Unitary $\mathcal{R}$-Matrices: Representations of Deformations of the Braid Group Arising From The BMW Algebra And Racks

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Dedication

To Cameron, to my family, and to my friends. This is as much your achievement as it is mine, for it would not have been possible without all of your support.

Abstract

In this thesis we look at $\mathcal{R}$-matrix representations of the BMW algebra and the Bloop group, a new group arising from the quandalisation process of racks. Both the BMW algebra and the Bloop group have diagrammatic presentations involving adding loops to the braid group, although the structures of these loops are different.

We define the contractive $\mathcal{R}$-matrices, a restriction of which form a representation of the BMW algebra, and show that they are stable under equivalence. We show that they have a maximum of 3 eigenvalues and we classify all 2-dimensional examples. Then we utilise a Markov trace to deduce restrictions on the possible values of the contraction constant c. In particular, we show that $|c|^{-2}$ is the Jones Index $[\rho_R(\mathcal{B}_\infty) : \varphi(\rho_R(\mathcal{B}_\infty))]$ and we deduce the form of the contractive $\mathcal{R}$-matrices for each value in the discrete range of the Jones Index.

The rack-induced $\mathcal{R}$-matrices are shown to be closely tied with the racks from which they are derived. In particular, isomorphic racks induce equivalent $\mathcal{R}$-matrices (though the opposite is not necessarily true), and if two rack-induced $\mathcal{R}$-matrices are equivalent then the quandalisations of their underlying racks also produce equivalent $\mathcal{R}$-matrices. We show that quandle-induced $\mathcal{R}$-matrices are equivalent if and only if their coloring numbers are equal for all oriented links. The quandalisation process of racks is the inspiration for the Bloop Group developed in this thesis, and we develop an $\mathcal{R}$-matrix representation of this new group.

These developments contribute to ongoing area of research of the classification of unitary $\mathcal{R}$-matrices, which has applications in many areas including quantum groups, knot theory and topological quantum computing. Further research in this area would include the analysis of $\mathcal{R}$-matrix representations of other structures.

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Thank you to my in-laws, Michaela and Martin Lamb, for all of your acceptance and support of me.

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Conventions and Notation

Here we denote conventions and notations used throughout this thesis.

Chapter 1:

- R will denote an $\mathcal{R}$-matrix, i.e. a solution to the quantum Yang-Baxter equation (1.2).
- We are only considering unitary $\mathcal{R}$-matrices, as defined in Equation (1.5).
- $\mathcal{R}(d)$ will denote the set of all unitary $\mathcal{R}$-matrices.
- V will denote a finite-dimensional vector space of dimension d.
- 1_V will denote the identity of V.
- We are working over the complex numbers $\mathbb{C}$.
- All diagrams are to be read top-to-bottom.
- $A*$ will denote the adjoint of an operator A.
- When dealing with multiple tensor copies of kets we use the short-hand notation $R|x_1, x_2\rangle := R(|x_1\rangle \otimes |x_2\rangle)$.
- $\mathcal{B}_n$ will denote the Braid group on n strands, defined in Section 1.2.
- $\varphi(R)$ will denote the canonical shift endomorphism, as defined in Equation (1.9).
- R_k will denote the shift endomorphism being applied $k-1$ times, i.e. $R_k = \varphi^{k-1}(R)$.

- ρ_R will denote the representation of the Braid group induced by R, as defined in (1.12).

- $\tau(R)$ will denote the normalised trace of R, defined by (1.14).

- When taking partial traces we can take the left or right partial trace, as for unitary $\mathcal{R}$-matrices these are equivalent [13].

- τ_R will denote the character of R, as defined in Definition 18.

- $R \sim S$ will denote the equivalence of R and S. Our notion of equivalence is defined in Definition 20.

- $\mathcal{B}(\mathscr{H})$ will denote the set of bounded operators on a Hilbert space $\mathscr{H}$.

- $\mathscr{F}'$ will denote the commutant of a set $\mathscr{F} \subset \mathcal{B}(\mathscr{H})$ of bounded operators, as defined in (1.18).

- $[\mathscr{N} : \mathscr{M}]$ will denote the Jones index of a type II_1 subfactor $\mathscr{N} \subseteq \mathscr{M}$, as defined in Equation (1.19).

<u>Chapter 2:</u>

- $\mathcal{C}_n(r, q)$ will denote the BMW algebra, as defined in Definition 27.

- $\mathcal{C}(d)$ will denote the set of all contractive $\mathscr{R}$-matrices with contractive eigenvalue α, as defined in Definition 29.

- $TL_n(\delta)$ will denote the Temperley-Lieb algebra, defined in Definition 33.

<u>Chapter 3:</u>

- r will denote a solution to the set-theoretic Yang Baxter equation, as defined in Definition 2.

- (X, λ) and $(X, \triangleright)$ will both denote a rack. Both notations are used as sometimes one notation is clearer than the other. Racks are defined in Definition 37.

- Rack tables are to be read as the element in the i^{th} row and j^{th} column denoting $i \triangleright j$.

Chapter 1

Introduction

The Yang-Baxter equation is deceptively simplistic, defined on $V \otimes V \otimes V$, where V is a vector space with identity 1_V, for some $R \in \text{End}(V \otimes V)$:

$$(R \otimes 1_V)(1_V \otimes R)(R \otimes 1_V) = (1_V \otimes R)(R \otimes 1_V)(1_V \otimes R) \tag{1.1}$$

It was first introduced by Yang in 1967 [34] in the context of studying 1-dimensional Bose gas. Baxter later independently developed the equation in 1972 [5] whilst working on the eight-vertex lattice model in statistical mechanics. Since then it has seen applications in quantum groups developed by Drinfeld and Jimbo in 1988 [15], in the representation theory of braids by Wenzl in 1990 [33], and more recently in the field of topological quantum computing by Kauffman [23] to name but a few applications.

Although it may appear on the surface to be a straightforward equation, the solutions to the Yang-Baxter equation (known as $\mathcal{R}$-matrices) are notoriously difficult to find. Recent development in the literature has looked at finding and classifying solutions of the Yang-Baxter equation up to various notions of equivalence. In 2019 Lechner, Pennig and Wood [25] classified all involutive ($R^2 = 1$) solutions up to an equivalence where two $\mathcal{R}$-matrices are equivalent iff they have the same character and dimension.

In this thesis we explore the braid group $\mathcal{B}_n$, the BMW algebra $\mathcal{C}_n(r, q)$, and the Bloop group $\mathcal{B}\ell_n$ - a new group inspired by the quandalisation of racks. In particular we analyse the $\mathcal{R}$-matrix representations of these structures and examine their properties to advance the ongoing effort to classify all unitary $\mathcal{R}$-matrices up to equivalence (in terms of having the same dimension and character).

In Chapter 1 we define the Yang-Baxter equation and its solutions, known as $\mathcal{R}$-matrices. We then look at the braid group and how these $\mathcal{R}$-matrices form a representation of it. We examine the traces and partial traces of these matrices, and detail the notions of equivalence we use in this thesis. Finally, we discuss how $\mathcal{R}$-matrices can be used to induce a type II_1 von-Neumann factor, which is utilised in Chapter 2 by considering a Jones Index.

In Chapter 2 we look at the BMW algebra in detail, and we define a new class of $\mathcal{R}$-matrices named contractive $\mathcal{R}$-matrices, a restriction upon whom form a representation of the BMW algebra. We then analyse the properties of contractive $\mathcal{R}$-matrices, including showing that they have at most 3 eigenvalues, are stable under equivalence, and that they satisfy a number of relations. We discuss when the normalised trace is a Markov trace, and utilise this to form restrictions on the possible values of the contraction constant. In particular, we show that $|c|^{-2}$ is equal to the Jones Index of a factor and subfactor induced from a contractive $\mathcal{R}$-matrix, and classify examples of contractive $\mathcal{R}$-matrices by the value of its related Jones Index in the discrete range.

In Chapter 3 we look at the set-theoretic Yang-Baxter equation and analyse $\mathcal{R}$-matrices that arise from linearising its non-degenerate solutions. These $\mathcal{R}$-matrices are shown in the literature to be equivalent to those derived from racks. We analyse these rack-derived $\mathcal{R}$-matrices and consider their various properties. In particular, we look at quandles (a particular type of rack that has a trivial square map), and analyse the quandalisation process through the lens of $\mathcal{R}$-matrices. This analysis led to the development of the Bloop group, for which we create an $\mathcal{R}$-matrix representation. We restrict even further to Alexander quandles, a type of quandle extremely useful in knot theory, and analyse when Alexander-quandle-induced $\mathcal{R}$-matrices are equivalent. Finally, we show that, given 2 racks with equivalent $\mathcal{R}$-matrices, the quandle-induced $\mathcal{R}$-matrices arising from the quandalisations of the racks are equivalent.

In Chapter 4 we conclude the thesis, highlighting the main results and discussing future implications of this body of work.

1.1 The Yang-Baxter Equation

In this section we discuss the Yang-Baxter equation (YBE) and its solutions, known as $\mathcal{R}-$matrices. We restrict ourselves to considering unitary ($RR^* = R^*R = 1$) $\mathcal{R}-$matrices and denote this subset of solutions by $\mathcal{R}(d)$, where d is the dimension of the underlying vector space V.

1.1.1 The Yang-Baxter Equations

The most significant form of the Yang-Baxter Equation (YBE) is the Quantum YBE.

Definition 1. ***The Quantum Yang-Baxter Equation (QYBE)***: *Let V be a finite-dimensional vector space of dimension d with identity operator 1_V, and let $R \in End(V \otimes V)$. Then the Quantum YBE is given by*

$$(R \otimes 1_V)(1_V \otimes R)(R \otimes 1_V) = (1_V \otimes R)(R \otimes 1_V)(1_V \otimes R) \tag{1.2}$$

The solutions to the Quantum YBE are called $\mathcal{R}$-matrices.

Example 1. *The following matrix is an $\mathcal{R}$-matrix.*

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The calculation to show this is long but elementary, so it is left out here.

There are several forms of the YBE. It can also be defined on copies of a set $X \times X \times X$ as opposed to a vector space - this is the set-theoretic YBE.

Definition 2. ***The Set-Theoretic Yang-Baxter Equation***. *The set-theoretic YBE is defined on $X \times X \times X$, where X is a set, and is given by*

$$(r \times 1_X)(1_X \times r)(r \times 1_X) = (1_X \times r)(r \times 1_X)(1_X \times r) \tag{1.3}$$

where 1_X is the identity map on X.

Throughout this thesis we will describe the $\mathcal{R}$-matrix solutions to the Quantum YBE by utilising "Dirac" notation, also known as "bra-ket" notation.

1.1.2 Dirac Notation

An $\mathcal{R}$-matrix can be considered as a bounded operator acting on a Hilbert Space, $R \in \mathcal{B}(\mathscr{H})$. In this subsection we briefly define Hilbert spaces, bounded operators and give an overview of the Dirac (AKA "bra-ket") notation used throughout this thesis to describe $\mathcal{R}$-matrices.

Definition 3. ***Hilbert Space:*** *A Hilbert space $\mathscr{H}$ is a complete*[1] *inner product space.*

A "ket" $|x\rangle$ represents a column vector in a Hilbert space. For example, in a 2-dimensional space,

$$|x\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

A "bra" $\langle x|$ is the Hermitian conjugate of a ket. For example, for the above ket the corresponding bra is given by

$$\langle x| = (|x\rangle)^\dagger = \begin{pmatrix} x_1^* & x_2^* \end{pmatrix}$$

where x_1^* and x_2^* are the complex conjugates of x_1 and x_2 respectively.

The inner product of a bra and a ket is a complex number defined by

$$\langle x|y\rangle := \sum_i x_i^* y_i$$

The norm of an inner product space, in particular a Hilbert space, is defined by

$$||x|| := \langle x|x\rangle$$

An operator R acts on kets $|x\rangle$ and can be represented as a matrix in Dirac notation. This action is denoted by $A|x\rangle$.

The adjoint of an operator A is denoted A^* and is defined by

$$\langle Ax|y\rangle = \langle x|A^*y\rangle$$

Example 2. *Let A be any matrix with real coefficients, i.e.*

$$A \in M_{d\times d}, a_ij \in \mathbb{R} \quad \forall i, j$$

Then the adjoint A^* is given by the transpose A^T.

[1] The limit of every Cauchy sequence is contained in the space.

When we are dealing with multiple tensor copies of kets, such as an $\mathcal{R}$-matrix which operates on End $(V \otimes V)$, we use the following short-hand notation

$$R|x_1, x_2\rangle := R(|x_1\rangle \otimes |x_2\rangle)$$

Definition 4. ***Bounded operators on a Hilbert space:*** *The bounded Hilbert Space, $\mathcal{B}(\mathscr{H})$, is the set of all bounded operators on the Hilbert space $\mathscr{H}$. When paired with the norm $(\mathcal{B}(\mathscr{H}), \|\cdot\|)$ it forms a C^*-algebra*

$$\|A\| = \sup_{\psi \in \mathscr{H}} \frac{\|A\psi\|}{\|\psi\|}$$

Any $\mathcal{R}$-matrix is a bounded operator on a Hilbert space.

1.1.3 Types of Solution To The Yang-Baxter Equation

There are various special cases of $\mathcal{R}$-matrices, whose classifications we will explore further in this thesis.

Definition 5. ***Involutive $\mathcal{R}$-matrix:*** *An involutive $\mathcal{R}-$matrix is a matrix $R \in$ End$(V \otimes V)$ that satisfies the YBE* (1.2) *and has the following property*

$$R^2 = 1_V \tag{1.4}$$

where 1_V is the identity of the underlying vector space V.

The involutive $\mathcal{R}$-matrices were completely classified by Lechner, Pennig and Wood [25] in 2019 under the notion of equivalence defined by $R \sim S$ if and only if they have the same dimension and character.

In this thesis we consider the special case of unitary $\mathcal{R}$-matrices, $R \in \mathcal{R}(d)$, where $d := \dim V$ is the dimension of R. These are a particularly useful subset of Yang-Baxter solutions - for example, the braiding of anyons for the construction of quantum gates in topological quantum computing requires the describing $\mathcal{R}$-matrices to be unitary [11].

Definition 6. ***Unitary $\mathcal{R}$-matrix:*** *A unitary $\mathcal{R}-$matrix is a matrix $R \in$ End$(V \otimes V)$ that satisfies the YBE* (1.2) *and has the following property*

$$RR^* = R^*R = 1_V \tag{1.5}$$

where R^* is the adjoint of R. The set of all unitary $\mathcal{R}$-matrices is denoted $\mathcal{R}(d)$, where $d :=$dim$V$ is defined to be the dimension of the $\mathcal{R}-$matrix.

For $d < \infty$, one can clearly see that an $\mathcal{R}$-matrix $R|\mathbf{x}\rangle$ is unitary iff, $\forall \mathbf{x}, \mathbf{y} \in V$,

$$\langle R\mathbf{x}|R\mathbf{y}\rangle = \langle \mathbf{x}|\mathbf{y}\rangle$$

Example 3. *Define the Flip matrix $F|x_1, x_2\rangle := |x_2, x_1\rangle$. This is a classic example of a unitary $\mathcal{R}-$matrix. We see that it satisfies the YBE:*

$$\begin{aligned}(F \otimes 1_V)(1_V \otimes F)(F \otimes 1_V)|x, y, z\rangle &= (F \otimes 1_V)(1_V \otimes F)|y, x, z\rangle \\ &= (F \otimes 1_V)|y, z, x\rangle \\ &= |z, y, x\rangle \\ &= (1_V \otimes F)|z, x, y\rangle \\ &= (1_V \otimes F)(F \otimes 1_V)|x, z, y\rangle \\ &= (1_V \otimes F)(F \otimes 1_V)(1_V \otimes F)|x, y, z\rangle\end{aligned}$$

As for unitarity, we see that

$$\begin{aligned}\langle Fx_1, x_2|Fy_1, y_2\rangle &= \langle x_2, x_1|y_2, y_1\rangle \\ &= \langle x_1, x_2|y_1, y_2\rangle\end{aligned}$$

Hence, $F \in \mathcal{R}(d)$.

1.2 The Braid Group

In this section we define the braid group $\mathcal{B}_n$ on n strands and consider its presentation in terms of generators and relations as well as its diagrammatic presentation established by Artin [3]. We examine how any braid $\sigma \in \mathcal{B}_n$ induces a permutation $\psi_\sigma \in S_n$. Finally, we recall how $\mathcal{R}$-matrices form a representation of $\mathcal{B}_n$.

1.2.1 Defining the Braid Group

We now define the braid group.

Definition 7. ***Braid Group*** $\mathcal{B}_n$*: The braid group on n strands ($n \in \mathbb{N}$, $n \geq 2$), denoted $\mathcal{B}_n$, is the group generated by the elementary braids*[2] *b_i, $i \in 1, ..., n-1$, with the following relations*

[2]The elementary braids are the "building blocks" of braid concatenations, i.e. a single overlap.

$$b_i b_j = b_j b_i \quad \forall |i-j| \geq 2 \tag{1.6}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \tag{1.7}$$

In [3] Artin showed that $\mathcal{B}_n$ has an equivalent presentation in terms of diagrams of n strands. A generator b_j is presented by an overlap of the j^{th} strand over the $j+1^{th}$ strand, as in Figure 1.1. Its inverse b_j^{-1} is presented as the overlap of the $j+1^{th}$ strand over the j^{th} strand, as in Figure 1.2. Elements of $\mathcal{B}_n$ are presented as concatenations of generator diagrams, an example of which is demonstrated in Figure 1.3. Note that in this thesis we use the convention of reading diagrams top-to-bottom.

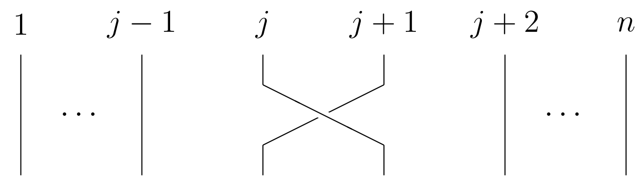


Figure 1.1: Diagrammatic representation of b_j, generator of $\mathcal{B}_n$.

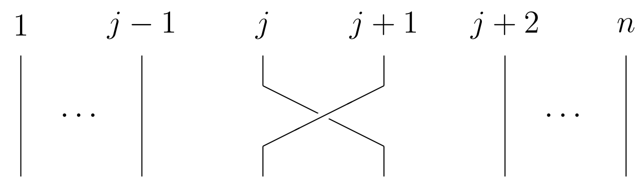


Figure 1.2: Diagrammatic representation of b_j^{-1}, generator of $\mathcal{B}_n$.

Example 4. *The braid $b_1 b_2^{-1} \in \mathcal{B}_4$ is presented diagrammatically as in Figure 1.3.*

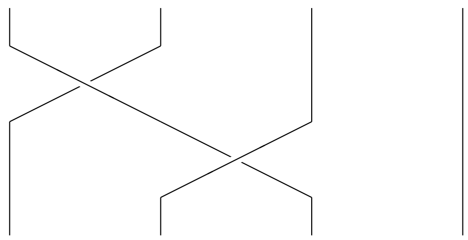


Figure 1.3: Diagrammatic presentation of $b_1 b_2^{-1} \in \mathcal{B}_4$

The braid Relations (1.6) and (1.7) are presented in Figures 1.4 and 1.5 respectively as elements of $\mathcal{B}_5$. Note that these diagrammatic equations can be seen to hold by allowing the strands to be manipulated by pulling on them in 2

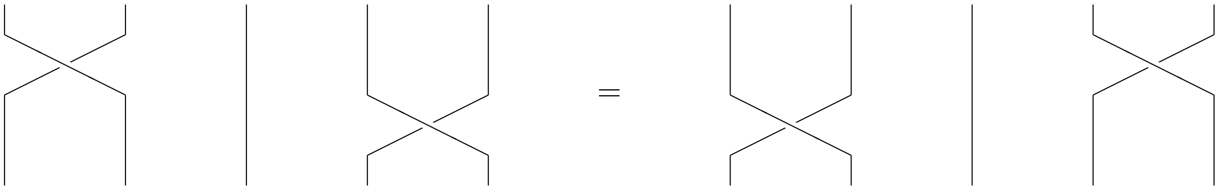


Figure 1.4: Diagrammatic presentation of equation (1.6) in $\mathcal{B}_5$

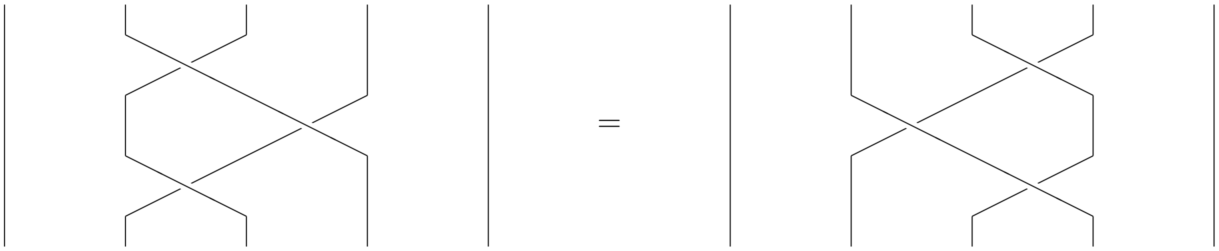


Figure 1.5: Diagrammatic presentation of equation (1.7) in $\mathcal{B}_5$

dimensions, with the end points on the top and bottom being fixed.

Any braid $\sigma \in \mathcal{B}_n$ can be considered as a braid in a larger braid group $\mathcal{B}_{n+1}$ by adding an additional identity strand to the right hand side, as in Figure 1.6.

Figure 1.6: Left: $b_1 \in \mathcal{B}_3$. Right: $b_1 \in \mathcal{B}_4$.

In this way, any braid can be considered as an element in any larger braid group. In particular, we may consider the infinite braid group $\mathcal{B}_\infty$ with a countably infinite number of strands, to which every braid belongs.

Definition 8. ***Infinite braid group*** $\mathcal{B}_\infty$: *The infinite braid group is the braid group* $\mathcal{B}_n$ *with an infinite amount of strands,* $n \to \infty$. *It is denoted* $\mathcal{B}_\infty$.

Example 5. *Figure 1.7 denotes* b_1 *as an element in the infinite braid group* $\mathcal{B}_\infty$.

Figure 1.7: $b_1 \in \mathcal{B}_\infty$.

The braid group has many interesting applications across mathematics, including modeling non-abelian anyons in topological quantum computing [11], braid cryptography [16], and a basis for knot theory (see Section 1.2.3).

1.2.2 Inducing Permutations from Braids

Any braid induces a permutation via the following surjective group homomorphism

$$\begin{aligned} \psi : \mathcal{B}_n &\to S_n \\ b_i &\mapsto \tau_i \end{aligned} \tag{1.8}$$

where $\tau_i \in S_n$ is the permutation $(i, i+1)$, i.e. swaps i and $i+1$ only. Note that $\tau_i^{-1} = \tau_i$, so b_i and b_i^{-1} both map to τ_i. Also, clearly every permutation can be written as the image of some braid under ψ. Hence this is a surjective but not injective map.

We denote the permutation induced by a braid $\sigma \in \mathcal{B}_n$ as $\psi_\sigma \in S_n$.

Diagrammatically this can be seen by "flattening"[3] the braid diagram and reading it in the opposite direction. For example:

Example 6. *Let* $\sigma = b_3 b_2^{-1} \in \mathcal{B}_5$.

Then $\psi_\sigma = \tau_3 \tau_2 \in S_5$, *as demonstrated in Figure 1.9.*

Explicitly this map is given by:

$$\begin{aligned} \psi_\sigma(1) &= 1 \\ \psi_\sigma(2) &= 4 \\ \psi_\sigma(3) &= 2 \\ \psi_\sigma(4) &= 3 \\ \psi_\sigma(5) &= 5 \end{aligned}$$

[3] I.e. ignoring the over/under-strands and only considering which element each element is mappted to.

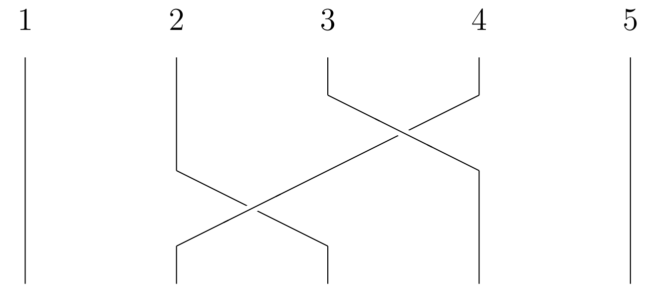


Figure 1.8: Diagrammatic representation of $b_3 b_2 \in \mathcal{B}_5$.

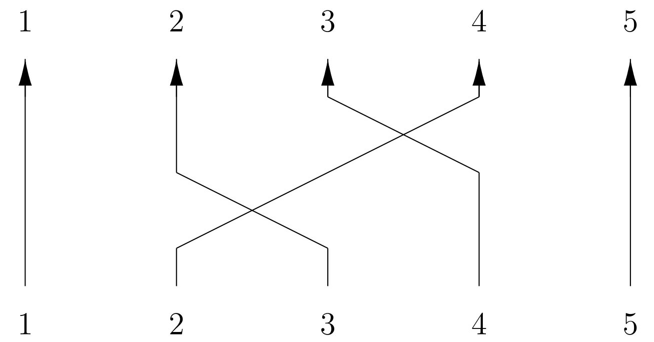


Figure 1.9: Diagrammatic representation of $\psi_{b_3 b_2} (= \tau_3 \tau_2) \in S_5$.

1.2.3 Closing braids

Braids can undergo "closure" to transform into links. In fact, it was shown by Alexander in 1923 that all links can be represented as a closed braid [1].

Definition 9. ***Closure of a braid:*** *The closure of a braid $b \in \mathcal{B}_n$, denoted $\hat{b}$, is an oriented link obtained by joining the top strands of a braid b to their respective bottom strands, as demonstrated in Figure 1.10. The orientation of the link arises from the braid being oriented top-to-bottom.*

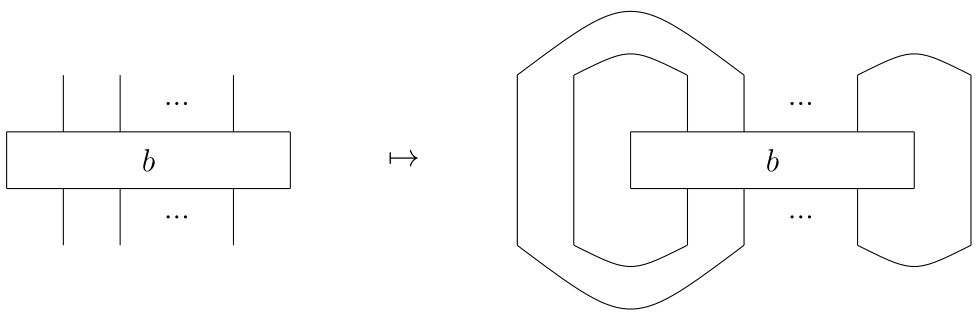


Figure 1.10: Closure of a braid $b \in \mathcal{B}_n$ to $\hat{b}$

Example 7. *The closure of* $\sigma = b_1 b_2 b_1 \in \mathcal{B}_3$ *is given in Figure 1.11. This is a so-called Hopf*[4] *link.*

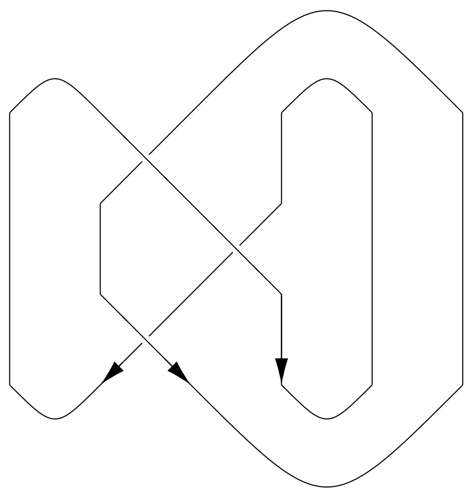


Figure 1.11: Closure of the braid $b_1 b_2 b_1$ to an oriented link.

Note that the correspondence between braids and links is not one-to-one; braids that can be transformed to each other under a sequence of the following Markov moves close to the same link [6].

Definition 10. ***Markov moves:*** *Let* $A, B \in \mathcal{B}_n$. *The Markov moves are given by*

$$\text{Markov move I (conjugation): } A \mapsto BAB^{-1}$$
$$\text{Markov move II (stabilization): } A \mapsto Ab_n^{\pm 1}$$

Note that the stabilisation results in a braid in $\mathcal{B}_{n+1}$.

Diagrammatically these moves are given by Figure 1.12.

1.2.4 Representation of $\mathcal{B}_n$ with $\mathcal{R}$-matrices

Any invertible $\mathcal{R}$-matrix $R \in \text{End}\,(V \otimes V)$ forms a representation of $\mathcal{B}_n$ by using the canonical shift endomorphism to operate on n tensor factors in the same pattern as a braid's overlaps on n strands.

[4] A link consisting of two circles linked together exactly once.

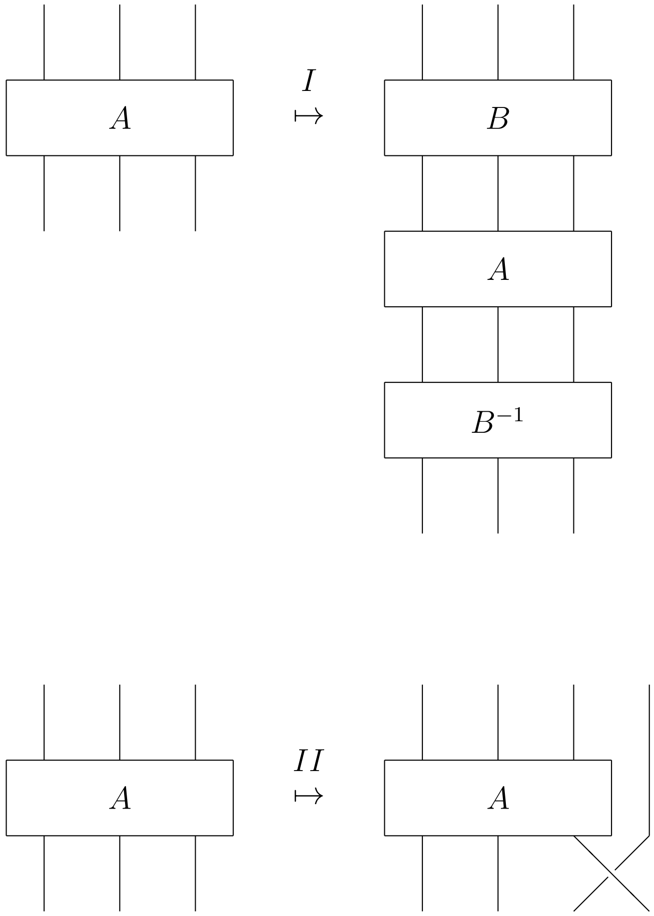


Figure 1.12: Markov moves I and II

Definition 11. ***Canonical shift endomorphism*** φ***:*** *Let* $R \in End\ (V \otimes V)$ *and consider it acting in* $V^{\otimes n}$ *(by considering* $R \otimes 1_V \otimes ... \otimes 1_V$*). Then the canonical shift endomorphism is defined by*

$$\begin{aligned} \varphi : V^{\otimes n} &\to V^{\otimes n} \\ R \underbrace{\otimes 1_V \otimes 1_V \otimes ... \otimes 1_V}_{n-2\ times} &\mapsto 1_V \otimes R \underbrace{\otimes 1_V \otimes ... \otimes 1_V}_{n-3\ times} \end{aligned} \tag{1.9}$$

We sometimes use the following shorthand notation for $R \in \mathcal{R}(d)$ operating on $V^{\otimes n}$:

$$R_k := \varphi^{k-1}(R) = \underbrace{1_V \otimes ... \otimes 1_V}_{k-1 \text{ times}} \otimes R \underbrace{\otimes 1_V \otimes ... \otimes 1_V}_{n-k-1 \text{ times}}$$

The Yang-Baxter endomorphism was defined by Cuntz [14] to be

$$\begin{aligned} \lambda_R : V^{\otimes \infty} &\to V^{\otimes \infty} \\ X &\mapsto \lim_{n \to \infty} R\varphi(R)\varphi^2(R)...\varphi^n(R) X \varphi^n(R^*)...\varphi^2(R^*)\varphi(R^*)R^* \end{aligned} \tag{1.10}$$

Using the fact that R satisfies the YBE and commutes with any operator that does not operate on the tensor factors it is itself operating on, we have that

$$\begin{aligned}\lambda_R(R) &= \lim_{n\to\infty} R\varphi(R)\varphi^2(R)...\varphi^n(R)X\varphi^n(R^*)...\varphi^2(R^*)\varphi(R^*)R^* \\ &= R\varphi(R)R\varphi(R^*)R^* \\ &= \varphi(R)R\varphi(R)\varphi(R^*)R^* \\ &= \varphi(R)\end{aligned}$$

It is important to highlight the following equation from the above:

$$\varphi(R) = \varphi(R)R\varphi(R)\varphi(R^*)R^* \tag{1.11}$$

This result will be particularly useful later on in this thesis. Note that this equation also holds for the adjoint R^* and for spectral projections of R, since they can be expressed as linear combinations of R.

