The paper [3] establishes *transference bounds* that link both areas in two *special cases*: for corner polyhedra and knapsacks with positive entries. Remarkably, this gives a drastic improvement on the previously known proximity bounds for knapsacks obtained in [6].

In this paper, we establish the first transference bounds that involve the integrality gap of integer linear programs and hold in the *general case*, addressing a future research question posed in [3]. The proofs explore a new geometric approach that combines Minkowski's geometry of numbers and box slicing inequalities.

Firstly, we will introduce the proximity and sparsity results most relevant to the contributions of this paper. Let $\boldsymbol{A} \in \mathbb{Z}^{m\times n}$ with $m < n$ and $\boldsymbol{b} \in \mathbb{Z}^m$. We define the polyhedron

$$P(\boldsymbol{A}, \boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$$

and assume that $P(\boldsymbol{A}, \boldsymbol{b})$ contains integer points. Given a cost vector $\boldsymbol{c} \in \mathbb{R}^n$, we consider the integer linear programming problem

$$\min\{\boldsymbol{c} \cdot \boldsymbol{x} : \boldsymbol{x} \in P(\boldsymbol{A}, \boldsymbol{b}) \cap \mathbb{Z}^n\}, \tag{1}$$

where $\boldsymbol{c} \cdot \boldsymbol{x}$ stands for the standard inner product. In this paper, we assume that (1) is feasible and bounded. A very successful and traditional approach for solving optimisation problems of the form (1) is based on solving their linear programming relaxations

$$\min\{\boldsymbol{c} \cdot \boldsymbol{x} : \boldsymbol{x} \in P(\boldsymbol{A}, \boldsymbol{b})\}, \tag{2}$$

obtained by dropping the integrality constraint. Subsequently, various methods are used to construct a feasible integer solution $\boldsymbol{z}^*$ to (1) from a fractional solution $\boldsymbol{x}^*$ of (2). In this setting, the proximity-type results play a central role by providing estimates for the size of a region containing $\boldsymbol{z}^*$ in terms of various parameters of the matrix $\boldsymbol{A}$.

We will now introduce the notation needed to state the proximity results. Let $[n] = \{1, \dots, n\}$, let $\binom{[n]}{k}$ be the set of all k-element subsets of $[n]$, and for $\tau = \{i_1, \dots, i_k\} \in \binom{[n]}{k}$ with $i_1 < \dots < i_k$, let $\boldsymbol{A}_\tau$ denote the $m \times k$ submatrix of $\boldsymbol{A}$ with columns indexed by τ. In the same manner, given $\boldsymbol{x} \in \mathbb{R}^n$, we will denote by $\boldsymbol{x}_\tau$ the vector $(x_{i_1}, \dots, x_{i_k})^\top$.

Without loss of generality, we assume that $\boldsymbol{A}$ has rank m and, for $1 \le r \le m$, denote by $\Delta_r(\boldsymbol{A})$ the maximum absolute value of an $r \times r$ subdeterminant of A, that is

$$\Delta_r(\boldsymbol{A}) = \max\{|\det \boldsymbol{B}| : \boldsymbol{B} \text{ is an } r \times r \text{ submatrix of } \boldsymbol{A}\}.$$

Further, by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ we denote the ℓ_1 and ℓ_∞ norms, respectively.

Let $\boldsymbol{x}^*$ be a vertex optimal solution for (2). The sensitivity theorems of Cook et al. [15] (see also Celaya et al. [13] for further improvements) imply existence of an optimal solution $\boldsymbol{z}^*$ to (1) with

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \le (n - m)\Delta_m(\boldsymbol{A}). \tag{3}$$

In the same setting, a more recent result of Eisenbrand and Weismantel [18] gives the estimate

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_1 \le m(2m\Delta_1(\boldsymbol{A}) + 1)^m, \tag{4}$$

which is, remarkably, independent of the dimension n.

Let $\mathrm{IP}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ and $\mathrm{LP}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ denote the optimal values of (1) and (2), respectively. The main focus of this paper is on estimating the *(additive) integrality gap* $\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ defined as

$$\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \mathrm{IP}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) - \mathrm{LP}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}).$$

The integrality gap is a fundamental proximity characteristic of the problem (1) extensively studied in the literature. Upper bounds for $\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ appear already in the work of Blair and Jeroslow [8, 9].

In view of (3), the additive integrality gap satisfies the bound

$$\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \le \|\boldsymbol{c}\|_1 (n - m)\Delta_m(\boldsymbol{A}). \tag{5}$$

The estimate (4), in its turn, gives the bound

$$\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \le \|\boldsymbol{c}\|_\infty m(2m\Delta_1(\boldsymbol{A}) + 1)^m. \tag{6}$$

In the special cases, such as the knapsack scenario $m = 1$ studied in [3, 6], stronger proximity bounds are known.

To introduce the sparsity results relevant to this paper, we will need the following notation. Given a vector $\boldsymbol{x} = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$, we will denote by $\mathrm{supp}(\boldsymbol{x})$ the *support* of $\boldsymbol{x}$, that is $\mathrm{supp}(\boldsymbol{x}) = \{i : x_i \neq 0\}$. To measure the size of the support, we use the 0-*norm* $\|\boldsymbol{x}\|_0 = |\mathrm{supp}(\boldsymbol{x})|$, widely used in the theory of compressed sensing [10, 11].

Let us consider the size of the support of solutions to the system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{x}$ is restricted to a structured domain $D \subset \mathbb{R}^n$. The sparsest solutions are optimal solutions of the optimisation problem

$$\min \{\|\boldsymbol{x}\|_0 \,:\, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \in D\}\,. \tag{7}$$

The 0-norm minimisation problem (7) is central in the theory of the compressed sensing, where for the classical choice $D = \mathbb{R}^n$ an appropriate linear programming relaxation of (7) provides a guaranteed approximation [10, 11]. The case $D = \mathbb{Z}^n_{\geq 0}$, relevant to the integer programming setting, has been extensively studied in [2, 4, 5, 17] and other works.

When $D = \mathbb{R}^n$ and $D = \mathbb{R}^n_{\geq 0}$, the tight upper bounds on (7) in terms of $\boldsymbol{A}$ are given by the rank of $\boldsymbol{A}$, which follows from basic linear algebra and the well-known Carathéodory's theorem from convexity, respectively. The paper Aliev et al. [2] introduced the *integer Carathéodory rank* $\mathsf{ICR}(\boldsymbol{A})$ of the matrix $\boldsymbol{A}$, defined as the tight upper bound on the optimisation problem (7) with $D = \mathbb{Z}^n_{\geq 0}$ in terms of $\boldsymbol{A}$. Specifically,

$$\mathsf{ICR}(\boldsymbol{A}) = \max_{\boldsymbol{y} \in \mathbb{Z}^n_{\geq 0}} \min\{\|\boldsymbol{x}\|_0 : \boldsymbol{x} \in P(\boldsymbol{A}, \boldsymbol{A}\boldsymbol{y}) \cap \mathbb{Z}^n\}.$$

Let

$$\Delta(\boldsymbol{A}) = \sqrt{\det \boldsymbol{A}\boldsymbol{A}^\top}.$$

Geometrically, $\Delta(\boldsymbol{A})$ is the m-dimensional volume of the parallelepiped determined by the rows of $\boldsymbol{A}$. Note that

$$\Delta(\boldsymbol{A}) = \sqrt{\sum_{\tau \in \binom{[n]}{m}} \det(A_\tau)^2} \tag{8}$$

by the Cauchy-Binet formula. Let further $\gcd(\boldsymbol{A})$ be the greatest common divisor of all $m \times m$ subdeterminants of $\boldsymbol{A}$.

