STRONG 1-BOUNDEDNESS, L^2-BETTI NUMBERS, ALGEBRAIC SOFICITY, AND GRAPH PRODUCTS

IAN CHARLESWORTH, ROLANDO DE SANTIAGO, BEN HAYES, DAVID JEKEL, SRIVATSAV KUNNAWALKAM ELAYAVALLI, AND BRENT NELSON

Abstract. We show that graph products of non trivial finite dimensional von Neumann algebras are strongly 1-bounded when the underlying $*$-algebra has vanishing first L^2-Betti number. The proof uses a combination of the following two key ideas to obtain lower bounds on the Fuglede–Kadison determinant of matrix polynomials in a generating set: a notion called "algebraic soficity" for $*$-algebras allowing for the existence of Galois bounded microstates with asymptotically constant diagonals; a probabilistic construction of the authors of permutation models for graph independence over the diagonal.

Key words: algebraic soficity, graph products, strong 1-boundedness
2020 Mathematics Subject Classification: (Primary) 46L54, 15B36, (Secondary) 20P05

1. Introduction

Finite dimensional approximations of infinite dimensional objects are a common theme in analysis, dynamics, and operator algebras. In the context of groups, they arise in both soficity and Connes embeddability of the group von Neumann algebra (sometimes referred to as hyperlinearity), which are the ability to be approximated by permutations or finite dimensional unitary matrices, respectively. Connes-embeddable tracial von Neumann algebras are those which admit matrix approximations in a weak sense. The quantum complexity result announced in [JNV$^+$20] implies that not all tracial von Neumann algebras have this property; among Connes-embeddable von Neumann algebras, some—such as free products—have an abundance of matrix approximations which can be constructed probabilistically through random matrix theory, while others—such as amenable von Neumann algebras, property (T) von Neumann algebras, or von Neumann algebras with Cartan subalgebras—have very few matrix approximations. More precisely, the latter are strongly 1-bounded in the sense of Jung, or have 1-bounded entropy $h(M) < \infty$ in the sense of Hayes [Hay18]. Von Neumann algebras with $h(M) = \infty$ enjoy strong indecomposability properties: for instance, they are unable to be decomposed nontrivially as a tensor product, a crossed product, or a join of amenable subalgebras with diffuse intersection; more generally, they cannot be decomposed as a join of subalgebras with finite 1-bounded entropy. Using 1-bounded entropy techniques to study the structure of II_1 factors (especially, free group factors) has recently been quite fruitful to approach open problems (see for instance [Hay22, BC22, BC23, CIKE22]).

Graph products of groups, defined by Green in [Gre90], are free products of groups indexed by the vertices of a graph, modulo the relations that Γ_v and Γ_w commute when v and w are adjacent vertices in the graph. Graph products of von Neumann algebras were introduced by Młotkowski in [Mło04] under a different name, then reintroduced and further studied by Caspers and Fima in [CF17]. From a probabilistic viewpoint, graph products give rise to a notion of "graph independence", which is a natural way to mix together classical independence and free independence [Mło04, SW16].

Preservation of Connes-embeddability by graph products was proved by Caspers [Cas16]. Collins and Charlesworth described how to construct random matrix approximations for a graph product out of given random matrix approximations for the individual algebras M_v [CC21]. But despite the matrix approximations being defined by similar techniques as for free products, it was not clear when graph products would have abundant matrix approximations in the sense that $h(M) = \infty$, because the matrix approximations were constructed in a subspace of $M_{N^k}(\mathbb{C})$

with much lower dimension than the ambient space. In this paper we make progress toward classifying when a graph product has $h(M) < \infty$, which can be summarized in the following theorem. Here items 2 and 3 give a complete characterization of when $h(M) < \infty$ for the case when M_v is diffuse for every v, whereas item 1 applies in the much more subtle case when each M_v is finite dimensional.

Theorem A (Section 5.1)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with $\#\mathcal{V} > 1$, and for each $v \in \mathcal{V}$, let (M_v, τ_v) be a tracial $*$-algebra. Let $(M, \tau) = \mathop{\Large\ast}_{v\in\mathcal{G}}(M_v, \tau_v)$ be their graph product over $\mathcal{G}$.*

(1) Suppose each M_v is finite dimensional, and the trace of every central projection in M_v is a rational number. Let A be the $*$-subalgebra of M generated by $\bigcup_{v\in\mathcal{V}} M_v$. If $\beta^1_{(2)}(A, \tau) = 0$, then M is strongly 1-bounded.

(2) If each M_v is diffuse and $\mathcal{G}$ is connected, then M is strongly 1-bounded (in fact has 1-bounded entropy at most zero).

(3) If each M_v is diffuse and Connes embeddable, and $\mathcal{G}$ is disconnected, then M is not strongly 1-bounded.

We remark that in (3) we show something stronger: there is a (potentially infinite) tuple x of self-adjoint elements of M so that $W^*(x) = M$ and $\delta_0(x) > 1$. As we show in Theorem 5.1, the second two items in the above theorem can be deduced quickly from the known robust properties of 1-bounded entropy. We turn our attention, instead, to the question of strongly 1-boundedness for graph products of finite dimensional algebras. This focus motivates the results in part (1), which is more subtle; developing the tools for its proof occupies the bulk of this paper, and in the end we are able to prove a more general statement in 5.4.

The final step in our proof is to invoke results of Jung [Jun03, Theorem 6.9] and Shlyakhtenko [Shl21, Theorem 3.2], which apply when an operator naturally arising from "generators and relations" has positive Fuglede–Kadison pseudo-determinant. If each (M_v, τ_v) were a group algebra, it would suffice to note that the graph product is sofic by [CHR14]: the relation matrix in question is a matrix over the rational group ring, and such matrices have positive Fuglede–Kadison pseudo-determinant by [ES05]. However, extending this to arbitrary $*$-algebras not arising from groups presents substantial challenges.

Our approach is to introduce a notion of *algebraic soficity* for tracial $*$-algebras inspired by soficity of groups. This notion of algebraic soficity ensures that if $y \in M_n(A)$ can be expressed as a matrix of polynomials in x with "nice" coefficients (e.g. rational, algebraic, etc.), then y has positive Fuglede–Kadison pseudo-determinant. Crucially, we establish that this notion of algebraic soficity is closed under graph products.

In order to prove positivity of these Fuglede–Kadison determinants we use our notion of algebraic soficity which, while akin to that of soficity of groups, is not a simple translation of the group case. One naïve approach, sufficient to force the appropriate determinants to be positive, would be to require matrix approximations for our generating tuple x with integer entries, chosen so that polynomials in these matrix approximations asymptotically have constant diagonals. However, this is impossible even for matrix algebras with matrix units as generators: there are very few projections with integral entries, and such projections do not have constant diagonals unless they are scalars. For similar reasons, it is too much to ask for tracial $*$-algebras which are only slight modifications of group algebras, such as group algebras twisted by an S^1-valued 2-cocycle or group measure-space constructions.

We relax this naïve approach by only requiring our matrix approximations to have algebraic integer entries. In order to obtain a lower bound on the pseudo-determinant of these approximations (and thus a lower bound on the Fuglede–Kadison pseudo-determinant of the limiting operator), we use an algebraic number theory argument analogous to [Tho08b, Theorem 4.3] which considers the Galois conjugates of a matrix with algebraic integer entries and converts upper bounds on the number of such Galois conjugates and of the operator norm of these conjugates into lower bounds on the pseudo-determinant. We call matrix approximations *Galois bounded microstates* when they have algebraic integer entries, a uniform bound on the number of their Galois conjugates, and a uniform bound on the operator norm of these Galois conjugates (see Definition 3.2 for the precise definition). These are the key to our notion of algebraic soficity: a tracial $*$-algebra (A, τ) is *algebraically sofic* when it has a generating tuple x which admits Galois bounded microstates with asymptotically constant diagonals. It turns out that all finite dimensional tracial *-algebras are algebraically sofic (see Theorem 3.11). From the above discussion, we realize that our proof of Theorem A1 reduces to the following two results.

Theorem B (Theorem 3.4)**.** *If (M, τ) is a tracial von Neumann algebra, and x is a generating tuple for M with a Galois bounded sequence of microstates, then for any matrix polynomial in x with algebraic coefficients, the Fuglede–Kadison pseudo-determinant is positive.*

The proof of Theorem B follows by adapting methods of Thom [Tho08b, Theorem 4.3]. In order to prove that graph products of finite dimensional algebras have Galois bounded microstates, we prove the following.

Theorem C (Theorem 4.2)**.** *Algebraic soficity is preserved by graph products.*

Theorem C is just a restatement of Theorem 4.2, which is proved in Section 4. As a consequence of Theorems C and B, we can replace "finite dimensional" in Theorem A (1) with "algebraically sofic," under a technical condition on traces (see Theorem 5.4 for more details).

Sofic groups themselves have seen numerous applications in recent years: their Bernoulli shift actions can be completely classified (by [Bow10, Bow12, Sew22]); they are known to satisfy the determinant conjecture [ES05](a conjecture arising in the theory of L^2-invariants) and consequently their L^2-torsion is well-defined [Lüc02]; they are known to admit a version of Lück approximation [ES05]; they satisfy Gottschalk's surjunctivity conjecture [Gro99]; and they are known to satisfy Kaplansky's direct finiteness conjecture [ES04]. In fact, any group for which one of these properties is known is also known to be sofic; it is a large open question whether or not every group is sofic. We refer the reader to [Bow18] for further applications of sofic groups, particularly to ergodic theory.

We expect that our new notion of algebraic soficity will have many similar applications in the theory of von Neumann algebras. Motivated by our work, and using our new notion of algebraic soficity, we make the following conjecture.

Conjecture D. *Let (M, τ) be a tracial von Neumann algebra. Assume that M has a weak*-dense, finitely presented, algebraically sofic, unital $*$-subalgebra A. Then (M, τ) is strongly 1-bounded if and only if $\beta^1_{(2)}(A, \tau) = 0$.*

The fact that if $\beta^1_{(2)}(A, \tau) = 0$ and (A, τ) is algebraically sofic, then (M, τ) is strongly 1-bounded is a consequence of [Shl21, Theorem 2.5] (see e.g. the proof of Theorem A (1) in Section 5.1). So the difficulty is in establishing the converse. Partial progress on this has already been made in [Shl09], and as discussed there an inherent part of the difficulty is in exponentiating a derivation to get a one-parameter family of deformations of M which "move in a free direction". This conjecture is already interesting to investigate when (M, τ) is the graph product of finite dimensional tracial von Neumann algebras, and A is the $*$-algebra generated by the vertex algebras.

Remark 1.1. The problem of studying the first L^2-Betti numbers for graph products of finite groups has been studied extensively in [DDJO07, DO01, DO12]. In particular the authors specify that there are algorithms to compute the first L^2 Betti numbers for certain graph products of finite groups. Combining this with [Shl21] should give examples of strongly 1-bounded group von Neumann algebras. One would expect that these algorithms would generalize to the setting of finite dimensional $*$-algebras, in which case one could use them in combination with Theorem A(1) to obtain examples of strongly 1-bounded von Neumann algebras not coming from groups.

One special case of Conjecture D that is worth studying is the case where $M = L(\Gamma)$ is the von Neumann algebra of a group Γ. In this case, we would expect that if Γ is sofic, then $L(\Gamma)$ is strongly 1-bounded if and only if $\beta^1_{(2)}(\Gamma) = 0$. This is of particular interest for graph products of groups, because of the aforementioned results that give an algorithmic approach to computing their first L^2-Betti number. For the special setting of group von Neumann algebras, Conjecture D would follow immediately if the following problem has an affirmative answer.

Problem 1.2. Suppose that Γ is a group with positive first L^2-Betti number. Is it true that $L(\Gamma)$ has a finite index subalgebra which decomposes as a free product of two tracial von Neumann algebras M_1, M_2?

Note that if a tracial von Neumann algebra is a nontrivial free product up to finite index, then it has no Cartan subalgebras ([Jun07], [Ioa15]). Note also that absence of Cartan for various subfamilies of groups with positive first L^2-Betti number has been obtained in the literature (see for instance [CS13, CSU13, Ioa12, PV14a, PV14b, Sin11]), using deformation/rigidity theory. An affirmative answer to Problem 1.2 would of course be a surprising structural property of group von Neumann algebras with positive first L^2-Betti number. However it is not a possibility that should be ruled out.

Acknowledgements. We thank Dimitri Shlyakhtenko for lively discussions; and we thank IPAM and the Lake Arrowhead Conference Center for hosting the Quantitative Linear Algebra long program second reunion conference, where some of these discussions took place. We thank the American Institute of Mathematics SQuaRES program for hosting us for a week each in April 2022 and April 2023 to collaborate on this project. BH was supported by the NSF grant DMS-2000105. IC was supported by long term structural funding in the form of a Methusalem grant from the Flemish Government. DJ was supported by postdoctoral fellowship from the National Science Foundation (DMS-2002826). BN was supported by NSF grant DMS-1856683.

Contents

2. Preliminaries

Definition 2.1. A *(simple undirected) graph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a finite set consisting of *vertices*, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is a set of *edges*. We insist that $\mathcal{E}$ is symmetric (i.e., $(x, y) \in \mathcal{E}$ if and only if $(y, x) \in \mathcal{E}$) and non-reflexive (i.e., $(x, x) \notin \mathcal{E}$ for any $x \in \mathcal{V}$; that is, we do not allow self-loops).

An *(undirected) multigraph* is the same except $\mathcal{E}$ is a multiset, allowing parallel edges.

