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Gradient-type generalizations of one-dimensional dynamical model of strain-limiting elasticity

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Abstract. In this work, we propose two different generalizations of one-dimensional strain-limiting elasticity model where the linearized strain is given as a nonlinear function of the stress. These formulations are called stress gradient-type and strain gradient-type generalizations, and their constitutive relations are presented in both differential form and integral form. One important feature of this framework is that contrary to the theory of strain-limiting elasticity, the propagation of linear stress waves becomes dispersive as a consequence of inclusion of stress or strain gradients. We study traveling stress wave solutions to the governing equations of the nonlinear models proposed in this work. For a sample case of the constitutive relation belonging to the stress gradient-type formulation, we obtain explicit expressions of smooth solitary wave solutions when the stress is small but finite. Finally, we show that, for both the stress gradient-type and strain gradient-type formulations, the propagation of small amplitude long waves is described by the well-known KdV equation with the same coefficients.

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1. Introduction

The implicit constitutive relations introduced to describe some response of elastic materials involve the stress **T** and the left (or right) Cauchy–Green strain tensor **B** (or **C**) as variables. An interesting subclass of implicit constitutive relations is given by the so-called strain-limiting models based on the assumption that the linearized strain remains bounded even for large values of stresses. The studies started with the investigation of elastostatic problems of strain-limiting materials [3,27,29] and then continued with some focus on wave propagation problems [5,28,30]. Studies also exist that have used the approach of Rajagopal to deal with research problems appearing in viscoelastic materials [7,14,25,26,32], cracks [13] and biological fibers [11]. As a final remark, it is worth mentioning that situations where the stresses are large but the strains are infinitesimal are quite rare in real-life applications. For a detailed and comprehensive discussion of this issue, we refer the reader to [19].

In this work, we aim to answer the question of whether it is possible to generalize the one-dimensional strain-limiting model describing elastic response of materials in two directions. First, by adding stress gradients to the implicit constitutive relation we get a stress gradient-type formulation of strain-limiting elasticity model. Then, in the linear case, we unearth the relation of such a formulation with the nonlocal elasticity theory of Eringen [10]. Second, by including strain gradients in the implicit constitutive relation we get a strain gradient-type formulation of strain-limiting model and, in the linear case, we establish the relation of such a formulation with the gradient elasticity theory of Altan and Aifantis [1].

Furthermore, it is very well-known that, in both classical elasticity and strain-limiting elasticity, the linear waves propagating in an infinite elastic medium are not dispersive, i.e., phase velocity is independent of wavenumber. We first observe that the inclusion of stress gradient or strain gradient terms in the implicit

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constitutive relation makes the wave propagation dispersive. Due to dispersive nature of the waves, the study of traveling wave solutions to the nonlinear governing equations of both of the generalizations introduced here might lead to solitary wave solutions. By considering a specific constitutive relation suggested in the strain-limiting theory literature, we also aim to obtain explicit form of such solitary wave solutions. For the stress gradient-type formulation, the study of traveling wave solutions yields a singular dynamical system that possesses nonsmooth wave solutions. Since we want to focus on investigating smooth solitary wave solutions only, we consider the regime where the stress is small but finite. To this end, we study the reduced equations obtained by taking just the first two terms of the Taylor expansion of the nonlinear term. We then present explicit forms of sech-type solitary wave solutions to the reduced equations. However, when we carry out a similar analysis for the strain gradient-type formulation, we identify the nonexistence of solitary wave solutions. Finally, for both stress gradient-type and strain gradient-type generalizations we show that the propagation of small amplitude long waves is governed by the Korteweg de Vries (KdV) equation that admits solitary wave solutions.

The remainder of the paper is organized as follows. To make our discussion self-contained, we briefly review the strain-limiting theory of one-dimensional elasticity in Sect. 2. In Sect. 3, a stress gradient-type formulation of strain-limiting elasticity is developed and then both smooth traveling wave solutions and the KdV approximation of unidirectional long waves are considered. A strain gradient-type formulation of the strain-limiting elasticity is then presented in Sect. 4, and a similar analysis to the one in Sect. 3 is carried out.

2. One-dimensional strain-limiting model

Consider a one-dimensional, homogeneous, elastic, infinite medium exhibiting small strains for large values of the stress. In the absence of external body forces, the equation of motion is given by

$$\rho u_{tt} = \sigma_x, \tag{2.1}$$

where ρ is the mass density of the medium, the scalar-valued function $u(x,t)$ is the displacement, and $\sigma(x,t)$ is the Cauchy stress. Here and throughout this work, the subscripts denote partial derivatives. For convenience, we now define the dimensionless quantities

$$\bar{x} = \frac{x}{l},\ \bar{t} = \frac{t}{l}\sqrt{\frac{\mu}{\rho}},\ \bar{u} = \frac{u}{l},\ \bar{\sigma} = \frac{\sigma}{\mu},$$

where l is a characteristic length and μ is a constant with the dimension of stress. In this case, dropping the notation with bars for convenience, equation (2.1) becomes $u_{tt} = \sigma_x$. Differentiating this equation with respect to x, we obtain

$$\epsilon_{tt} = \sigma_{xx}, \tag{2.2}$$

where ϵ denotes the linearized strain defined by $\epsilon = u_x$. In strain-limiting theory [17,18], the constitutive relation is given by

