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Almost Sure Central Limit Theorems for Parabolic/Hyperbolic Anderson Models with Gaussian Colored Noises

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Abstract
This short note is devoted to establishing the almost sure central limit theorem for the parabolic/hyperbolic Anderson models driven by colored-in-time Gaussian noises, completing recent results on quantitative central limit theorems for stochastic partial differential equations. We combine the second-order Gaussian Poincaré inequality with the method of characteristic functions of Ibragimov and Lifshits, effectively overcoming the challenge from the lack of Itô tools in this colored-in-time setting, and achieving results that are inaccessible with previous methods.

Keywords Almost sure central limit theorem · Hyperbolic Anderson model · Parabolic Anderson model · criterion of Ibragimov and Lifshits · Second-order Gaussian Poincaré inequality · Malliavin calculus · Space-time colored noises

Mathematics Subject Classification (2020) 60F15 · 60H30

1 Introduction

The classical central limit theorem (CLT) states that for a random sample of size n drawn from a population with mean zero and variance one, the sample mean M_n admits Gaussian fluctuation as the sample size n tends to infinity:

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$$\sqrt{n} M_n \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0,1).$$

Following this setting, the almost sure central limit theorem (ASCLT) in its simplest form asserts that one can observe a Gaussian behavior (asymptotically) along a generic trajectory via a logarithmic average: for almost every $\omega \in \Omega$

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{\sqrt{n} M_n(\omega)} \Longrightarrow \zeta \tag{1.1}$$

as $n \to \infty$, where δ_x denotes the Dirac mass at $x \in \mathbb{R}$, "$\Longrightarrow$" indicates the weak convergence of finite measures, and $\zeta \sim \mathcal{N}(0,1)$ stands for the standard Gaussian measure on $\mathbb{R}$ *throughout this note*. The first ASCLT was introduced by P. Lévy in his book [28, page 270] but remained largely unnoticed for several decades, until it was rediscovered by various researchers in probability and dynamical systems [1, 11, 12, 22, 27, 38, 40]. For a comprehensive historical account up to 2001, see also the work of Berkes and Csáki [9]. In recent years, several works [2, 8, 41, 42] have established almost sure (non-)central limit theorems using variants of Malliavin calculus in Gaussian, Poisson, and Rademacher settings.

On a different note, around 2018, Huang, Nualart, and Viitasaari initiated in [24] a study on (quantitative) central limit theorems for stochastic partial differential equations (SPDEs) driven by Gaussian noises. More precisely, they established a quantitative CLT for the spatial averages of the solution to a stochastic nonlinear heat equation with multiplicative Gaussian space-time white noise. Additionally, the first attempt of the similar topic for stochastic nonlinear wave equations was published in [18]. Since this pioneering work [24], there has been a rapidly growing literature on the spatial averages of SPDEs. For CLT results, see, e.g., [4, 10, 15, 16, 25, 35, 36], and for spatial ergodicity that precedes the CLTs, refer to [14, 37]. We refer the interested readers to [3, (incomplete) table on page 5] for an overview of relevant results in the Gaussian setting and [7] in the Lévy setting.

In a series of papers [29–31], Li and Zhang developed the ASCLTs for several SPDEs with Gaussian noises that are white in time. A key tool they employed is Malliavin calculus, particularly the Clark–Ocone formula, which replies heavily on the martingale structure resulting from the white-in-time nature of the Gaussian noises; see Remark 1.5-(ii) for more details. For a similar treatment applied on the hyperbolic Anderson model driven by space-time pure-jump Lévy white noise, refer to [6, Section 3.1]. However, this strategy using Clark–Ocone formula fails when attempting to establish the ASCLT results for cases with colored-in-time Gaussian noises as in, e.g., [4, 35]. Motivated by our joint work with Balan [6], we will use a combination of the method of characteristic functions of Ibragimov and Lifshits and the second-order Gaussian Poincaré inequality to establish the ASCLT results for hyperbolic and parabolic Anderson models (HAM/PAM) driven by space-time Gaussian colored noises. Such a combination was originally introduced in [6, Section 3.2] within the Lévy setting.

1.1 Framework

Consider the following two equations:

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u + u \diamond \dot{W}, \\ u(0,\cdot) \equiv 1; \end{cases} \tag{PAM}$$

and

$$\begin{cases} \partial_t^2 u = \Delta u + u \diamond \dot{W} \\ u(0,\cdot) \equiv 1 \quad \text{and} \quad \partial_t u(0,\cdot) \equiv 0; \end{cases} \tag{HAM}$$

where $\diamond$ denotes the Wick product, meaning that the corresponding stochastic integral is interpreted in the Skorohod sense, and $\dot{W}$ is a centered space-time Gaussian noise with correlation

$$\mathbb{E}\big[\dot{W}(t,x)\dot{W}(s,y)\big] = \gamma_0(t-s)\gamma_1(x-y)$$

satisfying certain conditions to ensure the existence and uniqueness of (random field) solutions. In this note, we consider (PAM) in any spatial dimension, while we restrict our analysis of (HAM) to dimensions $d = 1, 2$; see Remark 1.7 for relevant discussions. Let us first state the following standing hypotheses of this note.
(H1) $\gamma_0 : \mathbb{R} \to [0,\infty]$ is nonnegative-definite and locally integrable.
(H2) $\gamma_1 : \mathbb{R}^d \to [0,\infty]$ is nonnegative-definite such that $\gamma_1 = \mathcal{F}\mu$ is the Fourier transform of some nonnegative tempered measure μ, called the spectral measure, satisfying Dalang's condition ([17]):

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(d\xi) < \infty. \tag{1.2}$$

Definition 1.1 A random field $u = \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is called a solution to (PAM)(PAM) or (HAM), provided that for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, it holds almost surely that

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)u(s,y)W(ds,dy), \tag{1.3}$$

where $G = G^H$ (resp. G^W) denotes the heat kernel (resp. wave kernel)

$$G_t^H(x) := (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}, \quad \forall d \ge 1, \quad \text{and} \quad G_t^W(x) := \begin{cases} \frac{1}{2}\mathbf{1}_{\{|x|<t\}}, & d = 1, \\ \frac{1}{2\pi\sqrt{t^2-|x|^2}}\mathbf{1}_{\{|x|<t\}}, & d = 2; \end{cases} \tag{1.4}$$

and the stochastic integral is interpreted in the Skorohod sense. That is, the mild form (1.3) should be understood as $u(t,x) = 1 + \delta(G_{t-\bullet}(x-*)u(\bullet,*))$ with δ the divergence operator in Malliavin calculus; see Sect. 2.1 and see also [33, Section 1.3.2] and [34, Section 2.5]. Note that we follow the convention that $G_t = 0$ for $t \le 0$.

Due to the linearity in the unknown u, one can formally iterate the integral equation (1.3) to obtain an infinite series, which corresponds exactly to the *Wiener chaos expansion* of the solution (whenever it exists):

$$u(t,x) = 1 + \sum_{p=1}^{\infty} I_p(f_{t,x,p}), \tag{1.5}$$

where $I_p(f_{t,x,p})$ stands for the p-th multiple Wiener-Itô integral of the kernel $f_{t,x,p}$ given by

$$f_{t,x,p}(s_1,\ldots,s_p,z_1,\ldots,z_p) := \frac{1}{p!}\prod_{i=0}^{p} G_{t_{\tau(i+1)}-t_{\tau(i)}}\big(z_{\tau(i+1)} - z_{\tau(i)}\big),$$

with τ denoting the permutation on $\{1,\ldots,p\}$ such that $0 < t_{\tau(1)} < \cdots < t_{\tau_{(p)}} < t$, and by convention $(t_{\tau(0)}, z_{\tau(0)}) := (0,0)$ and $(t_{\tau(p+1)}, z_{\tau(p+1)}) := (t,x)$; see Sect. 2 for some preliminaries. The finiteness of the above series in $L^2(\Omega)$, or the validity of the chaos expansion (1.5), can be verified by computing the second moment of each multiple integral, which relies on the orthogonality relation (see, e.g., [34, Proposition 2.7.4]).

Remark 1.2 In fact, the above hypotheses **(H1)** and **(H2)** suffice to guarantee the unique existence of solutions to (PAM)(PAM) and (HAM); see [23, Theorem 3.2] for the heat case and see [5, Section 5] for the wave case. For the wave case, Balan and Song proved the unique existence of solutions to (HAM) on any spatial dimension. More precisely, **(H1)** and **(H2)** imply that (1.5) is the unique solution to (HAM) when $d \le 2$ (see [5, Theorem 5.2]), and under the additional assumption that the spectral measure μ is absolutely continuous with respect to the Lebesgue measure, (1.5) is the unique solution to (HAM) when $d \ge 3$ (see [5, Theorem 5.6]. The delicacy in higher dimensions comes from the fact that the corresponding wave kernel G^W on $\mathbb{R}^d$, for $d \ge 3$, is not a function anymore, so that the interpretation of product of G^W and the unknown u (and thus the interpretation of the multiple integral $I_p(f_{t,x,p})$) requires additional care. Such delicacy also makes it more difficult to establish the CLT results in higher dimensions (see [20, 21]).

