Int. J. Production Economics 285 (2025) 109612

Contents lists available at ScienceDirect

International Journal of Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Dynamic analysis of the proportional order-up-to policy with damped trend forecasts

Qinyun Li [a,*], Gerard Gaalman [b], Stephen M. Disney [c]

[a] Logistics Systems Dynamics Group, PARC Institute of Manufacturing, Logistics, and Inventory, Cardiff Business School, Cardiff University, Colum Drive, Cardiff, CF10 3EU, United Kingdom
[b] Department of Operations, Faculty of Economics and Business, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands
[c] Center for Simulation, Analytics, and Modelling, University of Exeter Business School, Streatham Court, Exeter, EX4 4PU, United Kingdom

ARTICLE INFO

Keywords:
Damped trend forecasting
Proportional OUT policy
Bullwhip
Inventory
Control theory
Eigenvalue analysis
Supply chain

ABSTRACT

We study the bullwhip behaviour in the proportional order-up-to (POUT) policy with non-stationary autoregressive integrated moving average (ARIMA) demand. We build a state-space model of the POUT policy where the damped trend forecasting method predicts ARIMA(1,1,2) demand. The POUT policy is closely related to the order-to-up (OUT) policy with the addition of a proportional feedback controller in the inventory and work-in-progress feedback loops. Our modelling approach allows us to derive and/or analyse the demand, order, and inventory variances. We also find the covariance between the demand forecast and the inventory forecast in an attempt to obtain the order variance. However, both the demand and the order variances are infinite under the non-stationary ARIMA(1,1,2) process. Thus, the traditional bullwhip measure (the ratio of the order variance divided by the demand variance) is indeterminate. Despite this difficulty, we can study the difference between the order variance and the demand variance for both the OUT and POUT policies. These differences are finite and their sign indicates whether a bullwhip effect has been generated or not. We find under non-stationary demand, the POUT policy's bullwhip behaviour contradicts some of the existing bullwhip theory. The POUT policy sometimes generates more bullwhip than the OUT policy, revealing that existing knowledge based on stationary demand should be used with caution in non-stationary demand environments. We validate our findings with an investigation of some ARIMA(1,1,2) time series from the M4 competition.

1. Introduction

The bullwhip effect, where the order variance is amplified as the orders proceed up the supply chain, has been observed in many industries for decades, Lee et al. (1997). Most analytics studies of this effect assume stationary random demand. Herein, we reveal existing bullwhip knowledge should be used with caution under non-stationary demand. Demand forecasting and ordering policies have been found to be two of the most important causes of the bullwhip effect, Wang and Disney (2016). The order-up-to (OUT) policy and the proportional order-up-to (POUT) policy are two of the most common ordering algorithms in the literature. The OUT policy is often used for regulating production and distribution in high-volume settings as it minimises inventory holding and backlog costs while maintaining customer service levels.

Researchers have quantified the bullwhip effect in supply chains with correlated demand processes. For example, Zhang (2004) study first-order auto-regressive, AR(1), demand. Whereas Luong and Phien (2007) consider second order AR demand, AR(2), demand. Alwan et al. (2003) studied the bullwhip effect resulting from the OUT replenishment policy with optimal forecasts for first-order autoregressive and moving average, ARMA(1,1), demand. Rostami-Tabar and Disney (2023) investigate the impact of a first-order integer auto-regressive, INAR(1), demand process on the bullwhip generated in the OUT policy. Findings from these studies all indicate the existence of the bullwhip effect in the OUT policy under different demand patterns, even with optimal forecasts. Despite that, the bullwhip effect can be avoided for some demand processes by the OUT policy, Gaalman et al. (2022). For example, the OUT policy with minimum mean squared error (MMSE) forecasting of some negatively correlated AR(1) demand process does not generate bullwhip, Alwan et al. (2003). Gaalman et al. (2022) reveal when ARMA(2,2) and ARMA(p,q) demand produces bullwhip that is, and is not, increasing in the lead time.

The POUT policy adds a proportional feedback controller to the OUT policy to alter the trade-off between inventory and capacity costs. The effectiveness of the POUT policy at reducing the bullwhip effect

* Corresponding author.
E-mail addresses: LiQY@cardiff.ac.uk (Q. Li), g.j.c.gaalman@rug.nl (G. Gaalman), s.m.disney@exeter.ac.uk (S.M. Disney).

https://doi.org/10.1016/j.ijpe.2025.109612
Received 30 April 2024; Received in revised form 9 February 2025; Accepted 23 March 2025
Available online 21 April 2025

is widely recognised. A steady stream of research on the POUT policy assumes stationary demand. Hosoda and Disney (2006) quantified and compared the bullwhip generated by the POUT and OUT policies with optimal forecasts for AR(1) demand. They found the order variance is increasing convex in the proportional feedback controller in most cases (except when the autoregressive parameter is near unity). The POUT policy with ARMA(p,q) demand was studied in Gaalman (2006). Gaalman and Disney (2009) studied the bullwhip effect generated by the POUT policy under ARMA(2,2) demand processes.

Less attention has been given to non-stationary demand. Of the few that do, Graves (1999) quantified the bullwhip effect for a first-order integrated moving average demand, IMA(0,1,1). Boute et al. (2022) considered non-stationary demand in a dual sourcing setting with a low cost, long lead time global supplier coupled with a local SpeedFactory. The more expensive, short lead time, local supplier allowed for tighter inventory control. The inventory benefit was sufficient to enable the re-shoring of a small proportion of the total demand to the high-cost SpeedFactory, even before production price parity was reached.

The exponential smoothing forecasting mechanism is popular in practice as it only has one parameter, and all future forecasts are constant (level) projections of the forecast of the next period's demand. Holt's method adds a linear trend to the future forecasts (and an extra smoothing parameter) to account for long-term linear trends in demand. The damped trend forecasting mechanism adds a further parameter to the forecasting mechanism that shapes future forecast projections. These projects can be *damped*, where future forecast projections level out geometrically of time, or they grow (decline) geometrically towards positive (negative) infinity, or they oscillate in a stable (as we will see in the 4 times series from the M4 competition in Fig. 1) or unstable manner. The damped trend forecasting mechanism contains, as special cases, 11 other forecasting methods, including exponential smoothing and Holt's method. The bullwhip effect can be avoided when the OUT policy employs the damped trend forecasting mechanism, Li et al. (2014). Bullwhip was eliminated when the forecasting parameters were selected from a special region in the parameter space under an independently and identically distributed (i.i.d.) demand, a feat never previously reported for the OUT policy. Li et al. (2023) further studied the system behaviour of the damped trend OUT policy, finding that its dynamic behaviour was equivalent to the POUT policy. Herein, we explore how optimal damped trend forecasts for ARIMA(1,1,2) demand perform.

While the damped trend forecasting method can be applied to any demand process, it is the optimal forecasting method for the ARIMA(1,1,2) process, Gardner and McKenzie (1985). However, OUT and POUT policy's bullwhip performance under ARIMA(1,1,2) demand remains under-explored. The auto-regressive, integrated, moving average, ARIMA(1,1,2), demand has one AR parameter, φ, one integration term, and two MA parameters, θ_1, and θ_2; the need to both capture long-term trends and filter short-term fluctuations presents a greater challenge for forecasting compared to simpler alternatives. Fig. 1 illustrates some example time series from the weekly time series of the M4 competition[1] (Makridakis et al., 2020) identified as ARIMA(1,1,2) demand processes by the `auto.arima` function in R. Two of the time series were excessively long, so we performed the ARIMA identification process on only the last 100 data points, see Panels (a) and (b) in Fig. 1. The other two time series only had 80 data points; we used all available data for Panels (c) and (d). In the title of each panel, you can find information on: the index of the weekly time series, the ARIMA(1,1,2) parameters identified by the `auto.arima` function in R, and the equivalent damped trend forecasting parameters.[2]

[1] The M4 data set is available from https://github.com/Mcompetitions/M4-methods.

[2] Using the relations $\alpha = (\theta_2 + \varphi)\varphi^{-1}$, $\beta = (\varphi^2 - \theta_2 - \theta_1\varphi)(\theta_2\varphi + \varphi^2)^{-1}$, and $\gamma = \varphi$, Li et al. (2023).

Another difficulty encountered when studying the ARIMA(1,1,2) process is its non-stationary nature. Non-stationary demand is characterised by a time-varying mean and variance, Gardner and McKenzie (1985). Over an infinite time horizon, this leads to infinite demand variance and infinite order variance. However, the inventory variance remains finite, Graves (1999). Of the few papers that study damped trend forecasting in a supply chain setting, Li et al. (2014, 2023), focus on using the damped trend forecasting method in suboptimal demand settings that result in finite demand and order variances. Thus, the complexity and challenges ARIMA(1,1,2) demand poses, particularly in the context of the bullwhip effect and inventory performance, remain significant. To address this inventory control challenge, we present a generalisable approach based on a state-space model and an analysis of the demand eigenvalues to understand the impact of the feedback controller under non-stationary demand.

The purpose of this paper is to study the performance of using the damped trend forecasting method to predict ARIMA(1,1,2) demand in both the OUT and POUT policies. We build state-space models of the policies and provide exact expressions for the order variance and the inventory variance in eigenvalue form. We measure the bullwhip effect as the difference between the order and demand variances, allowing us to study variance amplification under non-stationary demand. We compare the bullwhip in the two policies and show:

- The bullwhip effect can be larger in the POUT policy than in the OUT policy for certain types of ARIMA(1,1,2) demand with a short lead time, even when the proportional feedback controller lies in the region $0 \le f < 1$.
- For other types of ARIMA(1,1,2) demand, the POUT policy (with $0 \le f < 1$) always generates less bullwhip than the OUT policy, for all lead times.
- Under certain type of ARIMA(1,1,2) demand, the proportional controller f needs to be carefully tuned in relation to the demand pattern and the lead time in order for the POUT policy to generate less bullwhip than the OUT policy.

The remainder of the paper is organised as follows. In Section 2, we present a state space approach to model ARIMA(1,1,2) processes. Inventory policies are modelled in state-space form in Section 3, and the variances for orders and inventory are derived in Section 4. Section 5 investigates the bullwhip produced by the OUT and POUT policies. Section 6 explores the four demand patterns in Fig. 1 numerically. Section 7 concludes.

2. The demand and the forecast

The ARIMA(1,1,2) demand process,

$$d_{t+1} - d_t - \varphi(d_t - d_{t-1}) = \eta_{t+1} - \theta_1\eta_t - \theta_2\eta_{t-1}, \tag{1}$$

where d_{t+1} is the demand at time t+1 and η_{t+j} is an i.i.d. random process (white noise). We can also interpret the ARIMA(1,1,2) demand as a non-stationary (unstable) ARMA(2,2) process,

$$d_{t+1} - (1+\varphi)d_t - (-\varphi)d_{t-1} = \eta_{t+1} - \theta_1\eta_t - \theta_2\eta_{t-1}. \tag{2}$$

Several state space forms of ARMA processes exist; there is no unique form. We follow Gaalman (2006) and Gaalman and Disney (2009) and use a state y_t and a (left) canonical form of the system matrix $\mathbf{D}$,

$$\left.\begin{aligned} d_{t+1} &= \mathbf{M}y_{t+1} + \eta_{t+1} \\ y_{t+1} &= \mathbf{D}y_t + \mathbf{G}\eta_t \end{aligned}\right\}, \tag{3}$$

where

$$\mathbf{D} = \begin{pmatrix} 1+\varphi & 1 \\ -\varphi & 0 \end{pmatrix}, \; \mathbf{G} = \begin{pmatrix} 1+\varphi-\theta_1 \\ -\varphi-\theta_2 \end{pmatrix}, \; \mathbf{M} = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{4}$$

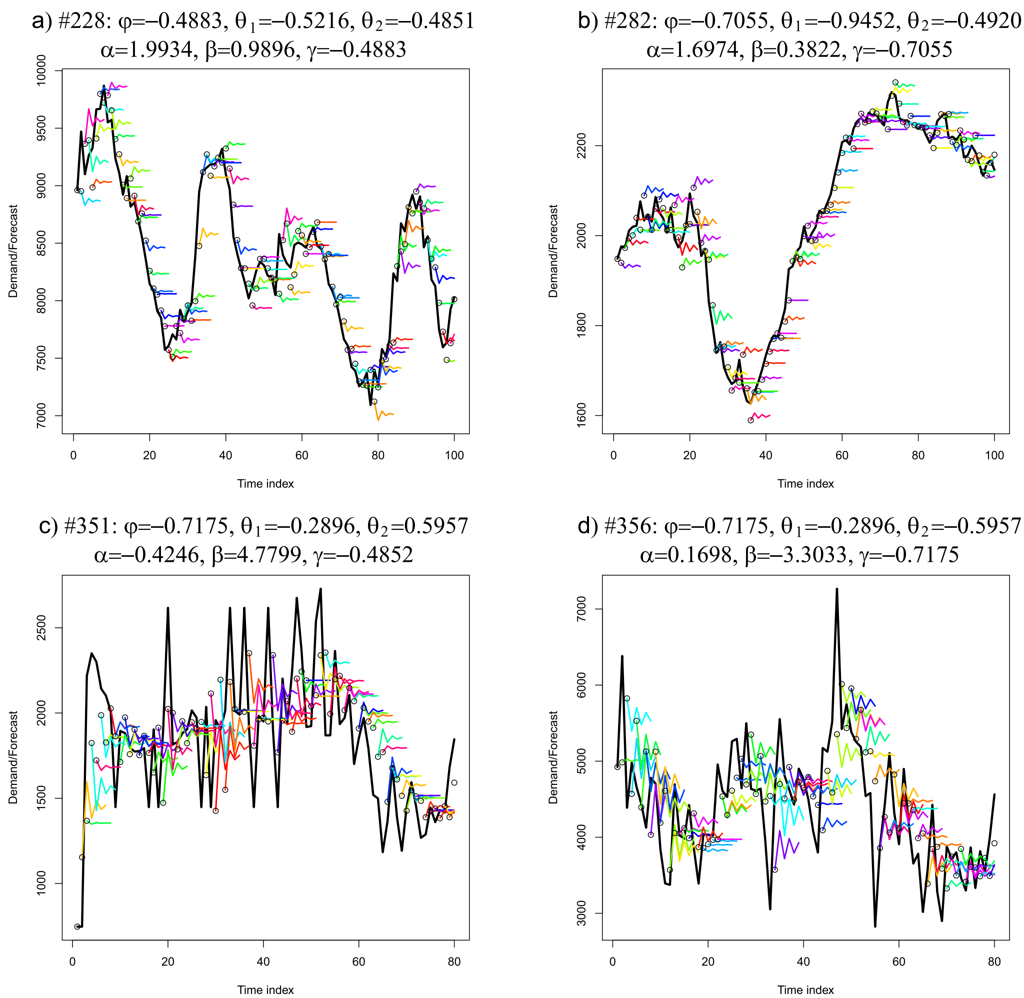

Fig. 1. Example ARIMA(1,1,2) time series plots and their optimal damped trend forecasts. Key: The black line is the demand, and the rainbow-coloured lines, originating at the circles, represent the five periods-ahead damped trend forecasts. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The system matrix $\mathbf{D}$ contains only auto-regressive coefficients; sometimes $\mathbf{D}$ is denoted as $\mathbf{D}^{\phi}$. The characteristic polynomial of $\mathbf{D}$ is

$$\det(\mathbf{D}-\lambda\mathbf{I}) = \lambda^2 - (1+\varphi)\lambda - (-\varphi) = (\lambda-\varphi)(\lambda-1) = \prod_{j=1}^{2}\left(\lambda-\lambda_j^{\phi}\right). \tag{5}$$