2.1. Graph products.

We will now define the graph product of von Neumann algebras, and some important related notions. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a collection of finite tracial von Neumann algebras (M_v, τ_v) for each $v \in \mathcal{V}$, the graph product will be constructed as finite von Neumann algebra containing a copy of each M_v in such a way that M_v and $M_{v'}$ are in tensor product position if $(v, v') \in \mathcal{E}$, and in free position otherwise.

Definition 2.2. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. We say that a word $(v_1, \ldots, v_n) \in \mathcal{V}^n$ is *$\mathcal{G}$-reduced* provided that whenever $i < k$ are such that $v_i = v_k$, there is some j with $i < j < k$ so that $(v_i, v_j) \notin \mathcal{E}$.

If $(v_1, \ldots, v_n) \in \mathcal{V}^n$ is such a word and $x_i \in M_{v_i}$, then saying the word is *not* $\mathcal{G}$-reduced is exactly saying that two x_i's from the same algebra could be permuted next to each other and multiplied, giving a shorter word.

Definition 2.3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, (M, τ) be a tracial von Neumann algebra, and for each $v \in \mathcal{V}$ let $M_v \subseteq M$ be a von Neumann sub-algebra. Then the algebras $(M_v)_{v\in\mathcal{V}}$ are said to be *$\mathcal{G}$-independent* if: M_v and M_w commute whenever $(v, w) \in \mathcal{E}$; and whenever $(v_1, \ldots, v_n)$ is a $\mathcal{G}$-reduced word and $x_1, \ldots, x_n \in M$ are such that $x_i \in M_{v_i}$ and $\tau(x_i) = 0$, we have $\tau(x_1 \cdots x_n) = 0$.

Conversely, given a collection (M_v, τ_v) of tracial von Neumann algebras, their *graph product* $(M, \tau) = \mathop{\star}_{v\in\mathcal{G}}(M_v, \tau_v)$ is the von Neumann algebra generated by copies of each M_v which are $\mathcal{G}$-independent, so that $\tau|_{M_v} = \tau_v$. (That the graph product exists and is unique was shown in [Mło04]). When the trace is clear from context, we may write simply $M = \mathop{\star}_{v\in\mathcal{G}} M_v$.

Notice that if $\mathcal{G} = (\mathcal{V}, \emptyset)$ then $\mathop{\star}_{v\in\mathcal{G}} M_v = \circledast_{v\in\mathcal{V}} M_v$; on the other hand, if $\mathcal{G}$ is a complete graph, then $\mathop{\star}_{v\in\mathcal{G}} M_v = \bigotimes_{v\in\mathcal{V}} M_v$.

2.2. **Laws in tracial von Neumann algebras.** A tracial von Neumann algebra is a pair (M, τ) where M is a von Neumann algebra and $\tau\colon M \to \mathbb{C}$ is a faithful, normal, tracial state. If $a \in M$ is a normal element, we let μ_a be the Borel probability measure supported on the spectrum of a defined by

$$\mu_a(E) = \tau(1_E(a)) \text{ for all Borel } E \subseteq \mathbb{C}.$$

We then necessarily have that

$$\tau(f(a)) = \int f \, d\mu_a$$

for all complex-valued, bounded Borel functions f defined on the spectrum of a.

Given an integer $r \geq 1$, we let $\mathbb{C}\langle T_1, T_1^*, T_2, \cdots, T_r, T_r^*\rangle$ be the algebra of noncommutative $*$-polynomials in r-variables (i.e. the universal $*$-algebra in r-variables). Given a $*$-algebra A and a tuple $a = (a_1, \cdots, a_r) \in A^r$, and $P \in \mathbb{C}\langle T_1, T_1^*, T_2, \cdots, T_r, T_r^*\rangle$, we use $P(a)$ for the image of P under the unique $*$-homomorphism $\mathbb{C}\langle T_1, T_1^*, T_2, \cdots, T_r, T_r^*\rangle \to A$ which sends T_j to a_j. For later use, if

$$P = (P_{ij})_{1\leq i\leq m, 1\leq j\leq n} \in M_{m,n}(\mathbb{C}\langle T_1, T_1^*, T_2, T_2^*, \cdots, T_r, T_r^*\rangle)$$

we define $P(x) \in M_{m,n}(A)$ by

$$(P(x))_{ij} = P_{ij}(x).$$

If (M, τ) is a tracial von Neumann algebra, and $x \in M^r$ is a tuple, we define its *law* ℓ_x to be the linear functional

$$\ell_x\colon \mathbb{C}\langle T_1, T_1^*, T_2, \cdots, T_r, T_r^*\rangle \to \mathbb{C}$$
$$P \mapsto \tau(P(x)).$$

If $n \in \mathbb{N}$ and $A \in M_n(\mathbb{C})$ then we let ℓ_A be the law of A with respect to the normalized tracial state tr_n on $M_n(\mathbb{C})$, namely

$$\mathrm{tr}_n(A) = \frac{1}{n}\sum_{j=1}^{n} A_{jj}.$$

Suppose we are given a sequence (M_n, τ_n) of tracial von Neumann algebras, and $a_n \in M_n^r$. If (M, τ) is a tracial von Neumann algebra and $a \in M^r$ we say that $\ell_{a_n} \to \ell_a$ if for all $P \in \mathbb{C}\langle T_1, T_1^*, T_2, \cdots, T_r, T_r^*\rangle$ we have

$$\ell_a(P) = \lim_{n\to\infty} \ell_{a_n}(P).$$

Laws are spectral measures are related by the following fact: suppose we are given

- (M_n, τ_n) are tracial von Neumann algebras,
- a $C > 0$ and an integer $r \geq 1$
- a sequence $a_n \in (M_n)^r_{s.a.}$ with $\|a_n\| \leq C$.
- a tracial von Neumann algebra (M, τ) and $a \in M^r_{s.a.}$

Then $\ell_{a_n} \to_{n\to\infty} \ell_a$ in law if and only if for every self-adjoint $P = P^* \in \mathbb{C}\langle T_1, T_1^*, \cdots, T_r, T_r^*\rangle$ we have $\mu_{P(a_n)} \to \mu_{P(a)}$ weak*. Moreover, if $r = 1$ these conditions are equivalent to saying that $\mu_{a_n} \to \mu_a$ weak*. The proof of this fact is an exercise in applying the Stone-Weierstrass theorem.

If (M, τ) is a tracial von Neumann algebra, and $x \in M_{m,n}(M)$, we define the **Fuglede–Kadison pseudo-determinant of** x by

$$\det{}^+_M(x) = \exp\left(n \int_{(0,\infty)} \log t \, d\mu_{|x|}(t)\right),$$

where $|x| = (x^*x)^{1/2}$, and $\mu_{|x|}$ is the spectral measure with respect to $\mathrm{tr}_n \otimes \tau$. Here we are following the usual convention that $\exp(-\infty) = 0$.

2.3. **Galois theory.** We fix some notation and recalling some of the fundamental concepts of Galois theory, specific to algebraic field extensions of $\mathbb{Q}$. Let $\overline{\mathbb{Q}}$ be the algebraic numbers in $\mathbb{C}$, this is a field by [DF04, Corollary 19 in Section 13.2] We will write $\mathcal{O}$ for the algebraic integers in $\mathbb{C}$; recall that $x \in \mathcal{O}$ if there is a monic $p \in \mathbb{Z}[T]$ so that $p(x) = 0$.

The **absolute Galois group of** $\mathbb{Q}$ is defined to be the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of all field automorphisms of $\overline{\mathbb{Q}}$ (note that such automorphisms automatically fix $\mathbb{Q}$). Each $x \in \overline{\mathbb{Q}}$ has finite orbit $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x := \{\sigma(x) : \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$; if we equip these sets with their discrete topologies, then $\prod_{x\in\overline{\mathbb{Q}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x)$ is compact by Tychonoff's Theorem, and contains $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Since $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is closed in this topology, it is a compact group. Note that a sequence $\sigma_n \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ converges to $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ if for every $x \in \overline{\mathbb{Q}}$ we have $\sigma_n(x) = \sigma(x)$ for all sufficiently large n.

Though we will not need it, we remark to the reader that the usual Galois correspondence between subgroups and subfields extends to this setting. Namely, the *closed* normal subgroups are in natural bijection with the Galois extensions of $\mathbb{Q}$, via the correspondence that sends a closed, normal subgroup H of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\mathrm{Fix}_H(\overline{\mathbb{Q}}) = \{x \in \overline{\mathbb{Q}} : \sigma(x) = x \text{ for all } \sigma \in H\}$. This correspondence naturally induces an isomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/H \cong \mathrm{Gal}(\mathrm{Fix}_H(F)/\mathbb{Q})$. In particular, if $[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : H] < +\infty$, then $\mathrm{Fix}_H(\overline{\mathbb{Q}})$ is a finite Galois extension with degree $[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : H]$.

We remind the reader here some of the core results of Galois theory and algebraic number theory, which we will use in Section 3.

(1) If $x \in \overline{\mathbb{Q}}$, then $x \in \mathbb{Q}$ if and only if $\sigma(x) = x$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (see [Lan02, Theorem 1.2]),
(2) the algebraic integers form a subring of $\overline{\mathbb{Q}}$ (see [DF04, Corollary 24 in Section 15.3]),
(3) $\mathcal{O} \cap \mathbb{Q} = \mathbb{Z}$ (see [DF04, Proposition 28 in Section 15.3]).

3. Galois bounded microstates and algebraic soficity

In this section we introduce the concepts of Galois bounded sequences of microstates and algebraic soficity. The motivation is to find an analogue of soficity which is better adapted to $*$-algebras not necessarily arising from groups, which will still be sufficient to give us bounds on certain Fuglede–Kadison pseudodeterminants of operators arising from such algebras.

The generators of a sofic group admit microstates among the permutation matrices, where all the entries are 0 or 1. This suffices to prove that their group algebras satisfy the determinant conjecture [ES05] (in turn implying that L^2-torsion of modules over them is well-defined [Lüc02]), as well as Lück approximation [ES05]. They are thus of fundamental importance in the study of L^2-invariance. We will see that the same sort of control can be obtained when a $*$-algebra has generators admitting microstates whose entries, rather than being integers, are algebraic integers all living in a fixed finite extension of $\mathbb{Q}$. We make these idea precise in Definitions 3.2 and 3.7.

As motivating examples, we show below that $M_n(\mathbb{C})$ is algebraically sofic (despite not being a group von Neumann algebra), as is $L(\Gamma)$ for any sofic group Γ.

3.1. **Galois bounded microstates and the Fuglede–Kadison determinant.** Given $A \in M_N(\overline{\mathbb{Q}})$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we write $\sigma(A)$ to mean the matrix obtained by applying σ to each entry of A. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we let $\widetilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be given by $\widetilde{\sigma}(z) = \overline{\sigma(\overline{z})}$. Note that if $A \in M_N(\overline{\mathbb{Q}})$, then

$$\sigma(A^*) = \widetilde{\sigma}(A)^*. \tag{1}$$

This will be used frequently in this section. For a matrix $A \in M_N(\overline{\mathbb{Q}})$, we use

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot A = \{\sigma(A) : \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}.$$

The following lemma allows us to use number theory to obtain lower estimates on pseudo-determinants of finite dimensional matrices. This lemma will then motivate a special type of microstates approximation sequence whose existence implies positivity of Fuglede–Kadison pseudo-determinants.

Lemma 3.1. *Suppose $A \in M_N(\mathcal{O})$. Set*

$$C = \max_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \|\sigma(A)\|,$$

$$d = \#(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot A)$$

Then

$$\det{}^+(A)^{1/N} \geq C^{-d^2+1}.$$

Proof. Let $\Omega = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot (A^*A)$. Following [Tho08b, Theorem 4.3], set

$$B = \bigoplus_{S \in \Omega} S.$$

Then the characteristic polynomial of B is

$$k_B = \prod_{S \in \Omega} k_S,$$

where k_S is the characteristic polynomial of S. Let $r \in \mathbb{N} \cup \{0\}$ be such that $k_{A^*A}(T) = T^r p$, where $p \in \mathcal{O}[T]$ has $p(0) \neq 0$. Then, for $S \in \Omega$, we have $k_S(T) = T^r p_S(t)$ with $p_S \in \mathcal{O}[T]$, and $p_S(0) \neq 0$. Hence

$$k_B(T) = T^{r\#\Omega} \prod_{T \in \Omega} p_S(T).$$

Set

$$q = \prod_{S \in \Omega} p_S.$$

Since k_B is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and has algebraic integer coefficients, we know that its coefficients are in $\mathbb{Q} \cap \mathcal{O} = \mathbb{Z}$. It follows that $q \in \mathbb{Z}[T]$ as well. Further $q(0) \neq 0$. Thus $q(0) \in \mathbb{Z} \setminus \{0\}$ so that

$$1 \leq |q(0)| = \det{}^+(A)^2 \prod_{S \in \Omega \setminus \{A^*A\}} |p_S(0)|.$$

For $S \in \Omega$, we know that $p_S(0)$ is the product of the nonzero eigenvalues of S. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we define $\widetilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by $\widetilde{\sigma}(z) = \overline{\sigma(\overline{z})}$. Then,

$$\|\sigma(A^*A)\| = \|\widetilde{\sigma}(A)^*\sigma(A)\| \leq C^2.$$

This estimate implies that $|p_S(0)| \leq C^{2N}$ for every $S \in \Omega$. So

$$1 \leq \det{}^+(A)^2 C^{2(\#\Omega - 1)N}.$$

Moreover, (1) implies that

$$\Omega \subseteq \{(\sigma(A))^*\phi(A) : \sigma, \phi \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\} \subseteq \{S_1^* S_2 : S_1, S_2 \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot A\}.$$

Thus $\#\Omega \leq d^2$, and this completes the proof. $\square$

The preceding lemma suggests the following definition.