In the following, we present several results on (quantitative) CLTs for spatial averages, which will serve as the basis for establishing the ASCLTs.

Theorem 1.3 *Let the above hypotheses* **(H1)** *and* **(H2)** *hold and we make further assumptions on the temporal/spatial correlation kernels:*

(a) for any $\varepsilon > 0$*,*

$$\int_0^{\varepsilon} \gamma_0(r)dr > 0; \tag{1.6}$$

(b) the spatial correlation kernel γ_1 *satisfies one of the following properties:*

(b1) $\gamma_1 \in L^1(\mathbb{R}^d)$ *with* $\|\gamma_1\|_{L^1(\mathbb{R}^d)} > 0$*,*
(b2) $\gamma_1(z) = |z|^{-\alpha}$ *with some* $\alpha \in (0, 2 \wedge d)$*.*

Let u *denote the solution* (1.5) *to* (PAM)(PAM) *for any spatial dimension* d *or* (HAM) *with* $d \leq 2$*. Fix any* $t_0 \in (0, \infty)$ *throughout this note. We define for any* $R > 0$ *that*

$$F_R = F_R(t_0) := \int_{|x| \leq R} [u(t_0, x) - 1]dx \quad \text{and} \quad \widehat{F}_R := \tfrac{1}{\sigma_R} F_R \tag{1.7}$$

with $\sigma_R := \sqrt{\operatorname{Var}(F_R)} > 0$ *for each* $R > 0$*. Then, the following quantitative CLTs hold for any* $R > 0$*:*

$$d_{\mathrm{TV}}(\widehat{F}_R, Z) \leq C \times \begin{cases} R^{-d/2} & \text{for (PAM)(PAM) with (b1)} \quad (\textbf{case 1}) \\ R^{-\alpha/2} & \text{for (PAM)(PAM) with (b2)} \quad (\textbf{case 2}) \\ R^{-d/2} & \text{for (HAM) with (b1)} \quad (\textbf{case 3}) \\ R^{-\alpha/2} & \text{for (HAM) with (b2)} \quad (\textbf{case 4}), \end{cases} \tag{1.8}$$

where $Z \sim \mathcal{N}(0, 1)$*,* $d_{\mathrm{TV}}(X, Y)$ *stands for the total-variation distance between two real-valued random variables* X *and* Y*:*

$$d_{\mathrm{TV}}(X, Y) := \tfrac{1}{2} \sup \big|\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]\big|, \tag{1.9}$$

where the supremum running over all real-valued bounded measurable functions $h : \mathbb{R} \to \mathbb{R}$ *such that* $\|h\|_\infty \leq 1$*. Here, the implicit constant* C *does not depend on* R*. See [4, 35] for more details.*

The range of α in (b2) follows from (1.2), with the corresponding spectral measure $\mu(d\xi) = c_\alpha |\xi|^{\alpha - d}$ for some explicit constant c_α. The conditions (1.6) and "$\|\gamma_1\|_{L^1(\mathbb{R}^d)} > 0$" ensure that we are dealing with nontrivial space-time Gaussian noises and that the spatial integral F_R has strictly positive variance for each R. It is not difficult to prove $\sigma_R > 0$ for each $R > 0$. In fact, for $t_0 > 0$, $\mathbb{E}[u(t_0, x)^2] > 1$, in view of the chaos expansion (1.5) and the orthogonality relation of multiple integrals. The same argument also leads to $\mathbb{E}[u(t_0, x)u(t_0, y)] \geq 1$, so that with these two facts and the $L^2(\Omega)$-continuity of the solution, we get

$$\sigma_R^2 = \int_{|x|, |y| < R} \big(\mathbb{E}[u(t_0, x)u(t_0, y)] - 1\big)dxdy > 0.$$

For the asymptotic behavior of σ_R, see (3.10).

1.2 Main Result

Let us first state the definition of ASCLT.

Definition 1.4 A family $\{F_\theta : \theta \geq 1\}$ of real random variables is said to satisfy the ASCLT if for $\mathbb{P}$-almost every $\omega \in \Omega$, the map $\theta \mapsto F_\theta(\omega)$ is almost surely measurable and

$$\nu_T^\omega := \frac{1}{\log T} \int_1^T \delta_{F_\theta(\omega)} \frac{d\theta}{\theta} \Longrightarrow \zeta \quad \text{as } T \to +\infty, \tag{1.10}$$

where ζ stands for the standard Gaussian measure on $\mathbb{R}$.

In this note, we aim to establish an ASCLT for $\widehat{F}_R$ in (1.7). Thus, we choose the continuum parameter $R > 0$ in Definition 1.4, unlike the discrete parameter $n \in \mathbb{N}$ stated in (1.1). We present an equivalent statement of ASCLT in Remark 1.5-(ii).

Remark 1.5 (i) The *logarithmic average* in (1.1) can be replaced by any slowly varying analogue. For example, $1/k$ and $\log n$ in (1.1), can be substituted with any d_k and $D_n > 0$, respectively, provided that $D_n = d_1 + \cdots + d_n \uparrow \infty$ and $D_{n+1}/D_n \to 1$ as n tends to infinity. See also the discussion in [27, page 204]. Note that our main result (Theorem 1.6) remains valid when the logarithmic average is replaced by a slowly varying analogue in (1.10), although we retain the former for simplicity in presentation.

(ii) The validity of the ASCLT (Definition 1.4) for $\{F_\theta : \theta \geq 1\}$ is equivalent to the statement that for any bounded and Lipschitz continuous function $\phi : \mathbb{R} \to \mathbb{R}$, it holds almost surely that

$$\frac{1}{\log T} \int_1^T \frac{1}{\theta} \phi(F_\theta) d\theta \to \int_{\mathbb{R}} \phi(x) \zeta(dx),$$

as $T \to +\infty$; see [6, Remark 1.2]. Assuming CLT holds (i.e., $F_\theta \Longrightarrow \zeta$), it is equivalent to show that, almost surely,

$$\frac{1}{\log T} \int_1^T \frac{1}{\theta} H_\theta d\theta \to \int_{\mathbb{R}} \phi(x) \zeta(dx),$$

where $H_\theta := \phi(F_\theta) - \mathbb{E}[\phi(F_\theta)]$ is uniformly bounded. Taking advantage of the uniform boundedness of H_θ, it suffices to obtain a power decay like $|\mathbb{E}[H_\theta H_{\mathrm{w}}]| \leq C(\theta/\mathrm{w})^\beta$ for $\theta < \mathrm{w}$ with some $\beta > 0$ and $C \in (0, \infty)$. This can be easily accomplished using Clark–Ocone formula when the noise is white in time; see, e.g., [6, Section 3.1] and [29–31]. However, in our note, where the Gaussian noises are colored in time, this strategy of applying the Clark–Ocone formula fails.

Now we are ready to state our main result.

Theorem 1.6 *Under the assumptions of Theorem 1.3, the* ASCLT *holds for* $\left\{\widehat{F}_R : R \geq 1\right\}$.

Remark 1.7 (i) In this note, we focus exclusively on the wave equation in low spatial dimensions (i.e., $d \le 2$). As previously mentioned, in higher dimension, the wave kernels G^W, unlike the those in (1.4), are no longer functions, leading to extra difficulty in employing the Malliavin–Stein method to establish the desired CLTs. Recent works by M. Ebina in [20, 21] have successfully established these CLTs for the stochastic nonlinear wave equation on $\mathbb{R}^d$ with $d \ge 3$. It is then a natural extension to study the ASCLT in this high-dimensional setting.

(ii) For the heat equation, we simplify the presentation by focusing only on the regular case from [35], where the spatial correlation kernel is a nonnegative function. We do not address the rough case, which involves generalized functions for the spatial correlation kernel. For instance, in one-dimensional case, this includes the correlation whose spectral measure is $\mu(d\xi) = C_{H_1}|\xi|^{1-2H_1}$, and $\gamma_0(t) = |t|^{2H_0-2}$ for some explicit constant $C_{H_1} > 0$ and $0 < H_1 < 1/2 < H_0 < 1$ such that $H_0 + H_1 > 3/4$, or even more rough situations as discussed in [32]. We are optimistic that our strategy can be applicable in this rough case, while such an investigation would inevitably require those very technical estimates from [35].