Setting the polynomial equation in (5) to zero and solving for λ provides λ_j^{ϕ}, the poles of the demand process; λ_j^{ϕ} are the AR eigenvalues and are related to the AR coefficients. There are two AR eigenvalues (poles): $\lambda_1^{\phi}=\varphi$ and $\lambda_2^{\phi}=1$, Li et al. (2023).

Remark 1. The poles are distinct.

The conditional expectation of the demand can be found from the one-period-ahead forecast using an *a priori estimation*,

$$\left.\begin{aligned}\hat{d}_{t+1|t} &= \mathbf{M}\hat{y}_{t+1|t}\\ \hat{y}_{t+1|t} &= \mathbf{D}\hat{y}_{t|t-1} + \mathbf{K}_t(d_t - \hat{d}_{t|t-1})\end{aligned}\right\} \tag{6}$$

of the gain $\mathbf{K}$, which can be determined by the discrete-time matrix Ricatti equation. As the system's structure has only one error η_t, the gain $\mathbf{K}$ can be derived directly by minimising the variance of the state space forecast error $v_{t+1} = (y_{t+1} - \hat{y}_{t+1|t})$. Hyndman et al. (2008) call this a *single source of error* (SSOE) model.

The state space error at time t satisfies

$$v_{t+1} = (y_{t+1} - \hat{y}_{t+1|t}) = \mathbf{D}(y_t - \hat{y}_{t|t-1}) - \mathbf{K}(d_t - \hat{d}_{t|t-1}) + \mathbf{G}\eta_t, \tag{7}$$

resulting in the variance expression

$$\mathbb{V}[v_{t+1}] = (\mathbf{D}-\mathbf{KM})\mathbb{V}[v_t](\mathbf{D}-\mathbf{KM})^T + (\mathbf{G}-\mathbf{K})(\mathbf{G}-\mathbf{K})^T\mathbb{V}[\eta]. \tag{8}$$

This expression holds for each t, as infinite past demand observations are considered. Backward iteration reveals that $(\mathbf{D}-\mathbf{KM})$ is stable. At time t, the minimum variance of v_t exists when $\mathbf{K}=\mathbf{G}$ and can even be zero, $\mathbf{K}=\mathbf{G}=0$. The state space one-period-ahead forecast $\hat{y}_{t+1|t}$ then becomes

$$\hat{y}_{t+1|t} = \mathbf{D}\hat{y}_{t|t-1} + \mathbf{K}(d_t - \hat{d}_{t|t-1}) = \mathbf{D}\hat{y}_{t|t-1} + \mathbf{GM}(y_t - \hat{y}_{t|t-1}) = \mathbf{D}\hat{y}_{t|t-1} + \mathbf{G}\eta_t. \tag{9}$$

The $(\mathbf{D}-\mathbf{KM})$ matrix is

$$(\mathbf{D}-\mathbf{KM}) = (\mathbf{D}-\mathbf{GM}) = \begin{pmatrix}\theta_1 & 1\\ \theta_2 & 0\end{pmatrix} = \mathbf{D}^{\theta}. \tag{10}$$

Eq. (10) shows that $\mathbf{D}^{\theta}$ is always stable and invertible. Assuming the zeros are distinct, the MA characteristic polynomial is

$$\det(\mathbf{D}^{\theta}-\lambda\mathbf{I}) = \lambda^2 - \theta_1\lambda - \theta_2 = \prod_{j=1}^{2}\left(\lambda-\lambda_j^{\theta}\right) \tag{11}$$

Setting the polynomial equation in (11) to zero and solving for λ provides λ_j^{θ}, the zeros of the demand process; λ_j^{θ} are the MA eigenvalues and are related to the MA coefficients. The MA eigenvalues (zeros) are located at $\lambda_1^{\theta} = \frac{1}{2}\left(\theta_1 - \sqrt{\theta_1^2+4\theta_2}\right)$ and $\lambda_2^{\theta} = \frac{1}{2}\left(\theta_1 + \sqrt{\theta_1^2+4\theta_2}\right)$, Gaalman et al. (2022).

The companion matrix $\mathbf{D} = \mathbf{D}^{\phi}$ has two eigenvalues and two *left-hand eigenvectors*. This is written as $\mathbf{U}\mathbf{D}^{\phi} = \Lambda^{\phi}\mathbf{U}$, where Λ^{ϕ} is the (2×2) diagonal matrix of eigenvalues and $\mathbf{U}$ is the Vandermonde matrix that consists of two rows of two eigenvectors:

$$\Lambda^{\phi} = \begin{pmatrix} \lambda_1^{\phi} & 0 \\ 0 & \lambda_2^{\phi} \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{U} = \begin{pmatrix} (\lambda_1^{\phi})^1 & (\lambda_1^{\phi})^0 \\ (\lambda_2^{\phi})^1 & (\lambda_2^{\phi})^0 \end{pmatrix} = \begin{pmatrix} \varphi & 1 \\ 1 & 1 \end{pmatrix}. \tag{12}$$

These eigenvalues should be distinct (but they can be conjugate complex) otherwise, the inverse $\mathbf{U}^{-1}$ does not exist. If some eigenvalues are common, extra independent eigenvectors are required. These can be found using the Jordan form (see Kailath, 1979). An alternative to the Kailath's approach is to take the z-transformation of the demand process, and using partial fractions, identify the systems poles and zeros. The ARMA demand process is written as a transfer function of the ratio of output (zeros) to the input (poles) eigenvalues, Gaalman et al. (2022).

To write $\mathbf{D}^{\phi}$ as a function of the eigenvectors, we get

$$\mathbf{D}^{\phi} = \mathbf{U}^{-1}\Lambda^{\phi}\mathbf{U}. \tag{13}$$

We now need to determine $\mathbf{U}^{-1}$. There are many approaches available; we follow Kailath (1979) and Antsaklis and Michel (2005). Consider the matrix $\mathbf{V}$ of right eigenvectors ($\mathbf{V} \neq \mathbf{U}^{-1}$),

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ -(1+\varphi) & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_1^{\phi} & \lambda_2^{\phi} \end{pmatrix}. \tag{14}$$

Then,

$$\mathbf{U}^{-1} = \mathbf{V}\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}, \ s_l = \frac{1}{\prod_{\substack{j=1 \\ j \neq l}}^{2} \left(\lambda_l^{\phi} - \lambda_j^{\phi}\right)}. \tag{15}$$

Next the state space form of the demand process (3) will be transformed to an eigenvector form $(\mathbf{U}y_{t+1}) = \Lambda^{\phi}(\mathbf{U}\mathbf{D}^{\phi}y_t) + \mathbf{U}\mathbf{G}\eta_t$ and further simplified to $v_{t+1} = \Lambda^{\phi}v_t + \mathbf{G}^{\lambda}\eta_t$, where

$$\mathbf{G}^{\lambda} = \mathbf{U}\mathbf{G} = \begin{pmatrix} \prod_{j=1}^{2}\left(\lambda_1^{\phi} - \lambda_j^{\theta}\right) \\ \prod_{j=1}^{2}\left(\lambda_2^{\phi} - \lambda_j^{\theta}\right) \end{pmatrix} = \begin{pmatrix} (\varphi - \lambda_1^{\theta})(\varphi - \lambda_2^{\theta}) \\ (1 - \lambda_1^{\theta})(1 - \lambda_2^{\theta}) \end{pmatrix}. \tag{16}$$

Then, the demand is rewritten as $d_{t+1} = \mathbf{M}y_{t+1} + \eta_{t+1} = \mathbf{M}\mathbf{U}^{-1}(\mathbf{U}y_{t+1}) + \eta_{t+1}$. Let $\mathbf{M}^{\lambda} = \mathbf{M}\mathbf{U}^{-1}$, we get $d_{t+1} = \mathbf{M}^{\lambda}v_{t+1} + \eta_{t+1}$,

$$\mathbf{M}^{\lambda} = \mathbf{M}\mathbf{U}^{-1} = \begin{pmatrix} s_1 & s_2 \end{pmatrix}, \ s_1 = \frac{1}{\varphi - 1}, \ s_2 = \frac{1}{1-\varphi}. \tag{17}$$

Thus, the eigenvector form of ARIMA(1,1,2) is:

$$v_{t+1} = \Lambda^{\phi}v_t + \mathbf{G}^{\lambda}\eta_t, \tag{18}$$

$$d_{t+1} = \mathbf{M}^{\lambda}v_{t+1} + \eta_{t+1}. \tag{19}$$

Finally, we find the impulse response[3] for the ARIMA(1,1,2) demand in the eigenvector form to be

$$\tilde{d}_{t+1} = \mathbf{M}^{\lambda}\Lambda^{t}\mathbf{G}^{\lambda} = \sum_{j=1}^{2} r_j(\lambda_j^{\phi})^t, \ \text{where } r_i = \frac{\prod_{j=1}^{2}\left(\lambda_i^{\phi} - \lambda_j^{\theta}\right)}{\prod_{\substack{j=1 \\ j \neq i}}^{2}\left(\lambda_i^{\phi} - \lambda_j^{\phi}\right)}. \tag{20}$$

Our study uses the damped trend forecasting method to predict ARIMA(1,1,2) demand, as damped trend forecasts are optimal for the ARIMA(1,1,2) demand process, Gardner and McKenzie (1985). The impulse response for j-step ahead demand forecast is identical to the j-step ahead demand impulse

$$\tilde{d}_{t+1+j} = r_1(\varphi)^{t+j} + r_2, \tag{21}$$

where

$$r_1 = \frac{(\varphi - \lambda_1^{\theta})(\varphi - \lambda_2^{\theta})}{(\varphi - 1)}, \ r_2 = \frac{(1 - \lambda_1^{\theta})(1 - \lambda_2^{\theta})}{(1-\varphi)}. \tag{22}$$

Remark 2. This derivation assumes the AR eigenvalues are distinct, see Remark 1. It also shows the eigenvalues do not need to be stable ($-1 < \lambda_j^{\phi} < 1$ is required for stability).

For the OUT policy, Gaalman and Disney (2009) introduced the inventory gain component,

$$E[0] = 1; \ E[l] = \sum_{j=0}^{l} \tilde{d}_j = 1 + \sum_{j=0}^{l-1} \mathbf{M}(\mathbf{D}^j)\mathbf{G}, \tag{23}$$

which we also use here. We can then rewrite the ARIMA(1,1,2) demand impulse response (20) as

$$\begin{aligned} &\tilde{d}_0 = 1; \ \tilde{d}_t = \mathbf{M}\mathbf{D}^{t-1}\mathbf{G} = E[t] - E[t-1], \text{ and} \\ &E[t] = 1 + \mathbf{M}(\mathbf{I} - \mathbf{D})^{-1}(\mathbf{I} - \mathbf{D}^t)\mathbf{G}. \end{aligned} \tag{24}$$

Note, $\tilde{d}_{t+1} = (\mathbf{M}\mathbf{D}^t\mathbf{G})$.