$$\epsilon = f(\sigma), \tag{2.3}$$

where f is a nonlinear function of σ with $f(0) = 0$. If we use relation (2.3) in equation (2.2), we obtain a nonlinear equation in terms of the stress:

$$f(\sigma)_{tt} = \sigma_{xx}. \tag{2.4}$$

As expected, this equation models nondispersive propagation of stress waves if f is an increasing function of σ, that is, $f'(\sigma) > 0$ for all σ. (Here and henceforth, the prime symbol denotes differentiation with respect to the argument.) A typical example for f in (2.3) and (2.4) is given by

$$f(\sigma) = \frac{\sigma}{\sqrt{1+\sigma^2}}, \tag{2.5}$$

which is a special case of the constitutive relation given in [29]. The characteristic property of the constitutive relation corresponding to this choice of f is that the strain stays bounded when the stress becomes arbitrarily large. We also observe that for f given in (2.5) we have $f'(\sigma) = (1+\sigma^2)^{-3/2} > 0$ for all σ.

In the following two sections, we generalize the one-dimensional strain-limiting elasticity model in two separate directions by using stress gradient-type and strain gradient-type constitutive relations. We want to remark that as a result of these generalizations it would be necessary to pose some additional boundary conditions. However, the current study only considers waves propagating in infinite medium by assuming the field quantities and all their derivatives tend to zero at infinity. Therefore, we do not intend to focus on the additional boundary conditions in this work.

3. A formulation based on stress gradients

In this section, we will introduce a stress gradient-type formulation of the one-dimensional strain-limiting elasticity, that is, we assume that the strain is a function of the stress and its higher-order derivatives:

$$\epsilon = f(\mathcal{L}\sigma), \tag{3.1}$$

where $f(0) = 0$ and $\mathcal{L}$ is a constant coefficient linear differential operator with respect to x. In this case, (2.2) takes the form

$$[f(\mathcal{L}\sigma)]_{tt} = \sigma_{xx}, \tag{3.2}$$

while substituting the linear constitutive relation $\epsilon = \mathcal{L}\sigma$ into (2.2) yields

$$\mathcal{L}\sigma_{tt} = \sigma_{xx}. \tag{3.3}$$

It then follows that if α denotes the Green's function of the linear differential operator $\mathcal{L}$, the constitutive equation $\epsilon = \mathcal{L}\sigma$ can be written as a convolution integral in the form

$$\sigma(x,t) = \int_{-\infty}^{\infty} \alpha(|x-y|)\epsilon(y,t)dy. \tag{3.4}$$

We note that the constitutive relation (3.4) coincides with the one given by Eringen in [10] within the context of nonlocal theory of elasticity, where the kernel α represents the long-range interatomic interactions.

Obviously, when $\mathcal{L}$ is the identity operator, (3.3) describes linear propagation of nondispersive waves whose phase velocity is equal to 1 or -1. On the other hand, when $\mathcal{L}$ is not the identity operator, the phase velocity of linear waves changes with wavenumber as we will see below through examples. The operator $\mathcal{L}$ can be taken to be quite general. However, we focus on simple forms of $\mathcal{L}$ for computational purposes. Some examples of $\mathcal{L}$ are as follows.

Example 1. Let $\mathcal{L} = 1 - \partial_x^2$ for which the Green's function is given by $\alpha(|x|) = \frac{1}{2}e^{-|x|}$. In order to show the dispersive nature of wave propagation, for the moment let us assume the linear constitutive relation $\epsilon = f(\mathcal{L}\sigma) = \mathcal{L}\sigma$, that is, $\epsilon = (1-\partial_x^2)\sigma = \sigma - \sigma_{xx}$. This can be written as a convolution integral in the form (3.4). For this choice of $\mathcal{L}$, equation (3.3) becomes a fourth-order linear differential equation for σ given as $\sigma_{tt} - \sigma_{xx} - \sigma_{xxtt} = 0$, whose dispersion relation is given by

$$\frac{\omega^2}{k^2} = \frac{1}{1+k^2}, \tag{3.5}$$

where k and ω represent wavenumber and frequency, respectively. We note that the phase velocity is bounded for large values of k. At this point we remind the reader that $\sigma_{tt} - \sigma_{xx} - \sigma_{xxtt} = 0$, which models dispersive wave propagation, also appears in many studies on elastic wave propagation. For instance, we refer the reader to two recently published articles [15], [9] where the same equation was studied.

Example 2. Let $\mathcal{L} = 1+\partial_x^4$ for which the Green's function is given by $\alpha(|x|) = \frac{1}{2\sqrt{2}}e^{-\frac{|x|}{\sqrt{2}}}\big(\cos\frac{|x|}{\sqrt{2}}+\sin\frac{|x|}{\sqrt{2}}\big)$. Again when $\epsilon = f(\mathcal{L}\sigma) = \mathcal{L}\sigma$, that is, $\epsilon = (1+\partial_x^4)\sigma = \sigma + \sigma_{xxxx}$, an integral representation of the constitutive relation is given by (3.4). For this particular $\mathcal{L}$, (3.3) becomes a sixth-order linear differential equation for σ given as $\sigma_{tt} - \sigma_{xx} + \sigma_{xxxxtt} = 0$, with the dispersion relation