We now recall how $\mathcal{R}$-matrices form a representation of the braid group.

Proposition 1. *Let $R \in End(V \otimes V)$ and let $\varphi : R \mapsto 1_V \otimes R$ define the canonical shift endomorphism on $V^{\otimes\infty}$. Then the following map defines a representation of the braid group*

$$\begin{aligned}\rho_R : \mathcal{B}_n &\to GL(V^{\otimes n}) \\ b_j &\mapsto \varphi^{j-1}(R)\end{aligned} \tag{1.12}$$

Proof. It is trivial to see that, for all $|i-j| \geq 2$,

$$\varphi^i(R)\varphi^j(R) = \varphi^j(R)\varphi^i(R)$$

This is because $\varphi^i(R)$ operates on tensor factors i and $i+1$, and thus will commute with any operator that does not operate on these factors.

It is clear to see that the representation of the second braid equation is the Yang-Baxter equation, which is satisfied by the definition of an $\mathcal{R}$-matrix. □

This representation allows us to diagrammatically represent equations of $\mathcal{R}$-matrices.

Using $\mathcal{R}$-matrices to represent groups and algebras is the main technique used throughout this thesis, as it allows us to analyse the original structure as well as to classify types of $\mathcal{R}$-matrices by the structure that they arise from.

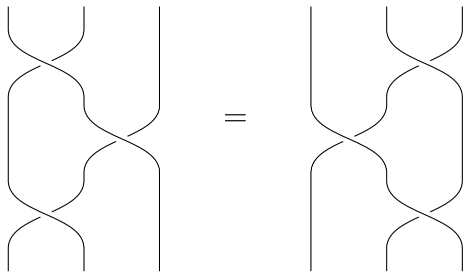


Figure 1.13: Diagrammatic presentation of the Yang-Baxter equation (1.2)

1.3 Traces and Equivalence of $\mathcal{R}$-Matrices

In this section we examine traces and partial traces, and explore various notions of equivalence of $\mathcal{R}$-matrices.

1.3.1 Traces and Partial Traces of $\mathcal{R}$-Matrices

We first define traces on a general matrix, then examine the various types of traces used in analysing $\mathcal{R}$-matrices.

Definition 12. ***Trace of a matrix:*** *Let V be a finite-dimensional vector space with basis B_V and $M \in End\ (V^{\otimes n})$. Taking the trace in the x_k^{th} tensor space is defined by*

$$Tr_{x_k}(M) := \sum_{x_k \in B_V} \langle x_1, ..., x_k, ..., x_n | M | x_1, ..., x_k, ..., x_n \rangle$$

where $\langle i|j\rangle = \delta_{i,j}$, where δ is the Kronecker delta.

Taking the full trace is defined by

$$Tr_{x_1,...,x_n}(M) := (\underbrace{Tr \otimes Tr \otimes ... \otimes Tr}_{n\ times})(M) = \sum_{x_1,...,x_n \in B_V} \langle x_1, ..., x_n | M | x_1, ..., x_n \rangle$$

Therefore the trace of an $\mathcal{R}$-matrix $R \in \mathcal{R}(d)$ that operates over the vector space V is given by:

$$Tr(R) := \sum_{x,y \in V} \langle x, y | R | x, y \rangle \tag{1.13}$$

since $R \in \text{End}\ (V \otimes V)$.

When dealing with $\mathcal{R}$-matrices it is often useful to normalise this trace with the dimension of the $\mathcal{R}$-matrix.

Definition 13. ***Normalised trace of an $\mathcal{R}$-matrix:*** *Let $R \in \mathcal{R}(d)$ operate over the vector space V, which has dimension d. Then the normalised trace of R is defined by:*

$$\tau(R) := \frac{1}{d^2} Tr(R) = \frac{1}{d^2} \sum_{x,y \in B_V} \langle x, y | R | x, y \rangle \tag{1.14}$$

The normalised trace τ is the main trace used in this thesis, but given a space such as a von-Neumann algebra (see section 1.5) there could be many different notions of a trace. We now define what it is to be a trace in a general von-Neumann algebra.

Definition 14. ***Trace:*** *A trace $tr_{\mathscr{A}}$ on a von Neumannn algebra $\mathscr{A}$, is a map $tr_{\mathscr{A}} : \mathscr{A} \to \mathbb{C}$ such that*

$$tr_{\mathscr{A}}(AB) = tr_{\mathscr{A}}(BA) \quad \forall A, B \in \mathscr{A}$$

This is known as the "tracial property",

Note that the trace on a given von Neumann algebra is always unique [13].

A trace may also have additional properties, such as positivity and faithfulness.

Definition 15. ***Positive Trace:*** *A trace $tr_{\mathscr{A}}$ on a von Neumannn algebra $\mathscr{A}$ is said to be positive iff the trace of any positive operator $A \in \mathscr{A}$, such that $A^*A \geq 0$), is positive, i.e.*

$$tr_{\mathscr{A}}(A^*A) \geq 0 \quad \forall A \in \mathscr{A}$$

Definition 16. ***Faithful Trace:*** *A trace $tr_{\mathscr{A}}$ on a von Neumannn algebra $\mathscr{A}$ is said to be faithful iff the trace is only 0 when the operator is 0, i.e.*

$$tr_{\mathscr{A}}(A) = 0 \implies A = 0 \quad \forall A \in \mathscr{A}$$

These two properties of traces can be combined into one equation. We say that $tr_{\mathscr{A}}$ is positive and faithful iff

$$tr_{\mathscr{A}}(A^*A) = 0 \iff A = 0 \quad \forall A \in \mathscr{A}$$

It is clear that our $\mathcal{R}$-matrix trace is positive and faithful. Indeed, for any $A \in \mathcal{R}_d$, we have that:

$$
\begin{aligned}
Tr(A^*A) = 0 &\implies \sum_{x,y} \langle x, y | A^*A | x, y\rangle = 0 \\
&\implies \sum_{x,y} \langle A(x,y) | A(x,y)\rangle = 0 \\
&\implies A = 0
\end{aligned}
$$

And:

$$
\begin{aligned}
A = 0 \implies Tr(A^*A) &= Tr(0^*0) \\
&= \sum_{x,y} \langle x, y | 0 | x, y\rangle \\
&= \sum_{x,y} \langle x, y | 0\rangle \\
&= 0
\end{aligned}
$$

This is especially obvious for unitary $\mathcal{R}$-matrices, as we have that $RR^* = 1_V \neq 0$ unless $R = 0$.

On $\mathcal{R}$-matrices an important concept is considering the trace on only one tensor factor. Taking the trace on only the left or right tensor factor of an $\mathcal{R}$-matrix is referred to as taking the partial trace.

Definition 17. ***Partial traces of an $\mathcal{R}$-matrix:*** *The left partial trace of an $\mathcal{R}$-matrix $R \in End\ (V \otimes V)$ is given by*

$$
Lptr(R) = (Tr \otimes 1_V)(R)
$$

The right partial trace is defined by

$$
Rptr = (1_V \otimes Tr)(R) \tag{1.15}
$$

In [13] it is proven that for all unitary $\mathcal{R}$-matrices the left and right partial traces coincide. Since in this thesis we solely focus on unitary $\mathcal{R}$-matrices, we simply write *ptr* to mean either.

The trace of a product of $\mathcal{R}$-matrices can be considered diagrammatically by looking at the closure of the related braid diagram and summing over all labels in the underlying vector space. The partial trace can be considered by only closing the left or right strand, depending upon if you are considering the left or right partial trace.

For example, if we label $\rho_R(b_1)$ in the following way we see that its trace is given by the closure of the braid and its right partial trace is given by the closure of the right strand.

Example 8. *Let* $R|x,y\rangle := |r_1(x,y), r_2(x,y)\rangle \in \mathcal{R}(d)$. *Its trace and (right) partial trace are given by*

$$Tr(R) = \sum_{x,y} \langle x, y | r_1(x,y), r_2(x,y)\rangle$$
$$ptr(R) = \sum_{y} |r_2(x,y)\rangle\langle y|$$

Now look at the conditions imposed from the closure of b_1 *in Figure 1.14.*

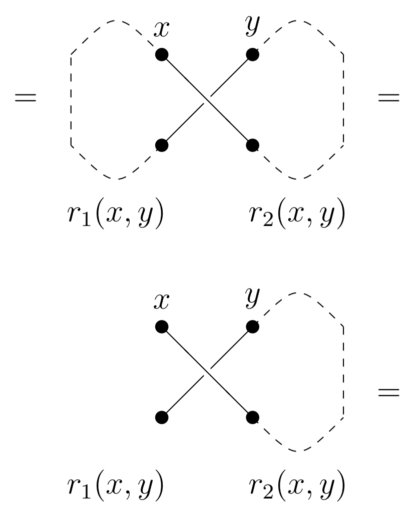


Figure 1.14: Top: closure of both strands of b_1 (diagrammatic presentation of taking the trace). Bottom: closure of the right strand of b_1 (diagrammatic presentation of taking the right partial trace).

By summing over all $x, y \in V$, *we see that closing the braid is exactly taking the trace and closing one strand is exactly taking the partial trace of the* $\mathcal{R}$*-matrix.*

The above notion holds true in general and is particularly useful for products of $\mathcal{R}$-matrices.

The normalised trace τ of an $\mathcal{R}$-matrix arising from a representation of a brad group is called the character.

Definition 18. ***Character:*** *Let $R \in \mathcal{R}(d)$ and let ρ_R denote the braid group representation induced by R. Then, the character of R, denoted τ_R, is defined by:*

$$\tau_R := \tau \circ \rho_R \tag{1.16}$$

1.3.2 Notions of Equivalence of $\mathcal{R}$-Matrices

The classification of $\mathcal{R}$-matrices up to equivalence is a large ongoing area of research. There are various notions of equivalence of $\mathcal{R}$-matrices, some of which have stronger conditions than others.

Definition 19. ***Unitary Equivalence:*** *Two $\mathcal{R}$-matrices $R, S \in \mathcal{R}(d)$ are said to be unitarily equivalent, denoted $R \approx S$, iff $\exists$ some unitary $U \in End\ (V \otimes V)$ such that*

$$R = USU^*$$

Here, U is called a (unitary) "intertwiner".

The following notion of equivalence, that of equivalence of braid representations, is one of the strongest notions of equivalence. This is the notion of equivalence we consider in this thesis. In [13] it is shown that equivalence of braid representations can be defined as follows.

Definition 20. ***Equivalence of $\mathcal{R}$-matrices:*** *Two $\mathcal{R}$-matrices $R, S \in \mathcal{R}(d)$ are said to be* ***equivalent*** *iff they have the same dimension and character, i.e.*

$$R \sim S \iff dim\ (R) = dim\ (S)\ and\ \tau_R = \tau_S$$

where τ_R is defined in Definition 18.

Equivalence is a stronger condition than unitary equivalence. In fact, equivalence implies unitary equivalence. Below we state and prove a proposition regarding notions of equivalence.

Proposition 2. *Let $R \in \mathcal{R}(d)$ be a unitary $\mathcal{R}$-matrix of dimension d, and let U be a unitary $d \times d$ matrix.*
Suppose R commutes with $U \otimes U$. Then $S \in \mathcal{R}(d)$ and $R \sim S$, where

$$S = (\mathbb{1} \otimes U)R(\mathbb{1} \otimes U^*)$$

Proof. This statement is equivalent to stating that the representations ρ_R^n and ρ_S^n of the braid Group $\mathcal{B}_n$ are unitarily equivalent for all $n \in \mathbb{N}$ [25]. Explicitly, this means finding a unitary matrix $V_n \in \text{End}(V^{\otimes n})$ such that

$$V_n \rho_R^n(b_i) V_n^* = \rho_S^n(b_i) \quad \text{for } i = 1, ..., n-1$$

where b_i are the generators of $\mathcal{B}_n$.

We claim that an appropriate intertwiner is given by

$$V_n := \mathbb{1} \otimes U \otimes U^2 \otimes ... \otimes U^{n-1}$$

Indeed,

$$\begin{aligned} V_n \rho_R(b_i) V_n^* &= (\mathbb{1} \otimes U \otimes ... \otimes U^{n-1})(\mathbb{1} \otimes ... \otimes R \otimes ... \otimes \mathbb{1})(\mathbb{1} \otimes U^* \otimes ... \otimes U^{*n-1}) \\ &= \mathbb{1} \otimes ... \otimes \mathbb{1} \otimes \Big((U^{i-1} \otimes U^i) R (U^{*i-1} \otimes U^{*i}) \Big) \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \end{aligned}$$

The centre section can be re-written as

$$\begin{aligned} (U^{i-1} \otimes U^i) R (U^{*i-1} \otimes U^{*i}) &= (\mathbb{1} \otimes U)(U \otimes U)^{i-1} R (U^* \otimes U^*)^{i-1} (\mathbb{1} \otimes U^*) \\ &= (\mathbb{1} \otimes U) R (\mathbb{1} \otimes U^*) \\ &= S \end{aligned}$$

Thus,

$$V_n \rho_R(b_i) V_n^* = \rho_S(b_i)$$

And so V_n intertwines ρ_R^n and ρ_S^n as required.

As for S satisfying the YBE, it is clear from the definition of S, the unitarity of U and the fact that R satisfies the YBE that this is the case.

□

1.4 Spectral Projections of $\mathcal{R}$-Matrices

In this section we briefly define spectral projection of matrices.

Let $R \in M_d$ be a $d \times d$ matrix with spectrum $\sigma(R) = \{\beta_0, \beta_1, \beta_2, ..., \beta_{n-1}\}$ so that R has precisely n distinct eigenvalues, where $n \leq d$. Then we may decompose R in terms of its spectral projections in the following way

$$R = \sum_{m=0}^{n-1} \beta_m Q_m$$

where Q_m is the spectral projection associated to eigenvalue β_m, for all $m = 0, 1, ..., n-1$.

Recall that all spectral projections satisfy the following equations for all i.

$$\begin{aligned}
&Q_i^2 = Q_i^* = Q_i \\
&Q_i Q_j = \delta_{ij} Q_i \\
&Q_i R = R Q_i = \beta_i Q_i \\
&\sum_i Q_i = 1_{V \otimes V}
\end{aligned}$$

where * denotes the adjoint and δ_{ij} is the Kronecker delta.

Using these properties we may derive a formula for any spectral projection Q_i of R in terms of a polynomial in R:

$$Q_i = \prod_{m=0,\dots,n=1(\neq i)} \frac{1}{\beta_i - \beta_m}(R - \beta_m \cdot \mathbb{1}) \tag{1.17}$$

This polynomial will become relevant in Chapter 2.

1.5 Inducing Factors from $\mathcal{R}$-Matrices

In this section we introduce Hilbert spaces, von Neumann algebras, factors, subfactors and the Jones Index [18]. We introduce $\mathscr{L}_R := \rho_R(\mathcal{B}_\infty)''$, which is a type II_1 factor induced from an $\mathcal{R}$-matrix.

Let $\mathcal{B}(\mathscr{H})$ denote the set of bounded operators on a Hilbert space $\mathscr{H}$. We now define some concepts on this space.

Definition 21. ***von Neumann algebra:*** *A von Neumann algebra is a unital *-subalgebra of $\mathcal{B}(\mathscr{H})$ that is closed in the weak operator topology, i.e. for every net sequence $\langle A_n\psi, \phi\rangle$, where A_n is a sequence operator and ψ, ϕ are vectors in $\mathscr{H}$, its limit $\lim_{n\to\infty}\langle A_n\psi, \phi\rangle = \langle A\psi, \phi\rangle$ is contained in the von Neumann algebra.*

Definition 22. ***(Von Neumann) Factor:*** *A factor is a von Neumann algebra with trivial centre*

There are four main types of factor, I_n, II_1, II_∞ and III_λ. The most important type of factors for this area of research is the II_1 factor.

Definition 23. ***Type*** II_1 ***factor:*** *a type* II_1 *factor is an infinite-dimensional von Neumann factor that has a unique trace that is positive, normalised and faithful.*

Definition 24. ***Subfactor:*** *A subfactor is a factor that is fully contained in another factor. i.e. a subfactor is a subalgebra that is also a factor.*

The commutant, in particular the bicommutant $\mathscr{F}'' := (\mathscr{F}')'$, is especially important in the theory of von Neumann algebras as it allows us to construct a von Neumann algebra from any unital $*$-subalgebra of $\mathcal{B}(\mathscr{H})$ using the bicommutant theorem.

Definition 25. ***Commutant:*** *The commutant of a set* $\mathscr{F} \subset \mathcal{B}(\mathscr{H})$ *of bounded operators is the set of all bounded operators that commute with every element in the subset* $\mathscr{F}$*. The commutant is denoted* $\mathscr{F}'$ *and is defined by*

$$\mathscr{F}' := \{T \in \mathcal{B}(\mathscr{H}) : TS = ST \quad \forall S \in \mathscr{F}\} \tag{1.18}$$

We now state the Bicommutant theorem.

Theorem 1. ***Bicommutant theorem:*** *Let* $\mathscr{F} \subset B(\mathscr{H})$ *be a unital* $*$*-subalgebra of* $\mathcal{B}(\mathscr{H})$*. Then*

$$\mathscr{F}'' = \bar{\mathscr{F}}^{WOT}$$

Where WOT is the weak operator topology.[4]

Thus, the bicommutant of a subset of bounded Hilbert operators is a von Neumann algebra.

Example 9. *Let* $R \in \mathcal{R}(d)$ *and define* $\mathscr{L}_R := \rho_R(\mathcal{B}_\infty)''$*. This is a von Neumann algebra by the bicommutant theorem. It is a type* II_1 *factor as it is infinite-dimensional and* τ *satisfies the necessary tracial conditions.*

Note that this is clearly a subfactor of $\varphi(\mathscr{L}_R)$*.*

Given a factor and a subfactor one can consider their Jones Index, which was originally intended to be a type of measure of the relative size of the subfactor in the factor.

Definition 26. ***Jones Index:*** *The Jones index of a type II_1 subfactor $\mathscr{N} \subseteq \mathscr{M}$ is defined by*

$$[\mathscr{N} : \mathscr{M}] := dim_{\mathscr{N}}(L^2(\mathscr{M})) \tag{1.19}$$

i.e. the dimension of $\mathscr{M}$ considered as an $\mathscr{N}$-module.

The Jones Index is used as a way of differentiating different subfactors, and its values have some interesting restrictions.

It is shown in [21] that the Jones index can only take the following values for type II_1 factors

$$[\mathscr{N} : \mathscr{M}] = \{4cos^2(\pi/n) : n = 3, 4, 5...\} \cup [4, \infty) \tag{1.20}$$

The lowest value the Jones index can take is 1, but this only occurs if $\mathcal{M} = \mathcal{N}$.

This result, along with the other restrictions on the Jones Index in [13], will be utilised in Section 2.6.3 to form restrictions on the contraction constant.

Chapter 2

$\mathcal{R}$-matrix Representations of the BMW Algebra

In this chapter we explore the BMW algebra $\mathcal{C}_n(r,q)$ and define the "contractive" $\mathcal{R}$-matrices, of which a restriction forms a representation of the BMW algebra. We then discuss examples and properties of the contractive $\mathcal{R}$-matrices, especially looking at restrictions on the possible values the contraction constant c can take.

The BMW algebra $\mathcal{C}_n(r,q)$ is a deformation of the Brauer algebra established in the 1980s [7]. It has various applications, most notably in knot theory as it has strong connections to the Kauffmann polynomial, as well as quantum groups, statistical mechanics, and topological quantum field theory.

Some exploration of $\mathcal{R}$-matrix representations has appeared in the literature [19], but this has focused on the skew-invertible BMW-type $\mathcal{R}$-matrices and follows a different line of inquiry as in this thesis, as it does not focus on the classification of examples of unitary $\mathcal{R}$-matrices up to equivalence.

2.1 The BMW Algebra

In this section we describe the BMW algebra in terms of generators and relations as well as its diagrammatic presentation.

Definition 27. ***The BMW Algebra*** $\mathcal{C}_n(r,q)$***:*** *The BMW algebra (Birman-Murakami-Wenzl algebra)* $\mathcal{C}_n(r,q)$*, where* $n \in \mathbb{N}$ *and* $r, q \in \mathbb{T}$[1]*,* $q \neq q^{-1}$ *is generated by* $g_1, g_2, ..., g_{n-1}$ *with the relations*

[1] $\mathbb{T}$ is used to denote the unit circle

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{2.1}$$

$$g_i g_j = g_j g_i \quad \text{if } |i-j| \geq 2 \tag{2.2}$$

$$e_i g_i = r^{-1} e_i \tag{2.3}$$

$$e_i g_{i-1}^{\pm 1} e_i = r^{\pm 1} e_i \tag{2.4}$$

where e_i is defined by

$$e_i = 1 - \frac{1}{q - q^{-1}} \left(g_i - g_i^{-1} \right) \tag{2.5}$$

In [7] it is shown that $\mathcal{C}_n(r, q)$ has a diagrammatic presentation. The diagrams of its generators are shown in Figure 2.1 and Figure 2.2.

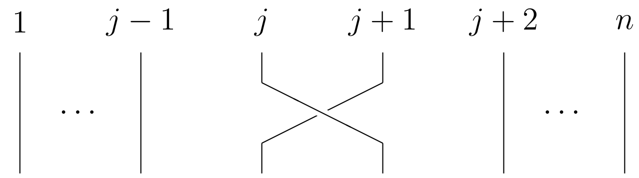


Figure 2.1: Diagrammatic representation of g_j in $\mathcal{C}_n(r, q)$

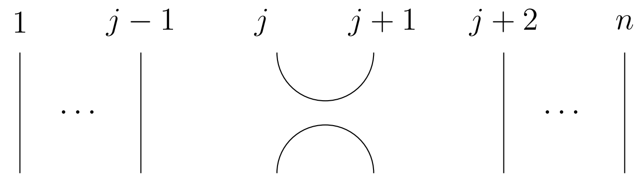


Figure 2.2: Diagrammatic representation of e_j in $\mathcal{C}_n(r, q)$

Example 10. *Let us look at an example in $\mathcal{C}_3(r, q)$. By Equation* (2.4)*, we have that*

$$e_2 g_1 e_2 = r e_2$$

Diagrammatically this is presented by Figure 2.3.

We now make some observations that will later motivate our choices in the representation of this algebra.

Note that the first two equations of the definition of the BMW algebra, (2.1) and (2.2), are precisely the braid equations.

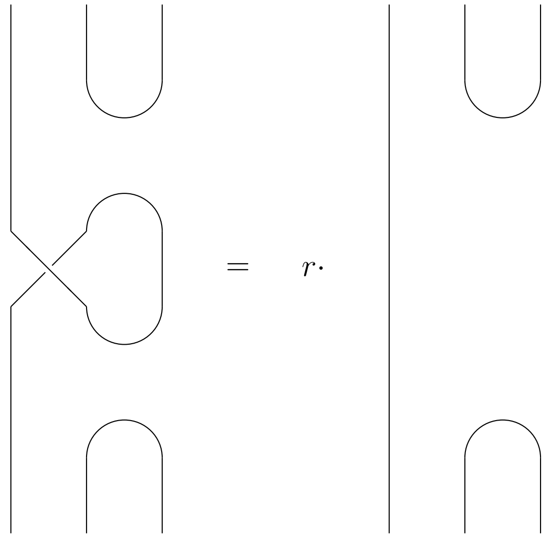


Figure 2.3: Diagrammatic presentation of $e_2 g_1 e_2 = re_2$ in $\mathcal{C}_3(r,q)$, the BMW algebra on 3 strands

Remark 1. *See also that Equation* (2.3) *is an eigenvalue equation for* e_i*.*

Also, note that multiplying Equation (2.5) *(the defining equation for* e_i*) by* e_i *yields*

$$\begin{aligned} e_i^2 &= e_i - \frac{1}{q-1^{-1}}(e_i g_i - e_i g_i^{-1}) \\ &= e_i - \frac{1}{q-1^{-1}}(r^{-1}e_i - re_i) \\ &= e_i\Big(1 + \frac{r-r^{-1}}{q-q^{-1}}\Big) \\ &= xe_i \end{aligned} \tag{2.6}$$

where $x := (1 + \frac{r-r^{-1}}{q-q^{-1}})$.

This has interesting implications in the diagrammatic presentation of the BMW algebra, as laid out in the following example. This will form part of the motivation of considering the Temperley-Lieb algebra in Section 2.2.3.

Example 11. *Consider* $e_1^2 \in \mathcal{C}_3(r,q)$*. By equation* (2.6)*, we have that*

$$e_1^2 = xe_1$$

Diagrammatically this is shown in Figure 2.4. This shows that x *is analogous to the loop parameter in the Temperley-Lieb algebra, which we explore in more detail in Section 2.2.3.*

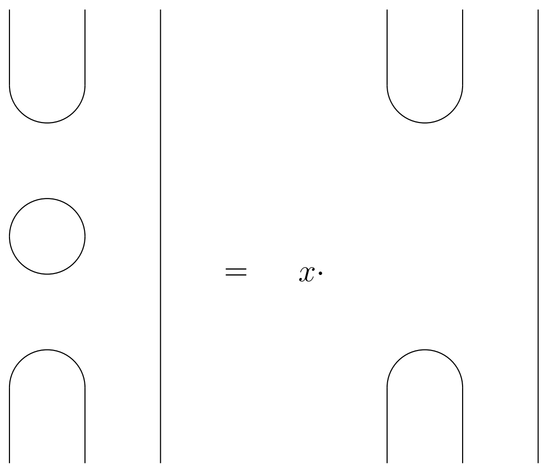


Figure 2.4: Diagrammatic presentation of $e_1^2 = xe_1$ in $\mathcal{C}_3(r,q)$.

Equation (2.6) means that e_i is a multiple of the characteristic idempotent p_i belonging to the characteristic value r^{-1} of g_i. Since equation (2.3) is exactly an eigenvalue equation, we see that e_i operates analogously to a multiple of a spectral projection.

Recall that we assume that r and q lie on the unit circle, i.e. $r, q \in \mathbb{T}$. This is in order to force a trace to be positive by ensuring its weight-vector components are non-negative. This will then induce a well-defined inner product $\langle a, b\rangle := tr(b^*a)$ on a C^*-algebra.

These observations motivate us to form an $\mathcal{R}$-matrix representation of $\mathcal{C}_n(r,q)$ in Section 2.2.4. First, we define a class of $\mathcal{R}$-matrices that satisfy the Contraction Relation, inspired by equation (2.4).

2.2 Contractive R-Matrices

In this section we define contractive $\mathcal{R}$-matrices and define a restriction on them to form a representation of the BMW algebra. We show that contractive $\mathcal{R}$-matrices must have 3 eigenvalues and we explore some examples.

2.2.1 The Contraction Relation

In this section we define the Contraction Relation, which we then use to define contractive $\mathcal{R}$-matrices. The contraction relation is a generalisation of Equation (2.4) in terms of $\mathcal{R}$-matrices.

Definition 28. ***Contraction relation****: The contraction relation is an equation in terms of a d-dimensional $\mathcal{R}$-matrix R and a spectral projection of R denoted by P. The contraction relation is defined by:*

$$\varphi(P)R\varphi(P) = c \cdot \varphi(P) \tag{2.7}$$

where φ is the canonical shift endomorphism $\varphi : R \otimes 1 \mapsto 1 \otimes R$ and $c \in \mathbb{C}$.

Restrictions of the possible values of c are explored in Section 2.6.

We now define a set of $\mathcal{R}$-matrices defined by whether or not they are a solution to the contraction relation.

Definition 29. ***Contractive R-matrix****: A contractive $\mathcal{R}$-matrix is a unitary $\mathcal{R}$-matrix $R \in \mathcal{R}(d)$ that has a spectral projection P such that R and P satisfy the contraction relation.*

We define the set of all d-dimensional contractive R-matrices as $\mathcal{C}_\alpha(d)$, where α is the contractive eigenvalue.

The set of all contractive R-matrices with arbitrary eigenvalues is defined to be $\mathcal{C}(d) = \cup_\alpha \mathcal{C}_\alpha(d)$.[2]

We formally define the projection P, its associated eigenvalue α and the constant c in the contraction relation below.

Definition 30. ***Contractive Projection:*** *A contractive projection P of a contractive $\mathcal{R}$-matrix $R \in \mathcal{R}(d)$ is a spectral projection of R such that R and P satisfy the contraction relation $\varphi(P)R\varphi(P) = c \cdot \varphi(P)$ for some $c \in \mathbb{C}$.*

Definition 31. ***Contractive Eigenvalue:*** *A contractive eigenvalue α of a contractive $\mathcal{R}$-matrix $R \in \mathcal{R}(d)$ is an eigenvalue of R such that its associated spectral projection P is a contractive projection of R.*

Definition 32. ***Contractive Constant:*** *A contractive constant c of a contractive $\mathcal{R}$-matrix $R \in \mathcal{R}(d)$ is a complex number $c \in \mathbb{C}$ such that, for some contractive projection P of R, the contraction relation $\varphi(P)R\varphi(P) = c \cdot \varphi(P)$ is satisfied.*

2.2.2 Eigenvalue Properties of Contractive $\mathcal{R}$-Matrices

Before forming a representation of the BMW algebra, we first show that the spectrum of a contractive $\mathcal{R}$-matrix must have the following spectrum structure:

$$\sigma(R) = \{\alpha, \beta, \gamma\} \quad \alpha, \beta, \gamma \in \mathbb{T} \quad \forall R \in \mathcal{C}(d)$$

[2]We note that this has the structure of a vector space.

