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ORIGINAL PAPER

On solutions of fractional nonlinear Fokker-Planck equation

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Abstract
In this work, the exact solutions of time fractional Fokker-Planck equation are investigated using the symmetry approach. Also, the convergence of the reported solutions is proved along with the graphical interpretation of the obtained solutions.

Keywords Fractional calculus (primary) · Fractional nonlinear Fokker-Planck equation · Erdélyi-Kober operator · Exact solutions · Symmetry analysis

Mathematics Subject Classification 26A33 (primary) · 34A08 · 35R11 · 76M60

1 Introduction

The importance of studying the fractional differential equations (FDEs) is evident due to their global behaviour and their ability to model highly complex systems with infinite variance. This is very challenging and is highly significant for its applications in modelling diverse fields of science and engineering eg. material science, biology, pollution control, artificial intelligence, image processing, fluid mechanics, biomathematics etc [1, 8, 12–14, 20]. Therefore, the exact solutions of these equations have gained a lot of interest from the researchers. The solutions of many significant fractional partial differential equations (PDEs) have already been reported in literature including Burgers equation [29, 32], KdV equation [28, 31], Hirota-Satsuma equations [28], Broer-Kaup system [28].

In this paper, our goal is to investigate the exact solutions of the fractional nonlinear Fokker-Planck equation. The nonlinear Fokker-Planck equations have found applications in various fields such as plasma physics, surface physics, astrophysics, the physics of polymer fluids and particle beams, nonlinear hydrodynamics, theory

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of electronic circuitry and laser arrays, engineering, biophysics and psychology [4, 23]. The Fokker-Planck equation is used to describe the Brownian motion of a single particle in an external potential and it gives an excellent approximation near the free energy minimum [4, 16]. The importance of the equation is evident from the fact that the maximum entropy principle for Fokker-Planck gives the Student distribution which is widely used in financial mathematics [2, 7]. The solution of the Fokker-Planck equation is a powerful tool that allows one to follow at each instant the direction of a gradient flux of the associated free energy functional by a discrete time formulation [4, 16]. The fractional version of the Fokker-Planck equation being more generalized allows its application to wider range of models which motivated us for this study. The considered nonlinear fractional generalized Fokker-Planck equation [4, 23] is as follows:

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(p(x,t)\right)^\mu = -\frac{\partial}{\partial x}\left[F(x)\left(p(x,t)\right)^\mu\right] + D\frac{\partial^2}{\partial x^2}\left(p(x,t)\right)^\nu, \tag{1.1}$$

where D is a dimensionless diffusion like constant, $F(x) = -\frac{dV(x)}{dx}$ is a dimensionless external force associated with the potential $V(x)$. Here $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Riemann-Liouville fractional derivative of order α defined as follows:

Definition 1 The partial Riemann-Liouville fractional derivative of order $\alpha \geq 0$ with respect to t is defined by [21, 26]

$$\frac{\partial^\alpha f(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1+[\alpha]-\alpha)}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{f(x,s)}{(t-s)^{\alpha-[\alpha]}}\mathrm{d}s, & t > 0, [\alpha] < \alpha < [\alpha]+1, \\ \frac{\partial^n f}{\partial t^n}, & \alpha = n \in \mathbb{N}. \end{cases} \tag{1.2}$$

Taking $\mu = 1$, $\nu = 2$ and $F(x) = 1$ in (1.1), we will solve the following nonlinear Fokker-Planck equation:

$$\frac{\partial^\alpha p(x,t)}{\partial t^\alpha} = -\frac{\partial p(x,t)}{\partial x} + 2D\left(\left(\frac{\partial p(x,t)}{\partial x}\right)^2 + p(x,t)\frac{\partial^2 p(x,t)}{\partial x^2}\right). \tag{1.3}$$

The main aim of the work is to investigate the exact solutions of fractional generalized Fokker-Planck equation (1.1). To the best of our knowledge, the Lie symmetry analysis and solutions of fractional equation (1.3) have not been discussed in literature proving the novelty of the current work.