Let us comment a bit our strategy and postpone the details to Sect. 3. We will apply the powerful Ibragimov–Lifshits criterion to prove Theorem 1.6.

Proposition 1.8 ([26, Ibragimov–Lifshits criterion]) *A family of real-valued random variables* $\{F_\theta\}_{\theta \ge 1}$ *satisfies the* ASCLT *if* $\theta \mapsto F_\theta$ *is measurable almost surely, and the following inequality holds*

$$\sup_{|s| \le T} \int_2^\infty \frac{\mathbb{E}\big[|\mathbf{K}_t(s)|^2\big]}{t \log t} dt < \infty, \tag{1.11}$$

for any finite $T > 0$*, where*

$$\mathbf{K}_t(s) := \frac{1}{\log t} \int_1^t \frac{1}{\theta}\big(e^{isF_\theta} - e^{-s^2/2}\big) d\theta, \quad t \in (1, \infty). \tag{1.12}$$

In the original paper of Ibragimov and Lifshits [26], the criterion is proved for the discrete-time version. For the proof of the above continuum version, see [6, Proposition 3.3]. Note that the Ibragimov–Lifshits criterion is not limited to the Gaussian limit (i.e., ASCLT), as one can see from the original paper and also from the recent application [2]. We expect that it will be useful in establishing almost sure noncentral limit theorem in the SPDE context, for example, an almost sure noncentral limit theorem in the framework of [19].

The said criterion requires us essentially to establish logarithmic decay in the second moment of the difference of characteristic functions of the (random) probability ν_T^ω (1.10) and standard Gaussian measure. By some simple algebra, we only need to bound the total-variation distances $d_{\rm TV}(F_\theta, \mathcal{N}(0,1))$ and $d_{\rm TV}(\frac{F_\theta - F_{\rm w}}{\sqrt{2}}, \mathcal{N}(0,1))$. While not claiming any originality, we state below an abstract result that concludes this discussion.

Proposition 1.9 *A family* $\{F_\theta : \theta \geq 1\}$ *satisfy the* ASCLT *if*

$$d_{\mathrm{TV}}(F_\theta, \mathcal{N}(0,1)) \leq C_1 \theta^{-\beta_1} \tag{1.13}$$

and

$$d_{\mathrm{TV}}\Big(\frac{F_\theta - F_{\mathrm{w}}}{\sqrt{2}}, \mathcal{N}(0,1)\Big) \leq C_2\big(\theta^{-\beta_2} + (\theta/\mathrm{w})^{\beta_3}\big) \ \ \textit{for } \theta < \mathrm{w}, \tag{1.14}$$

where C_1, C_2 *are constants that do not depend on* θ *and* w, *while* $\beta_i > 0$ *for* $i = 1, 2, 3$. *The above result remains valid, if the total-variation distances in* (1.13)-(1.14) *are replaced by* 1*-Wasserstein distances.*

Here, the 1-Wasserstein distance $d_{\mathrm{Wass}}(X, Y)$ of two real-valued random variables X, Y is defined by

$$d_{\mathrm{Wass}}(X, Y) := \sup_{\|h'\|_\infty \leq 1} \big|\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]\big|, \tag{1.15}$$

where the supremum in (1.15) runs over all 1-Lipschitz functions $h : \mathbb{R} \to \mathbb{R}$. In some contexts (e.g., [6]), it is more natural to use the bounds in 1-Wasserstein distances in place of (1.13)-(1.14). We postpone the proof of Proposition 1.9 to Sect. 2.

The rest of this note is organized as follows: We collect a few preliminaries in Sect. 2 and we present the proof of Theorem 1.6 in Sect. 3.

2 Preliminaries

In Sect. 2.1, we present a few basics on Malliavin calculus, while in Sect. 2.2, we record a criterion of Ibragimov and Lifshits.

Assume that all probabilistic objects in this note are defined on a rich enough probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We write $\|\bullet\|_p$ to denote the $L^p(\Omega)$-norm for $p \in [1, \infty]$ and we write $a(R) \lesssim b(R)$ for $\limsup_{R\to+\infty} a(R)/b(R) < +\infty$, and $a(R) \sim b(R)$ for

$$0 < \liminf_{R\to+\infty} a(R)/b(R) \leq \limsup_{R\to+\infty} a(R)/b(R) < +\infty,$$

for any nonnegative functions a and b.

Suppose the Gaussian noise $\dot{W}$ is defined as in the Introduction such that both hypotheses **(H1)** and **(H2)** hold. Under these conditions, we can rigorously built the isonormal framework needed for developing the L^2 theory of Malliavin calculus. Let $C_c(\mathbb{R}_+ \times \mathbb{R}^d)$ denote the set of all real continuous functions on $\mathbb{R}_+ \times \mathbb{R}^d$ with compact support. We define the following inner product on $C_c(\mathbb{R}_+ \times \mathbb{R}^d)$:

$$\langle h_1, h_2 \rangle_{\mathcal{H}} := \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} h_1(r, y) h_2(r', y') \gamma_0(r - r') \gamma_1(y - y') dr dr' dy dy' \tag{2.1}$$

for any $h_1, h_2 \in C_c(\mathbb{R}_+ \times \mathbb{R}^d)$. Let $\mathcal{H}$ denote the closure of $C_c(\mathbb{R}_+ \times \mathbb{R}^d)$ under the above inner product (2.1). A random field $W = \{W(h) : h \in \mathcal{H}\}$ is an isonormal Gaussian process over the Hilbert space $\mathcal{H}$, if W is a centered Gaussian family with covariance given by

$$\mathbb{E}[W(h_1)W(h_2)] = \langle h_1, h_2\rangle_{\mathcal{H}}.$$

Let $\sigma\{W\}$ denote the σ-algebra generated by the noise $\dot{W}$. That is, $\sigma\{W\}$ is the σ-algebra generated by $\{W(h) : h \in C_c(\mathbb{R}_+ \times \mathbb{R}^d)\}$. The well-known Wiener-Itô chaos decomposition (see, e.g., [33, Theorem 1.1.2]) asserts that $L^2(\Omega, \sigma\{W\}, \mathbb{P})$ can be decomposed into mutually orthogonal closed subspaces (called Wiener chaoses):

$$L^2(\Omega, \sigma\{W\}, \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathbb{C}_n^W, \tag{2.2}$$

where $\mathbb{C}_0^W \simeq \mathbb{R}$ is the set of constant random variables, $\mathbb{C}_n^W$ is called n-th Wiener chaos that consists of all multiple integrals of order n; see, e.g., [34, Section 2.7]. The n-th multiple integral operator I_n is a bounded linear operator from the n-th tensor product $\mathcal{H}^{\otimes n}$ to $\mathbb{C}_n^W$ with the following orthogonality relation

$$\mathbb{E}[I_n(f)I_m(g)] = \mathbf{1}_{\{n=m\}} n! \langle \widetilde{f}, \widetilde{g}\rangle_{\mathcal{H}^{\otimes n}} \tag{2.3}$$

with $\widetilde{f}$ denoting the canonical symmetrization of f; see, e.g., [34, Appendix B] for the Hilbert space notation. Alternative to (2.2), we can express the Wiener-Itô chaos decomposition as follows: for any $F \in L^2(\Omega, \sigma\{W\}, \mathbb{P})$, there exist *symmetric* kernels $f_n \in \mathcal{H}^{\otimes n}$ such that

$$F = \mathbb{E}[F] + \sum_{n\geq 1} I_n(f_n). \tag{2.4}$$

The membership of F in $L^2(\Omega)$ is equivalent to the finiteness of $\sum_{n\geq 1} n! \| f_n \|^2_{\mathcal{H}^{\otimes n}}$. With this chaos expansion, we can define several Malliavin operators in a convenient manner.

2.1 Basic Malliavin Calculus

Let us first define several relevant Malliavin operators using the above chaos expansions (2.4), and present results specifically tailored to the solutions to (PAM)(PAM) and (HAM).

• **Malliavin derivative operator**. For $k \in \{1, 2\}$, we let $\mathbb{D}^{k,2}$ denote the set of all square-integrable random variables F with the Wiener-Itô decomposition as in (2.4), such that

$$\sum_{n\geq 1} n! n^k \| f_n \|^2_{\mathcal{H}^{\otimes n}} < \infty.$$

For any $F \in \mathbb{D}^{2,2}$ with the Wiener-Itô decomposition as in (2.4), the Malliavin derivative DF and second-order Mallivin derivative $D^2 F$ are given by

$$DF = \sum_{n\geq 1} n I_{n-1}(f_n) \quad \text{and} \quad D^2 F = \sum_{n\geq 2} n(n-1) I_{n-2}(f_n),$$

which are random vectors in $\mathcal{H}$ and $\mathcal{H}^{\otimes 2}$, respectively. Note that the Hilbert space $\mathcal{H}$ may contain generalized functions, so that the random 'function' DF may not be valued pointwise in general. The same comment applies to $D^2 F$, and when $D^2 F$ is indeed a function, $D_{s,y} D_{r,z} F$ is symmetric in (s, y) and (r, z) so that often we state bounds only for $r < s$; see, e.g., (2.5) below. When $u(t, x)$ denotes the solution to (PAM)(PAM) or (HAM) in this note, we have the following results.