The results of Aliev et al. [4] imply the general bound

$$\mathsf{ICR}(\boldsymbol{A}) \leq m + \left\lfloor \log_2 \left(\frac{\Delta(\boldsymbol{A})}{\gcd(\boldsymbol{A})} \right) \right\rfloor. \tag{9}$$

In what follows, we denote by $\mathrm{cone}(\boldsymbol{A})$ the cone generated by the columns of the matrix $\boldsymbol{A}$. Recall that a cone C is *pointed* if $C \cap (-C) = \{\boldsymbol{0}\}$. Assuming that $\mathrm{cone}(\boldsymbol{A})$ is pointed, let

$$q(\boldsymbol{A}) = \min_j \sqrt{\sum_{\substack{\sigma \subset [n]: \\ |\sigma| = m,\, j \in \sigma}} (\det(\boldsymbol{A}_\sigma))^2}, \tag{10}$$

where the minimum in (10) is taken over the indices j which correspond to the extreme rays of cone$(\boldsymbol{A})$. Note that $q(\boldsymbol{A}) \leq \Delta(\boldsymbol{A})$ by (8). Theorem 3 in [2] shows that (9) can be strengthened by replacing $\Delta(\boldsymbol{A})$ with $q(\boldsymbol{A})$. That is the bound

$$\mathsf{ICR}(\boldsymbol{A}) \leq m + \left\lfloor \log_2 \left(\frac{q(\boldsymbol{A})}{\gcd(\boldsymbol{A})} \right) \right\rfloor \tag{11}$$

holds.

We will now state the main results of this paper. Observe that the integrality gap is positive homogeneous of degree one in $\boldsymbol{c}$, that is for $t > 0$

$$\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, t\boldsymbol{c}) = t\,\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}). \tag{12}$$

In what follows, we use the notation $\|\cdot\|_2$ for the ℓ_2 norm. In view of (12), we may assume without loss of generality that $\boldsymbol{c}$ is a unit vector, that is $\|\boldsymbol{c}\|_2 = 1$.

Given a set $K \subset \mathbb{R}^n$ we will denote by conv(K) the convex hull of K. The polyhedron

$$P_I(\boldsymbol{A}, \boldsymbol{b}) = \mathrm{conv}(P(\boldsymbol{A}, \boldsymbol{b}) \cap \mathbb{Z}^n)$$

is traditionally referred to as the *integer hull* of $P(\boldsymbol{A}, \boldsymbol{b})$. Given an optimal solution $\boldsymbol{z}^*$ to (1) which is a vertex of the integer hull $P_I(\boldsymbol{A}, \boldsymbol{b})$, we obtain transference bounds that link the integrality gap IG$(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ with the size of the support of $\boldsymbol{z}^*$.

Theorem 1 *Let $\boldsymbol{A} \in \mathbb{Z}^{m\times n}$, $m < n$, be a matrix of rank m, $\boldsymbol{b} \in \mathbb{Z}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ be a unit cost vector. Suppose that (1) is feasible and bounded. Let $\boldsymbol{z}^*$ be an optimal solution to (1) which is a vertex of $P_I(\boldsymbol{A}, \boldsymbol{b})$. Then*

$$\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \leq \frac{s}{2^{s-m-1}} \cdot \frac{\Delta(\boldsymbol{A})}{\gcd(\boldsymbol{A})}, \tag{13}$$

where $s = \|\boldsymbol{z}^*\|_0$.

Hence, we obtain an upper bound for the integrality gap that drops *exponentially* in the size of the support of any optimal solution to (1) which is a vertex of the integer hull $P_I(\boldsymbol{A}, \boldsymbol{b})$. In this vein, we remark that [3] gives optimal transference bounds for positive knapsacks and corner polyhedra that connect the ℓ_∞-distance proximity and the size of the support of integer feasible solutions. Theorem 1 applies in the general case $m \geq 1$ and a different setting; it connects the integrality gap and the size of the support of optimal solutions to (1).

Next, we obtain from Theorem 1 transference bounds in terms of $\Delta_m(\boldsymbol{A})$ and $\Delta_1(\boldsymbol{A})$.

Corollary 2 *Assume the conditions of Theorem 1. Then the bounds*

$$\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \leq \frac{s\binom{s+m}{m}^{1/2}}{2^{s-m-1}} \cdot \frac{\Delta_m(\boldsymbol{A})}{\gcd(\boldsymbol{A})} \tag{14}$$

and

$$\mathrm{IG}(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}) \le \frac{s(s+m)^{m/2}}{2^{s-m-1}} \cdot \frac{(\Delta_1(\boldsymbol{A}))^m}{\gcd(\boldsymbol{A})} \tag{15}$$

hold.

For a unit cost vector $\boldsymbol{c}$, the bound (14) improves on (5) when $s \ge 4m$, and the bound (15) improves on (6) when $s \ge 6m$.

The proof of Theorem 1 makes use of results from convex geometry and Minkowski's geometry of numbers. Exploring this geometric approach, we obtain the following new proximity and sparsity bounds. Firstly, we estimate the ℓ_2-distance from a vertex of the polyhedron $P(\boldsymbol{A},\boldsymbol{b})$ a nearest integer point in $P(\boldsymbol{A},\boldsymbol{b})$.

Theorem 3 *Let $\boldsymbol{A} \in \mathbb{Z}^{m\times n}$, $m < n$, be a matrix of rank m, $\boldsymbol{b} \in \mathbb{Z}^m$, and suppose that $P(\boldsymbol{A},\boldsymbol{b})$ contains integer points. Let $\boldsymbol{x}^*$ be a vertex of $P(\boldsymbol{A},\boldsymbol{b})$. There exists an integer point $z^* \in P(\boldsymbol{A},\boldsymbol{b})$ such that*

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_2 \le \frac{\Delta(\boldsymbol{A})}{\gcd(\boldsymbol{A})} - 1\,. \tag{16}$$

We remark that for certain matrices $\boldsymbol{A}$ the bound (16) is smaller than the ℓ_2-norm proximity bounds $\sqrt{n}(n-m)\Delta_m(\boldsymbol{A})$ and $m(2\,m\Delta_1(\boldsymbol{A})+1)^m$ that can be derived from [13] and [18], respectively. It is sufficient to observe that the ratio $\Delta(\boldsymbol{A})/\Delta_m(\boldsymbol{A})$ can be arbitrarily close to one and the ratio $\Delta(\boldsymbol{A})/\Delta_1(\boldsymbol{A})$ can be arbitrarily small. For instance, we can consider the sequence of matrices $\boldsymbol{A}_t = (t\boldsymbol{I}_m|\boldsymbol{B})$ where $\boldsymbol{I}_m$ is the $m\times m$ identity matrix and $\boldsymbol{B}$ is a fixed $m\times(n-m)$ integer matrix. Then $\Delta(\boldsymbol{A}_t)/\Delta_m(\boldsymbol{A}_t) \to 1$ as $t\to\infty$. Further, the quantity $\Delta(\boldsymbol{A})$ is the determinant of the lattice generated by the rows of $\boldsymbol{A}$. The same lattice can be generated by the rows of a matrix $\widetilde{\boldsymbol{A}}$ with arbitrarily large $\Delta_1(\widetilde{\boldsymbol{A}})$. Hence the ratio $\Delta(\widetilde{\boldsymbol{A}})/\Delta_1(\widetilde{\boldsymbol{A}})$ can be made arbitrarily small.

Applying Theorem 3 to a vertex optimal solution $\boldsymbol{x}^*$ of (2) we obtain the following estimate.

Corollary 4 *Let $\boldsymbol{A} \in \mathbb{Z}^{m\times n}$, $m < n$, be a matrix of rank m, $\boldsymbol{b} \in \mathbb{Z}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ be a unit cost vector. Suppose that (1) is feasible and bounded. Then the bound*

$$\mathrm{IG}(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}) \le \frac{\Delta(\boldsymbol{A})}{\gcd(\boldsymbol{A})} - 1 \tag{17}$$

holds.

For a unit cost vector $\boldsymbol{c}$, the bound (17) improves (5) when $\Delta(\boldsymbol{A})/\gcd(\boldsymbol{A}) < (n-m)\Delta_m(\boldsymbol{A})+1$ and improves (6) when $\Delta(\boldsymbol{A})/\gcd(\boldsymbol{A}) < mn^{-1/2}(2\,m\Delta_1(\boldsymbol{A})+1)^m+1$.

Next, we use the geometric tools developed for the proof of Theorem 1 to obtain a new estimate for the integer Carathéodory rank that generalises Theorem 4 in [2] and strengthens the bound (11).