The present work is organized as follows. Section 2 discusses the group classification of the considered fractional equation. Section 3 consists of the investigation of novel solutions of the equation and Section 4 contains the graphical interpretation of the obtained solutions. In Section 5, the convergence of the obtained power series solutions is discussed and the last section gives the concluding remarks of the present study.

2 Group classification of Fokker-Planck equation with $\nu = 2$

Let us recall the Lie symmetry method for fractional PDEs with one dependent variable u and two independent variables (x, t) [9, 11, 22, 25, 27, 28, 33, 34]. Consider a time fractional PDE in the following form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F(x, t, u, u_x, u_{xx}, ...), \tag{2.1}$$

where $\alpha > 0$ and subscripts denote the partial derivatives. Assume the invariance of (2.1) under one parameter Lie group of transformations given by

$$\begin{aligned}
&\Big\{\tilde{x}, \tilde{t}, \tilde{u}\Big\} = \Big\{x, t, u\Big\} + \epsilon\Big\{\xi(x, t, u) + \tau(x, t, u) + \eta(x, t, u)\Big\} + O(\epsilon^2),\\
&\frac{\partial^\alpha \tilde{u}}{\partial \tilde{t}^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon \eta^{\alpha,t} + O(\epsilon^2),\\
&\frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x} + \epsilon \eta^x + O(\epsilon^2),\\
&\vdots
\end{aligned} \tag{2.2}$$

where (ξ, τ, η) is the set of infinitesimals, and $\eta^{\alpha,t}, \eta^x, \eta^{xx}, \cdots$ represent prolongation operators of order α, 1, 2 and so on, respectively [3, 19, 27]. The corresponding infinitesimal symmetry generator is given by

$$X = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u}. \tag{2.3}$$

The α^{th} order extended prolongation $\eta^{\alpha,t}$ related to Riemann-Liouville fractional derivative (1.2) is given by [27, 28]

$$\begin{aligned}
\eta^{\alpha,t} &= \frac{\partial^\alpha \eta}{\partial t^\alpha} + \Big(\eta_u - \alpha \mathrm{D}_t \tau\Big)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1}\mathrm{D}_t^{n+1}\tau\right]\\
&\quad \times \partial_t^{\alpha-n} u - \sum_{n=1}^{\infty}\binom{\alpha}{n}\mathrm{D}_t^n \xi \, \partial_t^{\alpha-n} u_x + \mu,
\end{aligned} \tag{2.4}$$

where $\eta_u = \frac{\partial \eta}{\partial u}$, D_t represents the total derivative operator and

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}. \tag{2.5}$$

Substituting the prolongations and equating the coefficients of various partial derivatives of dependent variables to zero results in an over-determined system of linear differential equations in the symmetries ξ, τ, η, called the set of determining equations.

Integrating the determining equations gives the infinitesimals ξ, τ, η involving some arbitrary constants or arbitrary functions. Solving the corresponding characteristic equations obtained from associated vector fields to find the dependent variables u in terms of new variables. Therefore, one can transform the considered PDE into a reduced fractional ODE in the new variables, which is easier to solve leading to solutions of the original fractional PDE.

By application of the Lie symmetry method [3, 11, 27], the invariance of the PDE (1.3) under the transformations (2.1) gives the following invariance criterion:

$$\eta^{\alpha,t} + \eta^{x} - 2D\left(2p_x\eta^{x} + p(x,t)\eta^{xx} + \eta p_{xx}\right) = 0. \tag{2.6}$$

