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Vibration-Based Crack Detection in Plates Using Natural Frequency Degradation.

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Abstract. Structural cracks reduce the performance and service life of buildings, bridges, and aircraft structures, leading to catastrophic failures resulting in economic losses and fatalities. To avoid such consequences, regular health monitoring and maintenance is required, especially for critical structures that carry high levels of dynamic and fatigue loading and whose failure would be catastrophic. Many techniques are available for structural health monitoring, including visual inspection, ultrasonic testing, acoustic emission, magnetic field, and vibrational-based methods. Any damage causes degradation in the natural frequencies and vibration modes of a structure, which are considered in vibration-based methods to characterize the damage. The focus of this research is to develop a more efficient method for the detection and characterisation of arbitrarily oriented surface cracks in isotropic plates, in terms of five parameters, namely the longitudinal and transverse location, length, depth and orientation. To achieve this objective, an analytical solution based on strain energy is used to generate synthetic data that quantifies changes in the natural frequencies for different crack locations and intensities based on noise-free simulation. The inverse problem, i.e. the determination of the crack parameters based on measured changes in natural frequencies can then be solved based on the use of synthetic data with a gradient based optimisation technique.

Keywords – Strain Energy, Crack, Natural Frequencies, Noise-free simulation, Gradient-based optimisation.

1. Introduction

Delaminations and cracks occurring in plate structures under service conditions can degrade their performance and lifespan. In buildings, bridges and aircraft, such damage can cause catastrophic failure, economic loss and even loss of life. To avoid such consequences, structures should be subject to regular health monitoring. Several non-destructive testing methods are available for this assessment which are able monitor the presence and extent of damage and enable its effects to be predicted. These include visual inspection, ultrasonic testing [1], magnetic field [2], eddy current [3], radiography [4], and vibration analysis. In case of ultrasonic guided wave propagation, minute damages could be detected using high frequency excitations with the capability of large area scanning, with less numbers of traducers [5], [6].

Vibration-based monitoring techniques can lead to less intrusive automated structural monitoring with reduced overheads. In the case of vibration analysis, changes to natural frequencies [7] or mode shapes [8] between undamaged and damaged structures can be used to determine the location and extent of damage. The environmental and operational conditions can affect the stiffness and mass distribution in the structures, which leads to change in the dynamic parameters of the structure[9].

In the case of damage detection using the dynamic parameters of the structure, studies can be divided in two major categories[10]; a) modelling the known damage and obtaining changes occurring in the dynamic parameters. This can be considered as a direct or forward problem. b) obtaining damage parameters from the dynamic response of the structure. This can be considered as the inverse problem. to avoid ill-position in inverse problems, the selection of dynamic parameters and methods of optimization can be considered for regularisation.

There are several methods for modelling damage based on analytical methods [11], [12], [13], [14] as well as finite element methods [15], [16], [17], [18]. Methods based on updating the geometric stiffness matrix [19], the use of neural networks [20], and various optimization techniques [21], [22] have been used to solve the inverse problem.

Damage indices based on changes in the mode shape between damaged and intact structures have been developed, to achieve non-destructive damage localisation for beam-like structures [23]. Such studies are further extended by using changes in strain energy for beam and plate like structures [24]

Frequency response functions (FRF) have been used, for example, on a three-bay frame structure for crack localization, using the FRF data instead of the modal data to determine damage vectors indicating the location and extent of the damage. However, this approach has a few limitations, particularly in the case of small surface and internal cracks, due to the technique's sensitivity to noise and measurement errors [25]. The FRF curvature method has been used to locate damage in beam structures by converting a continuous system into a lumped mass system (Sampaio et al.[25]) and calculating the curvature at each location across a mode shape [26]. Damage locations are identified by large differences between FRF curvatures of intact and damaged structures for each location. Similar methods have been implemented to find delamination's in composite structures [27]. In other studies, plate-like structures have been analyzed using mode shape curvature and strain frequency response functions via the finite element method to create damage indices, which provide damage locations and severities [[28]. Care must be taken while preparing the index, and lower frequency modes should be selected.