Proposition 2.1 *Under the assumptions of Theorem 1.6, we have* $u(t, x) \in \mathbb{D}^{2,4} \supset \mathbb{D}^{2,2}$*, meaning that* $|u(t, x)| + \|Du(t, x)\|_{\mathcal{H}} + \|D^2 u(t, x)\|_{\mathcal{H}\otimes\mathcal{H}} \in L^4(\Omega)$*. Moreover, the map* $(r, y) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto D_{r,y} u(t, x) \in \mathbb{R}$ *is indeed a (random) function such that for any finite* $p \geq 2$ *and for almost every* $0 < r < s < t \leq T$*, and* $x, y \in \mathbb{R}^d$*,*

$$\left\| D_{s,y} u(t, x) \right\|_p \lesssim_T G_{t-s}(x - y) \quad \text{and} \quad \left\| D_{r,z} D_{s,y} u(t, x) \right\|_p \lesssim_T G_{t-s}(x - y) G_{s-r}(y - z), \tag{2.5}$$

where the above implicit constants in $\lesssim_T$ *do not depend on* (r, s, t) *but depend on* T*; see [35, Theorem 3.1] and [4, Theorem 1.3] for more details.*

For notational convenience, we say a random variable F satisfies the property **(P)** if $F \in \mathbb{D}^{2,2}$ and almost surely, $DF \in |\mathcal{H}|$ and $D^2 F \in |\mathcal{H}^{\otimes 2}|$ meaning that

$$\begin{cases} (r, y) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto |D_{r,y} F| \in \mathbb{R} \text{ belongs to } \mathcal{H} \\ (r, y, s, z) \in (\mathbb{R}_+ \times \mathbb{R}^d)^2 \mapsto |D_{s,z} D_{r,y} F| \in \mathbb{R} \text{ belongs to } \mathcal{H}^{\otimes 2}. \end{cases} \tag{P}$$

It is not difficult, via direct computations, to show that the solution $u(t, x)$ in Proposition 2.1 and the corresponding spatial integrals (1.7) satisfy the property **(P)**.

• **Ornstein–Uhlenbeck operators**. For $F \in \mathbb{D}^{2,2}$ written as in (2.4), we define $LF = \sum_{n\geq 1} -n I_n(f_n)$. Let $F \in L^2(\Omega)$ written as in (2.4). Suppose that $\mathbb{E}[F] = 0$, we define $L^{-1} F = \sum_{n\geq 1} -\frac{1}{n} I_n(f_n)$. Here, L is called the Ornstein–Uhlenbeck operator associated to the noise $\dot{W}$ and its domain coincides with $\mathbb{D}^{2,2}$. The operator L^{-1} is called the pseudo-inverse of L due to the fact that $L^{-1} L F = F - \mathbb{E}[F]$ for $F \in \mathbb{D}^{2,2}$, and $LL^{-1} F = F$ for $F \in L^2(\Omega, \sigma\{W\}, \mathbb{P})$ with zero mean. We define $P_t = e^{tL}$ for $t \in \mathbb{R}_+$, which is called the Ornstein–Uhlenbeck semigroup and satisfies the contraction property:

$$\|P_t F\|_p \leq \|F\|_p \tag{2.6}$$

for any $F \in L^p(\Omega, \sigma\{W\}, \mathbb{P})$ with $p \in [1, \infty)$. It is not difficulty to see that for F as in (2.4), $P_t F = \mathbb{E}[F] + \sum_{n\geq 1} e^{-tn} I_n(f_n)$. Then, using the orthogonality relation (2.3), we get $\|P_t F\|_2^2 = |\mathbb{E}[F]|^2 + \sum_{n\geq 1} e^{-2tn} \|I_n(f_n)\|_2^2 \leq \|F\|_2^2$ with equality when and only when F is a constant or $t = 0$. This gives a proof of (2.6) for $p = 2$. The general case can be easily proved by using Mehler formula (see, e.g., [34, Proposition 2.8.6]).

Using the chaos expansion, it is not difficult to show that $-D_\bullet L^{-1} F = \int_0^\infty e^{-t} P_t D_\bullet F dt$, from which, with Minkowski's inequality and (2.6), we can have

$$\begin{aligned} \|D_\bullet L^{-1} F\|_p &\leq \int_0^\infty e^{-t} \|P_t D_\bullet F\|_p dt \\ &\leq \int_0^\infty e^{-t} \|D_\bullet F\|_p dt = \|D_\bullet F\|_p \end{aligned} \tag{2.7}$$

for any $p \in [1, \infty)$, whenever $D_\bullet F$ is a function.

• **Integration-by-parts formula and chain rule.** The divergence operator δ is the adjoint operator for D, which can be characterized by the following integration-by-part formula:

$$\mathbb{E}[\langle DF, u\rangle_{\mathcal{H}}] = \mathbb{E}[F\delta(u)] \tag{2.8}$$

for any $F \in \mathbb{D}^{1,2}$ and $u \in \operatorname{dom}(\delta)$. Here, $\operatorname{dom}(\delta)$ is the set of random vector $u \in L^2(\Omega; \mathcal{H})$ such that there is some finite constant C_u satisfying $|\mathbb{E}[\langle DF, u\rangle_{\mathcal{H}}]| \leq C_u \|F\|_2$ for any $F \in \mathbb{D}^{1,2}$. It is not difficult to prove via chaos expansion that $L = -\delta D$ on $\mathbb{D}^{2,2}$ (see, e.g., [33, Proposition 1.4.3]). For $\phi : \mathbb{R} \to \mathbb{R}$ Lipschitz, differentiable and $F \in \mathbb{D}^{1,2}$, it is known that $\phi(F) \in \mathbb{D}^{1,2}$ with $D\phi(F) = \phi'(F)DF$; see [33, Proposition 1.2.3]. Then, we can easily derive the following formula:

$$\mathbb{E}[G\phi(F)] = \mathbb{E}\big[\langle DF, -DL^{-1}G\rangle_{\mathcal{H}} \phi'(F)\big] \tag{2.9}$$

for any differentiable and Lipschitz function ϕ, $F \in \mathbb{D}^{1,2}$, and $G \in L^2(\Omega)$ with $\mathbb{E}[G] = 0$. Indeed, using $G = LL^{-1}G$ and the above chain rule with (2.8) and $L = -\delta D$, we get

$$\begin{aligned} \mathbb{E}[G\phi(F)] &= \mathbb{E}[-\delta DL^{-1} G\phi(F)] = \mathbb{E}[\langle -DL^{-1}G, D\phi(F)\rangle_{\mathcal{H}}] \\ &= \mathbb{E}[\langle -DL^{-1}G, DF\rangle_{\mathcal{H}} \phi'(F)]. \end{aligned}$$

Note that taking $\phi(x) = x$, we get

$$\mathrm{Cov}(F, G) = \mathbb{E}\big[\langle DF, -DL^{-1}G\rangle_{\mathcal{H}}\big] \tag{2.10}$$

for centered random variable G with finite second moment and $F \in \mathbb{D}^{1,2}$.

Now we state a key bound in this note, which arises in the so-called improved second-order Gaussian Poincaré inequality; see [4, 39].

Proposition 2.2 ([4, Proposition 1.9]) *Recall the notation in **(P)**. Let F_1, F_2 be centered random variables in $\mathbb{D}^{2,4}$ such that $DF_j \in |\mathcal{H}|$ and $D^2F_j \in |\mathcal{H}^{\otimes 2}|$ almost surely for $j = 1, 2$, (i.e., F_1, F_2 satisfy the property **(P)**. Then,*

$$\mathrm{Var}\big(\langle DF_1, -DL^{-1}F_2\rangle_{\mathcal{H}}\big) \lesssim \mathcal{A}(F_1, F_2) + \mathcal{A}(F_2, F_1),$$

where

$$\begin{aligned}\mathcal{A}(F_1, F_2) := \int_{\mathbb{R}_+^6 \times \mathbb{R}^{6d}} & dr dr' ds ds' d\theta d\theta' dz dz' dy dy' d\mathrm{w} d\mathrm{w}' \\ & \times \gamma_0(s-s')\gamma_0(r-r')\gamma_0(\theta-\theta')\gamma(z-z')\gamma(y-y')\gamma(\mathrm{w}-\mathrm{w}') \\ & \times \|D_{r,z}D_{\theta,\mathrm{w}}F_1\|_4 \|D_{s,y}D_{\theta',\mathrm{w}'}F_1\|_4 \|D_{r',z'}F_2\|_4 \|D_{s',y'}F_2\|_4.\end{aligned}$$

2.2 Proof of Proposition 1.9

We conclude this section with the proof of Proposition 1.9.