Substituting the expressions for prolongation operators results in the following set of determining equations:

$$\begin{aligned}
&\frac{\partial \xi}{\partial t} = \frac{\partial \tau}{\partial x} = 0,\\
&\frac{\partial \xi}{\partial u} = \frac{\partial \tau}{\partial u} = 0,\\
&\frac{\partial^2 \eta}{\partial u^2} = 0,\\
&\eta + \alpha u\frac{\partial \tau}{\partial t} - 2u\frac{\partial \xi}{\partial x} = 0,\\
&2\frac{\partial^2 \eta}{\partial u \partial t} + (1-\alpha)\frac{\partial^2 \tau}{\partial t^2} = 0,\\
&\alpha\frac{\partial \tau}{\partial t} - 4uD\frac{\partial^2 \eta}{\partial u \partial x} - 4D\frac{\partial \eta}{\partial x} + 2uD\frac{\partial^2 \xi}{\partial x^2} - \frac{\partial \xi}{\partial x} = 0,\\
&3\frac{\partial^3 \eta}{\partial u \partial t^2} + (2-\alpha)\frac{\partial^3 \tau}{\partial t^3} = 0.
\end{aligned} \tag{2.7}$$

Solving this system, the obtained symmetries are as follows:

$$\xi = \alpha x c_1 + c_2, \quad \tau = c_1 t + c_3, \quad \eta = \alpha u c_1, \tag{2.8}$$

where c_1, c_2, c_3 are arbitrary constants. Due to the Riemann-Liouville fractional derivative operator and preservation of its structure under transformations (2.2), $\tau(x,t)|_{t=0} = 0$ must hold, which gives $c_3 = 0$. The corresponding vector fields for constants c_1, c_2 are as follows:

$$\begin{aligned}
V_1 &= \alpha x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \alpha u\frac{\partial}{\partial u},\\
V_2 &= \frac{\partial}{\partial x}.
\end{aligned} \tag{2.9}$$

For V_1, the characteristic equations are given by

$$\frac{\mathrm{d}x}{\alpha x} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}u}{\alpha u}, \tag{2.10}$$

which lead to the following similarity solutions:

$$z = xt^{-\alpha}, \quad u = t^{\alpha} f(z). \tag{2.11}$$

Before discussion of the reduction of considered fractional PDE into a fractional ordinary differential equation (ODE), let us recall the Erdélyi-Kober operators [3, 13, 15, 17, 18]. Erdélyi-Kober operator is a fractional operation introduced by Arthur Erdélyi (1940) and Hermann Kober (1940). As we know, $\left(\mathcal{P}_{\delta}^{\zeta,\alpha}\right)$ is the left-hand sided Erdélyi-Kober fractional differential operator defined as follows:

$$\begin{aligned} \left(\mathcal{P}_{\delta}^{\zeta,\alpha} h\right)(z) &:= \prod_{j=0}^{m-1}\left(\zeta + j - \frac{1}{\delta} z \frac{\mathrm{d}}{\mathrm{d}z}\right)(\mathcal{K}_{\delta}^{\zeta+\alpha, m-\alpha} h)(z), \qquad z > 0, \delta > 0, \alpha > 0, \\ m &= \begin{cases} [\alpha] + 1 & \text{if } \alpha \notin \mathbb{N}, \\ \alpha & \text{if } \alpha \in \mathbb{N}, \end{cases} \end{aligned} \tag{2.12}$$

where

$$\left(\mathcal{K}_{\delta}^{\zeta,\alpha} h\right)(z) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_{1}^{\infty} (s-1)^{\alpha-1} s^{-(\zeta+\alpha)} h(zs^{\frac{1}{\delta}}) \mathrm{d}s & \text{if } \alpha > 0, \\ h(z) & \text{if } \alpha = 0 \end{cases} \tag{2.13}$$

is the left-hand sided Erdélyi-Kober fractional integral operator. Also, $\left(\mathcal{D}_{\delta}^{\zeta,\beta}\right)$ is the right-hand sided Erdélyi-Kober fractional differential operator defined by

$$\begin{aligned} \left(\mathcal{D}_{\delta}^{\zeta,\beta} h\right)(z) &:= \prod_{j=1}^{m}\left(\zeta + j + \frac{1}{\delta} z \frac{\mathrm{d}}{\mathrm{d}z}\right)(\mathcal{I}_{\delta}^{\zeta+\beta, m-\beta} h)(z), \qquad z > 0, \delta > 0, \beta > 0, \\ m &= \begin{cases} [\beta] + 1 & \text{if } \beta \notin \mathbb{N}, \\ \beta & \text{if } \beta \in \mathbb{N}, \end{cases} \end{aligned} \tag{2.14}$$