This paper introduces an analytical solution for changes to the natural frequencies and the corresponding inverse problem in a simply supported isotropic plate structure with a surface crack based on a strain energy approach [29], in which the crack is modelled as linear, rotational springs. This method has been used earlier. However, some modifications have been implemented regarding accuracy regarding the calculation of bending moment and additional torsional effect [29]. The considered crack parameters for the inverse problems are length, location, orientation, and depth.

To solve the inverse problem, synthetic data has been simulated for a cracked plate. The synthetic data allows the calculation of changes in the square of the natural frequencies between intact and damaged plates. This data is then used to solve the inverse problem based on a set of equations developed based on the modes of the first six natural frequencies with a gradient-based optimization approach. The sets of trial values used for this optimization are determined using Latin hypercube samples.

In this paper, Section 2 describes the formulation of the crack as a linear rotational spring for both beams and plates. Section 3 determines the relationship between changes in natural frequencies for different modes and the respective strain energy, along with the developed equations to solve the inverse problem. Section 4 provides results for the numerical example of an isotropic square plate.

2. Crack Compliance

Caddemi and Calio developed a closed-form solution for a column with a crack [7]. A similar approach was used for a cracked beam [30] and a frame [31], modelling the crack using a rotational spring with intact axial stiffness. As an alternative, the effect of a crack in a small-scale bridge with a moving load has been modelled using a linear variation in the thickness of the beam section near the damage [32].

Labib et al.[33] used the formulation in (Eq.(1)) for compliance of the crack shown in Figure 1 and applied the Wittrick-William algorithm to find the natural frequencies of a beam with multiple cracks.

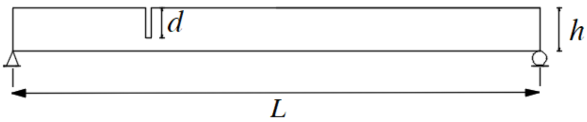

Figure 1.Simply supported beam with a crack at midpoint with depth d.

$$\lambda^* = \frac{h}{L} C(d/h) \tag{1}$$

Here, λ^* is the dimensionless local compliance, *EI* is the bending stiffness, *L* is the length of beam, *h* is the depth of the crack and $C(d/h)$ is a dimensionless function which can be expressed in the form:

$$C(d/h) = \frac{(d/h)[2-(d/h)]}{0.9[1-(d/h)]^2} \tag{2}$$

For an isotropic plate structure, Luo et al. [18] used the finite element method to model a crack in any element with a modified version of this compliance - compliance per unit length (Eq.(3) from [33]).to perform the finite element analysis, the effect of the crack was distributed to the nodal points of the elements through which the crack passed.

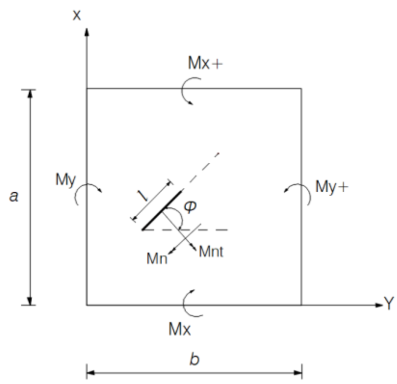

Figure 2. Plate with an arbitrary crack.

In this paper, compliance per unit length is utilized for a plate to develop an analytical expression. In this case, the plate is assumed as a single element to formulate the additional strain energy due to the rotation produced by the crack. As shown in Figure 2, a single crack runs from (x_1, y_1) to (x_2, y_2) and makes an angle 'φ' with respect to the y-coordinate. The relationship between the crack depth ratio and compliance can be expressed as follows.

$$C_n = \frac{h}{D} C(d/h) \frac{l}{a^2} \tag{3}$$

Where, d is the depth of crack, D is the modulus of rigidity for the isotropic plate, l is the length of the crack, and h is the thickness of the plate.

