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SHORT NOTE

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Mean relative error and standard relative deviation

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Abstract
Statistics on the natural log scale, such as differences, regression coefficients, and standard deviations, frequently arise when analyzing log-transformed data or modeling binary, count, or time-to-event data. The statistical properties of log-transformed estimators (for example, log odds ratio estimators) frequently appear in statistical articles. Understanding the magnitude of these quantities can be useful. Remarkably, many can be readily interpreted, without exponentiation. In this note, we introduce four new interpretations. While a log-scale standard deviation can be interpreted as an approximate coefficient of variation describing variation about an arithmetic mean, we argue it can be more useful to interpret it exactly as a "standard relative deviation" describing variation about a geometric mean. For positive-valued estimators, we show how the standard error of a log-transformed estimator can be interpreted as the "standard relative error" describing the precision of the untransformed estimator. We also show how the bias and root mean squared error of a the log-transformed estimator can be interpreted as the "mean relative error" and "root mean squared relative error" of the untransformed estimator. We illustrate the

usefulness of our interpretations for (i) the usual scale parameter of a lognormal distribution, (ii) statistical properties of a log odds ratio estimator, and (iii) a single measure of the precision of an odds ratio estimator which agrees with its usual, asymmetric 95% confidence interval.

KEYWORDS

bias, coefficient of variation, geometric mean, geometric standard deviation, geometric standard error, lognormal, mean squared relative error

1 | INTRODUCTION

We regularly see log-scale quantities. Log-scale means, differences, regression coefficients, standard deviations (SDs) and standard errors (SEs) arise when analyzing log-transformed data or modeling binary, count or time-to-event data (for example, models for log odds, log expected count, or log hazard). Tables and graphs of the statistical properties of log-transformed estimators (for example, log odds ratio estimator) frequently appear in statistical articles. Given how often such quantities appear, understanding the magnitude (not just the sign) of these quantities can be useful. Cole (2000) showed that many can be readily interpreted, without exponentiation. In this note, we introduce new ways to readily interpret a log-scale SD, as well as the SE, bias, and mean squared error (MSE) of a log-transformed estimator (for example, log odds ratio estimator).

Central to all our interpretations is a particular definition of a relative difference. The log-scale difference $\log B - \log A$ (where log denotes the natural logarithm) has been independently discovered and promoted as the best definition of a relative difference between two positive numbers, A and B, in economics (Törnqvist, Vartia, & Vartia, 1985), medical statistics (Cole, 2000; Cole & Altman, 2017a, 2017b) and psychophysics (Graff, 2014). It is a relative difference (Cole, 2000; Törnqvist et al., 1985) because

$$\log B - \log A = \log(B/A) = (B - A)/\mathrm{L}(A, B),$$

where $\mathrm{L}(A, B)$ is the logarithmic mean of A and B, which is defined as $(B - A)/\log(B/A)$ when $A \neq B$, or A when $A = B$. It has a symmetry property (swapping A and B changes the sign but not the magnitude) as well as an additive property (the relative difference between A and C can be written as the sum of the relative difference between A and B and the relative difference between B and C). Throughout this note, we are referring to this definition of a relative difference when using the terms relative difference, relative deviation, and relative error.

2 | NEW INTERPRETATIONS

2.1 | Standard relative deviation

Some positive-valued distributions are often specified by their log-scale mean and log-scale SD. This is the case for the distribution of multiplicative random effects in many settings (such as a lognormal frailty model for time-to-event data). In these settings, the log-scale mean is zero—equivalently, the geometric mean is 1. In other settings (for example, a distribution of ratios), the geometric mean is estimated. Before introducing a new way to describe variation about the geometric mean of a random variable Y, $\mathrm{G}(Y) = \exp\{\mathrm{E}(\log Y)\}$ (Hines, 2006), we first take a close look at the definition of the SD.