where

$$\left(\mathcal{I}_{\delta}^{\zeta,\beta} h\right)(z) := \begin{cases} \frac{1}{\Gamma(\beta)} \int\limits_{0}^{1} (1-s)^{\beta-1} s^{\zeta} h(zs^{\frac{1}{\delta}}) \mathrm{d}s & \text{if } \beta > 0, \\ h(z) & \text{if } \beta = 0 \end{cases} \tag{2.15}$$

is the right-hand sided Erdélyi-Kober fractional integral operator. In view of the diverse applications of these operators [30], many authors have used these operators for solving significant fractional PDEs. In this study, only the left-hand sided Erdélyi-Kober fractional differential and integral operators are used for the symmetry reduction.

The reduction of (1.3) into a fractional ODE, is described by the following assertion:

Theorem 1 *The similarity transformation $u(x,t) = t^{\alpha} f(z)$ with the similarity variable $z = xt^{-\alpha}$ reduce the time fractional Fokker-Planck equation (1.3) into the nonlinear fractional ODE given by*

$$\left(\mathcal{P}^{1,\alpha}_{\frac{1}{\alpha}} f\right)(z) = -f'(z) + 2D(f'(z)^2 + f(z)f''(z)), \tag{2.16}$$

where $\left(\mathcal{P}^{\zeta,\alpha}_{\delta}\right)$ is the Erdélyi-Kober fractional differential operator.

Proof For calculating $\frac{\partial^{\alpha} u}{\partial t^{\alpha}}$, let $n-1 < \alpha < n$, $n = 1, 2, 3, \ldots$ then by the definition of Riemann -Liouville fractional derivative, it can be proved that

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{n}}{\partial t^{n}}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} s^{\alpha} f(xs^{-\alpha})ds\right]. \tag{2.17}$$

Let $p = \frac{t}{s}$, then the above expression is transformed into the following:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{n}}{\partial t^{n}}\left[\frac{t^n}{\Gamma(n-\alpha)}\int_1^{\infty} (p-1)^{n-\alpha-1} p^{-(n+1)} f(zp^{\alpha})dp\right]. \tag{2.18}$$

In view of the definition of Erdélyi-Kober fractional integral operator it can be written as

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{n}}{\partial t^{n}}\left[t^n \left(\mathcal{K}^{1+\alpha, n-\alpha}_{\frac{1}{\alpha}} f\right)(z)\right]. \tag{2.19}$$

For further simplification, considering $\psi(z) \in C^1(0,\infty)$ with respect to $z = xt^{-\alpha}$ such that the following holds true:

$$t\frac{\partial}{\partial t}\psi(z) = t\frac{\mathrm{d}}{\mathrm{d}z}\psi(z)\frac{\partial z}{\partial t} = tx(-\alpha)t^{-\alpha-1}\frac{\mathrm{d}}{\mathrm{d}z}\psi(z) = -\alpha z\frac{\mathrm{d}}{\mathrm{d}z}\psi(z).$$

Therefore, (2.19) takes the following form:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-1}\left(n - \alpha z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}^{1+\alpha, n-\alpha}_{\frac{1}{\alpha}} f\right)(z)\right]. \tag{2.20}$$

Continuing in this manner, leads to the following expression:

$$\begin{aligned}\frac{\partial^{\alpha} u}{\partial t^{\alpha}} &= \prod_{j=0}^{n-1}\left(1 + j - \alpha z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}^{1+\alpha, n-\alpha}_{\frac{1}{\alpha}} f\right)(z), \\ &= \left(\mathcal{P}^{1,\alpha}_{\frac{1}{\alpha}} f\right)(z),\end{aligned} \tag{2.21}$$

by using the definition of Erdélyi-Kober fractional differential operator. So the required differential is given as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left(\mathcal{P}_{\frac{1}{\alpha}}^{1,\alpha} f\right)(z). \tag{2.22}$$

