Advances in Mathematics 475 (2025) 110325

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Grothendieck-Verdier module categories, Frobenius algebras and relative Serre functors

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ARTICLE INFO

Article history:
Received 17 December 2024
Received in revised form 24 April 2025
Accepted 26 April 2025
Available online 14 May 2025
Communicated by Ross Street

Keywords:
Grothendieck-Verdier categories
*-Autonomous categories
Module categories
Frobenius algebras

ABSTRACT

We develop the theory of module categories over a Grothendieck-Verdier category $\mathcal{C}$, i.e. a monoidal category with a dualizing object and hence a duality structure more general than rigidity. Such a category comes with two monoidal structures $\otimes$ and $\circledast$ which are related by non-invertible morphisms and which we treat on an equal footing. Quite generally, non-invertible structure morphisms play a dominant role in this theory. In any Grothendieck-Verdier module category $\mathcal{M}$ we find two distinguished subcategories $\widehat{\mathcal{M}}^{\circledast}$ and $\widehat{\mathcal{M}}^{\otimes}$, which can be characterized by certain structure morphisms being actually invertible. The internal Hom $A_m := \underline{\mathrm{Hom}}(m,m)$ of an object m in $\widehat{\mathcal{M}}^{\otimes}$ that is a $\mathcal{C}$-generator is an algebra such that mod-A_m is equivalent to $\mathcal{M}$ as a module category. Crucially, the subcategories $\widehat{\mathcal{M}}^{\circledast}$ and $\widehat{\mathcal{M}}^{\otimes}$ are precisely those on which a relative Serre functor can be defined. This relative Serre functor furnishes an equivalence $S \colon \widehat{\mathcal{M}}^{\otimes} \longrightarrow \widehat{\mathcal{M}}^{\circledast}$, and

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https://doi.org/10.1016/j.aim.2025.110325

any isomorphism $m \xrightarrow{\cong} S(m)$ endows the algebra A_m with the structure of a Grothendieck-Verdier Frobenius algebra.

Contents

1. Introduction

Monoidal categories arise in a wide variety of contexts, for instance as cobordism categories and as representation categories of diverse algebraic structures. Accordingly, monoidal categories have found numerous applications, ranging from representation theory to quantum topology and logic. Rigidity is a natural duality property for an object in a monoidal category. A rigid monoidal category is a monoidal category in which every object has rigid duals; such categories are pervasive in quantum topology.

To motivate the results of this paper, consider the following subclass of rigid categories: Let $\Bbbk$ be an algebraically closed field. A *finite tensor category* over $\Bbbk$ is a $\Bbbk$-linear abelian rigid monoidal category that is finite, i.e. is equivalent, as an abelian category, to the category of finite-dimensional modules over a finite-dimensional $\Bbbk$-algebra. The class of finite tensor categories includes e.g. categories of finite-dimensional modules over finite-dimensional Hopf algebras; for these, rigidity is rather directly inherited from basic properties of finite-dimensional vector spaces. Finite tensor categories are well understood [11]. In particular there is a satisfactory theory of exact module categories over finite tensor categories. This theory is deeply linked with rich algebraic notions like Drinfeld centers and relative Serre functors. In fact, the bicategory of exact module categories is as indispensable for the understanding of a finite tensor category as the category of modules is for the understanding of a ring.

On the other hand, for a general monoidal category rigidity is a highly restrictive property. For instance, it forces the tensor product of a finite tensor category to be an exact functor. It should therefore not come as a surprise that many concrete monoidal categories arising in representation theory, algebraic geometry or linear logic are not rigid. However, many interesting tensor categories still exhibit the more general duality structure of a star-autonomous or *Grothendieck-Verdier* duality. The primary purpose of this paper is to develop a theory of module categories over Grothendieck-Verdier categories. We are motivated by the desire to reach a deeper structural understanding of Grothendieck-Verdier categories, as well as by potential applications of module categories in the construction of two-dimensional conformal field theories. (The latter application actually requires pivotal structures, which we also touch upon.)

Roughly speaking, a Grothendieck-Verdier category is a monoidal category $(\mathcal{C}, \otimes)$ together with the additional structure of a *dualizing object* $K \in \mathcal{C}$ (see Definition 2.1), which entails a contravariant duality functor $G\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$ that is an anti-equivalence. Examples of Grothendieck-Verdier categories include suitable representation categories of Hopf algebroids [1] and of a large class of vertex algebras [2], as well as categories of finite-dimensional bimodules over finite-dimensional algebras, see e.g. the discussion in [13]. Apart from naturally arising as representation categories for large classes of algebraic structures, Grothendieck-Verdier categories also have several other appealing features. In particular, while their tensor product is not necessarily exact, the structure is still sufficiently restrictive so as to guarantee the existence of internal Homs. Owing to the fact that the duality functor G of a Grothendieck-Verdier category $(\mathcal{C}, \otimes, K)$ is not necessarily monoidal, $\mathcal{C}$ naturally comes with a second tensor product $\circledast$ that is left exact. This also explains why such categories have been considered in categorical approaches to logic, which require two monoidal operations "and" and "or" that are related by a negation G. Each of the two tensor products come with associator isomorphisms obeying respective pentagon identities. In addition they are linked by "mixed associators", called *distributors*, which, unlike ordinary associators, need not be isomorphisms, but still obey pentagon identities. Structure morphisms that are not isomorphisms will be ubiquitous in our discussion.

We now summarize the most significant results of this paper. We consider left module categories $(\mathcal{M}, \triangleright)$ over a Grothendieck-Verdier category $(\mathcal{C}, \otimes, K)$ for which the functors

$$c \triangleright - : \quad \mathcal{M} \longrightarrow \mathcal{M} \qquad \text{and} \qquad - \triangleright m : \quad \mathcal{C} \longrightarrow \mathcal{M} \tag{1.1}$$

have a right adjoint for every $c \in \mathcal{C}$ and $m \in \mathcal{M}$. (The action of $\mathcal{C}$ on itself, the regular module, gives an example of such a module category.) If $\mathcal{C}$ is even rigid, then this reduces to the class of exact module categories considered in [11]. We show in Proposition 3.5 that any $\otimes$-module category $(\mathcal{M}, \triangleright)$ also comes with a natural structure of a module category $(\mathcal{M}, \blacktriangleright)$ over the monoidal category $(\mathcal{C}, \circledast)$ with monoidal structure given by the left exact tensor product on $\mathcal{C}$. In fact, we take it as a guiding principle of our analysis that the two monoidal structures of a Grothendieck-Verdier category should be treated on an equal footing. In particular, constructions that are defined using one of the two monoidal structures, such as module categories and module functors, should exhibit corresponding features with respect to the other monoidal structure as well.

Next we study module functors. Given two module categories $\mathcal{M}$ and $\mathcal{N}$ over the same Grothendieck-Verdier category $\mathcal{C}$, these are functors $F\colon \mathcal{M} \to \mathcal{N}$ with additional constraint data. In our setting it is not reasonable to require that these constraints are isomorphisms. This leads us to consider four classes of module functors, being lax or oplax for either of the two actions $\blacktriangleright$ and $\triangleright$, respectively, constituting four a priori different bicategories. In Theorem 3.13 we show that two of these bicategories can be naturally identified. Our insights about module functors allow us to introduce, in Definition 3.17, *module distributors* which relate the two actions $\blacktriangleright$ and $\triangleright$. In Section 3.3, we use profunctors and adjoints to exhibit numerous relations between different classes of module functors; these are summarized in Table (3.58). Our theory allows us to exhibit a plentitude of six pentagon diagrams for Grothendieck-Verdier module categories, and 32 pentagon diagrams for Grothendieck-Verdier bimodule categories.

It is well known that Grothendieck-Verdier categories admit evaluation and coevaluation morphisms. Our constructions allow for a concise proof that these obey snake relations that involve the distributors. We finally reveal in Proposition 3.45 in which way the internal Hom (and coHom) functors possess natural structures of module functors.

The structure of an algebra in a Grothendieck-Verdier category is naturally defined with respect to the monoidal structure $(\mathcal{C}, \otimes)$, and the structure of a coalgebra with respect to $(\mathcal{C}, \circledast)$. It is then immediate that internal (co)Homs of Grothendieck-Verdier module categories provide examples. As is well known, Frobenius algebras in monoidal categories play a significant role in many different contexts (for a few of them see e.g. the list given in the introduction of [14]). This motivates us to investigate, in Section 4, also Frobenius algebras in Grothendieck-Verdier categories. We introduce three possible definitions of the notion of a Frobenius algebra and show in Theorem 4.14 that they are equivalent. Since both monoidal structures enter, this proof is substantially more involved than in the case of rigid tensor categories. We then consider the following issue: While it is obvious that for any algebra A in a Grothendieck-Verdier category $\mathcal{C}$ the category mod-A of right A-modules is a $\mathcal{C}$-module category, it turns out to be a much more subtle question whether a given $\mathcal{C}$-module category $\mathcal{M}$ can be represented as a category of modules over some algebra in $\mathcal{C}$. And indeed generically this is not possible (see Example 4.28 for a counterexample); we call those module categories which *can* be represented in this way *algebraic*.

When investigating this question we are led to the Definition 4.23 of two interesting subcategories of $\mathcal{M}$: one of these is the full subcategory $\widehat{\mathcal{M}}^{\otimes}$ of $\otimes$-*admissible* objects, consisting of those objects $m \in \mathcal{M}$ for which the functor $\underline{\mathrm{Hom}}(m, -)\colon \mathcal{M} \to \mathcal{C}$ is a strong $\triangleright$-module functor and has a right adjoint. Our principle to treat the two monoidal structures $\otimes$ and $\circledast$ on the same footing leads us to also consider the corresponding structures for the $\blacktriangleright$ -action. We thus obtain another subcategory $\widehat{\mathcal{M}}^{\circledast}$ of objects $m \in \mathcal{M}$ for which the functor $\underline{\mathrm{coHom}}(m,-)\colon \mathcal{M} \longrightarrow \mathcal{C}$ is a strong $\blacktriangleright$-module functor and has a left adjoint.

The categories $\widehat{\mathcal{M}}^{\otimes,\circledast}$ need not be abelian; nevertheless they are still nicely compatible with the monoidal structure on $\mathcal{C}$. As we show in Proposition 4.24, by restriction of the action of $\mathcal{C}$ on $\mathcal{M}$ the category $\widehat{\mathcal{M}}^{\otimes}$ is a left $\widehat{\mathcal{C}}^{\otimes}$-module category, while $\widehat{\mathcal{M}}^{\circledast}$ is a left $\widehat{\mathcal{C}}^{\circledast}$-module category. In particular, the subcategories $\widehat{\mathcal{C}}^{\otimes}$ and $\widehat{\mathcal{C}}^{\circledast}$ of a Grothendieck-Verdier category $\mathcal{C}$ are monoidal subcategories. With these subcategories at hand, we are in a position to characterize, in Proposition 4.26 and Theorem 4.32, algebraic module categories: objects $m \in \widehat{\mathcal{M}}^{\otimes}$ which are $\mathcal{C}$-generators are precisely those for which the internal End $A_m = \underline{\mathrm{Hom}}(m,m)$ is an algebra such that mod-A_m is equivalent to $\mathcal{M}$. A dual statement holds for $\mathcal{C}$-cogenerators $m \in \widehat{\mathcal{M}}^{\circledast}$, the coalgebra $C_m = \underline{\mathrm{coHom}}(m,m)$ and C_m-comodules.

We consider the existence of the subcategories $\widehat{\mathcal{M}}^{\otimes}$ and $\widehat{\mathcal{M}}^{\circledast}$ as one of the main insights of this paper. They constitute a profound structure with important consequences. In particular, they enter into the definition of *relative Serre functors* which we introduce in Section 5.1: two functors

$$S: \quad \widehat{\mathcal{M}}^{\otimes} \longrightarrow \mathcal{M} \qquad \text{and} \qquad \widetilde{S}: \quad \widehat{\mathcal{M}}^{\circledast} \longrightarrow \mathcal{M}\,, \tag{1.2}$$

characterized by

$$\underline{\mathrm{Hom}}(n,Sm) \cong G(\underline{\mathrm{Hom}}(m,n)) \qquad \text{and} \qquad \underline{\mathrm{Hom}}(\widetilde{S}m,n) \cong G(\underline{\mathrm{Hom}}(n,m))\,. \tag{1.3}$$

Note that it is a priori not at all clear that the subcategories $\widehat{\mathcal{M}}^{\otimes}$ and $\widehat{\mathcal{M}}^{\circledast}$ admit the isomorphisms required in (1.3). Theorem 5.7 then asserts that S and $\widetilde{S}$ provide an equivalence of categories between $\widehat{\mathcal{M}}^{\otimes}$ and $\widehat{\mathcal{M}}^{\circledast}$. This allows us to generalize the insights of [19] to show that every choice of isomorphism $p\colon m \longrightarrow Sm$ in $\mathcal{M}$ induces the structure of a Frobenius algebra on the algebra $\underline{\mathrm{Hom}}(m,m)$.

In the present paper we discuss pivotal Grothendieck-Verdier categories only in two specific contexts: In Section 4.2 we characterize *symmetric* Frobenius algebras in pivotal Grothendieck-Verdier categories, and at the end of Section 5.3 we explain how rigid duals and Grothendieck-Verdier duals are related in pivotal categories. In view of potential applications to modular functors and conformal field theories, pivotal Grothendieck-Verdier categories deserve a further study.

2. Preliminaries

We assume that the reader is familiar with the concept of rigid duality in monoidal categories. In the present section we review the notion of Grothendieck-Verdier duality, contrast it to rigid duality, and recall concepts and results from the theory of module categories over monoidal categories.

In our discussion we consider simultaneously two types of categories – linear and not necessarily linear ones. More explicitly, if no extra assumptions are made, then categories and functors will be arbitrary. If instead we deal with a linear category, then we assume that $\Bbbk$ is an arbitrary but fixed field and that the category is $\Bbbk$-linear without any further finiteness assumptions: a functor between linear categories is assumed to be linear, natural transformations are linear maps, and for a module category over a linear monoidal category the action functor is assumed to be bilinear. Following [11, Defs. 1.8.5 & 1.8.6], a *finite category* is for us a linear category that is equivalent as an abelian category to the category of finite-dimensional modules over a finite-dimensional $\Bbbk$-algebra.

Definition 2.1.

1. An object K in a monoidal category $(\mathcal{C}, \otimes, 1, \alpha, l, r)$ is said to be a *dualizing object* if, for every $y \in \mathcal{C}$, the functor $x \mapsto \mathrm{Hom}(x \otimes y, K)$ is representable by some object $Gy \in \mathcal{C}$ and the so defined contravariant functor $G\colon \mathcal{C} \longrightarrow \mathcal{C}$ is an anti-equivalence, so that there are isomorphisms
$$\varpi_{x,y}: \quad \mathrm{Hom}(x \otimes y, K) \xrightarrow{\cong} \mathrm{Hom}(x, Gy) \tag{2.1}$$
natural in $x, y \in \mathcal{C}$. G is called the *duality functor with respect to K*.
2. A *Grothendieck-Verdier category – or GV-category for short – is a monoidal category together with a choice of a dualizing object $K \in \mathcal{C}$.*

The assignment of the functor G on a morphism $f\colon x \longrightarrow y$ is obtained by noting that the pullback induces the natural transformation in the left column of the diagram

$$\begin{tikzcd} \mathrm{Hom}(-\otimes y, K) \arrow[r, "\varpi_{-,y}"] \arrow[d, "(\mathrm{id}\otimes f)^*"'] & \mathrm{Hom}(-, Gy) \arrow[d, "Gf_*"] \\ \mathrm{Hom}(-\otimes x, K) \arrow[r, "\varpi_{-,x}"] & \mathrm{Hom}(-, Gx) \end{tikzcd} \tag{2.2}$$

Transporting this natural transformation to the one in the right column of (2.2) defines, by the Yoneda lemma, the morphism Gf.

In general, $G(y) \otimes G(x)$ is not isomorphic to $G(x \otimes y)$. However, the covariant functor G^2 comes with a canonical monoidal structure [4, Prop. 5.2]. The choice of K is *structure*. The following result [4, Prop. 2.3] clarifies the freedom given by this structure:

Proposition 2.2. *Let $(\mathcal{C}, K)$ be a Grothendieck-Verdier category with duality functor $G \equiv G_K$.*

1. *The functor G is an anti-equivalence between the full subcategory of invertible objects U in $\mathcal{C}$ and the full subcategory of dualizing objects. Indeed, G satisfies $G(U) = K \otimes U^{-1}$ for any invertible object $U \in \mathcal{C}$.*
   Analogous statements hold for the functor G^{-1}, with $G^{-1}(U) = U^{-1} \otimes K$.
2. *If an object $U \in \mathcal{C}$ is invertible, then $G^2(U)$ is invertible as well, and one has a canonical isomorphism $K \otimes U^{-1} \xrightarrow{\cong} (G^2U)^{-1} \otimes K$.*

The duality functor G allows for the following definition, which is central to our work.

Definition 2.3. The functor $\circledast : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is defined by the mapping

$$x \circledast y := G^{-1}(Gy \otimes Gx) \tag{2.3}$$

on objects, and analogously on morphisms.

This functor provides a second monoidal structure on $\mathcal{C}$:

Proposition 2.4. *Let $(\mathcal{C}, \otimes, 1, \alpha, l^{\otimes}, r^{\otimes}, K)$ be a GV-category. Then the functor $\circledast : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ defined by (2.3) together with unit constraints*

$$l^{\circledast}_x := G^{-1}((r^{\otimes}_{Gx})^{-1}) \qquad \textit{and} \qquad r^{\circledast}_x := G^{-1}((l^{\otimes}_{Gx})^{-1}) \tag{2.4}$$

and associator

$$\alpha^{\circledast}_{x,y,z} := G^{-1}(\alpha^{-1}_{Gz,Gy,Gx}) \tag{2.5}$$

endows $\mathcal{C}$ with the structure of a monoidal category $(\mathcal{C}, \circledast, K, \alpha^{\circledast}, l^{\circledast}, r^{\circledast})$ with monoidal unit K.

One may also define a functor $\circledast' : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ by $x \circledast' y := G(G^{-1}y \otimes G^{-1}x)$. This is not independent, however: The $\otimes$-monoidal structure on the functor G^2 implies a canonical identification of $\circledast$ and $\circledast'$ [4, Sect. 4.1]. The functors $\otimes$ and $\circledast$ are isomorphic if and only if G is monoidal, and thus in particular if $\mathcal{C}$ is rigid. Also note that the functor G^2 is monoidal both for $\otimes$ and for $\circledast$.

If we change the choice of dualizing object from K to $\widetilde{K} := g \otimes K$ for an invertible object $g \in \mathcal{C}$, then the monoidal functor G^2 gets replaced by $\widetilde{G}^2$ defined as

$$\widetilde{G}^2(y) := g \otimes G^2(y) \otimes g^{-1} \tag{2.6}$$

for $y \in \mathcal{C}$. To see this, just note that the functor G^2 is determined by the isomorphism $\mathrm{Hom}(x \otimes y, K) \cong \mathrm{Hom}(G^2y \otimes x, K)$ for all $x, y \in \mathcal{C}$ and that we have

$$\begin{aligned}\mathrm{Hom}(x\otimes y,\widetilde{K}) &= \mathrm{Hom}(x\otimes y, g\otimes K) \cong \mathrm{Hom}(g^{-1}\otimes x\otimes y, K)\\ &\cong \mathrm{Hom}(G^2y\otimes g^{-1}\otimes x, K)\\ &\cong \mathrm{Hom}(g\otimes G^2y\otimes g^{-1}\otimes x, g\otimes K) = \mathrm{Hom}(\widetilde{G}^2y\otimes x,\widetilde{K})\end{aligned}\tag{2.7}$$

for $x, y\in\mathcal{C}$.

For $a, b, c\in\mathcal{C}$ we have

$$\mathrm{Hom}(a\otimes b, c) \cong \mathrm{Hom}(a, c\circledast Gb)\tag{2.8}$$

and

$$\mathrm{Hom}(a\otimes b, c) \cong \mathrm{Hom}(b, G^{-1}a\circledast c)\,.\tag{2.9}$$

Informally, G behaves like a right duality and G^{-1} like a left duality if we take the $\otimes$-tensor product in the domain and the $\circledast$-tensor product in the codomain of a morphism.

In general the two monoidal structures are different. However, there are mixed associators, called *distributors*, which relate them. These are coherent morphisms $(a\circledast b)\otimes c\longrightarrow a\circledast(b\otimes c)$ and $a\otimes(b\circledast c)\longrightarrow(a\otimes b)\circledast c$, for $a, b, c\in\mathcal{C}$; they are not necessarily isomorphisms. In Section 3.4 we will examine the distributors in detail and in particular prove the coherence diagrams satisfied by them.

Example 2.5. Let A be a finite-dimensional $\Bbbk$-algebra. The category A-bimod of finite-dimensional A-bimodules is a GV-category with the usual tensor product over A as the right exact tensor product and with the dualizing functor given by $G(m)=m^*$ for $m\in A$-bimod, with m^* denoting the linear dual [13]. If A is in addition commutative, then the category mod-A of finite-dimensional right A-modules is a full subcategory of A-bimod by regarding a right module as a bimodule in the natural way. Moreover, the GV-structure on A-bimod then induces the structure of a GV-category on mod-A.

Other examples of GV-categories include representation categories of vertex operator algebras [2] and of Hopf algebroids [1].

Remark 2.6. In an abelian GV-category $\mathcal{C}$, $x\in\mathcal{C}$ is projective if and only if $G(x)$ is injective: x is projective if and only if $\mathrm{Hom}(x,-)$ is exact, which is the case iff $\mathrm{Hom}(G(-),G(x))$ is exact, which (since G is an equivalence) is the case iff $\mathrm{Hom}(-,G(x))$ is exact, which is the case if and only if $G(x)$ is injective.

Crucial for our work is the connection between module categories and categories of modules.

Definition 2.7. Let $\mathcal{C}$ be a monoidal category. A *left module category over* $\mathcal{C}$ is a category $\mathcal{M}$ together with a functor

$$\triangleright:\quad \mathcal{C}\times\mathcal{M}\longrightarrow\mathcal{M}\,,\tag{2.10}$$

called the *action functor*, together with a natural family of isomorphisms $c \triangleright (d \triangleright m) \xrightarrow{\cong} (c \otimes d) \triangleright m$ for all $c, d \in \mathcal{C}$ and all $m \in \mathcal{M}$ and isomorphisms $1 \triangleright m \xrightarrow{\cong} m$ for all $m \in \mathcal{M}$, which obey the appropriate pentagon and triangle constraints.

Analogously one considers right module categories over $\mathcal{C}$. For $\mathcal{D}$ another monoidal category, a $(\mathcal{C}, \mathcal{D})$-bimodule category is a left $\mathcal{C}$- and a right $\mathcal{D}$-module category with an additional family of coherent isomorphisms that interchange the ordering of the two actions.

Definition 2.8. A (unital, associative) *algebra* in a monoidal category $(\mathcal{C}, \otimes)$ is an algebra object in $\mathcal{C}$, i.e. a triple (A, μ, η) with $A \in \mathcal{C}$, $\mu \in \mathrm{Hom}(A \otimes A, A)$ and $\eta \in \mathrm{Hom}(1, A)$ satisfying (including the associator $\alpha_{A,A,A} \colon (A \otimes A) \otimes A \longrightarrow A \otimes (A \otimes A)$ and unitors $l_A \colon 1 \otimes A \longrightarrow A$ and $r_A \colon A \otimes 1 \longrightarrow A$)

$$\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu) \circ \alpha_{A,A,A} \tag{2.11}$$

as morphisms in $\mathrm{Hom}((A \otimes A) \otimes A, A)$ and

$$\mu \circ (\eta \otimes \mathrm{id}_A) \circ l_A^{-1} = \mathrm{id}_A = \mu \circ (\mathrm{id}_A \otimes \eta) \circ r_A^{-1} \tag{2.12}$$

as morphisms in $\mathrm{End}(A)$.