3. Formulation of the inventory policy

The order o_t is placed at the end of period t (or the beginning of period $t+1$) with lead time k and a review period. This order is received and influences the end inventory at time $t+k+1$. Thus, like Gaalman (2006) and Gaalman and Disney (2009), we focus on the inventory state at $t+k+1$:

$$i_{t+k+1} = i_{t+k} + o_t - d_{t+k+1}. \tag{25}$$

k is a non-negative integer that represents the physical lead time, $k \in \mathbb{N}_0$. As state variables in the future need to be forecasted, we have

$$\hat{i}_{t+k+1|t+1} = \hat{i}_{t+k|t} + o_t - \hat{y}_{t+k+1|t} - E[k]\eta_{t+1}. \tag{26}$$

From (6) and (9), we can write the demand state forecast made at time $t+1$ for $k+1$ periods ahead:

$$\hat{y}_{t+k+2|t+1} = \mathbf{D}\hat{y}_{t+k+1|t} + \mathbf{D}^k\mathbf{G}\eta_{t+1}. \tag{27}$$

By this, the forecast state space system for inventory and demand, Gaalman and Disney (2009), can be written as

$$\begin{pmatrix} \hat{i}_{t+k+1|t+1} \\ \hat{y}_{t+k+2|t+1} \end{pmatrix} = \begin{pmatrix} 1 & -\mathbf{M} \\ 0 & \mathbf{D} \end{pmatrix} \begin{pmatrix} \hat{i}_{t+k|t} \\ \hat{y}_{t+k+1|t} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} o_t + \begin{pmatrix} -E[k] \\ \mathbf{D}^k\mathbf{G} \end{pmatrix} \eta_{t+1} \tag{28}$$

The order in the POUT policy is

$$o_t = \begin{pmatrix} -f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i}_{t+k|t} \\ \hat{d}_{t+k+1|t} \end{pmatrix}. \tag{29}$$

f is the proportional feedback controller. $0 \leq f < 2$ is required for stability, Disney (2008). When $f = 1$, the POUT policy (29) degenerates into the OUT policy.

As $\mathbf{M} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ for ARIMA(1,1,2) demand processes, substituting (29) into (28) yields the complete forecast state space recursion for the damped trend POUT policy:

$$\begin{pmatrix} \hat{i}_{t+k+1|t+1} \\ \hat{y}_{t+k+2|t+1} \end{pmatrix} = \begin{pmatrix} (1-f) & 0 \\ 0 & \mathbf{D} \end{pmatrix} \begin{pmatrix} \hat{i}_{t+k|t} \\ \hat{y}_{t+k+1|t} \end{pmatrix} + \begin{pmatrix} -E[k] \\ \mathbf{D}^k\mathbf{G} \end{pmatrix} \eta_{t+1}. \tag{30}$$

The eigenvalues of this system can be found from the determinant,

$$\det \begin{pmatrix} (1-f) - \lambda & 0 \\ 0 & \mathbf{D} - \mathbf{I}_m\lambda \end{pmatrix} = 0. \tag{31}$$

Solving $((1-f) - \lambda)(\mathbf{D} - \mathbf{I}_m\lambda) = 0$ gives us the eigenvalue $\lambda_1 = (1-f)$ and the two eigenvalues of the ARIMA(1,1,2) demand $\mathbf{D}$: λ_1^{ϕ} and λ_2^{ϕ}. Since $0 < f < 2$, $-1 < \lambda_1 = (1-f) < 1$ means the real λ_1 lies within the stability area, the unit circle in the complex plane.[4]

4. Expressions for variances

Before deriving the order and inventory variance expressions, we first study the variance of demand. The demand at time $t+k+1$, d_{t+k+1},

[3] We use a tilde to indicate an impulse response. The impulse response is the system's response to an impulse input. That is, $\eta_{t=0} = 1$ and $\eta_{t=\{1,2,3...\}} = 0$. The impulse response is also equal to the system's autocovariance function.

[4] Note: The alternative formulation of the POUT policy is given the literature: $o_t = \hat{d}_{t+k+1|t} + f\left(i^{\star} - i_t + \sum_{i=1}^{k} \hat{d}_{t+k+i|t} - \sum_{i=1}^{k} o_{t-i}\right)$. Here $i^{\star}$ is the target inventory (safety stock).

can be formulated as a function of the forecast $\hat{d}_{t+k+1|t}$ made at time t for d_{t+k+1},

$$d_{t+k+1} = \hat{d}_{t+k+1|t+k} + \eta_{t+k+1} = \hat{d}_{t+k+1|t} + \eta_{t+k+1} + \sum_{j=0}^{k-1} \mathbf{M}(\mathbf{D}^j)\mathbf{G}\eta_{t+k-j}. \quad (32)$$

Remark 3. As $\hat{d}_{t+k+1|t}$ is a function of $\{\eta_t, \eta_{t-1}, \eta_{t-2}, \dots\}$ and the error terms in (32) are a function of $\{\eta_{t+k+1}, \eta_{t+k}, \dots, \eta_{t+1}\}$, the components in (32) are uncorrelated.

Due to Remark 3, the demand variance in period $t+k+1$ is given by

$$\begin{aligned}\mathbb{V}[d_{t+k+1}] &= \mathbb{V}[\hat{d}_{t+k+1|t}] + \mathbb{V}[\eta]\left(1 + \sum_{j=0}^{k-1}(\mathbf{M}\mathbf{D}^j\mathbf{G})^2\right) \\ &= \mathbb{V}[\hat{d}_{t+k+1|t}] + \mathbb{V}[\eta]\left(1 + \sum_{j=0}^{k-1}(\tilde{d}_j)^2\right).\end{aligned} \quad (33)$$

Remark 4. As $\hat{d}_{t+k+1|t}$ contains an infinite number of error terms (Remark 3), the first component in (33), $\mathbb{V}[\hat{d}_{t+k+1|t}]$, is infinite. The second component in (33), $\sum_{j=0}^{k-1}(\tilde{d}_j)^2\mathbb{V}[\eta]$, is a finite sum. The demand variance $\mathbb{V}[d_{t+k+1}]$ is infinite.

4.1. Inventory variance

Eq. (30) shows that the inventory forecast $\hat{i}_{t+k+1|t+1} = (1-f)\hat{i}_{t+k|t} - E[k]\eta_{t+1}$ is stable. As $\mathbb{V}[\hat{i}_{t+k+1|t+1}] \equiv \mathbb{V}[\hat{i}_{t+k|t}]$, the variance of $\hat{i}_{t+k+1|t+1}$ is

$$\mathbb{V}[\hat{i}_{t+k+1|t+1}] = \left(\frac{1}{f(2-f)}\right)(E[k])^2\mathbb{V}[\eta]. \quad (34)$$

The inventory can be written as

$$i_{t+k+1} = \hat{i}_{t+k+1|t} + (i_{t+k+1} - \hat{i}_{t+k+1|t}), \quad (35)$$

where $(i_{t+k+1} - \hat{i}_{t+k+1|t}) = -\sum_{j=1}^{k+1}(d_{t+j} - \hat{d}_{t+j|t}) = -\sum_{j=1}^{k+1} E[k+1-j]\eta_{t+j}$. The component, $(i_{t+k+1} - \hat{i}_{t+k+1|t})$, is the inventory forecast error and is uncorrelated with the inventory forecast, Gaalman and Disney (2009). We then obtain the variance for the inventory forecast error

$$\mathbb{V}[i_{t+k+1} - \hat{i}_{t+k+1|t}] = \sum_{l=0}^{k}(E[l])^2\mathbb{V}[\eta], \quad (36)$$

where

$$E[l] = \frac{\theta_2 + \varphi^l\left(\varphi(\varphi-\theta_1)-\theta_2\right) + l(\varphi-1)(\theta_1+\theta_2-1) + \varphi(\theta_1-2)+1}{(\varphi-1)^2}. \quad (37)$$

In the OUT case, $\hat{i}_{t+k+1|t} = 0$. However in the POUT case, $\hat{i}_{t+k+1|t} \neq 0$. Consider (30) and (36), the inventory forecast variance in the POUT policy can be obtained

$$\mathbb{V}[\hat{i}_{t+k+1|t}] = (1-f)^2\mathbb{V}[\hat{i}_{t+k|t}] = \left(\frac{(1-f)^2}{f(2-f)}\right)(E[k])^2\mathbb{V}[\eta]. \quad (38)$$

The POUT policy's inventory variance then becomes

$$\mathbb{V}[i_{t+k+1}|\mathrm{POUT}] = \mathbb{V}[\eta]\left(\left(\frac{1}{f(2-f)}\right)(E[k])^2 + \sum_{l=0}^{k-1}(E[l])^2\right). \quad (39)$$

When $f=1$, $\mathbb{V}[i_{t+k+1}|\mathrm{OUT}] = E[k]^2 + \sum_{l=0}^{k-1}E[l]^2\mathbb{V}[\eta] = \sum_{l=0}^{k}E[l]^2\mathbb{V}[\eta]$, which concurs with the OUT case. The function $(f(2-f))^{-1}$ is positive and is convex in f with a minimum of 1 at $f=1$ and asymptotes to infinity at $f=0$ and $f=2$. Thus, the OUT policy's inventory variance is the minimal case of the POUT policy. The derivative, $d\left(\mathbb{V}[i_{t+k+1}|\mathrm{POUT}]\right)/df$ also confirms this; details are provided in Appendix A. The difference

$$\mathbb{V}[i_{t+k+1}|\mathrm{POUT}] - \mathbb{V}[i_{t+k+1}|\mathrm{OUT}] = \frac{(f-1)^2}{f(2-f)}(E[k])^2\mathbb{V}[\eta] > 0, \quad (40)$$

also verifies these claims.

4.2. Covariance of demand forecast and inventory forecast

The state space expression (30) shows that both $\hat{i}_{t+k+1|t+1}$ and $\hat{y}_{t+k+2|t+1}$ have the same error component η_{t+1}. Then we have

$$\mathrm{cov}[\hat{i}_{t+k+1|t+1}, \hat{y}_{t+k+2|t+1}] = ((1-f)\mathbf{D})\,\mathrm{cov}[\hat{i}_{t+k|t}, \hat{y}_{t+k+1|t}]. \quad (41)$$

The inventory is a scalar, thus an alternative expression is

$$\begin{aligned}&\mathrm{cov}[\hat{y}_{t+k+2|t+1}, \hat{i}_{t+k+1|t+1}] \\ &= ((1-f)\mathbf{D})\,\mathrm{cov}[\hat{y}_{t+k+1|t}, \hat{i}_{t+k|t}] - (\mathbf{D}^k\mathbf{G})E[k]\mathbb{V}[\eta].\end{aligned} \quad (42)$$

In the OUT case ($f=1$), $\mathrm{cov}[\hat{y}_{t+k+2|t+1}, \hat{i}_{t+k+1|t+1}] = -(\mathbf{D}^k\mathbf{G})E[k]\mathbb{V}[\eta]$. Given infinite past observations, $\mathrm{cov}[\hat{y}_{t+k+2|t+1}, \hat{i}_{t+k+1|t+1}] \equiv \mathrm{cov}[\hat{y}_{t+k+1,t}, \hat{i}_{t+k|t}]$ holds. Then we obtain

$$\mathrm{cov}[\hat{y}_{t+k+1|t}, \hat{i}_{t+k|t}] = -(\mathbf{I}_m - (1-f)\mathbf{D})^{-1}(\mathbf{D}^k\mathbf{G})E[k]\mathbb{V}[\eta]. \quad (43)$$

Thus, the covariance between the demand forecast and the inventory forecast is

$$\mathrm{cov}[\hat{d}_{t+k+1|t}, \hat{i}_{t+k|t}] = \mathbf{M}\mathrm{cov}[\hat{y}_{t+k+1|t}, \hat{i}_{t+k|t}] = -W[f,k]E[k]\mathbb{V}[\eta], \quad (44)$$

where

$$W[f,k] = \mathbf{M}((\mathbf{I}_m - (1-f)\mathbf{D})^{-1}\mathbf{D}^k)\mathbf{G}. \quad (45)$$

The $(\mathbf{I}_m - (1-f)\mathbf{D})^{-1}$ in (45) is the Woodbury matrix identity, Woodbury (1950), $(A-B)^{-1} = \sum_{j=0}^{\infty}(A^{-1}B)^jA^{-1}$. Let $A = \mathbf{I}_m$ and $B = (1-f)\mathbf{D}$, then

$$(\mathbf{I}_m - (1-f)\mathbf{D})^{-1} = \sum_{j=0}^{\infty}((1-f)\mathbf{D})^j. \quad (46)$$

Substituting (13) into the above equation, we have

$$(\mathbf{I}_m - (1-f)\mathbf{D})^{-1} = \mathbf{U}^{-1}\sum_{j=0}^{\infty}\begin{pmatrix}((1-f)\lambda_1^{\phi})^j & \\ 0 & ((1-f)\lambda_2^{\phi})^j\end{pmatrix}\mathbf{U}. \quad (47)$$

As $-1 < (1-f)\lambda_i^{\phi} < 1$, the infinite summation $\sum_{j=0}^{\infty}((1-f)\lambda_i^{\phi})^j = (1-(1-f)\lambda_i^{\phi})^{-1}$. Thus,

$$W[f,k] = \mathbf{M}((\mathbf{I}_m - (1-f)\mathbf{D})^{-1}\mathbf{D}^k)\mathbf{G} = \sum_{i=1}^{2}\frac{r_i(\lambda_i^{\phi})^k}{1-(1-f)\lambda_i^{\phi}}. \quad (48)$$

Substituting (22) into (48), the $W[f,k]$ in the POUT policy, when the damped trend mechanism produces optimal forecasts of the ARIMA (1,1,2) demand, becomes

$$W[f,k] = \left(\frac{(\varphi-\lambda_1^{\theta})(\varphi-\lambda_2^{\theta})}{(\varphi-1)}\right)\left(\frac{(\varphi)^k}{1-(1-f)\varphi}\right) + \frac{(1-\lambda_1^{\theta})(1-\lambda_2^{\theta})}{(1-\varphi)(1-(1-f))}. \quad (49)$$

4.3. Order variance

From (29), the order variance expression can be written as

$$\mathbb{V}[o_t] = \mathbb{V}[\hat{d}_{t+k+1|t}] - 2f\,\mathrm{cov}[\hat{d}_{t+k+1|t}, \hat{i}_{t+k|t}] + f^2\mathbb{V}[\hat{i}_{t+k|t}]. \quad (50)$$

We have shown that $\hat{d}_{t+k+1|t}$ and $\hat{i}_{t+k|t}$ are correlated. Substitution of the inventory forecast variance (34) and the covariance (44) results in

$$\mathbb{V}[o_t] = \mathbb{V}[\hat{d}_{t+k+1|t}] + 2fW[f,k]E[k]\mathbb{V}[\eta] + \left(\frac{f}{2-f}\right)(E[k])^2\mathbb{V}[\eta]. \quad (51)$$

Substituting (49) into (51), the order variance in the POUT policy becomes

$$\begin{aligned}\mathbb{V}[o_t] = \mathbb{V}[\hat{d}_{t+k+1|t}] &+ 2\sum_{i=1}^{2}\left(\frac{f}{1-(1-f)\lambda_i^{\phi}}\right)r_i(\lambda_i^{\phi})^kE[k]\mathbb{V}[\eta] \\ &+ \left(\frac{f}{2-f}\right)(E[k])^2\mathbb{V}[\eta].\end{aligned} \quad (52)$$

The order variance is also infinite due to the infinite $\mathbb{V}[\hat{d}_{t+k+1|t}]$. The remainder of the components in the order variance function are finite.