Theorem 5 *Let $\boldsymbol{A} \in \mathbb{Z}^{m\times n}$, $m < n$, be a matrix of rank m. Assume that* $\operatorname{cone}(\boldsymbol{A})$ *is pointed. Then*

$$\mathsf{ICR}(\boldsymbol{A}) \leq m + \left\lfloor \log_2 \left(\frac{\mu(\boldsymbol{A})}{\gcd(\boldsymbol{A})} \right) \right\rfloor, \tag{18}$$

where

$$\mu(\boldsymbol{A}) = \min_{j\in[n]} \sqrt{\sum_{\substack{\sigma\subset[n]:\\ |\sigma|=m,\, j\in\sigma}} (\det(\boldsymbol{A}_\sigma))^2}. \tag{19}$$

Theorem 4 in [2] gives the bound (18) in the knapsack scenario $m = 1$. The bound (11), in its turn, restricts the minimum in (19) to the indices $j \in [n]$ which correspond to the extreme rays of the cone generated by the columns of $\boldsymbol{A}$. The ratio $\mu(\boldsymbol{A})/q(\boldsymbol{A})$ can be arbitrarily small. For instance, consider the sequence of matrices $\boldsymbol{A}_t = (t\boldsymbol{I}_m|\boldsymbol{B})$ where $\boldsymbol{B}$ is a fixed $m \times (n-m)$ integer matrix with positive entries. Observe that $(\mu(\boldsymbol{A}_t))^2$ is a polynomial in t and its degree is $2(m-1)$ for sufficiently large t. Similarly, $(q(\boldsymbol{A}_t))^2$ is a polynomial in t of degree $2m$ for sufficiently large t. Hence $\mu(\boldsymbol{A}_t)/q(\boldsymbol{A}_t) \to 0$ as $t \to \infty$.

Note also that Proposition 3 in [2] implies that the bound (18) is optimal.

2 Volumes and linear transforms

In this section, we develop geometric tools needed for the proof of the transference bounds.

For a matrix $\boldsymbol{A} \in \mathbb{R}^{l\times r}$ we denote by $\operatorname{lin}(\boldsymbol{A})$ the linear subspace of $\mathbb{R}^l$ spanned by the columns of A. Given a set $M \subset \mathbb{R}^r$ we let $\boldsymbol{A}M = \{\boldsymbol{Ax} \in \mathbb{R}^l : \boldsymbol{x} \in M\}$ and use the notation $[\boldsymbol{A}] = \boldsymbol{A}[0,1]^r$. For a set $X \subset \mathbb{R}^l$ and a linear subspace L of $\mathbb{R}^l$, we denote by $X|L$ the orthogonal projection of X onto L. Further, $\operatorname{vol}_i(\cdot)$ denotes the i-dimensional volume.

Let S be an $(l-k)$-dimensional subspace of $\mathbb{R}^l$. Consider an orthonormal basis $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{l-k}, \boldsymbol{s}_{l-k+1}, \ldots, \boldsymbol{s}_l$ of $\mathbb{R}^l$ such that the first $l-k$ vectors form a basis of S. Let further $\boldsymbol{S}_{l-k} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{l-k}) \in \mathbb{R}^{l\times(l-k)}$ and $\boldsymbol{S}_k = (\boldsymbol{s}_{l-k+1}, \ldots, \boldsymbol{s}_l) \in \mathbb{R}^{l\times k}$.

Given a measurable set M in the subspace S, we are interested in the $(l-k)$-dimensional volume of its image $\boldsymbol{D}M$, where we assume that $\boldsymbol{D} \in \mathbb{R}^{l\times l}$ is an invertible matrix. The first result gives a general expression for $\operatorname{vol}_{l-k}(\boldsymbol{D}M)$ in terms of $\operatorname{vol}_{l-k}(M)$.

Lemma 1 *Let S be an $(l-k)$-dimensional subspace of $\mathbb{R}^l$. Let $M \subset S$ be measurable, $\boldsymbol{D} \in \mathbb{R}^{l\times l}$ be nonsingular and let the rows of $\boldsymbol{B} \in \mathbb{R}^{k\times l}$ form a basis of the subspace $(\boldsymbol{D}S)^\perp$, the orthogonal complement of the subspace $\boldsymbol{D}S = \operatorname{lin}(\boldsymbol{D}\boldsymbol{S}_{l-k})$. Then*

$$\operatorname{vol}_{l-k}(\boldsymbol{D}M) = |\det(\boldsymbol{D})| \sqrt{\frac{\det(\boldsymbol{B}\boldsymbol{B}^\top)}{\det(\boldsymbol{B}\boldsymbol{D}\boldsymbol{D}^\top\boldsymbol{B}^\top)}} \operatorname{vol}_{l-k}(M).$$

Proof By the elementary properties of volume, we have

$$\mathrm{vol}_{l-k}(\boldsymbol{D}M) = \mathrm{vol}_{l-k}([\boldsymbol{D}\boldsymbol{S}_{l-k}])\mathrm{vol}_{l-k}(M). \tag{20}$$

On the other hand, we have

$$\begin{aligned} |\det(\boldsymbol{D})| &= \mathrm{vol}_l([\boldsymbol{D}(\boldsymbol{S}_{l-k}, \boldsymbol{S}_k)]) \\ &= \mathrm{vol}_{l-k}([\boldsymbol{D}\boldsymbol{S}_{l-k}])\mathrm{vol}_k([\boldsymbol{D}\boldsymbol{S}_k]|\mathrm{lin}(\boldsymbol{D}\boldsymbol{S}_{l-k})^{\perp}) \\ &= \mathrm{vol}_{l-k}([\boldsymbol{D}\boldsymbol{S}_{l-k}])\mathrm{vol}_k([\boldsymbol{D}\boldsymbol{S}_k]|\mathrm{lin}(\boldsymbol{B}^{\top})). \end{aligned} \tag{21}$$

Now, according to the definition of projections, we have

$$[\boldsymbol{D}\boldsymbol{S}_k]|\mathrm{lin}(\boldsymbol{B}^{\top}) = \boldsymbol{B}^{\top}(\boldsymbol{B}\boldsymbol{B}^{\top})^{-1}\boldsymbol{B}[\boldsymbol{D}\boldsymbol{S}_k]$$

and hence, recalling that the columns of $\boldsymbol{S}_k$ are part of an orthonormal basis, we have

$$\begin{aligned} \mathrm{vol}_k([\boldsymbol{D}\boldsymbol{S}_k]|\mathrm{lin}(\boldsymbol{B}^{\top})) &= \sqrt{\det(\boldsymbol{S}_k^{\top}\boldsymbol{D}^{\top}\boldsymbol{B}^{\top}(\boldsymbol{B}\boldsymbol{B}^{\top})^{-1}\boldsymbol{B}\boldsymbol{B}^{\top}(\boldsymbol{B}\boldsymbol{B}^{\top})^{-1}\boldsymbol{B}\boldsymbol{D}\boldsymbol{S}_k)} \\ &= \sqrt{\det((\boldsymbol{B}\boldsymbol{B}^{\top})^{-1})}\sqrt{\det(\boldsymbol{B}\boldsymbol{D}\boldsymbol{S}_k\boldsymbol{S}_k^{\top}\boldsymbol{D}^{\top}\boldsymbol{B}^{\top})} \\ &= \sqrt{\frac{\det(\boldsymbol{B}\boldsymbol{D}\boldsymbol{D}^{\top}\boldsymbol{B}^{\top})}{\det(\boldsymbol{B}\boldsymbol{B}^{\top})}}. \end{aligned}$$

Substituting this expression in (21) gives along (20) the identity. □

The second result provides a lower bound for $\mathrm{vol}_{l-k}(\boldsymbol{D}M)$ that involves the eigenvalues of the matrix $\boldsymbol{D}^{\top}\boldsymbol{D}$ and $\mathrm{vol}_{l-k}(M)$.