The SD of a random variable Y is defined as the square root of the variance. More informatively, it could equally be defined as the average (root mean square (RMS) or quadratic mean) magnitude of a deviation from the arithmetic mean because

$$\mathrm{SD}(Y) = \sqrt{\mathrm{E}\left[\{Y - \mathrm{E}(Y)\}^2\right]} = \sqrt{\mathrm{E}\left[|Y - \mathrm{E}(Y)|^2\right]} = \mathrm{RMS}(|Y - \mathrm{E}(Y)|).$$

For a positive-valued random variable Y, we define the *standard relative deviation* (SRD) as the average (RMS) magnitude of a relative deviation from the geometric mean. The SD of $\log Y$ can be interpreted as the SRD of Y because

$$\mathrm{SRD}(Y) = \sqrt{\mathrm{E}\left[\{\log Y - \log \mathrm{G}(Y)\}^2\right]} = \sqrt{\mathrm{E}\left[\{\log Y - \mathrm{E}(\log Y)\}^2\right]} = \mathrm{SD}(\log Y).$$

2.2 | Statistical properties of positive-valued estimators of $\theta > 0$

Standard relative error

We define the *standard relative error* (SRE) of a positive-valued estimator $\widehat{\theta}$ as the SRD of its sampling distribution. The $\mathrm{SRE}(\widehat{\theta})$ describes the average (RMS) magnitude of a relative difference between $\widehat{\theta}$ and $\mathrm{G}(\widehat{\theta})$. The SE of $\log \widehat{\theta}$ can be interpreted as the SRE of $\widehat{\theta}$ because

$$\mathrm{SRE}(\widehat{\theta}) = \sqrt{\mathrm{E}\left[\{\log \widehat{\theta} - \log \mathrm{G}(\widehat{\theta})\}^2\right]} = \sqrt{\mathrm{E}\left[\{\log \widehat{\theta} - \mathrm{E}(\log \widehat{\theta})\}^2\right]} = \mathrm{SE}(\log \widehat{\theta}).$$

Mean relative error

The bias of $\widehat{\theta}$ is $\mathrm{Bias}(\widehat{\theta}, \theta) = \mathrm{E}(\widehat{\theta}) - \theta$. It is the same as the mean error (ME) of $\widehat{\theta}$: $\mathrm{ME}(\widehat{\theta}, \theta) = \mathrm{E}(\widehat{\theta} - \theta)$ (Morris, White, & Crowther, 2019). We define the *mean relative error* (MRE) of $\widehat{\theta}$ as $\mathrm{MRE}(\widehat{\theta}, \theta) = \mathrm{E}(\log \widehat{\theta} - \log \theta)$. It is the mean relative error between $\widehat{\theta}$ and θ. The bias of $\log \widehat{\theta}$ can be interpreted as the MRE of $\widehat{\theta}$ because

$$\mathrm{MRE}(\widehat{\theta}, \theta) = \mathrm{E}(\log \widehat{\theta} - \log \theta) = \mathrm{ME}(\log \widehat{\theta}, \log \theta) = \mathrm{Bias}(\log \widehat{\theta}, \log \theta).$$

Others sometimes use the term MRE to mean $\mathrm{E}\{(\widehat{\theta} - \theta)/\theta\} = \{\mathrm{E}(\widehat{\theta}) - \theta\}/\theta = \mathrm{Bias}(\widehat{\theta}, \theta)/\theta$, i.e., relative bias.

Mean squared relative error

We define the *mean squared relative error* of $\widehat{\theta}$ as $\mathrm{MSRE}(\widehat{\theta}, \theta) = \mathrm{E}[(\log \widehat{\theta} - \log \theta)^2]$, which is the same as $\mathrm{MSE}(\log \widehat{\theta}, \log \theta)$. We can therefore rewrite the familiar bias-variance decomposition of the MSE,

$$\mathrm{MSE}(\log\hat{\theta}, \log\theta) = \{\mathrm{ME}(\log\hat{\theta}, \log\theta)\}^2 + \{\mathrm{SE}(\log\hat{\theta})\}^2,$$

as

$$\mathrm{MSRE}(\hat{\theta}, \theta) = \{\mathrm{MRE}(\hat{\theta}, \theta)\}^2 + \{\mathrm{SRE}(\hat{\theta})\}^2.$$

We define the *root mean squared relative error* of $\hat{\theta}$ as $\mathrm{RMSRE}(\hat{\theta}, \theta) = \sqrt{\mathrm{E}[(\log\hat{\theta} - \log\theta)^2]}$. It is the average (RMS) magnitude of a relative error.