$$\frac{\omega^2}{k^2} = \frac{1}{1+k^4}. \tag{3.6}$$

Example 3. Let $\mathcal{L} = 1-\partial_x^2+\partial_x^4$ for which the Green's function is given by $\alpha(|x|) = \frac{1}{2\sqrt{3}}e^{-\frac{\sqrt{3}}{2}|x|}\big(\cos\frac{|x|}{2} + \sqrt{3}\sin\frac{|x|}{2}\big)$. The linear form of the constitutive equation (3.1) is $\epsilon = (1-\partial_x^2+\partial_x^4)\sigma = \sigma - \sigma_{xx} + \sigma_{xxxx}$. This constitutive relation also has the integral representation given by (3.4). The equation of motion (3.3) becomes again a sixth-order nonlinear differential equation for σ given as $\sigma_{tt} - \sigma_{xx} - \sigma_{xxtt} + \sigma_{xxxxtt} = 0$, with the dispersion relation

$$\frac{\omega^2}{k^2} = \frac{1}{1+k^2+k^4}, \tag{3.7}$$

where again the phase velocity is bounded for large values of k.

3.1. Traveling wave solutions

We now investigate traveling wave solutions of the nonlinear differential equation (3.2) corresponding to the stress gradient-type generalization of the one-dimensional strain-limiting model. We consider the traveling wave solutions of (3.2) in the form $\sigma = \sigma(\xi)$, $\xi = x - ct$ where c is the constant wave speed. We assume that σ and its derivatives tend to vanish as $\xi \to \pm\infty$. Substitution of the ansatz $\sigma = \sigma(\xi)$ into (3.2) yields an ordinary differential equation. After integrating this equation twice and using the conditions at infinity, we obtain

$$c^2 f(\mathcal{L}\sigma) = \sigma, \tag{3.8}$$

where $\mathcal{L}$ is now a constant coefficient differential operator that includes derivatives with respect to ξ. In the remaining part of this section, we focus on a specific form of the constitutive relation together with the operators $\mathcal{L}$ introduced above. By choosing f as in (2.5), equation (3.8) becomes

$$c^2\frac{\mathcal{L}\sigma}{\sqrt{1+(\mathcal{L}\sigma)^2}} = \sigma. \tag{3.9}$$

There are two implications of this equation. First, σ is bounded from above by c^2. Second, the sign of σ determines the sign of $\mathcal{L}\sigma$. Rewriting (3.9) in the form

$$c^2\mathcal{L}\sigma = \sigma\sqrt{1+(\mathcal{L}\sigma)^2},$$

taking the square of both sides of this equation and solving the resulting equation for $\mathcal{L}\sigma$, we get

$$\mathcal{L}\sigma = \pm\frac{\sigma}{\sqrt{c^4-\sigma^2}},$$

where $\sigma^2 < c^4$ due to the first implication. Since the sign of σ determines the sign of $\mathcal{L}\sigma$, here we should take the positive sign. We then have the nonlinear ordinary differential equation

$$\mathcal{L}\sigma - \frac{\sigma}{\sqrt{c^4-\sigma^2}} = 0. \tag{3.10}$$

This differential equation can be approached from the point of view of dynamical systems by rewriting as a system of coupled first-order differential equations. By means of a phase-plane analysis, one can prove the existence of homoclinic orbits which indicates the existence of solitary wave solutions of (3.10). Since the resulting dynamical system has singularities at $\sigma = \pm c^2$, nonsmooth solutions like peakons and cuspons may arise. This is a subject that deserves a separate study. The current study focuses on smooth traveling wave solutions with decay at infinity, and also obtaining them in analytical form. Due to these reasons, we study a regularized form of (3.10) instead of (3.10) itself. Therefore, to confine our problem to small but finite values of σ, we only consider the first two terms of the Taylor series expansion of the nonlinear term about $\sigma = 0$ and obtain the following regularized form of (3.10):

$$\mathcal{L}\sigma - \frac{1}{c^2}\sigma - \frac{1}{2c^6}\sigma^3 = 0. \tag{3.11}$$

We note that if σ is a solution to this equation, then so is $-\sigma$. We will study solutions of this equation by considering the three different forms of the operator $\mathcal{L}$.

Case 1: $\mathcal{L} = 1 - \partial_x^2$. In this case, (3.11) reduces to the second-order differential equation

$$\sigma'' - \frac{c^2-1}{c^2}\sigma + \frac{1}{2c^6}\sigma^3 = 0, \tag{3.12}$$

where $'$ denotes differentiation with respect to ξ. It follows from the Pohozaev-type identity (A.4) given in Appendix A that (3.12) has no solution with the zero conditions at infinity if $c^2 < 1$. To get the solution corresponding to the case $c^2 > 1$, we use the scaling transformation

$$\sigma(\xi) = 2c^2\sqrt{c^2-1}\,\psi(\zeta), \quad \zeta = \sqrt{\frac{c^2-1}{c^2}}\,\xi$$

in (3.12). This yields

$$\psi'' - \psi + 2\psi^3 = 0.$$

This equation has a solution given as $\psi(\zeta) = \operatorname{sech}\zeta$. Thus, equation (3.12) has a solitary wave solution given as

$$\sigma(\xi) = 2c^2\sqrt{c^2-1}\operatorname{sech}\left(\sqrt{\frac{c^2-1}{c^2}}\,\xi\right) \tag{3.13}$$

if the square of the wave speed c is greater than the square of the nondispersive phase velocity ± 1. Furthermore, by recalling $\sigma^2 < c^4$, we see that the square of the wave speed is also bounded from above by $5/4$. That is, the solitary wave given in (3.13) exists if $1 < c^2 < 5/4$. With the substitution $\xi = x - ct$, (3.13) provides an approximate solution to (3.2) for the particular f and $\mathcal{L}$.