Proof of Proposition 1.9 According to Proposition 1.8, the ASCLT holds for $\big\{F_\theta : \theta \geq 1\big\}$ if

$$\sup_{|s|\leq T} \int_2^\infty \frac{\mathbb{E}\big[|\mathbf{K}_t(s)|^2\big]}{t \log t} dt < \infty$$

for any finite $T > 0$, where $\mathbf{K}_t(s)$ is as in (1.12). Expanding $|\mathbf{K}_t(s)|^2$, we get

$$\begin{aligned}|\mathbf{K}_t(s)|^2 &= \frac{1}{(\log t)^2} \int_{[1,t]^2} \frac{1}{\theta \mathrm{w}} \big(e^{isF_\theta} - e^{-\frac{s^2}{2}}\big)\big(e^{-isF_\mathrm{w}} - e^{-\frac{s^2}{2}}\big) d\theta d\mathrm{w} \\ &= \frac{1}{(\log t)^2} \int_{[1,t]^2} \frac{1}{\theta \mathrm{w}} \big(e^{is(F_\theta - F_\mathrm{w})} + e^{-s^2} - e^{isF_\theta} e^{-\frac{s^2}{2}} - e^{-isF_\mathrm{w}} e^{-\frac{s^2}{2}}\big) d\theta d\mathrm{w} \\ &= \mathbb{I}_t(s) - e^{-\frac{s^2}{2}} \mathrm{II}_t(s),\end{aligned}$$

where

$$\begin{aligned}\mathbb{I}_t(s) &:= \frac{1}{(\log t)^2} \int_{[1,t]^2} \frac{1}{\theta \mathrm{w}} \left(e^{is(F_\theta - F_\mathrm{w})} - e^{-s^2}\right) d\theta d\mathrm{w}, \\ \mathrm{II}_t(s) &:= \frac{1}{\log t} \int_1^t \frac{1}{\theta} \big(e^{isF_\theta} + e^{-isF_\theta} - 2e^{-\frac{s^2}{2}}\big)\, d\theta.\end{aligned} \tag{2.11}$$

Therefore, it suffices to show that

$$\mathbf{A}_1(s) := \int_2^\infty \frac{\mathbb{E}\big[\mathbb{I}_t(s)\big]}{t \log t} dt \quad \text{and} \quad \mathbf{A}_2(s) := \int_2^\infty \frac{\mathbb{E}\big[\mathrm{II}_t(s)\big]}{t \log t} dt, \quad s \in [-T, T] \tag{2.12}$$

are both uniformly bounded for any given $T > 0$.

• **Estimation for $\mathbf{A}_2$.** Recall that $\mathbb{E}[e^{isY}] = e^{-\frac{s^2}{2}}$ with $Y \sim \mathcal{N}(0,1)$, and for any real random variable X,

$$\left|\mathbb{E}[e^{isX}] - \mathbb{E}[e^{isY}]\right| = \left|\mathbb{E}[e^{isX}] - e^{-\frac{s^2}{2}}\right| \le 4\, d_{\mathrm{TV}}(X, Y), \tag{2.13}$$

where the total-variation distance d_{TV} is defined as in (1.9). While using the local Lipschitz property of the complex exponentials, we have

$$\sup_{|s|\le T} \left|\mathbb{E}[e^{isX}] - \mathbb{E}[e^{isY}]\right| \le 2T\, d_{\mathrm{Wass}}(X, Y). \tag{2.14}$$

Therefore, it follows from (2.13)-(2.14) that

$$\left|\mathbb{E}\big[e^{isF_\theta} + e^{-isF_\theta} - 2e^{-\frac{s^2}{2}}\big]\right| \le \min\{8\, d_{\mathrm{TV}}\big(F_\theta, Y\big), 4T\, d_{\mathrm{Wass}}(F_\theta, Y)\}. \tag{2.15}$$

Therefore, it follows from (2.11), (2.15), and (1.13) that for any finite $T > 0$,

$$\sup\big\{|\mathbf{A}_2(s)| : s \in [-T, T]\big\} \le \int_2^\infty \frac{1}{t \log^2 t} \int_1^t \frac{1}{\theta^{1+\beta_1}} d\theta dt < \infty,$$

that is, $\sup\{|\mathbf{A}_2(s)| : s \in [-T, T]\} < \infty$ for any finite $T > 0$.

• **Estimation for $\mathbf{A}_1$.** Using the inequality,

$$\begin{aligned}\left|\mathbb{E}\big[e^{is(F_\theta - F_{\mathrm{w}})} - e^{-s^2}\big]\right| &= \left|\mathbb{E}\Big[e^{i\sqrt{2}s\left(\frac{F_\theta - F_{\mathrm{w}}}{\sqrt{2}}\right)} - e^{i\sqrt{2}sY}\Big]\right| \\ &\le \min\left\{4\, d_{\mathrm{TV}}\Big(\frac{F_\theta - F_{\mathrm{w}}}{\sqrt{2}}, Y\Big), 2\sqrt{2}T\, d_{\mathrm{Wass}}\Big(\frac{F_\theta - F_{\mathrm{w}}}{\sqrt{2}}, Y\Big)\right\},\end{aligned}$$

we can write, with dist $= d_{\mathrm{TV}}$ or d_{Wass},

$$\begin{aligned}\sup_{|s|\le T} \mathbf{A}_1(s) &\le \int_2^\infty \frac{1}{t(\log t)^3} \left(\int_{[1,t]^2} \frac{1}{\theta \mathrm{w}} \operatorname{dist}\big(\tfrac{F_\theta - F_{\mathrm{w}}}{\sqrt{2}}, Y\big)\, d\theta d\mathrm{w} \right) dt \\ &\le \int_2^\infty \frac{1}{t(\log t)^3} \left(\int_{1<\theta<\mathrm{w}<t} \frac{1}{\theta \mathrm{w}} \operatorname{dist}\big(\tfrac{F_\theta - F_{\mathrm{w}}}{\sqrt{2}}, Y\big)\, d\theta d\mathrm{w} \right) dt, \end{aligned} \tag{2.16}$$

which is finite due to the assumption (1.14).

Hence, the proof of Proposition 1.9 is completed. □

3 Proof of Theorem 1.6

In this section, we provide the proof of our main result Theorem 1.6. According to Proposition 1.9, we only need to check the conditions (1.13) and (1.14) for the spatial integrals $\widehat{F}_R$ (1.7) in place of F_θ.

Note that Theorem 1.3 implies that in any of the four cases in (1.8), there is some positive constant $b > 0$ such that

$$d_{\mathrm{TV}}(\widehat{F}_\theta, Y) \le C\theta^{-b}. \tag{3.1}$$

That is, the condition (1.13) is verified.

Next, we will show that there exist positive real numbers β_1 and β_2 such that

$$d_{\mathrm{TV}}\Big(\tfrac{\widehat{F}_\theta - \widehat{F}_{\mathrm{w}}}{\sqrt{2}}, Y\Big) \lesssim \theta^{-\beta_1} + (\theta/\mathrm{w})^{\beta_2} \tag{3.2}$$

for $1 < \theta < \mathrm{w} < \infty$. This constitutes the bulk of the proof.