3. Forward and inverse problems

Consider the thin rectangular plate of thickness h shown in Figure 2, having width a, length b and simply supported on all four sides. The plate is made from an isotropic material with Young's modulus E Poisson's ratio ν and density ρ and it vibrates at a natural frequency ω_{mn0} with a mode shape [34] given by Eq.(4). For harmonic vibration, we can describe this mode shape in the form given in Eq.(5).

$$w(x,y,t) = w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin(\omega_{mn0} t) \tag{4}$$

$$w(x,y) = w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{5}$$

Here, m and n are the number of half wavelengths in the x and y directions respectively. When $\omega t = n\pi$ the displacement is zero. The strain energy is zero and the kinetic energy takes its maximum value:

$$T_0 = \frac{1}{2}\rho h \omega_{mn0}^2 w_{mn}^2 \int_0^a \int_0^b \sin^2\left(\frac{m\pi x}{a}\right) \sin^2\left(\frac{n\pi y}{b}\right) dy\, dx \tag{6}$$

Similarly, when $\omega t = \left(n + \frac{1}{2}\right)\pi$, its kinetic energy becomes zero and the strain energy takes its maximum value. which can be expressed by Eq.(7) and for a simply supported plate [35].

$$U_0 = \frac{D}{2}\int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 dy\, dx \tag{7}$$

Here, flexural rigidity can be expressed by:

$$D = \frac{E h^3}{12(1-\nu^2)} \tag{8}$$

Rayleigh's Quotient [36] is used to obtain the eigenvalues in terms of square of the natural frequencies. (The law of Conservation of energy requires that $T_0 = U_0$) After performing algebra, the square of the natural frequencies for m and n modes of vibration can be expressed:

$$\omega_{mn0}^2 = \frac{D\pi^4}{\rho h}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \tag{9}$$

Now consider a crack of compliance per unit length 'C_n' running from (x_1, y_1) *to* (x_2, y_2) is introduced into the plate. Consider a small portion of the crack of length ds, where the bending moment perpendicular to the crack direction is M_n per unit length and the discontinuity of rotation across the crack is θ_n where:

$$\theta_n = C_n(M_n ds) \tag{10}$$

The incremental strain energy [29] associated with the discontinuous rotation in this small portion of the crack given by Eq.(11):

$$dU_d = \frac{1}{2}\theta_n(M_n ds) = \frac{1}{2}C_n(M_n ds)^2 \tag{11}$$

The bending moment M_n can be expressed [35] in the form of Eq.(12),

$$M_n = M_x(cos\varphi)^2 + M_y(sin\varphi)^2 + 2cos\varphi\, sin\,\varphi\, M_{xy} \tag{12}$$

where M_x , M_y and M_{xy} are the longitudinal, transverse and twisting bending moments respectively [35]. The strain energy associated with the discontinuous rotation for the whole crack can then be expressed by integrating over length of the crack.

$$U_d = \iint_0^l dU_d \tag{13}$$

The available mechanical strain energy at peak displacement can then be written as a subtraction between the strain energy of the undamaged plate and strain energy stored in the crack:

$$U_c = U_0 - U_d \tag{14}$$

We also know that the cracked plate vibrates with a reduced natural frequency ω_{mnc} and at zero displacement, has kinetic energy:

$$T_c = \frac{1}{2}\rho h \omega_{mnc}^2 w_{mn}^2 \int_0^a \int_0^b sin^2\left(\frac{m\pi x}{a}\right) sin^2\left(\frac{n\pi y}{b}\right) dy\, dx \tag{15}$$

According to the law of Conservation of energy, the total strain energy of cracked plate must be equal to the total kinetic energy of the same plate, which can be described by:

$$T_c = U_c \tag{16}$$

subtracting maximum kinetic energy and strain energy of the plate from both sides

$$T_c - T_0 = U_c - U_0 = -U_d \tag{17}$$

And substituting in the difference in the square of the natural frequencies:

$$\frac{1}{2}\rho h(\omega_{mnc}^2 - \omega_{mn0}^2) w_{mn}^2 \int_0^a \int_0^b sin^2\left(\frac{m\pi x}{a}\right) sin^2\left(\frac{n\pi y}{b}\right) dy\, dx = -U_d \tag{18}$$

Defining this difference:

$$\delta_{mn} = \omega_{mn0}^2 - \omega_{mnc}^2 \tag{19}$$

With the use of conservation of energy,w_{mn}^2 gets eliminated after performing algebra. The expression for δ_{mn} can be simplified for 'm 'and 'n 'modes of the plate using Eq.(20).