Lemma 2 *Let $M \subset S$ be measurable, $\boldsymbol{D} \in \mathbb{R}^{l\times l}$ nonsingular and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$ be the positive eigenvalues of $\boldsymbol{D}^{\top}\boldsymbol{D}$. Then*

$$\mathrm{vol}_{l-k}(\boldsymbol{D}M) \geq \left(\prod_{i=1}^{l-k}\sqrt{\lambda_i}\right)\mathrm{vol}_{l-k}(M).$$

Proof According to (20), we need to estimate

$$\mathrm{vol}_{l-k}([\boldsymbol{D}\boldsymbol{S}_{l-k}]) = \sqrt{\det(\boldsymbol{S}_{l-k}^{\top}\boldsymbol{D}^{\top}\boldsymbol{D}\boldsymbol{S}_{l-k})}.$$

Let $\boldsymbol{O} \in \mathbb{R}^{l\times l}$ be a matrix such that its rows form an orthonormal basis consisting of eigenvectors of $\boldsymbol{D}^{\top}\boldsymbol{D}$. For convenience, we will denote by $\mathrm{diag}(\lambda_i)$ the diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_l$ on the main diagonal. Then we have $\boldsymbol{D}^{\top}\boldsymbol{D} = \boldsymbol{O}^{\top}\mathrm{diag}(\lambda_i)\boldsymbol{O}$.

Let $[l] = \{1, \dots, l\}$ and let $\binom{[l]}{r}$ be the set of all r-element subsets of $[l]$. With $\widetilde{\boldsymbol{S}_{l-k}} = \boldsymbol{O}\boldsymbol{S}_{l-k}$ we get by the Cauchy-Binet formula

$$\begin{aligned}
\det(\boldsymbol{S}_{l-k}^\top \boldsymbol{D}^\top \boldsymbol{D}\boldsymbol{S}_{l-k}) &= \det(\widetilde{\boldsymbol{S}_{l-k}}^\top \operatorname{diag}(\lambda_i^{1/2})\operatorname{diag}(\lambda_i^{1/2})\widetilde{\boldsymbol{S}_{l-k}}) \\
&= \sum_{I\in\binom{[l]}{l-k}} \det(\operatorname{diag}(\lambda_i^{1/2})^I)^2(\det(\widetilde{\boldsymbol{S}_{l-k}}^I))^2 \\
&\geq \left(\prod_{i=1}^{l-k}\lambda_i\right) \sum_{I\in\binom{[l]}{l-k}} (\det(\widetilde{\boldsymbol{S}_{l-k}}^I))^2 \\
&= \left(\prod_{i=1}^{l-k}\lambda_i\right) \det(\widetilde{\boldsymbol{S}_{l-k}}^\top\widetilde{\boldsymbol{S}_{l-k}}) = \prod_{i=1}^{l-k}\lambda_i .
\end{aligned}$$

Here $\operatorname{diag}(\lambda_i^{1/2})^I$ is the $(l-k)\times(l-k)$ diagonal matrix with $\lambda_i^{1/2}$ indexed by I and $\widetilde{\boldsymbol{S}_{l-k}}^I$ is the $(l-k)\times(l-k)$ submatrix of $\widetilde{\boldsymbol{S}_{l-k}}$ with rows indexed by I. In the second to last identity we used again the Cauchy-Binet formula. □

From Lemma 2 we obtain the following corollary.

Corollary 6 *Let S be an $(l-k)$-dimensional subspace of $\mathbb{R}^l$ and let $\boldsymbol{D} = \operatorname{diag}(d_1,\dots,d_l)$ with $0 < d_1 \leq d_2 \leq \cdots \leq d_l$. Then*

$$\operatorname{vol}_{l-k}(\boldsymbol{D}(S\cap(-1,1)^l)) \geq 2^{l-k}\prod_{i=1}^{l-k} d_i . \tag{22}$$

Proof By Vaaler's cube slicing inequality [28], we have $\operatorname{vol}_{l-k}(S\cap(-1,1)^l) \geq 2^{l-k}$. Now the bound (22) immediately follows from Lemma 2. □

3 Proofs of the transference bounds

We will derive Theorem 1 from Corollary 4 and from the following result.

Theorem 7 *Let $\boldsymbol{A}\in\mathbb{Z}^{m\times n}$, with $n > m+1$, be a matrix of rank m, $\boldsymbol{b}\in\mathbb{Z}^m$ and $\boldsymbol{c}\in\mathbb{R}^n$ be a unit cost vector. Suppose that (1) is feasible and bounded. Let $\boldsymbol{z}^* = (z_1^*,\dots,z_n^*)^\top$ be an optimal solution of (1) which is a vertex of $P_I(\boldsymbol{A},\boldsymbol{b})$. Assuming without loss of generality $z_1^*\leq\cdots\leq z_n^*$, the bound*

$$\operatorname{IG}(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}) \leq \frac{(n-m)}{\prod_{i=1}^{n-m-1}(z_i^*+1)}\cdot\frac{\Delta(\boldsymbol{A})}{\gcd(\boldsymbol{A})} \tag{23}$$

holds.

Proof Let $\boldsymbol{c}|\ker(\boldsymbol{A})$ denote the orthogonal projection of the vector $\boldsymbol{c}$ on the kernel subspace $\ker(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$ of the matrix $\boldsymbol{A}$. Observe first that if $\boldsymbol{c}$ is orthogonal to $\ker(\boldsymbol{A})$, then $\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = 0$ and the bound (23) holds. Hence, we may assume without loss of generality that $\boldsymbol{c}|\ker(\boldsymbol{A})$ is a nonzero vector.

Suppose, to derive a contradiction, that the bound (23) does not hold. Then there exists a vertex $\boldsymbol{x}^*$ of $P(\boldsymbol{A}, \boldsymbol{b})$ and a vertex $\boldsymbol{z}^*$ of $P_I(\boldsymbol{A}, \boldsymbol{b})$ optimising (1), such that, assuming $z_1^* \leq \cdots \leq z_n^*$, we have

$$\boldsymbol{c} \cdot (\boldsymbol{z}^* - \boldsymbol{x}^*) > \frac{(n-m)}{\prod_{i=1}^{n-m-1}(z_i^* + 1)} \cdot \frac{\Delta(\boldsymbol{A})}{\gcd(\boldsymbol{A})}. \tag{24}$$

Let $d_i = z_i^* + 1, i \in [n]$, $\boldsymbol{D} = \mathrm{diag}(d_1, \ldots, d_n)$ and let $\boldsymbol{B} \in \mathbb{Z}^{(m+1)\times n}$ be the matrix obtained by adding the $(m+1)$-st row $\boldsymbol{c}^\top$ to the matrix $\boldsymbol{A}$. Let further $V = \ker(\boldsymbol{B})$. Consider the box section

$$K = V \cap (-d_1, d_1) \times \cdots \times (-d_n, d_n).$$

We can write $K = \boldsymbol{D}M$, where M is an $(n-m-1)$-dimensional section of the cube $(-1, 1)^n$. Hence, by Corollary 6,

$$\mathrm{vol}_{n-m-1}(K) \geq 2^{n-m-1} \prod_{i=1}^{n-m-1} d_i. \tag{25}$$

Consider the origin-symmetric convex set

$$L = \mathrm{conv}(\boldsymbol{x}^* - \boldsymbol{z}^*, K, \boldsymbol{z}^* - \boldsymbol{x}^*) \subset \ker(\boldsymbol{A}).$$

The set L is a bi-pyramid with apexes $\pm(\boldsymbol{x}^* - \boldsymbol{z}^*)$ and $(n-m-1)$-dimensional basis K. As $K \subset \ker(\boldsymbol{A}) \cap \ker(\boldsymbol{c})$ the height of $\boldsymbol{x}^* - \boldsymbol{z}^*$ over K is given by

$$\frac{\boldsymbol{c}|\ker(\boldsymbol{A}) \cdot (\boldsymbol{z}^* - \boldsymbol{x}^*)}{\|\boldsymbol{c}|\ker(\boldsymbol{A})\|_2} = \frac{\boldsymbol{c} \cdot (\boldsymbol{z}^* - \boldsymbol{x}^*)}{\|\boldsymbol{c}|\ker(\boldsymbol{A})\|_2}.$$

Hence, we have

$$\mathrm{vol}_{n-m}(L) = \frac{2\,\boldsymbol{c} \cdot (\boldsymbol{z}^* - \boldsymbol{x}^*)\,\mathrm{vol}_{n-m-1}(K)}{(n-m)\|\boldsymbol{c}|\ker(\boldsymbol{A})\|_2}.$$

Then, using (25) and (24) and noting that the assumption $\|\boldsymbol{c}\|_2 = 1$ implies $\|\boldsymbol{c}|\ker(\boldsymbol{A})\|_2 \leq 1$, we obtain the lower bound

$$\mathrm{vol}_{n-m}(L) > 2^{n-m} \frac{\Delta(\boldsymbol{A})}{\gcd(\boldsymbol{A})}. \tag{26}$$

Observe that the lattice $\Lambda(\boldsymbol{A}) = \ker(\boldsymbol{A}) \cap \mathbb{Z}^n$ has determinant

$$\det(\Lambda(\boldsymbol{A})) = \frac{\Delta(\boldsymbol{A})}{\gcd(\boldsymbol{A})} \tag{27}$$

(see e. g. [27, Chapter 1, §1]). The $(n-m)$-dimensional subspace $\ker(\boldsymbol{A})$ can be considered as a usual Euclidean $(n-m)$-dimensional space. Therefore, by (26), (27) and Minkowski's first fundamental theorem (in the form of Theorem II in Chapter III of [12]), applied to the set L and the lattice $\Lambda(\boldsymbol{A})$, there is a nonzero point $\boldsymbol{y} \in L \cap \Lambda(\boldsymbol{A})$.