Consider the following representation map of the BMW algebra.

$$\begin{aligned}\pi_R : \mathcal{C}_n(r,q) &\to \mathrm{End}(V^{\otimes n})\\ \pi_R(g_i) &:= \varphi^{i-1}(R)\\ \pi_R(e_i) &:= \varphi^{i-1}(xP)\end{aligned}$$

where

$$x = \Big(1 + \frac{r - r^{-1}}{q - q^{-1}}\Big)$$

It is shown in Theorem 3 that this does indeed form a representation of the BMW algebra with some restrictions.

For $\pi_R(g_i) := \varphi^{i-1}(R)$, we show that R will have at most 3 eigenvalues. This is because g_i has a maximum of 3 characteristic values. We can clearly see that r^{-1} is a characteristic value from equation (2.3). We claim that the other 2 characteristic values are q and $-q^{-1}$. This can be seen by inputting the defining equation for e_i (2.5) into the characteristic equation for r (2.3):

$$\begin{aligned}e_i g_i &= r^{-1} e_i\\ \Big(1 - \frac{g_i - g_i^{-1}}{q - q^{-1}}\Big) g_i &= r^{-1}\Big(1 - \frac{g_i - g_i^{-1}}{q - q^{-1}}\Big)\\ g_i(q - q^{-1}) - g_i^2 + 1 &= r^{-1}(q - q^{-1}) - r^{-1} g_i + r^{-1} g_i^{-1}\\ q g_i^2 - q^{-1} g_i^2 - g_i^3 + g_i &= r^{-1} q g_i - r^{-1} q^{-1} g_i - r^{-1} g_i^2 + r^{-1}\\ 0 &= g_i^3 + g_i^2(-r^{-1} + q^{-1} - q) + g_i(-1 + r^{-1} q - r^{-1} q^{-1}) + r^{-1}\\ &= (g_i - r^{-1})(g_i - q)(g_i + q^{-1})\end{aligned}$$

Therefore there are at most 3 characteristic values on g_i, namely r^{-1}, q, and $-q^{-1}$. We thus require $|\sigma(R)| \leq 3$.

Clearly $|\sigma(R)| = 1$ is the trivial case of the identity, so we do not consider this possibility.

As for $|\sigma(R)|$=2, all unitary $\mathcal{R}$-matrices of spectrum size 2 have already been classified. These matrices form a representative of the Temperley-Lieb algebra. We give a brief overview of Temperley-Lieb $\mathcal{R}$-matrices below in Section 2.2.3. We consider the spectrum of size 3 thereafter.

2.2.3 Temperley-Lieb

The Temperley-Lieb algebra is defined as follows.

Definition 33. ***Temperley-Lieb Algebra:*** *The Temperley-Lieb algebra* $TL_n(\delta)$ *is a unital algebra over* $\mathbb{C}$ *with generators* $T_1, ..., T_{n-1}$ *and relations*

$$\begin{aligned} T_i^2 &= \delta \cdot T_i \\ T_i T_{i+1} T_i &= T_i \\ T_i T_j &= T_j T_i \quad \forall |i-j| \geq 2 \end{aligned}$$

where $n \in \mathbb{N}$, $n \geq 2$ *is the number of strands and* $\delta \in \mathbb{C}$.

This has a diagrammatic presentation as in Figure 2.5.

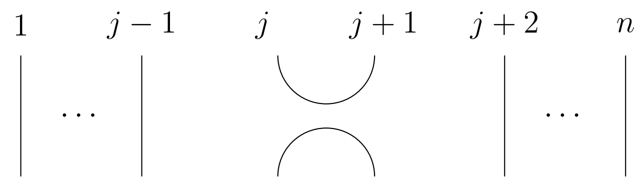


Figure 2.5: Diagrammatic presentation of T_j in the Temperley-Lieb algebra $TL_n(\delta)$.

Let us consider the element $S \in TL_n(\delta)$ defined by $T_i = \delta S_i$ for all $i = 1, 2, ..., n-1$. Then we have that

$$\begin{aligned} T_i^2 = \delta \cdot T_i &\implies \delta^2 S_i^2 = \delta \cdot \delta S_i \implies S_i^2 = S_i \\ T_i T_{i+1} T_i = T_i &\implies \delta^3 S_i S_{i+1} S_i = \delta S_i \implies S_i S_{i+1} S_i = \delta^{-2} S_i \\ T_i T_j = T_j T_i \quad \forall |i-j| \geq 2 &\implies S_i S_j = S_j S_i \quad \forall |i-j| \geq 2 \end{aligned}$$

Hence, the defining relations of the Temperley-Lieb algebra may be rewritten as

$$\begin{aligned} S_i^2 &= S_i \\ S_i S_{i+1} S_i &= \delta' S_i \\ S_i S_j &= S_j S_i \quad \forall |i-j| \geq 2 \end{aligned} \tag{2.8}$$

where $\delta' := \delta^{-2}$. We henceforth refer to δ' as the loop parameter, and Equation (2.8) as the Temperley-Lieb equation.

We consider a contractive $\mathcal{R}$-matrix R with precisely two eigenvalues[3] and denote the spectrum without loss of generality[4] to be

$$\sigma(R) = \{-1, q\}$$

Thus, the spectral decomposition of R is given by

$$R = -P + q(1_V - P) \tag{2.9}$$

Where 1_V is the unit of the vector space V on which $R \in V \otimes V$.

By the Yang-Baxter equation we have that

$$\begin{aligned}
&(-P + q(1_V - P)) \cdot \varphi(-P + q(1_V - P)) \cdot (-P + q(1_V - P)) = \\
&\quad \varphi(-P + q(1_V - P)) \cdot (-P + q(1_V - P)) \cdot \varphi(-P + q(1_V - P)) \\
\implies& P\varphi(P)P - \frac{q}{(1+q)^2}P = \varphi(P)P\varphi(P) - \frac{q}{(1+q)^2}\varphi(P)
\end{aligned}$$

By Theorem 7, P satisfies the Temperley-Lieb equations, i.e.

$$\begin{aligned}
P\varphi(P)P &= \delta' P \\
\varphi(P)P\varphi(P) &= \delta' \varphi(P)
\end{aligned}$$

Hence, we have that

$$\delta' = \frac{q}{(1+q)^2} \tag{2.10}$$

We will now use restrictions on the possible values of the loop parameter to limit the possible eigenvalues a contractive $\mathcal{R}$-matrix with $|\sigma(R)| = 2$ can have.

Theorem 2. *Let R be a contractive $\mathcal{R}$-matrix with contractive projection P.*

Suppose R has the following spectral decomposition:

$$R = -P + q(1_V - P)$$

Then we must have that

[3]This is sometimes referred to as the "Hecke" $\mathcal{R}$-matrices as these form a representation of the Hecke algebra, which is closely related to the Temperley-Lieb algebra.

[4]For a spectrum $\sigma(R) = \{p, q\}$ one can always multiply with $-p^{-1}$ to obtain this form of spectrum.

$$q \in \{e^{i\alpha} : \alpha = 2\cos^{-1}\left(\frac{d}{2\sqrt{n}}\right)\}$$

where $d = dim(V)$ *and* $n = Tr_{V\otimes V}(P)$.

Proof. Recall Equation (2.10):

$$\delta' = \frac{q}{(1+q)^2}$$

We shall use restrictions of this loop parameter to impose restrictions on q.

Firstly, note that $\delta' \in \mathbb{R}$. This is because

$$\begin{aligned} P \text{ projection } &\implies P\varphi(P)P = P^*\varphi(P^*)P^* \\ &\implies \delta' P = \bar{\delta}' P^* = \bar{\delta}' P \\ &\implies \delta' = \bar{\delta}' \end{aligned}$$

Secondly, $|\delta'| \leq 1$, since

$$\begin{aligned} &\|P\varphi(P)P\| = |\delta'| \cdot \|P\| \\ \implies &\|P\|\|\varphi(P)\|\|P\| \geq |\delta'|\|P\| \\ \implies &1 \geq |\delta'| \end{aligned}$$

We also see that δ is positive, as for any vector $\psi \in V$,

$$\begin{aligned} \langle \psi, P\varphi(P)P\psi \rangle &= \delta\langle \psi, P\psi \rangle \\ &= \langle P\psi, P\psi \rangle \\ &= \|P\psi\|^2 \geq 0 \end{aligned}$$

Since $q \in \mathbb{T}$, we re-write this eigenvalue in polar notation

$$q = e^{i\alpha}$$

Where $\alpha \in [0, 2\pi)$. We will go on to show the required restrictions of α. So we may rewrite Equation (2.10) as

$$\begin{aligned}\delta' &= \frac{e^{i\alpha}}{(1+e^{i\alpha})^2}\\ &= \frac{1}{(e^{-i\alpha/2}+e^{i\alpha/2})^2}\\ &= \frac{1}{4\cos^2(\frac{\alpha}{2})}\end{aligned}$$

Since the largest value cos can take is 1, we have that

$$\delta' \geq \frac{1}{4}$$

Finally, note that $q \neq -1$ means that we do not have an opposite pair of eigenvalues, and so τ_R is a Markov trace (see Section 2.5). We will utilise the Markov trace to show that $\delta' = \tau(P)$, but first we write a rearrangement of Equation (2.9)

$$P = -\frac{1}{1+q}(R - q1_V) \tag{2.11}$$

Now we consider taking the Markov trace of the Temperley-Lieb equation

$$\begin{aligned}\delta'\tau(P) &= \tau(P\varphi(P)P)\\ &= \tau(P\varphi(P)) && \text{(Tracial)}\\ &= \frac{-1}{1+q}\tau(R\varphi(P)) + \frac{1}{1+q}\tau(\varphi(P)) && \text{(By (2.11))}\\ &= \frac{-1}{1+q}\tau(R)\tau(P) + \frac{1}{1+q}\tau(\varphi(P)) && \text{(Markov trace)}\\ &= \tau\Big(\frac{-1}{1+q}(R - q1_V)\Big)\tau(P)\\ &= \tau(P)\tau(P)\\ &= \tau(P)^2\end{aligned}$$

Hence, we have that

$$\delta' = \tau(P) := \frac{Tr_{V\otimes V}}{d^2} \in \mathbb{Q}$$

So, we have that

$$\delta' = \frac{1}{4\cos^2(\frac{\alpha}{2})} \in \mathbb{Q}$$

We denote $n = Tr_{V\otimes V}(P)$ and recall that $d = \dim(V)$. Then we have that

$$\begin{aligned}
\frac{1}{4\cos^2(\frac{\alpha}{2})} &= \frac{n}{d^2} \\
\implies \frac{d^2}{4n} &= \cos^2(\frac{\alpha}{2}) \\
\implies \alpha &= 2\cos^{-1}\left(\sqrt{\frac{d^2}{4n}}\right) \\
&= 2\cos^{-1}\left(\frac{d}{2\sqrt{n}}\right)
\end{aligned}$$

as required.

□

We do not delve into the classification of these $\mathcal{R}$-matrices as all Temperley-Lieb $\mathcal{R}$-matrices[5] are classified in [25].

2.2.4 Representing the BMW Algebra with R-matrices

Now that we have defined contractive $\mathcal{R}$-matrices, we can show that a restriction on this class of $\mathcal{R}$-matrices does indeed induce a representation of the BMW algebra. Having already considered the case of $|\sigma(R)| = 2$, we henceforth assume $|\sigma(R)| = 3$ and denote this spectrum to be

$$\sigma(R) = \{\alpha, \beta, \gamma\}$$

where $\alpha, \beta, \gamma \in \mathbb{T}$, since the characteristic values $r^{-1}, q, -q^{-1} \in \mathbb{T}$.

Theorem 3. *Let $R \in \mathcal{C}(d)$, with contractive projection P and contractive eigenvalue α. Let the spectrum of R be $\sigma(R) := \{\alpha, \beta, \gamma\}$ such that $\beta + \gamma \neq \alpha - \alpha^{-1}$.*

Then there exists a unique representation of the BMW algebra $\mathcal{C}_n(r, q)$ that is given by the representation map π_R, defined by:

$$\begin{aligned}
\pi_R : \mathcal{C}_n(r, q) &\to End(V^{\otimes n}) \\
\pi_R(g_i) &:= \varphi^{i-1}(R) \\
\pi_R(e_i) &:= \varphi^{i-1}(xP)
\end{aligned}$$

When

[5] $\mathcal{R}$-matrices with 2 eigenvalues and a spectral projection that satisfies the Temperley-Lieb equation.

$$c = \frac{\beta + \gamma}{1 + \alpha(\beta + \gamma - \alpha)}$$

Recalling that $x = (1 + \frac{r - r^{-1}}{q - q^{-1}})$.

Proof. For π_R to be a representation of the BMW algebra, it must satisfy the defining relations of the BMW algebra, namely

$$\begin{aligned}
&(1) \quad \pi_R(g_i)\pi_R(g_{i+1})\pi_R(g_i) = \pi_R(g_{i+1})\pi_R(g_i)\pi_R(g_{i+1}) \\
&(2) \quad \pi_R(g_i)\pi_R(g_j) = \pi_R(g_j)\pi_R(g_i) \quad \text{for } |i - j| \geq 2 \\
&(3) \quad \pi_R(e_i)\pi_R(g_i) = \frac{1}{r}\pi_r(e_i) \\
&(4) \quad \pi_R(e_i)\pi_R(g_{i-1})\pi_R(e_i) = r\pi_R(e_i)
\end{aligned}$$

Notice that (1) and (2) are immediately satisfied, since these are the braid equations. The use of $\mathcal{R}-$matrices to represent the braid group has been very extensively studied, so I do not replicate a proof here.

As for (3);

$$\begin{aligned}
\pi_R(e_i)\pi_R(g_i) &= \varphi^{i-1}(xP)\varphi^{i-1}(R) \\
&= \varphi^{i-1}(xPR) \\
&= \varphi^{i-1}(\alpha xP) \\
&= \alpha\varphi^{i-1}(xP) \\
&= \alpha\pi_R(e_i)
\end{aligned}$$

Thus, we require

$$\alpha = r^{-1}$$

As for (4);

$$\begin{aligned}
\pi_R(e_i)\pi_R(g_{i-1})\pi_R(e_i) &= \varphi^{i-1}(xP)\varphi^{i-2}(R)\varphi^{i-1}(xP) \\
&= x^2\varphi^{i-2}(\varphi(P)R\varphi(P)) \\
&= x^2\varphi^{i-2}(x\varphi(P)) \\
&= xc\varphi^{i-1}(xP) \\
&= xc\pi_R(e_i)
\end{aligned}$$

Thus, we require

$$xc = r \tag{2.12}$$

In [33], Wenzl's defining equation for e_i is given by

$$(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}$$

Applying π_R yields

$$\begin{aligned}
(q - q^{-1})(\mathbb{1} - xP) &= R - R^* \\
\implies (q - q^{-1})(R - x\alpha P) &= R^2 - \mathbb{1} \\
\implies x\alpha P &= R - \frac{1}{q - q^{-1}}(R^2 - \mathbb{1}) \\
\implies 0 &= \beta Q - \frac{1}{q - q^{-1}}(\beta^2 Q - Q) \\
&= Q(\beta - \frac{\beta^2 - 1}{q - q^{-1}}) \\
\implies 0 &= \beta^2 - (q - q^{-1})\beta - 1
\end{aligned}$$

This gives solutions $\beta = q, -q^{-1}$. An exactly analogous calculation yields $\gamma = q, -q^{-1}$. It is interesting to note that there are no distinguishable features of β, γ and their relative spectral projections.

This was to be expected as $\frac{1}{r}, q, -q^{-1}$ are the characteristic values of g_i and α, β, γ are the eigenvalues of R. Thus, we have:

$$\alpha = r^{-1} \quad \beta, \gamma = q, -q^{-1}$$

Plugging these into equation (2.12) yields:

$$\begin{aligned}
\left(1 + \frac{\alpha^{-1} - \alpha}{\beta + \gamma}\right)c &= \alpha^{-1} \\
\implies c &= \frac{1}{\alpha + \frac{1 - \alpha^2}{\beta + \gamma}} \\
&= \frac{\beta + \gamma}{1 + \alpha(\beta + \gamma - \alpha)}
\end{aligned}$$

Therefore, π_R is a representation of the BMW algebra with the above restriction on the constant factors. □

2.3 Equivalence of Contractive $\mathcal{R}$-Matrices

In this section, we show that contractive $\mathcal{R}$-matrices are stable under equivalence. We then provide some 2-dimensional examples of contractive $\mathcal{R}$-matrices.

Firstly, we show that unit scalar multiples of contractive $\mathcal{R}$-matrices are contractive $\mathcal{R}$-matrices.

Proposition 3. *Let $R \in \mathcal{C}_\alpha(d)$. Then, for all $\lambda \in \mathbb{T}$, $\lambda R \in \mathcal{C}_\alpha(d)$.*

Proof. Since $R := \alpha P + \beta Q + \gamma Q'$, we have that $\lambda R = \lambda\alpha P + \lambda\beta Q + \lambda\gamma Q'$, and so if P is a spectral projection of R then it is a spectral projection of λR.

Now, in order to show that $R \in \mathbb{C}_\alpha(d) \implies \lambda R \in \mathbb{C}_\alpha(d)$ we simply need to show how λ interacts with the contraction relation.

$$\begin{aligned} \varphi(P)\lambda R\varphi(P) &= \lambda\varphi(P)R\varphi(P) \\ &= \lambda c \cdot \varphi(P) \end{aligned}$$

Since $\lambda \in \mathbb{T}$, the contraction constant remains of the same magnitude, so it will still satisfy the same requirements. Thus, λR is a contractive R-matrix.

□

We now show a brief proposition that will be used to show that contractivity is stable under equivalence in Theorem 4. Note that we describe this proposition in terms of $\mathcal{R}$-matrices, but it does hold true for matrices in general.

Proposition 4. *Let $R \in \mathcal{R}(d)$ be unitarily equivalent to S, i.e. $R = USU^*$. Then $P^R_\lambda = UP^S_\lambda U^*$ where P^R_λ is the spectral projection of eigenvalue λ in R and P^S_λ is the spectral projection of eigenvalue λ in S.*

Proof. Firstly, recall that $P^R_\lambda := f_\lambda(R)$ is a spectral projection of R iff

$$\begin{aligned} &f_\lambda(\lambda) = 1 \\ &f_\lambda(\gamma) = 0 \quad \text{for all } \gamma \in \sigma(R)\ ,\ \gamma \neq \lambda \end{aligned}$$

Thus, each spectral projection can be expressed as

$$P^R_\lambda = \prod_{\substack{\gamma\in\sigma(R)\\ \gamma\neq\lambda}} \frac{R - \gamma\mathbb{1}}{\lambda - \gamma}$$

Thus, $R \sim S$ implies

$$
\begin{aligned}
P_\lambda^R &= \prod_{\substack{\gamma\in\sigma(R)\\ \gamma\neq\lambda}} \frac{R-\gamma\mathbb{1}}{\lambda-\gamma} \\
&= \prod_{\substack{\gamma\in\sigma(S)\\ \gamma\neq\lambda}} \frac{USU^*-\gamma\mathbb{1}}{\lambda-\gamma} \\
&= U\Big(\prod_{\substack{\gamma\in\sigma(S)\\ \gamma\neq\lambda}} \frac{S-\gamma\mathbb{1}}{\lambda-\gamma}\Big)U^* \\
&= UP_\lambda^S U^*
\end{aligned}
$$

□

We now show that contractive $\mathcal{R}$-matrices are stable under equivalence.

Theorem 4. *Let $R \in \mathcal{C}_\alpha(d)$ and $R \sim S$. Then, $S \in \mathcal{C}_\alpha(d)$.*

Proof. To begin, we recall that two equivalent matrices have the same spectrum.

To show that the contraction relation holds, recall that if $R \sim S$ then there exists some unitary intertwiner $U \in \mathrm{End}(V \otimes V \otimes V)$ such that

$$
\begin{aligned}
S \otimes 1_V &= U(R \otimes 1_V)U^* \\
1_V \otimes S &= U(1_V \otimes R)U^*
\end{aligned}
$$

By Proposition 4, this also means that

$$
1_V \otimes P_S = U(1_V \otimes P_R)U^*
$$

where P_S, P_R is the contractive projection in the spectral decomposition of S and R respectively.

Now consider the LHS of the contraction relation for S:

$$
\begin{aligned}
\varphi(P_S)S\varphi(P_S) &= (1_V \otimes P_S) \otimes (S \otimes 1_V)(1 \otimes P_S) \\
&= U(1_V \otimes P_R)U^*U(R \otimes 1_V)U^*U(1_v \otimes P_R)U^* \\
&= U(1_V \otimes P_R)(R \otimes 1_V)(1_V \otimes P_R)U^* \\
&= c \cdot U(1_V \otimes P_R)U^* \\
&= c \cdot (1_V \otimes P_S) \\
&= c \cdot \varphi(P_S)
\end{aligned}
$$

Thus, S satisfies the contraction relation, and so S is a contractive matrix. □

Now that we have shown being contractive is stable under our notion of equivalence, we may classify all 2-dimensional contractive $\mathcal{R}$-matrices, utilising the work of Conti and Lechner [13].

Theorem 5. *Let $R \in \mathcal{C}_\alpha(2)$, i.e. a contractive unitary $\mathcal{R}$-matrix with $d = 2$. Then R is equivalent to a multiple of one of the three following $\mathcal{R}$-matrices:*

$$\mathbb{1}_4$$

$$\begin{pmatrix} x & & & \\ & & 1 & \\ & 1 & & \\ & & & x \end{pmatrix} \quad \textit{where } x \in \mathbb{T}$$

$$\begin{pmatrix} & & & z \\ & y & & \\ & & y & \\ \frac{1}{z} & & & \end{pmatrix} \quad \textit{where } y, z \in \mathbb{T}$$

Proof. In [13], Conti and Lechner classify all unitary $\mathcal{R}(2)$ matrices to be equivalent to one of the following four forms:

$$R_1 = a \cdot \mathbb{1} \qquad a \in \mathbb{T}$$

$$R_2 = \begin{pmatrix} a & & & \\ & & b & \\ & c & & \\ & & & d \end{pmatrix} \qquad a, b, c, d \in \mathbb{T}$$

$$R_3 = \begin{pmatrix} & & & a \\ & b & & \\ & & b & \\ c & & & \end{pmatrix} \qquad b, a \cdot c \in \mathbb{T}$$

$$R_4 = \frac{a}{\sqrt{2}} \begin{pmatrix} 1 & 1 & & \\ -1 & 1 & & \\ & & 1 & -1 \\ & & 1 & 1 \end{pmatrix} \qquad a \in \mathbb{T}$$

The spectra of these matrices are:

$$\sigma(R_1) = \{a\}$$
$$\sigma(R_2) = \{a, d, \pm\sqrt{bc}\}$$
$$\sigma(R_3) = \{b, \pm\sqrt{ac}\}$$
$$\sigma(R_4) = \{\frac{a}{\sqrt{2}}(1-i), \frac{a}{\sqrt{2}}(1+i)\}$$

Recall that we insist on having a maximum of 3 eigenvalues. We begin by looking at R_1 and R_3, then provide a restriction of R_2 to reduce its number of eigenvalues to 3.

$\underline{R_1}$

It is clear that the identity satisfies the contraction relation.

$\underline{R_3}$

We begin by rescaling R_3 via multiplication of a factor $\frac{1}{\sqrt{ac}}$ to be in the form

$$R_3 = \begin{pmatrix} & & & z \\ & y & & \\ & & y & \\ \frac{1}{z} & & & \end{pmatrix}$$

where $y, z \in \mathbb{T}$.

This rescales the spectrum to be

$$\sigma(R_3) = \{y, 1, -1\}$$

The spectral projection of eigenvalue 1 of R is given by

$$P_1 = \frac{1}{2}\begin{pmatrix} 1 & & & z \\ & 0 & & \\ & & 0 & \\ \overline{z} & & & 1 \end{pmatrix}$$

Then

$$(\mathbb{1} \otimes P_1)(R_3 \otimes \mathbb{1})(\mathbb{1} \otimes P_1) = \frac{1}{4}\begin{pmatrix} 1 & & & z & & & & \\ & 0 & & & & & & \\ & & 0 & & & & & \\ \overline{z} & & & 1 & & & & \\ & & & & 1 & & & z \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & \overline{z} & & & 1 \end{pmatrix}$$
$$= \frac{1}{2}(\mathbb{1} \otimes P_1)$$

Using the "right into left" tensor product convention.

Therefore $R_3 \in \mathcal{C}_1(2)$. Note that we can also take -1 to be the contractive eigenvalue but not y, since this would mean that q is associated to 1 or -1, meaning that $q - q^{-1} = 0$, which we cannot have by the definition of a BMW matrix.

$\underline{R_4}$

The spectral decomposition of R_4 is

$$R_4 = \frac{a}{\sqrt{2}}\begin{pmatrix} 1 & 1 & & \\ -1 & 1 & & \\ & & 1 & -1 \\ & & 1 & 1 \end{pmatrix} + \frac{a^2}{2}(1-i)\begin{pmatrix} 1-i & & & \\ & 1-i & & \\ & & 1+i & \\ & & & 1+i \end{pmatrix}$$

Testing the contraction equation for each spectral projection shows that neither are contractive projections. Therefore $R_4 \notin \mathcal{C}(2)$.

$\underline{R_2}$

Now for R_2. Firstly, we show that the sub-family of R_2 whereby $b = c$ is equivalent to the case when $b \neq c$. Using Proposition 2, notice R_2 commutes with $U \otimes U$, where

$$U = \begin{pmatrix} \sqrt{b} & \\ & \sqrt{c} \end{pmatrix}$$

and

$$(\mathbb{1}\otimes U)R_2(\mathbb{1}\otimes U^*) = \begin{pmatrix} a & & & \\ & & \sqrt{bc} & \\ & \sqrt{bc} & & \\ & & & d \end{pmatrix}$$

Thus, by Therorem 2, the 2 cases are equivalent and we may continue to just work with the subfamily of R_2 matrices where $b = c$.

We rescale R_2 to be in the form

$$R_2 = \begin{pmatrix} w & & & \\ & & 1 & \\ & 1 & & \\ & & & x \end{pmatrix}$$

This re-scales the spectrum to be

$$\sigma(R_2) = \{w, x, 1, -1\}$$

There are 3 distinct ways to restrict R_2 to have precisely 3 eigenvalues;

$$w = x(\neq \pm 1) \quad w = 1 \quad w = -1$$

Note that $x = \pm 1$ is equivalent to $w = \pm 1$.

We restrict by letting $w = x$ (see remarks below for further details) so that our final R_2 matrix becomes

$$R_2 = \begin{pmatrix} x & & & \\ & & 1 & \\ & 1 & & \\ & & & x \end{pmatrix} \quad |x| = 1, x \neq \pm 1$$

The spectral projection of eigenvalue 1 of R_2 is given by

$$Q_1 = \frac{1}{2}\begin{pmatrix} 0 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & 0 \end{pmatrix}$$

Then

$$(\mathbb{1} \otimes Q_1)(R_2 \otimes \mathbb{1})(\mathbb{1} \otimes Q_1) = \frac{x}{2} \begin{pmatrix} 0 & & & & & & & \\ & 1 & 1 & & & & & \\ & 1 & 1 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 1 & 1 & \\ & & & & & 1 & 1 & \\ & & & & & & & 0 \end{pmatrix}$$

$$= \frac{y}{2}(\mathbb{1} \otimes Q_1)$$

Thus, R_2 in this restricted form satisfies the contraction relation.

Since we have exhausted all possibilities of matrices in $\mathcal{R}(2)$, all $\mathcal{C}(2)$ matrices must be in one of the forms we have shown above.

□

Some remarks about the $\mathcal{C}(2)$ matrices:

1. The only possible contractive eigenvalues for both R_2 and R_3 are ± 1. The spectral projections associated to x and y in R_2 and R_3 respectively do not satisfy the contraction relation.

2. The restriction we make of R_2 (namely $w = x$) is in fact the only restriction of R_2 that results in a matrix which satisfies the contraction relation with any of its eigenvalues' spectral projections.

3. All of these spectra contain 1 and -1, i.e. have an opposite pair of eigenvalues. A further area of research would be to search for examples in higher dimensions without opposite pairs of eigenvalues to ensure the normalised trace is Markov (see Section 2.5).

2.4 Results for Contractive R-matrices

In this section we prove some interesting results relating to R, P and their shifted versions.

Firstly, we prove a couple of useful results that hold for general $\mathcal{R}$-matrices.

Proposition 5. *Let R be an $\mathcal{R}$-matrix. Then for any $n \in \mathbb{N}$ the following equations hold.*

$$\varphi(R^n)R\varphi(R) = R\varphi(R)R^n \tag{2.13}$$

$$R^n\varphi(R)R = \varphi(R)R\varphi(R^n) \tag{2.14}$$

Proof. The proofs of these 2 statements are analogous, so we just show (2.13). The result is clear for $n = 1$ as this is simply the YBE. Thus, let $n \geq 2$ and consider the following, noting that we underline the places where the Yang-Baxter equation is being used for clarity:

$$\begin{aligned}
\varphi(R^n)R\varphi(R) &= \varphi(R^{n-1})\underline{\varphi(R)R\varphi(R)} \\
&= \varphi(R^{n-1})R\varphi(R)R \\
&= \varphi(R^{n-2})\underline{\varphi(R)R\varphi(R)}R \\
&= \varphi(R^{n-2})R\varphi(R)R^2 \\
&= \varphi(R^{n-3})\underline{\varphi(R)R\varphi(R)}R^2 \\
&= \varphi(R^{n-3})R\varphi(R)R^3 \\
&= \ldots \\
&= R\varphi(R)R^n
\end{aligned}$$

as required. □

Theorem 6. *Let R be an R-matrix and Q be any of its spectral projections*[6]*. Then the following equations hold.*

$$\varphi(Q)R\varphi(R) = R\varphi(R)Q \tag{2.15}$$

$$Q\varphi(R)R = \varphi(R)R\varphi(Q) \tag{2.16}$$

Proof. Firstly we show (2.15).

[6] Note that the following equations do hold for $Q = P$ but they do not require the selection of any particular spectral projection.

$$\begin{aligned}
\varphi(Q)R\varphi(R) &= \varphi(\sum_j a_j R^j)R\varphi(R) \quad \text{Using equation (1.17)}\\
&= \sum_j a_j \varphi(R^j)R\varphi(R)\\
&= \sum_j a_j R\varphi(R)R^j\\
&= R\varphi(R)\sum_j a_j R^j\\
&= R\varphi(R)Q
\end{aligned}$$

The proof for (2.16) is analogous to the above proof.

□

We now look at relations of contractive $\mathcal{R}$-matrices.

Theorem 7. *Let $R \in \mathcal{C}_\alpha(d)$ and let P be the spectral projection of R associated to the eigenvalue α. Then the following equations hold.*

$$\varphi(R)P\varphi(P) = \alpha c \cdot R^{-1}\varphi(P) \tag{2.17}$$

$$P\varphi(R)P = c \cdot P \tag{2.18}$$

$$R\varphi(P)P = \alpha c \cdot \varphi(R^{-1})P \tag{2.19}$$

$$\overline{\alpha c} \cdot P\varphi(R)R = P\varphi(P) \tag{2.20}$$

$$P\varphi(P)P = |c|^2 P \tag{2.21}$$

$$\varphi(P)P\varphi(P) = |c|^2\varphi(P) \tag{2.22}$$

$$P = \frac{1}{\alpha + \overline{\alpha}\beta\gamma - (\beta+\gamma)} R + \beta\gamma R^{-1} - (\beta+\gamma)\mathbb{1} \tag{2.23}$$

Proof. To show (2.17) we prove the equivalent relation

$$R\varphi(R)P\varphi(P) = \alpha c \cdot \varphi(P)$$

$$
\begin{aligned}
R\varphi(R)P\varphi(P) &= \varphi(P)R\varphi(R)\varphi(P) && \text{by (2.15)}\\
&= \varphi(P)R\varphi(RP)\\
&= \alpha\cdot\varphi(P)R\varphi(P)\\
&= \alpha c\cdot\varphi(P)
\end{aligned}
$$

as required.