Case 2: $\mathcal{L} = 1 + \partial_x^4$. In this case, (3.11) reduces to the fourth-order differential equation

$$\sigma^{(4)} + \frac{c^2-1}{c^2}\sigma - \frac{1}{2c^6}\sigma^3 = 0. \tag{3.14}$$

It follows from the Pohozaev-type identity (B.4) given in Appendix B that if $c^2 < 1$, (3.14) has no solution with the zero conditions at infinity. On the other hand, an explicit, exact solution of (3.14) is not available for $c^2 > 1$. However, it is interesting to note that the traveling wave solutions of the Rosenau equation $v_t + v_x + v_{xxxxt} + \frac{1}{2}(v^3)_x = 0$ satisfy (3.14) again if a suitable scaling transformation is used. A rigorous proof of the existence of solitary wave solutions for a more general form of (3.14) is given in [6]. An interesting feature of the solitary wave profiles numerically obtained in [6,8] for (3.14) is that their tails are non-monotone. This is different from what we observed in the previous case since the sech-type solitary wave solution given in (3.13) has monotonic tails.

Case 3: $\mathcal{L} = 1 - \partial_x^2 + \partial_x^4$. In this case, (3.11) reduces to the fourth-order equation

$$\sigma^{(4)} - \sigma'' + \frac{c^2-1}{c^2}\sigma - \frac{1}{2c^6}\sigma^3 = 0. \tag{3.15}$$

It follows from the Pohozaev-type identity (C.5) given in Appendix C that if $c^2 < 1$, (3.15) has no solution with the zero conditions at infinity. A formula for the exact solution of (3.15) is not available for

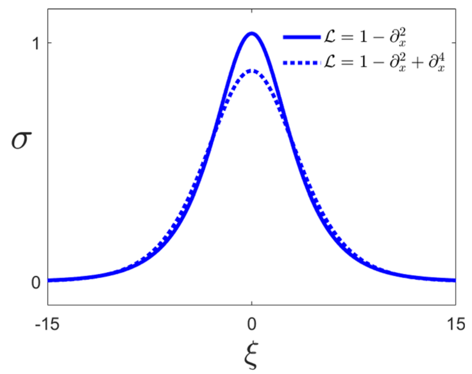

Fig. 1. Solitary wave profiles corresponding to $c^2 = 25/21$ for $\sigma(\xi)$ given in (3.13) and (3.16) which are obtained when $\mathcal{L} = 1 - \partial_x^2$ and $\mathcal{L} = 1 - \partial_x^2 + \partial_x^4$, respectively

an arbitrary value of c^2 such that $c^2 > 1$. However, there is a specific value of c^2 for which (3.15) has a sech-type solitary wave solution. This particular solution is of the form

$$\sigma(\xi) = \sqrt{\frac{3}{5}\left(\frac{25}{21}\right)^3} \operatorname{sech}^2\left(\frac{\xi}{\sqrt{20}}\right), \tag{3.16}$$

which corresponds to the value $c^2 = 25/21$. Figure 1 shows the profiles of the solitary wave solutions given in (3.13) and (3.16) for $c^2 = 25/21$. From the figure we see that there is no significant difference between the profiles except the difference in the amplitudes.

3.2. The long-wave approximation

In this section, we will perform the long-wave approximation to right-going wave solutions of equation (3.2) for a general function f when the operator $\mathcal{L}$ is chosen as $\mathcal{L} = 1 - \partial_x^2$ as in Example 1 above. In this case, equation (3.2) becomes

$$\big(f(\sigma - \sigma_{xx})\big)_{tt} = \sigma_{xx}, \tag{3.17}$$

and the linear dispersion relation (3.5) implies $\omega = \pm k/\sqrt{1+k^2}$. Choosing the plus sign, we approximate the phase $kx - \omega t$ in the form

$$kx - \omega t = k(x-t) + \frac{1}{2}k^3 t + \ldots$$

for small wavenumbers. We then introduce the slow variables (η, τ) as

$$\eta = \delta^{1/2}(x-t), \quad \tau = \delta^{3/2} t,$$

where δ is a positive small parameter measuring the smallness of the wavenumber k. By performing the coordinate transformation $(x,t) \to (\eta, \tau)$, we can rewrite (3.17) in the form

$$\big(\partial_{\eta\eta} - 2\delta\partial_\eta\partial_\tau + \delta^2\partial_{\tau\tau}\big) f(\sigma - \delta\sigma_{\eta\eta}) = \sigma_{\eta\eta}. \tag{3.18}$$