Suppose first that $\boldsymbol{c} \cdot \boldsymbol{y} = 0$. Consider the points $\boldsymbol{y}^+ = \boldsymbol{z}^* + \boldsymbol{y}$ and $\boldsymbol{y}^- = \boldsymbol{z}^* - \boldsymbol{y}$. We have $\boldsymbol{y}^+, \boldsymbol{y}^- \in (\boldsymbol{z}^* + K)$ and, consequently, $\boldsymbol{y}^+, \boldsymbol{y}^- \in P(\boldsymbol{A}, \boldsymbol{b})$. Further, $\boldsymbol{z}^*$ is the midpoint of the segment with endpoints $\boldsymbol{y}^+$ and $\boldsymbol{y}^-$, contradicting the choice of $\boldsymbol{z}^*$ as a vertex of the integer hull $P_I(\boldsymbol{A}, \boldsymbol{b})$.

It remains to consider the case $\boldsymbol{c} \cdot \boldsymbol{y} \neq 0$. Since L is origin-symmetric, we may assume without loss of generality that $\boldsymbol{c} \cdot \boldsymbol{y} < 0$. Observe that the point $\boldsymbol{y}^+ = \boldsymbol{z}^* + \boldsymbol{y}$ is in the set $\operatorname{conv}(\boldsymbol{z}^* + K, \boldsymbol{x}^*)$ and hence $\boldsymbol{y}^+ \in P(\boldsymbol{A}, \boldsymbol{b})$. Now, it is sufficient to notice that $\boldsymbol{c} \cdot \boldsymbol{y}^+ < \boldsymbol{c} \cdot \boldsymbol{z}^*$, contradicting the optimality of $\boldsymbol{z}^*$. □

3.1 Proof of Theorem 1

Given $I = \{i_1, \ldots, i_k\} \subset [n]$ with $i_1 < i_2 < \cdots < i_k$, we denote by $\mathbb{R}^I$ the k-dimensional real space with coordinates indexed by I. The complement of I in $[n]$ will be denoted as $\bar{I}$.

Let $\boldsymbol{x}^*$ be a vertex optimal solution to (2). Clearly, we may assume that $\boldsymbol{x}^* \neq \boldsymbol{z}^*$. Let $I \subset [n]$ denote the set of indices i for which at least one of z_i^*, x_i^* is non-zero. Let $\hat{\boldsymbol{A}}_I$ be any integer matrix whose rows constitute a basis of the lattice of all integer points contained inside the subspace spanned by the rows of the matrix $\boldsymbol{A}_I$. Let $\hat{\boldsymbol{b}} = \hat{\boldsymbol{A}}_I \boldsymbol{z}_I^*$, so that $\hat{\boldsymbol{A}}_I \boldsymbol{x}_I = \hat{\boldsymbol{b}}$ and $\boldsymbol{A}_I \boldsymbol{x}_I = \boldsymbol{b}$ describe the same affine subspace in $\mathbb{R}^I$. Let $\hat{m}, \hat{n}$ denote the dimensions of $\hat{\boldsymbol{A}}_I$, so that $\hat{\boldsymbol{A}}_I$ has $\hat{m}$ rows and $\hat{n} = |I|$. Note that if $\boldsymbol{y} \in \mathbb{R}^{\hat{m}}$ is any vector for which $\boldsymbol{y}^\top \hat{\boldsymbol{A}}_I$ is integral, then $\boldsymbol{y}$ itself is integral by construction of $\hat{\boldsymbol{A}}_I$. It follows that $\gcd(\hat{\boldsymbol{A}}_I) = 1$ by [25, Corollary 4.1c].

We consider a new linear program

$$\min\{\boldsymbol{c}_I \cdot \boldsymbol{x}_I : \hat{\boldsymbol{A}}_I \boldsymbol{x}_I = \hat{\boldsymbol{b}}, \boldsymbol{x}_I \geq 0\}. \tag{28}$$

Note that $\boldsymbol{x}_I^*$ and $\boldsymbol{z}_I^*$ are optimal fractional and integral solutions, respectively, and that $\boldsymbol{z}_I^*$ is a vertex of the integer hull of $P(\hat{\boldsymbol{A}}_I, \hat{\boldsymbol{b}})$. Note also that

$$\Delta(\hat{\boldsymbol{A}}_I) \leq \frac{\Delta(\boldsymbol{A})}{\gcd(\boldsymbol{A})}. \tag{29}$$

The bound (29) follows from Lemma 2.3 in [5]. Geometrically, it is sufficient to observe that the quantity on the left is a divisor of the volume of the orthogonal projection of a parallelepiped whose volume is given by the quantity on the right.

If $\hat{n} = \hat{m} + 1$, then the bound (13) immediately follows from the bound (17) in Corollary 4. Otherwise, suppose that $\hat{n} > \hat{m} + 1$. We have then that $\hat{\boldsymbol{A}}_I$, $\hat{\boldsymbol{b}}$, $\boldsymbol{c}_I / \|\boldsymbol{c}_I\|_2$, $\boldsymbol{x}_I^*$, $\boldsymbol{z}_I^*$ satisfy the hypotheses of Theorem 7. We therefore get

$$\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \mathrm{IG}(\hat{\boldsymbol{A}}_I, \hat{\boldsymbol{b}}, \boldsymbol{c}_I) \leq \|\boldsymbol{c}_I\|_2 \cdot \frac{\hat{n} - \hat{m}}{\prod_i (z_i^* + 1)} \cdot \Delta(\hat{\boldsymbol{A}}_I), \tag{30}$$

where the product in the denominator is over the $\hat{n} - \hat{m} - 1$ smallest coordinates of $\boldsymbol{z}_I^*$.

Now, $\hat{n}$ is equal to $\|\boldsymbol{z}^* + \boldsymbol{x}^*\|_0$. Also, $\boldsymbol{x}^*$ has support of size at most $\hat{m}$, since $\|\boldsymbol{x}^*\|_0 = \left\|\boldsymbol{x}_I^*\right\|_0$ and $\boldsymbol{x}_I^*$ is a vertex of the new linear program (28). Thus we get

$$\hat{n} - \hat{m} \leq \left\|\boldsymbol{z}^* + \boldsymbol{x}^*\right\|_0 - \left\|\boldsymbol{x}^*\right\|_0 \leq \left\|\boldsymbol{z}^*\right\|_0 = s. \tag{31}$$

On the other hand, we have the following lower bound for the product in the denominator of (30):

$$\prod_i (z_i^* + 1) \geq 2^{\hat{n} - \hat{m} - 1 - \left|I \setminus \operatorname{supp}(\boldsymbol{z}_I^*)\right|} = 2^{s - \hat{m} - 1} \geq 2^{s - m - 1}. \tag{32}$$

Combining together (30), (31), (32), and the fact that $\|\boldsymbol{c}_I\|_2 \leq 1$, we get

$$\mathrm{IG}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \leq \frac{s}{2^{s-m-1}} \cdot \Delta(\hat{\boldsymbol{A}}_I). \tag{33}$$

The desired conclusion (13) then follows from (29) and (33).