We will shortly show (2.18), which we note is similar to the contraction relation. We first state a re-arrangement of (2.16).

$$
\begin{aligned}
P\varphi(R)R &= \varphi(R)R\varphi(P)\\
\implies \varphi(R^*)P\varphi(R)R &= R\varphi(P)\\
\implies \varphi(R^*)P\varphi(R) &= R\varphi(P)R^*\\
\implies \varphi(R^*)P &= R\varphi(P)R^*\varphi(R^*)
\end{aligned}
$$

Using the contraction relation (1.11) and the above re-arrangement of (2.16), we have that

$$
\begin{aligned}
P\varphi(R)P &= P\{R\varphi(R)R\varphi(R^*)R^*\}P && \text{by (1.11)}\\
&= \alpha\overline{\alpha}P\varphi(R)R\varphi(R^*)P\\
&= P\varphi(R)R^2\varphi(P)R^*\varphi(R^*)\\
&= \varphi(R)R\varphi(P)R\varphi(P)R^*\varphi(R^*)\\
&= c\cdot\varphi(R)R\varphi(P)R^*\varphi(R^*)\\
&= c\cdot P\varphi(R)RR^*\varphi(R^*)\\
&= c\cdot P
\end{aligned}
$$

Now for (2.19), we show the equivalent equation $\varphi(R)R\varphi(P)P = \alpha c\cdot P$.

$$
\begin{aligned}
\varphi(R)R\varphi(P)P &= P\varphi(R)RP\\
&= \alpha P\varphi(R)P\\
&= \alpha c\cdot P
\end{aligned}
$$

Next we look at (2.20). Consider the Yang-Baxter endomorphism defined in Equation (1.10). We have that

$$\lambda_{R^*}(P) = \varphi(P) = R^*\varphi(R^*)P\varphi(R)R$$

Also, P being a spectral projection of R implies that it is also a spectral projection of R^* since the adjoint simply conjugates the eigenvalues. Therefore if the contraction relation holds for R and P then it holds for R^* and P with conjugation of the constant, i.e.

$$P\varphi(R^*)P = \bar{c}P$$

Now to show (2.20)

$$\begin{aligned} P\varphi(P) &= P(R^*\varphi(R^*)P\varphi(R)R) \\ &= \overline{\alpha}P\varphi(R^*)P\varphi(R)R \\ &= \overline{\alpha c}P\varphi(R)R \end{aligned}$$

Finally we look at the Temperley-Lieb equations (2.21) and (2.22).

We use (1.11) and both the contraction relation and its adjoint equivalent to show (2.21):

$$\begin{aligned} P\varphi(P)P &= PR\varphi(R)P\varphi(R^*)R^*P \\ &= \alpha\overline{\alpha}(P\varphi(R)P)\varphi(R^*)P \\ &= cP\varphi(R^*)P \\ &= c\bar{c}P \\ &= |c|^2P \end{aligned}$$

To show (2.22) we first rearrange the equation for $\varphi(P)$, equation (1.11):

$$\begin{aligned} &\varphi(P) = R\varphi(R)P\varphi(R^*)R^* \\ \Longrightarrow\ &R^*\varphi(P)R = \varphi(R)P\varphi(R^*) \\ \Longrightarrow\ &\varphi(R^*)R^*\varphi(P)R\varphi(R) = P \end{aligned}$$

Now, using this and both the contraction relation and its adjoint equation, we show (2.22):

$$\begin{aligned}\varphi(P)P\varphi(P) &= \varphi(P)(\varphi(R^*)R^*\varphi(P)R\varphi(R))\varphi(P)\\ &= \alpha\overline{\alpha}(\varphi(P)R^*\varphi(P))R\varphi(P)\\ &= \overline{c}\varphi(P)R\varphi(P)\\ &= \overline{c}c\varphi(P)\\ &= |c|^2\varphi(P)\end{aligned}$$

Finally, we show the equation for expanding P in terms of R, R^{-1}, and $\mathbb{1}$.

Recall

$$\begin{aligned}R &= \alpha P + \beta Q + \gamma Q'\\ R^{-1} &= \overline{\alpha}P + \overline{\beta}Q + \overline{\gamma}Q'\end{aligned}$$

Now, consider

$$\begin{aligned}R + \beta\gamma R^{-1} &= (\alpha + \overline{\alpha}\beta\gamma)P + (\beta+\gamma)(Q+Q')\\ &= (\alpha + \overline{\alpha}\beta\gamma)P + (\beta+\gamma)(\mathbb{1} - P)\\ &= (\alpha + \overline{\alpha}\beta\gamma - (\beta+\gamma)P + (\beta+\gamma)\mathbb{1}\end{aligned}$$

A simple rearrangement gives us our required equation.

□

These last 2 equations are the Temperley-Lieb equations. These were utilised in Section 2.2.3 and we analyse these in further detail in Section 2.6 where we explore the significance of $|c|^2$ as the loop parameter.

2.5 Markov Traces

The normalised trace τ having the Markov trace property is utilised in this thesis to derive restrictions on the contractive constant c. In this section we define a Markov trace, discuss some results from the literature regarding when the normalised trace is Markov, and show that if a positive faithful normalised Markov trace exists on $\rho_R(\mathcal{B}_\infty)$ then it is unique.

Definition 34. ***Markov Trace**: A Markov trace in the representation of the braid group is defined as a trace $m : \rho_R(\mathbb{C}\mathcal{B}_\infty) \to \mathbb{C}$ such that*

$$m(x\varphi^n(R)) = m(x)m(R) \tag{2.24}$$

for any $x \in \rho_R(\mathbb{C}\mathcal{B}_\infty)$, $n \in \mathbb{N}$.[17]

It is shown in [24] that for any $R \in \mathcal{R}(d)$ with no pair of opposite eigenvalues[7], the normalised trace τ_R is a Markov trace. In Proposition 7 we also show that for the discrete range of a certain Jones Index, defined by Equation (2.34), the normalised trace is Markov.

Throughout this thesis we assume that τ_R has the Markov property by either enforcing no pairs of opposite eigenvalues or the Jones Index defined in Equation (2.34) being 2 or 3. Further research could entail analysing exactly when else the normalised trace is Markov, which could perhaps generalise the results laid out in this thesis to include further cases.

We show now that if there exists a positive faithful normalised trace on $\rho_R(\mathbb{C}\mathcal{B}_\infty)$, then it is unique.[8] We first show a short proposition that will be used in our uniqueness proof later, adapted from an analogous proof shown for the BMW algebra in [8].

Proposition 6. *Any element of $\rho_R(\mathbb{C}\mathcal{B}_n)$ may be written as linear combinations of elements of the form*

$$w = a\chi b \tag{2.25}$$

where $a, b \in \rho_R(\mathbb{C}\mathcal{B}_{n-1})$ and $\chi \in \{\varphi^{n-2}(R), \varphi^{n-2}(P), \mathbb{1}\}$.

Proof. We will use the method of proof by induction.

Firstly we look at $n = 2$. In this case, our algebra is $\rho_R(\mathbb{C}\mathcal{B}_2)$, which simply consists of polynomials in R. Thus, for this base case, we need only to show that any power of R may be written as linear combinations of $\mathbb{1}, R$, and P. To do this, recall equation (2.23)

[7] By "opposite pair" we mean $\sigma_i = -\sigma_j$ for any i, j in the spectrum $\sigma(R) = \{\sigma_1, ..., \sigma_n\}$.

[8] This does follow from the fact that a trace on any II_1 factor is unique [13], but we still state an explicit proof.

$$
\begin{aligned}
& R = (\alpha + \overline{\alpha}\beta\gamma - (\beta+\gamma))P - \beta\gamma R^* + (\beta+\gamma)\mathbb{1} \\
\implies & R^2 = \alpha(\alpha + \overline{\alpha}\beta\gamma - (\beta+\gamma))P - \beta\gamma\mathbb{1} + (\beta+\gamma)R \\
\implies & R^3 = \alpha^2(\alpha + \overline{\alpha}\beta\gamma - (\beta+\gamma))P - \beta\gamma R + (\beta+\gamma)R^2 \\
& \quad = \alpha^2(\alpha + \overline{\alpha}\beta\gamma - (\beta+\gamma))P - \beta\gamma R + (\beta+\gamma)(\alpha(\alpha + \overline{\alpha}\beta\gamma - (\beta+\gamma))P \\
& \qquad - \beta\gamma\mathbb{1} + (\beta+\gamma)R) \\
& \quad \text{etc.}
\end{aligned}
$$

This gives a recursion relation for any power of R, namely

$$R^i = \alpha^{i-1}(\alpha + \overline{\alpha}\beta\gamma - (\beta+\gamma))P - \beta\gamma R + (\beta+\gamma)R^{i-1}$$

where

$$R = (\alpha + \overline{\alpha}\beta\gamma - (\beta+\gamma))P - \beta\gamma R^* + (\beta+\gamma)\mathbb{1}$$

So, $n = 2$ is shown, and we move on to the induction.

We note that any element of $\rho_R(\mathbb{C}\mathcal{B}_n)$ can be written as linear combinations of elements of the form

$$
\begin{aligned}
w &= w_0 y_0 w_1 y_1 ... w_r y_r \\
&= w_0 y_0 w_1 y_1 z \qquad (z = w_2 y_2 ... w_r y_r)
\end{aligned}
$$

where $w_i \in \rho_R(\mathbb{C}\mathcal{B}_{n-1})$ and $y_i \in \{\varphi^{n-2}(R), \varphi^{n-2}(P)\}$.

Note that y_i is not assumed to be able to take the value $\mathbb{1}$, since if this were the case one would have $w_j \mathbb{1} w_{j+1} = w_j w_{j+1}$, which we can combine to make $\tilde{w_j}$.

Now, by induction on n, we have that

$$w_1 = v_0 s v_1$$

where $v_0, v_1 \in \rho_R(\mathbb{C}\mathcal{B}_{n-2})$ and $s \in \{\varphi^{n-3}(R), \varphi^{n-3}(P)\}$

Since v_0 and v_1 only operate on $n-2$ strings, and y_0 and y_1 operate on the $n-1^{th}$ and n^{th} strings, our v's must commute with our y's. So, w becomes

$$
\begin{aligned}
w &= w_0 y_0 (v_0 s v_1) y_1 z \\
&= w_0 v_0 y_0 s y_1 v_1 z
\end{aligned}
$$

Induction shows that we can always "push" the braids on $n-1$ or less strands to the outside of w. Now all that remains to show is that the middle part, $y_0 s y_1$, is always in the set $\{\varphi^{n-2}(R), \varphi^{n-2}(P), \mathbb{1}\}$.

Let $A, B, C \in \{R, P\}$. Note that $y_0 s y_1 = \varphi^{n-2}(A)\varphi^{n-3}(B)\varphi^{n-2}(C) = \varphi^{n-3}(\varphi(A)B\varphi(C))$, so equivalently we need to show that $\varphi(A)B\varphi(C)$ reduces to something in the form $A'\varphi(B')C'$ where $B' \in \{R, P, \mathbb{1}\}$.

There are precisely 8 possibilities for A, B, and C. Namely

A	B	C
R	R	R
R	R	P
R	P	R
R	P	P
P	R	R
P	P	R
P	R	P
P	P	P

<u>$\varphi(R)R\varphi(R)$</u>
By Yang-Baxter,

$$\varphi(R)R\varphi(R) = R\varphi(R)R$$

<u>$\varphi(R)R\varphi(P)$</u>
By (2.17),

$$\varphi(R)R\varphi(P) = \alpha c \cdot R^{-1}\varphi(P)$$

<u>$\varphi(R)P\varphi(R)$</u>
Since we are dealing with linear combinations anyway, it is enough to show that $\varphi(R)P\varphi(\mathbb{1}), \varphi(R)P\varphi(R^*)$ and $\varphi(R)P\varphi(P)$ can be reduced to the required form, as $\varphi(R)$ can be written as a linear combination of $\mathbb{1}.R^*$, and P by (2.23).

$$\varphi(R)P\varphi(\mathbb{1}) \text{ is already in the required form}$$
$$\varphi(R)P\varphi(P) = \alpha c R^{-1}\varphi(P) \text{ by (2.17)}$$

As for $\varphi(R)P\varphi(R^*)$, recall the equation for the shift endomorphism

$$\begin{aligned} \varphi(P) &= R\varphi(R)P\varphi(R^*)R^* \\ \implies \varphi(R)P\varphi(R^*) &= R^*\varphi(P)R \end{aligned}$$

<u>$\varphi(R)P\varphi(P)$</u>
Similarly, P can be expressed in terms of linear combinations of R, R^*, and $\mathbb{1}$.

$$\begin{aligned} \varphi(R)P\varphi(\mathbb{1}) &\text{ is already in the required form} \\ \varphi(R)P\varphi(R) &\text{ is shown above} \\ \varphi(R)P\varphi(R^*) &\text{ is shown above} \end{aligned}$$

<u>$\varphi(P)R\varphi(R)$</u>
Using linear combinations again, this time on $\varphi(P)$,

$$\begin{aligned} \varphi(R)R\varphi(R) &\text{ shown above} \\ \varphi(\mathbb{1})R\varphi(R) &\text{ is already in the required form} \\ \varphi(R^*)R\varphi(R) &= R\varphi(R)R^* \text{ is clear from the YBE (1.2)} \end{aligned}$$

<u>$\varphi(P)P\varphi(R)$</u>
Expanding $\varphi(P)$ in terms of its linear combinations,

$$\begin{aligned} \varphi(\mathbb{1})P\varphi(R) &\text{ is already in the required form} \\ \varphi(R)P\varphi(P) &\text{ is shown above} \\ \varphi(R^*)P\varphi(P) &= \varphi(R^*)\alpha cR\varphi(R)P\varphi(P)P\varphi(P) \text{ by (2.17)} \\ &= \alpha cR\varphi(R)R^*P^2\varphi(P)P \text{ by Temperley-Lieb (2.21) and Yang-Baxter (1.2)} \\ &= \alpha^2 cR\varphi(R)P\varphi(P)P \\ &\text{and } \varphi(R)P\varphi(P) \text{ is shown above} \end{aligned}$$

<u>$\varphi(P)R\varphi(P)$</u>

$$\varphi(P)P\varphi(P) = c \cdot \varphi(P) \text{ by the contraction relation (2.7)}$$

<u>$\varphi(P)P\varphi(P)$</u>

$$\varphi(P)P\varphi(P) = P\varphi(P)P \text{ by Temperley-Lieb (2.21)}$$

Thus, by induction, we have shown that every element of $\rho_R(\mathbb{C}\mathcal{B}_n)$ may be written as linear combinations of elements of the form $w = A\chi B$ where $A, B \in \rho_R(\mathbb{C}\mathcal{B}_{n-1})$ and $\chi \in \{\varphi^{n-2}(R), \varphi^{n-2}(P), \mathbb{1}\}$.

$\square$

We now show a theorem about the uniqueness of positive faithful normalised Markov traces.

Theorem 8. *If there exists a positive faithful normalised Markov trace on* $\rho_R(\mathbb{C}\mathcal{B}_\infty)$*, then it is unique.*

Proof. Assume $m : \rho_R(\mathbb{C}\mathcal{B}_\infty) \to \mathbb{C}$ to be a positive faithful normalised Markov trace, i.e.

$$\begin{aligned}
\text{Normalised: } & m(\mathbb{1}) = 1 & \\
\text{Linear: } & m \text{ is linear} & \\
\text{Positive and faithful: } & m(a^*a) = 0 \iff a = 0 & \text{for all } a \in \rho_R(\mathbb{C}\mathcal{B}_\infty) \\
\text{Markov: } & m(x\varphi^{n-1}(R)) = m(x)m(R) & \text{for all } x \in \rho_R(\mathbb{C}\mathcal{B}_n) \\
\text{Tracial: } & m(ab) = m(ba) & \text{for all } a, b \in \rho_R(\mathbb{C}B_\infty)
\end{aligned}$$

Note that m being Markov for R immediately implies that it is Markov for R^*, i.e. $m(x\varphi^{n-1}(R^*)) = m(x)m(R^*)$ for all $x \in \rho_R(\mathbb{C}B_n)$. We can see this by taking the conjugate of the Markov equation:

$$m(x\varphi(R^*) = \overline{m(x\varphi(R))} = \overline{m(x)m(R)} = m(\overline{x})m(\varphi(R^*))$$

To show that m, if it exists, is unique on $\rho_R(\mathbb{C}B_\infty)$, we show that it is uniquely fixed on

$$\mathbb{1}, R, R^n, \varphi^l(R) \quad \text{for all } n \in \mathbb{Z}, l \in \mathbb{N}$$

Note that we need not show it is fixed on R^* as $\rho_R(\mathbb{C}\mathcal{B}_\infty) = \rho_{R^*}(\mathbb{C}\mathcal{B}_\infty)$

Applying m to the contraction relation yields

$$\begin{aligned}
m(\varphi(P)R\varphi(P)) &= m(c \cdot \varphi(P)) \\
\implies m(\varphi(P)^2 R) &= cm(\varphi(P)) \\
\implies m(\varphi(P))m(R) &= cm(\varphi(P)) \\
\implies m(R) &= c
\end{aligned} \tag{2.26}$$

Note that $m(\varphi(P)) \neq 0$ since m is faithful and P is positive.

By assumption m is Hermitian, so this implies

$$m(R^*) = \overline{c}$$

To show that m is uniquely fixed on all powers (positive and negative) of R, recall that

$$R^n = \sum_i (\lambda_i)^n P_{\lambda_i}$$

where $i = 1, ..., |spec(R)|$, λ_i are the eigenvalues of R and P_{λ_i} is the spectral projection associated to the eigenvalue λ_i.

Therefore, since m is linear, we may show that m is fixed for all powers of R by showing that it is fixed for all spectral projections of R.

Applying m to the Temperley-Lieb relation yields

$$\begin{aligned} m(\varphi(P)P\varphi(P)) &= m(|c|^2\varphi(P)) \\ \implies m(P)m(\varphi(P)) &= |c|^2 m(\varphi(P)) \\ \implies m(P) &= |c|^2 \end{aligned} \tag{2.27}$$

As for the other spectral projections, recall that the sum of spectral projections equal the identity. Applying m to this yields

$$\begin{aligned} m(P + Q + Q') &= m(\mathbb{1}) \\ \implies |c|^2 + m(Q) + m(Q') &= 1 \end{aligned}$$

Also, applying m to the spectral decomposition of R gives us

$$\begin{aligned} m(\alpha P + \beta Q + \gamma Q') &= m(R) \\ \implies \alpha|c|^2 + \beta m(Q) + \gamma m(Q') &= c \end{aligned}$$

Solving the above 2 equations as a system of simultaneous equations gives

$$m(Q) = \frac{(\gamma - \alpha)|c|^2 + c - \gamma}{\beta - \gamma} \tag{2.28}$$

$$m(Q') = 1 - |c|^2 - \frac{(\gamma - \alpha)|c|^2 + c - \gamma}{\beta - \gamma} \tag{2.29}$$

Thus we have shown that if m exists, it is uniquely fixed on all R^n, $n \in \mathbb{Z}$.

Now all that is left is to show that it exists on shifts of R, i.e. $\varphi^m(R)$ for all $m \in \mathbb{N}$. We do this by using induction on $\rho_R(\mathbb{C}\mathcal{B}_2)$ to show that if m exists here, it will also exist on any higher number of strings.

We have already shown that m is uniquely fixed on $\mathbb{1}$ (by assumption) and all powers of R. This covers every element of $\rho_R(\mathbb{C}\mathcal{B}_2)$, giving us the base step of our induction.

As we have shown in equation (2.25), any element w of $\rho_R(\mathbb{C}\mathcal{B}_n)$ may be written as linear combinations of elements of the form

$$w = a\chi b$$

where $a, b \in \rho_R(\mathbb{C}\mathcal{B}_{n-1})$ and $\chi \in \{\varphi^{n-2}(R), \varphi^{n-2}(P), \mathbb{1}\}$.

Applying m to this gives

$$\begin{aligned} m(w) &= m(a\chi b) \\ &= m(ba\chi) \\ &= m(ba)m(\chi) \end{aligned}$$

This last step holds regardless of which value χ takes, since it's true by definition if $\chi = \varphi^{n-2}(R)$ and since m is linear and P can be expressed as a linear combination of R and R^* it also must hold for $\chi = P$. It is trivial if $\chi = \mathbb{1}$.

Thus, if m exists, it is uniquely fixed for all elements of $\rho_R(\mathbb{C}\mathcal{B}_\infty)$ as we have defined above.

□

2.6 Contraction Constant

In this section, we use various properties, including the Markov trace, the Temperley-Lieb equation, and the Jones index, to find restrictions on the possible values of the contraction constant c.

Throughout this section we assume $\tau : \rho_R(\mathcal{B}_\infty) \to \mathbb{C}$ is a normalised Markov trace. To enforce a Markov trace we assume that we do not have an opposite pair of eigenvalues, i.e.

$$\alpha \neq -\beta$$
$$\alpha \neq -\gamma$$
$$\beta \neq -\gamma$$

2.6.1 Rationality of the Loop Parameter

Recall that τ is the normalised Markov trace. Then

$$|c|^2 = \tau(P) = \frac{Tr_{V \otimes V}(P)}{d^2} \in \mathbb{Q}$$

Also, note that $|c|^2$ being the loop parameter in the Temperley-Lieb relation, which in our case consists of spectral projections, automatically implies $0 \leq |c|^2 \leq 1$, since

$$\begin{aligned} \|P\varphi(P)P\| &= \||c|^2 P\| \\ \implies \|P\|\|\varphi(P)\|\|P\| &\geq |c|^2 \|P\| \\ \implies 1 &\geq |c|^2 \quad (\|P\| = 1 \text{ for spectral projections}) \end{aligned}$$

Actually, we cannot have $|c|^2 = \tau(P) = 1$ since this occurs iff $P \equiv \mathbb{1}$ which is a special case that can be considered seperately. Thus,

$$1 > |c|^2 \in \mathbb{Q} \tag{2.30}$$

2.6.2 Skein Relations

In this section we investigate the restrictions induced from a Skein-type relation.

A Skein relation is an equation of the form

$$R + R^* = z(E + \mathbb{1}) \tag{2.31}$$

where $E = \frac{1}{|c|} P$ satisfies the following relations that form an equivalence class on "hooks" on the diagram monoid D_n [22]

$$\begin{aligned} E^2 &= \frac{1}{|c|^2} E = E \frac{1}{|c|^2} \\ E\varphi(E)E &= E \end{aligned}$$

Calculating the LHS of the Skein relation in our 3-eigenvalue setting yields

$$R + R^* = (\alpha + \overline{\alpha})P + (\beta + \overline{\beta})Q + (\gamma + \overline{\gamma})Q'$$

We make the simplifying assumption that $\beta\gamma = 1$, i.e. $\gamma = \overline{\beta}$ (since they lie on the unit circle), since as discussed before we can rotate the spectrum of R. We calculate

$$\begin{aligned} R + \beta\gamma R^* &= (\alpha + \beta\gamma\overline{\alpha})P + (\beta + \gamma)(Q + Q') \\ \iff R + R^* &= (\alpha + \overline{\alpha} - (\beta + \overline{\beta})|c|E + (\beta + \overline{\beta})\mathbb{1} \end{aligned}$$

This becomes of Skein form when

$$\begin{aligned} |c|(\alpha + \overline{\alpha} - (\beta + \overline{\beta}) &= \beta + \overline{\beta} \\ \iff |c| &= \frac{\beta + \overline{\beta}}{\alpha + \overline{\alpha} - (\beta + \overline{\beta})} \end{aligned}$$

Note that this denominator $\neq 0$, as

$$\begin{aligned} \alpha + \overline{\alpha} - (\beta + \overline{\beta}) = 0 &\implies \alpha + \overline{\alpha} = \beta + \overline{\beta} \\ &\implies \alpha_1 + \alpha_2 i + \alpha_1 - \alpha_2 i = \beta_1 + \beta_2 i + \beta_1 - \beta_2 i \\ &\implies 2\alpha_1 = 2\beta_1 \\ &\implies \alpha_1 = \beta_1 \end{aligned}$$

where $\alpha := \alpha_1 + \alpha_2 i$ and $\beta = \beta_1 + \beta_2 i$ $(\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R})$.

If $\alpha_1 = \beta_1$ (i.e. $\mathrm{Re}(\alpha) = \mathrm{Re}(\beta)$) then we would have $\alpha = \overline{\beta}$ as all eigenvalues lie on the unit circle. However, we have already rotated the spectrum so that $\beta = \overline{\gamma}$, so $\alpha = \overline{\beta} \implies \alpha = \gamma$ which contradicts our assumption of having distinct eigenvalues. Therefore the above denominator $\neq 0$.

Now, since α and β are on the unit circle, they may be re-written as

$$\alpha = e^{ia} \quad \beta = e^{ib}$$

where $a, b \in [0, 2\pi]$.

Then this becomes

$$\begin{aligned}|c| &= \frac{e^{ib}+e^{-ib}}{e^{ia}+e^{-ia}-(e^{ib}+e^{-ib})}\\ &= \frac{cos(b)}{cos(a)-cos(b)}\end{aligned} \tag{2.32}$$

This denominator is never 0 since we have distinct eigenvalues. There are 2 possibilities to make $cos(a) = cos(b)$: either $a = b$, which does not occur since $\alpha \neq \beta$, or $a = 2\pi - b$, which does not occur since $\alpha \neq \overline{\beta}$.

We divide top and bottom by $cos(b)$, which is not 0 since this would give $\beta = 1$, making $\gamma = \overline{\beta} = 1 = \beta$ which cannot occur as we have distinct eigenvalues.

Thus,

$$|c| = \frac{1}{\frac{cos(a)}{cos(b)} - 1}$$

This is a strong restriction on the values that c can take. In particular we can use it to restrict the values of β.

Recall that $0 < |c| < 1$. This means that there are 2 possible cases:

$$\begin{aligned}(1)\underline{cos(a) > cos(b)} &\implies 0 < 2cos(b) < cos(a)\\ (2)\underline{cos(b) > cos(a)} &\implies 0 > 2cos(b) > cos(a)\end{aligned}$$

Since $cos \in [-1, -1]$ we must therefore have that

$$|cos(b)| \leq 1/2$$

On the unit circle, this means that β (and γ since $\beta = \overline{\gamma}$) can only exist in the range highlighted in Figure 2.6.

2.6.3 Jones Index

In this section we discuss the relationship between the contraction constant and the Jones index.

Recall the following type II_1 factor

$$\mathscr{L}_R := \rho_R(B_\infty)'' \tag{2.33}$$

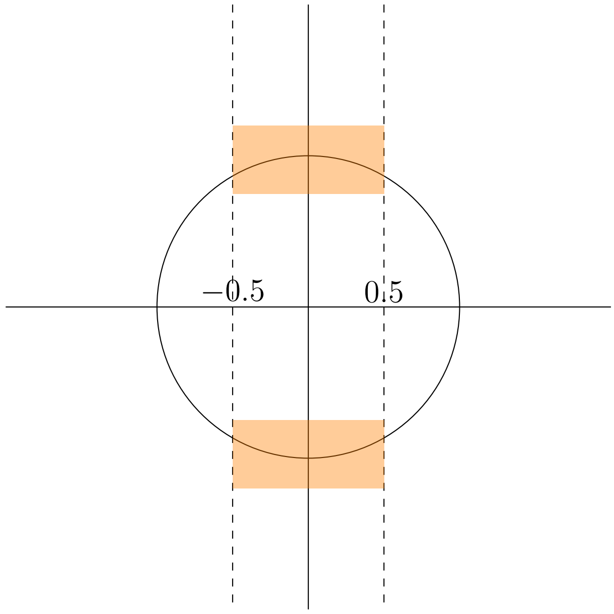


Figure 2.6: Range of possible values of β

We consider the following Jones index

$$I_R := [\mathscr{L}_R : \varphi(\mathscr{L}_R)] \tag{2.34}$$

Theorem 9. *Let I_R be the Jones Index as above and let E be the (unique) trace-preserving conditional expectation. Then the following relation holds [27]*

$$I_R := [\mathscr{L}_R : \varphi(\mathscr{L}_R)] = \left(inf\{||E(P)|| : P(\neq 0) \text{ is a projection on } \mathscr{L}_R\}\right)^{-1} \tag{2.35}$$

Let us suppose that $\tau : \mathscr{L}_R \to \mathbb{C}$, the normalised trace, has the Markov property.

Proposition 7. *Let R be a contractive $\mathcal{R}$-matrix. Suppose $I_R < 4$. Then the τ-preserving conditional expectation coincides with τ on R and all P_λ (spectral projections of R), i.e.*

$$E(R) = \tau(R) \cdot \mathbb{1} \text{ and } E(P_\lambda) = \tau(P_\lambda) \cdot \mathbb{1} \tag{2.36}$$

Proof. Let $I_R < 4$. Then by Jones [21], we must have that the relative commutant is trivial, i.e.

$$\mathscr{L}_R \cap \varphi(\mathscr{L}_R)' = \mathbb{C} \cdot 1$$

In [13] Equation (3.17), Conti and Lechner show that the conditional expectation must always be in the relative commutant, i.e.

$$E(R) \in \mathscr{L}_R \cap \varphi(\mathscr{L}_R)' = \mathbb{C} \cdot 1$$

Since the conditional expectation is trace-preserving, we therefore must have that

$$E(R) = \tau(R) \cdot 1$$

In particular, since P_λ can be expressed as a polynomial in R (1.17), we have that

$$E(P_\lambda) = \tau(P_\lambda) \cdot 1$$

As required. □

We now show a key lemma demonstrating the connection between the contractive constant and the Jones Index.

Lemma 1. *Let I_R be the Jones Index relating to the contractive $\mathcal{R}-$matrix R as above, and let c be the contractive constant of R. Then,*

$$I_R = |c|^{-2}$$

Proof. Note that in [13] it is shown that

$$E(R) = \varphi \circ \phi_R(R)$$

and that τ is a Markov trace $\iff \phi_R(R) = \tau(R) \cdot 1$. Hence, by the above theorem, we have that for $I_R < 4$ the normalised trace τ is Markov.

From (2.35) we have the follwing, where P_λ is any spectral projection of R:

$$I_R^{-1} \leq \tau(P_\lambda)$$

Taking in particular $P_\lambda = P$ (i.e. choosing the contractive spectral projection), for which $\tau(P) = |c|^2$, we therefore have

$$|c|^{-2} \leq I_R$$

Also recall from [13] that

$$I_R \leq |\tau(R)|^{-2} (= |c|^{-2})$$

Therefore $|c|^{-2} \leq I_R \leq |c|^{-2}$, and so

$$I_R = |c|^{-2}$$

□

The above equation is an extremely important result, as it allows us to restrict the possible values of c in many ways.

There are a number of inequalities already known about I_R [13]:

$$\begin{aligned} I_R &\leq d^2 \text{ (where d is the dimension of the base space)} \\ I_R &\leq |\tau(R)|^{-2} \\ |\sigma(R_I)| &\leq I_R \\ |\sigma(ptr(R))|^2 &\leq I_R \\ 1/\tau(P_\lambda) &\leq I_R \text{ for any spectral projection } P_\lambda \text{ of } R \end{aligned}$$

where P_λ is any spectral projection of R.