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$$\delta_{mn} = \frac{2U_d}{\rho h\, w_{mn}^2 \int_0^a \int_0^b sin^2\left(\frac{m\pi x}{a}\right) sin^2\left(\frac{n\pi y}{b}\right) dy\, dx} \tag{20}$$

Note the double integral in the denominator of Eq.(20) is over the whole area of the plate and takes the value $(ab/4)$. Using this equation a set of data regarding the differences between the natural frequencies of intact and damaged plates, for different modes, for an individual or set of cracks, located in arbitrary orientations, can be calculated analytically. The mode shapes are assumed to be the same for the uncracked and cracked plate in this method[37].

We can then use this data to solve the inverse problem, locating the cracks in a plate structure, by optimising the set of crack parameters, to provide set of changes in natural frequencies corresponding to those measured using a gradient based method. The full inverse problem will require consideration of measurement noise however in this paper, we solve the simplified noise free problem.

Optimisation is based on minimisation of the error function expressed in Eq.(*21*). Here $\bar{\delta}_{mn}$ is set of difference of squared natural frequencies between intact and cracked plate for first few fundamental natural frequencies, obtained for unknown crack parameters. This data can be obtained from real structure or simulations. In Eq.(*21*), the numerical value in the numerator is higher, which are supposed to be scaled for minimisation. To scale the numerator, the normalisation constants (δ_{mn}^{sc}) are used. The subscript '*m*' and 'n' represent the mode number. In this paper $\bar{\delta}_{mn}$ is obtained by using simulation based on Eq.(20), for the plate with known parameters of crack (location and depth).

$$\boldsymbol{f}(e) = \sum_{m,n} \left\{ \frac{\left(\delta_{mn} - \bar{\delta}_{mn}\right)}{\delta_{mn}^{sc}} \right\}^2 \tag{21}$$

A numerical example presented in next section 4, shows how the crack parameters can be obtained through gradient based method.

4. Numerical Example and Results

The supported isotropic square plate with a crack as, shown in Figure 3, and the parameters presented bellow in Table 1.

Table 1. Properties of plate and with crack parameters.

Properties	
Young's Modulus (*E*)	1.10 x 10^{11} Nm^{-2}
Modulus of rigidity (*D*)	10.0733 Nm
Mass Density (ρ)	4480 Kgm^{-3}
Poisson's ratio (ν)	0.3
Length (*a*)	0.1 m
Breadth (*b*)	0.1 m
Thickness (h)	0.001m
Depth of crack (d)	0.2 *h*
Compliance (C_n)	1.9864 x 10^{-4} rad $N^{-1}m^{-1}$

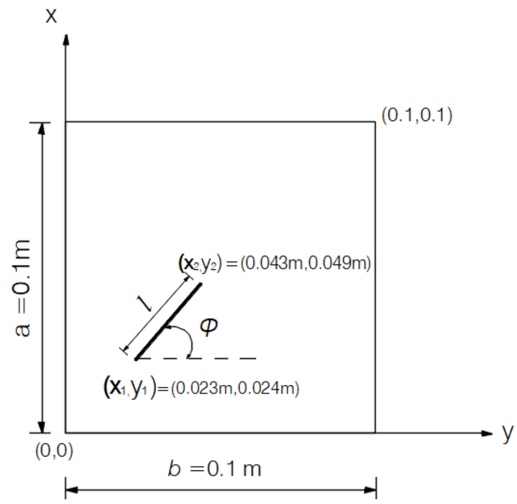

Figure 3. Plate with predefined crack used for inverse problem.