3.2 Proof of Corollary 2

Let I and $\hat{\boldsymbol{A}}_I$ be as in the proof of Theorem 1. We will derive the bounds (14) and (15) from the bound (33). Choose any $J \subset [n] \setminus I$ that is minimal with respect to the property that $\boldsymbol{A}_{I \cup J}$ has rank m. Thus, $|J| = m - \hat{m}$, and

$$|I \cup J| = \hat{n} + (m - \hat{m}) \leq s + \hat{m} + (m - \hat{m}) = s + m. \tag{34}$$

We have, by Lemma 2.3 in [5],

$$\Delta(\hat{\boldsymbol{A}}_I) \leq \frac{\Delta(\boldsymbol{A}_{I \cup J})}{\gcd(\boldsymbol{A}_{I \cup J})} \leq \frac{\Delta(\boldsymbol{A}_{I \cup J})}{\gcd(\boldsymbol{A})}. \tag{35}$$

By (34) and the Cauchy-Binet formula, we have

$$\Delta(\boldsymbol{A}_{I \cup J}) \leq \binom{|I \cup J|}{m}^{1/2} \Delta_m(\boldsymbol{A}_{I \cup J}) \leq \binom{s+m}{m}^{1/2} \Delta_m(\boldsymbol{A}). \tag{36}$$

Combining (33), (35), and (36), we obtain the bound (14). On the other hand, if we use (34) and Hadamard's inequality, we get

$$\Delta(\boldsymbol{A}_{I\cup J}) \leq (\sqrt{|I \cup J|} \cdot \Delta_1(\boldsymbol{A}_{I\cup J}))^m \leq (\sqrt{s+m} \cdot \Delta_1(\boldsymbol{A}))^m. \tag{37}$$

Combining (33), (35), and (37), we obtain the bound (15).

4 Proof of the ℓ_2-distance proximity bound

First, we will prove two lemmas needed for the proof of Theorem 3.

Let $\boldsymbol{y} \in \mathbb{R}^n$ and let

$$C^n(\boldsymbol{y}) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{y}\|_\infty < 1\}$$

be an open cube in $\mathbb{R}^n$ with edge length 2 centered at the point $\boldsymbol{y}$. Given two points $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ we will consider an open set $D(\boldsymbol{u}, \boldsymbol{v})$ defined as

$$D(\boldsymbol{u}, \boldsymbol{v}) = \operatorname{conv}(C^n(\boldsymbol{u}), C^n(\boldsymbol{v})).$$

Lemma 3 *Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n_{\geq 0}$. Then $D(\boldsymbol{u}, \boldsymbol{v}) \cap \mathbb{Z}^n = D(\boldsymbol{u}, \boldsymbol{v}) \cap \mathbb{Z}^n_{\geq 0}$.*

Proof Suppose, to derive a contradiction, that there exists an integer point $\boldsymbol{z} = (z_1, \ldots, z_n)^\top \in D(\boldsymbol{u}, \boldsymbol{v})$ such that $z_j \leq -1$ for some $j \in [n]$. Then there exist points $\boldsymbol{x} = (x_1, \ldots, x_n)^\top \in C^n(\boldsymbol{u})$ and $\boldsymbol{y} = (y_1, \ldots, y_n)^\top \in C^n(\boldsymbol{v})$ such that for some $\lambda \in [0, 1]$

$$z_j = \lambda x_j + (1 - \lambda) y_j.$$

Therefore, since $x_j > -1$ and $y_j > -1$, we must have $z_j > -1$. □

Next, we consider an origin-symmetric open convex set $E = E(\boldsymbol{u}, \boldsymbol{v})$ defined as

$$E = \operatorname{conv}(C^n(\boldsymbol{u} - \boldsymbol{v}), C^n(\boldsymbol{v} - \boldsymbol{u})).$$

Notice that

$$E = (D(\boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{v}) \cup (-D(\boldsymbol{u}, \boldsymbol{v}) + \boldsymbol{v}). \tag{38}$$

Lemma 4 *Suppose that $\boldsymbol{u}, \boldsymbol{v} \in P(\boldsymbol{A}, \boldsymbol{b})$. Then the bound*

$$\operatorname{vol}_{n-m}(E \cap \ker(\boldsymbol{A})) \geq 2^{n-m}(1 + \|\boldsymbol{u} - \boldsymbol{v}\|_2) \tag{39}$$

holds.

Proof First, we will separately consider the case $n = m + 1$. Then $\ker(\boldsymbol{A})$ has dimension one and, noticing that $\boldsymbol{u} - \boldsymbol{v} \in \ker(\boldsymbol{A})$ we can write

$$\mathrm{vol}_1(E \cap \ker(\boldsymbol{A})) = \mathrm{vol}_1(C^n(\boldsymbol{0}) \cap \ker(\boldsymbol{A})) + 2\|\boldsymbol{u} - \boldsymbol{v}\|_2.$$

Since $\mathrm{vol}_1(C^n(\boldsymbol{0}) \cap \ker(\boldsymbol{A})) \geq 2$, we obtain the bound (39).

For the rest of the proof we assume that $n > m + 1$. If $\boldsymbol{u} = \boldsymbol{v}$, then $E = C^n(\boldsymbol{0})$ and the bound (39) immediately follows from Vaaler's cube slicing inequality [28]. Hence, we may also assume without loss of generality that $\boldsymbol{u} - \boldsymbol{v} \neq \boldsymbol{0}$.

Let

$$S = C^n(\boldsymbol{0}) \cap \ker(\boldsymbol{A}) \cap \ker((\boldsymbol{u} - \boldsymbol{v})^\top).$$

The set S is a section of the open cube $C^n(\boldsymbol{0})$. Since $\boldsymbol{u} - \boldsymbol{v} \in \ker(\boldsymbol{A})\backslash\{\boldsymbol{0}\}$, the section S has dimension $n - m - 1$. Let further

$$S^+ = \{\boldsymbol{x} \in C^n(\boldsymbol{0}) \cap \ker(\boldsymbol{A}) : (\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{x} > 0\}$$

and

$$S^- = \{\boldsymbol{x} \in C^n(\boldsymbol{0}) \cap \ker(\boldsymbol{A}) : (\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{x} < 0\}.$$

By construction, S^+, S^- and S do not overlap and $C^n(\boldsymbol{0}) \cap \ker(\boldsymbol{A}) = S^+ \cup S^- \cup S$. Further, by Vaaler's cube slicing inequality [28], we have

$$\mathrm{vol}_{n-m}(S^+) + \mathrm{vol}_{n-m}(S^-) = \mathrm{vol}_{n-m}(C^n(\boldsymbol{0}) \cap \ker(\boldsymbol{A})) \geq 2^{n-m} \tag{40}$$

and

$$\mathrm{vol}_{n-m-1}(S) \geq 2^{n-m-1}. \tag{41}$$

Observe that $E \cap \ker(\boldsymbol{A})$ contains the sets $S^+ + \boldsymbol{u} - \boldsymbol{v}$, $S^- + \boldsymbol{v} - \boldsymbol{u}$ and the cylinder $\mathrm{conv}(\boldsymbol{u} - \boldsymbol{v} + S, \boldsymbol{v} - \boldsymbol{u} + S)$. These three sets do not overlap and, using (40) and (41), we have

$$\begin{aligned}\mathrm{vol}_{n-m}(E \cap \ker(\boldsymbol{A})) \geq\ & \mathrm{vol}_{n-m}(S^+) + \mathrm{vol}_{n-m}(S^-) \\ & + \mathrm{vol}_{n-m}(\mathrm{conv}(\boldsymbol{u} - \boldsymbol{v} + S, \boldsymbol{v} - \boldsymbol{u} + S)) \\ =\ & \mathrm{vol}_{n-m}(C^n(\boldsymbol{0}) \cap \ker(\boldsymbol{A})) + 2\,\mathrm{vol}_{n-m-1}(S)\,\|\boldsymbol{u} - \boldsymbol{v}\|_2 \\ \geq\ & 2^{n-m}(1 + \|\boldsymbol{u} - \boldsymbol{v}\|_2).\end{aligned}$$