Definition 3.2. Let $n(k)$ be a sequence of natural numbers. Let $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$. We say that $X^{(k)} \in M_{n(k)}(\mathbb{C})$ is **Galois bounded** if

- the entries of $X^{(k)}$ are algebraic integers;
- $\sup_k \max_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \|\sigma(X^{(k)})\| < +\infty$; and
- $\sup_k \#(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot X^{(k)}) < +\infty$.

If $X^{(k)} \in M_{n(k)}(\mathbb{C})^r$ we say that it is **Galois bounded** if $(X_j^{(k)})_{k=1}^{\infty}$ is Galois bounded for all $j = 1, \cdots, r$. If (M, τ) is a tracial von Neumann algebra, and if $x \in M^r$ has $\ell_{X^{(k)}} \to \ell_x$, then we say that $X^{(k)}$ are a **Galois bounded sequence of microstates** for x.

Recalling the correspondence between finite Galois extensions and finite index normal subgroups of the absolute Galois group discussed in §2.3, one can rephrase being Galois bounded in the following way. A sequence $(X^{(k)})_{k=1}^{\infty} \in \prod_k M_{n(k)}(\overline{\mathbb{Q}})$ is Galois bounded if and only if there is a sequence F_k of subfields of $\mathbb{C}$ which are finite Galois extensions of $\mathbb{Q}$ such that:

- we have $X^{(k)} \in M_{n(k)}(F_k \cap \mathcal{O})$,
- $\sup_k [F_k : \mathbb{Q}] < +\infty$,
- $\sup_k \max_{\sigma \in \mathrm{Gal}(F_k/\mathbb{Q})} \|\sigma(X^{(k)})\| < +\infty$.

In fact, it is possible to rephrase all of our proofs in this framework without any reference to the absolute Galois group. However, phrasing everything in terms of the absolute Galois group makes the setup cleaner and simplifies the proofs of closure of Galois bounded elements under various operations such as multiplication, adjoints, and conjugation by permutations.

Proposition 3.3. *Let S be the set of Galois bounded sequences in $\prod_k M_{n(k)}(\mathbb{C})$.*

(1) S is a subring of $\prod_k M_{n(k)}(\mathcal{O})$ which is closed under adjoints, and contains all sequences of the form $(\alpha 1_{n(k)})_{k=1}^{\infty}$ for $\alpha \in \mathcal{O}$.
(2) S is invariant under the conjugation action of $\prod_k S_{n(k)}$ on $\prod_k M_{n(k)}(\overline{\mathbb{Q}})$.
(3) Suppose $X = (X_k)_k \in S$ and $m(k)$ any sequence of integers. If $(Y_k)_k \in \prod_k M_{m(k)}(\overline{\mathbb{Q}})$ is Galois bounded, we have that $(X_k \otimes Y_k)_k$ is Galois bounded.
(4) If $(X^{(k)})_k \in S^r$, then for all $P \in \overline{\mathbb{Q}} \langle T_1, T_1^*, \cdots, T_r, T_r^* \rangle$ we have

$$\liminf_{k \to \infty} \det{}^+(P(X^{(k)}))^{1/n(k)} > 0.$$

Proof. (1): That the norm boundedness condition is closed under sums and products follows from the facts $\mathcal{O}$ is a ring, that each σ induces an automorphism of $M_{n(k)}(\mathcal{O})$, and that the operator norm is submultiplicative. For the last condition, note that for $A, B \in S$, we have that

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot (A^{(k)} B^{(k)}) \subseteq \{\sigma(A^{(k)}) \phi(B^{(k)}) : \sigma, \phi \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\},$$

with a similar result for the sum. That S contains all constant algebraic integer multiples of the identity is an exercise. Finally, to see that S is closed under adjoints, let $A = (A^{(k)})_{k=1}^{\infty} \in S$. Equation (1) implies that $((A^{(k)})^*)_{k=1}^{\infty} \in S$. The desired result follows.

(2): This follows from the fact the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $M_{n(k)}(\mathcal{O})$ commutes with the conjugation action of $S_{n(k)}$.

(4): By scaling, we may assume that $P \in \mathcal{O} \langle T_1, T_1^* \cdots, T_r, T_r^* \rangle$. By (1), we know that $(P(X^{(k)}))_{k=1}^{\infty} \in S$. Set

$$C = \sup_k \max_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \|\sigma(P(X^{(k)}))\| < +\infty,$$

$$d = \sup_k \# \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot P(X^{(k)}) < +\infty.$$

For each k, we then have by Lemma 3.1,

$$\det{}^+(P(X^{(k)}))^{1/n(k)} \geq C^{-d^2+1}.$$

Taking limit infimums of both sides completes the proof.

(3): Using (1) we may reduce to the case that Y_k is the $m(k) \times m(k)$ identity matrix. This case is an exercise using, for example, that

$$\sigma \cdot (A \otimes 1_m) = (\sigma(A) \otimes 1_m)$$

for $m, n \in \mathbb{N}$ and $A \in M_n(\overline{\mathbb{Q}})$.

□

We now obtain Theorem B from the introduction as a corollary of Proposition 3.3.

Theorem 3.4 (Theorem B). *If (M, τ) is a tracial von Neumann algebra, and x is a generating tuple for M with a Galois bounded sequence of microstates, then for any matrix polynomial in x with algebraic coefficients, the Fuglede–Kadison pseudo-determinant is positive.*

Proof. Suppose that $x = (x_1, \cdots, x_r)$, and let $P \in M_{m,n}(\overline{\mathbb{Q}}\langle T_1, \cdots, T_r\rangle)$. Since

$$P^*P \in M_n(\overline{\mathbb{Q}}\langle T_1, T_1^*, \cdots, T_r, T_r^*\rangle)$$

we may, and will, assume that case $m = n$. Let E_{ij} be the standard matrix units in $M_n(\mathbb{C})$. It follows by Proposition 3.3 (3) that the new tuple

$$\widetilde{x} = ((x_l \otimes E_{ij})_{1 \leq i,j \leq n, 1 \leq l \leq r}, (1 \otimes E_{ij})_{1 \leq i,j \leq n})$$

has a Galois bounded sequence of microstates. Let

$$I = [r] \times [n] \times [n] \sqcup [n] \times [n].$$

For $1 \leq i, j \leq n$ we use $(\varnothing, i, j)$ for the element of I which correspond to (i, j) in the copy of $[n] \times [n]$ inside I. Suppose $P = \sum_{i,j} P_{ij} \otimes E_{ij}$, then as

$$P(x) = \sum_{ij} P_{ij}((x_l \otimes E_{ii})_{l=1}^r)(1 \otimes E_{ij}),$$

we have

$$P(x) = \sum_{i,j} Q_{ij}(\widetilde{x}),$$

where $Q_{ij} \in \overline{\mathbb{Q}}\left\langle T_\beta, T_\beta^* : \beta \in I\right\rangle$ is given by

$$Q_{ij} = P_{ij}((T_{(l,i,i)})_{l=1}^r)T_{\varnothing,i,j}.$$

This construction allows us to reduce to $n = 1$, by replacing x with $\widetilde{x}$. Hence we may, and will, assume that $n = 1$.

Let $(X^{(k)})_k$ be a Galois bounded sequence of microstates for x. The fact that $(X^{(k)})_k$ are microstates for x implies that $\mu_{|P(X^{(k)})|} \to \mu_{|P(x)|}$ weak*. Thus, by weak*-semicontinuity of integrating the logarithm and (4),

$$\begin{aligned}\log \det{}_M^+(P(x)) = \int_{(0,\infty)} \log(t)\, d\mu_{|P(x)|}(t) &\geq \liminf_{k\to\infty} \int_{(0,\infty)} \log(t)\, d\mu_{|P(X^{(k)})|}(t) \\ &= \liminf_{k\to\infty} \log \det{}^+(P(X^{(k)}))^{1/n(k)} > -\infty.\end{aligned}$$

□

For later use, we record the fact that the existence of Galois boundedness passes to direct sums.

Lemma 3.5. *Let $(M_j, \tau_j), j = 1, 2$ be tracial von Neumann algebras. Suppose that $x_j \in M_j^{r_j}$ for some r_1, r_2 and each $j = 1, 2$. Suppose that $(n_j(k))_{k=1}^\infty$ are sequences of natural numbers for $j = 1, 2$. Assume we are given for $j = 1, 2$ microstates sequences $X_j^{(k)} \in M_{n_j(k)}(\mathbb{C})^{r_j}$ for x_j. Finally, assume that $(t_{k,j})_{k=1}^\infty, j = 1, 2$ are sequence of integers so that*

$$\alpha = \lim_{k\to\infty} \frac{t_{k,1}n_1(k)}{t_{k,1}n_1(k) + t_{k,2}n_2(k)}$$

exists. Then $((X_1^{(k)})^{\oplus t_{k,1}} \oplus 0, 0 \oplus (X_2^{(k)})^{\oplus t_{k,2}})$ converges in law to the law of $(x_1 \oplus 0, 0 \oplus x_2)$ regarded as an element in $\alpha(M_1, \tau_1) \oplus (1-\alpha)(M_2, \tau_2)$. In particular, if there are Galois bounded *sequences of microstates for x_1, x_2 then for every $0 \leq \alpha \leq 1$, there are Galois bounded sequence of microstates for $(x_1 \oplus 0, 0 \oplus x_2)$ regarded as an element in $\alpha(M_1, \tau_1) \oplus (1-\alpha)(M_2, \tau_2)$.*

3.2. **Algebraic soficity.**

Definition 3.6. A sequence of matrices $X^{(k)} \in M_{n(k)}(\mathbb{C})$ is called **asymptotically constant on the diagonal** if

$$\lim_{k\to\infty} \left\| \Delta_{n(k)}[X^{(k)}] - \mathrm{tr}_{n(k)}[X^{(k)}]1 \right\|_2 = 0,$$

where $\Delta_{n(k)}$ is the conditional expectation onto the diagonal subalgebra of $M_{n(k)}(\mathbb{C})$.

Definition 3.7. Given a tracial von Neumann algebra (M, τ) we say that a tuple $x = (x_i)_{i \in I}$ in M^I is **algebraically sofic** if for any finite $F \subseteq I$, there is a sequence of microstates $(X_i^{(k)})_{i \in F}$ for $x\big|_F$ that is Galois bounded (Definition 3.2), such that $P(X^{(k)})$ is asymptotically constant on the diagonal for every $*$-polynomial P. We say that M is **algebraically sofic** if it has an algebraically sofic generating tuple.

If (M, τ) is a tracial von Neumann algebra, and $x \in M^I$ is algebraically sofic, we remark that for any set J and any $P \in \mathcal{O}\langle T_i, T_i^* : i \in I\rangle^J$ we have that $P(x)$ is algebraically sofic. The name derives from the case of soficity of groups, as defined by Gromov [Gro99] and named by Weiss [Wei00]. Soficity can be phrased in terms of microstates: a group Γ is sofic if and only for every finite $F \subseteq \Gamma$ there is a sequence $\sigma_k \in S_{n(k)}^F$ which, when viewed as matrices, form microstates for F. If we equip $S_{n(k)}$ with the metric

$$d(\sigma, \pi) = \frac{1}{n(k)} |\{j : \sigma(j) \neq \pi(j)\}|,$$

then if Σ, Π are the matrices corresponding to σ, π we have

$$d(\sigma, \pi) = \frac{1}{2} \|\Sigma - \Pi\|_2^2.$$

If $F \subseteq \Gamma$ is finite, with $e \in F$ and if $\sigma_k \in S_{n(k)}^F$ is a microstates sequence for F, then for every $g \in F$ we have

$$\mathrm{tr}(\sigma_{k,g}) \to_{k\to\infty} \delta_{g=e}$$

Since

$$\|\Delta_{n(k)}(\sigma_{k,g}) - \mathrm{tr}(\sigma_{k,g})\|_2^2 = \|\Delta_{n(k)}(\sigma_{k,g})\|_2^2 - \mathrm{tr}(\sigma_{k,g})^2 = \mathrm{tr}(\sigma_{k,g})(1 - \mathrm{tr}(\sigma_{k,g}))$$

being a sequence of microstates forces σ_k to be asymptotically constant on the diagonal. Thus soficity of Γ implies that every tuple in Γ is algebraically sofic, when we view $\Gamma \leq \mathcal{U}(L(\Gamma))$. We record this observation in the following proposition.

Proposition 3.8. *If Γ is a sofic group, then $\{u_g : g \in \Gamma\} \subseteq L(\Gamma)$ is algebraically sofic. In particular, $L(\Gamma)$ is algebraically sofic.*

In the definition of algebraic soficity, we retain having asymptotically constant diagonals, but we relax the requirement of being a permutation (ill-adapted to a nongroup setting). We instead require entries which are algebraic integers and whose entries have a "size of integrality" (both in absolute value and in terms of how large of a field extension they live) that is controlled. The intuition behind this relaxation is that the fact that permutation matrices have integer entries, and the integrality of permutations is used in the proofs of many applications of soficity.

We want to show that finite-dimensional tracial $*$-algebras are algebraically sofic, and to this end, we first show that $M_n(\mathbb{C})$ is algebraically sofic using the following group-measure-space construction.

Proposition 3.9. *Let Γ be a finite abelian group. Let $(u_\chi)_{\chi \in \widehat{\Gamma}}$ be the canonical unitaries in $L(\widehat{\Gamma})$. Consider the action α of Γ on $L(\widehat{\Gamma})$ by $\alpha_g(u_\chi) = \chi(g)^{-1} u_\chi$ for all $g \in \Gamma, \chi \in \widehat{\Gamma}$.*

(1) We have $L(\widehat{\Gamma}) \rtimes \Gamma \cong M_{|\Gamma|}(\mathbb{C})$.