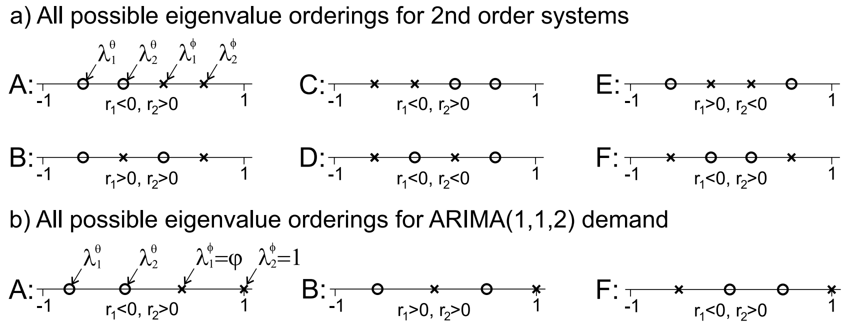

Fig. 2. The possible eigenvalue orderings. Panel a: All possible eigenvalue ordering for second order systems. Panel b: Possible eigenvalue orderings for ARIMA(1,1,2).

5. Comparison of bullwhip effect produced by the OUT and POUT policies

The order impulse response of the OUT policy under ARIMA(1,1,2) demand consists of two components. When $t = 0$, $\tilde{o}_0$ is the sum of the first $k+1$ demand impulses; when $t > 0$, the order impulse is identical to the $(k+1)$ period-ahead demand impulse $\tilde{o}_t = \tilde{d}_{t+k+1}$,

$$\tilde{o}_t = \begin{cases} \sum_{t=0}^{k+1} \tilde{d}_t = E[k+1], & \text{if } t = 0, \\ \tilde{d}_{t+k+1} = r_1\varphi^{t+k} + r_2, & \text{if } t > 0. \end{cases} \tag{53}$$

As the demand variance for ARIMA(1,1,2) is infinite, we measure bullwhip as the difference between the order variance and the demand variance, $CB[k] = \mathbb{V}[o_t] - \mathbb{V}[d_{t+k+1}]$. This leads to the cancellation of the infinite component, $\mathbb{V}[\hat{d}_{t+k+1|t}]$, in the functions $\mathbb{V}[o_t]$ and $\mathbb{V}[d_{t+k+1}]$, leaving a finite bullwhip measure for analysis:

$$CB[k|\text{OUT}] = \mathbb{V}[\eta]\left(2(r_2 + r_1\varphi^k)E[k] + (E[k])^2 - \left(1 + \sum_{j=0}^{k-1}(\tilde{d}_j)^2\right)\right). \tag{54}$$

For the POUT policy, the bullwhip difference can be written as

$$\begin{aligned} CB[k|\text{POUT}] &= \mathbb{V}[\eta]\left(2fW[f,k]E[k] + \left(\frac{f}{2-f}\right)(E[k])^2 \right. \\ &\quad \left. - \left(1 + \sum_{j=0}^{k-1}(\tilde{d}_j)^2\right)\right) \\ &= \mathbb{V}[\eta]\left(2\sum_{i=1}^{2}\left(\frac{f}{1-(1-f)\lambda_i^{\phi}}\right)r_i(\lambda_i^{\phi})^k E[k] \right. \\ &\quad \left. + \left(\frac{f}{2-f}\right)(E[k])^2 - \left(1 + \sum_{j=0}^{k-1}(\tilde{d}_j)^2\right)\right). \end{aligned} \tag{55}$$

To compare the bullwhip between POUT and OUT policies, we measure $CB[k|\text{OUT}] - CB[k|\text{POUT}] = \mathbb{V}[o_t|\text{OUT}] - \mathbb{V}[o_t|\text{POUT}]$. Using (54) and (55),

$$CB[k|\text{OUT}] - CB[k|\text{POUT}] = 2(1-f)E[k]\left(\frac{E[k]}{2-f} + \frac{r_1(1-\varphi)\varphi^k}{1-(1-f)\varphi}\right)\mathbb{V}[\eta]. \tag{56}$$

Note, $\lambda_1^{\phi} = \varphi$, thus, when $\varphi < 0$ an odd-even lead time effect can be directly observed from (54), (55), and (56). Further note when $f = 1$, $CB[k|\text{OUT}] - CB[k|\text{POUT}] = 0$

Li et al. (2023) show that ARIMA(1,1,2) eigenvalues (the poles and zeros) have three orderings, A, B, and F, out of 6 possible cases, see Fig. 2, Gaalman et al. (2022). Note, the zeros of Type A and B, λ_1^{θ}, λ_2^{θ}, can be negative or positive. With knowledge of (22), we can deduce the following Lemma.

Lemma 5.1. *For Type A ARIMA(1,1,2) demand, $r_2 > 0$ and $r_1 < 0$; for Type B demand, $r_2 > 0$ and $r_1 > 0$; and for Type F, $r_2 > 0$ and $r_1 < 0$.*

Proof. The proof of Lemma 5.1 can be found in Gaalman et al. (2022). □

We will now examine each Type (A, B, and F) of the ARIMA(1,1,2) demand process in detail, studying their impulse response and investigating the OUT and POUT policy's bullwhip performance.

5.1. Type A ARIMA(1,1,2) demand

When the eigenvalues are ordered zero-zero-pole-pole, we have Type A ARIMA(1,1,2) demand.

Lemma 5.2. *The demand impulse response $\tilde{d}_t$ is always positive for Type A ARIMA(1,1,2) processes.*

Proof. Observe from Fig. 2 (Type A), we know that the distance between $\lambda_2^{\phi} = 1$ and λ_1^{θ} is greater than the distance between $\lambda_1^{\phi} = \varphi$ and λ_1^{θ}, that is $(1-\lambda_1^{\theta}) > (\varphi-\lambda_1^{\theta}) > 0$. Similarly, we have $(1-\lambda_2^{\theta}) > (\varphi-\lambda_2^{\theta}) > 0$. Rewriting (20) as

$$\tilde{d}_{t+1} = \frac{(1-\lambda_1^{\theta})(1-\lambda_2^{\theta}) - (\varphi-\lambda_1^{\theta})(\varphi-\lambda_2^{\theta})\varphi^t}{(1-\varphi)}, \tag{57}$$

and knowing that $|\varphi^t| < 1$ and $\tilde{d}_0 = 1$, $\forall t \in \mathbb{N}_0$, $\tilde{d}_{t+1} > 0$ is proved. □

For Type A, Lemma 5.2 holds for both positive and negative φ, recall $-1 < (\varphi = \lambda_1^{\phi}) < 1$. A positive demand impulse response means the OUT policy creates a bullwhip effect that always increases in the lead time, Gaalman et al. (2022). When $\varphi > 0$ the Type A_1 is present in the taxonomy of Gaalman et al. (2022); when $\varphi < 0$ Type A_{2i} is present. The other instances of Type A second order systems are not possible as always $\lambda_2^{\phi} = 1$ for ARIMA(1,1,2) demand.

Lemma 5.3. *For Type A ARIMA(1,1,2) demand, $E[k] > 0$.*

Proof. This can be proved directly by recalling $E[k] = \sum_{t=0}^{k} \tilde{d}_t$ and $\tilde{d}_t > 0$. □

Gaalman et al. (2022) finds the demand impulse for ARMA(2,2) is positive for the Type A_1 case $\left(\lambda_1^{\phi} = \varphi\right) > 0$ but is not always positive for $(\lambda_1^{\phi} = \varphi) < 0$. We show for any Type A ARIMA(1,1,2) demand, the demand impulse is always positive. The difference for negative φ is due to the second ARIMA(1,1,2) pole being at unity, $\lambda_2^{\phi} = 1$. Then, the OUT's order impulse response $\tilde{o}_t = E[k+1] > 0$. When $t > 0$, $\tilde{o}_t = \tilde{d}_{t+k+1} > 0$, as assured by Lemma 5.2. Therefore, we conclude the OUT policy's order impulse response is always positive for Type A ARIMA(1,1,2) demand.

For Type A ARIMA(1,1,2) demand, the proportional controller in the POUT policy value needs to be carefully tuned based on the eigenvalues of the demand process in order to reduce bullwhip. As $r_1 < 0$, $E[k] > 0$, it is possible that (56) is negative (that is, it is possible that $CB[k|\text{POUT}] > CB[k|\text{OUT}]$). Solving $CB[k|\text{OUT}] - CB[k|\text{POUT}] > 0$ (see (56)) for f while $0 \le f < 2$, we obtain,

Table 1
The value of the lower bound in f for different k within the Type A ARIMA(1,1,2) parameter sets.

Parameters			Lead time k										
φ	θ_1	θ_2	0	1	2	3	4	5	6	7	8	9	10
−0.6	−1.4	−0.5	0	0	0	0	0	0	0	0	0	0	0
−0.1	−1.77	−0.78	0.25	0	0	0	0	0	0	0	0	0	0
0.5	0.2	0.1	0	0	0	0	0	0	0	0	0	0	0
0.75	0.1	0.05	0.53	0	0	0	0	0	0	0	0	0	0
0.9	0.3	0.01	0.67	0.25	0.08	0.01	0	0	0	0	0	0	0
0.99	0.4	0.1	0.65	0.31	0.18	0.11	0.08	0.06	0.04	0.03	0.02	0.02	0.01

Theorem 5.4. *Define f' as*

$$f' = \frac{(\varphi-1)\left(E[k]+2r_1\varphi^k\right)}{E[k]\varphi + r_1(\varphi-1)\varphi^k}. \tag{58}$$

For positive φ, if $f|_{0\le f<1} > f'$ or $f|_{1<f<2} < f'$, then $CB[k|POUT] < CB[k|OUT]$. When φ is negative, $f|_{0\le f<1} > f'$ for $k=0$. When $k>0$, any $0 \le f < 1$ ensures $CB[k|POUT] < CB[k|OUT]$.

Proof. The proof of Theorem 5.4 is given in Appendix B. □

Expanding out $E[k]$ and simplifying f',

$$f' = \frac{(\varphi-1)\left(\theta_2+(2\varphi-1)\varphi^k\left(\varphi^2-\theta_1\varphi-\theta_2\right)+k(\varphi-1)(\theta_1+\theta_2-1)+(\theta_1-2)\varphi+1\right)}{((\varphi-1)\varphi+1)\varphi^k\left(\varphi^2-\theta_1\varphi-\theta_2\right)+\varphi(\theta_2+k(\varphi-1)(\theta_1+\theta_2-1)+(\theta_1-2)\varphi+1)}. \tag{59}$$

Both the numerator and denominator of (59) contain a term that is linear in k, and also a term that is either: decreasing or increasing in k when $\varphi > 0$, or is oscillating in k when $\varphi < 0$, or is zero when $\varphi = 0$. Note, when $\varphi = 0$ the ARIMA(1,1,2) demand degenerates into an ARIMA(0,1,2) process. Some numerical examples of the lower bound in f for some Type A ARIMA(1,1,2) demand processes are shown in Table 1.

We also derive a sufficient condition of the lead time k, that ensures $CB[k|\text{POUT}] < CB[k|\text{OUT}]$ for Type A ARIMA(1,1,2) demand with any $0 \le f < 1$:

$$k_{\min} > \frac{\varphi-2}{\varphi-1} - \frac{W\left[\varphi^{\frac{1}{1-\varphi}+2}\log[\varphi](1-\varphi)^{-1}\right]}{\log[\varphi]}. \tag{60}$$

Here, $W[\cdot]$ is the Lambert W function, Disney and Warburton (2012). The feasibility of the POUT policy can be evaluated by considering both the demand and the organisation's lead time (60). Theorem 5.4 informs the selection of a proportional feedback controller value that targets the desired bullwhip performance. The POUT policy may not always be dynamically superior to the OUT policy. However, Theorem 5.4 and (60) can be used to define a POUT policy that surpasses the OUT policy's bullwhip performance.

Fig. 3 illustrates an example of Type A_1 ARIMA(1,1,2) demand process when $\varphi = 0.9$, $\theta_1 = -1.7$, and $\theta_2 = -0.72$; note the signed log scale on the y-axis. The corresponding eigenvalues in Panel a are $\lambda_1^\phi = 0.9$, $\lambda_2^\phi = 1$, $\lambda_1^\theta = -0.9$, and $\lambda_2^\theta = -0.8$. In Panel b, the demand impulse originates from unity and increases. Tyspkin's relation (Tsypkin, 1964; Disney and Towill, 2003; Boute et al., 2022), reveals both the demand and order variances are infinite (as they are equal to the sum of the squared impulse response over all non-negative t).