Hence, we obtain the bound (39). □

4.1 Proof of Theorem 3

We will say that $B \subset [n]$ is a *basis* of $\boldsymbol{A}$ if $|B| = m$ and the submatrix $\boldsymbol{A}_B$ is nonsingular. Take any vertex $\boldsymbol{x}^* \in P(\boldsymbol{A}, \boldsymbol{b})$. There is a basis B of $\boldsymbol{A}$ such that, denoting by N the complement of B in $[n]$, we have

$$\boldsymbol{x}_B^* = \boldsymbol{A}_B^{-1}\boldsymbol{b} \text{ and } \boldsymbol{x}_N^* = \boldsymbol{0}_N.$$

Choose an integer point $\boldsymbol{z}^* \in P(\boldsymbol{A}, \boldsymbol{b})$ with the minimum possible distance between the points $\boldsymbol{x}_N^* = \boldsymbol{0}_N$ and $\boldsymbol{z}_N^*$. Then

$$\|\boldsymbol{z}_N^*\|_2 = \min\{\|\boldsymbol{y}_N\|_2 : \boldsymbol{y} \in P(\boldsymbol{A}, \boldsymbol{b}) \cap \mathbb{Z}^n\}. \tag{42}$$

Suppose, to derive a contradiction, that the bound (16) does not hold for the point $\boldsymbol{z}^*$. Then, using (27),

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_2 > \frac{\Delta(\boldsymbol{A})}{\gcd(\boldsymbol{A})} - 1 = \det(\Lambda(\boldsymbol{A})) - 1. \tag{43}$$

Recall that we denote by $\Lambda(\boldsymbol{A})$ the lattice formed by all integer points in the kernel subspace of the matrix $\boldsymbol{A}$.

The lower bound (43) and Lemma 4 imply that for $E = E(\boldsymbol{x}^*, \boldsymbol{z}^*)$ we have

$$\mathrm{vol}_{n-m}(E \cap \ker(\boldsymbol{A})) > 2^{n-m} \det(\Lambda(\boldsymbol{A})). \tag{44}$$

The $(n-m)$-dimensional subspace $\ker(\boldsymbol{A})$ can be considered as a usual Euclidean $(n-m)$-dimensional space. Noting the bound (44), Minkowski's first fundamental theorem (in the form of Theorem II in Chapter III of [12]) implies that the set $E \cap \ker(\boldsymbol{A})$ contains nonzero points $\pm \boldsymbol{z}$ of the lattice $\Lambda(\boldsymbol{A})$. Using (38), we may assume without loss of generality that we have $\boldsymbol{z} \in D(\boldsymbol{x}^*, \boldsymbol{z}^*) - \boldsymbol{z}^*$. Therefore, the point $\boldsymbol{w} = \boldsymbol{z} + \boldsymbol{z}^*$ is in the set $D(\boldsymbol{x}^*, \boldsymbol{z}^*) \cap (\ker(\boldsymbol{A}) + \boldsymbol{z}^*)$. By Lemma 3, $\boldsymbol{w} \in \mathbb{Z}^n_{\geq 0}$ and, hence, $\boldsymbol{w} \in P(\boldsymbol{A}, \boldsymbol{b})$.

Next, we will show that $\|\boldsymbol{w}_N\|_2 < \|\boldsymbol{z}_N^*\|_2$, contradicting (42). Notice first that for any $\boldsymbol{x} \in P(\boldsymbol{A}, \boldsymbol{b})$ we have $\boldsymbol{x}_B = \boldsymbol{A}_B^{-1}(\boldsymbol{b} - \boldsymbol{A}_N \boldsymbol{x}_N)$. Hence $\boldsymbol{w}_N = \boldsymbol{z}_N^*$ implies that $\boldsymbol{w} = \boldsymbol{z}^*$. Therefore, we may assume that $\boldsymbol{w}_N \neq \boldsymbol{z}_N^*$.

Take any index $j \in N$. Since $\boldsymbol{w} \in D(\boldsymbol{x}^*, \boldsymbol{z}^*)$, we have $w_j \leq z_j^*$. Hence there is at least one index $j_0 \in N$ with $w_{j_0} < z_{j_0}^*$. Therefore $\|\boldsymbol{w}_N\|_2 < \|\boldsymbol{z}_N^*\|_2$ and we obtain a contradiction with (42).

5 Proof of Theorem 5

We will derive Theorem 5 from the following result.

Theorem 8 *Let $\boldsymbol{A} \in \mathbb{Z}^{m \times n}$, $m < n$, be a matrix of rank m and let $\boldsymbol{b} \in \mathbb{Z}^m$. Assume that the polyhedron $P(\boldsymbol{A}, \boldsymbol{b})$ contains integer points and that for some index $j \in [n]$*

the linear program

$$\max\{x_j : \boldsymbol{x} = (x_1, \ldots, x_n)^\top \in P(\boldsymbol{A}, \boldsymbol{b})\} \tag{45}$$

is bounded. Let $\boldsymbol{x}^*$ *be a vertex optimal solution to* (45) *with a basis* γ*. Then either* $\boldsymbol{x}^*$ *is integer, or there exists an integer point* $\boldsymbol{z} = (z_1, \ldots, z_n)^\top \in P(\boldsymbol{A}, \boldsymbol{b})$ *such that, letting* $\tau = \operatorname{supp}(\boldsymbol{z}) \cup \gamma \cup \{j\}$*, the bound*

$$\prod_{i \in [n] \setminus \{j\}} (z_i + 1) \le (\gcd(\boldsymbol{A}_\tau))^{-1} \sqrt{\sum_{\substack{\sigma \subset \tau: \\ |\sigma| = m, j \in \sigma}} (\det(\boldsymbol{A}_\sigma) \prod_{i \in \sigma \setminus \{j\}} (z_i + 1))^2} \tag{46}$$

holds.

Proof of Theorem 8 Let $\boldsymbol{x}^*$ be a non-integer vertex optimal solution to (45) with a basis γ. Note that any integer point $\boldsymbol{z} \in P(\boldsymbol{A}, \boldsymbol{b})$ has $\operatorname{supp}(\boldsymbol{z}) \cap \bar{\gamma} \neq \emptyset$. Let $\xi = \gamma \cup \{j\}$. If $\xi = [n]$ then we set $\tilde{\boldsymbol{z}} = \boldsymbol{0} \in \mathbb{Z}^n$. Otherwise choose an integer point $\tilde{\boldsymbol{z}} \in P(\boldsymbol{A}, \boldsymbol{b})$ with the following property. The size of support of $\tilde{\boldsymbol{z}}$ corresponding to the complementary set $\bar{\xi}$ has the minimum possible value, that is

$$\|\tilde{\boldsymbol{z}}_{\bar{\xi}}\|_0 = \min\{\|\boldsymbol{y}_{\bar{\xi}}\|_0 : \boldsymbol{y} \in P(\boldsymbol{A}, \boldsymbol{b}) \cap \mathbb{Z}^n\}.$$

Next, we choose a vertex $\boldsymbol{z}$ of the convex hull F of the optimal integer solutions of the integer linear program

$$\max\{y_j : \boldsymbol{y} \in P(\boldsymbol{A}, \boldsymbol{b}) \cap \mathbb{Z}^n \text{ and } \operatorname{supp}(\boldsymbol{y}) \subset \operatorname{supp}(\tilde{\boldsymbol{z}}) \cup \xi\}. \tag{47}$$

Let

$$\tau = \operatorname{supp}(\boldsymbol{z}) \cup \xi = \{i_1, \ldots, i_k\}.$$

In what follows, we will denote by $\mathbb{R}^\tau$ the coordinate subspace of $\mathbb{R}^n$ with coordinates indexed by τ.

Consider the subspace $G = \ker(\boldsymbol{A}_\tau)$ of $\mathbb{R}^\tau$. Since the vertex $\boldsymbol{x}^*$ is not integer, we have $\gamma \subsetneq \tau$. Consequently, $|\tau| = k > m$ and hence, noticing that $\operatorname{rank}(\boldsymbol{A}_\tau) = m$, we have $\dim(G) = k - m \ge 1$.

Let $u_i = z_i + 1$ for $i \in \tau \setminus \{j\}$ and let $u_j = t$, where $t \ge 1$ is an arbitrarily chosen real number. Let further $\boldsymbol{D} = \operatorname{diag}(u_{i_1}, \ldots, u_{i_k})$. We will first show that

$$u_{i_1} \cdots u_{i_k} \le \frac{\Delta(\boldsymbol{A}_\tau \boldsymbol{D})}{\gcd(\boldsymbol{A}_\tau)}. \tag{48}$$

Subsequently, we will use (48) to derive the desired bound (46).