(2) *Endow* $L(\widehat{\Gamma}) \rtimes \Gamma$ *with its unique tracial state* τ. *For* $g \in \Gamma$, *let* v_g *be the canonical unitaries in* $L(\widehat{\Gamma}) \rtimes \Gamma$ *implementing the action of* Γ. *Let*

$$\pi \colon L(\widehat{\Gamma}) \rtimes \Gamma \to B(L^2(L(\widehat{\Gamma}) \rtimes \Gamma))$$

be the GNS representation coming from τ. *Then:*

(a) $\{u_\chi v_g \colon \chi \in \widehat{\Gamma},\ g \in G\}$ *is an orthonormal basis of* $L^2(L(\widehat{\Gamma}) \rtimes \Gamma)$;

(b) *if* D *is the MASA in* $B(L^2(L(\widehat{\Gamma}) \rtimes \Gamma))$ *generated by the rank one projections onto* $\mathbb{C}u_\chi v_g$, *for* $\chi \in \widehat{\Gamma}$ *and* $g \in \Gamma$, *then*

$$\mathbb{E}_D \circ \pi = \tau;$$

(c) *the matrix entries of* $\pi(u_\chi v_g)$ *with respect to* $(u_\theta v_h)_{\theta \in \Gamma, h \in G}$ *are elements of* $\{0\} \cup \{\phi(k) : \phi \in \widehat{\Gamma}, k \in \Gamma\}$.

(d) *For* $1 \leq i, j \leq |\Gamma|$, *let* E_{ij} *be the standard matrix units of* $M_{|\Gamma|}(\mathbb{C})$. *Let* $K = \{\phi(k) : \phi \in \widehat{\Gamma}, k \in \Gamma\}$. *Then the isomorphism given in* (1) *can be chosen so that the matrix entries of* $\pi(E_{ij})$ *with respect to* $(u_\theta v_h)_{\theta \in \Gamma, h \in \Gamma}$ *lie in* $\frac{1}{|\Gamma|}\mathbb{Z}[K]$, *for* $1 \leq i, j \leq \Gamma$.

Proof. (1): The Fourier transform induces an isomorphism $L(\widehat{\Gamma}) \cong \ell^\infty(\Gamma)$ which conjugates the action of Γ on $L(\widehat{\Gamma})$ to the shift action of Γ on $\ell^\infty(\Gamma)$. This induces an isomorphism $\ell^\infty(\Gamma) \rtimes \Gamma \cong L(\widehat{\Gamma}) \rtimes \Gamma$, where the action of Γ on $\ell^\infty(\Gamma)$ is the shift action. The algebra $\ell^\infty(\Gamma) \rtimes \Gamma$ is generated by the family of matrix units $\{\delta_g u_{gh^{-1}} \delta_h \colon g, h \in \Gamma\}$ and is therefore isomorphic to $M_{|\Gamma|}(\mathbb{C})$.

(2): The fact that $\{u_\chi v_g \colon \chi \in \hat{\Gamma},\ g \in \Gamma\}$ are pairwise orthogonal is a direct computation. We leave it as an exercise to verify that

$$\langle \pi(u_\chi v_g) u_\theta v_h, u_\phi v_k \rangle = \theta(g) \delta_{\chi\theta = \phi} \delta_{gh = k},$$

for all $\chi \in \widehat{\Gamma}, g \in \Gamma$. This implies that

$$\mathbb{E}_D(\pi(u_\chi v_g)) = \tau(u_\chi v_g) 1$$

for all $\chi \in \widehat{\Gamma}, g \in \Gamma$. Since such elements span $L(\widehat{\Gamma}) \rtimes \Gamma$, it follows that $\mathbb{E}_D \circ \pi = \tau$. Part (**??**) follows from the above computation For part (2d), note that the fact that $(u_\theta v_h)_{\theta,h}$ is an orthonormal basis implies that

$$E_{ij} = \sum_{\chi, g} \mathrm{tr}(E_{ij}(u_\chi v_g)^*) u_\chi v_g.$$

As shown above, the matrix entries of $\pi(u_\chi v_g)$ with respect to the basis $(u_\theta v_h)_{\theta,h}$ are in $\mathbb{Z}[K]$, so the above expansion completes the proof. □

We start by recording how algebraic soficity behaves under tensor products.

Proposition 3.10. *For* $j = 1, 2$ *let* (M_j, τ_j) *be tracial von Neumann algebras and* $x_j \in M_j^{r_j}$ *algebraically sofic tuples. Then* $(x_1 \otimes 1, 1 \otimes x_2)$ *is algebraically sofic.*

Proof. Let $X_j^{(k)} \in M_{n_j(k)}(\mathcal{O})$ be Galois bounded microstates for x_j so that polynomials in $X_j^{(k)}$ are asymptotically constant on the diagonal. By Proposition 3.3 (3) we know that $(X_1^{(k)} \otimes 1_{n_2(k)}, 1_{n_1(k)} \otimes X_2^{(k)})$ is Galois bounded. Monomials in $(X_1^{(k)} \otimes 1_{n_2(k)}, 1_{n_1(k)} \otimes X_2^{(k)})$ are asymptotically constant on the diagonal, and thus polynomials in $(X_1^{(k)} \otimes 1_{n_2(k)}, 1_{n_1(k)} \otimes X_2^{(k)})$ are asymptotically constant on the diagonal. □

This result on tensor products can also be used to show that algebraic soficity is preserved under finite direct sums with rational weights. We show in the next section that the direct sum of two algebraically sofic algebras without rational weights can fail to be algebraically sofic (see Corollary 3.17).

Theorem 3.11. *Suppose that (M_j, τ_j), $j = 1, 2$ are algebraically sofic, and let $q \in (0,1) \cap \mathbb{Q}$. Let $M = M_1 \oplus M_2$ equipped with the trace*

$$\tau(a_1, a_2) = q\tau_1(a_1) + (1 - q)\tau_2(a_2).$$

Then (M, τ) is algebraically sofic. In particular, finite dimensional tracial von Neumann algebras where every central projection has trace in $\mathbb{Q}$ are algebraically sofic.

Proof. The "in particular" part follows from the fact that every finite-dimensional von Neumann algebra is a direct sum of matrix algebras.

Note that $\{\phi(k) : k \in \mathbb{Z}/n\mathbb{Z}, \phi \in \widehat{(\mathbb{Z}/n\mathbb{Z})}\} = \mathbb{Z}[e^{2\pi i/n}]$, and $e^{2\pi i/n}$ is algebraic integer. So the fact that $M_n(\mathbb{C})$ is algebraically sofic follows from Proposition 3.9 applied to $\Gamma = \mathbb{Z}/n\mathbb{Z}$. For later use, we note the following specific consequence. For $1 \le i, j \le n$, let E_{ij} be the standard matrix units of $M_n(\mathbb{C})$. Then Proposition 3.9 shows that $(nE_{i,j})_{i,j}$ is an algebraically sofic tuple.

Now let $(A_j, \tau_j)_{j=1,2}$ be tracial $*$-algebras. Let $s_j \in A_j^{r_j}$ be a generating tuple for $A_j, j = 1, 2$ such that there exists Galois bounded microstates $(X_j^{(k)})_{k=1}^{\infty} \in \prod_k M_{n_j(k)}(\mathbb{C})$ as in the definition of algebraic soficity.

Let $A = A_1 \oplus A_2$ be endowed with the trace

$$\tau(a_1, a_2) = t\tau_1(a_1) + (1 - t)\tau_2(a_2)$$

for some $t \in \mathbb{Q} \cap (0, 1)$. Write $t = \frac{k}{n}$ with $n \in \mathbb{N}$ and $0 < k < n$. We use the embedding

$$\pi \colon A_1 \oplus A_2 \to M_n(\mathbb{C}) \otimes A_1 \otimes A_2$$

given by $\pi(a_1, a_2) = \left(\sum_{i=1}^{k} E_{ii}\right) \otimes a_1 \otimes 1 + \left(\sum_{i=k+1}^{n} E_{ii}\right) \otimes 1 \otimes a_2$. It thus suffices to note that Propositions 3.10 and 3.3 (1) implies that

$$n\left(\left(\sum_{i=1}^{k} E_{ii}\right) \otimes s_1 \otimes 1, n\left(\sum_{i=k+1}^{n} E_{ii}\right) \otimes 1 \otimes s_2\right)$$

is an algebraically sofic tuple. □

3.3. **Tracial $*$-algebras which are not algebraically sofic.** In this section, we show that certain $*$-algebras can fail to be algebraically sofic. In fact, we show that any self-adjoint element which is algebraically sofic (regarded as a 1-tuple) must have transcendental trace. Using, we can show that if we equip $A = M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C})$ with a trace which has a central projection with transcendental trace, then every algebraic sofic element of A must be a scalar multiple of the identity.

Our starting point is the following result of Thom.

Lemma 3.12 (Lemma 3.1 of [Tho08b])**.** *Fix $k \in \mathbb{N}$ and $C \in [0, +\infty)$. Let $T_{k,c}$ be the set of polynomials in $\mathbb{Z}[t]$ of degree at most k and whose roots in $\mathbb{C}$ all have modulus at most C. Then $T_{k,C}$ is finite.*

For our purposes, it will be best to rephrase this as follows.

Lemma 3.13. *Fix $k \in \mathbb{N}$ and $C \in [0, +\infty)$. Let $S_{k,C}$ be the set of algebraic integers in $\mathbb{C}$ which have at most k Galois conjugates, all of which have modulus at most C. Then $S_{k,c}$ is finite.*

Proof. Let $T_{k,c}$ be as in Lemma 3.12. Then

$$S_{k,c} = \bigcup_{p \in T_{k,c}} p^{-1}(\{0\}),$$

so $S_{k,C}$ is a finite union of finite sets.

□

Corollary 3.14. *Let (M, τ) be a tracial von Neumann algebra and suppose that $x \in M_{s.a.}$ is algebraically sofic. Then $\tau(x)$ is an algebraic integer.*

Proof. Let $S_{k,c}$ be as in Lemma 3.13. Let $X^{(N)} \in M_{k(N)}(\mathcal{O})$ be a Galois bounded sequence of microstates which witness that x is algebraically sofic. Since $X^{(N)}$ has asymptotically constant diagonal entries and the average of these entries converges to $\tau(x)$ we may choose $j(N) \in \{1, \cdots, K(N)\}$ with

$$X^{(N)}_{j(N),j(N)} \to_{N\to\infty} \tau(x).$$

By definition of Galois boundedness, there is a $C \in [0, +\infty)$ and a $k \in \mathbb{N}$ with $X^{(N)}_{j(N),j(N)} \in S_{k,C}$ for all N. By Lemma 3.13, we have that $\tau(x) \in S_{k,C}$ and so $\tau(x)$ is an algebraic integer.

□

Theorem 3.15. *Let (M, τ) be a tracial von Neumann algebra and $x \in M$ algebraically sofic and self-adjoint. Then all the eigenvalues of x are algebraic integers.*

Proof. Let $X^{(N)} \in M_{k(N}(\mathcal{O})$ be a sequence of Galois bounded microstates which witness algebraic soficity. By passing to a subsequence, we may assume that there is an $r \in \mathbb{N}$ with

$$r = |\{\sigma(X^{(N)}) : \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}.$$

For each $N \in \mathbb{N}$, choose $\sigma_{0,N}, \cdots, \sigma_{r-1,N} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma_{0,N} = \mathrm{id}$ and

$$\{\sigma(X^{(N)}) : \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\} = \{\sigma_{j,N}(X^{(N)}) : j \in \{0, \cdots, r-1\}\}.$$

Set

$$Y^{(N)} = \bigoplus_{j=0}^{r-1} \sigma(Y^{(j)}).$$

Passing to a further subsequence we may assume that $\mu_{Y^{(N)}}$ weak*-converges to a probability measure μ. Let $\mu_{X^{(N)}}, \mu_{Y^{(N)}}$ be the spectral measures of $X^{(N)}, Y^{(N)}$. Since the characteristic polynomial of $Y^{(N)}$ is invariant under the absolute Galois group, we know it is an integer and thus $\mu_{Y^{(N)}}$ is an atomic measure supported on algebraic integers, and by Galois boundedness it is supported in a uniformly bounded set. Thus μ is an integer measure in the sense of [Tho11]. Let μ_x be the spectral measure of x. Since $\mu_{X^{(N)}} \le r\mu_{Y^{(N)}}$ for every N we have that $\mu_x \le r\mu$. If $\lambda \in \mathbb{C}$ is not an algebraic integer, then since μ is an integer measure it follows from [Tho11, Theorem 2.8] that

$$\mu_x(\{\lambda\}) \le r\mu(\{\lambda\}) = 0.$$

□

Corollary 3.16. *Let (M, τ) be a tracial von Neumann algebra and $x \in M$ algebraically sofic and self-adjoint. If the spectral measure μ_x of x is atomic, then $\mu_x(\{\lambda\})$ is algebraic for every $\lambda \in \mathbb{C}$.*

Proof. The case where λ is transcendental follows from the above Theorem. So suppose that λ is algebraic. Define a polynomial

$$F_\lambda(t) = \prod_{\beta \in \mathrm{spec}(x), \beta \neq \lambda} (t - \beta).$$

Note that F_λ has algebraic coefficients. Then

$$1_\lambda(x) = \prod_{\beta \in \mathrm{spec}(x), \beta \neq \lambda} (\lambda - \beta)^{-1} F_\lambda(x).$$

So

$$\mu_x(\{\lambda\}) = \prod_{\beta \in \mathrm{spec}(x), \beta \neq \lambda} (\lambda - \beta)^{-1} \tau(F_\lambda(x)).$$

We have that $\tau(F_\lambda(x)) \in \overline{\mathbb{Q}}$ by the preceding theorem, since x^k is algebraically sofic for all $k \in \mathbb{N} \cup \{0\}$.