3 | EXAMPLES

3.1 | Describing lognormal continuous data

The distribution of incubation periods for a number of infectious diseases appears lognormal (Sartwell, 1950). Data on $n = 67$ incubation periods of chickenpox ranged from 11 to 20 days (Gordon & Meader, 1929). The arithmetic mean and the SD of the untransformed data were 15.1 days and 2.0 days respectively. After log-transforming the data, the arithmetic mean and the SD were 2.708 and 0.135 respectively.

One way to describe the distribution of incubation periods is to say the distribution was approximately lognormal with geometric mean exp(2.708) days = 15.0 days and SRD 13.5%. The latter quantity gives a sense of the variation about the geometric mean incubation period. Here, the average (RMS) magnitude of a relative deviation from 15.0 days is 13.5%.

Approximately 68% of the distribution lies in the interval (13.1, 17.2) days. The bounds of this interval, which can be calculated as $\exp(2.708 \pm 0.135)$ days, differ from the geometric mean by relative differences of $\pm 13.5\%$. Similarly, approximately 95% of the distribution lies in the interval (11.5, 19.6) days. The bounds of this interval differ from the geometric mean by relative differences of $\pm 1.96 \times 13.5\% = 26.5\%$.

The coefficient of variation of a positive-valued random variable Y is

$$\mathrm{CV}(Y) = \frac{\mathrm{SD}(Y)}{\mathrm{E}(Y)} = \frac{\mathrm{RMS}(|Y - E(Y)|)}{E(Y)} = \mathrm{RMS}\left(\left|\frac{Y - E(Y)}{E(Y)}\right|\right).$$

Here, an estimate is CV = 2.0/15.1 = 13.4%, which is similar to the estimate of the SRD (13.5%). Interpretation of SRD focuses on variation about the geometric mean, while the interpretation of a CV focuses on variations about the arithmetic mean: for example, the average (RMS) magnitude of a deviation from the arithmetic mean here was 13.4% of the arithmetic mean. However, for a lognormal random variable, the CV is a transformation of the SRD, specifically $\mathrm{CV}(Y) = \sqrt{\exp[\{\mathrm{SRD}(Y)\}^2] - 1}$, so it can be argued that the CV is also a measure of variation about the geometric mean of a lognormal distribution.

3.2 | Precision of estimates of odds ratios

Imagine a study where 100 participants received standard care and another 100 participants received an intervention. Now suppose that 50/100 and 30/100 participants in the standard care

TABLE 1 Selected logistic regression output from `Stata`.

	Log odds scale		Odds scale	
	Coefficient (SE)	**95% Confidence interval**	**Odds ratio (SE)**	**95% Confidence interval**
intervention	−0.85 (0.30)	(−1.42, −0.27)	0.43 (0.13)	(0.24, 0.77)
intercept	0.00 (0.20)	(−0.39, 0.39)		

and intervention groups respectively had a bad outcome at follow-up, and we want to quantify how the odds of a bad outcome differ between groups. Our focus in this example is on the reporting of an interpretable single measure of precision for an estimate of an odds ratio, which is here estimated to be $\hat{\theta} = 0.43$. (The principle applies equally to many other ratio estimates, including hazard ratios and ratios from generalized linear models where a log link is used.)

When fitting a logistic regression model, the output may be given on the log odds scale or the odds scale (Table 1). It is rare that software provides a single measure of precision for an estimate of an odds ratio. For example, `R` does not do so, but `Stata` does.

The SE associated with the odds ratio estimate in Table 1 is an estimate of an approximate SE derived using the delta method, which makes use of a first-order Taylor series expansion (Feiveson, 2024). Here, it is $0.43 \times 0.30 = 0.13$. It is a rather strange measure of the precision of an estimated odds ratio, given (i) the multiplicative nature of odds ratios and (ii) that this SE is not used in the accompanying calculation of its 95% CI (which is instead based on antilogs of the Wald 95% CI for the log odds ratio).