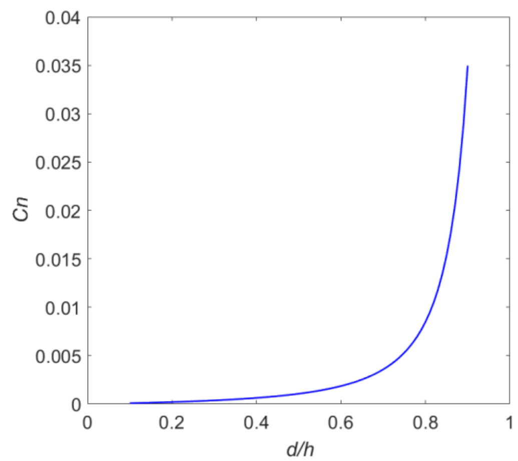

Figure 4. Variation of Compliance (*Cn*) with respect to Crack -depth ratio (*d/h*)

Table 2 This shows the differences between the squared natural frequencies of intact and cracked plates for the first six fundamental modes of vibration using Eq.(20). This synthetic data is noise-free.

Table 2. obtained synthetic data ($\overline{\delta}_{mn}$) for optimization using Table 1 and Figure 3

Modes $(\boldsymbol{m}, \boldsymbol{n})$	(1,1)	(1,2)	(2,1)	(2,2)	(1,3)	(3,1)
$\overline{\boldsymbol{\delta}}_{\boldsymbol{mn}}(\boldsymbol{rad^2 s^{-2}})$	1417.7	7712.81	12891.23	19795.21	19731.44	21271.12

Further, this data is then used in Eq.(21) to develop a set of six equations for the first six fundamental modes of vibrations ('m' and 'n').

These equations are solved using the gradient-based optimization available in FMINCON (an optimization toolbox available in MATLAB). To obtain results, an initial five trial values are required.

for the crack parameters, namely the x and y coordinates of the two ends of the crack and the crack depth ratio.

Table 3. Range of trial values used to solve inverse problem for a plate shown in Figure 3.

Parameters	x_1	y_1	x_2	y_2	d/h
Upper Limit (U_L)	0.1	0.1	0.1	0.1	0.99
Lower Limit (L_L)	0	0	0	0	0.01

. Using more than one set of trial values, increases the likelihood of finding a global rather than local solution. In this case, ten sets of trial values are used. The selection of the ten sets is based on the Latin hypercube approach.

The solutions found by using Eq.(21) with the data in Table 2. obtained synthetic data ($\overline{\delta}_{mn}$) for optimization using Table 1 and Figure 3 are presented in Table 4. We can see that set numbers 1,2,7,8 and 9 converge for a tolerance of 10^{-6}. On the other hand, set numbers 3,4,5,6 and 10 converge with higher errors.

Table 4. final values of crack parameters and error function for 10 set of trial values.

Set No.	x_1	y_1	x_2	x_2	d/h	$f(e)$
1	0.0229	0.024	0.043	0.049	0.2	1.33×10^{-6}
2	0.043	0.049	0.023	0.024	0.2	6.94×10^{-7}
3	0.057	0.048	0.077	0.025	0.23	1.86×10^{-2}
4	0.061	0.052	0.079	0.069	0.324	3.45×10^{-2}
5	0.042	0.051	0.023	0.074	0.2199	1.44×10^{-2}
6	0.078	0.027	0.058	0.048	0.24	1.51×10^{-2}
7	0.056	0.051	0.077	0.076	0.199	1.32×10^{-6}
8	0.022	0.024	0.043	0.049	0.2	1.21×10^{-6}
9	0.077	0.076	0.057	0.051	0.2	1.38×10^{-6}
10	0.044	0.003	0.022	0.0462	0.079	6.12×10^{-2}

The first four parameters for all the sets are plotted in Figure 5. The original crack lies below the parameters obtained by Set No.1,2 and 8. Set Nos.7 and 9 can be seen to be symmetric to the original location of the crack. Due to the symmetry, the crack obtained through sets 7 and 9 will make the same contribution to the change in natural frequencies for the same modes of vibrations. The crack locations obtained through set numbers 5 and 6 are very close to the symmetric locations of the crack but do not match the real crack location or its symmetric location.