□

Corollary 3.17. *Suppose that* $k_1, k_2 \in \mathbb{N}$ *and that* $\gcd(k_1, k_2) = 1$. *Let* $\alpha \in \mathbb{C}$ *be transcendental. Let* $A = M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C})$ *equipped with a trace*

$$\tau(x_1, x_2) = \alpha \operatorname{tr}(x_1) + (1 - \alpha) \operatorname{tr}(x_2).$$

If $x \in A$ *is algebraically sofic with respect to* τ, *then* $x \in \mathbb{C}1$. *In particular,* A *is not algebraically sofic.*

Proof. Since $x + x^*$ and $i(x - x^*)$ are algebraically sofic if x is, we may assume that x is self-adjoint. Consider the spectral measure μ_x of x. Write $x = (x_1, x_2)$. By Theorem 3.15 and Corollary 3.16, μ_x is an atomic measure concentrated on algebraic integers and $\mu_x(\{\lambda\})$ is algebraic for every $\lambda \in \mathbb{C}$. Let $\lambda \in \mathbb{C}$, and let

$$t_i = \frac{\dim(\ker(x_i - \lambda))}{k_i} \in \mathbb{Q}.$$

Then

$$\mu_x(\{\lambda\}) = \alpha(t_1 - t_2) + t_2.$$

Note that α is transcendental, whereas $t_1, t_2, \mu_x(\{\lambda\})$ are algebraic. Since algebraic numbers form a field, this forces $t_1 = t_2$. Our assumptions on k_1, k_2 thus forces that either

$$\frac{\dim(\ker(x_1 - \lambda))}{k_1} = \frac{\dim(\ker(x_2 - \lambda))}{k_2} = 0$$

or

$$\frac{\dim(\ker(x_1 - \lambda))}{k_1} = \frac{\dim(\ker(x_2 - \lambda)}{k_2} = 1.$$

Since this holds for all λ and μ_x is a probability measure, this forces μ_x to be a Dirac mass. Thus $x \in \mathbb{C}1$. □

4. Algebraic soficity preserved by graph products

In this section, we show that the graph product of algebraically sofic tracial ∗-algebras is algebraically sofic. In order to obtain the Galois bounded microstates for the graph product from Galois bounded microstates for the individual algebras, we use a construction based on conjugation by random permutation matrices from [CdSH+] (stated as Theorem 4.1 below); this is the analog of Charlesworth and Collins' construction in the unitary case [CC21], and the proof uses a similar technique as in the free case studied by [ACD+21].

To model graph products, we will need to force certain matrices to commute with each other, and certain matrices to be asymptotically free. As in [CC21], we will accomplish this by taking the models in a tensor product of several copies of $M_N(\mathbb{C})$, with matrices having only scalar components in certain tensor factors; in this way we can ensure that matrices which are meant to commute do so. Heuristically, the index set of this tensor product will be a finite set of *strings.* Given a subalgebra of this larger product formed by replacing some of the tensor factors with copies of $\mathbb{C}I_N$, we will think of its elements as corresponding to collections of beads on the strings where the algebra has a non-trivial factor. Two algebras commute, then, if the beads representing their elements can slide past each other on this collection of strings. For more detail on this picture, refer to [CC21, §3.2] or more generally [CN10].

The information of which tensor factors of a matrix are allowed to be non-scalar is determined by the vertex it corresponds to. We will choose our set of strings and the assignments of vertices to sets of strings in such a way that matrices will share a string in common precisely when the graph product structure insists that the algebras they are modelling should be freely independent. Given a prescribed finite graph $\mathcal{G}$ it is always possible to choose a set $\mathcal{S}$ and a relation -• with this with this property; one approach was given in [CC21, Section 3.1].

The matrices produced by our construction will all live in $M_N(\mathbb{C})^{\otimes \mathcal{S}}$. The inputs to the construction are deterministic matrices $X_j^{(N)}$ which are each assigned a certain vertex $\chi(j)$, such that $X_j^{(N)}$ is $M_N(\mathbb{C})^{\otimes \mathcal{S}_v}$, viewed as a subalgebra of $M_N(\mathbb{C})^{\otimes \mathcal{S}}$ in the standard way. Each matrix $X_j^{(N)}$ with $\chi(j) = v$ will be conjugated by a random permutation matrix $\Sigma_v^{(N)}$ in $\bigotimes_{\mathcal{S}_v} M_N(\mathbb{C})$

to produce a new random matrix $\underline{\underline{X}}_j^{(N)}$ in $M_N(\mathbb{C})^{\otimes \mathcal{S}}$. When we apply this construction in the proof of Theorem C, $X_j^{(N)}$ will be a matrix approximation for some element of $A_{\chi(j)}$, more specifically some polynomial evaluated on microstates for our chosen generators of $A_{\chi(j)}$.

Theorem 4.1 is a statement about certain polynomials in $\underline{\underline{X}}_j^{(N)}$ given by $\mathcal{G}$-reduced words with respect to the graph $\mathcal{G}$ (see Definition 2.2). The following theorem is a special case and slight reformulation of the main theorem of [CdSH+].

Theorem 4.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a simple graph with vertex set $\mathcal{V}$, $\mathcal{S}$ be a finite set, and $\multimap$ be as above so that $\mathcal{S}_v \cap \mathcal{S}_{v'} = \varnothing$ if and only if $(v, v') \in \mathcal{E}$.*

For $N \in \mathbb{N}$, let $\Delta_{N^{\#\mathcal{S}}}$ be the conditional expectation onto the diagonal $*$-subalgebra D_N of $\bigotimes_{\mathcal{S}} M_N(\mathbb{C})$.

Let N_k be a sequence of natural numbers with $N_k \to \infty$. Let $\chi : [m] \to \mathcal{V}$ be such that $\chi(1) \cdots \chi(m)$ is a $\mathcal{G}$-reduced word, and for $i = 1, \dots, m$ and $k \in \mathbb{N}$, let $X_i^{(k)} \in \bigotimes_{\mathcal{S}_{\chi(i)}} M_{N_k}(\mathbb{C})$ be a deterministic matrix, with $\sup_{k,i,j} \left\| X_i^{(k)} \right\| < \infty$.

Further, let $\left\{ \Sigma_v^{(N)} : v \in \mathcal{V} \right\}$ be a family of independent uniformly random permutation matrices, with $\Sigma_v \in \bigotimes_{\mathcal{S}_v} M_N(\mathbb{C})$, and write

$$\underline{\underline{X}}_i^{(k)} = \left(\Sigma_{\chi(i)}^{(k)} \right)^* X_i^{(k)} \Sigma_{\chi(i)}^{(k)} \otimes I_{N_k}^{\otimes \mathcal{S} \setminus \mathcal{S}_{\chi(i)}} \in \bigotimes_{\mathcal{S}} M_N(\mathbb{C}).$$

Then

$$\lim_{k \to \infty} \left\| \Delta_{N_k^{\#\mathcal{S}}} [(\underline{\underline{X}}_1^{(k)} - \Delta_{N^{\#\mathcal{S}}}[\underline{\underline{X}}_1^{(k)}]) \dots (\underline{\underline{X}}_k^{(k)} - \Delta_{N^{\#\mathcal{S}}}[\underline{\underline{X}}_m^{(k)}])] \right\|_2 = 0 \text{ almost surely.} \quad (2)$$

Note in [CdSH+], $\mathcal{C}$ rather than $\mathcal{V}$ is used for the set of vertices of $\mathcal{G}$. In the notation of [CdSH+], we have taken the diagonal matrices $\Lambda_{i,j}^{(n)}$ to be identity. Moreover, rather than having matrices $X_{i,j}^{(N)}$ with $j = 1, \dots, \ell(i)$, we have a single matrix $X_i^{(N)}$ (we take $\ell(i) = 1$). We used m here rather than k to denote the length of the word. Finally, we rather than having a sequence $\underline{\underline{X}}^{(N)}$ of matrices of size $N^{\#\mathcal{S}}$, we consider a sequence $\underline{\underline{X}}^{(N_k)}$ of matrices of size $N_k^{\#\mathcal{S}}$; the theorem clearly still holds in this setting, since the proof is based on computing expectations and analyzing their dependence on N, for which can simply substitute N_k.

We are now ready to prove Theorem C, which we restate here.

Theorem 4.2 (Theorem C)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite simple graph and let (A_v, τ_v) for a $v \in \mathcal{V}$ be a family of tracial $*$-algebras. If each (A_v, τ_v) is algebraically sofic, then so is $\mathop{\ast}_{v \in \mathcal{G}} (A_v, \tau_v)$.*

Proof. Suppose that (A_v, τ_v) for $v \in \mathcal{V}$ are algebraically sofic, and let us prove that the graph product $\mathop{\ast}_{v \in \mathcal{G}} (A_v, \tau_v)$ is algebraically sofic.

For each v, fix a generating tuple y_v for A_v. Fix Galois bounded sequences of microstates $\tilde{Y}_v^{(k)}$ in $M_{N_{v,k}}(\mathbb{C})$ for y_v. Let $N_k = \prod_{v \in \mathcal{V}} N_{v,k}$, and let $Y_v^{(k)} = \tilde{Y}_v^{(k)} \otimes I_{N_k / N_{k,v}}$. Note that $Y_v^{(k)}$ is a Galois bounded sequence of microstates for y_v, but these microstates now come from the same matrix algebra $M_{N_k}(\mathbb{C})$ for all vertices v.

Let $\mathcal{S}$ be a finite set and $\multimap$ a relation between $\mathcal{S}$ and $\mathcal{V}$ so that for $v_1, v_2 \in \mathcal{V}$, $(v_1, v_2) \in \mathcal{E}$ if and only if $\mathcal{S}_{v_1} \cap \mathcal{S}_{v_2} = \emptyset$. For each $v \in \mathcal{V}$, fix some $s_v \in \mathcal{S}_v$. Let $\left\{ \Sigma_v^{(k)} : v \in \mathcal{V} \right\}$ be a family of independent uniformly random permutation matrices, with $\Sigma_v \in \bigotimes_{\mathcal{S}_v} M_{N_k}(\mathbb{C})$. Let

$$Z_v^{(k)} = \left[(\Sigma_v^{(k)})^t (X_v^{(k)} \otimes I_N^{\otimes \mathcal{S}_v \setminus \{s_v\}}) \Sigma_v^{(k)} \right] \otimes I_N^{\otimes \mathcal{S} \setminus \mathcal{S}_v}.$$

Let y and $Z^{(k)}$ be the tuples obtained by concatenating the tuples y_v and $Z_v^{(k)}$ respectively, over all $v \in \mathcal{V}$. It is immediate that each random outcome of $(Z^{(k)})_{k \in \mathbb{N}}$ is Galois bounded.

It remains to show that almost surely $Z^{(k)}$ is a microstate sequence for y and has asymptotically constant diagonal. Being a microstate sequence means that for every non-commutative polynomial p, we have

$$\lim_{k\to\infty} |\operatorname{tr}_{N_k^{\#\mathcal{S}}}(p(Z^{(k)})) - \tau(p(y))| = 0,$$

while being asymptotically constant on the diagonal means that

$$\lim_{k\to\infty} \left\| \Delta_{N_k^{\#\mathcal{S}}}[p(Z^{(k)}] - \operatorname{tr}_{N_k^{\#\mathcal{S}}}(p(Z^{(k)}))I_{N_k^{\#\mathcal{S}}} \right\|_2 = 0.$$

In fact, the combination of these two conditions is equivalent to

$$\lim_{k\to\infty} \left\| \Delta_{N_k^{\#\mathcal{S}}}[p(Z^{(k)}] - \tau(p(y))I_{N_k^{\#\mathcal{S}}} \right\|_2 = 0; \tag{3}$$

this follows from the triangle inequality and the fact that $\operatorname{tr}_{N_k^{\#\mathcal{S}}} \circ \Delta_{N_k^{\#\mathcal{S}}} = \operatorname{tr}_{N_k^{\#\mathcal{S}}}$. By linearity, it suffices to check (3) for a spanning set of polynomials. Recall [CF17, Remark 2.7] that polynomials in y are spanned by 1 and polynomials of the form

$$p(z) = (p_1(z_{\chi(1)}) - \tau(p_1(y_{\chi(1)}))) \dots (p_\ell(z_{\chi(\ell)}) - \tau(p_\ell(y_{\chi(\ell)}))) \tag{4}$$

for $\mathcal{G}$-reduced words $\chi(1)\dots\chi(\ell)$, with $\ell \geq 1$. The claim (3) is immediate when $p = 1$. Thus, it remains to show (3) in the case when p has the form (4), and note that in this case the term $\tau(p(y))$ in (3) vanishes by graph independence of $(y_v)_{v\in\mathcal{V}}$. Hence, our goal (3) reduces to showing that almost surely

$$\lim_{k\to\infty} \left\| \Delta_{N_k^{\#\mathcal{S}}}[(p_1(Z_{\chi(1)}^{(k)}) - \tau(p_1(y_{\chi(1)}))) \dots (p_\ell(Z_{\chi(\ell)}^{(k)}) - \tau(p_\ell(y_{\chi(\ell)})))] \right\|_2 = 0. \tag{5}$$