Dually, a (counital, coassociative) *coalgebra* in $(\mathcal{C}, \otimes, 1, \alpha, l, r)$ is a triple (C, Δ, ε) with $C \in \mathcal{C}$, $\Delta \in \mathrm{Hom}(C, C \otimes C)$ and $\varepsilon \in \mathrm{Hom}(C, 1)$ such that $(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\alpha_{C,C,C})^{-1} \circ (\mathrm{id}_C \otimes \Delta) \circ \Delta$ as morphisms in $\mathrm{Hom}(C, (C \otimes C) \otimes C)$ and $l_C \circ (\varepsilon \otimes \mathrm{id}_C) \circ \Delta = \mathrm{id}_C = r_C \circ (\mathrm{id}_C \otimes \varepsilon) \circ \Delta$ as morphisms in $\mathrm{End}(C)$.

Similarly one defines for a (co)algebra in $\mathcal{C}$ the notion of a (co)module. It is well known that for any algebra $A \in \mathcal{C}$ the category mod-A of right A-modules is a left $\mathcal{C}$-module category. The category comod-C of right C-comodules over a coalgebra $C \in \mathcal{C}$ is a left $\mathcal{C}$-module category, too.

3. GV-module categories

In this section we develop the theory of module categories over GV-categories. According to our guiding principle, module categories and module functors should exhibit similar features with respect to both monoidal structures of a GV-category.

3.1. Module categories and module functors for GV-categories

Definition 3.1. Let $\mathcal{C}$ be a GV-category. A *left GV-module category* is a left module category over $(\mathcal{C}, \otimes)$ with action functor $\triangleright \colon \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$, such that the functors

$$c \triangleright - : \quad \mathcal{M} \longrightarrow \mathcal{M} \qquad \text{and} \qquad - \triangleright m : \quad \mathcal{C} \longrightarrow \mathcal{M} \tag{3.1}$$

have a right adjoint for all objects $c \in \mathcal{C}$ and $m \in \mathcal{M}$.

Remark 3.2. This definition could be formulated for an arbitrary monoidal category $\mathcal{C}$, as the GV-structure of $\mathcal{C}$ is not used at all. Nevertheless the separate terminology 'GV-module category' is legitimate: it is justified by the results that we will obtain below, such as Proposition 3.5. Also note that a functor that has a right adjoint preserves colimits and is thus in particular cocontinuous. Demanding the existence of adequate adjoints allows us to work simultaneously in the set-theoretic and linear setting, while in the latter case it generalizes certain exactness assumptions, see Lemma 3.4.

Remark 3.3. Twisting an action of a monoidal category by a monoidal endofunctor yields again an action. In the case of a GV-category we can in particular twist the action by any even power of the functor G. Thus with any GV-module category over a GV-category $\mathcal{C}$ there comes a whole family of GV-module categories over $\mathcal{C}$.

The internal Hom functor $\underline{\mathrm{Hom}}$ is the right adjoint of the functor $- \triangleright m$, which exists by the assumptions in Definition 3.1:

$$\mathrm{Hom}_{\mathcal{M}}(c \triangleright m, m') \cong \mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(m, m')) \tag{3.2}$$

for $c \in \mathcal{C}$ and $m, m' \in \mathcal{M}$. These will be examined in detail in Section 3.5.

In case $\mathcal{C}$ is a rigid finite category, then for a module category $\mathcal{M}$ one requires exactness in the first variable (see Definition 7.3.1 of [11]); exactness in the second variable is then automatic (Exercise 7.3.2 in [11]). Indeed, existence of the respective adjoints follows in this case as well:

Lemma 3.4. *If $\mathcal{C}$ is a rigid finite category, then a GV-module category over $\mathcal{C}$ is the same as a $\mathcal{C}$-module category in the sense of [11, Def. 7.3.1], i.e. the action functor $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ is exact in the first variable.*

Proof. Let $\mathcal{M}$ be a module category, in the sense of [11], over a rigid finite category. A functor between finite categories has a right/left adjoint if and only if it is right/left exact (see e.g. Corollary 2.3 in [12]). Thus if $- \triangleright n : \mathcal{C} \longrightarrow \mathcal{M}$ is exact, then it has in particular a right adjoint. Hence by [11, Exc. 7.3.2], the functors $c \triangleright - : \mathcal{M} \longrightarrow \mathcal{M}$ have a right adjoint. Conversely, if the module category $\mathcal{M}$ is a GV-module category, then the functors $- \triangleright n$ have a right adjoint and are thus right exact. Further, the isomorphisms

$$\mathrm{Hom}(m, c \triangleright n) \cong \mathrm{Hom}(c^\vee \triangleright m, n) \cong \mathrm{Hom}(c^\vee, \underline{\mathrm{Hom}}(m, n)) \cong \mathrm{Hom}({}^\vee\underline{\mathrm{Hom}}(m, n), c) \tag{3.3}$$

for $m, n \in \mathcal{M}$ and $c \in \mathcal{C}$ show that $- \triangleright n$ also has a left adjoint and is thus also left exact. (Note that in (3.3) rigidity is only used in the first step.) $\square$

For a GV-module category, by definition for every $c \in \mathcal{C}$ the endofunctor $\rho^{\otimes}_{\mathcal{M}}(c) := c \triangleright -$ of $\mathcal{M}$ has a right adjoint. In the special case $\mathcal{M} = \mathcal{C}$, i.e. the regular module category of $\mathcal{C}$ acting on itself by $\otimes$, according to the isomorphisms (2.9) we already have such a right adjoint, namely $G^{-1}(c) \circledast -$; by convention we will work with this particular right adjoint. In all other cases we fix some right adjoint, which we denote by $H_c \colon \mathcal{M} \to \mathcal{M}$, i.e. we have a natural family

$$\mathrm{Hom}_{\mathcal{M}}(c \triangleright m, n) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{M}}(m, H_c(n)) \tag{3.4}$$

of isomorphisms for $m, n \in \mathcal{M}$. The endofunctor H_c defines via $c \longmapsto H_{Gc}(-)$ a functor

$$\rho^{\circledast}_{\mathcal{M}} : \quad \mathcal{C} \longrightarrow \mathrm{Fun}_{\mathrm{l.e.}}(\mathcal{M}, \mathcal{M}) \tag{3.5}$$

from $\mathcal{C}$ to the category of left exact endofunctors of $\mathcal{M}$, i.e. $\rho^{\circledast}_{\mathcal{M}}(c)(m) = H_{Gc}(m)$. The functor H_{Gc} appearing here furnishes an action of the monoidal category $(\mathcal{C}, \circledast)$ on $\mathcal{M}$:

Proposition 3.5. *Let $\mathcal{M}$ be a left GV-module category over a GV-category $(\mathcal{C}, \otimes, K)$. Then the bifunctor $\blacktriangleright \colon \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ defined by*

$$c \blacktriangleright m := H_{Gc}(m) \tag{3.6}$$

for $c \in \mathcal{C}$ and $m \in \mathcal{M}$ has a left adjoint (and thus preserves limits, and is in particular continuous) in each variable and defines a left module category structure over $(\mathcal{C}, \circledast)$.

Proof. The left adjoint of $c \blacktriangleright -$ is by definition the endofunctor $Gc \triangleright -$. For the adjoint in the second variable we use the family of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}}(m, c \blacktriangleright n) &= \mathrm{Hom}_{\mathcal{M}}(m, H_{Gc}(n)) \cong \mathrm{Hom}_{\mathcal{M}}(Gc \triangleright m, n) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(Gc, \underline{\mathrm{Hom}}(m, n)) \cong \mathrm{Hom}_{\mathcal{C}}(G^{-1}\underline{\mathrm{Hom}}(m, n), c) \end{aligned} \tag{3.7}$$

to conclude that $- \blacktriangleright n$ has $G^{-1}\underline{\mathrm{Hom}}(-, n)$ as left adjoint. The module category structure is established as follows. The module constraint of $\blacktriangleright$ is the natural family of isomorphisms that are obtained by the Yoneda lemma from the isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}}(m, b \blacktriangleright (c \blacktriangleright n)) &\cong \mathrm{Hom}_{\mathcal{M}}(Gb \triangleright m, c \blacktriangleright n) \cong \mathrm{Hom}_{\mathcal{M}}(Gc \triangleright (Gb \triangleright m), n) \\ &\cong \mathrm{Hom}_{\mathcal{M}}((Gc \otimes Gb) \triangleright m, n) \\ &\cong \mathrm{Hom}_{\mathcal{M}}(m, G^{-1}(Gc \otimes Gb) \blacktriangleright n) = \mathrm{Hom}_{\mathcal{M}}(m, (b \circledast c) \blacktriangleright n) \end{aligned} \tag{3.8}$$

for $m, n \in \mathcal{M}$ and $b, c \in \mathcal{C}$. Moreover, the Yoneda embedding transports the pentagon relation for the module constraint of $\triangleright$ to the pentagon relation for the module constraint of $\blacktriangleright$. $\square$

By construction we have the adjunction formula

$$\mathrm{Hom}_{\mathcal{M}}(m, G^{-1}c \blacktriangleright m') \cong \mathrm{Hom}_{\mathcal{M}}(c \triangleright m, m') \,. \tag{3.9}$$

The resulting module structure depends on the choice of right adjoints. However, we have

Lemma 3.6. *The module categories over* $(\mathcal{C}, \circledast)$ *for different choices of right adjoints* (3.4) *are canonically equivalent.*

Proof. Let $\mathcal{M} = (\mathcal{M}, \triangleright)$ be a left GV-module category over $(\mathcal{C}, \otimes)$, and let H_{Gc} and $\widetilde{H}_{Gc}$, for $c \in \mathcal{C}$, be two families of right adjoints (3.4); this provides us with two $(\mathcal{C}, \circledast)$-module categories, which we denote by $(\mathcal{M}, \blacktriangleright)$ and $(\mathcal{M}, \widetilde{\blacktriangleright})$, with respective module structures $c \blacktriangleright m = H_{Gc}(m)$ and $c \mathbin{\widetilde{\blacktriangleright}} m = \widetilde{H}_{Gc}(m)$. The following considerations show that the identity functor $\mathrm{id}_{\mathcal{M}}$ has a canonical structure of a module functor and thus provides an equivalence of module categories: By the uniqueness of right adjoints we get for every $c \in \mathcal{C}$ a unique natural isomorphism $\varphi_c \colon H_{Gc} \longrightarrow \widetilde{H}_{Gc}$ with components $\varphi_{c,m} \colon c \blacktriangleright m = H_{Gc}(m) \longrightarrow \widetilde{H}_{Gc}(m) = c \mathbin{\widetilde{\blacktriangleright}} m$ for all $m \in \mathcal{M}$. These morphisms are natural in $m \in \mathcal{M}$ as well as natural in $c \in \mathcal{C}$: For any morphism $f \colon c \longrightarrow d$ in $\mathcal{C}$ we are given a natural transformation $\psi_f = (G(f) \triangleright -) \colon (G(d) \triangleright -) \Longrightarrow (G(c) \triangleright -)$, which defines two natural transformations

$$\psi_f^* : \quad H_{Gc} \Longrightarrow H_{Gd} \qquad \text{and} \qquad \widetilde{\psi}_f^* : \quad \widetilde{H}_{Gc} \Longrightarrow \widetilde{H}_{Gd} \tag{3.10}$$

between the respective right adjoints. By a general fact about the isomorphism relating different right adjoints, for all $c, d \in \mathcal{C}$ and all $f \colon c \longrightarrow d$ we have the equality

$$\widetilde{\psi}_f^* \cdot \varphi_c = \varphi_d \cdot \psi_f^* \tag{3.11}$$

between vertical composites of natural transformations. This directly implies that the isomorphisms $\varphi_{c,m} \colon c \blacktriangleright m \longrightarrow c \mathbin{\widetilde{\blacktriangleright}} m$ are also natural in $c \in \mathcal{C}$. To conclude that (id, φ) is a module functor from $(\mathcal{M}, \blacktriangleright)$ to $(\mathcal{M}, \widetilde{\blacktriangleright})$, it thus remains to verify the coherence diagrams for φ. These readily follow from the uniqueness of the natural isomorphism that relates two different right adjoints. $\square$

For the opposite category of a GV-module category over $\mathcal{C}$ we get analogously right $\mathcal{C}$-actions:

Proposition 3.7. *Let* $\mathcal{C}$ *be a GV-category and* $\mathcal{M}$ *a left GV-module category over* $\mathcal{C}$*. Then* $\mathcal{M}^{\mathrm{opp}}$ *is a right GV-module category over* $(\mathcal{C}, \otimes)$ *and a right module category over* $(\mathcal{C}, \circledast)$*, with left and right exact action bifunctors given by*

$$\begin{aligned} \triangleleft : \quad \mathcal{M}^{\mathrm{opp}} \times \mathcal{C} &\longrightarrow \mathcal{M}^{\mathrm{opp}} \,, \\ (m, c) &\longmapsto G^{-1}c \blacktriangleright m \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \blacktriangleleft : \quad \mathcal{M}^{\mathrm{opp}} \times \mathcal{C} &\longrightarrow \mathcal{M}^{\mathrm{opp}} \,, \\ (m, c) &\longmapsto G^{-1}c \triangleright m \,, \end{aligned} \tag{3.12}$$

respectively, for $c \in \mathcal{C}$ *and* $m \in \mathcal{M}$*.*

Proof. First note that we only need to define the $\triangleleft$ -action; the $\blacktriangleleft$-action is then fixed by the adjunction $\mathrm{Hom}_{\mathcal{M}^{\mathrm{opp}}}(m', m \blacktriangleleft Gc) \cong \mathrm{Hom}_{\mathcal{M}^{\mathrm{opp}}}(m' \triangleleft c, m)$ which follows directly from (3.9). Further, the proposed bifunctor $\triangleleft$ is well defined due to G being contravariant, and it inherits the asserted right and left exactness from $\blacktriangleright$. Again we determine the module constraint with the help of the Yoneda lemma. We have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}}(m, n \triangleleft (b \triangleleft c)) &\cong \mathrm{Hom}_{\mathcal{M}}(m, G^{-1}c \blacktriangleright (G^{-1}b \blacktriangleright n)) \\ &\cong \mathrm{Hom}_{\mathcal{M}}(m, (G^{-1}c \circledast G^{-1}b) \blacktriangleright n) \\ &\cong \mathrm{Hom}_{\mathcal{M}}(m, G^{-1}(b \otimes c) \blacktriangleright n) \cong \mathrm{Hom}_{\mathcal{M}}(m, n \triangleleft (b \otimes c)) \end{aligned} \tag{3.13}$$

for $b, c \in \mathcal{C}$ and $m, n \in \mathcal{M}$. The pentagon axiom for this action constraint follows from the injectivity of the Yoneda embedding and the fact that it holds for the constraints of $\blacktriangleright$. $\square$

Remark 3.8.

1. In accordance with our guiding principle, if $(\mathcal{C}, \otimes, 1, K)$ is a GV-category, then $(\mathcal{C}, \circledast, K, 1)$ is an 'op-GV-category', for which the primary monoidal structure is left exact and has K as monoidal unit, and for which the defining isomorphisms are $\mathrm{Hom}(1, b \circledast c) \xrightarrow{\cong} \mathrm{Hom}(Gc, b)$ in place of (2.1). Similarly there is a notion of op-GV-module category over $(\mathcal{C}, \circledast)$, for which the action functors that are analogous to (3.1) are required to admit left adjoints. Analogously to Proposition 3.5 there is then an associated GV-module category over $(\mathcal{C}, \otimes)$. In this terminology, $(\mathcal{M}^{\mathrm{opp}}, \blacktriangleleft)$ is a right op-GV-module category over $(\mathcal{C}, \circledast)$.
2. Recall from Remark 3.3 that for a module category $(\mathcal{M}, \triangleright)$ also $G^{2n}(-) \triangleright m$ is a $\mathcal{C}$-action on $\mathcal{M}$. Since G^2 is also a monoidal autoequivalence of $(\mathcal{C}, \circledast)$, an analogous statement applies to $\blacktriangleright$: for any integer n, also $G^{2n}(-) \blacktriangleright m$ defines a $(\mathcal{C}, \circledast)$-action on $\mathcal{M}$.

Example 3.9. $\mathcal{C}$ is a $(\mathcal{C}, \otimes)$-module – the regular module – with action given by the right exact tensor product $\otimes$. The corresponding left exact action is given by the $\circledast$-tensor product.

For $\mathcal{M}$ a left module category over a GV-category $\mathcal{C}$ we write the associated endofunctors of $\mathcal{M}$ that result from the left action as

$$\begin{aligned} \mathrm{L}_c^{\triangleright}: \quad & \mathcal{M} \longrightarrow \mathcal{M}, \\ & m \longmapsto c \triangleright m \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{L}_c^{\blacktriangleright}: \quad & \mathcal{M} \longrightarrow \mathcal{M}, \\ & m \longmapsto c \blacktriangleright m. \end{aligned} \tag{3.14}$$

The basic adjunction (3.9) for the left $\mathcal{C}$-GV-module category $\mathcal{M}$ then reads

$$\mathrm{Hom}_{\mathcal{M}}(m, \mathrm{L}^{\blacktriangleright}_{c}(n)) \cong \mathrm{Hom}_{\mathcal{M}}(\mathrm{L}^{\triangleright}_{Gc}(m), n)\,. \tag{3.15}$$

The counit and unit of this adjunction are

$$\mathrm{ev}_{c,m}: \quad G(c) \triangleright (c \blacktriangleright m) \longrightarrow m \qquad \text{and} \qquad \mathrm{coev}_{c,m}: \quad m \longrightarrow c \blacktriangleright (Gc \triangleright m)\,, \tag{3.16}$$

while

$$\begin{tikzcd} c \blacktriangleright m \arrow[r, "\mathrm{coev}_{c,c\blacktriangleright m}"'] \arrow[rr, bend left, "\mathrm{id}"] & c \blacktriangleright (G(c) \triangleright (c \blacktriangleright m)) \arrow[r, "c\blacktriangleright \mathrm{ev}_{c,m}"'] & c \blacktriangleright m \end{tikzcd} \tag{3.17}$$

and

$$\begin{tikzcd} Gc \triangleright m \arrow[r, "Gc\,\triangleright\, \mathrm{coev}_{c,m}"'] \arrow[rr, bend left, "\mathrm{id}"] & Gc \triangleright (c \blacktriangleright (Gc \triangleright m)) \arrow[r, "\mathrm{ev}_{c,Gc\triangleright m}"'] & Gc \triangleright m \end{tikzcd} \tag{3.18}$$

are the snake identities of the adjunction.

The $\blacktriangleright$-action provides us with distinguished natural isomorphisms $\mathrm{L}^{\blacktriangleright}_{c} \circ L^{\blacktriangleright}_{d} \xrightarrow{\cong} L^{\blacktriangleright}_{c\otimes d}$ for $c, d \in \mathcal{C}$. Likewise, the $\triangleright$-action gives distinguished natural isomorphisms $\mathrm{L}^{\triangleright}_{Gd} \circ L^{\triangleright}_{Gc} \xrightarrow{\cong} L^{\triangleright}_{G(c\otimes d)}$. We denote the right adjoint of a functor F by $F^{\mathrm{r.a.}}$. Using that for composable functors F_1 and F_2 that have right adjoints there is a commuting diagram $(F_1 \circ F_2) \circ (F_1 \circ F_2)^{\mathrm{r.a.}} \rightrightarrows \mathrm{id}$ of counits, we thus have a commuting diagram

$$\begin{tikzcd} G(c \otimes d) \triangleright ((c \otimes d) \blacktriangleright m) \arrow[r, "\mathrm{ev}_{c\otimes d,m}"] \arrow[d, "\cong"'] & m \\ Gd \triangleright (Gc \triangleright (c \blacktriangleright (d \blacktriangleright m))) \arrow[r, "Gd\triangleright \mathrm{ev}_{c,d\blacktriangleright m}"] & Gd \triangleright (d \blacktriangleright m) \arrow[u, "\mathrm{ev}_{d,m}"'] \end{tikzcd} \tag{3.19}$$

A similar coherence diagram holds for the collection of morphisms $\mathrm{coev}_{c,m}$.

3.2. Conjugated pairs of lax and oplax module functors

The following provides a meaningful notion of GV-module functors:

Definition 3.10. Let $\mathcal{C}$ be a GV-category and let $(\mathcal{M}, \triangleright)$ and $(\mathcal{N}, \triangleright')$ be left GV-module categories over $(\mathcal{C}, \otimes)$.

A *lax $\triangleright$-module functor* from $\mathcal{M}$ to $\mathcal{N}$ is a functor $F\colon \mathcal{M} \to \mathcal{N}$ together with a family of morphisms $f_{c,m}\colon c \triangleright' F(m) \to F(c \triangleright m)$ for $c \in \mathcal{C}$ and $m \in \mathcal{M}$ that obeys coherence conditions in the form of the commutativity of the pentagon and triangle diagrams

$$
\begin{tikzcd}
& F((c\otimes d)\triangleright m) \arrow[dl, "F(\alpha_{c.d.m})"', "\cong"] & \\
F(c\triangleright(d\triangleright m)) & & (c\otimes d)\triangleright' F(m) \arrow[ul, "f_{c\otimes d,m}"'] \arrow[d, "\alpha'_{c,d,F(m)}", "\cong"'] \\
c\triangleright' F(d\triangleright m) \arrow[u, "f_{c,d\triangleright m}"] & & c\triangleright'(d\triangleright' F(m)) \arrow[ll, "\mathrm{id}_c\triangleright' f_{d,m}"']
\end{tikzcd}
\tag{3.20}
$$

and

$$
\begin{tikzcd}
F(1\triangleright m) \arrow[dr, "F(\lambda_m)"', "\cong"] & & 1\triangleright' F(m) \arrow[ll, "f_{1,m}"'] \arrow[dl, "\lambda'_{F(m)}", "\cong"'] \\
& F(m) &
\end{tikzcd}
\tag{3.21}
$$

(with α, λ and α', λ' the module constraints for the actions $\triangleright$ and $\triangleright'$, respectively) with respect to the action of $(\mathcal{C},\otimes)$.

An *oplax* $\triangleright$*-module functor* is a functor $F\colon \mathcal{M}\longrightarrow\mathcal{N}$, together with an analogous coherent family of morphisms $f_{c,m}\colon F(c\triangleright m)\longrightarrow c\triangleright' F(m)$.

Similarly, a *lax* (*oplax*) $\blacktriangleright$*-module functor* is a functor $F\colon \mathcal{M}\to\mathcal{N}$, together with an analogous family of morphisms that are coherent with respect to the action $\blacktriangleright$ of $(\mathcal{C},\otimes)$.