In addition, we observe the OUT's order impulse response is larger than the demands impulse response. This implies the OUT policy generates the bullwhip effect, confirming our previous analysis. The OUT's order impulse response is a tick shape, initially rising above the POUT's order impulse response before falling below it. However, using the figure alone, it is unclear whether the POUT policy will generate more or less bullwhip than the OUT policy. Instead, we must resort to the test in Theorem 5.4.

Fig. 3c illustrates the difference in the bullwhip between these two policies; note, the signed log scale on the y-axis. For $k = 0$, $CB[k|\text{OUT}] - CB[k|\text{POUT}] < 0$ for all $0 \le f < 1$. That is, the POUT policy always generates more bullwhip than the OUT policy for Type A ARIMA(1,1,2) demand when $k = 0$ and $0 \le f < 1$. For $k = 5$, as predicted by (60), any POUT policy with proportional controller $0 \le f < 1$ exhibits less bullwhip than the OUT policy. Fig. 3c also suggests when $1 < f < 2$, the POUT policy exhibits limited efficacy at mitigating the bullwhip effect compared to the OUT policy across a wide range of lead-times (only for $k = 0$ does the POUT policy exhibit less bullwhip that the OUT policy; even than, it is only for a small range of $f > 1$.) In addition, unlike the i.i.d. or AR(1) demand cases where $f \to 0$ reduces the bullwhip significantly, a lower f value might result in higher bullwhip in the POUT policy than the OUT policy for Type A ARIMA(1,1,2) demand with small lead times. These observations corroborate our prior analysis, underscoring the critical role of the lead time and the proportional feedback controller at mitigating the bullwhip effect within the POUT policy. Fig. 3d shows the impact of lead time on the bullwhip effect, confirming Panels b and c. Both policies generate bullwhip that increases in the lead time.

5.2. Type B ARIMA(1,1,2) demand

When the eigenvalues are ordered zero-pole-zero-pole, we have Type B ARIMA(1,1,2) demand.

Lemma 5.5. *The demand impulse response $\tilde{d}_t$ is always positive for Type B ARIMA(1,1,2) processes with non-negative φ. When $\varphi < 0$, $\varphi > -r_2/r_1$ ensures demand impulse is always positive. When $\varphi < 0$ and $\varphi < -r_2/r_1$ the demand impulse response is initially oscillating positive and negative, before becoming and remaining positive.*

Proof. For non-negative φ, this is the B_1 case in Gaalman et al. (2022). Here, $(1-\lambda_1^\theta)(1-\lambda_2^\theta) > 0$ and $(\varphi-\lambda_1^\theta)(\varphi-\lambda_2^\theta) < 0$. These facts, in conjunction with $\tilde{d}_0 = 1$, reveal $\tilde{d}_t > 0$. For negative φ, as $-\varphi < 1$, the positivity of the demand impulse is determined by the sign of $\tilde{d}_{t+1}$ at $t = 1$. If $(\lambda_1^\phi/\lambda_2^\phi)^1 > -r_2/r_1$, $\tilde{d}_2 > 0$ and all subsequent $\tilde{d}_{t+1} > 0$. This is equivalent to the B_{2ia} case in Gaalman et al. (2022). □

The always positive demand impulse implies that the OUT policy generates bullwhip that increases in the lead time for Type B ARIMA(1,1,2) demand. When $\varphi < -r_2/r_1$, the case B_{2ib} in Gaalman et al. (2022) is present. In this case, demand is initially oscillating positive and negative, implying the OUT policy exhibits an odd-even lead time effect in the bullwhip effect. That is, and increase in the lead time can result in a larger or smaller amount of bullwhip, depending on the parity of the lead time. When the lead time is sufficiently large, the bullwhip always increases in the lead time. The other B cases in Gaalman et al. (2022) are not possible under ARIMA(1,1,2) demand.

Lemma 5.6. *For Type B ARIMA(1,1,2) demand, $E[k] > 0$.*

Proof. Expanding (23),

$$E[k] = 1 + kr_2 + \frac{1-\varphi^k}{1-\varphi}r_1. \tag{61}$$

It is easy to notice that $\frac{1-\varphi^k}{1-\varphi} \ge 0$. Considering $\{r_1, r_2\} > 0$ for Type B ARIMA(1,1,2) demand and $k \ge 0$, $E[k] > 0$. □

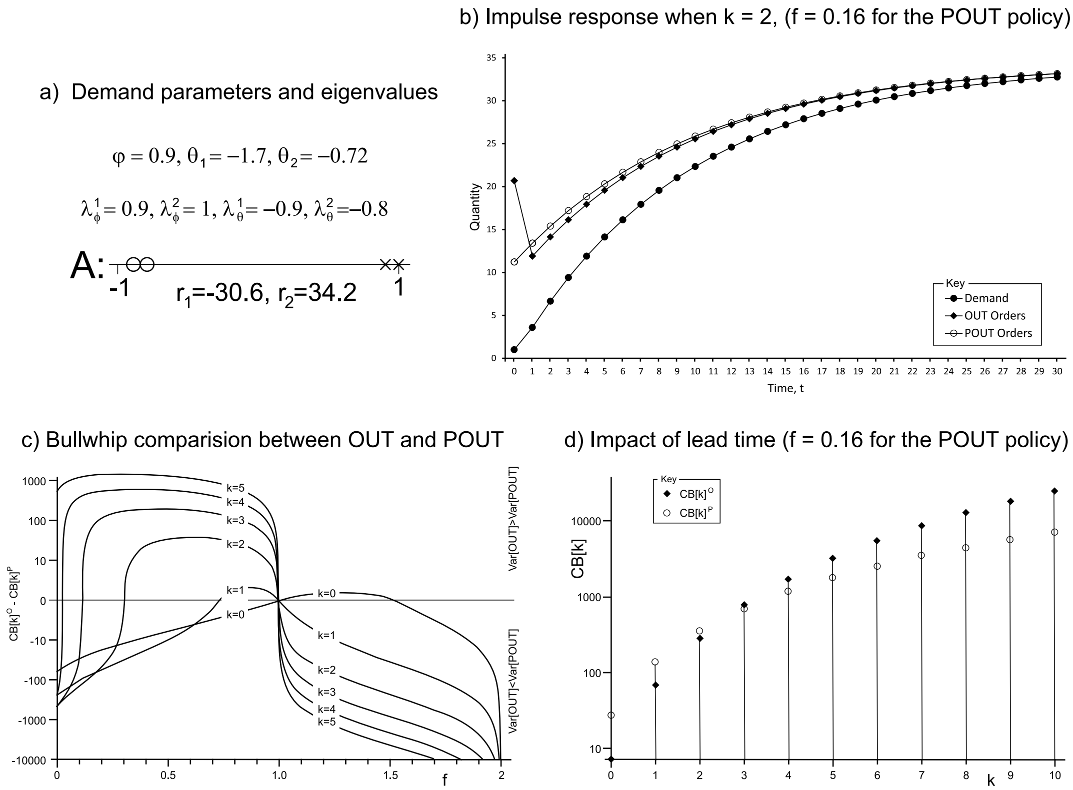

Fig. 3. Bullwhip analysis for Type A_1 ARIMA(1,1,2) demand.

Lemma 5.6 implies d_0 is increasing in k.

Lemma 5.7. *For Type B ARIMA(1,1,2) demand, the lower bound of $E[k] + 2r_1\varphi^k$ is $\frac{1}{2} + kr_2$; that is $E[k] + 2r_1\varphi^k \geq \frac{1}{2} + kr_2$.*

Proof.

$$E[k] + 2r_1\varphi^k = 1 + \frac{r_1\left(1 - 2\phi^{k+1} + \phi^k\right)}{1-\phi} + kr_2 \tag{62}$$

$$= 1 + \frac{(\varphi - \lambda_1^\theta)(\lambda_2^\theta - \varphi)(1 - 2\phi^{k+1} + \phi^k)}{(1-\varphi)^2} + kr_2 \tag{63}$$

For Type B ARIMA(1,1,2) demand, $0 < (\varphi - \lambda_1^\theta)(\lambda_2^\theta - \varphi) < 1$. When $\varphi > 0$, $\frac{1-2\phi^{k+1}+\phi^k}{(1-\varphi)^2} > 1$. When $-1 \leq \varphi < 0$ and $k = 0$, $1 \leq \frac{1-2\phi^{k+1}+\phi^k}{(1-\varphi)^2} < 2$. When $-1 \leq \varphi < 0$ and k is even, $0 < \frac{1-2\phi^{k+1}+\phi^k}{(1-\varphi)^2} < 1$. When $-1 \leq \varphi < 0$ and k is odd, $-0.5 \leq \frac{1-2\phi^{k+1}+\phi^k}{(1-\varphi)^2} < 1$. Thus, we can deduce that $\left(1 + \frac{(\varphi-\lambda_1^\theta)(\lambda_2^\theta-\varphi)(1-2\phi^{k+1}+\phi^k)}{(1-\varphi)^2}\right) \geq 0.5$. $E[k]+2r_1\varphi^k \geq \frac{1}{2}+kr_2$ is proved. □

Both Lemmas 5.6 and 5.7 provide important properties of Type B ARIMA(1,1,2) demand that will be used in the following Theorem.

Theorem 5.8. *For $\varphi > 0$ or for $\varphi < 0$ with even lead time k, $0 \leq f < 1$ leads to $CB[k|POUT] < CB[k|OUT]$, while $1 < f < 2$ results in $CB[k|POUT] > CB[k|OUT]$. For $\varphi < 0$ with odd lead time k, it is always that $CB[k|POUT] < CB[k|OUT]$ if $0 \leq f < 1$.*

Proof. When $\varphi > 0$ or $\varphi < 0$ with even lead time k, $\varphi^k > 0$. In addition, $1 - (1-f)\varphi > 0$, and $2 - f > 0$ when $0 \leq f < 2$. We also have shown, for Type B ARIMA(1,1,2) demand, $r_1 > 0$ and via Lemma 5.6, $E[k] > 0$. Then, the sign of $CB[k|\text{OUT}] - CB[k|\text{POUT}] > 0$ (the sign of (56)) is only determined by the sign of $(1-f)$. Thus, when $0 \leq f < 1$, $CB[k|\text{OUT}] - CB[k|\text{POUT}] > 0$, and when $1 < f < 2$, $CB[k|\text{OUT}] - CB[k|\text{POUT}] < 0$ are proved for Type B ARIMA(1,1,2) demand with positive φ as well as negative φ with even lead time k.

Next, we study the Type B ARIMA(1,1,2) demand case with $\varphi < 0$ and odd lead time k. When $f = 1$, (56) $= 0$. When $f = 0$, (56) reduces to

$$CB[k|\text{OUT}] - CB[k|\text{POUT}] = 2E[k]\left(\frac{E[k]}{2} + r_1\varphi^k\right)\mathbb{V}[\eta]. \tag{64}$$

Consider $k \in \mathbb{N}_0$ and $r_2 > 0$ for Type B ARIMA(1,1,2) demand, and Lemma 5.7, we conclude $CB[k|\text{OUT}]-CB[k|\text{POUT}] > 0$ at $f = 0$. Then, we investigate the sign of (56) in the open interval $0 < f < 1$. Rewrite (56) as

$$CB[k|\text{OUT}] - CB[k|\text{POUT}] = 2(1-f)E[k]\left(\frac{h[f]}{g[f]}\right)\mathbb{V}[\eta], \tag{65}$$

where

$$\frac{h[f]}{g[f]} = \frac{f\left(-E[k]\phi - r_1\varphi^{k+1} + r_1\varphi^k\right) + (\varphi - 1)\left(E[k] + 2r_1\varphi^k\right)}{f^2\varphi + f(1-3\varphi) + 2\phi - 2}. \tag{66}$$

$g[f]$ and $h[f]$ are univariate polynomials of f with real coefficients. Let $v[g]$ denote the number of sign variations in the sequence of the coefficients of $g[f]$. Also, let $v_\tau[g]$ denote the number of sign variations in the sequence of the coefficients of the polynomial $g[f+\tau]$. Thus, $g[f+0] = g[f]$. $g[0] = -2 + 2\varphi < 0$ when $-1 < \varphi < 0$; $g[1] = -1$, and $v_0[g] - v_1[g] = 2 - 2 = 0$. Budan's Theorem suggests there are no real roots for the polynomial $g[f]$ in the open interval $(0,1)$ and $g[f] < 0$. $h[0] = (\varphi - 1)\left(E[k] + 2r_1\varphi^k\right) < 0$, due to Lemma 5.7.

$$h[1] = r_1(\varphi - 1)\varphi^k - E[k]$$
$$= -\left(1 + \frac{(\varphi - \lambda_1^\theta)(\lambda_2^\theta - \varphi)(1 - 2\varphi^{k+1} + \varphi^{k+2})}{(1-\varphi)^2} + kr_2\right). \tag{67}$$

As $-0.5 \leq \frac{1-2\varphi^{k+1}+\varphi^{k+2}}{(1-\varphi)^2} < 1$ for $-1 \leq \varphi < 0$ and k even, $h[1] < 0$. $v_0[h] - v_1[h] = 0 - 0 = 0$. Budan's Theorem suggests there are no real