We now expand the dependent variable σ in power series of δ which gives

$$\sigma = \delta\sigma_1 + \delta^2\sigma_2 + \ldots.$$

Here $\sigma_i, i = 1, 2, \ldots,$ are coefficient functions independent of δ. If we use the relation

$$\sigma - \delta\sigma_{\eta\eta} = \delta\sigma_1 + \delta^2\big(\sigma_2 - (\sigma_1)_{\eta\eta}\big) + \ldots,$$

and Taylor series expansion for f in terms of δ, equation (3.18) becomes

$$\partial_{\eta\eta}f(0)+\delta\Big(-2\big(f(0)\big)_{\eta\tau}+\left(\frac{df}{d\delta}\Big|_{\delta=0}\right)_{\eta\eta}\Big)$$
$$+\delta^2\Big(f(0)_{\tau\tau}-2\left(\frac{df}{d\delta}\Big|_{\delta=0}\right)_{\eta\tau}+\frac{1}{2}\left(\frac{d^2f}{d\delta^2}\Big|_{\delta=0}\right)_{\eta\eta}\Big)+\ldots=\delta(\sigma_1)_{\eta\eta}+\delta^2(\sigma_2)_{\eta\eta}+\ldots.$$

Equating the coefficients of the corresponding powers of δ, we get a hierarchy of equations. Since $f(0)=0$, the zeroth-order equation is identically satisfied. Since $\frac{df}{d\delta}\big|_{\delta=0}=f'(0)\sigma_1$, we get $f'(0)(\sigma_1)_{\eta\eta}=(\sigma_1)_{\eta\eta}$ at order δ, which implies $f'(0)=1$. Similarly, since

$$\frac{d^2f}{d\delta^2}\Big|_{\delta=0}=f''(0)\sigma_1^2+2(\sigma_2-(\sigma_1)_{\eta\eta})$$

at order δ^2, we obtain

$$-2(\sigma_1)_{\eta\tau}+\frac{1}{2}\big(f''(0)\sigma_1^2+2(\sigma_2-(\sigma_1)_{\eta\eta})\big)_{\eta\eta}=(\sigma_2)_{\eta\eta}.$$

Canceling $(\sigma_2)_{\eta\eta}$ on both sides and assuming that σ_1 and its derivatives converge to 0 as η tends to $\pm\infty$, we integrate this equation with respect to η. This yields the well-known KdV equation

$$(\sigma_1)_\tau-\frac{1}{2}f''(0)\sigma_1(\sigma_1)_\eta+\frac{1}{2}(\sigma_1)_{\eta\eta\eta}=0, \tag{3.19}$$

with the condition $f''(0)\neq 0$ to be satisfied. The sign of $f''(0)$ determines the characteristic of equation (3.19). We refer to it as the focusing or defocusing KdV equation when $f''(0)<0$ or $f''(0)>0$, respectively. Moreover, if $\sigma_1(\eta,\tau)$ is a solution of the focusing equation, then $-\sigma_1(\eta,\tau)$ is a solution of the defocusing equation. The above equation is valid under the assumptions that $f'(0)=1$ and $f''(0)\neq 0$. If we have $f''(0)=0$ for a particular f, changing the scaling in the above approximation and using a similar approach we may get the modified KdV equation with cubic nonlinearity instead of the KdV equation with quadratic nonlinearity. It is interesting to note that $f''(0)=0$ for the constitutive relation given in (2.5). On the other hand, just as an example, for the constitutive relation

$$f(\sigma)=\frac{\sigma}{1-\gamma\sigma}, \tag{3.20}$$

given in [30], where γ is a constant, we have $f''(0)=2\gamma$.

To find traveling solutions of (3.19), we use the scaling transformation

$$\sigma_1(\eta,\tau)=-\frac{4c}{f''(0)}\psi(\zeta),\quad \zeta=\sqrt{\frac{c}{2}}(\eta-c\tau)$$

in (3.19) where the constant $c>0$ represents the wave speed. Integrating the resulting equation with respect to ζ and using the assumptions $\psi,\psi',\psi''\to 0$ as $\zeta\to\pm\infty$, we obtain the ordinary differential equation

$$\psi''-4\psi+4\psi^2=0. \tag{3.21}$$

This equation has a solution given as $\psi(\zeta)=\frac{3}{2}\mathrm{sech}^2\zeta$, which implies that equation (3.19) has a solitary wave solution in the form

$$\sigma_1(\eta,\tau)=-\frac{6c}{f''(0)}\mathrm{sech}^2\left(\sqrt{\frac{c}{2}}(\eta-c\tau)\right). \tag{3.22}$$

This solution is unique up to an arbitrary constant. As it is expected from (3.19), $f''(0)$ affects the amplitude of the solitary wave. We also note that the above solution leads to a tensile or compressive solitary wave if $f''(0)$ is negative or positive, respectively. In Fig. 2 profiles of solitary waves are presented for two typical values of $f''(0)$.

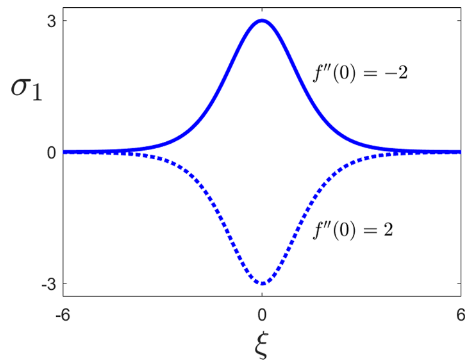

FIG. 2. Solitary wave profiles for $\sigma_1(\xi)$ given in (3.22) where $f''(0) = -2$ and $f''(0) = 2$