Now we assumed that $Y_v^{(k)}$ is a microstate sequence for y that is asymptotically constant on the diagonal, and $Z_v^{(k)}$ is obtained from $Y_v^{(k)}$ by tensoring with the identity and conjugating by a permutation matrix, and so

$$\lim_{k\to\infty} \left\| \Delta_{N_k^{\#\mathcal{S}}}[p_j(Z_{\chi(j)}^{(k)})] - \tau(p_j(y))I_{N_k^{\#\mathcal{S}}} \right\|_2 = \lim_{k\to\infty} \left\| \Delta_{N_k}[p_j(Y_{\chi(j)}^{(k)})] - \tau(p_j(y))I_{N_k} \right\|_2 = 0.$$

Thus, by swapping out each $\tau(p_j(y))$ term (5) for $\Delta_{N_k^{\#\mathcal{S}}}[p_j(Z_{\chi(j)}^{(k)})]$, using the fact that $p_j(Z_{\chi(j)}^{(k)})$ is uniformly bounded in operator norm as $k \to \infty$, we obtain

$$\begin{aligned}\lim_{k\to\infty} \Big\| &\Delta_{N_k^{\#\mathcal{S}}}[(p_1(Z_{\chi(1)}^{(k)}) - \tau(p_1(y_{\chi(1)}))) \dots (p_\ell(Z_{\chi(\ell)}^{(k)}) - \tau(p_\ell(y_{\chi(\ell)})))] \\ &- \Delta_{N_k^{\#\mathcal{S}}}[(p_1(Z_{\chi(1)}^{(k)}) - \Delta_{N_k^{\#\mathcal{S}}}[p_1(Z_{\chi(1)}^{(k)})]) \dots (p_\ell(Z_{\chi(\ell)}^{(k)}) - \Delta_{N_k^{\#\mathcal{S}}}(p_\ell(Z_{\chi(\ell)}^{(k)})))] \Big\| = 0,\end{aligned}$$

so now the claim (5) to be proved reduces to

$$\lim_{k\to\infty} \left\| \Delta_{N_k^{\#\mathcal{S}}}[(p_1(Z_{\chi(1)}^{(k)}) - \Delta_{N_k^{\#\mathcal{S}}}[p_1(y_{\chi(1)})]) \dots (p_\ell(Z_{\chi(\ell)}^{(k)}) - \Delta_{N_k^{\#\mathcal{S}}}(p_\ell(Z_{\chi(\ell)}^{(k)})))] \right\|_2 = 0. \tag{6}$$

Now we can apply Theorem 4.1, taking

$$X_j^{(k)} = p_j(Y_{\chi(j)}^{(k)}) \otimes I_N^{\otimes \mathcal{S}_v \setminus \{s_v\}},$$

so that

$$\underline{\underline{X}}_j^{(k)} = (\Sigma_{\chi(j)}^{(k)})^*(p_j(Y_{\chi(j)}) \otimes I_N^{\otimes \mathcal{S}_v \setminus \{s_v\}})\Sigma_{\chi(j)}^{(k)} = p_j((\Sigma_{\chi(j)}^{(k)})^*(Y_{\chi(j)} \otimes I_N^{\otimes \mathcal{S}_v \setminus \{s_v\}})\Sigma_{\chi(j)}^{(k)}) = p_j(Z_{\chi(j)}^{(k)}).$$

Thus, Theorem 4.1 implies that (6) holds, which completes the proof. □

5. Strong 1-boundedness for graph products

Strong 1-boundedness is a von Neumann algebraic property introduced by Jung in [Jun07]. It implies the lack of a robust space of microstates up to conjugacy for any generating set of a von Neumann algebra. This typically is achieved when the von Neumann algebra is hyperfinite (see in connection, [Jun03, Jun06]) or admits algebraic rigidity in the form of abundant commutation (see [Ge98, GS00, Voi99]) or existence of diffuse regular hyperfinite subalgebras (see [Voi95, Hay18]), or even in the analytic setting of Property (T) which allows for discretizing the microstate space (see [JS07, HJKE21]). On the other hand, strong 1-boundedness implies that every generating set has microstates free entropy dimension $\delta_0(x) = 1$, hence the free group factors are not strongly 1-bounded. Hayes refined this notion by extracting a numerical invariant, implicit in [Jun07], for von Neumann algebras called the 1-bounded entropy h (see [Hay18]). This is the main framework in which the modern theory of strong 1-boundedness is carried out. Non-strongly 1-bounded algebras often exhibit indecomposability relative to strongly 1-bounded subalgebras, which can be used to prove non-isomorphism results or rule out possible structural properties. As a precise example, non-strongly 1-bounded algebras cannot be generated by two strongly 1-bounded subalgebras with diffuse intersection. Another application is a free absorption theorem for strongly 1-bounded subalgebras in free products ([HJNS21]).

Such indecomposability phenomena in the setting of groups in many instances can be encapsulated in L^2-invariants, such as the first L^2-Betti number (see [Lüc02]). This cohomological invariant has been of extreme use in the analytic study of groups, and has been increasingly incorporated as far as possible into the study of von Neumann algebras due to its rich applications (see [CS05, Pet09]). Having positive first L^2-Betti number automatically implies the lack of the sort of algebraic rigidity described above in the group level. See [PT11] for such results. The relationship between the first ℓ^2-Betti number and free entropy theory is a difficult subject that has been heavily investigated ([CS05, Jun, Shl21, HJKE]). Strong 1-boundedness for Connes-embeddable group von Neumann algebras is believed to coincide with vanishing first ℓ^2-Betti number for the group. However, this has been checked only in certain cases, particularly in one direction as outlined in [Shl09, Shl21], and remains a challenging open problem.

Given a tracial von Neumann algebra (M, τ) and N a von Neumann subalgebra of M, the **1-bounded entropy of N in the presence of M is denoted $h(N : M)$.** We set $h(M) = h(M : M)$ and call this the **1-bounded entropy of M**. Roughly speaking, the quantity $h(N : M)$ is a measurement of "how many" finite-dimensional approximations of N there are which extend to M, We will not need the technical definition of 1-bounded entropy, and refer the reader to [Hay18, Definition 2.2 and Definition A.2] for the precise definition. We enumerate below the most essential properties of this quantity for our purposes:

(1) (see [HJKE21, §2.3.3]) $h(N_1 : M_1) \leq h(N_2 : M_2)$ if $N_1 \subset N_2 \subset M_2 \subset M_1$.
(2) (see [Hay18, Lemma A.12]) $h(N_1 \vee N_2 : M) \leq h(N_1 : M) + h(N_2 : M)$ if $N_1, N_2 \subset M$ and $N_1 \cap N_2$ is diffuse. In particular, $h(N_1 \vee N_2) \leq h(N_1) + h(N_2)$.
(3) (see [Hay18]) $h(N_1 : N_2) \leq h(W^*(\mathcal{N}_{N_2}(N_1)) : N_2)$ if $N_1 \subset N_2$ is diffuse.
(4) If $N \subseteq M$ and N is hyperfinite, then $h(N : M) \leq 0$.

We will also need Voiculescu's microstates free entropy dimension $=\delta_0(x)$ of a self-adjoint tuple x in a tracial von Neumann algebra, define by Voiculescu [Voi96]. We will need to allow x to be an infinite tuple, as opposed to a finite tuple in Voiculescu's original definition. It is well known to experts how to extend the definition to this setting, for a precise discussion see e.g. the discussion in Section 4 of [HJKE21]. We use $\underline{\delta}_0(x)$ for the version of microstates free entropy dimension where we replace a limit supremum in the definition with a limit infimum.

In contrast to the rest of the paper, we will need to restrict ourselves to self-adjoint generating tuples. For an integer $r \in \mathbb{N}$, we let $\mathbb{C}\langle S_1, \cdots, S_r \rangle$ be the algebra of noncommutative polynomials in abstract variables $S_1, \cdots, S_r$. We give $\mathbb{C}\langle S_1, \cdots, S_r \rangle$ the unique $*$-structure which makes each X_j self-adjoint. Given a von Neumann algebra M and a $x \in M^r_{s.a.}$ there is a unique $*$-homomorphism

$$\mathrm{ev}_x \colon \mathbb{C}\langle S_1, \cdots, S_r \rangle \to M$$

satisfying $\mathrm{ev}_x(S_j) = x_j$. We set $P(x) = \mathrm{ev}_x(P)$ for $P \in \mathbb{C}\langle S_1, \cdots, S_r \rangle$. We will use $\mathcal{O}\langle S_1, \cdots, S_r \rangle$, $\overline{\mathbb{Q}}\langle S_1, \cdots, S_r \rangle$ etc. for the noncommutative polynomials in r-variables whose coefficients are in $\mathcal{O}, \overline{\mathbb{Q}}$ etc.

5.1. **Proof of Theorem A.** We are now ready to prove Theorem A. For simplicity, we treat its parts 2 and 3 separately from part 1.

Theorem 5.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with $\#\mathcal{V} > 1$, and for each $v \in \mathcal{V}$, let (M_v, τ_v) be a tracial $*$-algebra. Let $(M, \tau) = ✻_{v \in \mathcal{G}}(M_v, \tau_v)$.*

(1) If each M_v is diffuse and $\mathcal{G}$ is connected, then M is strongly 1*-bounded (in fact has* 1*-bounded entropy at most zero).*
(2) If each M_v is diffuse and Connes embeddable, and $\mathcal{G}$ is disconnected, then there is an index set I and a generating tuple $x \in M_{s.a.}^I$ so that $\delta_0(x) > 1$. In particular, M is not strongly 1*-bounded.*

Proof. (1): Since $\mathcal{G}$ is connected, we can find a walk $v_1, v_2, \cdots, v_k$ which visits every vertex of $\mathcal{G}$ at least once. Let us denote by $M_{\leq j}$ the algebra generated by $M_{v_1}, \ldots, M_{v_j}$ within M. We claim that $h(M_{\leq j}) \leq 0$ for all $j \geq 2$.

Because M_{v_1} and M_{v_2} are diffuse, we may choose diffuse abelian subalgebras $A_1 \leq M_{v_1}$ and $A_2 \leq M_{v_2}$. Using Properties (1), (3), (4) of 1-bounded entropy,

$$\begin{aligned} h(M_{v_1} \vee A_2) &= h(M_{v_1} \vee A_2 : M_{v_1} \vee A_2) \\ &\leq h(W^*(\mathcal{N}_{M_{v_1} \vee A_2}(A_2)) : M_{v_1} \vee A_2)) \\ &\leq h(A_2 : M_{v_1} \vee A_2) \\ &\leq 0. \end{aligned}$$

Similarly, $h(A_1 \vee M_{v_2}) \leq 0$. As

$$M_{v_1} \vee A_2 \cap (A_1 \vee M_{v_2}) \supseteq A_1 \vee A_2,$$

we know that $M_{v_1} \vee A_2 \cap (A_1 \vee M_{v_2})$ is diffuse. Thus, by Property (2) of 1-bounded entropy:

$$h(M_{\leq 2}) \leq h(M_{v_1} \vee A_2) + h(A_1 \vee M_{v_2}) \leq 0.$$

For the general case, note for every $2 \leq i < n$, we have $(M_{v_i} \vee M_{v_{i+1}}) \cap M_{\leq i} \supseteq M_{v_i}$, which is diffuse. Thus by Property (2) of 1-bounded entropy:

$$h(M_{\leq i+1}) \leq h(M_{v_i} \vee M_{v_{i+1}}) + h(M_{\leq i}) \leq h(M_{\leq i}),$$

the last inequality following from an argument identical to the case of $M_{\leq 2}$. We thus inductively see that $h(M) = h(M_{\leq n}) \leq 0$.

(2): Let $\mathcal{V}_1, \ldots, \mathcal{V}_l$ be the connected components of $\mathcal{G}$, and note that $l \geq 2$ by assumption. Let M_i be the graph product corresponding to the subgraph induced by $\mathcal{V}_i$. Then the $(M_i)_{i=1}^l$ are freely independent. Let $x_i \in (M_i)_{s.a.}^{J_i}$ be a generating tuple. Set $J = \bigsqcup_i J_i$, and let $x \in M_{s.a.}^J$ be defined by $x\big|_{J_i} = x_i$. Since each M_v is embeddable, we know that M_i is embeddable by [Cas16]. Since M_i is diffuse, this implies by the proof of [Jun03, Corollary 4.7]) that $\underline{\delta}_0(x_i) \geq 1$. Thus, by the proof of [Voi98b][Remark 4.8],

$$\delta_0(x) = \delta_0(x_1) + \sum_{i=1}^{l} \underline{\delta}_0(x) \geq l > 1,$$

the last inequality following as $\mathcal{V}$ is disconnected. □

To deduce strong 1-boundedness from vanishing first L^2-Betti number, we will apply the results in [Shl21] which require positive of certain Fuglede–Kadison pseudo-determinants associated to our relations. To get this positivity, we will use Theorem B which requires polynomials with algebraic coefficients. This will force us to reduce general relations among generators for our tracial von Neumann algebras to only relations that have algebraic coefficients. For this, the following lemma will be useful.

Lemma 5.2. *Let (M, τ) be a tracial von Neumann algebra and $x = (x_1, \cdots, x_r) \in M_{s.a.}^r$. Suppose that for all $P \in \overline{\mathbb{Q}} S_1, \cdots, S_r$ we have that $\tau(P(x)) \in \overline{\mathbb{Q}}$. Let $\mathrm{ev}_x \colon \mathbb{C}\langle S_1, \cdots, S_r\rangle \to M$ be the $*$-homomorphism $\mathrm{ev}_x(P) = P(x)$. Then:*
(a) $\ker(\mathrm{ev}_x)$ is the complex linear span of $\ker(\mathrm{ev}_x) \cap \overline{\mathbb{Q}}\langle S_1, \cdot, S_r\rangle$.
(b) If $\ker(\mathrm{ev}_x)$ is finitely generated as a two-sided ideal, then there is a finite set

$$F \subseteq \overline{\mathbb{Q}}\langle S_1, \cdots, S_r\rangle$$

which generates $\ker(\mathrm{ev}_x)$ as a two-sided ideal.