Consider the box section

$$K = G \cap (-u_{i_1}, u_{i_1}) \times \cdots \times (-u_{i_k}, u_{i_k}).$$

There exists a subspace S of $\mathbb{R}^\tau$ with $\dim(S) = \dim(G) = k - m$ such that

$$K = \boldsymbol{D}Q,$$

where $Q = S \cap \{\boldsymbol{x} \in \mathbb{R}^\tau : -1 < x_i < 1, i \in \tau\}$ is a section of an open cube in $\mathbb{R}^\tau$. Using Lemma 1, we have

$$\mathrm{vol}_{k-m}(K) = u_{i_1} \cdots u_{i_k} \frac{\Delta(\boldsymbol{A}_\tau)}{\Delta(\boldsymbol{A}_\tau \boldsymbol{D})} \mathrm{vol}(Q).$$

By Vaaler's cube slicing inequality [28], we have

$$\mathrm{vol}_{k-m}(Q) \geq 2^{k-m}. \tag{49}$$

Suppose, to derive a contradiction, that

$$u_{i_1} \cdots u_{i_k} > \frac{\Delta(\boldsymbol{A}_\tau \boldsymbol{D})}{\gcd(\boldsymbol{A}_\tau)}. \tag{50}$$

Then, using (49) and (50), we get

$$\mathrm{vol}_{k-m}(K) > 2^{k-m} \frac{\Delta(\boldsymbol{A}_\tau)}{\gcd(\boldsymbol{A}_\tau)} = 2^{k-m} \det(\Lambda(\boldsymbol{A}_\tau)),$$

where we use (27) for the last equality. Therefore, by Minkowski's first fundamental theorem, the box section K contains nonzero integer points $\pm \boldsymbol{y} \in \Lambda(\boldsymbol{A}_\tau)$. Adding the zero coordinates indexed by $[n]\backslash\tau$ to $\pm\boldsymbol{y}$, we obtain the n-dimensional integer points $\pm\tilde{\boldsymbol{y}}$. Let $\boldsymbol{y}^+ = \boldsymbol{z} + \tilde{\boldsymbol{y}}$ and $\boldsymbol{y}^- = \boldsymbol{z} - \tilde{\boldsymbol{y}}$. Observe that by construction of K the points $\boldsymbol{y}^+$ and $\boldsymbol{y}^-$ have nonnegative coordinates indexed by $[n]\backslash\{j\}$. Furthermore, at least one of the points $\boldsymbol{y}^+$ and $\boldsymbol{y}^-$ has its j-th coordinate greater than or equal to z_j. Assume without loss of generality that this point is $\boldsymbol{y}^+$. Then $\boldsymbol{y}^+ \in P(\boldsymbol{A}, \boldsymbol{b})$ and we will consider two cases.

Suppose first that

$$y_j^+ > z_j.$$

In this case, we get a contradiction with the optimality of $\boldsymbol{z}$ for (47).

Now, consider the case $y_j^+ = y_j^- = z_j$. But then both points $\boldsymbol{y}^+$ and $\boldsymbol{y}^-$ are in F and $\boldsymbol{z} = (\boldsymbol{y}^+ + \boldsymbol{y}^-)/2$. This contradicts the choice of $\boldsymbol{z}$ as a vertex of the polyhedron F. Therefore (50) does not hold and hence (48) is justified.

Dividing both sides of (48) by $u_j = t$ and noticing that, by the Cauchy-Binet formula,

$$\Delta(\boldsymbol{A}_\tau \boldsymbol{D}) = \sqrt{\sum_{\substack{\sigma \subset \tau: \\ |\sigma| = m}} (\det(\boldsymbol{A}_\sigma) \prod_{i \in \sigma} (z_i + 1))^2},$$

we obtain

$$\prod_{i\in[n]\setminus\{j\}}(z_i+1)=\prod_{i\in\tau\setminus\{j\}}u_i\le(\gcd(\boldsymbol{A}_\tau))^{-1} \times\sqrt{\sum_{\substack{\sigma\subset\tau:\\|\sigma|=m,j\in\sigma}}(\det(\boldsymbol{A}_\sigma)\prod_{i\in\sigma\setminus\{j\}}(z_i+1))^2+O(t^{-2})}, \tag{51}$$

where the implicit constant in the $O(t^{-2})$ term on the right hand side of (51) depends on $\boldsymbol{A}$ and $\boldsymbol{z}$ only. Since the bound (51) holds for arbitrarily large t, it implies (46). □

Now we are ready to prove Theorem 5.

Proof of Theorem 5 Let us choose any $\boldsymbol{y}\in\mathbb{Z}^m_{\ge0}$ such that

$$\mathsf{ICR}(\boldsymbol{A})=\min\{\|\boldsymbol{x}\|_0:\boldsymbol{x}\in P(\boldsymbol{A},\boldsymbol{A}\boldsymbol{y})\cap\mathbb{Z}^n\}$$

and let $\boldsymbol{b}=\boldsymbol{A}\boldsymbol{y}$. It is sufficient to show that there exists an integer point $\boldsymbol{z}\in P(\boldsymbol{A},\boldsymbol{b})$ with

$$\|\boldsymbol{z}\|_0\le m+\log_2\left(\frac{\mu(\boldsymbol{A})}{\gcd(\boldsymbol{A})}\right). \tag{52}$$

Since cone$(\boldsymbol{A})$ is pointed, $P(\boldsymbol{A},\boldsymbol{b})$ is bounded. Further, if $P(\boldsymbol{A},\boldsymbol{b})$ has an integer vertex $\boldsymbol{x}^*$, then $\|\boldsymbol{x}^*\|_0\le m$ and the bound (52) is satisfied for $\boldsymbol{z}=\boldsymbol{x}^*$. Hence we may assume that all vertices of $P(\boldsymbol{A},\boldsymbol{b})$ are non-integer.

Choose any $j\in[n]$ such that

$$\mu(\boldsymbol{A})=\sqrt{\sum_{\substack{\sigma\subset[n]:\\|\sigma|=m,j\in\sigma}}(\det(\boldsymbol{A}_\sigma))^2}$$

and apply Theorem 8 with $\boldsymbol{A}$, $\boldsymbol{b}$ and the index j. Let $\boldsymbol{z}$ be an integer point in $P(\boldsymbol{A},\boldsymbol{b})$ that satisfies (46).

Suppose first that $m\ge2$. Let $z_{i_1},\ldots,z_{i_{m-1}}$ be the $m-1$ largest entries of the point $\boldsymbol{z}_{[n]\setminus\{j\}}$ and let $\nu=\{i_1,\ldots,i_{m-1}\}$. Then, dividing (46) by the product $\prod_{i\in\nu}(z_i+1)$, we get

$$\begin{aligned}\prod_{i\in[n]\setminus\{\nu\cup\{j\}\}}(z_i+1)&\le(\gcd(\boldsymbol{A}_\tau))^{-1}\\&\times\sqrt{\sum_{\substack{\sigma\subset\tau:\\|\sigma|=m,j\in\sigma}}\left(\det(\boldsymbol{A}_\sigma)\left(\prod_{i\in\sigma\setminus\{j\}}(z_i+1))/(\prod_{i\in\nu}(z_i+1)\right)\right)^2}\\&\le\frac{\mu(\boldsymbol{A})}{\gcd(\boldsymbol{A}_\tau)}.\end{aligned} \tag{53}$$

When $m = 1$, we set $\nu = \emptyset$ and immediately obtain (53). Observe next that, using (53), we have

$$2^{\|z\|_0} \leq 2^m \prod_{i\in[n]\setminus\{\nu\cup\{j\}\}} (z_i + 1) \leq 2^m \frac{\mu(\boldsymbol{A})}{\gcd(\boldsymbol{A}_\tau)}. \tag{54}$$

Noticing that $\gcd(\boldsymbol{A}_\tau) \geq \gcd(\boldsymbol{A})$ and taking the logarithm with base two of (54), we obtain the bound (18).

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