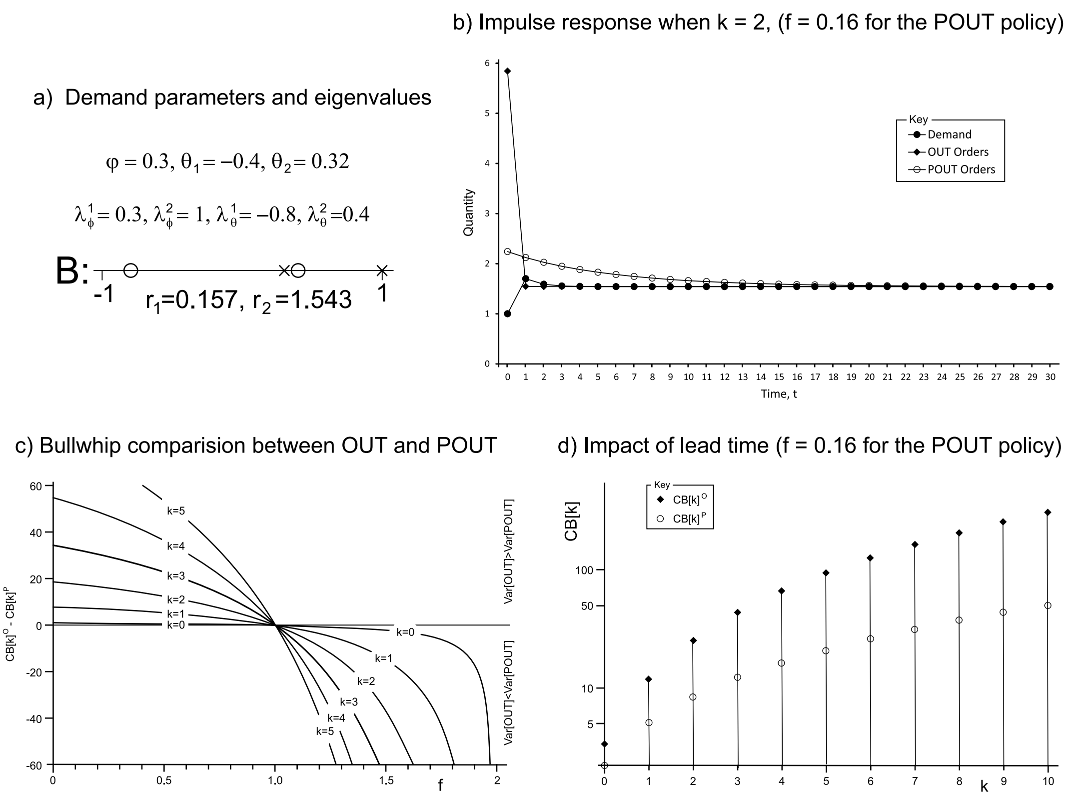

Fig. 4. Bullwhip analysis for Type B_1 ARIMA(1,1,2) demand.

roots for the polynomial $h[f]$ in the interval $0 \le f < 1$ and $h[f] < 0$. Since $\{h[f], g[f]\} < 0$ in the interval $0 < f < 1$, we know (66) > 0, thus, (65) > 0. This means $CB[k|\text{OUT}] - CB[k|\text{POUT}] > 0$ for Type B ARIMA(1,1,2) demand with negative φ and odd k when $0 \le f < 1$. □

Fig. 4 presents an example of Type B_1 ARIMA(1,1,2) demand where $\varphi = 0.3$, $\theta_1 = -0.4$, and $\theta_2 = 0.32$. In this case, we have one zero on the left $\lambda_1^{\theta} = -0.8$, followed by a pole $\lambda_1^{\phi} = 0.3$, the second zero $\lambda_2^{\theta} = 0.4$, and the second pole $\lambda_2^{\phi} = 1$, see Panel a.

Fig. 4b illustrates the demand impulse starts from unity, increases, and then drops to a fixed level. The POUT policy order impulse response starts from a higher value than the demand impulse response, decreases and approaches the demand impulse response over time. The OUT policy order impulse response starts from a much higher value, then drops significantly to slightly under the demand impulse response, which it then approaches over time. Thus, we expect both POUT and OUT to generate bullwhip, as their order variances will likely be larger than the demand variance. The POUT policy generates less bullwhip than the OUT policy, as the OUT policy will be penalised by its very large impulse response at time $t = 0$.

Fig. 4c confirms that the bullwhip difference is always positive when the proportional controller f is between 0 and 1. There is always less bullwhip in the POUT policy than the OUT policy when facing Type B ARIMA(1,1,2) demand for any lead time. While $1 < f < 2$, OUT policy exhibits less bullwhip than the POUT policy. We also find the difference $CB[k|\text{OUT}] - CB[k|\text{POUT}]$ is the most positive when $f = 0$. As $CB[k|\text{OUT}]$ is a constant over f, this means the bullwhip produced by the POUT policy, $CB[k|\text{POUT}]$, has a minimum for $f = 0$. This behaviour has also been observed under i.i.d. demand and AR(1) demand in the literature. Fig. 4d shows both the OUT and POUT policies' bullwhip is increasing in the lead time (note, the log scale on the y-axis).

5.3. Type F ARIMA(1,1,2) demand

When the eigenvalues are ordered pole-zero-zero-pole, we have Type B ARIMA(1,1,2) demand.

Lemma 5.9. *For Type F ARIMA(1,1,2) demand with $\varphi > 0$, the demand impulse response is positive if $\tilde{d}_1 = 1 + \varphi - \theta_1 > 0$ (this is equivalent to $1 + \varphi > \lambda_1^{\theta} + \lambda_2^{\theta}$).*

Proof. When $\varphi > 0$, the F_1 case in Gaalman et al. (2022) is present. Gaalman et al. (2022) proves that the demand impulse is always positive if $\tilde{d}_1 > 0$ (in which case we have case F_{1b}). If $\tilde{d}_1 < 0$ then case F_{1a} is present. Case F_{1a} initially has a negative demand impulse, but after one change in sign, the demand impulse will become and remain positive. The demand at time $t = 1$, $\tilde{d}_1$ is positive if $1+\varphi-\theta_1 > 0$. This is equivalent to the stated condition $1 + \varphi > \lambda_1^{\theta} + \lambda_2^{\theta}$. □

If $\varphi < 0$ and $\tilde{d}_1 < 0$, the demand alternates in sign with $\tilde{d}_{t+1|\text{odd } t} > 0$ and $\tilde{d}_{t+1|\text{even } t} > 0$; after some time, the demand impulse becomes and remains positive, case F_{2ia}. If $\varphi < 0$ and $\tilde{d}_1 > 0$, the demand impulse is always positive, case F_{2ib}, Gaalman et al. (2022). The F_{2ii} and F_3 cases are not possible. Complex conjugate poles have an impulse response that oscillates between positive and negative values and bullwhip does not always increase in the lead time.

For Type F_{1b} with $\varphi > 0$ and $1 + \varphi > \lambda_1^{\theta} + \lambda_2^{\theta}$, the demand impulse is always positive, as shown in Lemma 5.9. In this case, $(\varphi - \lambda_1^{\theta})(\varphi - \lambda_2^{\theta}) +$

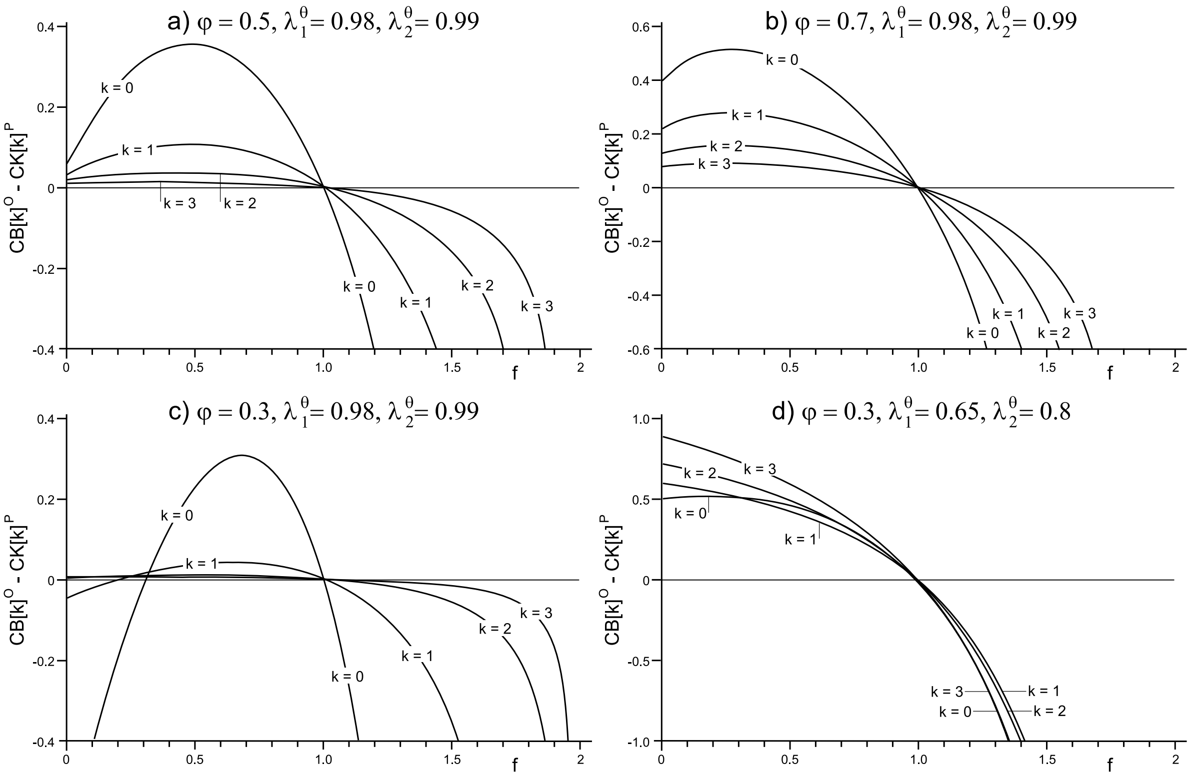

Fig. 5. Bullwhip comparison for four Type F_1 ARIMA(1,1,2) demands with $\varphi > 0$.

$\varphi > 0$. This means

$$\frac{(\varphi-1)\left(E[k]+2r_1\varphi^k\right)}{E[k]\varphi+r_1(\varphi-1)\varphi^k} < 1 \tag{68}$$

is needed for $CB[k|\text{POUT}] < CB[k|\text{OUT}]$. The LHS of (68) is always negative and decreasing in k. Thus, $0 \le f < 1$ leads to $CB[k|\text{POUT}] < CB[k|\text{OUT}]$.

When $\varphi > 0$ and $1+\varphi < \lambda_1^\theta + \lambda_2^\theta$, the Type F_{1a} demand impulse response is not always positive. However, $E[k]$ is positive. $CB[k|\text{POUT}]$ can be less than or greater than $CB[k|\text{OUT}]$. $f|_{0\le f<1} > f'$ is required for $CB[k|\text{POUT}] < CB[k|\text{OUT}]$.

It is interesting to note when $\lambda_1^\theta \to 1$ and $\lambda_2^\theta \to 1$, $CB[k|\text{OUT}] - CB[k|\text{POUT}]$ is concave in $0 \le f < 1$. This suggests reducing f value might not always reduce the bullwhip in the POUT policy. A POUT policy with a small f value might result in poor bullwhip performance. More specifically, under the above conditions ($\lambda_1^\theta \to 1$ and $\lambda_2^\theta \to 1$), $CB[k|\text{POUT}] < CB[k|\text{OUT}]$ always holds for $\varphi \ge 0.5$, see Fig. 5a and b, and $CB[k|\text{POUT}] > CB[k|\text{OUT}]$ is likely to happen when $0 < \varphi < 0.5$, see Fig. 5c. In addition, when λ_1^θ and λ_2^θ are not close to 1, any $0 \le f < 1$ leads to $CB[k|\text{POUT}] < CB[k|\text{OUT}]$, for example, see Fig. 5d.

Theorem 5.10. *When $\lambda_1^\theta \to 1$ and $\lambda_2^\theta \to 1$, $CB[k|POUT] > CB[k|OUT]$ for negative φ, $CB[k|POUT] < CB[k|OUT]$ for $\varphi > 0.5$, $CB[k|POUT]$ can be less than or greater than $CB[k|OUT]$ for $0 < \varphi \le 0.5$.*

Proof. When $\lambda_1^\theta \to 1$ and $\lambda_2^\theta \to 1$, (56) reduces to

$$CB[k|\text{OUT}] - CB[k|\text{POUT}] = \frac{h[f]}{g[f]}$$
$$= \frac{2\varphi^{2k}\left(f^2\left(\varphi^2-\varphi+1\right) - f\left(3\varphi^2-4\varphi+2\right)+2\varphi^2-3\varphi+1\right)}{f^2\varphi+f(1-3\varphi)+2(\varphi-1)}. \tag{69}$$

Via the same approach used in Theorem 5.8, given the interval $[0,1)$ for f, $g[0] = -2(1-\phi)$, $g[1] = -1$. $v_0[g]-v_1[g] = 1-1 = 0$ for $0 < \varphi < 1$. Budan's Theorem suggests that the polynomial $g[f]$ has no real roots in the open interval $(0,1)$. As $g[0] = -2(1-\phi) < 0$ and $g[1] = -1$, we assert that $g[f] < 0$ for positive φ. In addition, we have $v_0[g]-v_1[g] = 2-2 = 0$ for $\varphi < 0$. Via Budan's Theorem, we assert that $g[f] < 0$ for negative φ too.

We study $h[f]$ in the same manner. For $\varphi < 0$, $v_0[h]-v_1[h] = 2-2 = 0$, $h[0] > 0$, and $h[1] = 0$, suggesting $h[f] > 0$. For $0 < \varphi \le 0.5$, $v_0[h]-v_1[h] = 2-0 = 2$, meaning that $h[f]$ has zero, or one, or two real roots in the open interval $(0,1)$. There is only one f value that satisfies $h[f] = 0$ with $0 < \varphi < 1$ and $0 < f < 1$. These facts reveal $h[f]$ has one real root. As $h[0] > 0$ when $0 < \varphi < 0.5$, $h[0] = 0$ when $\varphi = 0.5$, $h'[0] < 0$, $h[1] = 0$, and $h'[1] > 0$, we assert that $h[f]$ is non-negative at $f = 0$, decreases in f, becomes negative, until reaching a stationary point, and then increases in f with a negative value, approaches to zero when $f \to 1$. Therefore, $h[f]$ is non-negative at small f values and is negative at large f values.