Note that no assumption on the existence of adjoints of such functors is made. To a monoidal category with exact tensor product one associates a bicategory with module categories as objects and strong module functors as 1-morphisms. If $\mathcal{C}$ is instead a GV-category, we can naturally associate to it four different bicategories $\text{GV-Mod}^{\triangleright,\text{lax}}(\mathcal{C})$, $\text{GV-Mod}^{\triangleright,\text{oplax}}(\mathcal{C})$, $\text{GV-Mod}^{\blacktriangleright,\text{lax}}(\mathcal{C})$ and $\text{GV-Mod}^{\blacktriangleright,\text{oplax}}(\mathcal{C})$:

Lemma 3.11. *Let $\mathcal{C}$ be a GV-category. GV-module categories as objects, lax, respectively oplax, $\triangleright$-module functors as 1-morphisms, and $\triangleright$-module natural transformations as 2-morphisms form a bicategory* $\text{GV-Mod}^{\triangleright,lax}(\mathcal{C})$*, respectively* $\text{GV-Mod}^{\triangleright,oplax}(\mathcal{C})$*. Analogously, GV-module categories, lax, respectively oplax, $\blacktriangleright$-module functors and $\blacktriangleright$-module natural transformations form a bicategory* $\text{GV-Mod}^{\blacktriangleright,lax}(\mathcal{C})$*, respectively (*$\text{GV-Mod}^{\blacktriangleright,oplax}(\mathcal{C})$*).*

Proof. The composition of lax $\triangleright$-module functors is canonically a lax $\triangleright$-module functor; analogous arguments apply in the oplax case and for $\blacktriangleright$-module functors. The claim thus follows from the fact that, for $B\mathcal{D}$ the delooping of any monoidal category $\mathcal{D}$, the bicategory $\mathrm{Lax}(B\mathcal{D},\mathcal{C}at)$ of strong 2-functors, lax natural 2-transformations and modifications is canonically isomorphic to the bicategory of $\mathcal{D}$-module categories, lax module functors and module natural transformations. The case of oplax module functors is treated analogously. $\square$

According to our guiding principle a functor that has a compatibility with the $\triangleright$-action should have a compatibility with the corresponding $\blacktriangleright$-action as well. The details depend, however, crucially on whether the considered compatibility is strong, lax or oplax. In the situation at hand, for a functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ between left GV-module categories $\mathcal{M}$ and $\mathcal{N}$, lax $\triangleright$-module functor structures and oplax $\blacktriangleright$-module functor structures on F are in bijection with each other. This can be seen as follows. Given a collection of morphisms $f^{\triangleright}_{c,m}\colon c \triangleright F(m) \to F(c \triangleright m)$ for $c \in \mathcal{C}$ and $m \in \mathcal{M}$, define the *conjugated* collection of morphisms $f^{\blacktriangleright}_{c,m}\colon F(c \blacktriangleright m) \longrightarrow c \blacktriangleright F(m)$ as the family whose members are the composites

$$
\begin{aligned}
F(c \blacktriangleright m) \xrightarrow{\mathrm{coev}_{c,F(c\blacktriangleright m)}} c \blacktriangleright (Gc \triangleright F(c \blacktriangleright m)) \xrightarrow{c\,\blacktriangleright\, f^{\triangleright}_{Gc,c\blacktriangleright m}} c \blacktriangleright F(Gc \triangleright (c \blacktriangleright m))\\
\xrightarrow{c\,\blacktriangleright\, F(\mathrm{ev}_{c,m})} c \blacktriangleright F(m)
\end{aligned}
\tag{3.22}
$$

with the morphisms ev and coev as defined in (3.16). Note that different choices of adjunction data lead to different conjugated functors; however, these can be shown to be related by the functors from Lemma 3.6. Under the adjunction

$$
\mathrm{Hom}(F(c \blacktriangleright m), c \blacktriangleright F(m)) \cong \mathrm{Hom}(Gc \triangleright (F(c \blacktriangleright m)), F(m)) \tag{3.23}
$$

the composite (3.22) corresponds to the morphism

$$
Gc \triangleright (F(c \blacktriangleright m)) \xrightarrow{f^{\triangleright}} F(Gc \triangleright (c \blacktriangleright m)) \xrightarrow{F(\mathrm{ev}_{c,m})} F(m)\,. \tag{3.24}
$$

Conversely, for a collection of morphisms $g^{\blacktriangleright}_{c,m}\colon F(c \blacktriangleright m) \to c \blacktriangleright F(m)$, we define the conjugated collection of morphisms $g^{\triangleright}_{Gc,m}\colon Gc \triangleright F(m) \to F(Gc \triangleright m)$ by

$$
\begin{aligned}
Gc \triangleright F(m) \xrightarrow{Gc\triangleright F(\mathrm{coev}_{c,m})} Gc \triangleright F(c \blacktriangleright (Gc \triangleright m)) \xrightarrow{g^{\blacktriangleright}} Gc \triangleright (c \blacktriangleright F(Gc \triangleright m))\\
\xrightarrow{\mathrm{ev}_{c,F(Gc\triangleright m)}} F(Gc \triangleright m)\,.
\end{aligned}
\tag{3.25}
$$

Proposition 3.12. *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between left GV-module categories $\mathcal{M}$ and $\mathcal{N}$.*

1. *A collection of morphisms $f^{\triangleright}_{c,m}\colon c \triangleright F(m) \longrightarrow F(c \triangleright m)$, for $c \in \mathcal{C}$ and $m \in \mathcal{M}$, defines the structure of a lax $\triangleright$-module functor on F if and only if the conjugated collection $f^{\blacktriangleright}_{c,m}$ as given in (3.22) defines the structure of an oplax $\blacktriangleright$-module functor on F.*
   We denote F with the latter oplax structure by $F^{\triangleright\mapsto\blacktriangleright}$.
2. *A collection of morphisms $g^{\blacktriangleright}_{c,m}\colon F(c \blacktriangleright m) \longrightarrow c \blacktriangleright F(m)$ defines the structure of an oplax $\blacktriangleright$-module functor on F if and only if the conjugated collection $g^{\triangleright}_{c,m}$ as given in (3.25) defines the structure of a lax $\triangleright$-module functor on F.*
   We denote F with the latter oplax structure by $F^{\blacktriangleright\mapsto\triangleright}$.

3. *The prescriptions* (3.22) *and* (3.25) *are mutually inverse, in the sense that*

$$((F, f^{\triangleright})^{\triangleright\mapsto\blacktriangleright})^{\blacktriangleright\mapsto\triangleright} = (F, f^{\triangleright}) \qquad \textit{and} \qquad ((F, g^{\blacktriangleright})^{\blacktriangleright\mapsto\triangleright})^{\triangleright\mapsto\blacktriangleright} = (F, g^{\blacktriangleright})\,. \tag{3.26}$$

Proof. Suppose that $f^{\triangleright}_{c,m} \colon c \triangleright F(m) \to F(c \triangleright m)$ is a lax $\triangleright$-module functor structure on F. Let $f^{\blacktriangleright}$ be the corresponding structure under the adjunction given in Equation (3.24). For objects $x, y \in \mathcal{C}$, the morphism

$$G(x \otimes y) \triangleright F((x \otimes y) \blacktriangleright m) \xrightarrow{f^{\triangleright}} F(G(x \otimes y) \triangleright ((x \otimes y) \blacktriangleright m)) \xrightarrow{F(\mathrm{ev})} F(m) \tag{3.27}$$

corresponds under the adjunction (3.23) to the morphism $F((x \otimes y) \blacktriangleright m) \xrightarrow{f^{\blacktriangleright}_{x \otimes y, m}} (x \otimes y) \blacktriangleright F(m)$, while the composite

$$\begin{aligned} G(x \otimes y) \triangleright F((x \otimes y) \blacktriangleright m) &\xrightarrow{\cong} G(y) \triangleright (G(x) \triangleright F(x \blacktriangleright (y \blacktriangleright m))) \\ &\xrightarrow{Gy \triangleright f^{\triangleright}} G(y) \triangleright F(Gx \triangleright (x \blacktriangleright (y \blacktriangleright m))) \\ &\xrightarrow{Gy \triangleright F(\mathrm{ev})} G(y) \triangleright F(y \blacktriangleright m) \\ &\xrightarrow{f^{\triangleright}} F(G(y) \triangleright (y \blacktriangleright m)) \xrightarrow{F(\mathrm{ev})} F(m) \end{aligned} \tag{3.28}$$

corresponds under the adjunction to the composite

$$\begin{aligned} F((x \otimes y) \blacktriangleright m) \xrightarrow{\cong} F(x \blacktriangleright (y \blacktriangleright m)) \xrightarrow{f^{\blacktriangleright}} x \blacktriangleright F(y \blacktriangleright m) &\xrightarrow{x \blacktriangleright f^{\blacktriangleright}} x \blacktriangleright (y \blacktriangleright F(m)) \\ &\xrightarrow{\cong} (x \otimes y) \blacktriangleright F(m)\,. \end{aligned} \tag{3.29}$$

Using that $f^{\triangleright}$ is coherent and that the diagram (3.19) commutes, it follows that the morphisms (3.27) and (3.28) are equal, and thus that $f^{\blacktriangleright}$ satisfies the pentagon diagram that is required for the structure of an oplax $\blacktriangleright$-module functor. For the corresponding triangle diagram, note that for $c = K$ we can choose as the adjunction data $\mathrm{ev}_{K,m}$ and $\mathrm{coev}_{K,m}$ in (3.16) a combination of the unitors. When doing so, then clearly the triangle identity for $f^{\triangleright}$ implies the triangle identity for $f^{\blacktriangleright}$. It follows that $f^{\blacktriangleright}$ defines an oplax $\blacktriangleright$-module functor structure on F. Starting with an oplax $\blacktriangleright$-module functor structure $g^{\blacktriangleright}$, one sees analogously that (3.25) defines a lax $\triangleright$-module functor structure on F. Using the triangle identities (3.17) and (3.18) we conclude that the two constructions are mutually inverse. □

According to this proposition we can equip every lax $\triangleright$-module functor with a corresponding conjugated oplax $\blacktriangleright$-module functor structure. This assignment is functorial:

Theorem 3.13. *Let $\mathcal{C}$ be a GV-category. The assignments in Proposition 3.12 are functorial in the following sense:*

1. *Let $\eta\colon F \Longrightarrow H$ be a natural transformation between functors $F, H\colon \mathcal{M} \longrightarrow \mathcal{N}$. Given lax $\triangleright$-module functor structures on F and H, η is a $\triangleright$-module natural transformation if and only if it is an oplax $\blacktriangleright$-module natural transformation for the conjugated oplax $\blacktriangleright$-structures on F and H.*
2. *Let $J\colon \mathcal{N} \longrightarrow \mathcal{O}$ be a functor to another left GV-module category $\mathcal{O}$. Given lax $\triangleright$-module functor structures on F and J, the composite of the conjugated oplax $\blacktriangleright$-module functor structures of F and J equals the conjugated oplax $\blacktriangleright$-module functor structure of $J \circ F$, i.e.*

$$J^{\triangleright\mapsto\blacktriangleright} \circ F^{\triangleright\mapsto\blacktriangleright} = (J \circ F)^{\triangleright\mapsto\blacktriangleright}. \tag{3.30}$$

The correspondence between lax $\triangleright$-module functors and oplax $\blacktriangleright$-module functors extends to an equivalence

$$\text{GV-Mod}^{\triangleright,\text{lax}}(\mathcal{C}) \xrightarrow{\simeq} \text{GV-Mod}^{\blacktriangleright,\text{oplax}}(\mathcal{C}) \tag{3.31}$$

of bicategories which is the identity on objects and on 2-morphisms as well as the identity on the functors that underly the module functors.

Proof. Assume that $\eta\colon F \Longrightarrow H$ is a $\triangleright$-module natural transformation between lax $\triangleright$-module functors $(F, f^{\triangleright})$ and $(H, h^{\triangleright})$. Then the diagram

$$\begin{tikzcd}
Gx \triangleright F(x \blacktriangleright m) \arrow[rr] \arrow[dr, "f^{\triangleright}_{Gx,x\blacktriangleright m}"'] \arrow[ddd, "Gx \triangleright \eta_{x\blacktriangleright m}"'] & & F(m) \arrow[ddd, "\eta_m"] \\
& F(Gx \triangleright (x \blacktriangleright m)) \arrow[ur, "F(\mathrm{ev}_{x,m})"'] \arrow[d, "\eta_{Gx\triangleright(x\blacktriangleright m)}"] & \\
& H(Gx \triangleright (x \blacktriangleright m)) \arrow[dr, "H(\mathrm{ev}_{x,m})"] & \\
Gx \triangleright H(x \blacktriangleright m) \arrow[rr] \arrow[ur, "h^{\triangleright}_{Gx,x\blacktriangleright m}"] & & H(m)
\end{tikzcd} \tag{3.32}$$

commutes: The triangles define the horizontal arrows; the left quadrilateral commutes because η is a $\triangleright$-module natural transformation; and the right quadrilateral commutes by naturality. By (3.23), the resulting commutativity of the outer square shows that η is a $\blacktriangleright$-module natural transformation for the oplax $\blacktriangleright$-module functor structures on F and H.

To prove the second assertion, consider the oplax $\blacktriangleright$-module functor structure on $J^{\triangleright\mapsto\blacktriangleright} \circ F^{\triangleright\mapsto\blacktriangleright}$. If the composite (3.22) (for F and for J, respectively) is inserted for the two arrows in $JF(x \blacktriangleright m) \longrightarrow J(x \blacktriangleright F(m)) \longrightarrow x \blacktriangleright JF(m)$ and the triangle identity (3.17) is used in the middle, we arrive at the $\blacktriangleright$-module functor structure of $(JF)^{\triangleright\mapsto\blacktriangleright}$.

Thus the two structures coincide. The equivalence of bicategories follows now directly from these two statements and Proposition 3.12. □

Remark 3.14. The two bicategories in Theorem 3.13 are distinguished by the existence of this correspondence: There is no analogous biequivalence between $\text{GV-Mod}^{\triangleright,\text{oplax}}(\mathcal{C})$ and $\text{GV-Mod}^{\blacktriangleright,\text{lax}}(\mathcal{C})$. This can be traced back to the fact that, generically, the functors $c \triangleright -$ admit only right adjoints.

We can give a symmetric characterization of the 1-morphisms in the bicategories of Theorem 3.13:

Definition 3.15. Let $\mathcal{M}$ and $\mathcal{N}$ be left GV-module categories over a GV-category $\mathcal{C}$.

1. A *GV-module functor* $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ is a functor with a conjugate pair of lax $\triangleright$- and oplax $\blacktriangleright$-module functor structures.
2. For two GV-module functors $F, H\colon \mathcal{M} \longrightarrow \mathcal{N}$, a *GV*-module natural transformation $\eta\colon F \Rightarrow H$ is a natural transformation that is equivalently a lax $\triangleright$-module natural transformation or an oplax $\blacktriangleright$-module natural transformation.

Clearly there is a bicategory $\text{GV-Mod}_{\mathcal{C}}$ of GV-module categories, GV-module functors and GV-module natural transformations, which is equivalent to the two bicategories in Theorem 3.13.

Now recall the counit and unit (3.16) of the adjunction (3.15). For later use we record compatibilities between the two weak module structures for a GV-functor:

Lemma 3.16. *Let $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a GV-functor between left $\mathcal{C}$-GV-module categories. The diagrams*

$$\begin{tikzcd} F(m) \arrow[r, "\mathrm{coev}_{c,F(m)}"] \arrow[d, "F(\mathrm{coev}_{c,m})"'] & c \blacktriangleright (Gc \triangleright F(m)) \arrow[d, "c \blacktriangleright f^{\triangleright}_{Gc,m}"] \\ F(c \blacktriangleright (Gc \triangleright m)) \arrow[r, "f^{\blacktriangleright}_{c,Gc\triangleright m}"] & c \blacktriangleright F(Gc \triangleright m) \end{tikzcd} \tag{3.33}$$

and

$$\begin{tikzcd} Gc \triangleright F(c \blacktriangleright m) \arrow[r, "Gc \triangleright f^{\blacktriangleright}_{c,m}"] \arrow[d, "f^{\triangleright}_{Gc,c\blacktriangleright m}"'] & Gc \triangleright (c \blacktriangleright F(m)) \arrow[d, "\mathrm{ev}_{c,Fm}"] \\ F(Gc \triangleright (c \blacktriangleright m)) \arrow[r, "F(\mathrm{ev}_{c,m})"] & F(m) \end{tikzcd} \tag{3.34}$$

commute for all $m \in \mathcal{M}$ and $c \in \mathcal{C}$.

Proof. Consider in (3.33) the composite $f^{\blacktriangleright}_{c,Gc\triangleright m} \circ F(\mathrm{coev}_{c,m})\colon F(m) \longrightarrow c \blacktriangleright F(Gc \triangleright m)$. Using the definition (3.22) of $f^{\blacktriangleright}$ in terms of $f^{\triangleright}$ and the naturality of the coevaluation it follows that this morphism is equal to the composite

$$\begin{aligned} F(m) \xrightarrow{\;\mathrm{coev}_{c,Fm}\;} c \blacktriangleright (Gc \triangleright Fm) &\xrightarrow{\;c\,\blacktriangleright\, f^{\triangleright}_{Gc,m}\;} c \blacktriangleright F(Gc \triangleright m) \\ &\xrightarrow{\;c\,\blacktriangleright\, F(Gc\,\triangleright\,\mathrm{coev}_{c,m})\;} c \blacktriangleright F(Gc \triangleright (c \blacktriangleright (Gc \triangleright m))) \\ &\xrightarrow{\;c\,\blacktriangleright\, F(\mathrm{ev}_{c,Gc\triangleright m})\;} c \blacktriangleright F(Gc \triangleright m)\,. \end{aligned} \tag{3.35}$$

By the triangle identity (3.18) this is, in turn, equal to the composite of the other two arrows in (3.33). The commutativity of the diagram (3.34) is seen analogously. □

Next we note that, given a GV-module $\mathcal{M}$ over a GV-category $\mathcal{C}$, for every object $m \in \mathcal{M}$ we have a strong $\triangleright$-module functor

$$\mathrm{R}^{\triangleright}_m : \quad {}^{\otimes}_{\mathcal{C}}\mathcal{C} \equiv {}_{\mathcal{C}}\mathcal{C} \longrightarrow {}_{\mathcal{C}}\mathcal{M}\,, \qquad c \longmapsto c \triangleright m \tag{3.36}$$

and a strong $\blacktriangleright$-module functor

$$\mathrm{R}^{\blacktriangleright}_m : \quad {}^{\circledast}_{\mathcal{C}}\mathcal{C} \longrightarrow {}_{\mathcal{C}}\mathcal{M}\,, \qquad c \longmapsto c \blacktriangleright m\,. \tag{3.37}$$

We are now in a position to introduce distributors for general GV-module categories. Combining the strong module functors $\mathrm{R}^{\triangleright}_m$ and $\mathrm{R}^{\blacktriangleright}_m$ defined in (3.36) and (3.37) we get

Definition 3.17. Let $\mathcal{M}$ be a left GV-module category over $\mathcal{C}$. The oplax $\blacktriangleright$-module structure of the functor $\mathrm{R}^{\triangleright}_m$ defines the *right module distributor*

$$\delta^{\mathrm{r}}_{x,y,m} : \quad (x \circledast y) \triangleright m \longrightarrow x \blacktriangleright (y \triangleright m)\,, \tag{3.38}$$

and the lax $\triangleright$-module structure of $\mathrm{R}^{\blacktriangleright}_m$ defines the *left module distributor*

$$\delta^{\mathrm{l}}_{x,y,m} : \quad x \triangleright (y \blacktriangleright m) \longrightarrow (x \otimes y) \blacktriangleright m\,, \tag{3.39}$$

for $x, y \in \mathcal{C}$ and $m \in \mathcal{M}$.

Module distributors for a right GV-module category are defined analogously. We will examine the module distributors in detail in Section 3.4.

With the help of the strong module functor $\mathrm{R}^{\blacktriangleright}_m$ we can show that the two weak module functor structures for a GV-module functor are compatible in the following sense:

Proposition 3.18. *Let $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a GV-module functor. The weak module functor structures $f^{\triangleright}$ and $f^{\blacktriangleright}$ of F obey four pentagon relations: Commutativity of the ordinary pentagon diagrams*

$$x \triangleright (y \triangleright F(m)) \mathrel{\substack{\longrightarrow\longrightarrow \\ \to\to\to}} F(x \triangleright (y \triangleright m)) \tag{3.40}$$

and

$$F(x \blacktriangleright (y \blacktriangleright m)) \mathrel{\substack{\longrightarrow\longrightarrow \\ \to\to\to}} x \blacktriangleright (y \blacktriangleright F(m)) \tag{3.41}$$

for $x, y \in \mathcal{C}$ *and* $m \in \mathcal{M}$*, each of which involves just one of the two actions, and commutativity of*

$$\begin{tikzcd}
x \triangleright F(y \blacktriangleright m) \arrow[r] \arrow[d] & F(x \triangleright (y \blacktriangleright m)) \arrow[r] & F((x \otimes y) \blacktriangleright m) \arrow[d] \\
x \triangleright (y \blacktriangleright F(m)) \arrow[rr] & & (x \otimes y) \blacktriangleright F(m)
\end{tikzcd} \tag{3.42}$$

and

$$\begin{tikzcd}
(x \circledast y) \triangleright F(m) \arrow[r] \arrow[d] & F((x \circledast y) \triangleright m) \arrow[r] & F(x \blacktriangleright (y \triangleright m)) \arrow[d] \\
x \blacktriangleright (y \triangleright F(m)) \arrow[rr] & & x \blacktriangleright F(y \triangleright m)
\end{tikzcd} \tag{3.43}$$

which involve both of them.

Proof. The two diagrams (3.40) and (3.41) are just the coherence diagrams for the individual weak module functors. For the third diagram consider, for $m \in \mathcal{M}$, the composite ${}_{\mathcal{C}}\mathcal{C} \xrightarrow{\mathrm{R}^{\blacktriangleright}_m} {}_{\mathcal{C}}\mathcal{M} \xrightarrow{F} {}_{\mathcal{C}}\mathcal{N}$ of module functors, which maps $x \in \mathcal{C}$ to $F(x \blacktriangleright m)$. On the other hand, the module functor $R^{\blacktriangleright}_{F(m)} \colon {}_{\mathcal{C}}\mathcal{C} \longrightarrow {}_{\mathcal{C}}\mathcal{N}$ maps c to $c \blacktriangleright F(m)$. The oplax $\blacktriangleright$-module structure of F provides a collection of morphisms $f^{\blacktriangleright}_{c,m} \colon F \circ R^{\blacktriangleright}_m(c) = F(c \blacktriangleright m) \longrightarrow c \blacktriangleright F(m) = R^{\blacktriangleright}_{F(m)}(c)$. These morphisms are coherent with respect to the $\blacktriangleright$-module structure, which is equivalent to the statement that $f^{\blacktriangleright}_{-,m} \colon F \circ \mathrm{R}^{\blacktriangleright}_m \longrightarrow \mathrm{R}^{\blacktriangleright}_{F(m)}$ is a $\blacktriangleright$-module natural transformation. By Theorem 3.13.1, $f^{\blacktriangleright}_{-,m}$ is a also a $\triangleright$-module natural transformation. The corresponding coherence diagram is precisely the diagram (3.42). Analogously, the lax $\triangleright$-structure $f^{\triangleright}_{x,m}$ of F provides an oplax $\blacktriangleright$-module natural transformation $\mathrm{R}^{\triangleright}_{F(m)} \longrightarrow F \circ R^{\triangleright}_m$, thus proving the commutativity of the diagram (3.43). □

Results analogous to those above hold for $\mathcal{C}$-right module categories and functors between them. Indeed, a right module category $\mathcal{M}$ can be seen as a left $\mathcal{C}^{\otimes\mathrm{opp}}$-module category, where $\mathcal{C}^{\otimes\mathrm{opp}}$ is the category $\mathcal{C}$ with reversed monoidal structure. Consider finally the case of a GV-bimodule category ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$: The categories $\mathcal{C}$ and $\mathcal{D}$ are GV-categories and $\mathcal{M}$ is a bimodule category that is a left $\mathcal{C}$- and right $\mathcal{D}$-GV-module category. Equivalently, $\mathcal{M}$ is a left $\mathcal{C} \times \mathcal{D}^{\otimes\mathrm{opp}}$-module category, where $\mathcal{D}^{\otimes\mathrm{opp}}$ has the opposite monoidal product. It is easily seen that $\mathcal{D}^{\otimes\mathrm{opp}}$ as well as $\mathcal{C} \times \mathcal{D}^{\otimes\mathrm{opp}}$ are GV-categories and that

$\mathcal{M}$ is a left $\mathcal{C}\times\mathcal{D}^{\otimes\mathrm{opp}}$-GV-module category. It therefore follows from Proposition 3.5 that $\mathcal{M}$ has the structure of a $(\mathcal{C},\circledast)$-$(\mathcal{D},\circledast)$-bimodule category. In particular we thus obtain

Lemma 3.19. *Let $F\colon \mathcal{M}\to\mathcal{N}$ be a functor between $(\mathcal{C},\mathcal{D})$-bimodule categories. For each lax $\triangleright$-bimodule functor structure on F there is a conjugated oplax $\blacktriangleright$-bimodule functor structure, and vice versa, such that analogous statements to those in Proposition 3.12 and Theorem 3.13 are valid.*

3.3. Adjoints of GV-module functors

We now discuss aspects of adjoints of GV-module functors. The structures we find will be used in the next subsection to obtain snake relations for GV-duality. In general the adjoint of a GV-module functor is not a GV-module functor. However, there are interesting transports of (weak) module functor structures to the adjoints. The first of these, which we call flipping transport is as follows:

Lemma 3.20 *(Flipping transport). Let $F\colon \mathcal{M}\longrightarrow\mathcal{N}$ be a left (op)lax $\triangleright$-module functor between module categories over $(\mathcal{C},\otimes)$ which admits a right adjoint $F^{\mathrm{r.a.}}\colon \mathcal{N}\longrightarrow\mathcal{M}$. Then $F^{\mathrm{r.a.}}$ is canonically a left (op)lax $\blacktriangleright$-module functor with respect to the left exact action $(\mathcal{C},\blacktriangleright)$. Analogously, if F is an (op)lax $\blacktriangleright$-module functor admitting a left adjoint $F^{\mathrm{l.a.}}$, then $F^{\mathrm{l.a.}}$ is an (op)lax $\triangleright$-module functor. Moreover, the $\triangleright$-module functor structure of F is strong if and only if the $\blacktriangleright$-module functor structure of $F^{\mathrm{r.a.}}$ is strong.*

Proof. Given a functor $F\colon \mathcal{M}\longrightarrow\mathcal{N}$ with right adjoint $F^{\mathrm{r.a.}}$, for $m,n\in\mathcal{M}$ and $c\in\mathcal{C}$ we have the composite isomorphisms

$$\mathrm{Hom}_{\mathcal{N}}(c\triangleright F(m),n) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{N}}(F(m),G^{-1}c\blacktriangleright n) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{M}}(m,F^{\mathrm{r.a.}}(G^{-1}c\blacktriangleright n)) \tag{3.44}$$

and

$$\mathrm{Hom}_{\mathcal{N}}(F(c\triangleright m),n) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{M}}(c\triangleright m,F^{\mathrm{r.a.}}(n)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{M}}(m,G^{-1}c\blacktriangleright F^{\mathrm{r.a.}}(n))\,. \tag{3.45}$$

Using them as horizontal arrows in the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{N}}(F(c\triangleright m),n) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(m,G^{-1}(c)\blacktriangleright F^{\mathrm{r.a.}}(n)) \arrow[d] \\ \mathrm{Hom}_{\mathcal{N}}(c\triangleright F(m),n) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{M}}(m,F^{\mathrm{r.a.}}(G^{-1}(c)\blacktriangleright n)) \end{tikzcd} \tag{3.46}$$

any one of the vertical arrows defines the other one. Moreover, the Yoneda lemma provides the corresponding structure on F or on its adjoint: In case that F is a lax $\triangleright$-module functor, the module constraint $f_{c,m}$ provides the left vertical arrow, so that the right vertical arrow gives a natural morphism $c\blacktriangleright F^{\mathrm{r.a.}}(n)\longrightarrow F^{\mathrm{r.a.}}(c\blacktriangleright n)$. By the injectivity of the Yoneda embedding, this satisfies its required coherence conditions because $f_{c,m}$ does. The oplax case is analogous, with the vertical arrows now pointing upwards. The last statement follows from the fact that in the commuting diagram (3.46) any one vertical arrow is an isomorphism if and only if the other one is an isomorphism. □

The second transport mechanism, to be called profunctor-transport, is obtained with the help of $\mathcal{C}$-module profunctors; it does not require the setting of GV-categories. Following [19, Def. 2.1] we give

Definition 3.21 *(Profunctor-transport).* Let $\mathcal{M}$ and $\mathcal{N}$ be left module categories over a linear monoidal category $\mathcal{C}$. A *$\mathcal{C}$-module profunctor* from $\mathcal{M}$ to $\mathcal{N}$ is a bilinear functor $H\colon \mathcal{M}^{\rm opp} \times \mathcal{N} \longrightarrow \mathrm{vect}$ together with a family

$$\theta_{m,n,c}: \quad H(m,n) \longrightarrow H(c \triangleright m, c \triangleright n) \tag{3.47}$$

of morphisms that is natural in $m \in \mathcal{M}$ and in $n \in \mathcal{N}$, is dinatural in $c \in \mathcal{C}$, and is coherent with respect to the monoidal structure and the monoidal unit 1 of $\mathcal{C}$, i.e. (suppressing associators and unitors) satisfies

$$\theta_{m,n,c\otimes d} = \theta_{d\triangleright m, d\triangleright n, c} \circ \theta_{m,n,d} \qquad \text{and} \qquad \theta_{m,n,1} = \mathrm{id}_{H(m,n)} \tag{3.48}$$

for all $m \in \mathcal{M}$, $n \in \mathcal{N}$ and $c, d \in \mathcal{C}$.

Given two $\mathcal{C}$-module profunctors H_1 and H_2 from $\mathcal{M}$ to $\mathcal{N}$, a *morphism* $\varphi\colon H_1 \longrightarrow H_2$ *of module profunctors* is a natural transformation that commutes with the respective (di)natural transformations.

Our main use of this notion is

Proposition 3.22. *[19, Lemma 2.3] Let $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ and $J\colon \mathcal{N} \longrightarrow \mathcal{M}$ be linear functors between $\mathcal{C}$-module categories. There are canonical bijections*

1. *between oplax module functor structures on F and $\mathcal{C}$-module profunctor structures on*

$$\mathrm{Hom}_{\mathcal{N}}(F(-),-): \quad \mathcal{M}^{\rm opp} \times \mathcal{N} \longrightarrow \mathrm{vect}\,; \tag{3.49}$$

2. *between lax module functor structures on J and $\mathcal{C}$-module profunctor structures on*

$$\mathrm{Hom}_{\mathcal{M}}(-,J(-)): \quad \mathcal{M}^{\rm opp} \times \mathcal{N} \longrightarrow \mathrm{vect}\,. \tag{3.50}$$

Proof. The proof is constructive: If $f_{c,m}\colon F(c \triangleright m) \longrightarrow c \triangleright F(m)$ is the oplax module structure on F, the $\mathcal{C}$-module profunctor structure on $\mathrm{Hom}_{\mathcal{N}}(F(-),-)$ is the composite

$$\theta_{m,n,c}: \quad \mathrm{Hom}(F(m),n) \xrightarrow{c\triangleright -} \mathrm{Hom}(c\triangleright F(m), c\triangleright n) \xrightarrow{f^*_{c,m}} \mathrm{Hom}(F(c\triangleright m), c\triangleright n)\,. \tag{3.51}$$

Conversely, given θ, the oplax module structure is the image of the identity under the map

$$\theta_{m,Fm,c}: \quad \mathrm{Hom}(Fm,Fm) \longrightarrow \mathrm{Hom}(F(c\triangleright m), c\triangleright F(m))\,. \tag{3.52}$$

These two constructions are clearly inverse, and it is straightforward to check that the respective coherence and naturality diagrams correspond to each other. The second bijection follows from the first by considering opposite categories. □

As an immediate consequence we obtain the profunctor-transport of weak module functor structures:

Corollary 3.23. *Let $F\colon \mathcal{M}\longrightarrow\mathcal{N}$ be a linear functor with right adjoint $F^{\mathrm{r.a.}}\colon \mathcal{N}\longrightarrow\mathcal{M}$. There is a canonical bijection between oplax $\mathcal{C}$-module functor structures on F and lax $\mathcal{C}$-module structures on $F^{\mathrm{r.a.}}$, such that the adjunction*

$$\varphi_{m,n}: \quad \mathrm{Hom}_{\mathcal{N}}(F(m),n) \longrightarrow \mathrm{Hom}_{\mathcal{M}}(m, F^{\mathrm{r.a.}}(n)) \tag{3.53}$$

is an isomorphism of $\mathcal{C}$-module profunctors.

In this situation, it is natural to expect a compatibility between the unit and counit of the adjunction and the module action. Note, however, that in general the weak module structures of F and $F^{\mathrm{r.a.}}$ go in opposite directions, so that it does not make sense to require the unit and counit to be module natural transformations. Instead, they have the following coherence properties with respect to the module action:

Lemma 3.24. *Let $F\colon \mathcal{M}\longrightarrow\mathcal{N}$ be an oplax module functor having a right adjoint, and let $\eta\colon \mathrm{id}\Longrightarrow F^{\mathrm{r.a.}}F$ and $\varepsilon\colon FF^{\mathrm{r.a.}}\Longrightarrow\mathrm{id}$ be the unit and counit of the adjunction (3.53). Then the diagrams*

$$\begin{tikzcd}
c\triangleright m \arrow[r, "\eta_{c\triangleright m}"] \arrow[dd, "="'] & F^{\mathrm{r.a.}}F(c\triangleright m) \arrow[d] \\
& F^{\mathrm{r.a.}}(c\triangleright F(m)) \\
c\triangleright m \arrow[r, "c\,\triangleright\,\eta_m"] & c\triangleright F^{\mathrm{r.a.}}F(m) \arrow[u]
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
c\triangleright FF^{\mathrm{r.a.}}(n) \arrow[r, "c\,\triangleright\,\varepsilon_n"] & c\triangleright n \arrow[dd, "="] \\
F(c\triangleright F^{\mathrm{r.a.}}(n)) \arrow[u] \arrow[d] & \\
F(F^{\mathrm{r.a.}}(c\triangleright n)) \arrow[r, "\varepsilon_{c\triangleright n}"] & c\triangleright n
\end{tikzcd} \tag{3.54}$$

commute for all $m\in\mathcal{M}$, $n\in\mathcal{N}$ and $c\in\mathcal{C}$, using the oplax and lax module functor structures of F and $F^{\mathrm{r.a.}}$, respectively.

Proof. Denote the lax module functor structure of F by f and the oplax structure of $F^{\text{r.a.}}$ by g. Since φ is an isomorphism of $\mathcal{C}$-module profunctors, the diagram

$$\begin{tikzcd}
\mathrm{Hom}(Fm,n) \arrow[r, "c\,\triangleright"] \arrow[d, "\varphi_{m,n}"] & \mathrm{Hom}(c\triangleright F(m), c\triangleright n) \arrow[r, "f^*"] & \mathrm{Hom}(F(c\triangleright m), c\triangleright n) \arrow[d, "\varphi_{c\triangleright m, c\triangleright n}"] \\
\mathrm{Hom}(m, F^{\text{r.a.}}(n)) \arrow[r, "c\,\triangleright"] & \mathrm{Hom}(c\triangleright m, c\triangleright F^{\text{r.a.}}(n)) \arrow[r, "g_*"] & \mathrm{Hom}(c\triangleright m, F^{\text{r.a.}}(c\triangleright n))
\end{tikzcd} \tag{3.55}$$

commutes for all m, n. Evaluating (3.55) at $n = F(m)$ gives the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(Fm,F(m)) \arrow[r, "c\,\triangleright"] \arrow[d, "\varphi_{m,F(m)}"] & \mathrm{Hom}(c\triangleright F(m), c\triangleright F(m)) \arrow[r, "f^*"] & \mathrm{Hom}(F(c\triangleright m), c\triangleright F(m)) \arrow[d, "\varphi_{c\triangleright m, c\triangleright F(m)}"] \\
\mathrm{Hom}(m, F^{\text{r.a.}}(F(m))) \arrow[r, "c\,\triangleright"] & \mathrm{Hom}(c\triangleright m, c\triangleright F^{\text{r.a.}}(F(m))) \arrow[r, "g_*"] & \mathrm{Hom}(c\triangleright m, F^{\text{r.a.}}(c\triangleright F(m)))
\end{tikzcd} \tag{3.56}$$

Taking the identity morphism in the upper left corner, this yields the left diagram in (3.54). The other diagram in (3.54) follows by duality. □

Profunctor transport is compatible with composition:

Lemma 3.25. *Let $F_1 \colon \mathcal{M} \longrightarrow \mathcal{N}$ and $F_2 \colon \mathcal{N} \longrightarrow \mathcal{O}$ be oplax $\mathcal{C}$-module functors between left $\mathcal{C}$-module categories. The canonical natural isomorphism $(F_2F_1)^{\text{r.a.}} \cong F_1^{\text{r.a.}} F_2^{\text{r.a.}}$ is an isomorphism of lax module functors.*

Proof. The claim follows from the definition of the transport: all morphisms in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}}(F_2F_1m, o) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathcal{M}}(m, (F_2F_1)^{\text{r.a.}}o) \arrow[d, "\varphi"] \\
\mathrm{Hom}_{\mathcal{N}}(F_1m, F_2^{\text{r.a.}}o) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{M}}(m, F_1^{\text{r.a.}}F_2^{\text{r.a.}}o),
\end{tikzcd} \tag{3.57}$$

are isomorphisms, and all of them except for possibly the one labeled φ are isomorphisms of $\mathcal{C}$-module profunctors. But since the diagram commutes, φ is in fact an isomorphism of $\mathcal{C}$-module profunctors, too. By the Yoneda Lemma, φ provides the canonical isomorphism $(F_2F_1)^{\text{r.a.}} \cong F_1^{\text{r.a.}} F_2^{\text{r.a.}}$, which is thus an isomorphism of lax module functors. □

We can summarize the constructions given above as follows. Let $\mathcal{C}$ be a GV-category, let $\mathcal{M}$ and $\mathcal{N}$ be left GV-module categories over $\mathcal{C}$, and let $F \colon \mathcal{M} \longrightarrow \mathcal{N}$ be a functor. The transport constructions are the rows in the following table:

(T1)	profunctor-transport	F oplax-$\triangleright$	$F^{\mathrm{r.a.}}$ lax-$\triangleright$
(T2)	profunctor-transport	F oplax-$\blacktriangleright$	$F^{\mathrm{r.a.}}$ lax-$\blacktriangleright$
(T3)	flipping transport	F oplax-$\triangleright$	$F^{\mathrm{r.a.}}$ oplax-$\blacktriangleright$
(T4)	flipping transport	F lax-$\triangleright$	$F^{\mathrm{r.a.}}$ lax-$\blacktriangleright$
(T5)	flipping transport	F strong-$\triangleright$	$F^{\mathrm{r.a.}}$ strong-$\blacktriangleright$

(3.58)

Each of these transport constructions can be applied from left to right as well as from right to left, i.e. if F has a left adjoint, then $F^{\mathrm{l.a.}}$ acquires transported structure from F. In parallel with Proposition 3.12, for a functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ this provides a pairing between lax $\triangleright$- and oplax $\blacktriangleright$-module structures:

Corollary 3.26. *Let $\mathcal{M}$ and $\mathcal{N}$ be left-module categories over a GV-category $\mathcal{C}$, and let $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a functor admitting a right adjoint. Then the transports via the right adjoint provide a bijection between lax $\triangleright$-module functor structures and oplax $\blacktriangleright$-module functor structures on F. In case that F admits a left adjoint, the transports via the left adjoint provide a (potentially different) bijection between lax $\triangleright$-module functor structures and oplax $\blacktriangleright$-module functor structures on F.*

Proof. Suppose that F is a lax $\triangleright$-module functor with a right adjoint $F^{\mathrm{r.a.}}$. Then by the transport (T4), $F^{\mathrm{r.a.}}$ is a lax $\blacktriangleright$-module functor, while by (T2) F acquires an oplax $\blacktriangleright$-module functor structure. Since both steps are bijections of structures, the structures are in bijection.

If F has a left adjoint, then the transports (T1) and (T3) provide again a bijection between the same structures: If F is a lax $\triangleright$-module functor, then by (T1) $F^{\mathrm{l.a.}}$ is a oplax $\triangleright$-module functor, and by (T3) F is an oplax $\blacktriangleright$-module functor. $\square$

In case that a lax $\triangleright$-module functor F admits both a right and a left adjoint, the transported structures coincide:

Proposition 3.27. *Let $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a lax $\triangleright$-module functor admitting a right adjoint. Then the oplax $\blacktriangleright$-module functor structure on F given in Proposition 3.12 coincides with the transported oplax $\blacktriangleright$-module functor structure given in Corollary 3.26. Analogously, if F admits a left adjoint, then both oplax $\blacktriangleright$-module functor structures coincide. In particular, if F has both a right and a left adjoint, then both transported module structures from Corollary 3.26 are the same.*

Proof. Assume that F is a lax $\triangleright$-module functor and has a right adjoint $F^{\mathrm{r.a.}}$. Under the adjunction $\mathrm{Hom}_{\mathcal{M}}(F(c \blacktriangleright m), c \blacktriangleright F(m)) \cong \mathrm{Hom}(Gc \triangleright F(c \blacktriangleright m), F(m))$, the conjugated oplax $\blacktriangleright$-module functor structure $f^{\blacktriangleright}$ of F is the composite

$$Gc \triangleright F(c \blacktriangleright m) \xrightarrow{f^{\triangleright}_{Gc,c \blacktriangleright m}} F(Gc \triangleright (c \blacktriangleright m)) \xrightarrow{F(\mathrm{ev}_{c,m})} F(m)\,. \tag{3.59}$$

On the other hand, the transported module functor structure $\widetilde{f}^{\blacktriangleright}$ is the image of the identity under the $(\mathcal{C},\otimes)$-profunctor structure $\mathrm{Hom}_{\mathcal{M}}(Fm,Fm) \xrightarrow{\theta_{m,Fm,c}} \mathrm{Hom}(F(c\blacktriangleright m), c\blacktriangleright F(m))$ that is obtained by the diagram

$$\begin{tikzcd}
\mathrm{Hom}(Fm,n) \arrow[rr, "\theta_{m,n,c}"] \arrow[d] & & \mathrm{Hom}(F(c\blacktriangleright m), c\blacktriangleright n) \\
\mathrm{Hom}(m,F^{\mathrm{r.a.}}(n)) \arrow[d, "c\blacktriangleright -"] & & \mathrm{Hom}(Gc\triangleright F(c\blacktriangleright m), n) \arrow[u] \\
\mathrm{Hom}(c\blacktriangleright m, c\blacktriangleright F^{\mathrm{r.a.}}(n)) \arrow[r] & \mathrm{Hom}(Gc\triangleright(c\blacktriangleright m), F^{\mathrm{r.a.}}(n)) \arrow[r] & \mathrm{Hom}(F(Gc\triangleright(c\blacktriangleright m)), n) \arrow[u, "f^{\triangleright}"']
\end{tikzcd} \tag{3.60}$$

Taking $n = F(m)$ and the identity morphism in the upper left corner yields the mappings

$$\begin{tikzcd}
\mathrm{id}_{Fm} \arrow[d, mapsto] & & \widetilde{f}^{\blacktriangleright}_{c,m} \\
\varepsilon_F(m) \arrow[d, mapsto] & & \mathrm{ev}_{c,m}\circ f^{\triangleright}_{Gc,c\blacktriangleright m} \arrow[u, mapsto] \\
c\blacktriangleright\varepsilon_F(m) \arrow[r, mapsto] & \varepsilon_F(m)\circ\mathrm{ev}_{c,m} \arrow[r, mapsto] & F(\mathrm{ev}_{c,m}) \arrow[u, mapsto]
\end{tikzcd} \tag{3.61}$$

($\varepsilon_F\colon \mathrm{id} \Longrightarrow F^{\mathrm{r.a.}}F$ is the unit of the adjunction). Thus in view of Equation (3.59) the two oplax $\blacktriangleright$-module functor structures coincide. The remaining statements follow by duality. □

3.4. Applications: distributors and duality

Recall the right and left module distributors from Definition 3.17. The fact that the module distributors come from weak module structures implies the commutativity of two pentagon diagrams for the right module distributor and of two pentagons for the left module distributor, while the two coherence diagrams involving both the right and the left module distributors follow from the two mixed pentagons in Proposition 3.18. Altogether we have the following coherences for the module distributors:

Proposition 3.28. *Let $\mathcal{M}$ be a left GV-module category over $\mathcal{C}$. The following six pentagon diagrams commute for all $x,y,z\in\mathcal{C}$ and $m\in\mathcal{M}$:*

1. *The pentagons*

$$((x\otimes y)\otimes z)\triangleright m \;\begin{smallmatrix}\longrightarrow\longrightarrow\\ \to\to\to\end{smallmatrix}\; x\blacktriangleright(y\blacktriangleright(z\triangleright m)) \tag{3.62}$$

and

$$(x \circledast y) \triangleright (z \triangleright m) \mathrel{\substack{\longrightarrow\longrightarrow \\ \longrightarrow\longrightarrow\longrightarrow}} x \blacktriangleright (y \triangleright (z \triangleright m)) \tag{3.63}$$

for the distributor δ^{r}*;*

2. *the pentagons*

$$(x \otimes y) \otimes (z \blacktriangleright m) \mathrel{\substack{\longrightarrow\longrightarrow \\ \longrightarrow\longrightarrow\longrightarrow}} (x \otimes (y \otimes z)) \blacktriangleright m \tag{3.64}$$

and

$$x \triangleright ((y \circledast z) \blacktriangleright m) \mathrel{\substack{\longrightarrow\longrightarrow \\ \longrightarrow\longrightarrow\longrightarrow}} (x \otimes y) \blacktriangleright (z \blacktriangleright m) \tag{3.65}$$

for the distributor δ^{l}*;*

3. *the two mixed pentagons*

$$x \triangleright ((y \circledast z)) \triangleright m) \mathrel{\substack{\longrightarrow\longrightarrow \\ \longrightarrow\longrightarrow\longrightarrow}} (x \otimes y) \blacktriangleright (z \triangleright m) \tag{3.66}$$

and

$$(x \circledast y) \triangleright (z \blacktriangleright m) \mathrel{\substack{\longrightarrow\longrightarrow \\ \longrightarrow\longrightarrow\longrightarrow}} (x \circledast (y \otimes z)) \blacktriangleright m\,. \tag{3.67}$$

Proof. The commutativity of the diagram (3.62) expresses the fact that δ^{r} is an oplax $\triangleright$-module structure on the functor $\mathrm{R}_m^{\triangleright}$; commutativity of (3.63) follows from (3.43) for $F = \mathrm{R}_m^{\triangleright}$. The commutativity of (3.64) expresses the fact that δ^{l} is a lax $\triangleright$-module structure on $\mathrm{R}_m^{\blacktriangleright}$, while commutativity of (3.65) follows from (3.42) for $F = \mathrm{R}_m^{\blacktriangleright}$. Finally, the diagram (3.66) is (3.42) for $F = \mathrm{R}_m^{\triangleright}$, while the diagram (3.67) is (3.43) for $F = \mathrm{R}_m^{\blacktriangleright}$. $\square$

We now specialize to the case of the regular $\mathcal{C}$-module category ${}_{\mathcal{C}}\mathcal{C}$. From the module distributors we then obtain natural isomorphisms

$$\delta^{\mathrm{r}}_{x,y,z} : \quad (x \circledast y) \otimes z \longrightarrow x \circledast (y \otimes z) \qquad \text{and} \qquad \delta^{\mathrm{l}}_{x,y,z} : \quad x \otimes (y \circledast z) \longrightarrow (x \otimes y) \circledast z\,. \tag{3.68}$$

By Proposition 3.28 these morphisms obey all coherence conditions that the left and right distributors in a linearly distributive category (see e.g. [6,18]) have to satisfy. Thus our result is stronger than Proposition 4.11 of [13], in which it was shown that all these conditions except for the two pentagon equations that involve both left and right distributors can be derived from the definition of distributors in terms of weak module functors.