4. A formulation based on strain gradients

In this section, we will introduce a strain gradient-type formulation of one-dimensional strain-limiting elasticity. We assume that a combination of the strain and its higher-order derivatives is a function of the stress in the form

$$\mathcal{L}\epsilon = f(\sigma), \tag{4.1}$$

where again $\mathcal{L}$ is a constant coefficient, linear differential operator with respect to x. If the Green's function of $\mathcal{L}$ is denoted by α, we can rewrite the above constitutive relation as a convolution integral in the form

$$\epsilon(x,t) = \int_{-\infty}^{\infty} \alpha(|x-y|) f(\sigma(y,t)) dy. \tag{4.2}$$

Using the constitutive relation (4.1) in (2.2), we get the equation of motion in the form

$$[f(\sigma)]_{tt} = \mathcal{L}\sigma_{xx}. \tag{4.3}$$

When we use the linear constitutive relation $\mathcal{L}\epsilon = \sigma$ in (2.2), we obtain the linearized equation of motion

$$\sigma_{tt} = \mathcal{L}\sigma_{xx}. \tag{4.4}$$

Some examples of the operator $\mathcal{L}$ are as follows:
Example 1: Let $\mathcal{L} = 1 - \partial_x^2$. In this case the linear constitutive relation takes the form $\epsilon - \epsilon_{xx} = \sigma$. We note that this relation coincides with the one given in [1] for a special theory of one-dimensional gradient elasticity. The linear equation (4.4) becomes $\sigma_{tt} - \sigma_{xx} + \sigma_{xxxx} = 0$ whose dispersion relation is given by

$$\frac{\omega^2}{k^2} = 1 + k^2. \tag{4.5}$$

We note that, contrary to (3.5) in the stress gradient-type formulation, the phase velocity becomes unbounded for large values of k.

Example 2. Let $\mathcal{L} = 1 + \partial_x^4$. Again, the linear equation (4.4) becomes $\sigma_{tt} - \sigma_{xx} - \sigma_{xxxxxx} = 0$, which has the dispersion relation as

$$\frac{\omega^2}{k^2} = 1 + k^4.$$

Example 3. Let $\mathcal{L} = 1 - \partial_x^2 + \partial_x^4$. The linear equation (4.4) takes the form $\sigma_{tt} - \sigma_{xx} + \sigma_{xxxx} - \sigma_{xxxxxx} = 0$, for which dispersion relation is given by

$$\frac{\omega^2}{k^2} = 1 + k^2 + k^4.$$

Again, the phase velocity becomes unbounded for large values of k.

4.1. Traveling wave solutions

We now investigate traveling wave solutions of the strain gradient-type strain-limiting model (4.3) for special cases of the operator $\mathcal{L}$ and the function f. We again consider the traveling wave solutions in the form $\sigma = \sigma(\xi)$, $\xi = x - ct$ where c is the constant wave speed and assume that σ and its derivatives tend to vanish as $\xi \to \pm\infty$. Using the ansatz $\sigma = \sigma(\xi)$ in (4.3) yields an ordinary differential equation. After integrating this equation twice and using the conditions at infinity, we obtain

$$c^2 f(\sigma) = \mathcal{L}\sigma, \tag{4.6}$$

where the derivatives in $\mathcal{L}$ are taken with respect to ξ. Substitution of f given in (2.5) into (4.6) yields

$$\mathcal{L}\sigma - c^2 \frac{\sigma}{\sqrt{1+\sigma^2}} = 0. \tag{4.7}$$

This implies that the sign of σ determines the sign of $\mathcal{L}\sigma$. Contrary to (3.10) in the stress gradient-type formulation, the above equation has no singular term. In order to make the comparison between the stress gradient-type and strain gradient-type formulations as fair as possible, we again confine ourselves to small but finite values of σ. Therefore, as in the previous section, we only consider the first two terms of the Taylor series expansion of the nonlinear term in (4.7), and we obtain

$$\mathcal{L}\sigma - c^2\sigma + \frac{c^2}{2}\sigma^3 = 0. \tag{4.8}$$

It should be noted that if σ is a solution to this equation, so is $-\sigma$. We now consider three different $\mathcal{L}$ operators introduced above.

Case 1: $\mathcal{L} = 1 - \partial_x^2$. Recall that for this form of $\mathcal{L}$ the kernel function in (4.2) is given by $\alpha(|x|) = \frac{1}{2}e^{-|x|}$. For this choice of $\mathcal{L}$, equation (4.8) becomes

$$\sigma'' + (c^2 - 1)\sigma - \frac{c^2}{2}\sigma^3 = 0. \tag{4.9}$$

It follows from the Pohozaev-type identity (A.4) given in Appendix A that if $c^2 < 1$, this equation has no solution with the zero conditions at infinity. On the other hand, using the scaling transformation

$$\sigma(\xi) = 2\sqrt{\frac{c^2-1}{c^2}}\,\psi(\zeta), \quad \zeta = \sqrt{c^2-1}\,\xi$$

for $c^2 > 1$ helps us to reduce (4.9) to a simpler form given by

$$\psi'' + \psi - 2\psi^3 = 0.$$

This equation has a periodic solution in the form $\psi = \sec\zeta$. We conclude that (4.9) does not admit solitary wave solutions for all speeds c. This is contrary to what we observed in the previous section for the stress gradient-type formulation with the same form of $\mathcal{L}$. Recall that, under the restriction $c^2 > 1$, the explicit expression of the solitary wave solution obtained for the same operator is given in (3.13). The main reason for this significant difference between the traveling wave solutions arising in stress gradient and strain gradient formulations of strain-limiting elasticity is that the linear dispersion relations of both formulations are completely different (see (3.5) and (4.5)). A similar difference between stress and strain gradient-type formulations of classical elasticity has been reported in [2] by carrying out a dispersion analysis of both models. In [2], it has been concluded that the model in [10] corresponding to stress gradient theory has a better performance in dynamics, and that the model in [1] corresponding to the strain gradient theory was originally proposed to model static problems rather than dynamical problems.