The bound in (3.2) is obtained using techniques analogous to those employed in deriving the bound in (3.1). We provide a brief outline of these techniques below. Observe that the random variable $\frac{\widehat{F}_\theta - \widehat{F}_{\mathrm{w}}}{\sqrt{2}}$ has mean zero and variance

$$V_{\theta,\mathrm{w}} := \mathrm{Var}\Big(\tfrac{\widehat{F}_\theta - \widehat{F}_{\mathrm{w}}}{\sqrt{2}}\Big) = 1 - \mathrm{Cov}\big(\widehat{F}_\theta, \widehat{F}_{\mathrm{w}}\big). \tag{3.3}$$

Thus, applying Stein's bound (see, e.g., [34, Theorem 3.3.1]), we get

$$d_{\mathrm{TV}}(G, Y) \le \sup \big|\mathbb{E}[G\phi(G) - \phi'(G)]\big| \quad \text{with } G := \tfrac{\widehat{F}_\theta - \widehat{F}_{\mathrm{w}}}{\sqrt{2}}, \tag{3.4}$$

where the above supremum runs over bounded, differentiable functions $\phi : \mathbb{R} \to \mathbb{R}$ with $\|\phi\|_\infty \le \sqrt{\pi/2}$ and $\|\phi'\|_\infty \le 2$. In view of (2.10), the inner product $\langle DG, -DL^{-1}G\rangle_{\mathcal{H}}$ has mean

$$\mathbb{E}[\langle DG, -DL^{-1}G\rangle_{\mathcal{H}}] = \mathbb{E}[G^2] = V_{\theta,\mathrm{w}}.$$

Then, it follows from (3.4), (3.3), and (2.9) with the Cauchy–Schwarz inequality that

$$\begin{aligned} d_{\mathrm{TV}}\Big(\tfrac{\widehat{F}_\theta - \widehat{F}_{\mathrm{w}}}{\sqrt{2}}, Y\Big) &\le 2\big|1 - V_{\theta,\mathrm{w}}\big| + 2\mathbb{E}\big|\langle DG, -DL^{-1}G\rangle_{\mathcal{H}} - V_{\theta,\mathrm{w}}\big| \\ &\le 2\big|\mathrm{Cov}(\widehat{F}_\theta, \widehat{F}_{\mathrm{w}})\big| + \sqrt{\mathrm{Var}\big(\big\langle D(\widehat{F}_\theta - \widehat{F}_{\mathrm{w}}), -DL^{-1}(\widehat{F}_\theta - \widehat{F}_{\mathrm{w}})\big\rangle_{\mathcal{H}}\big)}. \end{aligned} \tag{3.5}$$

Recalling our goal (3.2), it suffices to show that, for any $1 < \theta < \mathrm{w} < \infty$,

$$\big|\mathrm{Cov}(\widehat{F}_\theta, \widehat{F}_{\mathrm{w}})\big| \lesssim (\theta/\mathrm{w})^{\beta_2} \tag{3.6}$$

with $\beta_2 = \frac{d}{2}$ in **(case 1)** and **(case 3)** and $\beta_2 = \frac{\alpha}{2}$ in **(case 2)** and **(case 4)** specified in (1.8), where the above implicit constant does not depend on (θ, w). The bound (3.6) will be proved in Sect. 3.1. For the variance term in (3.5), it is essential to estimate

$$\mathrm{Var}\big(\big\langle D\widehat{F}_\theta, -DL^{-1}\widehat{F}_{\mathrm{w}}\big\rangle_{\mathcal{H}}\big) \tag{3.7}$$

for $\theta, \mathrm{w} \in (1, \infty)$, concerning the bilinearity of the inner product operation, the linearity of the operators D and L^{-1}, and the elementary inequality $\mathrm{Var}(X_1 + X_2) \leq 2\mathrm{Var}(X_1) + 2\mathrm{Var}(X_2)$ for any square-integrable random variables X_1, X_2. When $\theta = \mathrm{w}$, the estimate for (3.7) has been established in [35, Section 3.1] and [4, Section 4.2], where it is shown that, with an implicit constant independent of θ,

$$\mathrm{Var}\big(\langle D\widehat{F}_\theta, -DL^{-1}\widehat{F}_\theta\rangle_{\mathcal{H}}\big) \lesssim \begin{cases} \theta^{-d} & \text{in } \textbf{(case 1)} \text{ and } \textbf{(case 3)} \\ \theta^{-\alpha} & \text{in } \textbf{(case 2)} \text{ and } \textbf{(case 4)}. \end{cases} \tag{3.8}$$

The derivation of (3.8) relies on the ideas around the so-called second-order Gaussian Poincaré inequality ([13, 39]), see Proposition 2.2. This inequality, utilized in [4, 35], will also play a crucial role when estimating the term (3.7) for $\theta \neq \mathrm{w}$. In Sect. 3.2, we will show for $1 < \theta < \mathrm{w}$:

$$\begin{aligned} &\mathrm{Var}\big(\langle D\widehat{F}_\theta, -DL^{-1}\widehat{F}_\mathrm{w}\rangle_{\mathcal{H}}\big) + \mathrm{Var}\big(\langle D\widehat{F}_\mathrm{w}, -DL^{-1}\widehat{F}_\theta\rangle_{\mathcal{H}}\big) \\ &\quad \lesssim \begin{cases} \theta^{-d} & \text{in } \textbf{(case 1)} \text{ and } \textbf{(case 3)} \\ \theta^{-\alpha} & \text{in } \textbf{(case 2)} \text{ and } \textbf{(case 4)}. \end{cases} \end{aligned} \tag{3.9}$$

Therefore, the claim (3.2) follows immediately from (3.5), (3.6), (3.8), and (3.9). Hence, the proof is complete. □

It remains to justify (3.6) and (3.9).

3.1 Proof of (3.6)

In this subsection, we will show (3.6) for the four cases specified in (1.8). First, we present the precise asymptotic relation of the limiting variance of F_R in (1.7), as cited from [36, Theorems 1.6 and 1.7] for (PAM)(PAM) and [4, Theorem 1.4] for (HAM):

$$\sigma_R = \sqrt{\mathrm{Var}(F_R)} \sim \begin{cases} R^{\frac{d}{2}} & \text{in } \textbf{(case 1)} \text{ and } \textbf{(case 3)} \\ R^{d-\frac{\alpha}{2}} & \text{in } \textbf{(case 2)} \text{ and } \textbf{(case 4)}, \end{cases} \tag{3.10}$$

where we write $a_R \sim b_R$ to mean $0 < \liminf_{R\to+\infty} a_R/b_R \leq \limsup_{R\to+\infty} a_R/b_R < +\infty$.

• **(case 1) and (case 3)**. It follows from [36, Theorem 1.6 for PAM] for **(case 1)** and [4, Formulas (4.8) and (4.10) for HAM] for **(case 3)** that there exists a **nonnegative** function $\Phi_{s,t} \in L^1(\mathbb{R}^d)$ such that

$$\Phi_{s,t}(x - y) := \mathrm{Cov}\big(u(t, x), u(s, y)\big). \tag{3.11}$$

As a result, taking (3.10) into account, we get for $R' \geq R > 1$,

$$\begin{aligned} \left|\mathrm{Cov}(\widehat{F}_R, \widehat{F}_{R'})\right| &\leq (RR')^{-\frac{d}{2}} \left| \int_{|x|<R} dx \int_{|y|<R'} dy\, \mathrm{Cov}\big(u(t,x), u(t,y)\big) \right| \\ &\leq (RR')^{-\frac{d}{2}} \int_{|x|<R} dx \int_{\mathbb{R}^d} dz \Phi_{t,t}(z) \leq \left(\tfrac{R}{R'}\right)^{d/2}. \end{aligned}$$

This verifies (3.6) for **(case 1)** and **(case 3)**.

Note that in **(case 2)** and **(case 4)**, the function $\Phi_{s,t}$ in (3.11) does not belong to $L^1(\mathbb{R}^d)$. As a result, we need to carry out another approach to settle this difficulty.

• **(case 2) and (case 4)**. Recall that $\widehat{F}_R$ is a centered random variable in $\mathbb{D}^{1,2}$. Then, using an integration-by-part formula from [34, Theorem 2.9.1], we can first write

$$\mathbb{E}[\widehat{F}_R \widehat{F}_{R'}] = \mathbb{E}\big[\langle D\widehat{F}_R, -DL^{-1}\widehat{F}_{R'}\rangle_{\mathcal{H}}\big]$$

(see also (2.9)) and then apply the definition (2.1) and (2.7) with Fubini's theorem and Cauchy–Schwarz inequality to get

$$\begin{aligned} &\left|\mathbb{E}[\widehat{F}_R \widehat{F}_{R'}]\right| \\ &\leq \mathbb{E}\Big[\int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} |D_{r,y}\widehat{F}_R| \times |-D_{r',y'}L^{-1}\widehat{F}_{R'}| \gamma_0(r-r')\gamma_1(y-y') dr dr' dy dy'\Big] \\ &\leq \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \big\| D_{r,y}\widehat{F}_R \big\|_2 \times \big\| D_{r',y'}\widehat{F}_{R'} \big\|_2 \gamma_0(r-r')\gamma_1(y-y') dr dr' dy dy'. \end{aligned} \tag{3.12}$$

Using the basic property of Malliavin derivative operator and the bound in (2.5), we have

$$\begin{aligned} \big\| D_{r,y}\widehat{F}_R \big\|_2 &\leq \frac{1}{\sigma_R} \int_{|x|<R} \| D_{r,y} u(t_0, x)\|_2 dx \\ &\leq \frac{1}{\sigma_R} \int_{|x|<R} G_{t_0 - r}(x-y) dx. \end{aligned} \tag{3.13}$$

Therefore, combining (3.12) with (3.13) leads us to

$$\begin{aligned} \left|\mathbb{E}[\widehat{F}_R \widehat{F}_{R'}]\right| \leq \frac{1}{\sigma_R \sigma_{R'}} &\int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \int_{|x|<R} \int_{|x'|<R'} G_{t_0-r}(x-y) G_{t_0-r'}(x'-y') \\ &\times \gamma_0(r-r') |y-y'|^{-\alpha} dr dr' dy dy' dx dx', \end{aligned} \tag{3.14}$$

where we recall from (3.10) that $\sigma_R \sim R^{d-\frac{\alpha}{2}}$ and from (1.8) that $\gamma_1(z) = |z|^{-\alpha}$ for $\alpha \in (0, 2 \wedge d)$.