We can equally well treat $\mathcal{C}$ as a right GV-module category $\mathcal{C}_{\mathcal{C}}$. The lax $\triangleright$-module functor structure of $\mathrm{L}_y^{\blacktriangleleft} : \mathcal{C}_{\mathcal{C}} \longrightarrow \mathcal{C}_{\mathcal{C}}$ consists again of morphisms $\widetilde{\delta}^{\mathrm{r}}_{x,y,z} : (x \circledast y) \otimes z \longrightarrow x \circledast (y \otimes z)$, and analogously we obtain morphisms $\widetilde{\delta}^{\mathrm{l}}_{x,y,z} : x \otimes (y \circledast z) \longrightarrow (x \otimes y) \circledast z$. Again it follows that these satisfy all coherence diagrams for distributors. However, they do not constitute new structures:

Lemma 3.29. *For all objects x, y, z in a GV-category $\mathcal{C}$ we have $\widetilde{\delta}^{\mathrm{r}}_{x,y,z} = \delta^{\mathrm{r}}_{x,y,z}$ and $\widetilde{\delta}^{\mathrm{l}}_{x,y,z} = \delta^{\mathrm{l}}_{x,y,z}$.*

Proof. By construction, $\delta^{\mathrm{r}}_{x,y,z}$ is the image of the morphism $\mathrm{ev}_{x,y} \otimes \mathrm{id}_z$ under the adjunction

$$\mathrm{Hom}(G(x) \otimes ((x \circledast y) \otimes z), y \otimes z) \cong \mathrm{Hom}((x \circledast y) \otimes z, x \circledast (y \otimes z)) \,, \tag{3.69}$$

while under the adjunction

$$\mathrm{Hom}((G(x) \otimes (x \circledast y)) \otimes z, y \otimes z) \cong \mathrm{Hom}(G(x) \otimes (x \circledast y), (y \otimes z) \circledast G(z)) \tag{3.70}$$

it is mapped to $\widetilde{\mathrm{coev}}_{y,Gz} \circ \mathrm{ev}_{x,y}$. On the other hand, $\widetilde{\delta}^{\mathrm{r}}_{x,y,z}$ is the image of $\mathrm{id}_x \circledast \widetilde{\mathrm{coev}}_{y,Gz}$ under

$$\mathrm{Hom}(x \circledast y, (x \circledast (y \otimes z)) \circledast G(z)) \cong \mathrm{Hom}((x \circledast y) \otimes z, x \circledast (y \otimes z))) \,, \tag{3.71}$$

and under the adjunction

$$\mathrm{Hom}(x \circledast y, x \circledast ((y \otimes z) \circledast G(z))) \cong \mathrm{Hom}(G(x) \otimes (x \circledast y), (y \otimes z) \circledast G(z)) \tag{3.72}$$

it gets mapped to the morphism $\widetilde{\mathrm{coev}}_{y,Gz} \circ \mathrm{ev}_{x,y}$ as well. It thus follows that $\widetilde{\delta}^{\mathrm{r}}_{x,y,z} = \delta^{\mathrm{r}}_{x,y,z}$. The second statement is shown analogously. □

The distributors also possess another symmetry: applying the anti-equivalence G to $\delta^{\mathrm{r}}_{x,y,z}$, we obtain a morphism

$$(Gz \circledast Gy) \otimes Gx \cong G(y \otimes z) \otimes Gx \cong G(x \circledast (y \otimes z)) \longrightarrow G((x \circledast y) \otimes z) \cong Gz \circledast (Gy \otimes Gx) \,. \tag{3.73}$$

Proposition 3.30. *The distributors of a GV-category $\mathcal{C}$ satisfy*

$$G(\delta^{\mathrm{r}}_{x,y,z}) = \delta^{\mathrm{r}}_{Gz,Gy,Gx} \qquad \text{and} \qquad G(\delta^{\mathrm{l}}_{x,y,z}) = \delta^{\mathrm{l}}_{Gz,Gy,Gx} \tag{3.74}$$

for all $x, y, z \in \mathcal{C}$.

Proof. We equip G with the structure of a module functor, so that we can use the general results on lax/oplax module functor structures. The category $\mathcal{C}^{\mathrm{opp}}$ is a left $\mathcal{C}$-module category with action $x \triangleright \overline{y} = \overline{y \circledast G(x)}$ for $x \in \mathcal{C}$ and $\overline{y} \in \mathcal{C}^{\mathrm{opp}}$, and similarly it is a right module category with action $\overline{y} \triangleleft x = \overline{Gx \circledast y}$. By a direct computation one sees that the ${}_{\mathcal{C}}\mathcal{C}$-module endofunctor $G \circ \mathrm{R}^{\triangleleft}_{G^{-1}(x)} \circ G^{-1}$, with $\mathrm{R}^{\triangleleft}_{G^{-1}(x)}$ the left module endofunctor that maps $\overline{y}$ to $\overline{x \circledast y}$, is isomorphic to $R^{\triangleright}_{G(x)}$ as a $\triangleright$-module functor. These two functors are then also equivalent as oplax $\blacktriangleright$-module functors. This translates directly into the equality $G(\delta^{\mathrm{r}}_{x,y,z}) = \delta^{\mathrm{r}}_{Gz,Gy,Gx}$. The second equality in (3.74) follows analogously. □

As observed by Max Demirdilek, equipping $\mathcal{C}^{\mathrm{opp}}$ with the canonical GV-structure corresponding to the monoidal product $x \,\widetilde{\otimes}\, y = y \circledast x$ on objects $x, y \in \mathcal{C}^{\mathrm{opp}}$, the result in Proposition 3.30 shows that $G\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$ is a strong linear distributive functor in the sense of [7, Def. 1].

Module functors preserve the distributors in the following sense:

Proposition 3.31. *Let $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a strong $\triangleright$-module functor. Then F preserves the distributor δ^{r} in the sense that the pentagon*

$$\begin{tikzcd}
F((d \circledast c) \triangleright m) \arrow[r, "F(\delta^{\mathrm{r}}_{d,c,m})"] \arrow[d, "f^{\triangleright}_{d \circledast c, m}"'] & F(d \blacktriangleright (c \triangleright m)) \arrow[r, "f^{\blacktriangleright}_{d, c \triangleright m}"] & d \blacktriangleright F(c \triangleright m) \arrow[d, "d \blacktriangleright f^{\triangleright}_{c,m}"] \\
(d \circledast c) \triangleright F(m) \arrow[rr, "\delta^{\mathrm{r}}_{d,c,F(m)}"] & & d \blacktriangleright (c \triangleright F(m))
\end{tikzcd} \tag{3.75}$$

commutes for all $c, d \in \mathcal{C}$ and all $m \in \mathcal{M}$. A strong $\blacktriangleright$-module functor preserves the distributor δ^{l} in an analogous manner.

Proof. With the strong $\triangleright$-module functor structure on $R^{\triangleright}_m\colon {}^{\circledast}_{\mathcal{C}}\mathcal{C} \longrightarrow {}_{\mathcal{C}}\mathcal{M}$, the module functor constraint of F provides an isomorphism $R^{\triangleright}_{Fm} \cong F \circ R^{\triangleright}_m$ of module functors from $\mathcal{C}$ to $\mathcal{N}$. By Theorem 3.13 this isomorphism is also an isomorphism of the conjugated oplax $\blacktriangleright$-structures on these functors. This is precisely the commutativity of the diagram (3.75). □

Let now ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ be a GV-bimodule category over GV-categories $\mathcal{C}$ and $\mathcal{D}$. Since for every $c \in \mathcal{C}$ the functor $\mathrm{L}^{\triangleright}_c\colon \mathcal{M} \longrightarrow \mathcal{M}$ with $\mathrm{L}^{\triangleright}_c(m) = c \triangleright m$ is a $\mathcal{D}$-$\triangleleft$-module functor, we obtain from its oplax $\blacktriangleleft$-module structure the distributor

$$\delta^{\mathrm{l}}_{c,m,d}\colon \quad c \triangleright (m \blacktriangleleft d) \longrightarrow (c \triangleright m) \blacktriangleleft d\,. \tag{3.76}$$

Analogously we obtain the natural morphisms

$$\delta^{\mathrm{r}}_{c,m,d}\colon \quad (c \blacktriangleright m) \triangleleft d \longrightarrow c \blacktriangleright (m \triangleleft d)\,. \tag{3.77}$$

There are in total $8 \cdot 4 = 32$ pentagon diagrams for the distributors of a GV-bimodule category $\mathcal{M}$; they can be schematically symbolized as $\mathcal{M}\square\square\square$, $\square\mathcal{M}\square\square$, $\square\square\mathcal{M}\square$ and $\square\square\square\mathcal{M}$, with $\square \in \{\otimes, \circledast\}$. For example, the pentagon of type $\otimes\mathcal{M}\circledast\otimes$ is assembled from the two possible composite morphisms $(c \triangleright (m \blacktriangleleft d)) \triangleleft d' \Longrightarrow (c \triangleright m) \blacktriangleleft (d \otimes d')$ for all $c \in \mathcal{C}$, $d, d' \in \mathcal{D}$ and $m \in \mathcal{M}$.

Corollary 3.32. *Let $\mathcal{M}$ be a GV-bimodule category. Each of the 32 pentagon diagrams for the distributors of $\mathcal{M}$ commutes.*

Proof. The commutativity of the 16 pentagons of type $\mathcal{M}\Box\Box\Box$ or $\Box\Box\Box\mathcal{M}$ follows by regarding $\mathcal{M}$ just as a left or right module category. The commutativity of all 16 pentagons of type $\Box\mathcal{M}\Box\Box$ or $\Box\Box\mathcal{M}\Box$ follows from Proposition 3.18. The pentagon of type $\otimes\mathcal{M}\circledast\otimes$, for example, is obtained by applying Proposition 3.18 to the $\mathcal{D}$-$\triangleleft$-module functor $\mathrm{L}_c^{\triangleright}$. $\Box$

Next we introduce GV-analogues of the evaluation and coevaluation morphisms of a rigid duality. To this end we make use of the distinguished isomorphisms

$$\mathrm{Hom}(Gy \otimes y, K) \xrightarrow{\cong} \mathrm{Hom}(Gy, Gy) \tag{3.78}$$

and

$$\mathrm{Hom}_{\mathcal{C}}(y \otimes G^{-1}y, K) \cong \mathrm{Hom}_{\mathcal{C}}(y, GG^{-1}y) = \mathrm{Hom}(y, y)\,, \tag{3.79}$$

which are special cases of the defining isomorphisms (2.1) of the GV-structure.

Definition 3.33. The right and left *GV-evaluation morphisms*

$$\mathrm{ev}^{\mathrm{r}}_y : \quad Gy \otimes y \longrightarrow K \qquad \text{and} \qquad \mathrm{ev}^{\mathrm{l}}_y : \quad y \otimes G^{-1}y \longrightarrow K \tag{3.80}$$

are the pre-image of the identity morphisms id_{Gy} and id_y under the isomorphisms (3.78) and (3.79), respectively. Analogously, the right and left *GV-coevaluation morphisms*

$$\mathrm{coev}^{\mathrm{r}}_x := G(\mathrm{ev}^{\mathrm{l}}_x) : \quad 1 \longrightarrow x \circledast G(x) \tag{3.81}$$

and

$$\mathrm{coev}^{\mathrm{l}}_x := G^{-1}(\mathrm{ev}^{\mathrm{r}}_x) : \quad 1 \longrightarrow G^{-1}(x) \circledast x \tag{3.82}$$

are obtained from the isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(1, x \circledast Gy) \cong \mathrm{Hom}_{\mathcal{C}}(GK, G(y \otimes G^{-1}x)) \cong \mathrm{Hom}_{\mathcal{C}}(y \otimes G^{-1}x, K)\,. \tag{3.83}$$

We now show that when complemented with the distributors (3.68), the GV-evaluation and GV-coevaluation morphisms ev^{l}, ev^{r} and $\mathrm{coev}^{\mathrm{l}}$, $\mathrm{coev}^{\mathrm{r}}$ obey snake relations which involve distributors and generalize the familiar duality structure of a rigid monoidal category. For the monoidal case this was already observed in Theorem 4.5 of (the revised version of) [6]. The proof given there involves "a straightforward verification to check that *-autonomous categories are weakly distributive, though the diagrams can be pretty horrid." Our approach generalizes the statement to the case of module categories and provides a conceptual understanding of the diagrams. We work in the setting of a left $\mathcal{C}$-module category $\mathcal{M}$. Recall from Equation (3.16) the collection of morphisms $\mathrm{coev}_{c,m} \colon m \longrightarrow c \blacktriangleright (Gc \triangleright m)$. It follows from our convention concerning the adjunction

(3.4) for the case $\mathcal{M} = \mathcal{C}$ that in this case the composite $1 \xrightarrow{\mathrm{coev}_{c,1}} c \circledast (Gc \otimes 1) \cong c \circledast Gc$ coincides with the GV-coevaluation $\mathrm{coev}^{\mathrm{r}}_c \colon 1 \longrightarrow c \circledast Gc$, and analogously for $\mathrm{ev}_{c,K}$:

$$\mathrm{coev}_{c,1} = \mathrm{coev}^{\mathrm{r}}_c \qquad \text{and} \qquad \mathrm{ev}_{c,K} = \mathrm{ev}^{\mathrm{r}}_c \,. \tag{3.84}$$

The GV-(co)evaluations and the module distributors determine all (co)evaluations:

Lemma 3.34. *Let $\mathcal{M}$ be a GV-module category. For all $m \in \mathcal{M}$ and $c \in \mathcal{C}$ the diagrams*

$$\begin{tikzcd} m \arrow[r, "\mathrm{coev}_{c,m}"] \arrow[d, "\cong"'] & c \blacktriangleright (Gc \triangleright m) \\ 1 \triangleright m \arrow[r, "\mathrm{coev}_c \triangleright m"] & (c \circledast Gc) \triangleright m \arrow[u, "\delta^{\mathrm{r}}_{c,Gc,m}"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} Gc \triangleright (c \blacktriangleright m) \arrow[r, "\mathrm{ev}_{c,m}"] \arrow[d, "\delta^{\mathrm{l}}_{Gc,c,m}"'] & m \\ (Gc \otimes c) \blacktriangleright m \arrow[r, "\mathrm{ev}_c \blacktriangleright m"] & K \blacktriangleright m \arrow[u, "\cong"'] \end{tikzcd} \tag{3.85}$$

commute.

Proof. Applying Lemma 3.16 to the GV-module functor $R_m \colon \mathcal{C} \longrightarrow \mathcal{M}$ and the object $1 \in \mathcal{C}$ directly shows the commutativity of the diagram

$$\begin{tikzcd} R_m(1) = 1 \triangleright m \arrow[r, "\mathrm{coev}_{c,1 \triangleright m}"] \arrow[d, "R_m(\mathrm{coev}_{c,1})"'] & c \blacktriangleright (Gc \triangleright (1 \triangleright m)) \arrow[d, "\cong"] \\ (c \circledast (Gc \otimes 1)) \triangleright m \arrow[r, "\delta^{\mathrm{r}}_{c,Gc \otimes 1,m}"] & c \blacktriangleright ((Gc \otimes 1) \triangleright m) \end{tikzcd} \tag{3.86}$$

Inserting the unitors, this implies the commutativity of the first of the diagrams (3.85). The commutativity of the second diagram is shown analogously. □

It is worth pointing out that in view of the fact that the distributors are generically not isomorphisms, it is not obvious that the GV-(co)evaluations obey appropriate snake relations. However, with the help of Lemma 3.34 we can show that this is indeed the case:

Proposition 3.35 *(Snake relations). Let $\mathcal{C}$ be a GV-category. For every $c \in \mathcal{C}$ the diagrams*

$$\begin{tikzcd} c \arrow[r, "\mathrm{coev}^{\mathrm{r}}_c \otimes \mathrm{id}"'] \arrow[rrr, bend left, "\mathrm{id}_c"] & (c \circledast Gc) \otimes c \arrow[r, "\delta^{\mathrm{r}}_{c,Gc,c}"'] & c \circledast (Gc \otimes c) \arrow[r, "\mathrm{id} \circledast \mathrm{ev}^{\mathrm{r}}_c"'] & c \end{tikzcd} \tag{3.87}$$

and

$$\begin{tikzcd}
Gc \arrow[r, "\mathrm{id}\otimes\mathrm{coev}^{\mathrm{r}}_c"'] \arrow[rrr, bend left, "\mathrm{id}_{Gc}"] & Gc\otimes(c\circledast Gc) \arrow[r, "\delta^{\mathrm{l}}_{Gc,c,Gc}"'] & (Gc\otimes c)\circledast Gc \arrow[r, "\mathrm{ev}^{\mathrm{r}}_c\circledast\mathrm{id}"'] & Gc
\end{tikzcd} \tag{3.88}$$

commute.

Proof. By Lemma 3.34 the diagram

$$\begin{tikzcd}
c\circledast K\cong c \arrow[r, "\mathrm{coev}_{c,c\circledast K}"] \arrow[d, "\mathrm{coev}^{\mathrm{r}}_c\otimes 1"'] & c\circledast(Gc\rhd(c\circledast K)) \arrow[r, "c\circledast\mathrm{ev}_{c,K}"] \arrow[d, "\cong"] & c\circledast K\cong c \arrow[d, "="] \\
(c\circledast Gc)\otimes c \arrow[r, "\delta^{\mathrm{r}}"] & c\circledast(Gc\otimes c) \arrow[r, "c\circledast\mathrm{ev}^{\mathrm{r}}_c"] & c\circledast K
\end{tikzcd} \tag{3.89}$$

commutes. By Equation (3.17) the horizontal arrows in the first row compose to id_c. This yields commutativity of (3.87). Commutativity of (3.88) is shown analogously. □

Remark 3.36. In particular, the evaluation and coevaluation morphisms endow every object of a GV-category with the structure of a (left and right) *nuclear object* in the sense of Definition A.1 of [7].

As a consequence of these snake identities, we can express the defining isomorphisms (2.1) of $\mathcal{C}$ also as follows:

Proposition 3.37. *Let $\mathcal{C}$ be a GV-category, and let $\varpi_{x,y}\colon \mathrm{Hom}(x\otimes y,K)\xrightarrow{\cong}\mathrm{Hom}(x,Gy)$ be defined as in (2.1).*

1. *The image of a morphism $f\colon x\longrightarrow Gy$ under the isomorphism $\varpi^{-1}_{x,y}$ equals $\mathrm{ev}^{\mathrm{r}}_y\circ(f\otimes\mathrm{id})$.*
2. *The image of a morphism $\xi\colon x\otimes y\longrightarrow K$ under the isomorphism $\varpi_{x,y}$ equals the composite (omitting unitors)*

$$x\xrightarrow{\mathrm{id}\otimes\mathrm{coev}^{\mathrm{r}}_y} x\otimes(y\circledast Gy)\xrightarrow{\delta^{\mathrm{l}}}(x\otimes y)\circledast Gy\xrightarrow{\xi\circledast\mathrm{id}} Gy\,. \tag{3.90}$$

3. *For a morphism $f\colon x\longrightarrow y$, the morphism Gf is equal to the composite*

$$\begin{aligned} Gy\xrightarrow{\mathrm{id}\otimes\mathrm{coev}^{\mathrm{r}}_x} Gy\otimes(x\circledast Gx)\xrightarrow{\delta^{\mathrm{l}}}(Gy\otimes x)\circledast Gx& \\ \xrightarrow{(\mathrm{id}\otimes f)\circledast\mathrm{id}}(Gy\otimes y)\circledast Gx\xrightarrow{\mathrm{ev}^{\mathrm{r}}_y\circledast\mathrm{id}} Gy\,.& \end{aligned} \tag{3.91}$$

Proof. The first statement is just the usual expression of the adjunction in terms of the unit. The second statement follows by combining the first with the triangle identity

(3.87): Composing (3.90) with the isomorphism from the first part gives, by (3.87), the identity. The last statement follows by combining the first two with the definition of the functor G via the commuting diagram (2.2). □

Remark 3.38. It follows from Lemma 3.34 that we can extend part 2 of Proposition 3.37 as follows: for all $x, y, z \in \mathcal{C}$ the adjunction $\mathrm{Hom}(x \otimes y, z) \xrightarrow{\cong} \mathrm{Hom}(x, z \mathbin{⊛} Gy)$ can be described as a composition with the coevaluation $\mathrm{coev}^{\mathrm{r}}_y$ and an appropriate distributor.

We finally have a compatibility of the evaluation and coevaluation morphisms with the monoidal structure of $\mathcal{C}$:

Proposition 3.39. *Let $\mathcal{C}$ be a GV-category and $x, y \in \mathcal{C}$. The right coevaluation $\mathrm{coev}^{\mathrm{r}}_{x\otimes y}$ of the object $x \otimes y$ is equal to the composite*

$$\begin{aligned} 1 \xrightarrow{\mathrm{coev}^{\mathrm{r}}_x} x \mathbin{⊛} Gx \xrightarrow{(\mathrm{id} \otimes \mathrm{coev}^{\mathrm{r}}_y) \mathbin{⊛} \mathrm{id}} (x \otimes (y \mathbin{⊛} Gy)) \mathbin{⊛} Gx \xrightarrow{\delta^{\mathrm{l}} \mathbin{⊛} \mathrm{id}} (x \otimes y) \mathbin{⊛} Gy \mathbin{⊛} Gx \\ \xrightarrow{\cong} (x \otimes y) \mathbin{⊛} G(x \otimes y)\,. \end{aligned} \tag{3.92}$$

Analogous expressions are valid for the left coevaluation and for the evaluations.

Proof. To show the claim for the right coevaluation, notice that the diagram

$$\begin{tikzcd} \mathrm{Hom}(x \otimes y, z) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Hom}(1, z \mathbin{⊛} G(x \otimes y)) \\ \mathrm{Hom}(x, z \mathbin{⊛} Gy) \arrow[d, "\cong"'] & \\ \mathrm{Hom}(1, (z \mathbin{⊛} Gy) \mathbin{⊛} Gx) \arrow[r, "\cong"] & \mathrm{Hom}(1, z \mathbin{⊛} (Gy \mathbin{⊛} Gx)) \arrow[uu, "\cong"'] \end{tikzcd} \tag{3.93}$$

commutes: The isomorphism from the upper to the middle row can be expressed as the composite

$$\begin{aligned} \mathrm{Hom}(x \otimes y, z) \xrightarrow{\varpi^{-1}_{x\otimes y, G^{-1}z}} \mathrm{Hom}((x \otimes y) \otimes G^{-1}z, K) \xrightarrow{\cong} \mathrm{Hom}(x \otimes (y \otimes G^{-1}z), K) \\ \xrightarrow{\varpi_{x, y\otimes G^{-1}z}} \mathrm{Hom}(x, G(y \otimes G^{-1}z)) = \mathrm{Hom}(x, z \mathbin{⊛} Gy) \end{aligned} \tag{3.94}$$

with the isomorphisms ϖ as defined in (2.1); commutativity of (3.93) is a consequence of the resulting cancellation in the composition of the two downwards arrows. Now when taking $z = x \otimes y$ and the identity morphism in the upper left corner, the isomorphism in the upper row yields the coevaluation $\mathrm{coev}^{\mathrm{r}}_{x\otimes y}$, while by Remark 3.38 the other path gives the composite (3.92). The other cases are shown analogously. □

3.5. Internal Homs and weak module functors

The definition of GV-module category ensures the existence of internal Homs and coHoms as suitable adjoints of action functors. We now discuss various aspects of these functors.

Definition 3.40. Let $\mathcal{C}$ be a GV-category and $(\mathcal{M}, \triangleright)$ a left GV-module category over $(\mathcal{C}, \otimes)$, and let $m \in \mathcal{M}$. Then the *internal Hom* $\underline{\mathrm{Hom}}(m, -)$ is the right adjoint of the functor from $\mathcal{C}$ to $\mathcal{M}$ that maps objects as $c \longmapsto c \triangleright m$. The *internal coHom* $\underline{\mathrm{coHom}}(m, -)$ is the left adjoint of the functor from $\mathcal{C}$ to $\mathcal{M}$ that maps objects as $c \longmapsto c \blacktriangleright m$. In more detail, we have isomorphisms

$$\mathrm{Hom}_{\mathcal{M}}(c \triangleright m, m') \cong \mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(m, m')) \tag{3.95}$$

and

$$\mathrm{Hom}_{\mathcal{M}}(m', c \blacktriangleright m) \cong \mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{coHom}}(m, m'), c) \tag{3.96}$$

for all $c \in \mathcal{C}$ and $m, m' \in \mathcal{M}$.

In view of the definition of the action $\blacktriangleright$ in terms of $\triangleright$ (see Proposition 3.5), it is not surprising that the internal coHom can be expressed in terms of the internal Hom:

Lemma 3.41. *Let $\mathcal{M}$ be a GV-module category. Then the internal coHom can be expressed as*

$$\underline{\mathrm{coHom}}(m, m') \cong G^{-1}(\underline{\mathrm{Hom}}(m', m)) \tag{3.97}$$

for all $m, m' \in \mathcal{M}$.