Case 2: $\mathcal{L} = 1 + \partial_x^4$. In this case, (4.8) reduces to the fourth-order equation

$$\sigma^{(4)} - (c^2 - 1)\sigma + \frac{c^2}{2}\sigma^3 = 0.$$

It follows from the Pohozaev-type identity (B.4) given in Appendix B that if $c^2 < 1$, this equation has no solution with the zero conditions at infinity. As in the previous case, for $c^2 > 1$ we expect periodic solutions only.

Case 3: $\mathcal{L} = 1 - \partial_x^2 + \partial_x^4$. In this case, (4.8) reduces to the fourth-order equation

$$\sigma^{(4)} - \sigma'' - (c^2 - 1)\sigma + \frac{c^2}{2}\sigma^3 = 0.$$

It follows from the Pohozaev-type identity (C.4) given in Appendix C that if $c^2 < 1$, this equation has no solution with the zero conditions at infinity. Again, we can only expect periodic solutions for $c^2 > 1$.

4.2. Long wave approximation

In this section, we will perform long-wave approximation to right-going wave solutions of equation (4.3) when the operator $\mathcal{L}$ is chosen as $\mathcal{L} = 1 - \partial_x^2$. In this case, the equation becomes

$$f(\sigma)_{tt} = \sigma_{xx} - \sigma_{xxxx}. \tag{4.10}$$

The linear dispersion relation gives $\omega = \pm k\sqrt{1 + k^2}$ and we choose the plus sign. Approximating the phase $kx - \omega t$ for small wave numbers and introducing the slow variables (η, τ) as $\eta = \delta^{1/2}(x - t)$ and $\tau = \delta^{3/2}t$ again, as a result of coordinate transformation $(x, t) \to (\eta, \tau)$ equation (4.10) becomes

$$\left(\partial_{\eta\eta} - 2\delta\partial_\eta\partial_\tau + \delta^2\partial_{\tau\tau}\right)f(\sigma) = \sigma_{\eta\eta} - \delta\sigma_{\eta\eta\eta\eta}. \tag{4.11}$$

Expanding the dependent variable σ in power series of δ as before and using the Taylor series expansion for f in terms of δ, equation (4.11) becomes

$$\begin{aligned}
\partial_{\eta\eta}f(0) + \delta\Big(-2\big(f(0)\big)_{\eta\tau} + \left(\frac{df}{d\delta}\Big|_{\delta=0}\right)_{\eta\eta}\Big)& \\
+\delta^2\Big(f(0)_{\tau\tau} - 2\left(\frac{df}{d\delta}\Big|_{\delta=0}\right)_{\eta\tau} + \frac{1}{2}\left(\frac{d^2 f}{d\delta^2}\Big|_{\delta=0}\right)_{\eta\eta}\Big) + \dots& \\
= \delta(\sigma_1)_{\eta\eta} + \delta^2\Big((\sigma_2)_{\eta\eta} - (\sigma_1)_{\eta\eta\eta\eta}\Big) + \dots.&
\end{aligned}$$

As before, we equate the coefficients of the corresponding powers of δ. Since $f(0) = 0$, the zeroth-order equation is identically satisfied. At order δ we get $f'(0)(\sigma_1)_{\eta\eta} = (\sigma_1)_{\eta\eta}$ which implies $f'(0) = 1$ as in the stress gradient case. At order δ^2 we obtain

$$-2(\sigma_1)_{\eta\tau} + \frac{1}{2}f''(0)(\sigma_1^2)_{\eta\eta} + (\sigma_2)_{\eta\eta} = (\sigma_2)_{\eta\eta} - (\sigma_1)_{\eta\eta\eta\eta}.$$

Canceling $(\sigma_2)_{\eta\eta}$ on both sides as well as differentiating with respect to η, we obtain the same KdV equation (3.19). That is, in the long-wave approximation the same behavior is observed for both stress gradient-type and integral-type formulations. Obviously, the solitary wave solution given in (3.22) is also valid for dispersive wave propagation within the context of integral-type formulation of strain-limiting theory.

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Appendix A

In this appendix, we derive Pohozaev-type identities [16] for the second-order differential equation

$$y'' + a_0 y + a_1 y^3 = 0, \quad y = y(x), \quad -\infty < x < \infty, \tag{A.1}$$

where a_0 and a_1 are constants. Those identities will be used to deduce nonexistence of traveling wave solutions. Throughout this appendix, we assume that $y(x)$ and all its derivatives vanish as $x \to \pm\infty$.