For $0.5 < \varphi < 1$, $v_0[h]-v_1[h] = 1-0 = 1$, and there is no f value that satisfies $h[f] = 0$ with $0.5 < \varphi < 1$ and $0 < f < 1$. These facts reveal there are no real roots in the interval $0 \le f < 1$. In addition, we have $h[0] < 0$ and $h[1] = 0$. Thus, $h[f] < 0$ and we can conclude, when $\varphi < 0$, (69) < 0; when $0 < \varphi \le 0.5$, (69) ≤ 0 for small f values and (69) > 0 for large f values; when $0.5 < \varphi < 1$, (69) < 0. □

The POUT bullwhip behaviour contradicts much of the existing bullwhip theory. The literature often recommends switching from the OUT policy to the POUT policy in order to reduce bullwhip. Our results show the POUT policy's order variance is sometimes larger than the OUT policy's order variance when demand is non-stationary, even when $0 \le f < 1$. This highlights that existing knowledge based on stationary demand should be used with caution in non-stationary demand environments.

6. Numerical explorations

We now revisit the four ARIMA(1,1,2) time series from the M4 competition dataset (Makridakis et al., 2020) illustrated in Fig. 1 to verify some of our theoretical results. In the header of Table 2 we

Table 2
Comparison of simulated and theoretical bullwhip measures for the OUT policy under the four real demands in Fig. 1.

Index	228			282			351			356		
n	100			100			80			80		
φ	−0.4883			−0.7055			−0.4852			−0.7175		
θ_1	−0.5216			−0.9452			−0.0453			−0.2896		
θ_2	−0.4851			−0.492			0.6912			0.5957		
λ_ϕ^1	−0.4883			−0.7055			−0.4852			−0.7175		
λ_ϕ^2	1			1			1			1		
λ_θ^1	$-0.278-0.6458i$			$-0.4726-0.5152i$			−0.8543			−0.9301		
λ_θ^2	$-0.278+0.6458i$			$-0.4726+0.5152i$			0.8090			0.6405		
α	1.9934			1.6974			−0.4246			0.1698		
β	0.9896			0.3822			4.7799			−3.033		
γ	−0.4851			−0.7055			−0.4852			−0.7175		
$\mathbb{V}[d]$	403129.8			38265.6			164929.1			621714.2		
$\mathbb{V}[\eta]$	37362.1			1013.43			130591.8			418311.4		
Type	F2ib			F2ib			B2ia			B2ia		
Lead time k	BWR	$CB[k]^S$	$CB[k]^T$	BWR	$CB[k]^S$	$CB[k]^T$	BWR	$CB[k]^S$	$CB[k]^T$	BWR	$CB[k]^S$	$CB[k]^T$
0	1.20	2.19	2.07	1.07	2.59	2.48	1.81	1.03	1.12	1.81	1.20	1.14
1	1.77	8.34	8.18	1.26	9.66	9.48	2.05	1.32	1.38	2.37	2.03	2.04
2	2.62	17.45	17.18	1.53	19.90	19.63	2.84	2.32	2.41	3.62	3.89	3.86
3	3.85	30.79	30.50	1.93	35.30	34.99	3.47	3.11	3.20	4.67	5.46	5.46
4	5.39	47.32	46.98	2.42	53.68	53.33	4.33	4.21	4.30	6.31	7.90	7.87
5	7.28	67.74	67.40	3.04	77.11	76.77	5.22	5.32	5.41	7.86	10.20	10.21
6	9.49	91.62	91.27	3.75	103.76	103.41	6.23	6.61	6.70	9.89	13.22	13.20
7	12.05	119.21	118.89	4.58	135.19	134.87	7.32	7.99	8.06	11.93	16.25	16.25
8	14.94	150.36	150.08	5.50	170.10	169.82	8.52	9.49	9.56	14.36	19.85	19.85
9	18.16	185.16	184.94	6.55	209.54	209.33	9.80	11.11	11.16	16.87	23.59	23.60
10	21.72	223.56	223.42	7.69	252.69	252.54	11.18	12.85	12.88	19.71	27.81	27.81
11	25.61	265.58	265.54	8.95	300.18	300.14	12.65	14.71	14.71	22.69	32.23	32.24
12	29.84	311.22	311.30	10.31	351.52	351.59	14.21	16.68	16.65	25.95	37.08	37.08
13	34.41	360.48	360.69	11.78	407.09	407.30	15.87	18.78	18.71	29.38	42.18	42.18
14	39.31	413.36	413.72	13.36	466.60	466.97	17.62	20.99	20.88	33.07	47.66	47.66

have gathered some descriptive statics for the four demand patterns. Here the index denotes which time series is being considered in the weekly demand subset of the M4 dataset. n is the number of data points considered (see our discussion on Fig. 1). The identified ARIMA(1,1,2) parameters (φ, θ_1, and θ_2) are then noted. The ARIMA(1,1,2) parameters were found using the `auto.arima` function in R. Following, the four eigenvalues of the ARIMA(1,1,2) demand process (λ_1^ϕ, λ_2^ϕ, λ_1^θ, and λ_2^θ) are listed, together with the three damped trend forecasting parameters (α, β, γ), Li et al. (2023). The demand variance ($\mathbb{V}[d]$) and the variance of the forecast error ($\mathbb{V}[\eta]$) is given.[5] Finally, in the header of the table, we note the type of the demand pattern identified by the order of eigenvalues.

In the main body of Table 2 we have collected some bullwhip measures for different lead times when the OUT policy with damped trend forecasts is used to set production targets. The bullwhip ratio $BWR = \mathbb{V}[o]/\mathbb{V}[d]$ was generated by an Excel simulation. As the time series had a finite length, the bullwhip ratio was finite. The simulated $CB[k]$ was determined by $CB[k]^S = (\mathbb{V}[o] - \mathbb{V}[d])/\mathbb{V}[\eta]$ using the data from the Excel simulation; the theoretical $CB[k]^T$ was determined by (54). We can see the simulated $CB[k]^S$ is remarkably close to the theoretical $CB[k]^T$. More importantly, the increasing-in-the-lead-time-bullwhip-behaviour for demands 228 and 282 corroborate our results on the nature of Type F2ib ARIMA demand. Furthermore, the results for demands 351 and 356 corroborate our results on the nature of Type B2ia in Lemma 5.5.

Table 3 documents the bullwhip differences, $CB[k|\text{OUT}] - CB[k|\text{POUT}]$, for each of the time series for two example values of f. Table 3 also concurs with our previous analysis. None of the weekly M4 time series that were identified as ARIMA(1,1,2) had surprising POUT policy bullwhip behaviour. That is, the POUT policy with $0 < f < 1$ resulted in less bullwhip than the OUT policy. One wonders how common ARIMA(1,1,2) demands with surprising bullwhip behaviour are in practice?

[5] $\mathbb{V}[\eta]$ was found by minimising the sum of squared error between the simulated and theoretical $CB[k]$ values for each time series.

7. Concluding remarks

We have contributed to the understudied area of the bullwhip effect under non-stationary demand patterns by investigating the bullwhip effect when optimal forecasts for ARIMA(1,1,2) demand processes are used in both the OUT and POUT inventory replenishment policies. We quantify the bullwhip for all possible ARIMA(1,1,2) demand processes under the OUT and POUT policies. Prior research has demonstrated that the bullwhip can be reduced and even eliminated by using the proportional controller. However, our analysis reveals that conventional values for the proportional controller parameter ($0 \le f < 1$) can, in certain scenarios (Type A and Type F), lead to a more pronounced bullwhip effect compared to the OUT policy when demand is non-stationary.

For Type A ARIMA(1,1,2) demand, the bullwhip in the OUT policy is always increasing in the lead time. In order to mitigate the bullwhip effect, the POUT policy's proportional controller, f, deviates from the conventional $f \in [0,1]$ interval. While $f \to 0$ can effectively dampen demand variability under stationary conditions, this approach may paradoxically exacerbate the bullwhip effect in the presence of some non-stationary ARIMA(1,1,2) demand processes. We also noticed that there is a minimum lead time when the feedback controller is always effective. The amplitude of harmonic frequencies in a Bode plot for ARIMA(1,1,2) demand tends to infinity as the frequency tends to zero. While the POUT policy with a small f near zero is very effective at reducing the amplitude of high-frequency harmonics, it amplifies some low-frequency harmonics. For some short lead times, this low-frequency amplification is sufficient to produce bullwhip (bullwhip is proportional to the area under the squared frequency response). This line of reasoning was not reported in this paper but is worthy

Table 3
$CB[k|\text{OUT}] - CB[k|\text{POUT}]$ for the four weekly ARIMA(1,1,2) demand time series in M4 dataset.

Index	228		282		351		356	
k	$f = 0.666$	$f = 1.5$	$f = 0.666$	$f = 1.5$	$f = 0.666$	$f = 1.5$	$f = 0.666$	$f = 1.5$
0	0.16	−1.00	0.43	−1.73	0.70	−2.30	0.65	−2.64
1	2.06	−6.96	3.11	−11.73	0.93	−3.88	1.00	−4.53
2	5.49	−19.67	7.56	−30.05	1.36	−5.02	1.80	−7.34
3	10.78	−37.74	14.59	−57.35	1.76	−6.74	2.49	−10.47
4	17.68	−62.29	23.34	−92.91	2.27	−8.47	3.56	−14.46
5	26.37	−92.58	34.62	−137.41	2.80	−10.54	4.60	−18.86
6	36.73	−129.06	47.70	−190.26	3.42	−12.76	5.95	−24.04
7	48.83	−171.46	63.21	−251.97	4.08	−15.23	7.34	−29.70
8	62.64	−219.95	80.63	−322.10	4.81	−17.90	8.98	−36.08
9	78.17	−274.42	100.38	−401.02	5.59	−20.79	10.70	−42.98
10	95.41	−334.94	122.13	−488.43	6.44	−23.90	12.64	−50.57
11	114.37	−401.47	146.13	−584.56	7.35	−27.23	14.69	−58.71
12	135.05	−474.03	172.20	−689.25	8.31	−30.78	16.93	−67.51
13	157.45	−552.61	200.47	−802.61	9.34	−34.55	19.31	−76.89
14	181.57	−637.22	230.83	−924.55	10.42	−38.53	21.86	−86.91

of investigation in future research. The conclusions drawn for Type F ARIMA(1,1,2) demand exhibit similar characteristics to those observed for Type A. For Type B demand, encompassing both positive φ and negative φ with even lead times, the POUT policy generates less bullwhip than the OUT policy, provided the proportional controller $f \in [0, 1]$. Conversely, values of $f > 1$ produces a greater bullwhip effect under the POUT policy than under the OUT policy.

Existing knowledge of the bullwhip effect derived from stationary demand models should be applied cautiously in non-stationary demand scenarios. The viability of the POUT policy under non-stationary demand can be assessed through a two-pronged approach, considering both the eigenvalues of the demand process and the lead time. For supply chains characterised by long lead times and frequent replenishment cycles, the POUT policy is a robust approach to mitigating the bullwhip effect. In supply chains with short lead times, or where the lead time approximates the replenishment cycle, practitioners should ascertain (from historical demand data) whether the demand pattern aligns with Type A or Type F. If so, the boundaries we have identified for the proportional feedback controller can guide the selection of an appropriate f value to reduce the bullwhip effect. If the identified demand pattern is classified as Type B, practitioners can leverage their existing understanding of the bullwhip effect for stationary demand, even though the underlying demand process is non-stationary.

The efficacy of supply chain management is predicated on a comprehensive understanding of demand, forecasting, production planning, and strategic considerations which extend beyond mere quantitative projections of future demand. While demand forecasting teams frequently produce numerical forecasts, a gap often exists in the characterisation of the underlying demand patterns, which are crucial for the optimisation of inventory control and supply chain management. This is particularly pertinent in the context of advanced control methodologies, such as POUT, where the tuning of the feedback controller needs a nuanced understanding of the demand characteristics. The absence of such knowledge within the supply chain management team can result in excessive bullwhip generated in the supply chain and/or sub-optimal inventories. Therefore, the integration of demand forecasting and supply chain management functions, specifically to facilitate the transmission of demand characteristics alongside numerical forecasts, is essential for enhancing operational performance and ultimately maximising organisational value. In addition, the integration of the controller, f, into the ERP system (which can be implemented into SAP's planning book with a User Defined Function) is crucial for realising the full potential of optimised inventory management. This allows for automated, data-driven calculation of replenishment decisions in real-time across the (potentially) many hundreds or thousands of stock-keeping units.

Four distinct ARIMA(1,1,2) demand patterns were identified within the M4 dataset have been analysed. Simulation based on these real-world data corroborates our theoretical findings. Further empirical investigation with additional real-world data, particularly for Type A and Type F demand, is warranted to further validate our theoretical analysis. Previous research on i.i.d. and AR(1) demand processes suggests that the bullwhip is increasing in the proportional controller value. While our findings concur with this finding under certain non-stationary demands, it does not always hold true. Exploring the POUT policy's bullwhip behaviour for other non-stationary demand processes is another promising avenue for further research.

CRediT authorship contribution statement

Qinyun Li: Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Formal analysis. **Gerard Gaalman:** Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Formal analysis. **Stephen M. Disney:** Writing – review & editing, Visualization, Validation, Methodology.

Acknowledgements

For the purpose of open access, the authors wish to apply for a Creative Commons Attribution (CC BY) licence to any Author Accepted Manuscript version arising from this submission.