Similarly, for $\alpha = n = 1, 2, 3, ...$, the following must hold:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left(\mathcal{P}_{\frac{1}{n}}^{1,n} f\right)(z). \tag{2.23}$$

Hence, the expression (2.22) holds for $n-1 < \alpha \leq n$. The result (2.16) of the theorem follows. $\square$

3 Power series solutions of reduced fractional ODE (2.16)

Assume the solutions of reduced fractional ODE (2.16) are as follows:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{3.1}$$

Therefore, this must satisfy the following expression:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\Gamma(1+\alpha-n\alpha)}{\Gamma(1-n\alpha)} a_n z^n &= -\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n \\ &+ 2D\left[\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n\right)\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n\right)\right. \\ &+ \left.\left(\sum_{n=0}^{\infty}a_n z^n\right)\left(\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}z^n\right)\right]. \end{aligned} \tag{3.2}$$

Comparing coefficients for $n = 0$ gives the following:

$$a_2 = \frac{1}{4Da_0}\left(a_1 - 2Da_1^2 + \Gamma(1+\alpha)a_0\right). \tag{3.3}$$

For $n \geq 1$, the following holds true:

$$\begin{aligned} a_{n+2} = \frac{1}{(n+1)(n+2)a_0} &\left(\frac{1}{2D}\frac{\Gamma(1+\alpha-n\alpha)}{\Gamma(1-n\alpha)}a_n + \frac{n+1}{2D}a_n\right. \\ &- \sum_{k=0}^{n}(k+1)(n-k+1)a_{k+1}a_{n-k+1} \end{aligned}$$

$$
-\sum_{k=1}^{n}(n-k+1)(n-k+2)a_k a_{n-k+2}\Bigg). \tag{3.4}
$$

Hence, the solution (3.1) can be written as follows:

$$
\begin{aligned}
f(z) &= a_0 + a_1 z + \left(\frac{1}{4Da_0}\left(a_1 - 2Da_1^2 + \Gamma(1+\alpha)a_0\right)\right)z^2 \\
&+ \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)a_0}\left(\frac{1}{2D}\frac{\Gamma(1+\alpha-n\alpha)}{\Gamma(1-n\alpha)}a_n\right. \\
&+ \frac{n+1}{2D}a_n - \sum_{k=0}^{n}(k+1)(n-k+1)a_{k+1}a_{n-k+1} \\
&\left. - \sum_{k=1}^{n}(n-k+1)(n-k+2)a_k a_{n-k+2}\right)z^{n+2}.
\end{aligned} \tag{3.5}
$$

Thus, the exact solutions of fractional Fokker-Planck equation can be written as

$$
\begin{aligned}
u(x,t) &= a_0 t^{\alpha} + a_1 x + \left(\frac{1}{4Da_0}\left(a_1 - 2Da_1^2 + \Gamma(1+\alpha)a_0\right)\right)x^2 t^{-\alpha} \\
&+ \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)a_0} \\
&\times\left(\frac{1}{2D}\frac{\Gamma(1+\alpha-n\alpha)}{\Gamma(1-n\alpha)}a_n + \frac{n+1}{2D}a_n\right. \\
&- \sum_{k=0}^{n}(k+1)(n-k+1)a_{k+1}a_{n-k+1} - \sum_{k=1}^{n}(n-k+1) \\
&\times(n-k+2)a_k a_{n-k+2}\Big)\, x^{n+2}t^{-(n+1)\alpha}.
\end{aligned} \tag{3.6}
$$

4 Graphs

The solutions (3.6) are interpreted graphically in figures presented in the text. The change in graphs of all the functions due to the variation in fractional order is clearly visible, as shown in the figures. Hence, the fractional order can be used to modify graphs of all the functions without changing the values of independent variables.