Therefore we have that

$$\begin{aligned} |c|^{-2} &\leq d^2 \\ |\sigma(R)| &\leq |c|^{-2} \\ |\sigma(ptr(R))|^2 &\leq |c|^{-2} \\ |c|^{-2} &\leq \tau(P_\lambda) \end{aligned}$$

where P_λ is any spectral projection of R.

Note that this final inequality has equality for $P_\lambda = P$, so all "other" spectral projections have a Markov trace greater than or equal to the contractive projection.

Recall that in the case of type II_1 factors, the Jones index can only take specific values [20], namely

$$I_R = \{4cos^2(\pi/n) : n = 3, 4, 5...\} \cup [4, \infty) \qquad (1.20)$$

Since $I_R = |c|^{-2}$, there are further restrictions on the value the Jones Index can take - in the discrete range, it may only take integer values.

Theorem 10. *Let* $I_R = [\mathscr{L}_R : \varphi(\mathscr{L}_R)]$*. Then*

$$I_R \in \{2, 3\} \cup [4, \infty)$$

Proof. Recall that $|c|^2 = \tau(P) \in \mathbb{Q}$. Therefore we must have that $I_R = |c|^{-2} \in \mathbb{Q}$. Focusing on the discrete range of I_R, this is equivalent to asking when $4cos^2(\pi/n) \in \mathbb{Q}$ for $n \in \mathbb{N}$.

Firstly, note that

$$\begin{aligned} cos^2(\pi/n) &= \Big(\frac{e^{i\pi/n} + e^{-i\pi/n}}{2}\Big)^2 \\ &= \frac{e^{2i\pi/n} + e^{-2i\pi/n} + 2}{4} \\ &= \frac{1}{2}\Big(\frac{e^{2i\pi/n} + e^{-2i\pi/n}}{2} + 1\Big) \\ &= \frac{1}{2}\Big(cos(\frac{2\pi}{n}) + 1\Big) \end{aligned}$$

For this to be rational, we need to consider when $cos(\frac{2\pi}{n})$ is rational.

By Niven's Theorem [26] [9] we have that

$$4cos^2(\frac{\pi}{n}) \in \{4, 3, 2\}$$

As required.

□

The lowest value the Jones index can take is 1, but this only occurs if $\mathcal{M} = \mathcal{N}$. The next value it can take is 2, so we therefore must have

$$2 \leq [\mathscr{L}_R : \varphi(\mathscr{L}_R)] \leq |\tau(R)|^{-2}$$

Now recall that in our contraction setting $\tau(R) = c$, so this becomes

$$2 \leq |c|^{-2} \implies |c| \leq \frac{1}{\sqrt{2}}$$

This is a strong restriction on the possible values that c can take.

[9] *If* θ/π *and* $\cos(\theta)$ *are rational, then* $\theta \in \{0, \pi/3, \pi/2\}$.

In the previous subsections we have shown the following restrictions on the value of c:

$$0 \leq |c| \leq \frac{1}{\sqrt{2}}$$
$$|c|^2 = \tau(P) \in \mathbb{Q}$$

Now, let us look at specific values of $I_R(= |c|^{-2})$ in the discrete range and deduce information about R.

2.7 Contractive $\mathcal{R}$-Matrices Classified By Their (Discrete) Jones Index

In this section we use the fact that $I_R = |c|^{-2}$ to deduce the form of a contractive $\mathcal{R}$-matrix for each discrete integer value of the Jones Index.

2.7.1 $I_R = 2$

For this case, we have that $|\sigma(R)| \leq 2$ as $I_R \leq |\sigma(R)|$. Note that the case $|\sigma(R)| = 1$ is trivial, so we only consider $|\sigma(R)| = 2$, which is the Temperley-Lieb case discussed in Section 2.2.3. We now state a formal theorem showing the exact form of a contractive $\mathcal{R}$-matrix of this type.

Theorem 11. *Let R be a contractive R-matrix with $I_R = 2$. Then R is of the form*

$$tR = -P \pm i(\mathbb{1} - P)$$

where $t \in \mathbb{T}$.

Proof. Since $|\sigma(R)| \leq I_R$, we must have that $|\sigma(R)| = 2$ (as $|\sigma(R)| = 1 \iff I_R = 1$) since this only occurs for R being a multiple of the identity matrix). Thus, we know R must be of the form

$$R = -P + \beta(\mathbb{1} - P)$$

Recall

$$\begin{aligned}\tau(R) &= c\\ \tau(P) &= |c|^2\\ I_R &= |c|^{-2}\end{aligned}$$

Thus, in our situation we have that $\tau(P) = 1/2$. Consider

$$\begin{aligned}c &= \tau(R)\\ &= \tau(-P + \beta(\mathbb{1} - P)\\ &= (1+\beta)|c|^2 + \beta\\ &= \frac{\beta - 1}{2}\end{aligned}$$

Since here we have that $|c| = \frac{1}{\sqrt{2}}$, so

$$|\frac{\beta - 1}{2}| = \frac{1}{\sqrt{2}}$$

Equivalently,

$$|\beta - 1| = \sqrt{2}$$

Now, as $|\beta| = 1$, we consider where the unit circle and the circle centred at 1 with radius $\sqrt{2}$ intersect by solving the following as a set of simultaneous equations:

$$\begin{aligned}x^2 + y^2 &= 1\\ (x-1)^2 + y^2 &= 2\end{aligned}$$

which have intersection $(0, \pm 1)$, i.e. $\beta = \pm i$.

This shows that for $I_R = 2$, we always have that R is of the form

$$R = -P + i(\mathbb{1} - P)$$

We have already shown that any matrix equivalent to a contractive R-matrix must also be contractive, so any scalar multiple will also be permitted. □

This is indeed unsurprising, as in [24], Lechner fully classifies Temperley-Lieb matrices in terms of, in his notation, $\tau_R(e_1)$. This is exactly our $\tau(P)$, and Lechner states all unitary R-matrices such that $\tau_R(e_1) = 1/2$, i.e. $I_R = 2$, are equivalent to

$$R = -e^{\frac{-i\pi}{4}} G_2 \boxtimes 1_k \quad \text{or} \quad R = (-e^{\frac{-i\pi}{4}} G_2 \boxtimes 1_k)^*$$

where G_2 is the Gaussian R-matrix of dimension 2.

These R-matrices have spectra $\{1, i\}$ and $\{1, -i\}$ respectively, and thus are equivalent to the form stated in the theorem above.

2.7.2 $I_R = 3$

The case for $I_R = 3$ is a little more complicated, since we can have $|\sigma(R)| = 2$ or $|\sigma(R)| = 3$.

2.7.2.1 $I_R = 3, |\sigma(R)| = 2$

In an analogous calculation to the proof of Theorem 6, R must be of the form

$$R = -P + \frac{1 + i\sqrt{3}}{2}(\mathbb{1} - P)$$

Again, Lechner [24] showed that all Temperley-Lieb matrices such that $\tau_R(e_1) = 1/3$ are equivalent to

$$R = iG_3 \boxtimes 1_k \quad \text{or} \quad R = (iG_3 \boxtimes 1_k)^*$$

where $k = dimV$.

These R-matrices have spectra $\{1, e^{\frac{i\pi}{3}}\}$ and $\{1, e^{-\frac{i\pi}{3}}\}$ respectively, and thus are equivalent to the form stated above.

2.7.2.2 $I_R = 3, |\sigma(R)| = 3$

This case seems to be difficult to find examples for. In this subsection we show that we cannot extend the $I_R = 3, |\sigma(R)| = 2$ case to have 3 eigenvalues as it results in contradiction. Future research could entail continuing to search for examples of this type.

Let R' be the $I_R = 3, |\sigma(R)| = 2$ solution, i.e.

$$R' := iG_3 \boxtimes 1_k = -P_{R'} + \frac{1 + i\sqrt{3}}{2}(\mathbb{1} - P_{R'})$$

where

$$G_3 := \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \xi^{k^2} U^k$$

U is a unitary operator defined by

$$U|k,l\rangle := (\xi)^{l-k}|k+1, l+1\rangle$$

and $\xi := e^{\frac{2\pi i}{3}}$ is a third root of unity.

This operator applied to some basis vectors e_k, e_l,[10] yields:

$$U(e_k \otimes e_l) = \xi^{l-k}(e_{k+1} \otimes e_{l+1})$$

In 3 dimensions U can be expressed explicitly as:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\xi^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi & 0 \\ 0 & 0 & -\xi & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & 0 & -\xi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We aim to construct an R from R' by using its spectral projection $P_{R'}$ and using other equations around R to solve for the other unknowns in the spectral decomposition of R, i.e. construct

$$R = \lambda_1 P + \lambda_2 Q + \lambda_3(\mathbb{1} - P - Q)$$

where $P := P_{R'}$.

Since P is a spectral projection of iG_3, which is a linear combination of powers of U's, it follows that there must be a way to write P as a linear combination of U's. Indeed, we find that any spectral projection with Markov trace $\frac{1}{3}$ that can be written in terms of U as defined above must be in the form described in the following theorem.

Theorem 12. *Let U be defined as $U|k,l\rangle := (\xi)^{l-k}|k+1, l+1\rangle$.*

Let Q be a spectral projection such that $\tau(Q) = \frac{1}{3}$, and suppose Q can be written as a linear combination of $\mathbb{1}, U, U^2$.

[10] where e_k operates as a 1 in the k^{th} space and 0's everywhere else.

Then, we must have that

$$Q = \gamma \cdot \frac{1}{3} \sum_{i=0}^{2} \xi^{ik} U^{ik}$$

where $\gamma \in \mathbb{T}$.

Proof. We begin by supposing that Q can be written as a linear combination of $\mathbb{1}, U, U^2$, i.e. is in the form

$$Q = a\mathbb{1} + bU + cU^2$$

Firstly, note that $\tau(Q) = \frac{1}{3}$ & $\tau(U) = 0$ immediately implies that $a = \frac{1}{3}$. Secondly, Q is a projection. By definition this gives us 2 relations:

1. $Q = Q^*$

2. $Q = Q^2$

We now see how these relations impact a, b, and c.

<u>1. $Q = Q^*$</u>

$$\begin{aligned} Q = Q^* \implies \frac{1}{3}\mathbb{1} + bU + cU^2 &= \frac{1}{3}\mathbb{1} + \bar{b}U^* + \bar{c}(U^*)^2 \\ &= \frac{1}{3}\mathbb{1} + \bar{b}U^2 + \bar{c}U \end{aligned}$$

Since $U^3 = \mathbb{1}$, and so $U^* = U^2$ and $(U^2)^* = U$.
Therefore

$$b = \bar{c}c = \bar{b}$$

<u>2. $Q = Q^2$</u>

$$\begin{aligned} Q = Q^2 \implies \frac{1}{3}\mathbb{1} + bU + cU^2 &= (\frac{1}{3}\mathbb{1} + bU + cU^2)^2 \\ &= (\frac{1}{9} + 2bc)\mathbb{1} + (\frac{2}{3}b + c^2)U + (\frac{2}{3}c + b^2)U^2 \end{aligned}$$

By equating coefficients in the above 2 equations we get that

$$|b| = \frac{1}{3} \text{ and } |c| = \frac{1}{3}$$

This gives precisely 3 possible scenarios (up to multiplication of a normed scalar):

$$b = \frac{1}{3}, c = \frac{1}{3}$$
$$b = \frac{1}{3}\xi, c = \frac{1}{3}\xi^2$$
$$b = \frac{1}{3}\xi^2, c = \frac{1}{3}\xi$$

Therefore Q must be in the above form.

□

We insist on the following labeling convention for the 3 possible projections:

$$Q_k = \sum_{i=0}^{2} \xi^{ik} U^i$$

Or, explicitly,

$$Q_0 = \mathbb{1} + U + U^2$$
$$Q_1 = \mathbb{1} + \xi U + \xi^2 U^2$$
$$Q_2 = \mathbb{1} + \xi^2 U + \xi U^2$$

Note that $P'_R = Q_0$. Recall that we wish to express R in terms of 3 spectral projections, one of which being P'_R. We claim that Q_k, $k \in \{0, 1, 2\}$ is the set of spectral projections of R, i.e.

$$R = \sum_{i=0}^{2} \lambda_i Q_i$$

Theorem 13. *Let Q_k, $k = 0, 1, 2$ be as described above. Then $\{Q_0, Q_1, Q_2\}$ is a complete set of spectral projections.*

Proof. For $\{Q_k : k = 0, 1, 2\}$ to be a complete set of spectral projections, the Q_k's must satisfy 3 properties:

1. $Q_k = Q_k^* = Q_k^2$ for all k

2. $\sum_{k=0}^{2} Q_k = \mathbb{1}$

3. $Q_j Q_k = \delta_{jk} Q_k$ where δ_{jk} is the Kronecker delta.

Property 1 is automatically satisfied by the construction of Q.

For property 2, recall that the sum of all roots of unity is 0 and consider

$$\begin{aligned}\sum_{k=0}^{2} Q_k &= \sum_{k=0}^{2} \frac{1}{3} \sum_{i=0}^{2} \xi^{ik} U^i \\ &= \sum_{i,k}^{2} \frac{1}{3} (\xi^k)^i U^i \\ &\text{this can only be non-zero for } i = 0 \\ &= \sum_{k=0}^{2} \frac{1}{3} 1^k \mathbb{1} \\ &= \mathbb{1}\end{aligned}$$

As for property 3:

$$\begin{aligned}9Q_j Q_k &= \sum_{n=0} \xi^{nj} U^n \cdot \sum_{m=0}^{2} \xi^{mk} U^m \\ &= \sum_{n,m}^{2} \xi^{nj+mk} U^{n+m} \\ &\text{we do an index change of } s = n + m \\ &= \sum_{s=0,n}^{2} \xi^{nj+(s-n)k} U^s \\ &= \sum_{s,n}^{2} (\xi^n)^{j-k} \xi^{sk} U^s \\ &\text{this is only non-zero for j=k since the sum of roots of unity is 0} \\ &= \sum_{s=0}^{2} \delta_{jk} \xi^{sk} U^s \\ &= 3\delta_{jk} Q_k\end{aligned}$$

□

So, we consider

$$R = \lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2$$

Before we continue we state a list of helpful equations:

Theorem 14. *Let ξ be a 3^{rd} root of unity and consider U, Q_k and R as above. We restate these definitions for ease:*

$$U|k,l\rangle := \xi^{l-k}|k+1,l+1\rangle$$
$$Q_k := \frac{1}{3}\sum_{i=0}^{2}\xi^{ik}U^i$$
$$R := \sum_{i=0}^{2}\lambda_i Q_i$$

Then, the following equations hold:

$$\begin{aligned}
U^i\varphi(U^j) &= \xi^{2ij}\varphi(U^j)U^i \\
U^j\varphi(Q_k) &= \varphi(Q_{k+2j})U^j \\
U^jQ_k &= \xi^{-kj}Q_k = Q_kU^j \\
Q_k\varphi(U^j) &= \varphi(U^j)Q_{k+2j} \\
\varphi(U^j)Q_k &= Q_{k-2j}\varphi(U^j) \\
\varphi(Q_i)Q_j\varphi(Q_k) &= \frac{1}{3}\xi^{(k-i)j}\varphi(Q_i)U^{k-i} \\
Q_i\varphi(Q_j)Q_k &= \frac{1}{3}\xi^{-(k-i)j}Q_i\varphi(U^{-(k-i)} \\
RU = UR &= \sum_{i=0}^{2}\xi^{-i}\lambda_iQ_i \\
\varphi(Q_k)R\varphi(Q_k) &= \frac{1}{3}\sum_i\lambda_iQ_k
\end{aligned}$$

Proof. 1. By definition,

$$\begin{aligned}
U^i|k,l,m\rangle &:= U^i(e_k\otimes e_l\otimes e_m) \\
&= \xi^{l-k}(e_{k+1}\otimes e_{l+1}\otimes e_m) \\
\varphi(U^j)|k,l,m\rangle &:= \varphi(U^j)(e_k\otimes e_l\otimes e_m) \\
&= \xi^{m-l}(e_k\otimes e_{l+1}\otimes e_{m+1})
\end{aligned}$$

Thus, consider

$$\begin{aligned}
U^i\varphi(U^j)(e_k\otimes e_l\otimes e_m) &= U^i\cdot\xi^{j(m-l)}(e_k\otimes e_{l+j}\otimes e_{m+j}) \\
&= \xi^{j(m-l)}\xi^{i(l+j-k}(e_{k+i}\otimes e_{l+i+j}\otimes e_{m+j}) \\
&= \xi^{-ki+l(i-j)+mj+ij}(e_{k+i}\otimes e_{l+i+j}\otimes e_{m+j})
\end{aligned}$$

Compare this with

$$\begin{aligned}\varphi(U^j)U^i(e_k \otimes e_l \otimes e_m) &= \varphi(U^j)\xi^{j(l-k)}(e_{k+i} \otimes e_{l+i} \otimes e_m) \\ &= \xi^{i(l-k)}\xi^{j(m-(l+i))}(e_{k+i} \otimes e_{l+i+j} \otimes e_{m+j}) \\ &= \xi^{-ki+l(i-j)+mj-ij}(e_{k+i} \otimes e_{l+i+j} \otimes e_{m+j})\end{aligned}$$

Thus,

$$U^i\varphi(U^j) = \xi^{2ij}\varphi(U^j)U^i$$

2.

$$\begin{aligned}U^i\varphi(Q_k) &= \frac{1}{3}U^i\sum_{n=0}^{2}\xi^{kn}\varphi(U^n) \\ &= \frac{1}{3}\sum_{n=0}^{2}\xi^{kn}U^i\varphi(U^n) \\ &= \frac{1}{3}\sum_{n=0}^{2}\xi^{kn+2in}\varphi(U^n)U^i \\ &= \frac{1}{3}\sum_{n=0}^{2}\xi^{n(k+2i)}\varphi(U^n)\cdot U^i \\ &= \varphi(Q_{k+2i})U^i\end{aligned}$$

3.

$$\begin{aligned}U^iQ_k &= \frac{1}{3}U^i\sum_{n=0}^{2}\xi^{kn}U^n \\ &= \frac{1}{3}\sum_{n=0}^{2}\xi^{kn}U^{i+n}\end{aligned}$$

We shift this index by i and use the new summation index $m = i + n$

$$\begin{aligned}&= \frac{1}{3}\sum_{m=0}^{2}\xi^{k(m-i)}U^m \\ &= \xi^{-ki}\cdot\frac{1}{3}\sum_{m=0}^{2}\xi^{km}U^m \\ &= \xi^{-ki}Q_k\end{aligned}$$

4. and 5.

$$
\begin{aligned}
Q_k\varphi(U^j) &= \frac{1}{3}\sum_{n=0}^{2}\xi^{kn}U^n\varphi(U^j) \\
&= \frac{1}{3}\sum_{n=0}^{2}\xi^{kn}\xi^{2nj}\varphi(U^j)U^n \\
&= \varphi(U^j)\cdot\frac{1}{3}\sum_{n=0}^{2}\xi^{n(k+2j)}U^n \\
&= \varphi(U^j)Q_{k+2j}
\end{aligned}
$$

6.

$$
\begin{aligned}
\varphi(Q_i)Q_j\varphi(Q_k) &= \frac{1}{3}\varphi(Q_i)\sum_{n=0}^{2}\xi^{nj}U^n\varphi(Q_k) \\
&= \frac{1}{3}\varphi(Q_i)\sum_{n=0}^{2}\xi^{nj}\varphi(Q_{k+2n})U^n \\
&= \frac{1}{3}\sum_{n=0}^{2}\xi^{nj}\varphi(Q_i)\varphi(Q_{k+2n})U^n
\end{aligned}
$$

Since the Q_k's are spectral projections, this can only be non-zero for $i = k - n$, i.e. $n = k - i$

$$= \frac{1}{3}\xi^{(k-i)j}\varphi(Q_i)U^{k-i}$$

7.

$$
\begin{aligned}
Q_i\varphi(Q_j)Q_k &= \frac{1}{3}Q_i\sum_{n=0}^{2}\xi^{nj}\varphi(U^n)Q_k \\
&= \frac{1}{3}Q_i\sum_{n=0}^{2}\xi^{nj}Q_{k-2n}\varphi(U^n) \\
&= \frac{1}{3}\sum_{n=0}^{2}\xi^{nj}Q_iQ_{k+n}\varphi(U^n)
\end{aligned}
$$

Since the Q_k's are spectral projections, this can only be non-zero for $i = k + n$, i.e. $n = i - k$

$$= \frac{1}{3}\xi^{(i-k)j}Q_i\varphi(U^{i-k})$$

8.

$$
\begin{aligned}
RU &= \sum_{i=0}^{2}\lambda_iQ_iU \\
&= \sum_{i=0}^{2}\lambda_i\xi^{-i}Q_i
\end{aligned}
$$

9.

$$\begin{aligned}\varphi(Q_k)R\varphi(Q_k) &= \sum_{i=0}^{2} \lambda_i \varphi(Q_k)Q_i\varphi(Q_k)\\ &= \frac{1}{3}\sum_{i=0}^{2} \lambda_i \cdot Q_k\end{aligned}$$

□

We now explore if $R = \sum_{i=0}^{2}\sum_{j=0}^{2} \lambda_i \xi^{ij} U^j$ is indeed an $\mathcal{R}$-matrix, as it does not necessarily follow that it would be from the current construction, and our ultimate goal is to seek $\mathcal{R}$-matrices.

For R to satisfy the Yang-Baxter equation, we must have that for any spectral projections Q_k, Q_l of R, the following equation is satisfied:

$$Q_k R\varphi(R)U\varphi(R^*)R^*Q_l = Q_k\varphi(U)Q_l$$

In particular this must be true for $k+2=l$. In this case, we have that

$$\begin{aligned} & Q_k R\varphi(R)U\varphi(R^*)R^*Q_{k+2} = Q_k\varphi(U)Q_{k+2}\\ \implies & \lambda_k\overline{\lambda_{k+2}}Q_k\varphi(R)U\varphi(R^*)Q_{k+2} = \varphi(U)Q_{k+2}\end{aligned}$$

Looking at the left hand side we have

$$\begin{aligned} LHS &= \lambda_k\overline{\lambda_{k+2}} \sum_{i=0,j=0}^{2} x_i\overline{x_j}Q_k\varphi(U^i)U\varphi(U^{-j})Q_{k+2}\\ &= \sum_{i,j} \lambda_k\lambda_{k+2}x_i\overline{x_j}\varphi(U^i)Q_{k+2i}UQ_{k+2+2j}\varphi(U^{-j})\\ &= \sum_{i,j} \lambda_k\lambda_{k+2}x_i\overline{x_j}\xi^{-(k+2i)}\varphi(U^i)Q_{k+2i}Q_{k+2+2j}\varphi(U^{-j})\\ & \text{This can only be non-zero for } k+2i=k+2+2k \iff i=1+j\\ &= \sum_{j} \lambda_k\overline{\lambda_{k+2}}x_{j+1}\overline{x_j}\xi^{-(k+2j+2)}\varphi(U^{j+1})Q_{k+2+2j}\varphi(U^{-j})\\ &= \sum_{j} \lambda_k\overline{\lambda_{k+2}}x_{j+1}\overline{x_j}\xi^{-(k+2+2j)}Q_k\varphi(U^{j+1})\varphi(U^{-j})\\ &= \sum_{j} \lambda_k\overline{\lambda_{k+2}}x_{j+1}\overline{x_j}\xi^{-(k+2+2j)}Q_k\varphi(U)\end{aligned}$$

Equating this with the RHS yields

$$\sum_j \lambda_k \overline{\lambda_{k+2}} x_{j+1} \overline{x_j} \xi^{-(k+2+2j)} Q_k \varphi(U) = Q_k \varphi(U)$$

Therefore we must have that

$$\lambda_k \overline{\lambda_{k+2}} \xi^{-(k+2+2j)} \sum_{j=0}^{2} x_{j+1} \overline{x_j} = 1$$

Regardless of the choice of k this equation must always hold if R satisfies the Yang-Baxter equation. Since the summation does not depend on k, we must have that $\lambda_k \overline{\lambda_{k+2}} \xi^{-(k+2+2j)}$ must always be equal regardless of the choice of k, i.e.

$$\lambda_0 \overline{\lambda_2} = \lambda_1 \overline{\lambda_0} \xi^{-1} = \lambda_2 \overline{\lambda_1} \xi^{-2}$$

If we rotate the spectrum such that $\lambda_0 = 1$, which we may do as we have not made any restriction thus far on the spectrum, then we have that

$$\lambda_1 = \lambda_1^2 \lambda_2 \xi^2 = \lambda_2^2 \xi$$

The first equality yields $\lambda_2 = \xi \overline{\lambda_1}$, which when substituted into the second equality gives

$$\begin{aligned} \lambda_1^2 (\xi \overline{\lambda_1}) \xi^2 &= (\xi \overline{\lambda_1})^2 \xi \\ \implies \lambda_1 &= \overline{\lambda_1}^2 \\ \implies \lambda_1^3 &= 1 \end{aligned}$$

Therefore λ_1 is a 3^{rd} root of unity. This gives 2 possibilities for the values of the eigenvalues:

$$\lambda_0 = 1 \quad \lambda_1 = \xi \implies \lambda_2 = \xi \overline{\lambda_1} = 1$$
$$\lambda_0 = 1 \quad \lambda_1 = \xi^2 \implies \lambda_2 = \xi \overline{\lambda_1} = \xi^2$$

Both of these possibilities lead to contradictions, as the first case would mean that $\lambda_2 = \lambda_0$ and the second case would mean that $\lambda_2 = \lambda_1$. Therefore it is not possible to construct an $\mathcal{R}-$matrix in this way.

2.8 Summary

In this section we briefly summarize the results in Chapter 2.

We defined the BMW algebra $\mathcal{C}_n(r,q)$ and used one of its defining relations (2.4) to define the contraction relation (2.7) in terms of $\mathcal{R}$-matrices and spectral projections. We then showed that any $\mathcal{R}$-matrix that forms a representation of the BMW algebra must have at most 3 eigenvalues.

After forming a restriction on c in terms of the eigenvalues of R, we show that the class of $\mathcal{R}$-matrices that satisfy the contraction relation with this restriction on c forms a representation of the BMW algebra. All examples in 2 dimensions are explored, utilising Conti and Lechner's classification [13].

We then prove some results for contractive $\mathcal{R}$-matrices, drawing on [7], and show that for any contractive $\mathcal{R}$-matrix, any equivalent $\mathcal{R}$-matrix is also contractive.

We go on to define Markov traces and and show that if there exists a positive, faithful and normalised Markov trace on $\rho(\mathbb{C}\mathcal{B}_\infty)$ then it is unique. We then utilize the Markov trace to form restrictions on the possible values of c.

Continuing with the search for restrictions on the contraction constant, we utilise the fact that $|c|^2$ is a Temperley-Lieb loop parameter to show that it must be rational and less than 1. We then use Skein relations to restrict the values of an eigenvalue β, which in turn limits the possible values of c.

Finally, we define a Jones index I_R such that $I_R = |c|^{-2}$. There are many known results about the restriction of a Jones index, which can then be applied to the contraction constant. We then show that for a contractive $\mathcal{R}$-matrix this Jones index can only take integer values in this discrete range. The chapter finishes with an attempt to classify all examples of contractive $\mathcal{R}$-matrices with Jones indices 2 and 3.

Chapter 3

Linearisation of the Set-Theoretic Yang-Baxter Equation

In this chapter we examine racks and show how they form solutions to the set-theoretic Yang-Baxter equation. In fact, all solutions to the set-theoretic Yang-Baxter equation are equivalent to a rack-derived solution, which motivates us to explore rack-derived solutions further.

These rack-derived solutions are linearised to solutions of the Quantum Yang-Baxter equation, and we establish a relationship between isomorphisms of racks and equivalence of their linearised $\mathcal{R}$-matrices.

We go on to define a new group called the Bloop group, inspired by a diagrammatic representation of the quandalisation process of racks. We establish a representation of this group and utilise this to prove a relationship between the equivalence of rack-induced $\mathcal{R}$-matrices and the equivalence of the $\mathcal{R}$-matrices induced from the quandalisations of these racks.

We delve into a special type of rack called an Alexander quandle, which has close ties to knot theory. We explore the concept of coloring a braid with a quandle and show that the coloring invariant presents a notion of equivalence of quandle-derived $\mathcal{R}$-matrices. We use this to show that the equivalence of Alexander-quandle-derived $\mathcal{R}$-matrices is entirely dependant upon the k-parameter of the Alexander quandles being equal or being inverses of one another.

3.1 The Set-Theoretic Yang-Baxter Equation

In this section we recall the definition of the set-theoretic YBE and define its non-degenerate solutions.

Recall Definition 2:

The Set-Theoretic Yang-Baxter Equation. *The set-theoretic YBE is defined on $X \times X \times X$, where X is a set, and is given by*

$$(r \times 1_X)(1_X \times r)(r \times 1_X) = (1_X \times r)(r \times 1_X)(1_X \times r)$$

where 1_X is the identity map on X.

We define solutions to the set-theoretic YBE in the following way.

Definition 35. ***Non-degenerate solution****: A non-degenerate solution to the set-theoretic YBE is a map $r : X \times X \to X \times X$ defined by*

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

such that r solves the set-theoretic Yang-Baxter equation, where λ_x and ρ_y are bijective functions on X for all $x, y \in X$.

Example 12. *Let G be a group. The "Venkov" solution is given by*

$$r(x, y) = (x, y^{-1}xy)$$

It is elementary to show that this is indeed a solution to the Set-Theoretic YBE.

3.2 The Bloop Group

In this section we define the Bloop group $\mathcal{B}\ell_n$ in terms of a semi-direct product of groups as well as its presentation in terms of braid and loop generators and their relations. We also give its diagrammatic presentation and highlight a key relation arising from pulling a loop under a strand.

Definition 36. ***Bloop Group*** $\mathcal{B}\ell_n$*: The Bloop Group $\mathcal{B}\ell_n$ is the group defined by the semi-direct product of tuples of integers and the Braid group:*

$$\mathcal{B}\ell_n = \mathbb{Z}^n \rtimes \mathcal{B}_n$$

with the following group law for all z_i, $z_j \in \mathbb{Z}^n$ and σ_i, $\sigma_j \in \mathcal{B}_n$

$$(z_i, \sigma_i) \cdot (z_j, \sigma_j) = (z_i + \alpha_{\psi_{\sigma_i}}(z_j), \sigma_i\sigma_j)$$

where $\rtimes$ denotes the semi-direct product, $\psi_{\sigma_i} \in S_n$ is the permutation induced by the braid σ_i as in equation (1.8) and $\alpha_{\psi_{\sigma_i}}$ is defined by

$$\alpha_{\psi_{\sigma_i}} : \mathbb{Z}^n \to \mathbb{Z}^n$$
$$(z_1, ..., z_n) \mapsto (z_{\psi_{\sigma_i}^{-1}(1)}, ..., z_{\psi_{\sigma_i}^{-1}(n)})$$

The $\alpha_{\psi_{\sigma_i}}$ map essentially swaps the i^{th} and $(i+1)^{th}$ entries of the element of $\mathbb{Z}^n$.

Example 13. *We show an example calculation in the Bloop group.*

$$\begin{aligned}((0,0,1), b_1) \cdot ((0,1,0), b_2) &= ((0,0,1) + \alpha_{\psi_{b_1}}((0,1,0)), b_1 b_2) \\ &= ((0,0,1) + (1,0,0), b_1 b_2) \\ &= ((1,0,1), b_1 b_2)\end{aligned}$$

One can derive a presentation of $\mathcal{B}\ell_n$ with generators b_j from the braid group and l_j having a diagrammatic representation as a loop on strand j as in Figures 3.1 and 3.2 respectively.