Proof. (a). Let $P \in \ker(\mathrm{ev}_x)$. Then there are monic monomials $m_1, \cdots, m_d$ and $\lambda_1, \cdots, \lambda_d \in \mathbb{C}$ with $P = \sum_{j=1}^d \lambda_j m_j$. Let $A \in M_d(\mathbb{C})$ be the matrix whose ij^{th} entry is $\tau(m_j(x)^* m_i(x))$. Since τ is a state, A is positive semidefinite. Let

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_d \end{bmatrix} \in \mathbb{C}^d.$$

By direct calculation,

$$\|P(x)\|_2^2 = \langle A\lambda, \lambda \rangle .$$

Since $P(x) = 0$ and A is positive semidefinite, we know that $A\lambda = 0$. Observe that A has algebraic entries, by assumption. Since A has algebraic entries, it follows from linear algebra that the kernel of A (regarded as a linear transformation on $\mathbb{C}^d$) has a basis $v_1, \cdots, v_s \in \overline{\mathbb{Q}}^d$. Choose complex numbers $\alpha_1, \cdots, \alpha_s$ so that

$$\lambda = \sum_{k=1}^{s} \alpha_k v_k$$

For $k = 1, \cdots, s$ write $v_k = (v_{kj})_{j=1}^d \in \overline{\mathbb{Q}}^d$ and set $P_k = \sum_{j=1}^d v_{kj} m_j$. Since $v_k \in \ker(A)$, we have that

$$\|P_k(x)\|_2^2 = \langle Av_k, v_k \rangle = 0.$$

So $P_k \in \ker(\mathrm{ev}_x) \cap \overline{\mathbb{Q}} \langle S_1, \cdots, S_r \rangle$ and

$$P = \sum_{k=1}^{s} \alpha_k P_k.$$

(b) Suppose that $F_1, \cdots, F_p \in \mathbb{C} \langle S_1, \cdots, S_r \rangle$ generate $\ker(\mathrm{ev}_x)$ as a two-sided ideal. By (a), we may find a $t \in \mathbb{N}$ and $\lambda_{ij} \in \mathbb{C}$, $F_{ij} \in \ker(\mathrm{ev}_x) \cap \overline{\mathbb{Q}} \langle S_1, \cdots, S_r \rangle$ for $1 \leq i \leq p, 1 \leq t \leq j$ so that

$$F_i = \sum_j \lambda_{ij} F_{ij}.$$

Then $\{F_{ij}\}_{1 \leq i \leq p, 1 \leq t \leq j}$ generate $\ker(\mathrm{ev}_x)$ as a two-sided ideal. □

We will also need to pass to direct sums of algebras for which the above lemma applies. For this we use the following lemma.

Lemma 5.3. *Let A_1, A_2 be two $*$-algebras which are generated by $x \in (A_1)_{s.a.}^{r_1}$ and $y \in (A_2)_{s.a.}^{r_2}$. Suppose that*

$$E_1 \subseteq \mathbb{C} \langle S_1, \cdots, S_{r_1} \rangle \qquad \text{and} \qquad E_2 \subseteq \mathbb{C} \langle T_1, \cdots, T_{r_2} \rangle$$

generate $\ker(\mathrm{ev}_x)$ *and* $\ker(\mathrm{ev}_y)$*, respectively, as two-sided ideals. Denote*

$$\tilde{x} := (1 \oplus 0, x_1 \oplus 0, \ldots, x_{r_1} \oplus 0)$$
$$\tilde{y} := (0 \oplus 1, 0 \oplus y_1, \ldots, 0 \oplus y_{r_2}).$$

Then $\ker(\mathrm{ev}_{\tilde{x},\tilde{y}}) \subset \mathbb{C} \langle S_0, S_1, \ldots, S_{r_1}, T_0, T_1, \ldots, T_{r_2} \rangle$ *is generated as a two-sided ideal by the union*

$$\begin{aligned} &\{S_0 P : P \in E_1\} \\ \cup &\{T_0 P : P \in E_2\} \\ \cup &\{S_i T_j : 0 \leq i \leq r_1,\ 0 \leq j \leq r_2\} \\ \cup &\{S_0 S_i - S_i, S_i S_0 - S_i : 1 \leq i \leq r_1\} \\ \cup &\{T_0 T_j - T_j, T_j T_0 - T_j : 1 \leq j \leq r_2\} \\ \cup &\{S_0 + T_0 - 1\}. \end{aligned}$$

Proof. Let J be the two-sided ideal in $\mathbb{C}\langle S_0,\dots,S_{r_1},T_0,\dots,T_{r_2}\rangle$ generated by the above union, and let $B:=\mathbb{C}\langle S_0,\cdots,S_{r_1},T_0,\dots,T_{r_2}\rangle/J$. Then $J\subseteq\ker(\mathrm{ev}_{\tilde{x},\tilde{y}})$ and this inclusion induces a unique homomorphism

$$\psi\colon B\to A_1\oplus A_2$$

satisfying $\psi\circ q=\mathrm{ev}_{\tilde{x},\tilde{y}}$, where q is the quotient map onto B. To prove the lemma, it suffices to show that this homomorphism is an isomorphism.

To see this, let $z_1=S_0+J, z_2=T_0+J$. Then z_1,z_2 are orthogonal projections which sum to 1. Observe that for all $P\in\mathbb{C}\langle S_1,\cdots,S_{r_1}\rangle$ we have

$$q(S_0P(S_1,\cdots,S_{r_1}))=q(P(S_0S_1,\cdots,S_0S_{r_1}))$$

Thus for $P\in E_1$ we have

$$q(P(S_0S_1,\cdots,S_0S_{r_1}))=q(S_0P(S_1,\cdots,S_{r_1}))=0.$$

Since E_1 generates $\ker(\mathrm{ev}_x)$ as a two-sided ideal and $A_1\cong\mathbb{C}\langle S_1,\cdots,S_{r_1}\rangle/\ker(\mathrm{ev}_{x_1})$, we may find a unique homomorphism $\phi_1\colon A_1\to z_1B$ satisfying $\phi_1(P(x))=S_0P+J$ for all $P\in\mathbb{C}\langle S_1,\cdots,S_{r_1}\rangle$. Similarly, we may find a unique homomorphism $\phi_2\colon A_2\to z_2B$ satisfying $\phi_2(P(y))=T_0P+J$ for all $P\in\mathbb{C}\langle T_1,\cdots,T_{r_2}\rangle$. The relations imposed on B imply that z_i acts as the identity on the image of ϕ_i. Since z_1,z_2 are orthogonal projections which sum to 1, this implies that the map $\phi\colon A_1\oplus A_2\to B$ defined by $\phi(a_1,a_2)=\phi_1(a_1)+\phi_2(a_2)$ is a homomorphism. Moreover, ϕ is the inverse to ψ. Thus ψ is an isomorphism, as desired. □

We are now ready to prove a general theorem from which we will quickly deduce Theorem A (1) as a corollary. For this we need the first L^2-Betti number of a $*$-subalgebra of a tracial von Neumann algebra. The L^2-Betti number of von Neumann algebras was first defined in [CS05, Definition 2.1] in terms of homology. Thom later gave a definition in terms of cohomology, see [Tho08a, Section 1].

Theorem 5.4 (Theorem A (1))**.** *Let $\mathcal{G}=(\mathcal{V},\mathcal{E})$ be a graph with $\#\mathcal{V}>1$, and for each $v\in\mathcal{V}$, let (M_v,τ_v) be a tracial $*$-algebra. Let $(M,\tau)=\ast_{v\in\mathcal{G}}(M_v,\tau_v)$. Suppose that for all $v\in\mathcal{V}$, we can write $M_v=\bigoplus_{i=1}^{g_v}M_{v,i}$ with $g_{v,i}\in\mathbb{N}$. Further assume that:*

- *$\tau(1_{M_{v,i}})\in\mathbb{Q}$, for all $v\in\mathcal{V}$ and $i=1,\cdots,g_{v,i}$,*
- *for all v and all $1\le i\le g_v$, there is a $x_{v,i}\in M_{v,i}^{r_{v,i}}$ which is algebraically sofic and generates $M_{v,i}$ as a von Neumann algebra,*
- *for all $v\in\mathcal{V},1\le i\le g_v$, $\tau_v(P(x_{v,i}))\in\overline{\mathbb{Q}}$ for all $P\in\overline{\mathbb{Q}}\langle T_1,T_1^*,\cdots,T_r,T_r^*\rangle$*
- *$\ker(\mathrm{ev}_{x_{v,i}})$ is finitely generated as a two-sided ideal in $\mathbb{C}\left\langle T_1,T_1^*,\cdots,T_{r_{v,i}},T_{r_{v,i}}^*\right\rangle$ for all $v\in\mathcal{V}$, $1\le i\le g_v$.*

Let A be the $*$-subalgebra of M generated by $\bigcup_{v\in V}\{x_{v,i,j}:1\le i\le g_v,1\le j\le r_{v,i}\}$. If $\beta^1_{(2)}(A,\tau)=0$, then M is strongly 1-bounded.

Proof. First, notice that by considering the real and imaginary parts of coordinates of $x_{v,i}$, we may assume that $x_{v,i}$ is a tuple of self-adjoint elements. Note that A is finitely presented, namely there is an $r\in\mathbb{N}$, a tuple $x\in A^r_{s.a.}$ so that:

(1) the evaluation homomorphism $\mathrm{ev}_x\colon\mathbb{C}\langle S_1,\cdots,S_r\rangle\to A$ given by $\mathrm{ev}_x(P)=P(x)$ is surjective,
(2) if $J=\ker(\mathrm{ev}_x)$, then J is finitely generated as a two-sided ideal say by $(F_1,\cdots,F_l)$.

In fact, we may choose x to be algebraically sofic and to choose $F_i\in\overline{\mathbb{Q}}\langle S_1,\cdots,S_r\rangle$. One way to see this is as follows.

By assumption, $\ker(\mathrm{ev}_{x_{v,i}})$ can be generated as a two-sided ideal by $(F_h^{(v,i)})_{h=1}^{k_{v,i}}$. Our assumptions on traces of algebraic polynomials in $x_{v,i}$ and Lemma 5.2 (b) implies we can choose $F_h^{(v,i)}$ to be in $\overline{\mathbb{Q}}\left\langle S_1,\cdots,S_{r_{v,i}}\right\rangle$. Set $r_v=g_v+\sum_i r_{v,i}$ and let

$$x_v=(1\oplus0^{\oplus g_v-1},0\oplus1\oplus0^{\oplus g_v-2},\cdots,0^{\oplus g_v-1}\oplus1,x_{v,1}\oplus0^{\oplus g_v-1},0\oplus x_{v,2}\oplus0^{\oplus g_v-2},\cdots,0^{\oplus g_v-1}\oplus x_{v,g_v}).$$

Let $r = \sum r_v$, $k_v = \sum_i k_{v,i}$ and $x \in M_{s.a.}^r$ be given by concatenating the x_v. We relabel the abstract variables $S_1, \cdots, S_r$ as $S_j^{(v)}$ with $v \in \mathcal{V}$ and $1 \le j \le r_v$, and set $S^{(v)} = (S_j^{(v)})_{1 \le j \le r_v}$. By iterated applications of Lemma 5.3, we can find a finite tuple $F^{(v)} \in (\overline{\mathbb{Q}} \langle S_1, \cdots, S_{r_v} \rangle)^{\oplus k_v}$ which generates $\ker(\mathrm{ev}_{x_v})$ as a two-sided ideal in $\mathbb{C} \langle S_1, \cdots, S_{r_v} \rangle$.

To generated $\ker(\mathrm{ev}_x)$, we need to take all the $F^{(v)}(S^{(v)})$ and also polynomials of the form

$$S_j^{(v_1)} S_p^{(v_2)} - S_p^{(v_2)} S_j^{(v_1)}$$

for all $(v_1, v_2) \in \mathcal{E}$, $1 \le j \le k_{v_1}$,$1 \le p \le k_{v_2}$. Then these polynomials generate $\ker(\mathrm{ev}_x)$ and have algebraic coefficients. Moreover, x is an algebraically sofic tuple by the proof of Theorem 3.11, and Theorem 4.2.