5 Convergence of solutions

Here, the convergence of the reported power series solutions is tested. As evident from the obtained solution (3.6) and the expression (3.4), the following holds true:

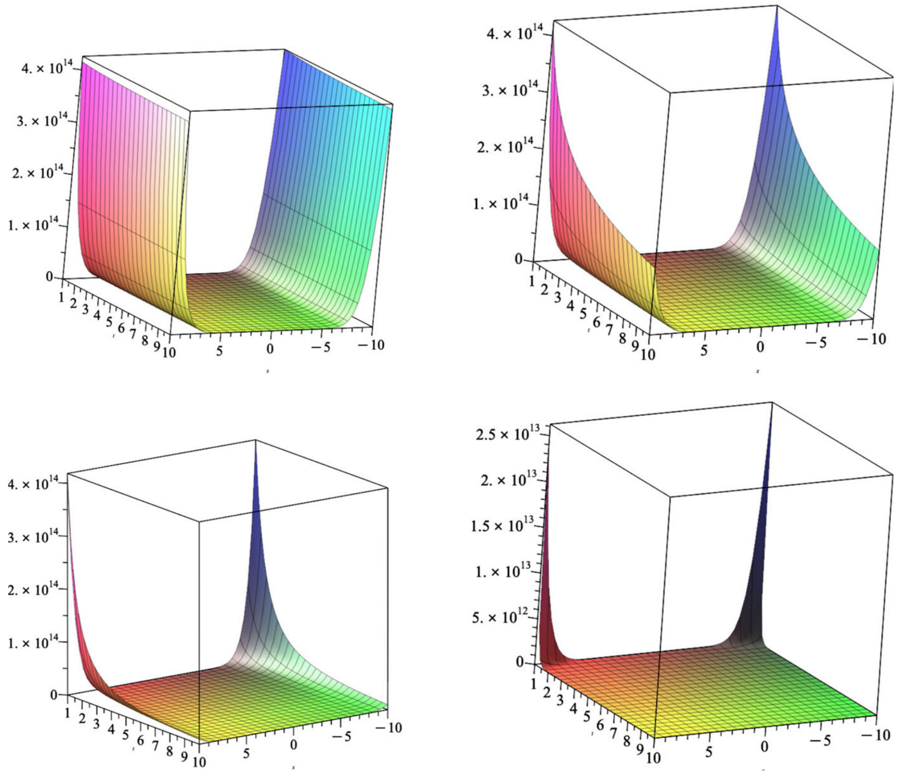

Fig. 1 Plots of $u(x, t)$ for $\alpha = 0.001, 0.05, 0.15, 0.95$ (left to right)

$$
\begin{aligned}
|a_{n+2}| \leq \frac{1}{|a_0|} \Bigg(& \frac{1}{2|D|} \frac{|\Gamma(1+\alpha-n\alpha)|}{|\Gamma(1-n\alpha)|} |a_n| + \frac{n+1}{2|D|} |a_n| \\
& - \sum_{k=0}^{n} (k+1)(n-k+1) |a_{k+1}| |a_{n-k+1}| \\
& - \sum_{k=1}^{n} (n-k+1)(n-k+2) |a_k| |a_{n-k+2}| \Bigg).
\end{aligned} \tag{5.1}
$$

Let us assume the following:

$$
A = max \left\{ \frac{1}{2|D|} \frac{|\Gamma(1+\alpha-n\alpha)|}{|\Gamma(1-n\alpha)|} |a_n|, \frac{n+1}{2|D|}, (k+1)(n-k+1), (n-k+1)(n-k+2) \right\}. \tag{5.2}
$$

It implies that, (5.1) can be written as follows:

$$
|a_{n+2}| \leq \frac{A}{|a_0|} \left(|a_n| - \sum_{k=0}^{n} |a_{k+1}| |a_{n-k+1}| - \sum_{k=1}^{n} |a_k| |a_{n-k+2}| \right). \tag{5.3}
$$