These generators relate to the semi-direct product presentation in the following way.

$$\begin{aligned}l_j &= (z_j, e) = ((0, ...0, \underbrace{1}_{j^{th} \text{ position}}, 0, ...0), e) \\ b_j &= (\mathbf{0}, b_j) = ((0, ..., 0), b_j)\end{aligned}$$

where e is the identity element of the braid group. We see that

$$(l_i, e) \cdot (\mathbf{0}, b_j) = (l_i + \alpha_{\psi_e}(\mathbf{0}), e b_j) = (l_i, b_j)$$

The group relations are given by:

$$b_k b_{k+1} b_k = b_{k+1} b_k b_{k+1} \tag{3.1}$$
$$b_j b_k = b_k b_j \quad \forall |j-k| \geq 2 \tag{3.2}$$
$$l_k l_j = l_j l_k \quad \forall j, k \tag{3.3}$$
$$l_k b_k = b_k l_{k+1} \quad \forall k \tag{3.4}$$
$$l_{k+1} b_k = b_k l_k \quad \forall k \tag{3.5}$$
$$l_k b_j = b_j l_k \quad \forall |j-k| \geq 2 \tag{3.6}$$

A key relation in $\mathcal{B}\ell_n$ is the ability to pull a loop under a strand:

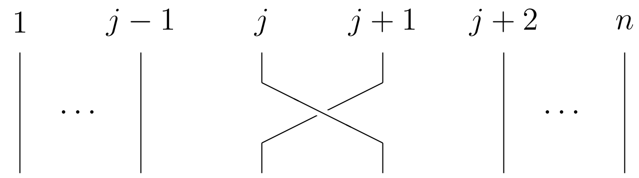


Figure 3.1: Diagrammatic representation of b_j, the "braid" generator of the Bloop group $\mathcal{B}\ell_n$

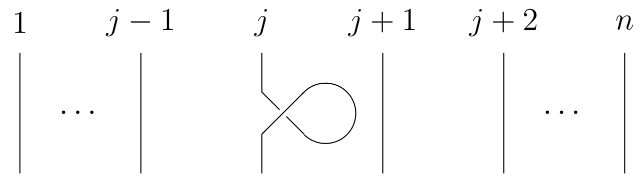


Figure 3.2: Diagrammatic representation of l_j, the "loop" generator of the Bloop group $\mathcal{B}\ell_n$.

$$l_k = b_k^{-1} l_{k+1} b_k \tag{3.7}$$

This equation arises from applying b_k^{-1} on the left of both sides of equation (3.5). It is demonstrated in Figure 3.3.

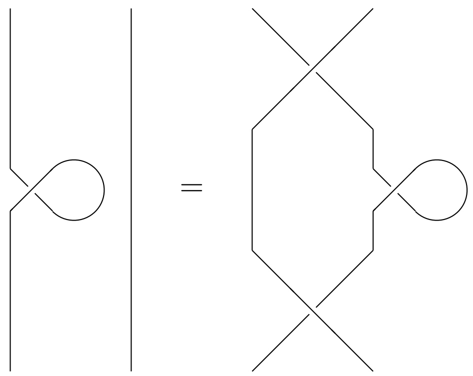


Figure 3.3: Diagrammatic presentation of equation (3.7)

The Bloop group was inspired by the quandalisation process of racks (see Section 3.4.3).

We may use $\mathcal{R}$-matrices to form a representation of $\mathcal{B}\ell_n$ as follows:

Proposition 8. *Let $\mathcal{B}\ell_n$ be the Bloop group on n strands, R be an $\mathcal{R}$-matrix in $End(V^{\otimes n})$ and $T := ptr(R)$ be the partial trace of R. Then the following map is a representation of $\mathcal{B}\ell_n$.*

$$\begin{aligned}\pi_{R,n} : \mathcal{B}\ell_n &\to End(V^{\otimes n}) \\ b_j &\mapsto \varphi^{j-1}(R) \\ l_j &\mapsto \varphi^{j-1}(T^*)\end{aligned}$$

Proof. We show that this forms a representation by considering the generators and relations. Recall the following relations of the Bloop group (**??**)

$$\begin{aligned}b_k b_{k+1} b_k &= b_{k+1} b_k b_{k+1} \\ b_j b_k &= b_k b_j \quad \forall |j-k| \geq 2 \\ l_k l_j &= l_j l_k \quad \forall j, k \\ l_k b_k &= b_k l_{k+1} \quad \forall k \\ l_{k+1} b_k &= b_k l_k \quad \forall k \\ l_k b_j &= b_j l_k \quad \forall |j-k| \geq 2\end{aligned}$$

The first two equations (3.1) and (3.2) are the relations for the braid group, and this representation is already well-established.

The third equation (3.3) is obvious as T operates on a single tensor factor.

The next two equations (3.4) and (3.5) are already proven in Lemma 2.

Finally, the last equation (3.6) is obvious by the properties of $V^{\otimes n}$.

□

3.3 Racks and Quandles

In this section we introduce racks, the square map Sq of a rack, and quandles - a subset of racks that have a trivial square map.

3.3.1 Racks

Definition 37. ***Racks****: A rack is a pair (X, λ) where X is a set and λ_i, $i \in X$, are bijective maps $\lambda_i : X \to X$, $j \mapsto \lambda_i(j)$ that satisfy the following* ***self-distributive property***

$$\lambda_i \lambda_j \lambda_i^{-1} = \lambda_{\lambda_i(j)} \quad \forall i, j \in X \tag{3.8}$$

An alternative notation for $\lambda_i(j)$ is the "triangle operator", defined by

$$i \triangleright j := \lambda_i(j)$$

This notation is sometimes clearer, for example we re-write the self-distributive property as

$$i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k) \tag{3.9}$$

Although racks do not need to be finite, we restrict to finite racks in order to have finite-dimensional linearisations (see Section 3.3.4).

Example 14. *Let X be a finite set and define $i \triangleright j = j \ \forall i \in X$. Then $(X, \triangleright)$ is a rack.*

Proof. Clearly $\lambda_i = id_X$ is bijective. As for the self-distributive property (3.9), we show it is satisfied $\forall i, j, k \in X$

$$i \triangleright (j \triangleright k) = i \triangleright k = k = j \triangleright k = (i \triangleright j) \triangleright (i \triangleright k)$$

□

Example 15. *Let G be a finite group and let $\lambda_i(j) = iji^{-1}$, where i^{-1} is the inverse of i with respect to the group operation of G. Then (G, λ) is a rack.*

Proof. Since G is a group, λ_i is clearly bijective. As for the self-distributive property (3.8),

$$\begin{aligned}
(\lambda_{\lambda_i(j)} \circ \lambda_i(k)) &= \lambda_{iji^{-1}}(iki^{-1}) \\
&= (iji^{-1})(iki^{-1})(iji^{-1})^{-1} \\
&= (iji^{-1})(iki^{-1})(ij^{-1}i^{-1}) \\
&= i(jkj^{-1})i^{-1} \\
&= \lambda_i(jkj^{-1}) \\
&= \lambda_i \circ \lambda_j(k)
\end{aligned}$$

□

Example 16. *Let X be a finite set and define $\lambda_i(j) = f(j)$, where $f : X \to X$ is a bijective map. Since all λ maps are the same, this clearly satisfies the required properties to be a rack. This is known as the permutation rack.*

Given a rack it is natural to consider its subracks. Subracks are explored in detail in Section 3.4.2.

3.3.2 The Sq Map

The square Map, denoted Sq, arises from considering $\lambda_x(x)$, $x \in (X, \triangleright)$. In particular it is used to define quandles (see Section 3.3.3) and its linearisation is the partial trace of a rack-derived $\mathcal{R}$-matrix (see Section 3.3.4). Deceptively simplistic, the Sq map has many interesting properties that are of use to us in this thesis.

Definition 38. *The* ***Square Map*** $Sq : (X, \triangleright) \to (X, \triangleright)$ *is the map defined by:*

$$Sq(x) := x \triangleright x = \lambda_x(x)$$

The Sq map is in the centralizer of the rack, which induces several interesting properties as seen below. Firstly, to show that Sq is in the centralizer, we introduce a few definitions.

Definition 39. ***Automorphism:*** *An automorphism $\alpha \in S_X$ of a rack $(X, \triangleright)$ is a map such that*

$$\alpha(x \triangleright y) = \alpha(x) \triangleright \alpha(y)$$

The set of all automorphisms of a rack is defined as

$$Aut_\lambda(X) := \{\alpha \in S_X : \alpha(x \triangleright y) = \alpha(x) \triangleright \alpha(y)\}$$

where S_X is the symmetric group on X.

Note that all λ_x in a rack are automatically automorphisms by their self-distributive property.

We see that $Sq \in Aut_\lambda(X)$, as

$$\begin{aligned}
Sq(x \triangleright y) &= Sq(\lambda_x(y)) \\
&= \lambda_{\lambda_x(y)}\lambda_x(y) \\
&= \lambda_x\lambda_y\lambda_x^{-1}\lambda_x(y) && \text{(By definition of a rack)} \\
&= \lambda_x\lambda_y(y) \\
&= \lambda_x\lambda_x\lambda_x^{-1}\lambda_y(y) \\
&= \lambda_{\lambda_x(x)}\lambda_y(y) \\
&= Sq(x) \triangleright Sq(y)
\end{aligned}$$

Note that $Sq \in S_X$ (the symmetric group of X) and so is bijective and, in particular, invertible.

The centralizer is made up of automorphisms of a rack that commute with the maps of the rack.

Definition 40. ***Centralizer:*** *The centralizer $C_\lambda(X)$ of a rack (X, λ_x) is the set of all automorphisms that commute with all inner automorphisms, i.e.*

$$C_\lambda(X) := \{\alpha \in Aut_X : \alpha\lambda_x = \lambda_x\alpha \quad \forall x \in X\}$$

Note that for all central automorphisms $\alpha \in C_\lambda(X)$, the following equations must hold for all $x, y \in X$. We repeat the equations in triangle and λ notations for clarity:

$$x \triangleright \alpha(y) = \alpha(x \triangleright y) = \alpha(x) \triangleright \alpha(y)$$
$$\lambda_x\alpha(y) = \alpha\lambda_x(y) = \lambda_{\alpha(x)}\alpha(y)$$

Since the centralizer also contains all inverses, the above equations will apply when we consider $y = \alpha^{-1}(z)$, which implies the following equations. Once again we repeat the equations in both notations for clarity:

$$x \triangleright z = \alpha^{-1}(x) \triangleright z = \alpha(x) \triangleright z$$
$$\lambda_x(z) = \lambda_{\alpha^{-1}(x)}(z) = \lambda_{\alpha(x)}(z)$$

We now prove that the Sq map is in the centralizer of the rack.

Proposition 9. *Let $(X, \triangleright)$ be a rack, and let $Sq : (X, \triangleright) \to (X, \triangleright)$ be the square mapping on this rack. Then Sq is in the centralizer of the rack, $Sq \in C_\triangleright(X)$.*

Proof. For the Sq map to be in the centralizer, we must have that for all $x, y \in X$:

$$\lambda_x \cdot Sq(y) = Sq \cdot \lambda_x(y) \quad \forall x \in X$$

By the self-distributive property we have that

$$\begin{aligned} \lambda_x \cdot Sq(y) &= x \triangleright (y \triangleright y) \\ &= (x \triangleright y) \triangleright (x \triangleright y) \\ &= Sq \cdot \lambda_x(y) \end{aligned}$$

Therefore the Sq map of a rack is always in its centralizer. □

Hence, the following equations hold $\forall x \in (X, \lambda)$ by equation (**??**).

$$\lambda_x = \lambda_{Sq(x)} \tag{3.10}$$

$$\lambda_x = \lambda_{Sq^{-1}(x)} \tag{3.11}$$

The Sq map plays a particularly important role in the representation theory of racks, as its linearisation is the partial trace of a rack-derived $\mathcal{R}$-matrix - see Section 3.3.4.

Racks that have trivial Sq map are an important class of racks called quandles.

3.3.3 Quandles

Quandles are a special type of rack that play an important role in knot theory (see Section 3.5). The process of transforming a rack into a quandle, called quandalisation, is the inspiration for the Bloop group (see Section 3.4.3). We now define quandles and explain the process of transforming a rack into a quandle, called "quandalisation".

Definition 41. ***Quandles:*** *A quandle is a rack (X, λ) with the additional property*

$$\lambda_x(x) = x \quad \forall x \in X$$

This property may be re-written as

$$x \triangleright x = x \quad \forall x \in X$$
$$Sq(x) = x \quad \forall x \in X$$

Example 17. *Examples 14 and 15 are quandles.*

Example 18. *Let G be an abelian group and let $x \triangleright y = y + y - x$ for $x, y \in G$. Then $(G, \triangleright)$ is a clearly a quandle, since $x + x - x = x$. This is known as the Takasaki quandle.*

From any rack we can produce a quandle via a process called "quandalisation". This process was first described by Brieskorn [9], and entails the following.

Proposition 10. *Let (X, λ) be a rack and define the following map for all $x \in X$:*

$$\tilde{\lambda_x} := \lambda_x \circ Sq^{-1}(x)$$

Then $(X, \tilde{\lambda})$ is a quandle.

Proof. We first prove that $(X, \tilde{\lambda})$ is a rack, then we show that its Sq map is trivial.

Recall that $(X, \tilde{\lambda})$ is a rack if and only if the following equation holds.

$$\tilde{\lambda_x} \circ \tilde{\lambda_y} \circ \tilde{\lambda_x}^{-1} = \tilde{\lambda}_{\tilde{\lambda_x}(y)}$$

We have that

$$\begin{aligned}
\tilde{\lambda}_x \circ \tilde{\lambda}_y \circ \tilde{\lambda}_x^{-1} &= \lambda_x Sq^{-1} \circ \lambda_y Sq^{-1} \circ (\lambda_x Sq^{-1})^{-1} \\
&= \lambda_x Sq^{-1} \circ \lambda_y Sq^{-1} \circ Sq\lambda_x^{-1} \\
&= \lambda_x \lambda_y \lambda_x^{-1} Sq^{-1} \\
&= \lambda_{\lambda_x(y)} Sq^{-1} \\
&= \lambda_{Sq(\lambda_x(y))} Sq^{-1} \\
&= \lambda_{\lambda_x \circ Sq(y)} Sq^{-1} \\
&= \lambda_{\tilde{\lambda}_x(y)} Sq^{-1} \\
&= \tilde{\lambda}_{\tilde{\lambda}_x(y)}
\end{aligned}$$

Using the fact that Sq is in the centralizer.

Therefore $(X, \tilde{\lambda})$ is a rack. We now show that its Sq map is trivial.

$$\begin{aligned}
\tilde{Sq}(x) &= \tilde{\lambda}_x(x) \\
&= \lambda_x \circ Sq^{-1}(x) \\
&= Sq^{-1} \circ \lambda_x(x) \\
&= Sq^{-1} \circ Sq(x) \\
&= x
\end{aligned}$$

Hence, $(X, \tilde{\lambda})$ is a quandle. ∎

Quandles, and racks in general, can be used to define $\mathcal{R}$-matrices.

3.3.4 Rack-Derived $\mathcal{R}$-Matrices and Linearisations of Non-Degenerate YBE Solutions

Given a rack (X, λ) one can produce a set-theoretic Yang-Baxter solution

$$r(x, y) = (\lambda_x(y), x) \tag{3.12}$$

Indeed, we see that

$$\begin{aligned} r_1r_2r_1(x, y, z) &= r_1r_2(\lambda_x(y), x, z) \\ &= r_1(\lambda_x(y), \lambda_x(z), x) \\ &= (\lambda_{\lambda_x(y)} \circ \lambda_x(z), \lambda_x(y), x) \\ &= (\lambda_x \circ \lambda_y(z),_x (y), x) \\ &= r_2(_x{\circ}\lambda_y(z), x, y) \\ &= r_2r_1(x, \lambda_y(z), y) \\ &= r_1r_2r_2(x, y, z) \end{aligned}$$

Utilising the rack property (3.8).

Given a set-theoretic Yang-Baxter solution we may linearise it to a quantum Yang-Baxter solution by defining the quantum solution as follows.

Definition 42. ***Linearised Solution:*** *A linearised solution to the quantum Yang-Baxter equation is defined by the following equation, where $r(x, y)$ is a non-degenerate solution to the set-theoretic Yang-Baxter equation:*

$$R|x, y\rangle := |r(x, y)\rangle = |\lambda_x(y), x\rangle$$

The trace of a rack-derived $\mathcal{R}$-matrix $R|x, y\rangle = |x \triangleright y, x\rangle$ is given by

$$\begin{aligned} Tr(R) &= \sum_{x,y} \langle x, y|R|x, y\rangle \\ &= \sum_{x,y} \langle x, y|x \triangleright y, x\rangle \\ &= \sum_{x,y} \langle x|x \triangleright y\rangle\langle y|x\rangle \\ &= \sum_{x} \langle x|x \triangleright x\rangle \\ &= \sum_{x} \langle x|Sq(x)\rangle \end{aligned}$$

The (right) partial trace is given by

$$\begin{aligned} ptr(R) &= \sum_{x,y} \langle x, y \rangle |x \triangleright y\rangle\langle x| \\ &= \sum_{x} |x \triangleright x\rangle\langle x| \\ &= \sum_{x} |Sq(x)\rangle\langle x| \end{aligned}$$

We denote this partial trace, the linearisation of the Sq map, by T.

$$T := ptr(R) = \sum_{x} |Sq(x)\rangle\langle x| \tag{3.13}$$

Since $Sq \in S_X$, i.e. is a permutation, this shows that the partial trace of a rack-derived $\mathcal{R}-$matrix is always a permutation matrix.

We note that $R \in \text{End}(V \otimes V)$, where V is the vector space spanned by X. This vector space must have an orthonormal basis (by having a suitable scalar product), and this ensures R is unitary. Indeed, we see that

$$\begin{aligned} \langle Rx_1x_2|Ry_1y_2\rangle &= \langle \lambda_{x_1}(x_2), x_1|\lambda_{y_1}(y_2), y_1\rangle \\ &= \langle \lambda_{x_1}(x_2), x_1|\lambda_{x_1}(y_2), y_1\rangle \\ &= \langle x_2, x_1|y_2, y_1\rangle \\ &= \langle x_1, x_2|y_1, y_2\rangle \end{aligned}$$

As $\lambda_{x_1}(x_2) = \lambda_{x_1}(y_2) \iff x_2 = y_2$, as λ_x is a bijective function for all $x \in X$ by the definition of a rack.

Since any rack can produce a solution to the set-theoretic Yang-Baxter equation, we have that any rack can produce a solution to the quantum Yang-Baxter equation:

$$R|x, y\rangle = |\lambda_x(y), x\rangle$$

Indeed, any linearised set-theoretic solution is equivalent to a rack-derived solution as above.

Proposition 11. *Let $r(x, y) := (\lambda_x(y), \rho_y(x))$, where λ_x and ρ_y are bijective functions for all x, y in a set X, be a non-degenerate solution to the set-theoretic*

Yang-Baxter equation with linearisation $R|x,y\rangle = |\lambda_x(y), \rho_y(x)\rangle$.

Then there exists a rack (X, λ') *which induces the quantum YBE solution* $R'|x,y\rangle = |\lambda'_x(y), x\rangle$ *such that* $R \sim R'$.

Proof. In order to show that $R \sim S$ we show that there is a unitary intertwiner U^n such that $U^n R_i = S_i U^n$.

We define the following intertwiner U^n, called the "guitar map", as in [2] Prop 5.4. Define

$$\begin{aligned} Q^n|x_1, ..., x_{n+1}\rangle &:= |\rho_{x_{n+1}}(x_1), ..., \rho_{x_{n+1}}(x_n), x_{n+1}\rangle \\ U^2|x_1, x_2\rangle &:= |\rho_{x_2}(x_1), x_2\rangle \\ U^n|x_1, ...x_n\rangle &:= Q^{n-1}(U^{n-1} \times \mathrm{id})|x_1, ..., x_n\rangle \end{aligned}$$

Note that since U^n does not rearrange any tensor factors, we can re-write it in the form

$$U^n|x_1, ..., x_n\rangle = |U_1^n(x_1), ..., U_i^n(x_i), ..., U_{n-1}^n(x_{n-1}), x_n\rangle \tag{3.14}$$

We complete this proof by induction. Firstly, see that for $n = 2$, the LHS is

$$\begin{aligned} U^2 R|x,y\rangle &= U^2|\lambda_x(y), \rho_y(x)\rangle \\ &= |\rho_{\rho_y(x)} \circ \lambda_x(y), \rho_y(x)\rangle \end{aligned}$$

For the RHS we have

$$\begin{aligned} SU^2|x,y\rangle &= S|\rho_y(x), y\rangle \\ &= |\rho_y(x) \triangleright y, \rho_y(x)\rangle \\ &= |\rho_{\rho_y(x)} \circ \lambda_{\rho_y^{-1} \circ \rho_y(x)}(y)\rangle \text{ by the previous proposition} \\ &= |\rho_{\rho_y(x)} \circ \lambda_x(y), \rho_y(x)\rangle \end{aligned}$$

As required.

Therefore we can assume for any $n(\geq 2) \in \mathbb{N}$ that $U^n R_i = S_i U^n$,[1] i.e.

[1]Where $R_i = \varphi^{i-1}(R)$, $S_i = \varphi^{i-1}(S)$

LHS

$$\begin{aligned} U^n R_i|x_1,...,x_n\rangle &= U^n|x_1,...,x_{i-1},\lambda_{x_i}(x_{i+1}),\rho_{x_{i+1}}(x_i),x_{i+2},...,x_n\rangle \\ &= |U_1^n(x_1),...,U_i^n \circ \lambda_{x_i}(x_{i+1}),U_{i+1}^n \circ \rho_{x_{i+1}}(x_i),...,x_n\rangle \end{aligned}$$

RHS

$$\begin{aligned} S_iU^n|x_1,...,x_n\rangle &= S_i|U_1^n(x_1),...,x_n\rangle \\ &= |U_1^n(x_1),...,U_{i-1}^n(x_{i-1}),U_i^n(x_i) \triangleright U_{i+1}^n(x_{i+1}),U_i^n(x_i),...,x_n\rangle \end{aligned}$$

Looking at the i^{th} and $i+1^{th}$ positions yields the following two equations:

$$U_i^n \circ \lambda_{x_i}(x_{i+1}) = U_i^n(x_i) \triangleright U_{i+1}^n(x_{i+1}) \tag{3.15}$$

$$U_{i+1}^n \circ \rho_{x_{i+1}}(x_i) = U_i^n(x_i) \tag{3.16}$$

$$\tag{3.17}$$

Now for the induction. Calculating the LHS and RHS of $U^{n+1}R_i = S_iU^{n+1}$ yields

LHS

$$\begin{aligned} U^{n+1}R_i|x_1,...,x_{n+1}\rangle &= U^{n+1}|x_1,...,\lambda_{x_i}(x_{i+1}),\rho_{x_{i+1}}(x_i),...,x_{n+1}\rangle \\ &= Q^{n-1}(U^n \otimes id)|x_1,...,\lambda_{x_i}(x_{i+1}),\rho_{x_{i+1}}(x_i),...,x_{n+1}\rangle \\ &= Q^{n-1}|U_1^n(x_1),...,U_i^n \circ \lambda_{x_i}(x_{i+1}),U_{i+1}^n) \circ \rho_{x_{i+1}}(x_i),...,x_{n+1}\rangle \\ &= |\rho_{x_{n+1}} \circ U_1^n(x_1),...,\rho_{x_{n+1}} \circ U_i^n \circ \lambda_{x_i}(x_{i+1}),\rho_{x_{n+1}} \circ U_{i+1}^n) \\ &\quad \circ \rho_{x_{i+1}}(x_i),...,x_{n+1}\rangle \end{aligned}$$

RHS

$$\begin{aligned} S_iU^{n+1}|x_1,...,x_{n+1}\rangle &= S_iQ^{n-1}(U^n \times id)|x_1,...,x_{n+1}\rangle \\ &= S_iQ^{n-1}|U_1^n(x_1),...,U_n^n(x_n),x_{n+1}\rangle \\ &= S_i|\rho_{x_{n+1}} \circ U_1^n(x_1),...,\rho_{x_{n+1}} \circ U_n^n(x_n),x_{n+1}\rangle \\ &= |\rho_{x_{n+1}} \circ U_1^n(x_1),...,(\rho_{x_{n+1}} \circ U_i^n(x_i)) \triangleright (\rho_{x_{n+1}} \circ U_{i+1}^n(x_{i+1})),\rho_{x_{n+1}} \\ &\quad \circ U_i^n(x_i),...,x_{n+1}\rangle \end{aligned}$$

Looking at the i^{th} position we require

$$\rho_{x_{n+1}} \circ U_i^n \circ \lambda_{x_i} = (\rho_{x_{n+1}} \circ U_i^n(x_i)) \triangleright (\rho_{x_{n+1}} \circ U_{i+1}^n(x_{i+1}))$$

Consider the RHS:

$$(\rho_{x_{n+1}} \circ U_i^n(x_i)) \triangleright (\rho_{x_{n+1}} \circ U_{i+1}^n(x_{i+1})) = \rho_{x_{n+1}}(U_i^n(x_i) \triangleright U_{i+1}^n(x_{i+1}) \\ = \rho_{x_{n+1}} \circ U_i^n \circ \lambda_{x_i}(x_{i+1}) \text{ by equation (3.15)}$$

As required. Now for the $i+1^{th}$ position, for which we require

$$\rho_{x_{n+1}} \circ U_{i+1}^n \circ \rho_{x_{i+1}}(x_i) = \rho_{x_{n+1}} \circ U_i^n(x_i)$$

Note that this is immediate from applying $\rho_{x_{n+1}}$ on the left to both sides of equation (3.16).

Therefore we have shown that $U^n R_i = S_i U^n$ for any $n \in \mathbb{N}$, i.e. that $R \sim S$.

□

Rack-derived $\mathcal{R}$-matrices are very closely tied to their originating racks. We now explore some of these properties.

3.4 Properties of Racks and Rack-Derived $\mathcal{R}$-Matrices

In this section we define what it means for two racks to be isomorphic. We show that the disjoint union of all subracks of a rack does satisfy the rack properties but is not isomorphic to the original rack. We show that isomorphic racks induce equivalent $\mathcal{R}$-matrices. We then examine the quandalisation process in terms of rack-induced $\mathcal{R}$-matrices. Finally, we show that given two racks that induce equivalent $\mathcal{R}$-matrices, their quandalisations must also produce equivalent $\mathcal{R}$-matrices.

3.4.1 Isomorphisms of Racks

We now define isomorphisms of racks, give examples, and show that if two racks are isomorphic then their linearisations are equivalent as $\mathcal{R}$-matrices.

Definition 43. ***Isomorphic racks:*** *Two racks (X, λ), (Y, μ) are said to be isomorphic if they are of the same size and their permutations are conjugate i.e.*

$$(X,\lambda) \simeq (Y,\mu) \iff |X| = |Y| \text{ and } \exists\, \pi \in S_X \text{ such that} \\ \lambda_i = \pi \circ \mu_{\pi^{-1}(i)} \circ \pi^{-1} \quad \forall i = 1, ..., |X| \tag{3.18}$$

All racks of size 3 are classified up to isomorphism in Appendix A.

If two racks are isomorphic, then their linearisations are equivalent as $\mathcal{R}$-matrices.

Theorem 15. *Let* $(X, \triangleright)$ *and* $(X', \triangleright')$ *be racks such that* $|X| = |X'| = n$ *with linearisations* $R_X|x, y\rangle := |x \triangleright y, x\rangle$ *and* $R_{X'}|x, y\rangle := |x \triangleright' y, x\rangle$ *respectively. Then*

$$(X, \triangleright) \simeq (X', \triangleright') \implies R_X \sim R_{X'}$$

Proof. By definition of the isomorphism $(X, \triangleright) \simeq (X', \triangleright')$, there must exist some $\pi \in S_X$ such that, for all $i \in 1, 2, ..., n$

$$\lambda_i = \pi \circ \mu_{\pi^{-1}(i)} \circ \pi^{-1}$$

Recall that

$$R_X \sim R_Y \iff \exists V_n : \rho^n_{R_X}(b_i) = V_n \rho^n_{R_Y}(b_i) V_n^* \quad \forall b_i \in \mathcal{B}_n$$

where $\rho^n_{R_X}(b_i)$ here is the representation of the generator $b_i \in \mathcal{B}_n$ induced by the $\mathcal{R}$-matrix $R_X|x, y\rangle := |\lambda_x(y), x\rangle$.

Let us define the unitary intertwiner V_n as the linearisation of the π map above, i.e.

$$V_n|x_1, ..., x_n\rangle := |\pi(x_1), ..., \pi(x_n)\rangle$$

Since π is a permutation this is clearly unitary with its adjoint given by

$$V_n^*|x_1, ..., x_n\rangle := |\pi^{-1}(x_1), ..., \pi^{-1}(x_n)\rangle$$

We now show that $\rho^n_{R_X}(b_i) = V_n \rho^n_{R_Y}(b_i) V_n^* \quad \forall b_i \in \mathcal{B}_n$.

$$\begin{aligned}
V_n \circ \rho^n_{R_Y}(b_i) \circ V_n^*|x_1, ..., x_n\rangle &= |V_n \circ \rho^n_{R_Y}(b_i)|\pi^{-1}(x_1), ..., \pi^{-1}(x_n)\rangle \\
&= V_n|\pi^{-1}(x_1), ..., \pi^{-1}(x_{i-1}), \mu_{\pi^{-1}(x_i)} \circ \pi^{-1}(x_{i+1}), \\
&\quad\quad \pi^{-1}(x_i), \pi^{-1}(x_{i+2}, ..., \pi^{-1}(x_n)\rangle \\
&= |x_1, ..., x_{i-1}, \pi \circ \mu_{\pi^{-1}(x_i)} \circ \pi^{-1}(x_{i+1}), \pi^{-1}(x_i), x_{i+2}, ..., x_n\rangle \\
&=: \rho^n_{R_X}(b_i)|x_1, ..., x_n\rangle
\end{aligned}$$

Hence, the required equation is satisfied for all $i \in 1, ..., n$ and so $R_X \sim R_Y$. □

3.4.2 Decomposability of Racks

We can create new racks from a given rack. For example, we can create subracks or consider the union of two racks.

Definition 44. ***Subrack:*** *A subrack is a rack (S, λ') that arises from another rack (X, λ) such that $X = S \cup S^{\perp}$ and the rack action λ' on S satisfies the self-distributive condition as in Equation* (3.8) *and is defined by:*

$$\lambda'_i(j) := \begin{cases} \lambda_i(j) & \textit{when } i \sim j \\ j & \textit{else} \end{cases}$$

where we denote $i \sim j \iff i, j \in S$ or $i, j \in S^{\perp}$.

Not every subset of the defining set of a rack will produce a subrack.

Example 19. *Consider the rack (X, λ) where $X = \{1, 2, 3\}$ and the rack operation is given by the table below. The entry $X_{i,j}$ in this table is to be read as $i \triangleright j$, where i is the row and j is the column of the entry. More details of this rack can be found in Appendix A.*

	1	*2*	*3*
1	*1*	*2*	*3*
2	*3*	*2*	*1*
3	*1*	*2*	*3*

Consider the subset $S = \{1, 2\}$. Then $\lambda'_2(1) = 3 \notin S$. Hence S cannot be a subrack as it is not closed.