Alternatively, one could report an estimated SRE of 30% associated with the odds ratio estimate. This number is already given in Table 1, and it agrees with the 95% CI for the odds ratio; the relative difference between the bounds of the 95% CI and the odds ratio estimate is $\pm 1.96 \times 30\% = 58\%$.

3.3 | Performance of an odds ratio estimator

Walter and Cook (1991) compared the statistical properties of several estimators of the odds ratio, θ, in single 2×2 contingency tables. The estimator we focus on is a modified version of the unconditional maximum likelihood estimator, $\hat{\theta} = (a + 0.5)(d + 0.5)/\{(b + 0.5)(c + 0.5)\}$, where a, b, c and d are the cell counts. We focus on a situation similar to the previous example: 100 participants in each of two groups, with true proportions 0.3 and 0.5 (so $\theta = 0.43$).

For this estimator in this situation, they reported $\text{Bias}(\log \hat{\theta}, \log \theta) = 0.00$ (i.e., between −0.005 and +0.005) and $\text{MSE}(\log \hat{\theta}, \log \theta) = 0.09$. Using these numbers, we can make the following interpretations. The $\text{MRE}(\hat{\theta}, \theta)$ is extremely small at 0% (i.e., between −0.5% and +0.5%). The $\text{RMSRE}(\hat{\theta}, \theta) = \sqrt{0.09} = 30\%$ meaning the average (RMS) magnitude of a relative error is 30%.

4 | DISCUSSION

To our knowledge, our interpretations are novel. The terms themselves have sometimes been used with other meanings and, conversely, other interpretations have been suggested for $\text{SD}(\log Y)$. The $\text{SD}(\log Y)$ is approximately equal to the CV (Julious & Debarnot, 2000). It could also be interpreted exactly as "a form of CV"—once the definition of CV is altered by using a particular generalization of the logarithmic mean, $\mathrm{L}(Y) = \mathrm{SD}(Y)/\mathrm{SD}(\log Y)$, as the denominator (Cole, 2000; Cole & Altman, 2017a; Cole & Kryakin, 2002). When the geometric mean is used as a measure of location, we prefer our SRD interpretation of $\mathrm{SD}(\log Y)$ over any CV interpretation because the former explicitly describes variation about the geometric (not arithmetic) mean.

The relative difference central to our interpretations takes values similar to the more familiar definition of a percentage difference when values are small. This is not the case when the relative difference is as large as $\pm \log 2 = \pm 0.69 = \pm 69\%$. This corresponds to numbers differing by a factor of 2. To understand relative differences of all sizes this is a useful fact to remember (Graff, 2014). When $\mathrm{SD}(\log Y)$ is large, one could instead calculate the geometric standard deviation, $\mathrm{GSD}(Y) = \exp\{\mathrm{SD}(\log Y)\}$ (Kirkwood, 1979).

Our SRE is one of three tightly related options available for describing uncertainty about ratio-based comparisons. For our example in Section 3.2, we mentioned one option is to say 30% is the estimated SRE associated with the odds ratio $\hat{\theta} = 0.43$ (intervention v standard care). Another option is to interpret the regression coefficient for the intervention in Table 1 as the relative difference in odds (Cole, 2000), here −85% (SE 30%, 95% CI: −142% to −27%). Although rarely used, a third option is to report the geometric standard error, $\mathrm{GSE}(\hat{\theta}) = \exp\{\mathrm{SE}(\log \hat{\theta})\}$ (Kirkwood, 1979), along with $\hat{\theta}$.

Almost all log-scale quantities are readily interpretable. Our hope is that this note enables researchers to better appreciate log-scale quantities including log-scale SDs and the SE, bias, and RMSE of log-transformed estimators, and that researchers are equipped with new tools for presenting and describing such quantities. Other commonly encountered settings where our interpretations may be useful include (i) regression models of log-transformed data (including time-to-event data), (ii) the distribution of random effects in various mixed models such as a mixed-effects logistic or Poisson regression model and (iii) random-effects meta-analysis of various ratios.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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