Assume a new power series given by

$$\tilde{A}(z) = \sum_{n=0}^{\infty} b_n z^n, \tag{5.4}$$

where $b_i = |a_i|$ only for $(n = 0, 1, 2)$. Also for $n = 1, 2, ..,$ the generalized expression b_{n+2} is given by

$$b_{n+2} = \frac{A}{b_0}\left(b_n - \sum_{k=0}^{n} b_{k+1}b_{n-k+1} - \sum_{k=1}^{n} b_k b_{n-k+2}\right). \tag{5.5}$$

Therefore, it holds that $|a_n| \leq b_n$ for all n. It shows that the series $\tilde{A}(z) = \sum_{n=0}^{\infty} b_n z^n$ is a majorant series of $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The convergence of $\tilde{A}(z)$ will prove the convergence of $f(z)$. Now, the following expression holds true:

$$\tilde{A}(z) = b_0 + b_1 z + b_2 z^2 + \frac{A}{b_0}\sum_{n=1}^{\infty}\left(b_n - \sum_{k=0}^{n} b_{k+1}b_{n-k+1} - \sum_{k=1}^{n} b_k b_{n-k+2}\right) z^{n+2}. \tag{5.6}$$

Let us consider the implicit function given by

$$\mathbf{A}(z, \tilde{A}) = \tilde{A} - b_0 - b_1 z - b_2 z^2 - \frac{A}{b_0}\left(z^2(\tilde{A} - b_0) - 3b_1 z(\tilde{A} - b_0 - b_1 z)\right). \tag{5.7}$$

The above expression is analytic in a neighbourhood of $(0, b_0)$ where $\mathbf{A}(0, b_0) = 0$ and $\frac{\partial \mathbf{A}(0,b_0)}{\partial \tilde{A}} \neq 0$. Therefore, by implicit function theorem [24], the series (5.4) has positive radius of convergence. Hence, the power series solution (3.6) is also convergent in neighbourhood of $(0, b_0)$.

6 Conclusions

In this paper, the Lie group classification of fractional order generalized Fokker-Planck equation is discussed for reducing it into fractional ODE in Erdélyi-Kober operators. By solving the reduced ODE with the help of power series method, the series solutions are reported successfully. The convergence of the series solutions is proved successfully. Also, the obtained solutions are depicted graphically for better understanding for some particular values of the fractional derivative α and arbitrary constants.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Anh, V.V., Leonenko, N.N., Ruiz-Medina, M.D.: Fractional-in-time and multifractional-in-space stochastic partial differential equations. Fract. Calc. Appl. Anal. **19**, 1434–1459 (2016). https://doi.org/10.1515/fca-2016-0074
2. Borland, L.: A theory of non-Gaussian option pricing. Quant. Finance **2**, 415–431 (2002)
3. Buckwar, E., Luchko, Y.: Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. J. Math. Anal. Appl. **227**, 81 (1998)
4. Frank, T.D.: Nonlinear Fokker-Planck Equations Fundamentals and Applications. Springer-Verlag, Berlin-Heidelberg (2005)
5. Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V.: Mittag-Leffler Functions. Related Topics and Applications. Springer-Verlag, Berlin-Heidelberg (2014)
6. Gupta, R.K., Singla, K.: Symmetry analysis of variable-coefficient time-fractional nonlinear systems of partial differential equations. Theoret. Math. Phys. **197**, 1737–1754 (2018)
7. Heyde, C.C., Leonenko, N.: Student Processes. Adv. Appl. Prob. **37**, 342–365 (2005)
8. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, River Edge (2000)
9. Inc, M., Yusuf, A., Aliyu, A.I., et al.: Lie symmetry analysis and explicit solutions for the time fractional generalized Burgers-Huxley equation. Opt. Quant. Electron. **50**, 94 (2018)
10. Janczura, J., Wylomanska, A.: Subdynamics of financial data from fractional Fokker-Planck equation. Acta Phys. Pol. B. **40**, 1341–1351 (2009)
11. Jefferson, G.F., Carminati, J.: FracSym: Automated symbolic computation of Lie symmetries of fractional differenial equations. Comput. Phys. Commun. **185**, 430–441 (2014)
12. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies. **204** Elsevier, Amsterdam (2006)
13. Kiryakova, V.: Generalized Fractional Calculus and Applications. Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Longman Group (1994)
14. Leonenko, N., Podlubny, I.: Monte Carlo method for fractional-order differentiation extended to higher orders. Fract. Calc. Appl. Anal. **25**, 841–857 (2022). https://doi.org/10.1007/s13540-022-00048-w
15. Liu, J.G., Zhang, Y.F., Wang, J.J.: Investigation of the time fractional generalized (2+1)-dimensional Zakharov-Kuznetsov equation with single-power law nonlinearity. Fractals **31**, 2350033 (2023)
16. Lucia, U., Girvino, G.: Fokker-Planck equation and thermodynamic system analysis. Entropy. **17**, 763–771 (2015)
17. Luchko, Y., Gorenflo, R.: Scale-invariant solutions of a partial differential equation of fractional order. Fract. Calc. Appl. Anal. **3**, 63–78 (1998)
18. Luchko, Y., Trujillo, J.J.: Caputo-type modification of the Erdélyi-Kober fractional derivative. Fract. Calc. Appl. Anal. **10**, 249–267 (2007)
19. Olver, P.J.: Applications of Lie Groups to Differential Equations. **107**, Springer-Verlag, New York (1993)
20. Ortigueira, M.D.: Fractional Calculus for Scientists and Engineers. Springer, Netherlands (2011)
21. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications. Vol 198 in Mathematics in Science and Engineering, Academic Press, San Diego (1999)