A rack is said to be decomposable if it contains any subracks.

Definition 45. ***Decomposable rack:*** *A rack (X, λ) is decomposable if $\exists$ a subset $S \subset X$ such that (S, λ') is a rack.*

One can always create a new rack from two existing racks by forming their disjoint union.

Proposition 12. *Let (X, λ) and (Y, η) be racks. Let $X \sqcup Y$ be the disjoint union of X and Y and define the following function for all $i, j \in X \sqcup Y$*

$$\mu_i(j) := \begin{cases} \lambda_i(j) & \textit{when } i, j \in X \\ \eta_i(j) & \textit{when } i, j \in Y \\ j & \textit{else} \end{cases}$$

Then $(X \sqcup Y, \mu)$ is a rack.

Proof. Clearly μ_i is bijective for all $i \in X \sqcup Y$ as it is a piece-wise function made up solely of bijective functions whose conditions cover the entire domain of μ_i.

It only remains to show that the self-distributive property holds, i.e.

$$\mu_i \mu_j(k) = \mu_{\mu_i(j)} \mu_i(k) \quad \forall i, j, k \in X \sqcup Y$$

There are four possibilities to consider:

1. $i, j, k \in X$
2. $j, k \in X$, $i \in Y$
3. $i, k \in X$, $j \in Y$
4. $i, j \in X$, $k \in Y$

Although one might consider the further cases of replacing X with Y in the above list, there is no material difference between the racks (X, λ) and (Y, η) so to consider these cases explicitly would simply be repeating the below proof and replacing λ with η.

Case 1

$$\begin{aligned} \mu_{\mu_i(j)} \mu_i(k) &= \lambda_{\lambda_i(j)} \lambda_i(k) \\ &= \lambda_i \lambda_j(k) \end{aligned}$$

Case 2

$$\begin{aligned} \mu_{\mu_i(j)} \mu_i(k) &= \mu_{\mu_i(j)}(k) \\ &= \mu_j(k) \\ &= \lambda_j(k) \\ &= \mu_i \lambda_j(k) \\ &= \mu_i \mu_j(k) \end{aligned}$$

Case 3

$$\begin{aligned}\mu_{\mu_i(j)}\mu_i(k) &= \mu_j\lambda_i(k)\\ &= \lambda_i(k)\\ &= \mu_i(k)\\ &= \mu_i\mu_j(k)\end{aligned}$$

Case 4

$$\begin{aligned}\mu_{\mu_i(j)}\mu_i(k) &= k\\ &= \mu_i\mu_j(k)\end{aligned}$$

Therefore the self-distributive property holds for all elements in the set, and so $(X \sqcup Y, \mu)$ is a rack. □

Given a decomposable rack (X, λ) such that the subsets S and $S^{\perp}$ both form subracks (S, ν) and $(S^{\perp}, \eta)$, it is not the case that the disjoint union of the subracks $(S \sqcup S^{\perp}, \mu)$ as defined in the above Proposition 12 is isomorphic to the original rack (X, λ). We provide a counter-example below.

Example 20. *Consider the rack (X, λ) where $X = \{1, 2, 3\}$ and λ is given by the table below*

X	1	2	3
1	1	3	2
2	1	3	2
3	1	3	2

Consider the subsets $S = \{1\}$ and $S^{\perp} = \{2, 3\}$. Their tables are given by

S	1	2	3
1	1	2	3
2	1	2	3
3	1	2	3

$S^{\perp}$	1	2	3
1	1	2	3
2	1	3	2
3	1	3	2

Their disjoint union is given by

$S \sqcup S^{\perp}$	1	2	3
1	1	2	3
2	1	3	2
3	1	3	2

By the table in Appendix A, this is not isomorphic to (X, λ).

3.4.3 Quandalisation in terms of R-matrices

Recall that if (X, λ) is a rack, then $(X, \lambda \circ Sq^{-1})$ is its quandalisation. Let us consider this in terms of $\mathcal{R}$-matrices by considering the linearisation of this process.

Let (X, λ) be a rack and let $(X, \tilde{\lambda})$ be its quandalisation, where $\tilde{\lambda_x} := \lambda_x \circ Sq^{-1}$ for all $x \in X$.

The set-theoretic solution derived from this rack is given by

$$r(x, y) = (\lambda_x(y), x)$$

The set-theoretic solution for the quandalisation of this rack is thus given by

$$\tilde{r}(x, y) := (\tilde{\lambda_x}(y), x) \tag{3.19}$$
$$= (\lambda_x \circ Sq^{-1}(y), x) \tag{3.20}$$
$$= (r \circ (1 \times Sq^{-1}))(x, y) \tag{3.21}$$

We now consider the linearisation of this.

Let us define the following operator T, which acts as the partial trace (notably it is the linearisation of the Sq map of a rack) on a single tensor factor.

Definition 46. T**:** *Let (X, λ) be a rack with a square map $Sq(x) := \lambda_x(x)$ for all $x \in X$. Then the T map is defined by:*

$$(T \otimes \mathbb{1})|x, y\rangle := |Sq(x), y\rangle$$
$$(\mathbb{1} \otimes T)|x, y\rangle := |x, Sq(y)\rangle$$

It is clear that its adjoint (inverse) is given by

$$(T^* \otimes \mathbb{1})|x, y\rangle := |Sq^{-1}(x), y\rangle$$

Hence, we have that the linearisation of equation (3.21) is given by

$$\tilde{R}|x, y\rangle := RT^*|x, y\rangle \tag{3.22}$$

Recalling that the partial trace is seen as closing a single strand of a braid, the quandalisation process is expressed in diagram form as in Figure 3.4.

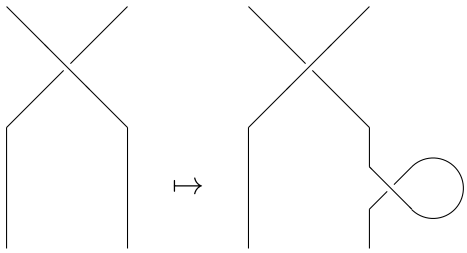


Figure 3.4: Diagrammatic presentation of quandalisation of a rack-induced $\mathcal{R}$-matrix

Note that this diagram is also the diagram of an element in the Bloop group, specifically $b_1 l_2 \in \mathcal{B}\ell_2$ - see Section 3.2.

3.4.4 Equivalence of Quandalisations

In this section we show that if two racks have equivalent linearisations then the linearisations of their quandalisations must also be equivalent. To do this we utilise our $\mathcal{R}$-matrix representation of the Bloop group (see Section 3.2).

Recall Equation (3.22), which describes the quandalisation process. We see that diagrammatically, as seen in Figure 3.4, this is simply adding a loop to the braid generator.

Two $\mathcal{R}$-matrices are equivalent iff their dimensions and characters are the same. In the quandalisation process the size of the rack does not change, so we need only check that the characters remain the same.

We aim to show that the quandalisations are equivalent by adding a loop after each braid generator to produce the quandle, pulling that loop to the far right, and "stretching" it out by cutting it open, pulling and then closing the braid. A simplified diagram of this can be seen in Figure 3.8. We then see that considering the character of the braid over the quandalisation of the rack is equal to considering the character of the quandalisation-deformed braid over the rack.

We begin by adding a loop to braid generators by mapping the braid group to the Bloop group with the "add-loop" map:

Definition 47. *The **add-loop function:** $\delta_n : \mathcal{B}_n \to \mathcal{B}\ell_n$ is defined by*

$$\begin{aligned} \delta_n : \mathcal{B}_n &\to \mathcal{B}\ell_n \\ b_j &\mapsto (l_{j+1}, b_j) \end{aligned} \tag{3.23}$$

where we extend to the whole Braid group as a group homomorphism.

We show that this satisfies the braid conditions.

Firstly, we consider

$$b_k b_{k+1} b_k = b_{k+1} b_k b_{k+1}$$

We have that

$$\begin{aligned} \delta_n(b_k b_{k+1} b_k) &= (l_{k+1}, b_k) \cdot (l_{k+2}, b_{k+1}) \cdot (l_{k+1}, b_k) \\ &= (l_{k+1}, b_k) \cdot (l_{k+2} + \alpha_{\psi_{b_{k+1}}}(l_{k+1}), b_{k+1} b_k) \\ &= (l_{k+1}, b_k) \cdot (l_{k+2} + l_{k+2}, b_{k+1} b_k) \\ &= (l_{k+1} + \alpha_{\psi_{b_k}}(l_{k+2} + l_{k+2}), b_k b_{k+1} b_k) \\ &= (l_{k+1} + l_{k+2} + l_{k+2}, b_k b_{k+1} b_k) \\ &= (l_{k+2} + l_{k+2} + l_{k+1}, b_{k+1} b_k b_{k+1}) \\ &= (l_{k+2} + \alpha_{\psi_{b_{k+1}}}(l_{k+1} + l_{k+2}), b_{k+1} b_k b_{k+1}) \\ &= (l_{k+2}, b_{k+1}) \cdot (l_{k+1} + l_{k+2}, b_k b_{k+1}) \\ &= (l_{k+2}, b_{k+1}) \cdot (l_{k+1} + \alpha_{\psi_{b_k}}(l_{k+2}), b_k b_{k+1}) \\ &= (l_{k+2}, b_{k+1}) \cdot (l_{k+1}, b_k) \cdot (l_{k+2}, b_{k+1}) \\ &= \delta_n(b_{k+1} b_k b_{k+1}) \end{aligned}$$

We now see that

$$b_k b_j = b_j b_k \quad \forall |j-k| \geq 2$$

This is quite clear, as:

$$\begin{aligned} \delta_n(b_k b_j) &= (l_{k+1}, b_k) \cdot (l_{j+1}, b_j) \\ &= (l_{k+1} + \alpha_{\psi_{b_k}}(l_{j+1}), b_k b_j) \\ &= (l_{k+1} + l_{j+1}, b_k b_j) && \text{(As } |j-k| \geq 2) \\ &= (l_{j+1} + l_{k+1}, b_j b_k) \\ &= (l_{j+1} + \alpha_{\psi_{b_j}}(l_{k+1}), b_j b_k) \\ &= (l_{j+1}, b_j) \cdot (l_{k+1}, b_k) \\ &= \delta_n(b_j b_k) \end{aligned}$$

The loop function takes an element of the braid group $b \in \mathcal{B}_n$ and maps it to the Bloop group $\mathcal{B}\ell_n$ by adding a loop after every braid generator.

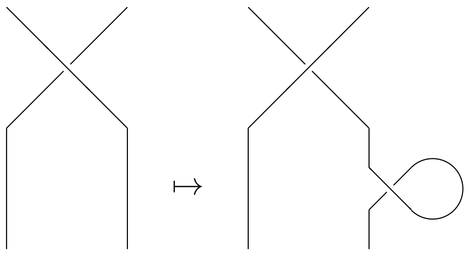


Figure 3.5: Add-loop function δ_n, as in equation (3.23)

Example 21. *The add-loop function δ_n adds a loop after every generator. For example*

$$\delta_n(b_1 b_2 b_1) = (l_2, b_1)(l_3, b_2)(l_2 b_1)$$

If we consider $\mathcal{B}\ell_n$ in terms of generators and relations as opposed to a semi-direct product, this can be re-written as

$$\delta_n(b_1 b_2 b_1) = b_1 l_2 b_2 l_3 b_1 l_2$$

Diagrammatically this can be seen as in Figure 3.6.

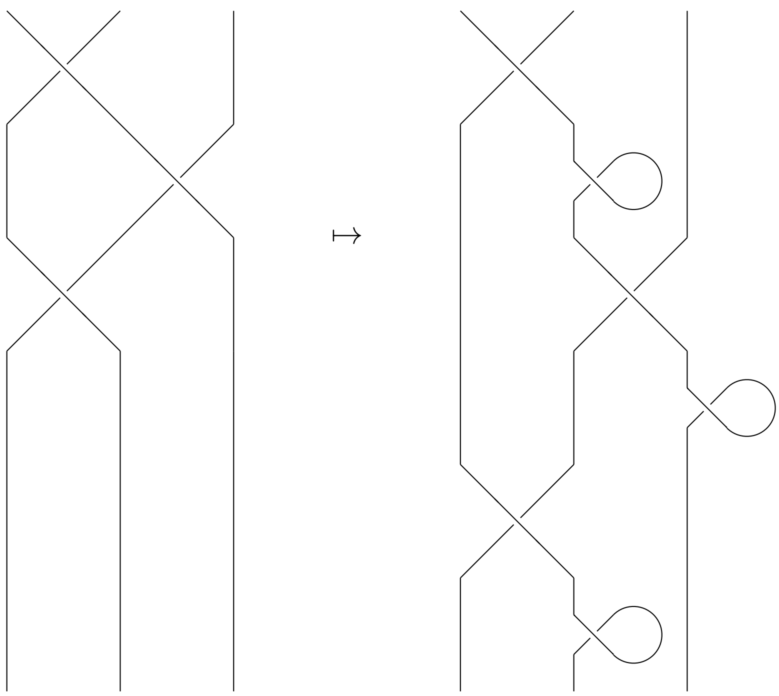


Figure 3.6: Diagrammatic presentation of $\delta_3(b_1 b_2 b_1)$

We now define un-twisting the loop.

Definition 48. *The **loop-untwist function** $\theta'_n : \mathbb{Z}^n \to \mathcal{B}_\infty$ is defined on a single loop by*

$$\begin{aligned} \theta'_n : \mathbb{Z}^n &\to \mathcal{B}_{n+1} \\ l_k &\mapsto b_k^{-1} b_{k+1}^{-1} ... b_{n-1}^{-1} b_n^{-1} b_{n-1} ... b_k \end{aligned}$$

Products of uniquely ordered[2] loops $l_{k_1} l_{k_2} ... l_{k_m}$, where $k_1 \geq k_2 \geq ... \geq k_m$, are mapped by

$$\theta'_n(l_{k_1} l_{k_2} ... l_{k_m}) := \theta'_n(l_{k_1}) \theta'_{n+1}(l_{k_2}) ... \theta'_{n+m-1}(l_{k_m})$$

The loop-untwist function is extended to take $\mathcal{B}\ell_n$ as its domain by performing the identity map on braid elements.

Definition 49. *The **bloop-untwist function** $\theta_n : \mathcal{B}\ell_n \to \mathcal{B}_{n+m}$ is defined on a generic bloop element by*

$$\begin{aligned} \theta_n : \mathcal{B}\ell_n &\to \mathcal{B}_\infty \\ (l, b) &\mapsto b \cdot \theta'_n(l) \end{aligned}$$

Remark 2. *Essentially, this function takes a bloop element, orders it to be in the form $\sigma l_{k_1}^{q_1} ... i_{k_m}^{q_m}$ where $k_1 \geq k_2 \geq ... \geq k_m$ and $\sigma \in \mathcal{B}_n$ and $q_j \in \mathbb{N}$ (any bloop can be re-written in this form using the exchange relations* **??***), then pulls the loop generators under any strands to the right using 3.7 and untwists them. Note that each untwist creates a new strand.*

This is demonstrated in Figure 3.7.

Definition 50. *The **quandler function** $\Phi_n : \mathcal{B}_n \to \mathcal{B}_\infty$ is defined by*

$$\Phi_n := \theta_n \circ \delta_n \tag{3.24}$$

[2]Since all loops commute you can always manipulate a concatenation of loops to be ordered this way.

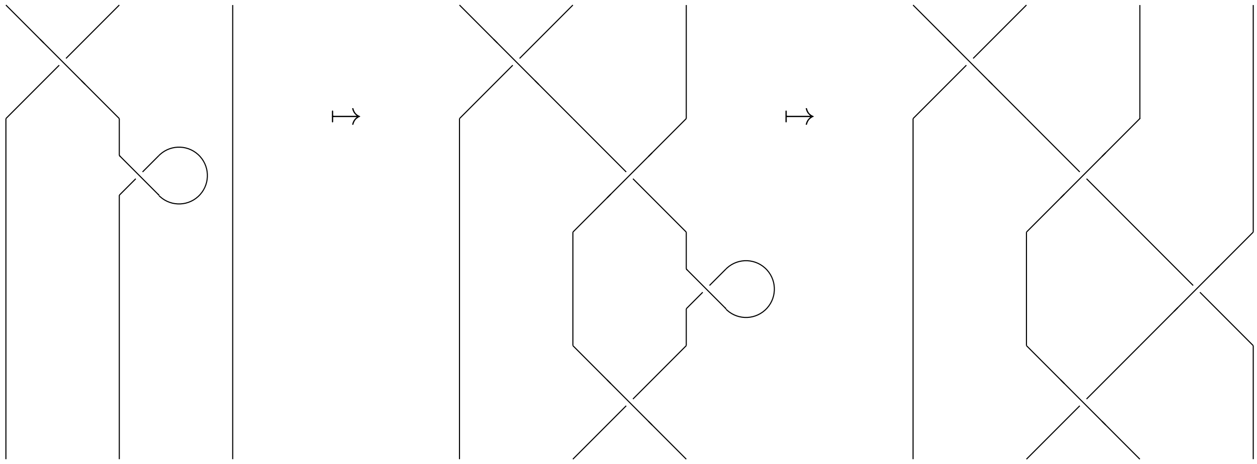


Figure 3.7: Bloop untwist of (l_2, b_1).

We now prove some lemmas of exchange relations, which will be used in proving a proposition about characters of quandles, and subsequently be used to prove the main theorem of this section. Note that these results holds in general for unitary $\mathcal{R}$-matrices.

Lemma 2. *Let $R \in V \otimes V$ be an $\mathcal{R}$- matrix arising from the linearisation of a set-theoretic Yang-Baxter solution and let T be its normalised partial trace. Denote by $\mathbb{1}$ the identity in End(V). Then*

$$(T \otimes \mathbb{1})R = R(\mathbb{1} \otimes T) \tag{3.25}$$

$$(\mathbb{1} \otimes T)R = R(T \otimes \mathbb{1}) \tag{3.26}$$

Proof. Recall that Sq, which linearises to T, has the following property

$$Sq(x \triangleright y) = Sq(x) \triangleright Sq(y) = x \triangleright Sq(y)$$

Now consider

$$\begin{aligned} ((Sq \times \mathrm{id}_X) \circ r)(a, b) &= (Sq \times \mathrm{id}_X)(a \triangleright b, a) \\ &= (Sq(a \triangleright b), a) \\ &= (a \triangleright Sq(b), a) \\ &= (r \circ (\mathrm{id}_X \times Sq)(a, b) \end{aligned}$$

Therefore

$$(Sq \times \mathrm{id}_X) \circ r = r \circ (\mathrm{id}_X \times Sq)$$

This linearises to the claimed result. Equation (3.26) is derived in an analogous manner. □

Lemma 3. *Let $R \in V \otimes V$ be an $\mathcal{R}$-matrix and let T be its normalised partial trace. Then*

$$R_j^{-1} R_{j+1}^{-1} ... R_{n-1}^{-1} T_n^* R_{n-1} ... R_j = T_j^*$$

Proof. Recall $(T \otimes \mathbb{1})R = R(\mathbb{1} \otimes T)$, i.e. $T_j R_j = R_j T_{j+1}$.

$$\begin{aligned}
R_j^{-1} ... R_{n-1}^{-1} T_n^* R_{n-1} ... R_j &= R_j^{-1} ... R_{n-2}^{-1} T_{n-1}^* R_{n-1}^{-1} R_{n-1} R_{n-2} ... R_j \\
&= R_j^{-1} ... R_{n-2}^{-1} T_{n-1}^* R_{n-2} ... R_j \\
&= R_j^{-1} ... R_{n-3}^{-1} T_{n-2}^* R_{n-2}^{-1} R_{n-2} ... R_j \\
&= ... \\
&= R_j^{-1} T_{j+1}^* R_j \\
&= T_j^* R_j^{-1} R_j \\
&= T_j^*
\end{aligned}$$

□

We now consider the $\mathcal{R}$-matrix representation of the Bloop group $\mathcal{B}\ell_n$. The following lemma shows that when we consider the closure of a Bloop, the loops can be pulled to the far right and then "unlooped", and this will not affect the closure of the Bloop. A simplified diagram of this process is shown in Figure 3.8.

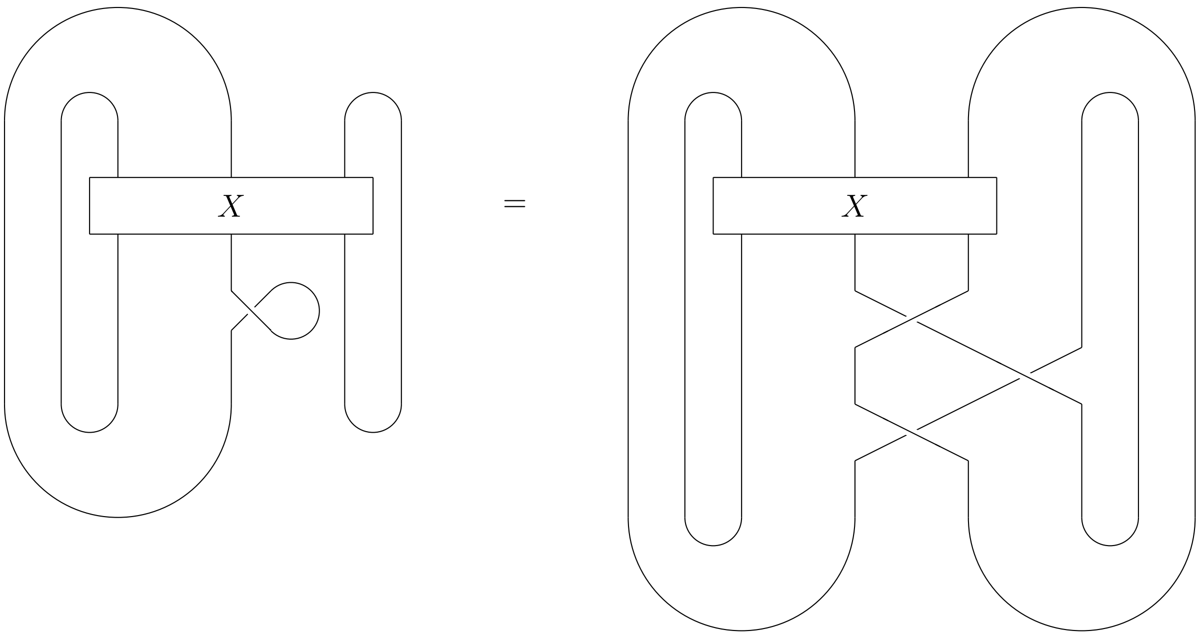


Figure 3.8: A simplified depiction of the equation proven in Lemma 4.

Lemma 4. *Let $X \in End(V \otimes V)$ and $T^{*q_j}_{k_j}$ be the normalised partial trace of a derived*[3] *$\mathcal{R}$-matrix R, where $k_j \in \mathbb{N}$ and $q_j \in \{\pm 1\}$ for all j. Let ρ be the Yang-Baxter representation of the Bloop group and let θ_n be the bloop-untwist function as described above. Let $X \in \mathcal{B}\ell_n$ be an arbitrary Bloop. Then, for all $m \in \mathbb{N}$:*

$$tr_{1...n}(XT^{*q_1}_{k_1}...T^{*q_m}_{k_n}) = tr_{1...n+m}(X\rho_n(\theta_n(l^{q_1}_{k_1}...l^{q_m}_{k_m}))$$

where $k_1 \geq k_2 \geq k_m$.

Proof. We show this statement with a proof by induction in m.

$m = 1$:

$$\begin{aligned} tr_{1...n+1}(X\rho(\theta_n(l^{q_1}_{k_1})) &= tr_{1...n+1}(XR^{-1}_{k_1}R^{-1}_{k_2}...R^{-1}_{k_{n-1}} \cdot R_n^{-q_1} \cdot R_{k_{n-1}}...R_{k_2}R_{k_1}) \\ &= tr_{1...n}(XR^{-1}_{k_1}R^{-1}_{k_2}...R^{-1}_{k_{n-1}} \cdot tr_{n+1}(R_n^{-q_1}) \cdot R_{k_{n-1}}...R_{k_2}R_{k_1}) \\ &= tr_{1...n}(XR^{-1}_{k_1}R^{-1}_{k_2}...R^{-1}_{k_{n-1}} \cdot T^{*q_1}_n \cdot R_{k_{n-1}}...R_{k_2}R_{k_1}) \\ &= tr_{1...n}(XT^{*q_1}_{k_1}) \end{aligned}$$

$m = j$

Hence, we may assume that

$$tr_{1...n}(XT^{*q_1}_{k_1}...T^{*q_j}_{k_j}) = tr_{1...(n+j)}(X\rho(\theta_n(l^{q_1}_{k_1}...l^{q_j}_{k_j})))$$

Now the for inductive step

$m = j + 1$

$$\begin{aligned} &tr_{1...(n+j+1)}(X\rho(\theta_n(l^{q_1}_{k_1}...l^{q_j}_{k_j}l^{q_{j+1}}_{k_{j+1}}))) \\ &= tr_{1...(n+j+1)}(X\rho(\theta_n(l^{q_1}_{k_1}...l^{q_j}_{k_j})\theta_{n+j}(l^{q_{j+1}}_{k_{j+1}}))) \\ &= tr_{1...(n+j+1)}(X\rho(\theta_n(l^{q_1}_{k_1}...l^{q_j}_{k_j}))R^{-1}_{k_{j+1}}...R_{k_{n+j-1}} \cdot R^{-q_{j+1}}_{k_{n+j}} \cdot R_{k_{n+j-1}}...R_{k_{j+1}}) \\ &= tr_{1...(n+j)}(X\rho(\theta_n(l^{q_1}_{k_1}...l^{q_j}_{k_j}))R^{-1}_{k_{j+1}}...R_{k_{n+j-1}} \cdot tr_{n+j+1}(R^{-q_{j+1}}_{k_{n+j}}) \cdot R_{k_{n+j-1}}...R_{k_{j+1}}) \\ &= tr_{1...(n+j)}(X\rho(\theta_n(l^{q_1}_{k_1}...l^{q_j}_{k_j}))R^{-1}_{k_{j+1}}...R_{k_{n+j-1}} \cdot T^{*-q_{j+1}}_{k_{n+j}} \cdot R_{k_{n+j-1}}...R_{k_{j+1}}) \\ &= tr_{1...(n+j)}(X\rho(\theta_n(l^{q_1}_{k_1}...l^{q_j}_{k_j}))T^{*-q_{j+1}}_{k_{j+1}}) \\ &= tr_{1...(n+j)}(X'\rho(\theta_n(l^{q_1}_{k_1}...l^{q_j}_{k_j})) \\ &= tr_{1...n}(X'T^{*q_1}_{k_1}...Tk_j{}^{*q_j}) \\ &= tr_{1...n}(XT^{*q_1}_{k_1}...Tk_j{}^{*q_j}T^{*q_{j+1}}_{k_{j+1}}) \end{aligned}$$

where $X' = T^{*-q_{j+1}}_{k_{j+1}}X$. □

[3] i.e. Derived from a rack.

Now for the key part of the proof - we show that considering the character of a braid over a quandlisation of a rack is equal to considering the character over the rack of the quandler-deformed braid - i.e. taking the normalized trace of the $\mathcal{R}$-matrix representation of the braid deformed by Φ_n (3.24) is the same as taking the normalized trace of the $\mathcal{R}$-matrix induced from a quandle.

Lemma 5. *Let (X, λ) be a rack and let $Q(X, \lambda)$ be its quandalisation. Then, for all $b \in B_n$,*

$$\chi_{Q(X,\lambda)}(b) = \chi_{(X,\lambda)}(\Phi_n(b)) \tag{3.27}$$

Proof. By linearising the quandalisation process, we see that if R is the linearisation of a solution of the set-theoretic Yang-Baxter equation from a rack, the linearisation of the solution derived from the quandalisation of that rack is given by

$$R_Q := R(1 \otimes T^*)$$

where T is the partial trace of R.

Let $b = b_{j_1}^{p_1} ... b_{j_m}^{p_m}$ be a generic element of $\mathcal{B}_n$. Then

$$\begin{aligned}
\chi_{Q(X,\lambda)}(b) &= tr(R_{j_1}^{p_1} T_{j_1'}^{*p_1'} ... R_{j_m}^{p_m} T_{j_m'}^{*p_m'}) \\
&= tr(R_{j_1}^{p_1} ... R_{j_m}^{p_m} T_{k_1}^{*q_1} T_{k_m}^{*q_m}) \\
&= tr(R_{i_1}^{p_1} ... R_{i_n}^{p_n} \rho_n(\theta_n(l_{k_1}^{q_1} ... l_{k_n}^{q_n}))) \\
&= \chi_{(X,\lambda)}(\Phi_n(b))
\end{aligned}$$

By the previous lemma.

□

Finally, we show the main result of this thesis using the above results.

Theorem 16. *Let (X, λ) and (X, μ) be racks such that their linearisations are equivalent as $\mathcal{R}$-matrices, i.e. $R_{(X,\lambda)} \sim R_{(X,\mu)}$*

Let $Q(X, \lambda)$ and $Q(X, \mu)$ be the quandalisations of of (X, λ) and (X, μ) respectively.

Then the linearisations of these quandles are also equivalent as $\mathcal{R}$-matrices, i.e.

$$R_{Q(X,\lambda)} \sim R_{Q(X,\mu)}$$

Proof. By our assumptions and lemma 16, we have that

$$\begin{aligned} & \chi_{(X,\lambda)}(b) &= \chi_{(X,\mu)}(b) \\ \implies & \chi_{(X,\lambda)}(\phi_n(b)) &= \chi_{(X,\mu)}(\phi_n(b)) \\ \implies & \chi_{Q(X,\lambda)}(b) &= \chi_{Q(X,\mu)}(b) \end{aligned}$$

□

We conjecture that the opposite case is not true, as the quandalisation process inherently "forgets" the rack map λ. Further research could entail searching for a counter-example.

We now consider a special case of a quandle, namely the Alexander quandle, and deduce when Alexander-quandle-derived $\mathcal{R}$-matrices are equivalent.

3.5 Alexander Quandles

In this section we introduce Alexander quandles - a special type of quandle that has close ties to knot theory. We examine how the quandle coloring of knots is linked to the Alexander matrix, and show that two rack-derived $\mathcal{R}$-matrices are equivalent iff their dimension and coloring invariants are equal. Finally, we show that given two Alexander quandles $Q(p, k)$ and $Q(p', k')$, their induced $\mathcal{R}$-matrices are equivalent iff $k = k'$ or $k = k'^{-1}$.

Definition 51. ***Alexander quandle:*** *The Alexander quandle $Q(p, k)$, where p is prime and $k \in \mathbb{Z}/p\mathbb{Z}$, is the quandle that consists of the set $X = \mathbb{Z}/p\mathbb{Z}$ and the operation*

$$x \triangleright y := \lambda_x(y) := x - k \cdot x + k \cdot y$$

We see that this is a rack. Clearly λ_x is bijective, in fact its inverse is given by

$$\lambda_x^{-1}(y) := x - k^{-1}x + k^{-1}y$$

It clearly satisfies the self-distributive property:

$$
\begin{aligned}
(x \triangleright y) \triangleright (x \triangleright z) &= (x - kx + ky) \triangleright (x - kx + kz) \\
&= (x - kx + ky) - k(x - kx + ky) + k(x - kx + kz) \\
&= (1 - k)x + (k - k^2)y + k^2 z \\
&= (x - kx + ky) - k^2 y + k^2 z \\
&= x - kx + k(y - ky + kz) \\
&= x \triangleright (y - ky + kz) \\
&= x \triangleright (y \triangleright z)
\end{aligned}
$$

It also clearly satisfies the quandle condition:

$$x \triangleright x = x - kx + kx = x$$

3.5.1 Linking Number and Seifert Surfaces

We now briefly define the linking number and the process of producing a Seifert Surface, which are later used to define a Seifert matrix in Definition 54.