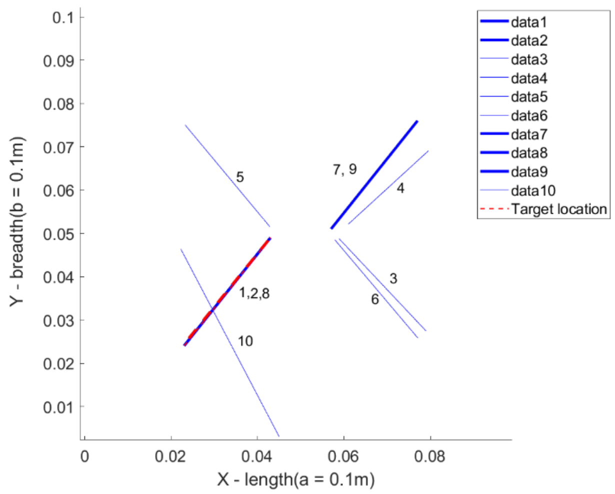

Figure 5.Optimised location of crack for 10 sets of trial values from Table 4.

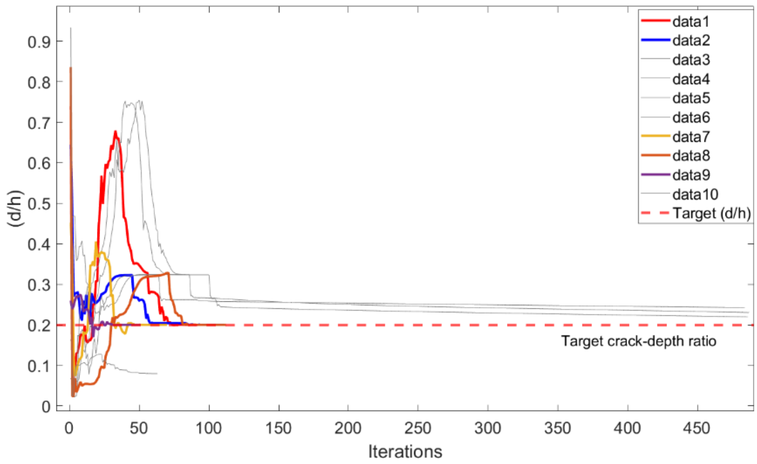

Figure 6 .Converge of crack depth-ratio crack depth vs iterations for 10 set of trial values.

The convergence of the crack depth ratios obtained from optimisation based on each of the 10 sets of trial values are shown in Figure 6. The 'Target depth' defines the crack depth ratio for the modelled crack. In this plot, we can again see that a few sets of trial values have converged based on the optimality criteria, which correspond to those which converged based on location parameters (sets number 1,2,7,8 and 9). The data that converge to the real crack-depth ratio (0.2) are indicated using thick lines.

5. Conclusion

A strain energy approach has been used to calculate the changes in natural frequencies in a simply supported isotropic plate structure, when a crack is introduced, based on modelling this crack as a rotational spring. This analytical approach is further used to generate data with which to solve the inverse problem and locate a crack based on a simulated set of 'measured' frequency degradation and a gradient-based optimization solver. Cracks with the same length and crack-depth ratio but symmetric location were also located since they contribute a similar amount of degradation in the natural frequencies. Results were obtained based on consideration of a noise-free situation. Overall, the developed analytical expression is helpful in terms of computational efficiency for crack modelling and obtaining objective function gradients with respect to damage parameters. Future work will incorporate the effect of measurement noise to assess the robustness of the approach for the real structure and analyse the practical approach.

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