22. Rahioui, M., El Kinani, H., Ouhadan, U.: Lie symmetry analysis and conservation laws for the time fractional generalized advection-diffusion equation. Comput. Applied Math. **42**, 50 (2023)
23. Risken, H.: The Fokker-Planck Equation: Methods of Solution and Applications. Springer, Berlin-Heidelberg (1996)
24. Rudin, W.: Principles of Mathematical Analysis. China Machine Press, Beijing, China (2004)
25. San, S., Yaşar, E.: On the Lie symmetry analysis, analytic series solutions, and conservation laws of the time fractional Belousov-Zhabotinskii system. Nonlinear Dynam. **109**, 2997–3008 (2022)
26. Samko, G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordan and Breach Science Publishers, Switzerland (1993)
27. Singla, K., Gupta, R.K.: On invariant analysis of some time fractional nonlinear systems of partial differential equations. I. J. Math. Phys. **57**, 101504 (2016)
28. Singla, K.: Existence of series solutions for certain nonlinear systems of time fractional partial differential equations. J. Geometry Phys. **167**, 104301 (2021)
29. Singla, K.: Investigation of exact solutions and conservation laws for nonlinear fractional (2+1)-dimensional Burgers system of equations. Reports Math. Phys. **92**, 75–83 (2023)
30. Sneddon, I.N.: The use in mathematical analysis of Erdélyi-Kober operators and some of their applications. In: Fractional Calculus and Its Applications, Proc. Internat. Conf. Held in New Haven, Lecture Notes in Math. 457, Springer, New York 37-75 (1975)
31. Wang, G.W., Liu, X.Q., Zhang, Y.Y.: Lie symmetry analysis to the time fractional fifth order KdV equation. Commun. Nonlinear Sci. Numer. Simul. **18**, 2321–2326 (2013)
32. Wang, G., Xu, T.: Invariant analysis and explicit solutions of the time fractional nonlinear perturbed Burgers equation. Nonlinear Anal. Modell. Control **20**, 570–584 (2015)
33. Yu, J., Feng, Y.: Lie symmetries, exact solutions and conservation laws of time fractional Boussinesq-Burgers system in ocean waves. Commun. Theoret. Phys. (2024). https://doi.org/10.1088/1572-9494/ad71ab
34. Yu, J., Feng, Y.: Group classification of time fractional Black-Scholes equation with time-dependent coefficients. Fract. Calc. Appl. Anal. **27**, 2335–2358 (2024). https://doi.org/10.1007/s13540-024-00339-4

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