Definition 52. ***Linking number:*** *The linking number $lk(L)$ of an oriented link is defined by*

$$lk(L) = \text{ number of positive crossings} - \text{ number of negative crossings}$$

where by convention we define a positive crossing and negative crossing as in Figure 3.9.

Figure 3.9: Left: Positive crossing in an oriented link. Right: Negative crossing in an oriented link

We now look at the Seifert surface of an oriented link.

The Seifert surface of a knot was first described by Seifert [29] and is produced with the following algorithm [30]. This algorithm is depicted in Figure 3.10.

1. Assign an orientation to all components of a link

2. Eliminate all crossings by connecting the incoming over-strand to the outgoing under-strand and the incoming under-strand to the outgoing under-strand. This will give a set of non-intersecting topological circles known as Seifert circles. Fill in these circles to create disks.

3. Connect the disks with twisted bands. Each twisted band corresponds to a crossing and twists in the opposite direction of the crossing.

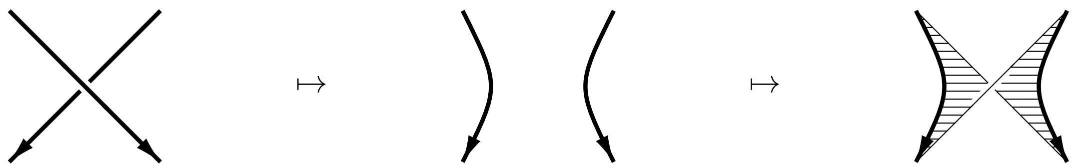


Figure 3.10: Seifert's algorithm for a band projection

Note that not all surfaces bounded by a knot arise from Seifert's algorithm. This algorithm was initially created to define the genus of a knot, an important invariant in knot theory. For our purposes, we need only to consider the linking number of the upside and downside curves that form the boundary of the Seifert surface, as this is what is used to define a Seifert Matrix in Definition 54.

3.5.2 Quandle Coloring of Knots and the Alexander Matrix

In this section we explore the quandle coloring of knots and define the Alexander matrix.

In [12] the following process for coloring knots by quandles is described.

Given a link, choose an orientation and label each arc with an element from the quandle. At each crossing perform the triangle operation as in Figure 3.11.

In order to get a consistent coloring one must have that each strand's labels are equal before and after each crossing, i.e. the following "coloring conditions" are satisfied.

Figure 3.11: Creation of quandle-coloring conditions

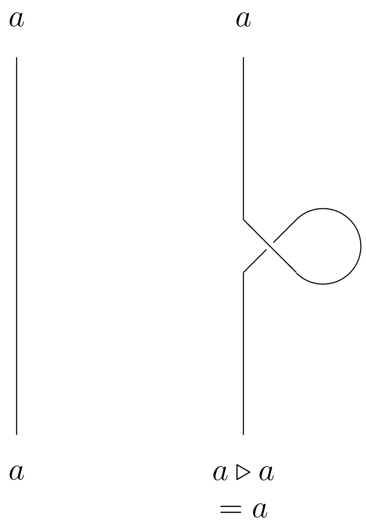


Figure 3.12: Reidemeister Move I

$$\text{red: } b = a \triangleright b$$
$$\text{green: } b = a \triangleright^{-1} b$$

Quandle-coloring of a knot is well-defined as it produces consistent colorings under the Reidemeister moves, as seen in Figure 3.12, Figure 3.13 and Figure 3.14.

Reidemeister move I (Figure 3.12) demonstrates that one can only color with quandles and not racks in general, as a self-crossings of a strand will always have the same color, i.e. we must have that $a \triangleright a = a$.

For each knot and quandle there may be many possibilities for the number of ways the knot can be colored.

Definition 53. ***Coloring number:*** *Given an oriented link L and a quandle Q the coloring number $Col_Q(L)$ is the number of non-trivial*[4] *ways a knot can be colored with a quandle.*

[4]Here by trivial we mean labeling every strand with the same element of the quandle.

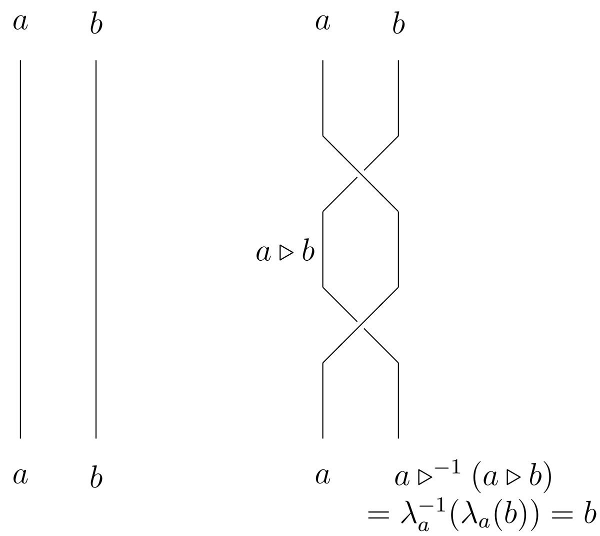


Figure 3.13: Reidemeister Move II

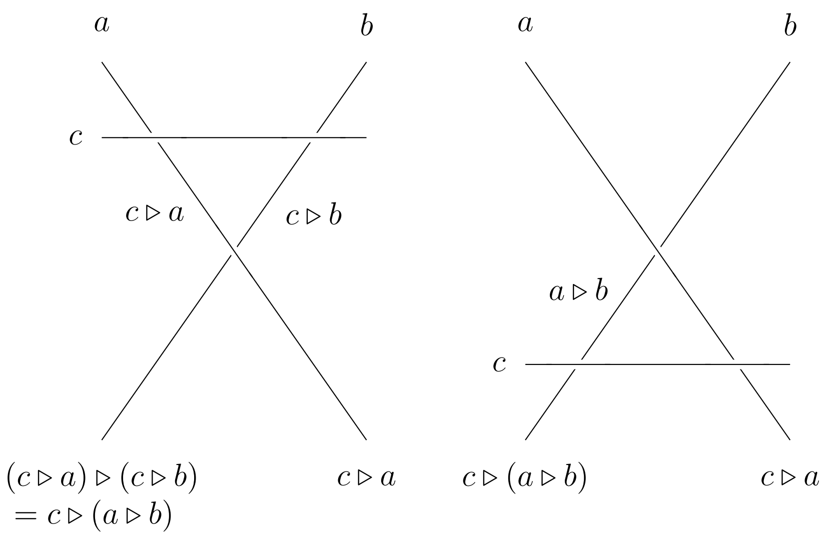


Figure 3.14: Reidemeister Move III

It is well-known that $Col_Q(K)$ is an invariant of knots [12]. In Section 3.5.3 we will show that the coloring invariant can be used as a definition of equivalence of $\mathcal{R}$-matrices.

Given an oriented link L and an Alexander quandle $Q(p, k)$, one can formulate the coloring conditions as a matrix in terms of k. This is the Alexander matrix, denoted $A_L(k)$. The Alexander matrix is defined in terms of the link's Seifert matrix, as defined below.

Definition 54. ***Seifert matrix:*** *The Seifert matrix $S(L)$ of an oriented link L is given by*

$$S(L) = (v_{jk}) = (lk(a_j^-, a_k))$$

where lk is the linking number and a_j^-, a_k are curves in S^-, S^+ respectively, where S^+ is the upside and S^- is the downside[5] *of the Seifert surface of the knot. [10]*

The Seifert matrix is not a knot invariant [28], but it does distinguish between different Seifert surfaces of a knot.

In [10] proposition 8.7 it is shown that all Seifert matrices V have the following form and all square matrices with even order of the following form are Seifert matrices:

$$V - V^T = Fl \tag{3.28}$$

where Fl is the "Flip"[6], given by

$$Fl = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}$$

The details of the Seifert matrix are not central to this thesis. We introduced the Seifert matrix and its form in order to properly define the Alexander matrix as follows.

[5] I.e. S^+ and S^- together form the boundary of the Seifert surface of the knot.

[6] Note that this is not the same "Flip" as in Example 3.

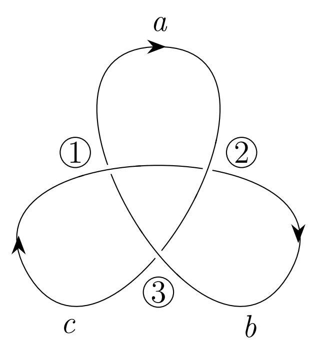
Definition 55. ***Alexander matrix:*** *The Alexander matrix of an oriented link L, that is colored by an Alexander quandle $Q(p,k)$ with Seifert matrix $S(L)$, is given by*

$$A_L(k) = k \cdot S(L) - S(L)^T \tag{3.29}$$

where $S(L)^T$ is the transpose of $S(L)$.

We now give a detailed example of how one can derive an Alexander matrix from an oriented link.

Example 22. *Consider the trefoil knot. We orient the knot and label each arc with elements of an Alexander quandle and each crossing as in the diagram below.*



The coloring conditions induce the following equations

$$\begin{aligned}
&① \quad a \triangleright c = b \implies a - ka + kc - b = 0\\
&② \quad b \triangleright a = c \implies b - kb + ka - c = 0\\
&③ \quad c \triangleright b = a \implies c - kc + kb - a = 0
\end{aligned}$$

We re-write this system of equations in matrix form:

$$\begin{pmatrix} 1-k & -1 & k \\ k & 1-k & -1 \\ -1 & k & 1-k \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since this system still counts the monochromatic colorings ($a = b = c$) we delete one column. We choose the c column.

$$\begin{pmatrix} 1-k & -1 \\ k & 1-k \\ -1 & k \end{pmatrix}$$

We see that the rows of this matrix are linearly dependant since

$$-\begin{pmatrix} -1 & k \end{pmatrix} = \begin{pmatrix} 1-k & -1 \end{pmatrix} + \begin{pmatrix} k & 1-k \end{pmatrix}$$

We therefore delete row 3 to yield.

$$\begin{pmatrix} 1-k & -1 \\ k & 1-k \end{pmatrix}$$

This is an[7] *Alexander matrix for the trefoil knot.*

The Alexander matrix has strong applications in knot theory, as its determinant yields the Alexander polynomial, denoted $\Delta(k)$ [28].

Example 23. *The Alexander polynomial of the trefoil knot is*

$$\Delta(k) = det \begin{pmatrix} 1-k & -1 \\ k & 1-k \end{pmatrix} = (1-k)^2 + k = k^2 - k + 1$$

Since by construction the Alexander matrix is the matrix form of the system of equations that are the coloring conditions, we have that for any oriented link L and Alexander quandle $Q = Q(p, k)$,

$$Col_Q(L) = |ker(A_L(k))| + p \tag{3.30}$$

i.e. The number of colorings is the number of solutions to the system of equations plus the trivial colorings.

3.5.3 Equivalence of Quandle-Derived Solutions

In this section we describe how the coloring invariant can be used to define equivalence of quandle-derived solutions by showing that two quandle-derived solutions have the same character iff their coloring invariants are equal.

Recall that the trace of a rack-derived $\mathcal{R}$-matrix $R|x, y\rangle = |x \triangleright y, x\rangle$ is given by (3.13)

[7] As we made a choice about what row and column to delete the Alexander matrix is not unique. This is just because this system is over-determined and does not affect the kernel (the importance of which will be clear shortly).

$$Tr(R) = \sum_x \langle x|Sq(x)\rangle$$

Recall also that the partial trace of a rack-derived $\mathcal{R}$-matrix, which we denote by T, is the linearisation of the square map

$$T := ptr(R) = \sum_x |Sq(x)\rangle\langle x|$$

Since $R_{k_1}...R_{k_n}$ may be considered as a representation of the braid $b_{k_1}...b_{k_n}$, traces of $\mathcal{R}-$matrices may be considered diagrammatically as closing the represented braid and summing when the conditions induced by the closure are satisfied.

Example 24. *Let $(X, \triangleright)$ be a rack and $R|x, y\rangle = |x \triangleright y, x\rangle$ be an $\mathcal{R}-$matrix on $V^{\otimes n}$.*

The trace of $R_1R_2R_1$ is given by

$$\begin{aligned}\sum_{x,y,z}\langle x, y, z|R_1R_2R_1|x, y, z\rangle &= \sum_{x,y,z}\langle x, y, z|R_1R_2|x \triangleright y, x, z\rangle \\ &= \sum_{x,y,z}\langle x, y, z|R_1|x \triangleright y, x \triangleright z, x\rangle \\ &= \sum_{x,y,z}\langle x, y, z|(x \triangleright y) \triangleright (x \triangleright z), x \triangleright y, x\rangle \\ &= \sum_{x,y,z}\langle x|(x \triangleright y) \triangleright (x \triangleright z)\rangle\langle y|x \triangleright y\rangle\langle z|x\rangle\end{aligned}$$

Diagrammatically this trace is presented by Figure 3.15.

As demonstrated by Example 24 it is clear that the conditions induced by closing the braid are exactly those in the bra-kets that contribute to the trace. If we sum up all the instances over the quandle in which the conditions are satisfied, i.e. we calculate the coloring number $Col_Q(\hat{b})$ (where $\hat{b}$ is the closure of the braid b), we see that this is exactly the trace.

This motivates the following theorem.

Theorem 17. *Let $b \in \mathcal{B}_n$ be an arbitrary braid, let $Q = Q(X, \triangleright)$ be a quandle such that $|X| = n$, and let $R|x, y\rangle = |x \triangleright y, x\rangle$ be the $\mathcal{R}-$matrix induced by Q. Then*

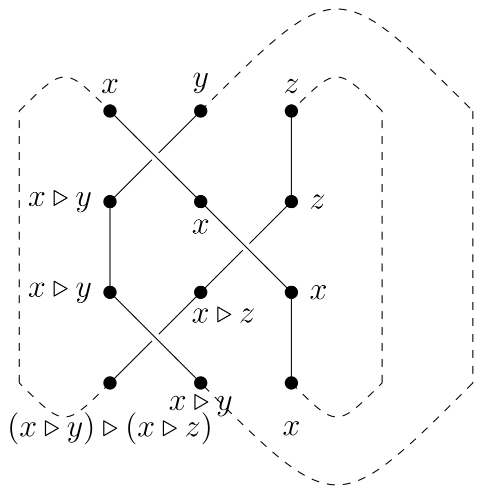


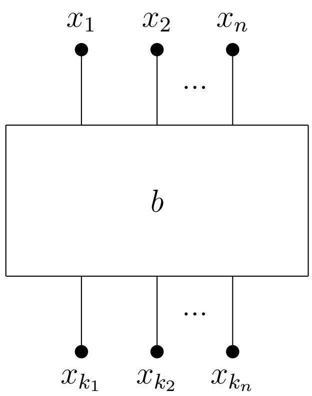
Figure 3.15: Diagrammatic presentation of $\mathrm{Tr}(R_1 R_2 R_1)$.

$$Tr(\rho_R(b)) = Col_Q(\hat{b}) \tag{3.31}$$

where $\hat{b}$ is the closure of the braid b to an oriented link.

Proof. Let $b = b_{j_1}^{q_1} ... b_{j_m}^{q_m} \in \mathcal{B}_n$ be an arbitrary braid.

The coloring conditions of the closure of the braid are deduced in the following way. The top of every strand is labelled x_j. For every crossing b_j in the braid b, one maps $(x_j, x_{j+1}) \mapsto (x_j \triangleright x_{j+1}, x_j)$. The strands are labelled at the bottom with x_{k_j}, which is equal to the result of applying the above mapping throughout the braid.



The coloring conditions are then given by

$$x_j = x_{k_j} \quad \forall j \in \{1, ..., n\}$$

where k_j is dependent upon the braid b.

The coloring invariant can be considered as the number of ways the link can be colored by the quandle such that all of these conditions are satisfied, i.e.

$$Col_Q(\hat{b}) = \sum_{x_1,...,x_n} \delta_{x_1,x_{k_1}}...\delta_{x_n,x_{k_n}}$$

Now, let $R_Q|x,y\rangle = R|x,y\rangle = |x \triangleright y, x\rangle$ be a rack-derived $\mathcal{R}-$matrix from the quandle Q. We now represent the braid b with R and calculate its trace

$$\begin{aligned} Tr(\rho_R(b)) &= \sum_{1...n} \langle x_1, ..., x_n | R_{j_1}^{q_1} ... R_{j_m}^{q_m} | x_1, ..., x_n \rangle \\ &= \sum_{1...n} \langle x_1, ..., x_n | R_{j_1}^{q_1} ... R_{j_{m-1}}^{q_{m-1}} | x_1, ..., x_{m-1}, x_m \triangleright x_{m+1}, x_m, x_{m+2}, ... x_n \rangle \\ &= \sum_{1...n} \langle x_1, ..., x_n | x_{k_1}, ..., x_{k_n} \rangle \\ &= \sum_{1..n} \delta_{x_1,x_{k_1}} ... \delta_{x_n,x_{k_n}} \end{aligned}$$

clearly $\rho_R(b)|x_1...x_j...x_n\rangle = |x_{k_1}...x_{k_j}...x_{k_n}\rangle$ since the mapping is the same as described with the coloring conditions.

Hence, for all $b \in \mathcal{B}_n$,

$$Tr(\rho_R(b)) = Col_Q(\hat{b})$$

□

Recall the following definition of equivalence, Definition 20:

Two $\mathcal{R}$-matrices $R, S \in \mathcal{R}(d)$ are said to be equivalent iff they have the same dimension and character, i.e.

$$R \sim S \iff \dim(R) = \dim(S) \text{ and } \tau_R = \tau_S$$

where $\tau_R := Tr \circ \rho_R$, i.e. the trace of the braid representation induced by R.

Hence, by Theorem 17, two quandle-derived quantum YBE solutions are equivalent iff they define the same coloring number. Note that having the same coloring number will automatically imply the $\mathcal{R}$-matrices have the same dimension, as in particular they will have the same coloring number of a trivial link, whose coloring number will always be the number of elements of the quandle.

Definition 56. ***Equivalence of derived solutions:*** *Let R, S be quandle-derived $\mathcal{R}$-matrices. They are said to be equivalent, denoted $R \sim S$, iff their related quandles produce the same number of colorings for an arbitrary link, i.e.*

$$R_{Q(X,\triangleright)} \sim S_{Q'(X,\triangleright')} \iff Col_Q(K) = Col_{Q'}(K) \text{ for all links } K$$

We now show that two Alexander quandles $Q(p,k)$ and $Q'(p',k')$ are equivalent if and only if $k' = k$ or $k' = k^{-1}$.

Note that we must have $p = p'$ for the size of these two R-matrices to be the same, which is necessary for them to have the same dimension and thus to be equivalent.

Theorem 18. *Let $Q(p,k)$ and $Q'(p,k')$ be Alexander quandles with p prime. Let $R_{Q(p,k)}$, $R_{Q'(p,k')}$ be the $Q(p,k)$- and $Q'(p,k')$-derived $\mathcal{R}$-matrices respectively. Then*

$$R_{Q(p,k)} \sim R_{Q'(p,k')} \iff k = (k')^{-1} \text{ or } k = k'$$

Proof. We begin by showing $\Longleftarrow$.

Let $S = S(L)$ be a Seifert matrix of an arbitrary knot L. Then, by equation (3.29) we have that

$$\begin{aligned}
|ker(A_L(k))| &= |ker(kS - S^T)| && \text{by definition} \\
&= |ker(k(S - k^{-1}S^T))| && \text{factoring out } k \\
&= |ker(-k(k^{-1}S^T - S))| && \text{times by factor} \\
&= |ker(k^{-1}S^T - S)| && \text{simplifying} \\
&= |ker(k^{-1}S - S^T)^T| && \text{transposition doesn't affect size of kernel} \\
&= |ker(k^{-1}S - S^T)| && \text{applying the transposition} \\
&= |ker(A_L(k')| && \text{by definition}
\end{aligned}$$

Therefore the cardinality of the kernel of $A_L(k)$ is always equal to that of $A_L(k')$. By equation (3.30) and the definition of equivalence, we have shown that if $k = k'$ or $k = (k')^{-1}$ then we must have that $R_{Q(p,k)} \sim R_{Q'(p,k')}$.

Now, to show $\Longrightarrow$.

To show this direction, we show the equivalent logic statement

$$k \notin \{k', (k')^{-1}\} \implies R_{Q(p,k)} \nsim R_{Q'(p,k')}$$

To show this we find an Alexander matrix such that, when the knot is quandle-colored by the quandles, the size of the matrix's kernels are different when $k \neq k', (k')^{-1}$.

Suppose we have a 2×2 Seifert matrix, i.e. $V \in M_2$ such that

$$V - V^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In particular, we consider the following specific example and show that $k \neq k'$ or $(k')^{-1} \implies R_{Q(p,k)} \sim R_{Q'(p,k')}$.

$$V = \begin{pmatrix} c^2+1 & c+1 \\ c & 1 \end{pmatrix}$$

Note that this is a Seifert matrix since

$$\begin{pmatrix} c^2+1 & c+1 \\ c & 1 \end{pmatrix} - \begin{pmatrix} c^2+1 & c \\ c+1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The Alexander matrix induced from this Seifert matrix is given by

$$\begin{aligned} A(k) &= kV - V^T \\ &= k\begin{pmatrix} c^2+1 & c+1 \\ c & 1 \end{pmatrix} - \begin{pmatrix} c^2+1 & c \\ c+1 & 1 \end{pmatrix} \\ &= (k-1)\begin{pmatrix} c^2+1 & c+\frac{k}{k-1} \\ c-\frac{1}{k-1} & 1 \end{pmatrix} \end{aligned}$$

We consider when the kernel is non-trivial, i.e. when it is not invertible.

$$det(A(k)) = (k-1)^2\Big(c^2+1-\Big(c+\frac{k}{k-1}\Big)\Big(c-\frac{1}{k-1}\Big)\Big) = 0$$

We may divide by $(k-1)$ since the case $k=1$ is the Flip and this is a special case that can be considered separately. This gives us

$$\begin{aligned} 0 &= \Big(c^2+1-\Big(c+\frac{k}{k-1}\Big)\Big(c-\frac{1}{k-1}\Big)\Big) \\ &= 1-\frac{c}{k-1}-\frac{ck}{k-1}+\frac{k}{(k-1)^2} \\ &= 1-c+\frac{k}{(k-1)^2} \end{aligned}$$

Re-written in terms of c yields

$$c = 1 + \frac{k}{(k-1)^2}$$

Now, one can work through the same process simply replacing k with k' to get

$$c = 1 + \frac{k'}{(k'-1)^2}$$

The c-value is the same in both cases since we are examining the same Seifert matrix of the same knot (in particular for this c value we will only have trivial colorings) so we may set the above two equations to be equal to one another.

$$\begin{aligned}
1 + \frac{k}{(k-1)^2} &= 1 + \frac{k'}{(k'-1)^2} \\
\frac{k}{(k-1)^2} &= \frac{k'}{(k'-1)^2} \\
k(k'-1)^2 &= k'(k-1)^2 \\
0 &= k(k'-1)^2 - k'(k-1)^2 \\
&= k(k'^2 - 2k' + 1) - k'(k^2 - 2k + 1) \\
&= k'^2 - 2k' + 1 - k'(k - 2 + \frac{1}{k})) \\
&= k'^2 - 2k' + 1 - k'k + 2k' - \frac{k'}{k} \\
&= k'2 - k'k - \frac{k'}{k} + 1 \\
0 &= (k - k')(\frac{1}{k} - k')
\end{aligned}$$

Thus, when $k = k'$ or $k = (k')^{-1}$ the determinant is 0 and we have a non-trivial kernel. If $k \neq k'$, $(k')^{-1}$ then the determinant is non-zero and thus the kernel is non-trivial.

By equation (3.31) we have that two $\mathcal{R}$−matrices cannot be equivalent if their respective Alexander matrices do not have the same size kernel. Hence, we have shown

$$k \neq k' \text{ or } (k')^{-1} \implies R_{Q(p,k)} \nsim R_{Q'(p,k')}$$

i.e.

$$R_{Q(p,k)} \sim R_{Q'(p,k')} \implies k = k' \text{ or } k = (k')^{-1}$$

□

3.6 Summary

In Chapter 3 we began by defining the set-theoretic YBE and a new construction called the Bloop Group, denoted $\mathcal{B}\ell_n$. We formed a representation of the Bloop group and later linked this to the quandalisation process of racks.

We then defined racks, quandles, and their linearisations to $\mathcal{R}$-matrices. We showed that any linearised non-degenerate YBE solution is equivalent to some rack-derived $\mathcal{R}$-matrix.

We examined the relationship between isomorphisms of racks and equivalence of their rack-derived $\mathcal{R}$-matrices, showing that isomorphic racks induce equivalent $\mathcal{R}$-matrices.

We analysed the decomposability of racks and showed that the union of a subrack and its compliment is indeed a rack but is not isomorphic to its original rack.

We utilised our representation of the Bloop group to show that given two racks with equivalent induced $\mathcal{R}$-matrices, the $\mathcal{R}$-matrices induced from their quandalisations must also be equivalent.

Finally, we look at a particular type of quandle called an Alexander quandle. This quandle is linked to knot theory and we utilise the coloring invariant of the closure of a braid as a notion of equivalence of rack-derived $\mathcal{R}$-matrices. In particular, we show that for Alexander quandles $Q(p, k)$ and $Q(p, k')$, their relative $\mathcal{R}$-matrices are equivalent iff $k = k'$ or $k = k'^{-1}$.

Chapter 4

Conclusion

In this thesis we focus on the furthering the research area of classification of unitary $\mathcal{R}$-matrices by looking at $\mathcal{R}$-matrix representations of algebraic structures.

We look at the BMW algebra and define contractive $\mathcal{R}$-matrices as an $\mathcal{R}$-matrix R that satisfies the following contraction relation with one of its spectral projections P.

$$\varphi(P)R\varphi(P) = c \cdot \varphi(P)$$

where φ is the canonical shift endomorphism and c is dubbed the contractive constant.

A restriction on the contractive $\mathcal{R}$-matrices is shown to represent the BMW algebra.

We show that contractive $\mathcal{R}$ matrices must have at most 3 eigenvalues. The 1-eigenvalue situation is trivial and the 2-eigenvalue situation is a Temperley-Lieb $\mathcal{R}$-matrix, which has already been fully classified, so we mainly consider the 3-eigenvalue situation.

We show that contractive $\mathcal{R}$-matrices are stable under equivalence and classify all 2-dimensional examples of contractive $\mathcal{R}$-matrices. We then show various relations for contractive $\mathcal{R}$-matrices inspired by Wenzl's work.

We rely on the concept of the normalised trace being Markov, which has been shown to always be the case when the spectrum does not contain a pair of opposite eigenvalues. It has also been shown that the normalised trace is Markov in the discrete range of the Jones Index. Further research could entail analysing other cases in which the normalised trace is Markov.

We show that the following relations hold for the contraction constant c:

$$\tau(R) = c$$
$$\tau(P) = |c|^2$$

when τ is Markov.

These results, along with other avenues of analysis, induce the following restrictions of the possible values that the contraction constant c can take

$$0 \leq |c| \leq \frac{1}{\sqrt{2}}$$
$$|c|^2 = \tau(P) \in \mathbb{Q}$$

Further research could entail additional restrictions on the possible values of c.

In this analysis we also see that $|c|^{-2} = [\rho_R(\mathcal{B}_\infty) : \varphi(\rho_R(\mathcal{B}_\infty))]$ where $[\cdot : \cdot]$ denotes the Jones Index. We go on to deduce the form of contractive $\mathcal{R}$-matrices corresponding to each discrete value of the Jones Index. We are unable to find an example for $I_R = 3$ and $|\sigma(R)| = 3$, but we do show that extending the $I_R = 3$, $|\sigma(R)| = 2$ case does not provide a required example. Finding an example of this case presents an opportunity for possible future research.

In Chapter 3 we look at racks, how they can be used to create solutions to the set-theoretic Yang-Baxter equation and how these can be linearised to solutions of the quantum Yang-Baxter equation (i.e. to $\mathcal{R}$-matrices). We show that isomorphic racks induce equivalent $\mathcal{R}$-matrices. A further avenue of research could entail finding a counter example to show that equivalent rack-derived $\mathcal{R}$-matrices do not necessarily come from isomorphic racks. We also examine the decomposability of racks, and show that the union of the subracks of a rack is not necessarily isomorphic to the original rack.

We examine the quandalisations of racks, in particular developing the Bloop group that is inspired by this process. We define an $\mathcal{R}$-matrix representation of the Bloop group and show that the quandalisation process translates to $\mathcal{R}$-matrices by the following map

$$\tilde{R} := RT^*$$

where R is a rack-derived $\mathcal{R}$-matrix, T^* is the adjoint of the partial trace of R and $\tilde{R}$ is the $\mathcal{R}$-matrix induced from the quandalisation of the rack that induced R.

The main result of this chapter is that for any two racks whose associated $\mathcal{R}$-matrices are equivalent, the $\mathcal{R}$-matrices produced from their quandalisations are also equivalent.

Finally, we consider a special case of quandle called an Alexander quandle. We consider quandle colorings of oriented links and show that two quandle-derived $\mathcal{R}$-matrices are equivalent if and only if their coloring numbers are equal for any oriented link. We then utilise this result to show that any two Alexander-quandle derived $\mathcal{R}$-matrices are equivalent if and only if their Alexander constants k are equal or inverses of one another.

Appendix A

Classification of all racks of size 3

Let $X = 1, 2, 3$ and consider the racks (X, λ_i), where λ_i is a bijective map for all $i \in X$ that satisfies the self-distributive property (3.8). Since they are bijective maps they can be considered to be permutations.

Every rack can be encoded in a table in the following way

	1	2	3
1	$\lambda_1(1)$	$\lambda_1(2)$	$\lambda_1(3)$
2	$\lambda_2(1)$	$\lambda_2(2)$	$\lambda_2(3)$
3	$\lambda_3(1)$	$\lambda_3(2)$	$\lambda_3(3)$

Table A.1: Tabulation of racks of size 3

Below we list all possible racks with $|X| = 3$, classifying by the notion of isomorphism as in Definition 43. Tables in the same row are in the same isomorphism class.

	1	2	3
1	1	2	3
2	1	2	3
3	1	2	3

	1	2	3
1	2	3	1
2	2	3	1
3	2	3	1

	1	2	3
1	3	1	2
2	3	1	2
3	3	1	2

	1	2	3
1	1	3	2
2	1	3	2
3	1	3	2

	1	2	3
1	2	1	3
2	2	1	3
3	2	1	3

	1	2	3
1	3	2	1
2	3	2	1
3	3	2	1

	1	2	3
1	1	2	3
2	1	2	3
3	2	1	3

	1	2	3
1	1	2	3
2	3	2	1
3	1	2	3

	1	2	3
1	1	3	2
2	1	2	3
3	1	2	3

	1	2	3
1	1	2	3
2	1	3	2
3	1	3	2

	1	2	3
1	3	2	1
2	1	2	3
3	3	2	1

	1	2	3
1	2	1	3
2	2	1	3
3	1	2	3

This table was produced in Python by a brute force approach.

This work builds off of existing numerical approaches, including the GAP libraries for the Yang-Baxter equation [32] and racks [31].

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