Let $F = (F_1, \cdots, F_l) \in \overline{\mathbb{Q}} \langle S_1, \cdots, S_r \rangle^{\oplus l}$. For $i = 1, \cdots, r$ define

$$(\partial_i F) \in M_{l,1}(\overline{\mathbb{Q}} \langle S_1, \cdots, S_r \rangle \otimes \overline{\mathbb{Q}} \langle S_1, \cdots, S_r \rangle)$$

by $(\partial_i F)_{j1} = \partial_i F_j$. Here $\partial_i : \overline{\mathbb{Q}} \langle S_1, \ldots, S_r \rangle \to \overline{\mathbb{Q}} \langle S_1, \ldots, S_r \rangle \otimes \overline{\mathbb{Q}} \langle S_1, \ldots, S_r \rangle$ are Voiculescu's free difference quotients (see [Voi98a, Section 2]), i.e., the unique derivations with $\partial_i(T_j) = \delta_{i=j} 1 \otimes 1$. Finally, set

$$D_F = \begin{bmatrix} S_1 \otimes 1 - 1 \otimes S_1 & S_2 \otimes 1 - 1 \otimes S_2 & \cdots & S_r \otimes 1 - 1 \otimes S_r \\ \partial_1 F & \partial_2 F & \cdots & \partial_r F \end{bmatrix}$$

Which is an element of $M_{l+1,r}(\overline{\mathbb{Q}} \langle T_1, \cdots, T_r \rangle \otimes \overline{\mathbb{Q}} \langle T_1, \cdots, T_r \rangle)$. It is then a folklore result (see e.g the proofs of [HJKE, Theorem 1.1], [BV18, Lemma 4.1]) that

$$\dim_{M \overline{\otimes} M^{op}}(\ker(D_F(x))) = \beta_{(2)}^1(A, \tau) = 0.$$

Thus if $\beta_{(2)}^1(A, \tau) = 0$, then $D_F(x)$ is injective, and so $\mu_{|D_F(x)|}(\{0\}) = 0$. Since x is algebraically sofic, it follows by Proposition 3.10 and Theorem B that

$$\det{}_M^+(D_F(x)) > 0.$$

Moreover, $F(x) = 0$ by construction. Hence it follows by [Shl21, Theorem 1.5], [HJKE, Theorem 1.2] that M is strongly 1-bounded. □

We remark that this theorem implies Theorem A (1) by taking each M_v to be finite-dimensional tracial algebras where every central projection has rational trace. Indeed in this case we may take each $M_{v,i}$ to be a matrix algebra. Now use the isomorphism $M_n(\mathbb{C}) \cong L(\mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}/n\mathbb{Z}$ from Proposition 3.9. The generators for $M_n(\mathbb{C})$ given in Proposition 3.9 (2) are algebraically sofic, by Proposition 3.9, and it is direct to check that monic monomials in these generators have algebraic traces.

References

[ACD+21] Benson Au, Guillaume Cébron, Antoine Dahlqvist, Franck Gabriel, and Camille Male, *Freeness over the diagonal for large random matrices*, The Annals of Probability **49** (2021), no. 1, 157 – 179.

[BC22] Serban Belinschi and Mireille Capitaine, *Strong convergence of tensor products of independent g.u.e. matrices*, 2022.

[BC23] Charles Bordenave and Benoit Collins, *Norm of matrix-valued polynomials in random unitaries and permutations*, 2023.

[Bow10] Lewis Bowen, *Measure conjugacy invariants for actions of countable sofic groups*, J. Amer. Math. Soc. **23** (2010), no. 1, 217–245. MR 2552252

[Bow12] L. Bowen, *Every countably infinite group is almost Ornstein*, Dynamical systems and group actions, Contemp. Math, vol. 567, Amer. Math. Soc, Providence, RI, 2012, pp. 67–78.

[Bow18] Lewis P. Bowen, *A brief introduction of sofic entropy theory*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 1847–1866. MR 3966832

[BV18] Michael Brannan and Roland Vergnioux, *Orthogonal free quantum group factors are strongly 1-bounded*, Adv. Math. **329** (2018), 133–156. MR 3783410

[Cas16] Martijn Caspers, *Connes embeddability of graph products*, Infinite Dimensional Analysis, Quantum Probability and Related Topics **19** (2016), no. 01, 1650004.

[CC21] Ian Charlesworth and Benoît Collins, *Matrix models for ε-free independence*, Archiv der Mathematik **116** (2021), no. 1, 585–605.

[CdSH+] Ian Charlesworth, Rolando de Santiago, Ben Hayes, David Jekel, Srivatsav Kunnawalkam Elayavalli, and Brent Nelson, *Random permutation matrix models for graph products*, in progress.

[CF17] Martijn Caspers and Pierre Fima, *Graph products of operator algebras*, J. Noncommut. Geom **11** (2017), no. 1, 367–411.

[CHR14] Laura Ciobanu, Derek F. Holt, and Sarah Rees, *Sofic groups: graph products and graphs of groups*, Pacific Journal of Mathematics **271** (2014), no. 1, 53–64.

[CIKE22] Ionut Chifan, Adrian Ioana, and Srivatsav Kunnawalkam Elayavalli, *Two non-elementarily equivalent II_1 factors without property gamma*, 2022.

[CN10] Benoît Collins and Ion Nechita, *Random quantum channels I: graphical calculus and the Bell state phenomenon*, Comm. Math. Phys. **297** (2010), no. 2, 345–370. MR 2651902

[CS05] Alain Connes and Dimitri Shlyakhtenko, *L^2-homology for von Neumann algebras*, J. Reine Angew. Math. **586** (2005), 125–168. MR 2180603

[CS13] Ionut Chifan and Thomas Sinclair, *On the structural theory of* II_1 *factors of negatively curved groups*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 1, 1–33 (2013).

[CSU13] Ionut Chifan, Thomas Sinclair, and Bogdan Udrea, *On the structural theory of II_1 factors of negatively curved groups, II: Actions by product groups*, Adv. Math. **245** (2013), 208–236. MR 3084428

[DDJO07] Michael W. Davis, Jan Dymara, Tadeusz Januszkiewicz, and Boris Okun, *Weighted L^2-cohomology of Coxeter groups*, Geom. Topol. **11** (2007), 47–138. MR 2287919

[DF04] David S. Dummit and Richard M. Foote, *Abstract algebra*, third ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004. MR 2286236

[DO01] Michael W. Davis and Boris Okun, *Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups*, Geom. Topol. **5** (2001), 7–74. MR 1812434

[DO12] ———, *Cohomology computations for Artin groups, Bestvina-Brady groups, and graph products*, Groups Geom. Dyn. **6** (2012), no. 3, 485–531. MR 2961283

[ES04] Gábor Elek and Endre Szabó, *Sofic groups and direct finiteness*, J. Algebra **280** (2004), no. 2, 426–434. MR 2089244

[ES05] ———, *Hyperlinearity, essentially free actions and L^2-invariants. The sofic property*, Math. Ann. **332** (2005), no. 2, 421–441. MR 2178069

[Ge98] Liming Ge, *Applications of free entropy to finite von Neumann algebras. II*, Ann. of Math. (2) **147** (1998), no. 1, 143–157. MR 1609522

[Gre90] Elisabeth Ruth Green, *Graph products of groups*, Ph.D. thesis, University of Leeds, 1990.

[Gro99] M. Gromov, *Endomorphisms of symbolic algebraic varieties*, J. Eur. Math. Soc. (JEMS) **1** (1999), no. 2, 109–197. MR 1694588

[GS00] Liming Ge and Junhao Shen, *Free entropy and property T factors*, Proc. Natl. Acad. Sci. USA **97** (2000), no. 18, 9881–9885. MR 1773835

[Hay18] Ben Hayes, *1-bounded entropy and regularity problems in von Neumann algebras*, Int. Math. Res. Not. IMRN (2018), no. 1, 57–137. MR 3801429

[Hay22] ———, *A random matrix approach to the Peterson-Thom conjecture*, Indiana Univ. Math. J. **71** (2022), no. 3, 1243–1297. MR 4448584

[HJKE] Ben Hayes, David Jekel, and Srivatsav Kunnawalkam Elayavalli, *Vanishing cohomology and strong* 1*-boundedness for von Neumann algebras*, J. Noncommut. Geom., to appear.

[HJKE21] ———, *Property (T) and strong* 1*-boundedness for von Neumann algebras*, 2021.

[HJNS21] Ben Hayes, David Jekel, Brent Nelson, and Thomas Sinclair, *A random matrix approach to absorption in free products*, Int. Math. Res. Not. IMRN (2021), no. 3, 1919–1979. MR 4206601

[Ioa12] Adrian Ioana, *Compact actions and uniqueness of the group measure space decomposition of* II_1 *factors*, J. Funct. Anal. **262** (2012), no. 10, 4525–4533. MR 2900475

[Ioa15] ———, *Cartan subalgebras of amalgamated free product* II_1 *factors*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 1, 71–130, With an appendix by Ioana and Stefaan Vaes. MR 3335839

[JNV+20] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen, *Mip*=re*, 2020.

[JS07] Kenley Jung and Dimitri Shlyakhtenko, *Any generating set of an arbitrary property T von Neumann algebra has free entropy dimension* ≤ 1, J. Noncommut. Geom. **1** (2007), no. 2, 271–279. MR 2308307

[Jun] Kenley Jung, *The rank theorem and L^2-invariants in free entropy: Global upper bounds*, arXiv:1602.04726.

[Jun03] ———, *The free entropy dimension of hyperfinite von Neumann algebras*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 5053–5089. MR 1997595

[Jun06] ———, *A hyperfinite inequality for free entropy dimension*, Proc. Amer. Math. Soc. **134** (2006), no. 7, 2099–2108. MR 2215780

[Jun07] ———, *Strongly* 1*-bounded von Neumann algebras*, Geom. Funct. Anal. **17** (2007), no. 4, 1180–1200. MR 2373014

[Lan02] Serge Lang, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR 1878556

[Lüc02] Wolfgang Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002. MR 1926649

[Mło04] W. Młotkowski, *λ-free probability*, Infinite Dimensional Analysis. Quantum Probability and Related Topics **7** (2004), no. 1, 27–41.

[Pet09] Jesse Peterson, *L^2-rigidity in von Neumann algebras*, Invent. Math. **175** (2009), no. 2, 417–433. MR 2470111

[PT11] Jesse Peterson and Andreas Thom, *Group cocycles and the ring of affiliated operators*, Invent. Math. **185** (2011), no. 3, 561–592. MR 2827095

[PV14a] Sorin Popa and Stefaan Vaes, *Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups*, Acta Math. **212** (2014), no. 1, 141–198. MR 3179609

[PV14b] ———, *Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups*, J. Reine Angew. Math. **694** (2014), 215–239. MR 3259044

[Sew22] Brandon Seward, *Bernoulli shifts with bases of equal entropy are isomorphic*, J. Mod. Dyn. **18** (2022), 345–362. MR 4459630

[Shl09] Dimitri Shlyakhtenko, *Lower estimates on microstates free entropy dimension*, Anal. PDE **2** (2009), no. 2, 119–146. MR 2547131

[Shl21] ———, *Von Neumann algebras of sofic groups with $\beta_1^{(2)} = 0$ are strongly 1-bounded*, J. Operator Theory **85** (2021), no. 1, 217–228.

[Sin11] Thomas Sinclair, *Strong solidity of group factors from lattices in $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$*, J. Funct. Anal. **260** (2011), no. 11, 3209–3221. MR 2776567

[SW16] Roland Speicher and Janusz Wysoczański, *Mixtures of classical and free independence*, Archiv der Mathematik **107** (2016), 445–453.

[Tho08a] Andreas Thom, *L^2-cohomology for von Neumann algebras*, Geom. Funct. Anal. **18** (2008), no. 1, 251–270. MR 2399103

[Tho08b] ———, *Sofic groups and Diophantine approximation*, Comm. Pure Appl. Math. **61** (2008), no. 8, 1155–1171. MR 2417890

[Tho11] ———, *Integer operators in finite von Neumann algebras*, J. Topol. Anal. **3** (2011), no. 4, 433–450. MR 2887671

[Voi95] Dan-Virgil Voiculescu, *Operations on certain non-commutative operator-valued random variables*, Recent advances in operator algebras, Astérisque, no. 232, Societe mathematique de France, 1995, pp. 243–275.

[Voi96] ———, *The analogues of entropy and of Fisher's information measure in free probability theory. III. The absence of Cartan subalgebras*, Geom. Funct. Anal. **6** (1996), no. 1, 172–199. MR 1371236

[Voi98a] D. Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability theory. V. Noncommutative Hilbert transforms*, Invent. Math. **132** (1998), no. 1, 189–227.

[Voi98b] Dan-Virgil Voiculescu, *A strengthened asymptotic freeness result for random matrices with applications to free entropy*, Internat. Math. Res. Not. IMRN (1998), no. 1, 41–63. MR 1601878

[Voi99] Dan Voiculescu, *Free entropy dimension ≤ 1 for some generators of property T factors of type* II_1, J. Reine Angew. Math. **514** (1999), 113–118. MR 1711283

[Wei00] Benjamin Weiss, *Sofic groups and dynamical systems*, Sankhyā Ser. A **62** (2000), no. 3, 350–359, Ergodic theory and harmonic analysis (Mumbai, 1999). MR 1803462

Cardiff University, School of Mathematics, Abacws, Senghennydd Rd, Cardiff CF24 4AG, United Kingdom
Email address: charlesworthi@cardiff.ac.uk
URL: https://www.ilcharle.com/

Department of Mathematics and Statistics, California State University Long Beach, 1250 N Bellflower Blvd, Long Beach, CA 90840
Email address: rolando.desantiago@math.csulb.edu
URL: https://www.math.purdue.edu/~desantir

Department of Mathematics, University of Virginia, 141 Cabell Drive, Kerchof Hall, Charlottesville, VA, 22904
Email address: brh5c@virginia.edu
URL: https://sites.google.com/site/benhayeshomepage/home

Department of Mathematics and Statistics, York University,
N520 Ross, 4700 Keele Street, Toronto, ON M3J 1P3
Email address: djekel@ucsd.edu
URL: http://davidjekel.com

Department of Mathematics, University of California,
San Diego, 9500 Gilman Drive # 0112, La Jolla, CA 92093
Email address: skunnawalkamelayaval@ucsd.edu
URL: https://sites.google.com/view/srivatsavke

Department of Mathematics, Michigan State University, 619 Red Cedar Road, C212 Wells Hall, East Lansing, MI 48824
Email address: brent@math.msu.edu
URL: https://users.math.msu.edu/users/banelson/