Appendix A. The inventory variance

Substituting in the relevant expressions for $E[k]$ and simplifying provides the following expression for (36):

$$\begin{aligned}\mathbb{V}[i_t] =&\mathbb{V}[\eta]\Bigg(\frac{((\theta_1-2)\varphi+\theta_2+\varphi^k(\varphi(\varphi-\theta_1)-\theta_2)+k(\varphi-1)(\theta_1+\theta_2-1)+1)^2}{f(2-f)(\varphi-1)^4}+\\ &\big(k^3(\varphi+1)(\varphi-1)^3(\theta_1+\theta_2-1)^2+3k^2(\varphi+1)(\varphi-1)^2(\theta_1+\theta_2-1)(\theta_1\varphi+\\ &\theta_1-\theta_2\varphi+3\theta_2-3\varphi+1)+k(\varphi^2-1)(\theta_1^2(-12\varphi^{k+1}+\varphi^2+4\varphi+1)-\\ &4\theta_1(\theta_2(3\varphi^{k+1}+3\varphi^k+\varphi^2-2\varphi-2)-3\varphi^{k+1}-3\varphi^{k+2}+2\varphi^2+2\varphi-1)+\\ &\theta_2^2(-12\varphi^k+\varphi^2-8\varphi+13)+2\theta_2(6\varphi^{k+2}+6\varphi^k+5\varphi^2-16\varphi+5)-\\ &12\varphi^{k+2}+13\varphi^2-8\varphi+1)+6(\varphi^k-1)(\varphi(\theta_1-\varphi)+\theta_2)(\theta_1\varphi(\varphi^k+1)+\\ &\theta_2(\varphi^k+2\varphi^2-1)-\varphi^{k+2}+\varphi^2-2)\big)(6(\varphi-1)^5(\varphi+1))^{-1}\Bigg)\end{aligned}\tag{A.1}$$

Differentiating (A.1) with respect to f provides

$$\frac{d\mathbb{V}[i_t]}{df}=\frac{2(f-1)\left(\theta_1\varphi+\theta_2+k(\varphi-1)(\theta_1+\theta_2-1)-\theta_1\varphi^{k+1}-\theta_2\varphi^k+\varphi^{k+2}-2\varphi+1\right)^2}{f^2(f-2)^2(\varphi-1)^4},\tag{A.2}$$

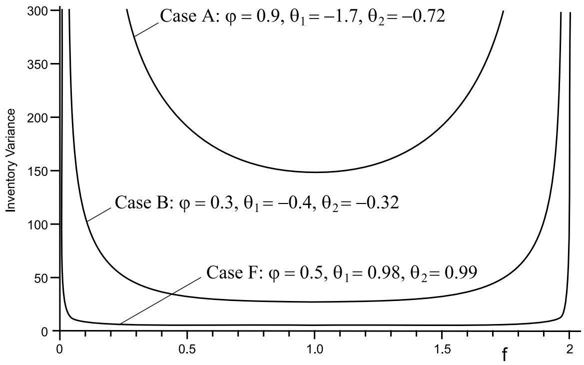

Fig. A.6. Variance of the inventory maintained by the POUT policy.

which clearly has a stationary point at $f = 1$. Fig. A.6 confirms this for three example ARIMA(1,1,2) demands corresponding to cases A, B and F.

Appendix B. Proof to Theorem 5.4

From (56), we can rewrite $CB[k|\text{OUT}] - CB[k|\text{POUT}] > 0$ as

$$\frac{2E[k](1-f)\left((1-\varphi)\left(E[k]+2r_1\varphi^k\right)+f\left(E[k]\varphi+r_1(\varphi-1)\varphi^k\right)\right)}{(2-f)(1-(1-f)\varphi)} > 0. \tag{B.1}$$

We know $E[k] > 0$ for Type A ARIMA(1,1,2) and $(2-f)(1-(1-f)\varphi) > 0$, thus, for (B.1) to hold, $(1-f)\left((1-\varphi)\left(E[k]+2r_1\varphi^k\right)+f\left(E[k]\varphi + r_1(\varphi-1)\varphi^k\right)\right) > 0$ is needed.

Now let us consider if $E[k]\varphi + r_1(\varphi-1)\varphi^k > 0$. After expanding $E[k]$, the following relation holds:

$$E[k]\varphi + r_1(\varphi-1)\varphi^k = \varphi + kr_2\varphi + r_1\left((\varphi-1)\varphi^k + \frac{\varphi\left(\varphi^k-1\right)}{\varphi-1}\right). \tag{B.2}$$

We wish to understand if the RHS of (B.2) is positive or negative:

$$\varphi + kr_2\varphi + r_1\left((\varphi-1)\varphi^k + \frac{\varphi\left(\varphi^k-1\right)}{\varphi-1}\right) > 0. \tag{B.3}$$

Subtracting φ from both sides provides,

$$kr_2\varphi + r_1\left((\varphi-1)\varphi^k + \frac{\varphi\left(\varphi^k-1\right)}{\varphi-1}\right) > -\varphi. \tag{B.4}$$

We wish to divide (B.4) by φ. When $\varphi > 0$ the following relation exists,

$$kr_2 + r_1\left((\varphi-1)\varphi^{k-1} + \frac{\varphi^k-1}{\varphi-1}\right) > -1. \tag{B.5}$$

When $\varphi < 0$ provides the alternative relation exists

$$kr_2 + r_1\left((\varphi-1)\varphi^{k-1} + \frac{\varphi^k-1}{\varphi-1}\right) < -1, \tag{B.6}$$

requiring us to bifurcate our analysis. First consider positive φ, the following relation holds:

$$k > (\varphi-1)\varphi^{k-1} + \frac{\varphi^k-1}{\varphi-1}. \tag{B.7}$$

This means the coefficient of r_2 is always greater than the coefficient of r_1 in (B.5). As $r_2 > 0$, $r_1 < 0$, and $r_2 > |r_1|$ for Type A ARIMA(1,1,2) demand, the relationship in (B.3), $E[k]\varphi + r_1(\varphi-1)\varphi^k > 0$, is proved. Then to ensure (B.1) is positive (that is, for the POUT policy to produce less bullwhip than the OUT policy) for $0 \le f < 1$, we have

$$f|_{0\le f<1} > \frac{(\varphi-1)\left(E[k]+2r_1\varphi^k\right)}{E[k]\varphi + r_1(\varphi-1)\varphi^k} \tag{B.8}$$

and for $1 < f < 2$

$$f|_{1<f<2} < \frac{(\varphi-1)\left(E[k]+2r_1\varphi^k\right)}{E[k]\varphi + r_1(\varphi-1)\varphi^k}. \tag{B.9}$$

Next, consider negative φ. $E[k]\varphi + r_1(\varphi-1)\varphi^k > 0$ indicates $f\left(E[k]\varphi + r_1(\varphi-1)\varphi^k\right) > 0$. In addition, $1-\varphi > 0$ always holds. The sign of $(1-\varphi)\left(E[k]+2r_1\varphi^k\right)+f\left(E[k]\varphi+r_1(\varphi-1)\varphi^k\right)$ is determined by the value of $E[k]+2r_1\varphi^k$. After expanding $E[k]$, the following relation is revealed:

$$E[k] + 2r_1\varphi^k = 1 + kr_2 + r_1\frac{\left(1+\phi^k-2\phi^{k+1}\right)}{1-\phi}. \tag{B.10}$$

When $k > 0$ and $\varphi < 0$,

$$k > \frac{\left(1+\phi^k-2\phi^{k+1}\right)}{1-\phi}. \tag{B.11}$$

As $r_2 > 0$, $r_1 < 0$, and $r_2 > |r_1|$ for Type A ARIMA(1,1,2) demand,

$$E[k] + 2r_1\varphi^k > 1 \tag{B.12}$$

is proved. This means $(1-\varphi)\left(E[k]+2r_1\varphi^k\right)+f\left(E[k]\varphi+r_1(\varphi-1)\varphi^k\right) > 0$. Thus, for $\varphi < 0$ and $k > 0$, $CB[k|\text{OUT}] - CB[k|\text{POUT}] > 0$ when $0 \le f < 1$; and $CB[k|\text{OUT}] - CB[k|\text{POUT}] < 0$ when $1 < f < 2$.

When $k = 0$, $CB[k|\text{OUT}] - CB[k|\text{POUT}]$ simplifies to

$$CB[k|\text{OUT}] - CB[k|\text{POUT}] = \frac{2(1-f)((2-f)r_1(1-\varphi)-(1-f)\varphi+1)}{(2-f)(1-(1-f)\varphi)}. \tag{B.13}$$

Substituting r_1 then solving $CB[k|\text{OUT}] - CB[k|\text{POUT}] = 0$ for f subject to the condition that $f \ne 1$,

$$f = \frac{2(\varphi-\lambda_1^\theta)(\varphi-\lambda_2^\theta)-(1-\varphi)}{(\varphi-\lambda_1^\theta)(\varphi-\lambda_2^\theta)+\varphi}. \tag{B.14}$$

Studying (B.14) for Type A ARIMA(1,1,2) demand with negative φ, we found (B.14) can only be less than 1, or greater than 2. Therefore,

$$f|_{0\le f<1} > \frac{2(\varphi-\lambda_1^\theta)(\varphi-\lambda_2^\theta)-(1-\varphi)}{(\varphi-\lambda_1^\theta)(\varphi-\lambda_2^\theta)+\varphi} \tag{B.15}$$

is required to ensure $CB[k|\text{OUT}] - CB[k|\text{POUT}] > 0$ for Type A ARIMA(1,1,2) demand when $k = 0$ and $\varphi < 0$. Note, (B.15) is equivalent to (B.8) when $k = 0$.

Data availability

No data was used for the research described in the article.

References

Alwan, L.C., Liu, J.J., Yao, D.-Q., 2003. Stochastic characterization of upstream demand processes in a supply chain. IIE Trans. 35, 207–219.

Antsaklis, P.J., Michel, A.N., 2005. Linear Systems. McGraw-Hill, New York.

Boute, R., Disney, S.M., Gijsbrechts, J., Mieghem, J.A.V., 2022. Dual sourcing and smoothing under non-stationary demand time series: Re-shoring with SpeedFactories. Manag. Sci. 68, 1039–1057.

Disney, S.M., 2008. Supply chain aperiodicity, bullwhip and stability analysis with Jury's inners. IMA J. Manag. Math. 19 (2), 101–116.

Disney, S.M., Towill, D.R., 2003. On the bullwhip and inventory variance produced by an ordering policy. Omega 31 (3), 157–167.

Disney, S.M., Warburton, R.D., 2012. On the lambert w function: Economic order quantity applications and pedagogical considerations. Int. J. Prod. Econ. 140 (2), 756–764.

Gaalman, G., 2006. Bullwhip reduction for ARMA demand: The proportional order-up-to policy versus the full-state-feedback policy. Autom. 42 (8), 1283–1290.

Gaalman, G.J.C., Disney, S.M., 2009. On bullwhip in a family of order-up-to policies with ARMA(2,2) demand and arbitrary lead-times. Int. J. Prod. Econ. 121, 454–463.

Gaalman, G., Disney, S.M., Wang, X., 2022. When bullwhip increases in the lead time: An eigenvalue analysis of ARMA demand. Int. J. Prod. Econ. 250, 108623.

Gardner, E.S., McKenzie, E., 1985. Forecasting trends in time series. Manag. Sci. 31 (10), 1237–1246.

Graves, S.C., 1999. A single-item inventory model for a non-stationary demand process. Manuf. Serv. Oper. Manag. 1 (1), 50–61.

Hosoda, T., Disney, S.M., 2006. The governing dynamics of supply chains: The impact of altruistic behavior. Autom. 42, 1301–1309.

Hyndman, R.J., Koehler, A.B., Ord, J.K., Snyder, R.D., 2008. Forecasting with Exponential Smoothing. Springer-Verlag Berlin Heidelberg.

Kailath, T., 1979. Linear Systems. Prentice-Hall Information and System Science Series, Prentice-Hall, New York.

Lee, H.L., Padmanabhan, V., Whang, S., 1997. Information distortion in a supply chain: The bullwhip effect. Manag. Sci. 50 (12), 546–558.

Li, Q., Disney, S.M., Gaalman, G.J.C., 2014. Avoiding the bullwhip effect using damped trend forecasting and the order-up-to replenishment policy. Int. J. Prod. Econ. 149, 3–16.

Li, Q., Gaalman, G.J.C., Disney, S.M., 2023. On the equivalence of the proportional and damped trend order-up-to policies: An eigenvalue analysis. Int. J. Prod. Econ. 265, 109005.

Luong, H.T., Phien, N.H., 2007. Measure of bullwhip effect in supply chains: The case of high order autoregressive demand process. European J. Oper. Res. 183 (1), 197–209.

Makridakis, S., Spiliotis, E., Assimakopoulos, V., 2020. The M4 competition: 100,000 time series and 61 forecasting methods. Int. J. Forecast. 36 (1), 54–74.

Rostami-Tabar, B., Disney, S.M., 2023. On the order-up-to policy with intermittent integer demand and logically consistent forecasts. Int. J. Prod. Econ. 257, 108763.

Tsypkin, Y., 1964. Sampling Systems Theory and its Applications. Vol. 1. Pergamon Press, Oxford, pp. 183—-192.

Wang, X., Disney, S.M., 2016. The bullwhip effect: Progress, trends and directions. European J. Oper. Res. 250 (3), 691–701.

Woodbury, M.A., 1950. Inverting modified matrices. Memo. Report/ Stat. Res. Group 42, 4.

Zhang, X., 2004. The impact of forecasting methods on the bullwhip effect. Int. J. Prod. Econ. 88, 15–27.