In the following, we will use the Fourier analysis to get fine estimates of the above spatial integral (3.14). Let us fix some notations: for an integrable function $g : \mathbb{R}^d \to$

$\mathbb{R}$, its Fourier transform $\widehat{g}$ is given by $\widehat{g}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} g(x)dx$. Recall the expressions of wave/heat kernels in (1.4) and we record below their Fourier transforms:

$$\widehat{G_t^H}(\xi) = e^{-\frac{t}{2}|\xi|^2} \quad \text{and} \quad \widehat{G_t^W}(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad \text{for all } t > 0, \xi \in \mathbb{R}^d. \tag{3.15}$$

Then, using Plancherel's theorem, we get from (3.14) with (3.10) that

$$\begin{aligned}\big|\mathbb{E}[\widehat{F}_R\widehat{F}_{R'}]\big| \lesssim& (RR')^{\frac{\alpha}{2}-d}\int_0^{t_0}\int_0^{t_0} drdr'\gamma_0(r-r')\int_{\mathbb{R}^d} d\xi \\ &\times \int_{|x|<R} dx \int_{|x'|<R'} dx' e^{-i\xi\cdot(x-x')}\widehat{G}_{t_0-r}(\xi)\widehat{G}_{t_0-r'}(\xi)|\xi|^{\alpha-d} \\ =& (\tfrac{R'}{R})^{\frac{\alpha}{2}}\int_0^{t_0}\int_0^{t_0} drdr'\gamma_0(r-r')\int_{\mathbb{R}^d} d\xi \\ &\times \int_{|x|<1} dx \int_{|x'|<1} dx' e^{-i\xi\cdot\left(x-\frac{R'}{R}x'\right)}\widehat{G}_{t_0-r}(\xi/R)\widehat{G}_{t_0-r'}(\xi/R)|\xi|^{\alpha-d},\end{aligned} \tag{3.16}$$

where in the last equality, we performed a change of variable $(x, x', \xi) \mapsto (Rx, R'x', \xi/R)$.

Next, we further bound (3.16) in **(case 2)** and **(case 4)** separately. For **(case 2)**, recalling (3.15), and using

$$|\xi|^{-2\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty d\theta e^{-\theta|\xi|^2}\theta^{\beta-1}$$

with $\beta = \frac{d-\alpha}{2} > 0$, and making a change of variables $(t_0 - r, t_0 - r') \mapsto (r, r')$, we can get from (3.16) that

$$\begin{aligned}\big|\mathbb{E}[\widehat{F}_R\widehat{F}_{R'}]\big| \lesssim& (\tfrac{R}{R'})^{\frac{\alpha}{2}}\int_0^{t_0}\int_0^{t_0} drdr'\gamma_0(r-r')\int_{|x|,|x'|<1} dxdx'\int_0^\infty d\theta\theta^{\frac{d-\alpha-2}{2}} \\ &\times \int_{\mathbb{R}^d} d\xi e^{-i(x-\frac{R'}{R}x')\cdot\xi-(\theta+\frac{r+r'}{2R^2})|\xi|^2} \\ =& (2\pi)^{\frac{d}{2}}(\tfrac{R}{R'})^{\frac{\alpha}{2}}\int_0^{t_0}\int_0^{t_0} drdr'\gamma_0(r-r')\int_{|x|,|x'|<1} dxdx' \\ &\times \int_0^\infty d\theta\theta^{\frac{d-\alpha-2}{2}}\big(\theta+\tfrac{r+r'}{2R^2}\big)^{-\frac{d}{2}}\exp\Big(-\frac{|x-\frac{R'}{R}x'|^2}{4(\theta+\frac{r+r'}{2R^2})}\Big),\end{aligned} \tag{3.17}$$

where in the last step, we use the Fourier transform of the heat kernel.

To deal with the integration in θ from (3.17), we decompose the region $(0,\infty)$ into two segments $(0,\frac{r+r'}{2R^2})$ and $(\frac{r+r'}{2R^2},\infty)$. Since $r, r' \in (0,t)$, it is easy to deduce that

$$\begin{aligned}
&\int_0^{\frac{r+r'}{2R^2}} d\theta\theta^{\frac{d-\alpha-2}{2}}\big(\theta+\tfrac{r+r'}{2R^2}\big)^{-\frac{d}{2}}\exp\Big(-\frac{|x-\frac{R'}{R}x'|^2}{4(\theta+\frac{r+r'}{2R^2})}\Big)\\
\le&\big(\tfrac{R^2}{r+r'}\big)^{\frac{d}{2}}\exp\Big(-\frac{|x-\frac{R'}{R}x'|^2}{4(r+r')/R^2}\Big)\int_0^{\frac{r+r'}{2R^2}} d\theta\theta^{\frac{d-\alpha-2}{2}} \qquad \text{with } \tfrac{d-\alpha-2}{2}>-1\\
\le&\big(\tfrac{R^2}{r+r'}\big)^{\frac{d}{2}}\exp\Big(-\frac{|x-\frac{R'}{R}x'|^2}{4(r+r')/R^2}\Big)\big(\tfrac{r+r'}{2R^2}\big)^{\frac{d-\alpha}{2}}\\
=&\big(\tfrac{R^2}{r+r'}\big)^{\frac{\alpha}{2}}\exp\Big(-\frac{|x-\frac{R'}{R}x'|^2}{4(r+r')/R^2}\Big)\le|x-\tfrac{R'}{R}x'|^{-\alpha},
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
\int_{\frac{r+r'}{2R^2}}^\infty d\theta\theta^{\frac{d-\alpha-2}{2}}\big(\theta+\tfrac{r+r'}{2R^2}\big)^{-\frac{d}{2}}\exp\Big(-\frac{|x-\frac{R'}{R}x'|^2}{4(\theta+\frac{r+r'}{2R^2})}\Big)&\le\int_{\frac{r+r'}{2R^2}}^\infty d\theta\theta^{\frac{-\alpha-2}{2}}e^{-\frac{|x-\frac{R'}{R}x'|^2}{8\theta}}\\
&\le\int_0^\infty d\theta\theta^{\frac{-\alpha-2}{2}}e^{-\frac{|x-\frac{R'}{R}x'|^2}{8\theta}}\le|x-\tfrac{R'}{R}x'|^{-\alpha}.
\end{aligned} \tag{3.19}$$

Therefore, we deduce from (3.17), (3.18), and (3.19) with the local integrability of γ_0 that

$$\begin{aligned}
\big|\mathbb{E}[\widehat{F}_R\widehat{F}_{R'}]\big|&\le\big(\tfrac{R}{R'}\big)^{\frac{\alpha}{2}}\int_0^{t_0}\int_0^{t_0}drdr'\gamma_0(r-r')\int_{|x|,|x'|<1}dxdx'|x-\tfrac{R'}{R}x'|^{-\alpha}\\
&=\big(\tfrac{R}{R'}\big)^{-\frac{\alpha}{2}}\int_{|x|<1}dx\int_{|x'|<1}dx'|\tfrac{R}{R'}x-x'|^{-\alpha}\le\big(\tfrac{R}{R'}\big)^{-\frac{\alpha}{2}},
\end{aligned} \tag{3.20}$$

where for the last step, we used the following elementary fact that

$$\sup_{z\in\mathbb{R}^d}\int_{|x|<1}|z-x|^{-\beta}dx<\infty, \quad \forall\beta\in(0,d). \tag{3.21}$$

Thus, the proof of (3.6) in **(case 2)** is complete.