Proof. This follows directly from the definition of $\blacktriangleright$, compare the calculation in (3.7). □

Being defined via the adjunctions (3.95) and (3.96), the internal (co)Hom functors come with canonical (co)evaluation morphisms

$$\underline{\mathrm{ev}}_{m,n} : \quad \underline{\mathrm{Hom}}(m, n) \triangleright m \longrightarrow n \tag{3.98}$$

and

$$\underline{\mathrm{coev}}_{m,n} : \quad m \longrightarrow \underline{\mathrm{coHom}}(n, m) \blacktriangleright n \,. \tag{3.99}$$

These, in turn, give rise to canonical multiplications

$$\underline{\mu}_{m,n,l} : \quad \underline{\mathrm{Hom}}(n, l) \otimes \underline{\mathrm{Hom}}(m, n) \longrightarrow \underline{\mathrm{Hom}}(m, l) \tag{3.100}$$

and comultiplications

$$\underline{\Delta}_{m,n,l}: \quad \underline{\mathrm{coHom}}(l,m) \longrightarrow \underline{\mathrm{coHom}}(n,m) \circledast \underline{\mathrm{coHom}}(l,n) \tag{3.101}$$

via the compositions (suppressing the associator)

$$\underline{\mathrm{ev}}_{n,l} \circ (\mathrm{id}_{\underline{\mathrm{Hom}}(n,l)} \triangleright \underline{\mathrm{ev}}_{m,n}): \quad \underline{\mathrm{Hom}}(n,l) \triangleright \underline{\mathrm{Hom}}(m,n) \triangleright m \longrightarrow \underline{\mathrm{Hom}}(n,l) \triangleright n \longrightarrow l \tag{3.102}$$

and

$$\begin{aligned}&(\mathrm{id} \blacktriangleright \underline{\mathrm{coev}}_{n,l}) \circ \underline{\mathrm{coev}}_{m,n}: \\ &\qquad m \longrightarrow \underline{\mathrm{coHom}}(n,m) \blacktriangleright n \longrightarrow \underline{\mathrm{coHom}}(n,m) \blacktriangleright \underline{\mathrm{coHom}}(l,n) \blacktriangleright l\,,\end{aligned} \tag{3.103}$$

respectively. The following result is standard:

Lemma 3.42. *The (co)multiplications (3.100) and (3.101) equip $\underline{\mathrm{Hom}}(m,m)$ with the structure of an associative algebra in $(\mathcal{C},\otimes)$ and $\underline{\mathrm{coHom}}(m,m)$ with the structure of a coassociative coalgebra in $(\mathcal{C},\circledast)$. Moreover, for all $m,l \in \mathcal{M}$ the objects $\underline{\mathrm{Hom}}(m,l)$ are canonically right $\underline{\mathrm{Hom}}(m,m)$-modules and the objects $\underline{\mathrm{coHom}}(l,m)$ are canonically right $\underline{\mathrm{coHom}}(m,m)$-comodules.*

In the case of the regular module category we can compute the internal Homs by making use of the isomorphisms $\mathrm{Hom}(c \otimes d, K) \cong \mathrm{Hom}(c, Gd) \cong \mathrm{Hom}(d, G^{-1}c)$. This gives

Lemma 3.43. *The internal (co)Homs of the regular module category $\mathcal{C}$ are given by*

$$\underline{\mathrm{Hom}}(c,d) \cong G(c \otimes G^{-1}d) = d \circledast Gc \qquad \text{and} \qquad \underline{\mathrm{coHom}}(c,d) \cong d \otimes G^{-1}c\,. \tag{3.104}$$

Further we have

Lemma 3.44. *Let $\mathcal{M}$ be a left GV-module category over a GV-category $\mathcal{C}$. Then the internal Hom satisfies*

$$\underline{\mathrm{Hom}}(b \triangleright m, c \blacktriangleright m') \cong c \circledast \underline{\mathrm{Hom}}(m,m') \circledast Gb \tag{3.105}$$

for $b,c \in \mathcal{C}$ and $m,m' \in \mathcal{M}$. In particular, the internal Hom is a strong module functor with respect to $\circledast$. Similarly, the internal coHom satisfies

$$\underline{\mathrm{coHom}}(c \blacktriangleright m, b \triangleright m') \cong b \otimes \underline{\mathrm{coHom}}(m,m') \otimes G^{-1}c \tag{3.106}$$

for $b,c \in \mathcal{C}$ and $m,m' \in \mathcal{M}$.

Proof. Making use of (3.9) and of the easily established adjunction formulas $\mathrm{Hom}_{\mathcal{C}}(Ga \otimes b, c) \cong \mathrm{Hom}_{\mathcal{C}}(b, a \circledast c)$ and

$$\mathrm{Hom}_{\mathcal{C}}(a \otimes G^{-1}b, c) \cong \mathrm{Hom}_{\mathcal{C}}(a, c \circledast b)\,, \tag{3.107}$$

for any $b, c, d \in \mathcal{C}$ and $m, n \in \mathcal{M}$ we obtain the sequence

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(d, \underline{\mathrm{Hom}}(b \triangleright m, c \blacktriangleright n)) &\cong \mathrm{Hom}_{\mathcal{M}}(d \triangleright b \triangleright m, c \blacktriangleright n)\\ &\cong \mathrm{Hom}_{\mathcal{M}}(Gc \triangleright d \triangleright b \triangleright m, n) \cong \mathrm{Hom}_{\mathcal{M}}((Gc \otimes d \otimes b) \triangleright m, n)\\ &\cong \mathrm{Hom}_{\mathcal{C}}(Gc \otimes d \otimes b, \underline{\mathrm{Hom}}(m, n)) \cong \mathrm{Hom}_{\mathcal{C}}(d, c \circledast \underline{\mathrm{Hom}}(m, n) \circledast Gb)\end{aligned} \tag{3.108}$$

of isomorphisms. Then (3.105) follows by the Yoneda lemma. Similarly, for internal coHoms we get

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{coHom}}(c \blacktriangleright m, b \triangleright n), d) &\cong \mathrm{Hom}_{\mathcal{M}}(b \triangleright n, d \blacktriangleright c \blacktriangleright m)\\ &\cong \mathrm{Hom}_{\mathcal{M}}(n, G^{-1}b \blacktriangleright d \blacktriangleright c \blacktriangleright m) \cong \mathrm{Hom}_{\mathcal{M}}(n, (G^{-1}b \circledast d \circledast c) \blacktriangleright m)\\ &\cong \mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{coHom}}(m, n), G^{-1}b \circledast d \circledast c)\\ &\cong \mathrm{Hom}_{\mathcal{C}}(b \otimes \underline{\mathrm{coHom}}(m, n) \otimes G^{-1}c, d)\,,\end{aligned} \tag{3.109}$$

which shows (3.106). □

In the following we use the symbol $\overline{m}$ for the object in $\mathcal{M}^{\mathrm{opp}}$ that corresponds to $m \in \mathcal{M}$, whereby e.g. the module structures on $\mathcal{M}^{\mathrm{opp}}$ from Proposition 3.7 read

$$\overline{m} \triangleleft c = \overline{G^{-1}c \blacktriangleright m} \qquad \text{and} \qquad \overline{m} \blacktriangleleft c = \overline{G^{-1}c \triangleright m}\,. \tag{3.110}$$

Proposition 3.45. *Let $\mathcal{M}$ be a left GV-module category over a GV-category $\mathcal{C}$. The internal Hom $\underline{\mathrm{Hom}}\colon \mathcal{M}^{\mathrm{opp}} \times \mathcal{M} \to \mathcal{C}$ carries the following (weak) module functor structures. For $m, n \in \mathcal{M}$ and $c \in \mathcal{C}$ there are*

1. *coherent morphisms*

$$c \otimes \underline{\mathrm{Hom}}(\overline{m}, n) \longrightarrow \underline{\mathrm{Hom}}(\overline{m}, c \triangleright n) \tag{3.111}$$

   which endow the functors $\underline{\mathrm{Hom}}(\overline{m}, -)\colon \mathcal{M} \to \mathcal{C}$ with the structure of lax $\triangleright$-module functors;
2. *coherent isomorphisms*

$$\underline{\mathrm{Hom}}(\overline{m}, c \blacktriangleright n) \xrightarrow{\cong} c \circledast \underline{\mathrm{Hom}}(\overline{m}, n) \tag{3.112}$$

   which endow the functors $\underline{\mathrm{Hom}}(\overline{m}, -)\colon \mathcal{M} \to \mathcal{C}$ with the structure of strong $\blacktriangleright$-module functors;

3. *coherent morphisms*

$$\underline{\mathrm{Hom}}(\overline{m},n)\otimes c \longrightarrow \underline{\mathrm{Hom}}(\overline{m}\triangleleft c,n) \tag{3.113}$$

which endow the functors $\underline{\mathrm{Hom}}(-,n)\colon \mathcal{M}^{\mathrm{opp}}\to\mathcal{C}$ *with the structure of lax* $\triangleleft$*-module functors;*

4. *coherent isomorphisms*

$$\underline{\mathrm{Hom}}(\overline{m}\blacktriangleleft c,n) \xrightarrow{\cong} \underline{\mathrm{Hom}}(m,n)\circledast c \tag{3.114}$$

which endow the functors $\underline{\mathrm{Hom}}(-,n)\colon \mathcal{M}^{\mathrm{opp}}\to\mathcal{C}$ *with the structure of strong* $\blacktriangleleft$*-module functors.*

In addition there are the following compatibilities: The corresponding lax and strong module functor structures in each argument form a conjugated pair of lax and oplax module functor structures. In particular the module functors $\underline{\mathrm{Hom}}(m,-)\colon \mathcal{M}\to\mathcal{C}$ *and* $\underline{\mathrm{coHom}}(m,-)\colon \mathcal{M}\to\mathcal{C}$ *are GV-module functors.*

Furthermore we have, for $m,n\in\mathcal{M}$ *and* $c,d\in\mathcal{C}$*:*

5. *The strong module functor structures commute, in the sense that the two obvious isomorphisms*

$$\underline{\mathrm{Hom}}(\overline{m}\blacktriangleleft c,d\blacktriangleright n) \overset{\cong}{\underset{\cong}{\rightrightarrows}} d\circledast\underline{\mathrm{Hom}}(\overline{m},n)\circledast c \tag{3.115}$$

are equal. Thus $\underline{\mathrm{Hom}}\colon \mathcal{M}^{\mathrm{opp}}\times\mathcal{M}\longrightarrow\mathcal{C}$ *is a strong* $\circledast$*-bimodule functor.*

6. *The lax module functor structures commute, in the sense that the two obvious morphisms*

$$d\otimes\underline{\mathrm{Hom}}(\overline{m},n)\otimes c \rightrightarrows \underline{\mathrm{Hom}}(\overline{m}\triangleleft c,d\triangleright n) \tag{3.116}$$

are equal. Thus $\underline{\mathrm{Hom}}\colon \mathcal{M}^{\mathrm{opp}}\times\mathcal{M}\longrightarrow\mathcal{C}$ *is a lax* $\otimes$*-bimodule functor.*

7. *The natural isomorphism* $\underline{\mathrm{Hom}}_{\mathcal{M}}(c\triangleright m,-)\cong\underline{\mathrm{Hom}}_{\mathcal{C}}(c,\underline{\mathrm{Hom}}_{\mathcal{M}}(m,-))\cong\underline{\mathrm{Hom}}_{\mathcal{M}}(m,-)\circledast Gc$ *obtained from the isomorphisms* (3.105) *is an isomorphism of lax* $\triangleright$*-module functors.*
Similarly, the natural isomorphism $\underline{\mathrm{coHom}}_{\mathcal{M}}(c\blacktriangleright m,-)\cong\underline{\mathrm{coHom}}_{\mathcal{C}}(c,\underline{\mathrm{coHom}}_{\mathcal{M}}(m,-))$ *is an isomorphism of oplax* $\blacktriangleright$*-module functors.*

Dual statements hold for the internal coHom.

Proof. Recall that the functor $\mathrm{R}_m^{\triangleright}\colon {}_{\mathcal{C}}\mathcal{C}\to{}_{\mathcal{C}}\mathcal{M}$ given in (3.36) is a strong $\triangleright$-module functor; its right adjoint is the functor $\underline{\mathrm{Hom}}(\overline{m},-)\colon {}_{\mathcal{C}}\mathcal{M}\to{}_{\mathcal{C}}\mathcal{C}$. The profunctor transport from Corollary 3.23 thus provides us with the lax $\triangleright$-module functor structure of

$\underline{\mathrm{Hom}}(\overline{m},-)$. The strong $\blacktriangleright$-module functor structure is obtained by the flipping transport from Lemma 3.20. The remaining structures are obtained with the help of Lemma 3.44 from the corresponding structures of the internal coHom. By Proposition 3.27, the resulting strong and lax module functor structures correspond to the lax/oplax functor structure pair of a GV-functor. By the proof of Lemma 3.44, the two strong module functor structures are compatible in the form of statement 5.

By the transport of structures in a GV-bimodule functor, as described in Lemma 3.19, it follows that the lax module functor structures are compatible as well. For statement 7, consider the strong $\triangleright$-module functors $R_c^{\triangleright}: \mathcal{C} \longrightarrow \mathcal{C}$ and $R_m^{\triangleright}: \mathcal{C} \longrightarrow \mathcal{M}$. The right adjoint lax module functor of their composite is $(R_m^{\triangleright} R_c^{\triangleright})^{\mathrm{r.a.}} \cong \underline{\mathrm{Hom}}(c \triangleright m,-)$. By Lemma 3.25, the isomorphism $(R_m^{\triangleright} R_c^{\triangleright})^{\mathrm{r.a.}} \cong (R_c^{\triangleright})^{\mathrm{r.a.}}(R_m^{\triangleright})^{\mathrm{r.a.}}$ is an isomorphism of lax module functors. This proves the claim concerning the internal Hom; the statement about the internal coHom follows analogously. □

In the case of $\mathcal{C}$ as a left module category over itself, the internal Hom is canonically identified with $\underline{\mathrm{Hom}}(x,y) \cong y \circledast Gx = R_{Gx}^{\circledast}(y)$. The lax $\triangleright$-module functor structure of $\underline{\mathrm{Hom}}(x,-)$ is given in Proposition 3.45. On the other hand, $R_{Gx}^{\circledast}: \mathcal{C} \longrightarrow \mathcal{C}$ is a strong $\blacktriangleright$-module functor with respect to the left action of $(\mathcal{C}, \circledast)$, hence by Proposition 3.12 it is also a lax $\triangleright$-module functor with the conjugated module functor structure. In this case the lax $\triangleright$-module functor structures can also serve as a means for defining the distributors of the GV-category $\mathcal{C}$ [13, Sect. 4]. The so obtained definition of the distributors is equivalent to the one given in Definition 3.17:

Corollary 3.46. *The natural isomorphism $\underline{\mathrm{Hom}}(x,-) \cong R_{Gx}^{\circledast}$ is an isomorphism of lax $\triangleright$-module functors.*

Proof. The left adjoint of $R_{Gx}^{\circledast}$ is the functor $R_x^{\otimes}: \mathcal{C} \longrightarrow \mathcal{C}$. Under the flipping transport, the strong $\blacktriangleright$-module functor structure of $R_{Gx}^{\circledast}$ corresponds to the strong $\triangleright$-module functor structure of $R_x^{\otimes}$. The profunctor transport of this structure yields, in turn, the lax $\triangleright$-module functor structure of $\underline{\mathrm{Hom}}(x,-)$. By Proposition 3.27, this structure coincides with the conjugated lax $\triangleright$-module functor structure of $R_{Gx}^{\circledast}$. □

Besides the internal Hom $\underline{\mathrm{Hom}} \equiv \underline{\mathrm{Hom}}^{\mathrm{r}}$ for the regular left module category, for which in the adjunction $\mathrm{Hom}(c \otimes d, d') \cong \mathrm{Hom}(c, \underline{\mathrm{Hom}}(d,d'))$ the right tensor factor changes place, analogously there is an internal Hom $\underline{\mathrm{Hom}}^{\mathrm{l}}$ for the regular right module category [1] for which the adjunction keeps the right tensor factor,

$$\mathrm{Hom}(c \otimes d, c') \cong \mathrm{Hom}(d, \underline{\mathrm{Hom}}^{\mathrm{l}}(c,c')) \,. \tag{3.117}$$

It computes as $\underline{\mathrm{Hom}}^{\mathrm{l}}(c,d) \cong G^{-1}(Gd \otimes c) = G^{-1}c \circledast d$. Analogously there is a second internal coHom $\underline{\mathrm{coHom}}^{\mathrm{l}}$. In accordance with our guiding principle, for every statement

[1] As compared to [5], in our notation the use of the superscripts l and r is interchanged.

involving $\underline{\mathrm{Hom}}$ or $\underline{\mathrm{coHom}}$ there is an analogous statement for $\underline{\mathrm{Hom}}^{\mathrm{l}}$, respectively $\underline{\mathrm{coHom}}^{\mathrm{l}}$. This observation applies not just to the regular right module, but likewise to every right GV-module category over $\mathcal{C}$. In particular there are evaluations

$$\underline{\mathrm{ev}}'_{m,n}: \quad m \otimes \underline{\mathrm{Hom}}^{\mathrm{l}}(m,n) \longrightarrow n\,, \tag{3.118}$$

and multiplication morphisms

$$\underline{\mu}'_{m,n,l}: \quad \underline{\mathrm{Hom}}^{\mathrm{l}}(m,n) \otimes \underline{\mathrm{Hom}}^{\mathrm{l}}(n,l) \longrightarrow \underline{\mathrm{Hom}}^{\mathrm{l}}(m,l) \tag{3.119}$$

via

$$\underline{\mathrm{ev}}'_{n,l} \circ (\underline{\mathrm{ev}}'_{m,n} \otimes \mathrm{id}_{\underline{\mathrm{Hom}}^{\mathrm{l}}(n,l)}): \quad m \otimes \underline{\mathrm{Hom}}^{\mathrm{l}}(m,n) \otimes \underline{\mathrm{Hom}}^{\mathrm{l}}(n,l) \longrightarrow m \otimes \underline{\mathrm{Hom}}^{\mathrm{l}}(m,l) \longrightarrow l \tag{3.120}$$

analogously to (3.98) and (3.100).

4. Frobenius algebras and admissible objects

4.1. Algebras and Frobenius algebras in GV-categories

Throughout this section $(\mathcal{C},\otimes) = (\mathcal{C},\otimes,1,\alpha,l,r,K)$ is a GV-category. Recall the notions of algebras and coalgebras in $\mathcal{C}$ as given in Definition 2.8.

Example 4.1. As seen in Lemma 3.42, for any object m in a GV-module category over $\mathcal{C}$, the internal End $\underline{\mathrm{Hom}}(m,m)$ carries the structure of an algebra in $(\mathcal{C},\otimes,1)$, while the internal coEnd $\underline{\mathrm{coHom}}(m,m)$ carries the structure of a coalgebra in $(\mathcal{C},\circledast,K)$.

Definition 4.2. In a GV-category $\mathcal{C}$ a *GV-algebra* is an algebra in the monoidal category $(\mathcal{C},\otimes,1,\alpha^{\otimes},l^{\otimes},r^{\otimes})$. A *GV-coalgebra* in $\mathcal{C}$ is a coalgebra in the monoidal category $(\mathcal{C},\circledast,K,\alpha^{\circledast},\, l^{\circledast},r^{\circledast})$.

Lemma 4.3. *In a GV-category $\mathcal{C}$, GV-algebras and GV-coalgebras are in bijection under both the dualizing functor G and its inverse G^{-1}.*

Proof. We show that if (A,μ,η) is a GV-algebra (that is an algebra in $(\mathcal{C},\otimes,1)$), then $G(A)$ naturally inherits the structure of a coalgebra in $(\mathcal{C},\circledast,K)$; the argument for $G^{-1}(A)$ is analogous. The reverse statement for GV-coalgebras then immediately follows from G being an antiequivalence. One checks that the morphism $\Delta\colon GA \xrightarrow{G(\mu)} G(A\otimes A) \xrightarrow{\cong} GA \circledast GA$ obtained via the composite isomorphism

$$\mathrm{Hom}(A\otimes A,A) \xrightarrow{\cong} \mathrm{Hom}(GA,G(A\otimes A)) \xrightarrow{\cong} \mathrm{Hom}(GA,GA\circledast GA) \tag{4.1}$$

is a coassociative comultiplication. Furthermore, the morphism $\varepsilon\colon GA \xrightarrow{\mathrm{id}\otimes\eta} GA\otimes A \xrightarrow{\mathrm{ev}_A} K$ is a counit for this comultiplication. □

Given the notions of algebras and coalgebras it is natural to also consider Frobenius algebras, which we do following Definition 2.3.2 of [10].

Definition 4.4. A *GV-Frobenius algebra* in $\mathcal{C}$ is a quintuple $(A, \mu, \eta, \Delta, \varepsilon)$ such that (A, μ, η) is an algebra in $\mathcal{C}$, (A, Δ, ε) is a GV-coalgebra in $\mathcal{C}$, and

$$(\mu \circledast \mathrm{id}_A) \circ \delta^{\mathrm{l}}_{A,A,A} \circ (\mathrm{id}_A \otimes \Delta) = \Delta \circ \mu = (\mathrm{id}_A \circledast \mu) \circ \delta^{\mathrm{r}}_{A,A,A} \circ (\Delta \otimes \mathrm{id}_A) \tag{4.2}$$

as morphisms in $\mathrm{Hom}(A \otimes A, A \circledast A)$.

A *morphism f of GV-Frobenius algebras* between GV-Frobenius algebras $(A, \mu^A, \eta^A, \Delta^A, \varepsilon^A)$ and $(B, \mu^B, \eta^B, \Delta^B, \varepsilon^B)$ is a morphism $f\colon A \to B$ that is compatible with the algebra and coalgebra structures, i.e. satisfies $\mu^B \circ (f \otimes f) = f \circ \mu^A$, $f \circ \eta^A = \eta^B$, $(f \circledast f) \circ \Delta^A = \Delta^B \circ f$ and $\varepsilon^B \circ f = \varepsilon^A$.

Remark 4.5. The two equalities postulated in Equation (4.2) are not independent. Indeed, it suffices to require that the left and right most expression are the same morphism. This fact is well known for Frobenius algebras in rigid monoidal categories. In the present setting the proof is considerably more subtle, since non-invertible distributors enter. A proof has been given in [8] with the help of a three-dimensional graphical calculus.

Note that this definition does not assume that the distributors $\delta^{\mathrm{l}}_{A,A,A}$ and $\delta^{\mathrm{r}}_{A,A,A}$ are isomorphisms. Just as with Frobenius algebras in a monoidal category we can also consider their morphisms.

Lemma 4.6. *Every morphism of GV-Frobenius algebras is an isomorphism.*

Proof. Let $f\colon A \to B$ be a morphism of GV-Frobenius algebras. It is not hard to check that the morphism

$$f^- := l^{\circledast}_A \circ \big((\varepsilon^B \circ \mu^B) \circledast \mathrm{id}_A\big) \circ \delta^{\mathrm{r}}_{B,B,A} \circ (\mathrm{id}_B \otimes (f \circledast \mathrm{id}_A)) \circ \big(\mathrm{id}_B \otimes (\Delta^A \circ \eta^A)\big) \circ (r^{\otimes}_B)^{-1} \tag{4.3}$$

is both left and right inverse to f. As compared to the calculation for 'ordinary' Frobenius algebras, the only new ingredient is the use of the naturality of the distributor δ^{r} in place of the naturality of the associator (which one would normally just suppress). □

Next we establish alternative characterizations of GV-Frobenius algebras that are equivalent to Definition 4.4 by preparing some suitable notions.

Definition 4.7. Let c be an object in a GV-category $\mathcal{C}$. A *GV-pairing* on c is a morphism $\kappa_c \in \mathrm{Hom}(c \otimes c, K)$; a *GV-copairing* on c is a morphism $\overline{\kappa}_c \in \mathrm{Hom}(1, c \circledast c)$. A GV-pairing on c is called *non-degenerate* if and only if there exists a GV-copairing $\overline{\kappa}_c$ that is side-inverse to κ, i.e. such that

$$
\begin{aligned}
& l_c^{\varotimes} \circ (\kappa_c \varotimes \mathrm{id}_c) \circ \delta^{\mathrm{l}}_{c,c,c} \circ (\mathrm{id}_c \otimes \overline{\kappa}_c) \circ r_c^{-1} \\
& \qquad = \mathrm{id}_c = r_c^{\varotimes} \circ (\mathrm{id}_c \varotimes \kappa_c) \circ \delta^{\mathrm{r}}_{c,c,c} \circ (\overline{\kappa}_c \otimes \mathrm{id}_c) \circ l_c^{-1}
\end{aligned} \tag{4.4}
$$

as morphisms in $\mathrm{End}(c)$.