Let us multiply (A.1) by xy' and integrate from $-\infty$ to ∞ to get

$$\int_{-\infty}^{\infty} xy'y''dx + a_0 \int_{-\infty}^{\infty} xyy'dx + a_1 \int_{-\infty}^{\infty} xy'y^3 dx = 0.$$

Using integration by parts and the zero conditions at infinity, we obtain the first identity as an integral equality given by

$$\int_{-\infty}^{\infty} \left((y')^2 + a_0 y^2 + \frac{a_1}{2} y^4\right) dx = 0. \tag{A.2}$$

Similarly, multiplying (A.1) by y and integrating over $-\infty < x < \infty$, we get

$$\int_{-\infty}^{\infty} yy''dx + a_0 \int_{-\infty}^{\infty} y^2 dx + a_1 \int_{-\infty}^{\infty} y^4 dx = 0.$$

Again, using integration by parts and the zero conditions at infinity, we obtain the second identity in the form

$$\int_{-\infty}^{\infty} \big(-(y')^2 + a_0 y^2 + a_1 y^4\big) dx = 0. \tag{A.3}$$

Combining (A.2) and (A.3) yields

$$\int_{-\infty}^{\infty} \left(2a_0 y^2 + \frac{3}{2} a_1 y^4\right) dx = 0. \tag{A.4}$$

This shows that if $a_0\, a_1 > 0$, equation (A.1) has no solution satisfying the boundary conditions at infinity.

Appendix B

We now obtain Pohozaev-type identities [16] for the fourth-order differential equation

$$y^{(4)} + a_0 y + a_1 y^3 = 0, \quad y = y(x), \quad -\infty < x < \infty, \tag{B.1}$$

where a_0 and a_1 are constants. Throughout this appendix, we assume that $y(x)$ and all its derivatives tend to zero as $x \to \pm\infty$.

We multiply (B.1) by xy' and integrate from $-\infty$ to ∞ to get

$$\int_{-\infty}^{\infty} xy'y^{(4)}dx + a_0 \int_{-\infty}^{\infty} xyy'dx + a_1 \int_{-\infty}^{\infty} xy^3y'dx = 0.$$

Using integration by parts and the zero conditions at infinity, we obtain the first identity as

$$\int_{-\infty}^{\infty} \big(3(y'')^2 - a_0 y^2 - \frac{a_1}{2} y^4\big) dx = 0. \tag{B.2}$$

Multiplying (B.1) by y and integrating over $-\infty < x < \infty$ yields

$$\int_{-\infty}^{\infty} yy^{(4)}dx + a_0 \int_{-\infty}^{\infty} y^2 dx + a_1 \int_{-\infty}^{\infty} y^4 dx = 0.$$

Again, using integration by parts and the zero conditions at infinity, we obtain the second identity in the form

$$\int_{-\infty}^{\infty} \big((y'')^2 + a_0 y^2 + a_1 y^4\big)\, dx = 0. \tag{B.3}$$

Combining (B.2) and (B.3) gives

$$\int_{-\infty}^{\infty} (4a_0 y^2 + \frac{7}{2} a_1 y^4) dx = 0. \tag{B.4}$$

This implies that if $a_0\, a_1 > 0$, then equation (B.1) has no solution satisfying the boundary conditions at infinity.

Appendix C

We now obtain Pohozaev-type identities [16] for the fourth-order differential equation

$$y^{(4)} + a_0 y'' + a_1 y + a_2 y^3 = 0, \quad y = y(x), \quad -\infty < x < \infty, \tag{C.1}$$

where a_0, a_1 and a_2 are constants. Throughout this appendix, we assume that $y(x)$ and all its derivatives tend to zero as $x \to \pm\infty$.

We multiply (C.1) by xy' and integrate from $-\infty$ to ∞ to get

$$\int_{-\infty}^{\infty} xy'y^{(4)}dx + a_0 \int_{-\infty}^{\infty} xy'y''dx + a_1 \int_{-\infty}^{\infty} xyy'dx + a_2 \int_{-\infty}^{\infty} xy^3y'dx = 0.$$

Using integration by parts and the zero conditions at infinity, we obtain the first identity as

$$\int_{-\infty}^{\infty} \left(6(y'')^2 - 2a_0(y')^2 - 2a_1y^2 - a_2y^4\right)dx = 0. \tag{C.2}$$

Multiplying (C.1) by y and integrating over $-\infty < x < \infty$ yields

$$\int_{-\infty}^{\infty} yy^{(4)}dx + a_0\int_{-\infty}^{\infty} yy''dx + a_1\int_{-\infty}^{\infty} y^2dx + a_2\int_{-\infty}^{\infty} y^4dx = 0.$$

Again, using integration by parts and the zero conditions at infinity, we obtain the second identity in the form

$$\int_{-\infty}^{\infty} \left((y'')^2 - a_0(y')^2 + a_1y^2 + a_2y^4\right) dx = 0. \tag{C.3}$$

If we eliminate y'' by combining (C.2) and (C.3), we get

$$\int_{-\infty}^{\infty} (-4a_0(y')^2 + 8a_1y^2 + 7a_2y^4)dx = 0. \tag{C.4}$$

This implies that when $a_0 < 0$, $a_1 > 0$, $a_2 > 0$, equation (C.1) has no solution satisfying boundary conditions at infinity. On the other hand, if we eliminate y' by combining (C.2) and (C.3), we obtain

$$\int_{-\infty}^{\infty} (-4(y'')^2 + 4a_1y^2 + 3a_2y^4)dx = 0. \tag{C.5}$$

This implies that if $a_1 < 0$ and $a_2 < 0$, then equation (C.1) has no solution satisfying the boundary conditions at infinity.

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