Now let us deal with **(case 4)**. For the wave kernel in $\mathbb{R}^d$ with $d\le 2$, it enjoys the following property that with $\mathbf{1}_R(x):=\mathbf{1}_{\{|x|<R\}}$,

$$(\mathbf{1}_R*G_r)(y):=\int_{\mathbb{R}^d}\mathbf{1}_{\{|x|<R\}}G_r(x-y)dx\le r\mathbf{1}_{R+r}(y), \tag{3.22}$$

which follows from the definition (1.4) of the wave kernel; see also [10, Lemma 2.1]. Note that the Fourier transform of $\mathbf{1}_R$ is a real-valued and rotationally symmetric function (see, e.g., [36, Lemma 2.1]). Then, using Plancherel's theorem and some Fourier calculations, we derive from (3.14) with (3.10) and a change of variables $(t_0 - r, t_0 - r') \to (r, r')$ that

$$\begin{aligned}\big|\mathbb{E}[\widehat{F}_R \widehat{F}_{R'}]\big| &\le (RR')^{\frac{\alpha}{2}-d} \int_0^{t_0}\int_0^{t_0} drdr'\gamma_0(r-r') \int_{\mathbb{R}^d} d\xi|\xi|^{\alpha-d}\widehat{G}_r(\xi)\widehat{G}_{r'}(\xi)\\ &\quad \times \left(\int_{|x|<R} dx \int_{|x'|<R'} dx' e^{-i(x-x')\cdot\xi}\right)\\ &= (RR')^{\frac{\alpha}{2}-d} \int_0^{t_0}\int_0^{t_0} drdr'\gamma_0(r-r') \int_{\mathbb{R}^d} d\xi|\xi|^{\alpha-d}\widehat{\mathbf{1}_R}(\xi)\widehat{\mathbf{1}_{R'}}(\xi)\widehat{G}_r(\xi)\widehat{G}_{r'}(\xi)\\ &= C_{\alpha,d}(RR')^{\frac{\alpha}{2}-d} \int_0^{t_0}\int_0^{t_0} drdr'\gamma_0(r-r') \int_{\mathbb{R}^{2d}} (\mathbf{1}_R * G_r)(y)(\mathbf{1}_{R'} * G_{r'})(y')\\ &\quad \times |y-y'|^{-\alpha}dydy',\end{aligned}$$

where the constant $C_{\alpha,d}$ comes from inverting the Fourier transform of the spectral measure $|\xi|^{\alpha-d}d\xi$. Now applying the inequality (3.22) with $r, r' \in (0, t_0)$ and utilizing the local integrability of γ_0, we get

$$\begin{aligned}\big|\mathbb{E}[\widehat{F}_R \widehat{F}_{R'}]\big| &\le t_0^2(RR')^{\frac{\alpha}{2}-d} \int_0^{t_0}\int_0^{t_0} drdr'\gamma_0(r-r') \int_{\mathbb{R}^{2d}} \mathbf{1}_{R+t_0}(y)\mathbf{1}_{R'+t_0}(y')\\ &\quad \times |y-y'|^{-\alpha}dydy'\\ &\lesssim (RR')^{\frac{\alpha}{2}-d} \int_{\mathbb{R}^{2d}} \mathbf{1}_{R+t_0}(y)\mathbf{1}_{R'+t_0}(y')|y-y'|^{-\alpha}dydy'\\ &\lesssim (RR')^{\frac{\alpha}{2}-d}(R+t_0)^d(R'+t_0)^{d-\alpha} \lesssim (\tfrac{R}{R'})^{\frac{\alpha}{2}},\end{aligned}$$

where we obtained the last second step with the same argument as in (3.20)-(3.21). Hence, the proof of (3.6) in **(case 4)** is finished. □

3.2 Proof of (3.9)

To prove (3.9), we first apply Proposition 2.2 with the bounds in (2.5) and (3.13): with $B_R := \{x \in \mathbb{R}^d : |x| < R\}$, we need to bound

$$\begin{aligned}\mathcal{A}(\widehat{F}_R, \widehat{F}_{R'}) \le& \sigma_R^{-2}\sigma_{R'}^{-2} \int_0^{t_0} dr \int_0^r d\theta \int_0^{t_0} dr' \int_0^{r'} d\theta' \int_{[0,t]^2} dsds' \int_{\mathbb{R}^{6d}} dzdz'dydy'dwdw'\\ &\times \gamma_0(s-s')\gamma_0(r-r')\gamma_0(\theta-\theta')\gamma(z-z')\gamma(y-y')\gamma(w-w')\\ &\times \int_{B_R^2} dx_1dx_2 G_{t-r}(x_1-z)G_{r-\theta}(z-w)G_{t-r'}(x_2-z')G_{r'-\theta'}(z'-w')\\ &\times \int_{B_{R'}^2} dx_3dx_4 G_{t-r'}(x_3-y)G_{t-s'}(x_4-y').\end{aligned}$$

In **(case 1)**, we simply enlarge the region B_R to $B_{R'}$. Then, applying the estimates for $\mathcal{A}^*$ appearing in [35, Section 3.1.1] and recalling from (3.10) that $\sigma_R = \sigma_R(t_0) \sim R^{\frac{d}{2}}$, one can easily conclude that

$$\mathcal{A}\big(\widehat{F}_R, \widehat{F}_{R'}\big) \leq R^{-d}.$$

For **(case 2)**, we follow the idea employed in [35, Section 3.1.2]. That is, for i.i.d. standard normal random variables $Z_1, \dots, Z_6$, we have

$$\begin{aligned}\mathcal{K} := &\int_{B_1^4} dx_1 \cdots dx_4 \mathbb{E}\Big[\Big|\frac{\sqrt{t-r}}{R}Z_1 - \frac{\sqrt{r-\theta}}{R}Z_2 - \frac{\sqrt{t-s}}{R}Z_3 + \frac{\sqrt{s-\theta'}}{R}Z_4 + x_1 - x_2\Big|^{-\alpha} \\ &\times \Big|\frac{\sqrt{t-s}}{R}Z_3 - \frac{\sqrt{t-s'}}{R'}Z_6 + x_2 - x_4\Big|^{-\alpha} \\ &\times \Big|\frac{\sqrt{t-r}}{R}Z_1 - \frac{\sqrt{t-r'}}{R'}Z_5 + x_1 - x_3\Big|^{-\alpha}\Big] \\ \leq &1,\end{aligned}$$

and thus, the expression

$$\begin{aligned} S_R &= S_R(t, r, r', s, s', \theta, \theta') \\ &:= \int_{B_R^2} dx_1 dx_2 \int_{B_{R'}^2} dx_3 dx_4 \int_{\mathbb{R}^{6d}} dz dz' dy dy' d\mathrm{w} d\mathrm{w}' \gamma(z-z')\gamma(y-y')\gamma(\mathrm{w}-\mathrm{w}') \\ &\quad \times G_{t-r}(x_1 - z) G_{r-\theta}(z - \mathrm{w}) G_{t-r'}(x_2 - z') G_{r'-\theta'}(z' - \mathrm{w}') G_{t-r'}(x_3 - y) \\ &\quad \times G_{t-s'}(x_4 - y') \\ &= R^{2d-2\alpha}(R')^{2d-\alpha}\mathcal{K} \leq R^{2d-2\alpha}(R')^{2d-\alpha}. \end{aligned}$$

Then, with $\sigma_R \sim R^{d-\frac{\alpha}{2}}$, we get

$$\mathcal{A}\big(\widehat{F}_R, \widehat{F}_{R'}\big) \leq \sigma_R^{-2}\sigma_{R'}^{-2} \int_0^{t_0} dr \int_0^r d\theta \int_0^{t_0} dr' \int_0^{r'} d\theta' \int_{[0,t]^2} ds ds' S_R \lesssim R^{-\alpha}.$$

Turning to the cases for (HAM), we first consider **(case 3)**. Just like **(case 1)**, with extending the integrating region in x_1, x_2 to $B_{R'}$, applying the estimates for $\mathcal{A}_R$ in [4, Page 809], we have
$\mathcal{A}\big(\widehat{F}_R, \widehat{F}_{R'}\big) \lesssim R^{-d}$; and for **(case 4)**, following a similar argument as in [4, Section 4.2.2], one can deduce that $\mathcal{A}\big(\widehat{F}_R, \widehat{F}_{R'}\big) \lesssim R^{-\alpha}$. The estimates for $\mathcal{A}(\widehat{F}_{R'}, \widehat{F}_R)$ are also very similar and thus omitted here. Therefore, we have

$$\mathcal{A}\big(\widehat{F}_R, \widehat{F}_{R'}\big) + \mathcal{A}\big(\widehat{F}_{R'}, \widehat{F}_R\big) \lesssim \begin{cases} R^{-d}, & \textbf{(case 1)} \text{ and } \textbf{(case 3)} \\ (R')^{-\alpha} < R^{-\alpha}, & \textbf{(case 2)} \text{ and } \textbf{(case 4)}. \end{cases}$$

The proof of (3.9) is then complete by invoking Proposition 2.2. □

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