It readily follows that the adjunctions $\mathrm{Hom}(c \otimes c, K) \cong \mathrm{Hom}(c, Gc) \cong \mathrm{Hom}(1, Gc \varotimes Gc)$ provide a bijection between the sets of GV-pairings for c and of GV-copairings for Gc. Under this bijection, non-degenerate (co)pairings correspond to each other.

Definition 4.8. Let A be an algebra in a GV-category $\mathcal{C}$. An *invariant* GV-pairing on A is a GV-pairing κ_A on A such that

$$
\kappa_A \circ (\mu \otimes \mathrm{id}_A) = \kappa_A \circ (\mathrm{id}_A \otimes \mu) \circ \alpha_{A,A,A} \tag{4.5}
$$

as morphisms in $\mathrm{Hom}((A \otimes A) \otimes A, K)$.

As is familiar from ordinary Frobenius algebras, invariant GV-pairings provide an alternative means of characterization:

Proposition 4.9. *For an algebra A in $\mathcal{C}$ there is a bijection between the GV-Frobenius algebra structures on A (in the sense of Definition 4.4) and the invariant non-degenerate pairings for A.*

Proof. Assume that $(A, \mu, \eta, \Delta, \varepsilon)$ is a GV-Frobenius algebra. Define a GV-pairing κ and a GV-copairing $\overline{\kappa}$ on A by $\kappa := \varepsilon \circ \mu$ and $\overline{\kappa} := \Delta \circ \eta$. By the associativity of μ, the GV-pairing κ is invariant. Moreover, the calculation

$$
\begin{aligned}
& l_A^{\varotimes} \circ (\kappa \varotimes \mathrm{id}_A) \circ \delta^{\mathrm{l}}_{A,A,A} \circ (\mathrm{id}_A \otimes \overline{\kappa}) \circ r_A^{-1} \\
& \qquad = l_A^{\varotimes} \circ (\varepsilon \varotimes \mathrm{id}_A) \circ (\mu \varotimes \mathrm{id}_A) \circ \delta^{\mathrm{l}}_{A,A,A} \circ (\mathrm{id}_A \otimes \Delta) \circ (\mathrm{id}_A \otimes \eta) \circ r_A^{-1} \\
& \qquad \overset{(4.2)}{=} l_A^{\varotimes} \circ (\varepsilon \varotimes \mathrm{id}_A) \circ \Delta \circ \mu \circ (\mathrm{id}_A \otimes \eta) \circ r_A^{-1} = \mathrm{id}_A
\end{aligned} \tag{4.6}
$$

shows that $\kappa = \kappa_A$ and $\overline{\kappa} = \overline{\kappa}_A$ satisfy the first of the equalities (4.4) for $c = A$. The second of those equalities follows analogously. Thus the GV-pairing κ is non-degenerate.

To show the converse, define, for (A, μ, η) an algebra in $\mathcal{C}$ and κ an invariant non-degenerate GV-pairing on A with side-inverse $\overline{\kappa}$,

$$
\Delta := (\mu \varotimes \mathrm{id}_A) \circ \delta^{\mathrm{l}}_{A,A,A} \circ (\mathrm{id}_A \otimes \overline{\kappa}) \circ r_A^{-1} \qquad \text{and} \qquad \varepsilon := \kappa \circ (\mathrm{id}_A \otimes \eta) \circ r_A^{-1}. \tag{4.7}
$$

Then with the help of the invariance property (4.5) of κ one sees that ε can alternatively be written as $\varepsilon = \kappa \circ (\eta \otimes \mathrm{id}_A) \circ l_A^{-1}$. Similarly, for brevity suppressing for the moment unitors as well as associators, one has

$$
\mu = (\kappa \varotimes \mathrm{id}_A) \circ \big((\mathrm{id}_A \otimes \mu) \varotimes \mathrm{id}_A \big) \circ \delta^{\mathrm{l}}_{A,A \otimes A,A} \circ (\mathrm{id}_A \otimes \delta^{\mathrm{l}}_{A,A,A}) \circ (\mathrm{id}_A \otimes \mathrm{id}_A \otimes \overline{\kappa}) \tag{4.8}
$$

which, in turn, implies that Δ can alternatively be written as

$$\Delta = (\mathrm{id}_A \circledast \mu) \circ \delta^{\mathrm{r}}_{A,A,A} \circ (\overline{\kappa} \otimes \mathrm{id}_A)\,. \tag{4.9}$$

The calculation

$$\begin{aligned}(\mathrm{id}_A \circledast \varepsilon) \circ \Delta &\equiv (\mathrm{id}_A \circledast \kappa) \circ \delta^{\mathrm{r}}_{A,A,A} \circ \big((\mu \circledast \mathrm{id}_A) \otimes \eta\big) \circ \delta^{\mathrm{l}}_{A,A,A} \circ (\mathrm{id}_A \otimes \overline{\kappa}) \\ &= \mu \circ (\mathrm{id}_A \otimes \eta) = \mathrm{id}_A\end{aligned} \tag{4.10}$$

then proves one of the counit properties. Analogously one shows that $(\varepsilon \circledast \mathrm{id}_A) \circ \Delta = \mathrm{id}_A$. Next we calculate, denoting for better distinction the two expressions (4.7) and (4.9) for Δ by different symbols Δ_1 and Δ_2, respectively,

$$\begin{aligned}(\mathrm{id}_A \circledast \Delta_1) \circ \Delta_2 &= (\mathrm{id}_A \circledast (\mu \circledast \mathrm{id}_A)) \circ (\mathrm{id}_A \circledast \delta^{\mathrm{l}}_{A,A,A}) \circ (\mathrm{id}_A \circledast (\mathrm{id}_A \otimes \overline{\kappa})) \\ &\qquad \circ (\mathrm{id}_A \circledast r_A^{-1}) \circ (\mathrm{id}_A \circledast \mu) \circ \delta^{\mathrm{r}}_{A,A,A} \circ (\overline{\kappa} \otimes \mathrm{id}_A) \circ l_A^{-1} \\ &= \big[\, \mathrm{id}_A \circledast \big[\big(\mu \circ (\mu \otimes \mathrm{id}_A)\big) \circledast \mathrm{id}_A\big]\big] \circ (\mathrm{id}_A \circledast \delta^{\mathrm{l}}_{A\otimes A,A,A}) \circ \delta^{\mathrm{r}}_{A,A\otimes A,A\circledast A} \\ &\qquad \circ (\delta^{\mathrm{r}}_{A,A,A} \otimes \mathrm{id}_{A\circledast A}) \circ ((\overline{\kappa} \otimes \mathrm{id}_A) \otimes \overline{\kappa}) \circ ((l_A^{-1} \otimes \mathrm{id}_A) \otimes \mathrm{id}_1) \circ r_A^{-1},\end{aligned} \tag{4.11}$$

where we use naturality of the distributors (as well as of the unitors). Similarly we obtain

$$\begin{aligned}(\Delta_2 \circledast \mathrm{id}_A) \circ \Delta_1 &= ((\mathrm{id}_A \circledast \mu) \circledast \mathrm{id}_A) \circ (\delta^{\mathrm{r}}_{A,A,A} \circledast \mathrm{id}_A) \circ ((\overline{\kappa} \otimes \mathrm{id}_A) \circledast \mathrm{id}_A) \\ &\qquad \circ (l_A^{-1} \circledast \mathrm{id}_A) \circ (\mu \circledast \mathrm{id}_A) \circ \delta^{\mathrm{l}}_{A,A,A} \circ (\mathrm{id}_A \otimes \overline{\kappa}) \circ r_A^{-1} \\ &= \big[\big[\, \mathrm{id}_A \circledast \big(\mu \circ (\mathrm{id}_A \otimes \mu)\big)\big] \circledast \mathrm{id}_A\big] \circ \big((\mathrm{id}_A \circledast \alpha^{\circledast}_{A,A,A}) \circledast \mathrm{id}_A\big) \\ &\qquad \circ \alpha^{\circledast}_{A,(A\otimes A)\otimes A,A} \circ (\delta^{\mathrm{r}}_{A,A\otimes A,A} \circledast \mathrm{id}_A) \circ \delta^{\mathrm{l}}_{A\circledast(A\otimes A),A,A} \\ &\qquad \circ (\delta^{\mathrm{r}}_{A,A,A} \otimes \mathrm{id}_{A\circledast A}) \circ ((\overline{\kappa} \otimes \mathrm{id}_A) \otimes \overline{\kappa}) \circ ((l_A^{-1} \otimes \mathrm{id}_A) \otimes \mathrm{id}_1) \circ r_A^{-1},\end{aligned} \tag{4.12}$$

where now besides naturality we also make use of the pentagon identity (3.64) for the left distributor, specialized to the regular GV-module category and $x = A \circledast A$ and $y = z = m = A$, followed by the pentagon identity (3.63) for the right distributor with $x = y = z = m = A$ ($\circledast$-multiplied with id_A). Invoking now the mixed pentagon identity (3.67) with $x = z = m = A$ and $y = A \otimes A$, and associativity of μ, it follows that

$$(\mathrm{id}_A \circledast \Delta) \circ \Delta = \alpha^{\circledast}_{A,A,A} \circ (\Delta \circledast \mathrm{id}_A) \circ \Delta\,, \tag{4.13}$$

i.e. that Δ is a coassociative comultiplication for the $\circledast$-tensor product.

The validity of the Frobenius relations (4.2) follows similarly as coassociativity, by expressing the coproduct Δ suitably either as in (4.7) or as in (4.9) and then making use of associativity of μ and of the properties of the distributors. $\square$

The result of Proposition 4.9 can be promoted to an equivalence of categories (in fact, by Lemma 4.6, of groupoids):

Definition 4.10. For $\mathcal{C}$ a GV-category, $\mathcal{A}lg^{\kappa}_{\mathcal{C}}$ is the following category: An object in $\mathcal{A}lg^{\kappa}_{\mathcal{C}}$ is a pair (A,κ) consisting of an algebra A in $\mathcal{C}$ and an invariant non-degenerate pairing κ on A. A morphism $f\colon (A,\kappa_A)\longrightarrow (B,\kappa_B)$ in $\mathcal{A}lg^{\kappa}_{\mathcal{C}}$ is an algebra morphism $f\colon A\longrightarrow B$ such that $\kappa_B\circ(f\otimes f)=\kappa_A$.

Proposition 4.11. *The category $\mathcal{A}lg^{\kappa}_{\mathcal{C}}$ is equivalent to the category of GV-Frobenius algebras in $\mathcal{C}$. In particular, $\mathcal{A}lg^{\kappa}_{\mathcal{C}}$ is a groupoid.*

Proof. Using the formulas relating the two equivalent definitions of Frobenius algebra one directly checks that an algebra morphism f in $\mathcal{A}lg^{\kappa}_{\mathcal{C}}$ is a coalgebra morphism (and thus a morphism of Frobenius algebras) iff $\kappa_B\circ(f\otimes f)=\kappa_A$. $\square$

A third equivalent definition of the notion of a GV-Frobenius algebra is obtained with the help of the following structure:

Definition 4.12. Let $A\in\mathcal{C}$ be an algebra in a GV-category $\mathcal{C}$. A *Frobenius form for A* is a morphism $\lambda\colon A\longrightarrow K$ such that the morphism

$$\begin{aligned}\Psi\equiv\Psi_\lambda:\quad & A\xrightarrow{(r^{\otimes}_A)^{-1}} A\otimes 1\xrightarrow{\mathrm{id}_A\otimes\mathrm{coev}^{\mathrm{r}}_A} A\otimes(A\odot G(A))\\ & \xrightarrow{\delta^{\mathrm{l}}_{A,A,GA}}(A\otimes A)\odot G(A)\xrightarrow{(\lambda\circ\mu)\odot\mathrm{id}_{G(A)}} K\odot G(A)\xrightarrow{l^{\odot}_{GA}} G(A)\end{aligned}\tag{4.14}$$

is invertible.

Proposition 4.13. *Let $A\in\mathcal{C}$ be an algebra in a GV-category $\mathcal{C}$. If λ is a Frobenius form for A, then the morphism*

$$\kappa:=\lambda\circ\mu\tag{4.15}$$

is a non-degenerate invariant GV-pairing on A. Conversely, for κ a non-degenerate invariant GV-pairing on A, the morphism

$$\lambda:=\kappa\circ(\mathrm{id}_A\otimes\eta)\tag{4.16}$$

is a Frobenius form for A.

Proof. Let λ be a Frobenius form and define κ by (4.15). By associativity of μ, κ is invariant. Further, set

$$\overline{\kappa}:=(\mathrm{id}_A\odot\Psi^{-1})\circ\mathrm{coev}^{\mathrm{r}}_A:\quad 1\longrightarrow A\odot A\tag{4.17}$$

with Ψ^{-1} the inverse of $\Psi\equiv\Psi_\lambda$. By naturality of $l^{\odot}$ and of δ^{l} we have

$$\mathrm{id}_A\equiv\Psi^{-1}\circ\Psi=l^{\odot}_A\circ(\kappa\odot\mathrm{id}_A)\circ\delta^{\mathrm{l}}_{A,A,A}\circ\big(\mathrm{id}_A\otimes[(\mathrm{id}_A\odot\Psi^{-1})\circ\mathrm{coev}^{\mathrm{r}}_A)]\big)\circ(r^{\otimes}_A)^{-1},\tag{4.18}$$

which means that $\overline{\kappa}$ satisfies the first of the side-inverseness equalities (4.4). Similarly, using naturality of r^{-1} and δ^{l} one sees that the identity $\Psi \circ \Psi^{-1} = \mathrm{id}_A$ amounts to $\overline{\kappa}$ satisfying the second of those equalities. Thus $\overline{\kappa}$ is non-degenerate.

Conversely, let κ be a non-degenerate invariant GV-pairing on A and define λ by (4.16). Then we have $\Psi = (\kappa \circledast \mathrm{id}_{GA}) \circ \delta^{\mathrm{l}}_{A,A.GA} \circ (\mathrm{id}_A \otimes \mathrm{coev}^{\mathrm{r}}_A)$. We claim that Ψ has a two-sided inverse given by

$$\Psi^{-1} = (\mathrm{ev}^{\mathrm{r}}_A \circledast \mathrm{id}_A) \circ \delta^{\mathrm{r}}_{GA,A,A} \circ (\mathrm{id}_{GA} \otimes \overline{\kappa}) \,. \tag{4.19}$$

That this morphism is a right inverse is seen by computing

$$\Psi \circ \Psi^{-1} = (\mathrm{ev}^{\mathrm{r}}_A \circledast \mathrm{id}_A) \circ \delta^{\mathrm{l}}_{GA,A,GA} \circ \big(\mathrm{id}_{GA} \otimes \big[(\mathrm{id}_A \circledast \kappa) \circ \delta^{\mathrm{r}}_{A,A,A} \circ (\overline{\kappa} \otimes \mathrm{id}_A)\big] \circledast \mathrm{id}_{GA}\big) \\ \circ (\mathrm{id}_{GA} \otimes \mathrm{coev}^{\mathrm{r}}_A) \tag{4.20}$$

and noticing that the term in square brackets in this expression equals id_A by the second of the equalities (4.4), so that after invoking the snake identity (3.88) we end up with id_{GA}. That the morphism (4.19) is also a left inverse of Ψ follows similarly with the help of the snake identity (3.87) and the first of the equalities (4.4). □

Together with Proposition 4.11 we have thus arrived at:

Theorem 4.14. *For a GV-category $\mathcal{C}$ the following three groupoids are equivalent:*

1. *The category of Frobenius algebras in $\mathcal{C}$ as described in Definition 4.4.*
2. *The category $\mathcal{A}lg^{\kappa}_{\mathcal{C}}$ introduced in Definition 4.10.*
3. *The category of pairs (A, λ) consisting of an algebra A in $\mathcal{C}$ and a Frobenius form λ on A.*

Remark 4.15. Replacing Ψ in the Definition 4.12 of a Frobenius form by the morphism

$$\Psi' := r^{\circledast}_{G^{-1}A} \circ (\mathrm{id}_{G^{-1}A} \circledast \kappa) \circ \delta^{\mathrm{r}}_{G^{-1}A,A,A} \circ (\mathrm{coev}^{\mathrm{l}}_A \otimes \mathrm{id}_A) \circ l_A^{-1} \tag{4.21}$$

from A to $G^{-1}A$ one obtains statements analogous to Proposition 4.13 in which the equality (4.16) is replaced by $\lambda = \kappa \circ (\eta \otimes \mathrm{id}_A)$.

4.2. Symmetric Frobenius algebras

Like in the rigid case there is a notion of symmetric Frobenius algebra, provided that the GV-category $\mathcal{C}$ is endowed with a pivotal structure.

Definition 4.16. [4, Def. 6.1] A *pivotal structure* on a GV-category $\mathcal{C}$ is a natural family of isomorphisms

$$\psi_{c,d}: \quad \mathrm{Hom}(c \otimes d, K) \xrightarrow{\cong} \mathrm{Hom}(d \otimes c, K) \tag{4.22}$$

for $c, d \in \mathcal{C}$ such that (suppressing associators)

$$\psi_{d,c} \circ \psi_{c,d} = \mathrm{id} \qquad \text{and} \qquad \psi_{b\otimes c,d} \circ \psi_{c\otimes d,b} \circ \psi_{d\otimes b,c} = \mathrm{id} \tag{4.23}$$

for $b, c, d \in \mathcal{C}$. A *pivotal GV-category* is a GV-category together with a choice of a pivotal structure.

As shown by the transformation behavior (2.6) of the functor G^2 under a change of dualizing object, the notion of a pivotal structure depends on the choice of dualizing object. Being pivotal is thus not a property of the underlying monoidal category only. In particular, the existence of a pivotal structure for one choice of dualizing object does not guarantee that a pivotal structure also exists for a different choice.

Pivotal structures on $\mathcal{C}$ are in bijection with monoidal isomorphisms $\pi\colon \mathrm{id}_{\mathcal{C}} \Rightarrow G^2$ for which $\pi_K\colon K \xrightarrow{\cong} G^2(K)$ coincides with the canonical isomorphism $K \xrightarrow{\cong} G1 = G^2G^{-1}1 \xrightarrow{\cong} G^2K$ [4, Prop. 6.7]. Conveniently, this monoidal isomorphism π allows us to construct Nakayama automorphisms:

Definition 4.17. Let A be a Frobenius algebra in a pivotal GV-category $\mathcal{C}$. Then the left and right *Nakayama automorphisms* of A are the invertible endomorphisms

$$\mho^{\mathrm{l}}_A := \Psi'^{-1} \circ \pi^{-1}_{G^{-1}A} \circ \Psi \qquad \text{and} \qquad \mho^{\mathrm{r}}_A := (\mho^{\mathrm{l}}_A)^{-1} = \Psi^{-1} \circ \pi_{G^{-1}A} \circ \Psi' \tag{4.24}$$

of A, respectively, with Ψ and Ψ' as defined in (4.14) and (4.21).

A *symmetric Frobenius algebra* A in $\mathcal{C}$ is a Frobenius algebra in $\mathcal{C}$ for which the Nakayama automorphisms are identities, $\mho^{\mathrm{l}}_A = \mathrm{id}_A$.

Proposition 4.18. *A Frobenius algebra A in a pivotal GV-category is symmetric if and only if the corresponding invariant GV-pairing κ is symmetric, in the sense that the equality*

$$\mathrm{ev}_A \circ \big[\mathrm{id}_A \otimes \big(\pi^{-1}_{G^{-1}A} \circ r^{\otimes}_{GA} \circ (\kappa \odot \mathrm{id}_{GA}) \circ \delta^{\mathrm{l}}_{A,A,GA} \circ (\mathrm{id}_A \otimes \mathrm{coev}_A)\big)\big] = \kappa \tag{4.25}$$

holds.

Proof. The claim follows directly by using the relation (4.19) between Ψ and κ and the corresponding relation for Ψ'. $\square$

Remark 4.19. (i) The proof is fully analogous to the rigid case, which is treated in Section 4 of [14].

(ii) Symmetric Frobenius algebras in *symmetric* linearly distributive categories have been considered, under the name *Girard monoids*, in [10]. In that case the definition of the algebra being symmetric can be expressed in terms of the (symmetric) braiding of the category.

4.3. GV-module categories versus categories of modules

A natural question to ask is under which conditions the category of modules over an algebra in a GV-category is a GV-module category, and conversely, under which conditions a GV-module category comes from an algebra. To investigate the first issue, recall from Lemma 4.3 that, given an algebra $A \in (\mathcal{C}, \otimes)$, the object $G(A)$ has a canonical structure of a coalgebra in $(\mathcal{C}, \circledast)$. The analogous result interrelating modules and comodules holds as well:

Lemma 4.20. *Let $\mathcal{C}$ be a GV-category, $A \in \mathcal{C}$ a GV-algebra and $m \in \mathcal{C}$. There are canonical bijections of*

1. *right A-actions on m and right $G(A)$-coactions on m;*
2. *left A-actions on m and left $G^{-1}(A)$-coactions on m;*
3. *right A-actions on m, left $G(A)$-coactions on $G(m)$ and left $G^{-1}(A)$-coactions on $G^{-1}(m)$;*
4. *left A-actions on m, right $G(A)$-coactions on $G(m)$ and right $G^{-1}(A)$-coactions on $G^{-1}(m)$.*

Proof. The bijection in Part 1 is given by the adjunction $\mathrm{Hom}(m \otimes A, m) \cong \mathrm{Hom}(m, m \circledast G(A))$. To see this, first note that the comultiplication Δ considered in the proof of Lemma 4.3 coincides with the image of μ under the sequence

$$\begin{aligned} \mathrm{Hom}(A \otimes A, A) \xrightarrow{\cong} \mathrm{Hom}(A, A \circledast GA) &\xrightarrow{\cong} \mathrm{Hom}(1, A \circledast GA \circledast GA) \\ &\xrightarrow{\cong} \mathrm{Hom}(GA, GA \circledast GA) \end{aligned} \tag{4.26}$$

of adjunctions. This follows directly from Part 3 of Proposition 3.37 together with Proposition 3.39. Then one sees that the associativity constraint on one side is mapped to the coassociativity constraint on the other. Part 2 follows analogously. Parts 3 and 4 are obtained by applying G or G^{-1} to (co)actions and using the fact that G is an antiequivalence. □

Proposition 4.21. *Let $\mathcal{C}$ be a GV-category with equalizers. For any algebra $A \in \mathcal{C}$ the category $\mathcal{M} = \mathrm{mod}\text{-}A$ is a left GV-module category over $\mathcal{C}$. The module distributors of $\mathrm{mod}\text{-}A$ are the distributors of $\mathcal{C}$.*

Proof. According to Definition 3.1 we must show that the functors $- \triangleright m$ and $c \triangleright -$ admit right adjoints, for $m \in \mathcal{M}$ and $c \in \mathcal{C}$, respectively. To see that $- \triangleright m$ has a right adjoint, define for $(n, \reflectbox{$\rho$})$ a right GA-comodule and (x, ρ) a left GA-comodule the object $n \circledast^A x$ as the equalizer

$$n \circledast^A x \xrightarrow{q_{n,x}} n \circledast x \underset{\mathrm{id}_n \circledast \rho}{\overset{\reflectbox{ρ} \circledast \mathrm{id}_x}{\rightrightarrows}} n \circledast GA \circledast x \,. \tag{4.27}$$

Then for $m, n \in \text{mod-}A$ consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(c \otimes m, n) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(c, n \circledast^A G(m)) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(c, q_{n,G(m)})"] \\
\mathrm{Hom}_{\mathcal{C}}(c \otimes m, n) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{C}}(c, n \circledast G(m))
\end{tikzcd} \tag{4.28}$$

where the left vertical arrow is obtained from the forgetful functor and the isomorphism in the bottom row is the adjunction (2.8). Using the universal property of the equalizer one sees that this diagram commutes, with the horizontal arrow in the top row an isomorphism. Thus we have

$$\mathrm{Hom}_A(c \otimes m, n) \cong \mathrm{Hom}_{\mathcal{C}}(c, n \circledast^A G(m)) \tag{4.29}$$

for $m, n \in \text{mod-}A$, which shows that $- \triangleright m$ has $\underline{\mathrm{Hom}}(m, -) = - \circledast^A G(m)$ as a right adjoint.

It remains to show that the action of $c \in \mathcal{C}$ on mod-A has a right adjoint. To this end we first note that for $c \in \mathcal{C}$ and $(m, \reflectbox{$\rho$}) \in \text{mod-}A$ the composite

$$(c \circledast m) \otimes A \xrightarrow{\delta^{\mathrm{r}}} c \circledast (m \otimes A) \xrightarrow{\mathrm{id} \circledast \reflectbox{ρ}} c \circledast m \tag{4.30}$$

furnishes a right A-module structure on the object $c \circledast m$. From the basic adjunction (2.9) we then directly get the desired adjunction

$$\mathrm{Hom}_A(c \otimes m, n) \cong \mathrm{Hom}_A(m, G^{-1}c \circledast n) \,. \tag{4.31}$$

Thus indeed $\mathcal{M} = \text{mod-}A$ is a GV-module category. The statement about the module distributors is obtained by applying Proposition 3.31 to the strong $\triangleright$-module functor $U\colon \text{mod-}A \to \mathcal{C}$ that forgets the A-module structure. □

Next we ask, conversely, when a GV-module category over a GV-category $\mathcal{C}$ comes from an algebra in $\mathcal{C}$. To analyze this issue, first recall that for an algebra $A \in (\mathcal{C}, \otimes)$ the $\mathcal{C}$-module category mod-A of right A-modules comes with a forgetful functor

$$U: \quad \mathcal{M} \simeq \text{mod-}A \longrightarrow \mathcal{C} \,. \tag{4.32}$$

The left adjoint of U is the induction functor $\mathrm{Ind}_A\colon \mathcal{C} \to \mathcal{M}$, mapping objects as $x \mapsto x \otimes A$. Dual statements hold for comodules and coinduction.

We also note that a $\mathcal{C}$-*generator* [9, Lemma 2.22] of a GV-module category $\mathcal{M}$ over $\mathcal{C}$ is an object $m_0 \in \mathcal{M}$ such that for every $m \in \mathcal{M}$ there exists an object $c \in \mathcal{C}$ with an epimorphism $c \triangleright m_0 \longrightarrow m$. Analogously we call an object $n_0 \in \mathcal{C}$ a $\mathcal{C}$-*cogenerator* of $\mathcal{M}$ if for every $m \in \mathcal{M}$ there exists an object $d \in \mathcal{C}$ with a monomorphism $m \longrightarrow d \blacktriangleright m_0$.

Lemma 4.22. *Any algebra $A = (A, \mu_A, \eta_A)$ in a monoidal category $(\mathcal{C}, \otimes)$ is a $\mathcal{C}$-generator of the $\mathcal{C}$-module category* mod-A *of right A-modules.*

Proof. For any object $m \in \text{mod-}A$ the A-action $m \otimes A \xrightarrow{\rho} m$ on m amounts to a morphism $\rho\colon \mathrm{Ind}_A(U(m)) \longrightarrow m$ which is a morphism in mod-A. This morphism has the unit as a right inverse and is thus an epimorphism. □

Now let $\mathcal{M}$ be a GV-module category. For any $m_0 \in \mathcal{M}$, the object $A_{m_0} := \underline{\mathrm{Hom}}(m_0, m_0) \in \mathcal{C}$ has a natural structure of an algebra in $\mathcal{C}$, and there is a natural functor

$$\begin{aligned} \underline{\mathrm{Hom}}(m_0, -) : \quad \mathcal{M} &\longrightarrow \text{mod-}A_{m_0} \,, \\ m &\longmapsto \underline{\mathrm{Hom}}(m_0, m) \,. \end{aligned} \tag{4.33}$$

Similarly there is a functor $\underline{\mathrm{coHom}}(m_0, -)$ from $\mathcal{M}$ to the category comod-C_{m_0} of C_{m_0}-comodules. Our goal is now to specify conditions under which these functors give us equivalences of GV-module categories.

To get in a position to do so we introduce a particular subclass of objects in a GV-module category $\mathcal{M}$, namely those for which the internal Hom is a strong module functor. This allows us to generalize the notion of dualizability to the module setting. We will use the abbreviations

$$\underline{\mathrm{Hom}}(m, -) =: Y_m \qquad \text{and} \qquad \underline{\mathrm{coHom}}(m, -) =: W_m \,. \tag{4.34}$$

Definition 4.23. Let $\mathcal{M}$ be a GV-module category over a GV-category $\mathcal{C}$.

1. An object $m \in \mathcal{M}$ is called $\otimes$-*admissible* if the lax $\triangleright$-module functor $Y_m\colon \mathcal{M} \longrightarrow \mathcal{C}$ is in fact a strong $\triangleright$-module functor and has a right adjoint.
2. An object $m \in \mathcal{M}$ is called $\circledast$-*admissible* if the oplax $\blacktriangleright$-module functor $W_m\colon \mathcal{M} \longrightarrow \mathcal{C}$ is in fact a strong $\blacktriangleright$-module functor and has a left adjoint.
3. We denote by $\widehat{\mathcal{M}}^{\otimes}$ and $\widehat{\mathcal{M}}^{\circledast}$ the full subcategories of $\mathcal{M}$ on the $\otimes$- and $\circledast$-admissible objects, respectively.
4. In particular, the subcategories $\widehat{\mathcal{C}}^{\otimes}$ and $\widehat{\mathcal{C}}^{\circledast}$ of $\mathcal{C}$ are the full subcategories on the $\otimes$- and $\circledast$-admissible objects that are obtained by regarding $\mathcal{C}$ as a left GV-module category over itself.

Note that the functor $Y_m = \underline{\mathrm{Hom}}(m, -)$ always has a left adjoint and that $W_m = \underline{\mathrm{coHom}}(m, -)$ has a right adjoint. In general, a strong module functor need not have an adjoint. For instance, a linear functor $F\colon \mathcal{M} \longrightarrow \mathrm{vect}$ from a linear category $\mathcal{M}$ is a strong vect-module functor, but having a left adjoint requires F to be representable.

Also note that by (3.104) we have

$$Y_1 = \underline{\mathrm{Hom}}_{\mathcal{C}}(1,-) \cong \mathrm{id}_{\mathcal{C}} \cong \underline{\mathrm{coHom}}_{\mathcal{C}}(K,-) = W_K \tag{4.35}$$

and hence

$$1 \in \widehat{\mathcal{C}}^{\otimes} \qquad \text{and} \qquad K \in \widehat{\mathcal{C}}^{\boldsymbol{\otimes}}. \tag{4.36}$$

Proposition 4.24.

1. *The subcategories $\widehat{\mathcal{C}}^{\otimes}$ and $\widehat{\mathcal{C}}^{\boldsymbol{\otimes}}$ of a GV-category $\mathcal{C}$ are monoidal subcategories.*
2. *Let $\mathcal{M}$ be a left GV-module category over $\mathcal{C}$. By restriction of the action of $\mathcal{C}$ on $\mathcal{M}$, the category $\widehat{\mathcal{M}}^{\otimes}$ is a left $\widehat{\mathcal{C}}^{\otimes}$-module category, while $\widehat{\mathcal{M}}^{\boldsymbol{\otimes}}$ is a left $\widehat{\mathcal{C}}^{\boldsymbol{\otimes}}$-module category.*
3. *Let $\mathcal{C}$ and $\mathcal{M}$ be in addition abelian. Then for any projective object $p \in \mathcal{C}$ and any $m \in \widehat{\mathcal{M}}^{\otimes}$ the object $p \triangleright m$ is projective, and for any injective object $q \in \mathcal{C}$ and any $n \in \widehat{\mathcal{M}}^{\boldsymbol{\otimes}}$ the object $q \blacktriangleright n$ is injective.*

Proof. According to Proposition 3.45.7, for all $b, c \in \mathcal{C}$ and all $m \in \mathcal{M}$ there are natural isomorphisms $\underline{\mathrm{Hom}}_{\mathcal{C}}(b \triangleright c, -) \cong \underline{\mathrm{Hom}}_{\mathcal{C}}(b, \underline{\mathrm{Hom}}_{\mathcal{C}}(c,-))$ and $\underline{\mathrm{Hom}}_{\mathcal{M}}(b \triangleright m, -) \cong \underline{\mathrm{Hom}}_{\mathcal{C}}(b, \underline{\mathrm{Hom}}_{\mathcal{M}}(m,-))$ which are isomorphisms of lax $\triangleright$-module functors. The composition of strong $\triangleright$-module functors having a right adjoint is again a strong $\triangleright$-module functor having a right adjoint. Hence together with (4.36), the first of those isomorphisms implies that $\widehat{\mathcal{C}}^{\otimes}$ is monoidal, while the second isomorphism shows that $\widehat{\mathcal{M}}^{\otimes}$ is a left $\widehat{\mathcal{C}}^{\otimes}$-module category. The statements about $\widehat{\mathcal{C}}^{\boldsymbol{\otimes}}$ and $\widehat{\mathcal{M}}^{\boldsymbol{\otimes}}$ follow in an analogous manner from Proposition 3.45 as well. Statement 3 follows in the projective case directly from the fact that the composition of the exact functors $\mathrm{Hom}_{\mathcal{C}}(p,-)$ and $\underline{\mathrm{Hom}}(m,-)$ is again exact; an analogous argument applies in the injective case. □

Remark 4.25. If in the definition of a projective (injective) object we think of a right (left) exact functor as a functor that preserves finite colimits (limits), then part 3 of Proposition 4.24 remains valid without the assumption that $\mathcal{C}$ and $\mathcal{M}$ are abelian.

We are now in a position to give sufficient conditions for a GV-module category to be equivalent to a category of modules or comodules.

Proposition 4.26. *Let $\mathcal{C}$ be a finite abelian GV-category and $\mathcal{M}$ a finite abelian GV-module over $\mathcal{C}$.*

1. *For $m_0 \in \mathcal{M}$, the functor $F := \underline{\mathrm{Hom}}(m_0,-)\colon \mathcal{M} \longrightarrow \text{mod-}A_{m_0}$ is an equivalence of $(\mathcal{C},\otimes)$-module categories if and only if $m_0 \in \widehat{\mathcal{M}}^{\otimes}$ and m_0 is a $\mathcal{C}$-generator of $\mathcal{M}$.*

2. *For $n_0 \in \mathcal{M}$, the functor $\underline{\mathrm{coHom}}(n_0,-)\colon \mathcal{M} \longrightarrow \text{comod-}C_{n_0}$ is an equivalence of $(\mathcal{C}, \otimes)$-module categories if and only if $n_0 \in \widehat{\mathcal{M}}^{\otimes}$ and n_0 is a $\mathcal{C}$-cogenerator of $\mathcal{M}$.*

Proof. We employ the strategy of [11, Thm 7.10.1]. We only show the first statement, the second follows by duality. Suppose that $m_0 \in \widehat{\mathcal{M}}^{\otimes}$ is a $\mathcal{C}$-generator.

1. Assume first that $m \in \mathcal{M}$ is of the form $m = c \triangleright m_0$ with some $c \in \mathcal{C}$. Using the fact that owing to $m_0 \in \widehat{\mathcal{M}}^{\otimes}$ the functor $\underline{\mathrm{Hom}}(m_0,-)$ is a strong module functor, we then have

$$F(m) = \underline{\mathrm{Hom}}(m_0, c \triangleright m_0) \cong c \otimes \underline{\mathrm{Hom}}(m_0, m_0) = c \triangleright A_{m_0} = \mathrm{Ind}_{A_{m_0}}(c) \,. \tag{4.37}$$

   Thus any object of $\mathcal{M}$ of the form $c \triangleright m_0$ is mapped by F to an induced A_{m_0}-module. Henceforth we drop the index m_0 and write $A := A_{m_0}$.
2. For all $m_1 \in \mathcal{M}$ of the form $c \triangleright m_0$ with $c \in \mathcal{C}$ and all $m_2 \in \mathcal{M}$ we have

$$\begin{aligned} \mathrm{Hom}_A(F(m_1), F(m_2)) &\cong \mathrm{Hom}_A(\mathrm{Ind}_A(c), F(m_2)) \cong \mathrm{Hom}_{\mathcal{C}}(c, UF(m_2)) \\ &= \mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(m_0, m_2)) \cong \mathrm{Hom}_{\mathcal{M}}(c \triangleright m_0, m_2) \\ &= \mathrm{Hom}_{\mathcal{M}}(m_1, m_2) \,. \end{aligned} \tag{4.38}$$

   Hence for such objects m_1 and m_2 the map $F\colon \mathrm{Hom}_{\mathcal{M}}(m_1, m_2) \longrightarrow \mathrm{Hom}_A(F(m_1), F(m_2))$ is an isomorphism.
3. By assumption, for every $m_1 \in \mathcal{M}$ there is an exact sequence $c_1 \triangleright m_0 \longrightarrow c_2 \triangleright m_0 \longrightarrow m_1 \longrightarrow 0$ for some $c_1, c_2 \in \mathcal{C}$. Since F is exact, the sequence $F(c_1 \triangleright m_0) \longrightarrow F(c_2 \triangleright m_0) \longrightarrow F(m_1) \longrightarrow 0$ is exact as well. Moreover, since for any $m \in \mathcal{M}$, the functor $\mathrm{Hom}_{\mathcal{M}}(-, m)$ is left exact, the rows of the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathcal{M}}(m_1, m_2) \arrow[r] \arrow[d, "F"] & \mathrm{Hom}_{\mathcal{M}}(c_1 \triangleright m_0, m_2) \arrow[r] \arrow[d, "F"] & \mathrm{Hom}_{\mathcal{M}}(c_2 \triangleright m_0, m_2) \arrow[d, "F"] \\ 0 \arrow[r] & \mathrm{Hom}_A(F(m_1), F(m_2)) \arrow[r] & \mathrm{Hom}_A(F(c_1 \triangleright m_0), F(m_2)) \arrow[r] & \mathrm{Hom}_A(F(c_2 \triangleright m_0), F(m_2)) \end{tikzcd} \tag{4.39}$$

   are exact. By step 2 the second and the third vertical arrow in this diagram are isomorphisms. Thus (the four-version of) the five-lemma implies that the first vertical arrow is an isomorphism as well. Thus the map $F\colon \mathrm{Hom}_{\mathcal{M}}(m_1, m_2) \longrightarrow \mathrm{Hom}_A(F(m_1), F(m_2))$ is an isomorphism for arbitrary $m_1, m_2 \in \mathcal{M}$.
4. For any $L \in \text{mod-}A$ there exists an object $c_1 \in \mathcal{C}$ with a surjection $\mathrm{Ind}_A(c_1) \longrightarrow L$ (e.g. one can take $c_1 = U(L)$). Thus there is an exact sequence

$$\mathrm{Ind}_A(c_2) \xrightarrow{f_A} \mathrm{Ind}_A(c_1) \longrightarrow L \longrightarrow 0 \tag{4.40}$$

   for some $c_2 \in \mathcal{C}$, by which L is written as the cokernel of a morphism of induced modules. Further, using first step 2 and then step 1 we obtain a composite isomorphism

$$\mathrm{Hom}_A(\mathrm{Ind}_A(c_2), \mathrm{Ind}_A(c_1)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{M}}(c_2 \triangleright m_0, c_1 \triangleright m_0)\,. \tag{4.41}$$

Denote by $f_{\mathcal{M}} \in \mathrm{Hom}_{\mathcal{M}}(c_2 \triangleright m_0, c_1 \triangleright m_0)$ the image of the morphism f_A from (4.40) under the linear map (4.41). Let $\tilde{m} \in \mathcal{M}$ be the cokernel of $f_{\mathcal{M}}$. Since F is exact, we have $F(\tilde{m}) \cong L$. We have thus shown that the strong module functor $F = \underline{\mathrm{Hom}}(m_0, -)$ is essentially surjective.

Suppose now conversely that $m_0 \in \mathcal{M}$ is such that the internal Hom functor $F = \underline{\mathrm{Hom}}(m_0, -)$ is an equivalence of $\mathcal{C}$-module categories. Then F is, by definition, a strong module functor and thus, as it is an equivalence, F is exact. Since the forgetful functor $U\colon \text{mod-}A_{m_0} \longrightarrow \mathcal{C}$ is exact, too, it follows that the composite UF is exact as well, and hence that $m_0 \in \widehat{\mathcal{M}}^{\otimes}$. Further, the object $A_{m_0} = F(m_0)$ is clearly a $\mathcal{C}$-generator in mod-A_{m_0}; thus, since F is an equivalence, $m_0 \in \mathcal{M}$ is a $\mathcal{C}$-generator in $\mathcal{M}$. □

Remark 4.27. An alternative proof of Proposition 4.26 can be obtained with the help of monadicity theorems. [2] First note that for every $m_0 \in \mathcal{M}$ we have an adjunction

$$\mathrm{Hom}_{\mathcal{M}}(c \triangleright m_0, -) \cong \mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(m_0, -))\,, \tag{4.42}$$

which induces a monad $c \mapsto \underline{\mathrm{Hom}}(m_0, c \triangleright m_0)$ on $\mathcal{C}$. Now for $m_0 \in \widehat{\mathcal{M}}^{\otimes}$, the internal Hom functor $\underline{\mathrm{Hom}}(m_0, -)$ is a strong module functor, so we have an isomorphism

$$\underline{\mathrm{Hom}}(m_0, - \triangleright m_0) \cong - \otimes \underline{\mathrm{Hom}}(m_0, m_0) = - \otimes A_{m_0} \tag{4.43}$$

of functors. This is in fact an isomorphism of monads on $\mathcal{C}$. Thus the comparison functor $H\colon \mathcal{M} \longrightarrow \text{mod-}A_{m_0}$ sends $m \in \mathcal{M}$ to the object $\underline{\mathrm{Hom}}(m_0, m)$ with the expected A_{m_0}-module structure. According to the crude monadicity theorem [3, p. 108], the comparison functor H is an equivalence of categories if the following conditions are satisfied: $U := \underline{\mathrm{Hom}}(m_0, -)$ has a left adjoint and reflects isomorphisms, $\mathcal{M}$ has coequalizers of those reflexive pairs (f, g) for which (Uf, Ug) is a coequalizer, and U preserves those coequalizers. These conditions are indeed met: The left adjoint of U is, by definition, the action functor $- \triangleright m_0$; assuming $\mathcal{M}$ to be abelian guarantees the existence of coequalizers; and because of $m_0 \in \widehat{\mathcal{M}}^{\otimes}$, U is exact and thus in particular preserves coequalizers. Finally, faithful functors between abelian categories reflect isomorphisms, and by the argument that proves the implication $(1) \Longrightarrow (2)$ of Lemma 2.22 in [9], the functor U is faithful.

It can happen that the subcategory $\widehat{\mathcal{M}}^{\otimes}$ does not contain a $\mathcal{C}$-generator, e.g. it can be zero. In this case the $\mathcal{C}$-module category $\mathcal{M}$ cannot be written as the category of modules over any algebra in $\mathcal{C}$. As an illustration, consider the following example:

[2] We thank the anonymous referee for sharing this proof with us.

Example 4.28. Consider the semisimple category $\mathcal{C}$ with two isomorphism classes of simple objects, represented by objects 1 and x, and with tensor product $x \otimes x = 0$ [9, Example 2.20]. The category $\mathcal{M} = \mathrm{vect}_{\Bbbk}$ becomes a $\mathcal{C}$-module by setting $x \triangleright \Bbbk := 0$ for the one-dimensional vector space $\Bbbk$. Now suppose that we had $\mathrm{vect}_{\Bbbk} = \text{mod-}A$ for some unital associative algebra A in $\mathcal{C}$. The underlying object of A is a direct sum $1^{\oplus n_1} \oplus x^{\oplus n_x}$ with $n_1 \geq 1$. For the regular right A-module A_A we thus have $x \triangleright A_A = x \otimes (1^{\oplus n_1} \oplus x^{\oplus n_x}) = x^{\oplus n_1} \neq 0$, which contradicts the action $x \triangleright 1 = 0$ on objects of $\mathrm{vect}_{\Bbbk}$. Hence $\mathrm{vect}_{\Bbbk}$ cannot be written as a category of right modules over an algebra in $\mathcal{C}$.

Next we determine the subcategory $\widehat{\mathrm{vect}_{\Bbbk}}^{\otimes}$. Since the relevant categories are semisimple, the linear functor $\underline{\mathrm{Hom}}(m, -)$ is exact for every $m \in \mathrm{vect}_{\Bbbk}$. The adjunctions

$$\begin{aligned} & \mathrm{Hom}_{\mathcal{C}}(1, \underline{\mathrm{Hom}}(\Bbbk, \Bbbk)) \cong \mathrm{Hom}_{\mathcal{M}}(\Bbbk, \Bbbk) \cong \Bbbk \\ \text{and} \qquad & \mathrm{Hom}_{\mathcal{C}}(x, \underline{\mathrm{Hom}}(\Bbbk, \Bbbk)) \cong \mathrm{Hom}_{\mathcal{M}}(x \triangleright \Bbbk, \Bbbk) = 0 \end{aligned} \tag{4.44}$$

show that $\underline{\mathrm{Hom}}(\Bbbk, \Bbbk) \cong 1$. Finally, by comparing the equalities $\underline{\mathrm{Hom}}(\Bbbk, x \triangleright \Bbbk) = \underline{\mathrm{Hom}}(\Bbbk, 0) = 0$ and $x \otimes \underline{\mathrm{Hom}}(\Bbbk, \Bbbk) \cong x \otimes 1 = x$ we conclude that $\underline{\mathrm{Hom}}(\Bbbk, -)$ cannot be a strong module functor, and hence that $\widehat{\mathrm{vect}_{\Bbbk}}^{\otimes}$ is zero.

Also note that, as a direct consequence of Proposition 5.9, the $\otimes$-admissible objects of $\mathcal{C}$, seen as a left module category over itself, are just the direct sums of copies of the monoidal unit 1.

The following result supplies us with specific objects in the subcategory $\widehat{\mathcal{M}}^{\otimes}$:

Lemma 4.29. *Let $\mathcal{M}$ be a GV-module category over $\mathcal{C}$. Assume that the subcategory $\widehat{\mathcal{M}}^{\otimes}$ of $\mathcal{M}$ contains a $\mathcal{C}$-generator, and thus in particular is not zero, so that we have an equivalence $\mathcal{M} \simeq \text{mod-}A$ for some algebra A in $\mathcal{C}$. Let U be the forgetful functor $U \colon \mathcal{M} \simeq \text{mod-}A \longrightarrow \mathcal{C}$, with left adjoint $\mathrm{Ind}_A \colon \mathcal{C} \longrightarrow \mathcal{M}$. For any $x \in \mathcal{C}$ we have an isomorphism*

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(\mathrm{Ind}_A(x), -) \cong \underline{\mathrm{Hom}}_{\mathcal{C}}(x, U-) \tag{4.45}$$

of $\mathcal{C}$-module functors.

Proof. This follows by direct calculation: we have

$$\mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(\mathrm{Ind}_A(x), m)) \cong \mathrm{Hom}_{\mathcal{C}}(c \otimes x, U(m)) \cong \mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(x, U(m))) \tag{4.46}$$

for all $x, c \in \mathcal{M}$ and all $m \in \mathcal{M}$. $\square$

Corollary 4.30. *In the situation of Lemma 4.29 we have $\mathrm{Ind}_A(x) \in \widehat{\mathcal{M}}^{\otimes}$ for every $x \in \widehat{\mathcal{C}}^{\otimes}$.*

Proof. Let $x \in \widehat{\mathcal{C}}^{\otimes}$. We must show that $\underline{\mathrm{Hom}}(\mathrm{Ind}_A(x), -)\colon \mathcal{M} \to \mathcal{C}$ is an exact strong $\mathcal{C}$-module functor. Since the forgetful functor $U\colon \mathcal{M} \to \mathcal{C}$ is an exact strong $\mathcal{C}$-module functor, this follows from Lemma 4.29. □

In view of the discussion above, it is desirable to have "enough" objects in the subcategory $\widehat{\mathcal{M}}^{\otimes}$, and there is the following option for making this idea precise:

Definition 4.31. We call a GV-module category $\mathcal{M}$ *algebraic* if it is equivalent to a GV-module category of the form mod-A for some algebra A in $\mathcal{C}$.

In terms of this notion, Proposition 4.26 amounts to

Theorem 4.32. *Let $\mathcal{M}$ be a GV-module over a GV-category $\mathcal{C}$. $\mathcal{M}$ is algebraic if and only if the subcategory $\widehat{\mathcal{M}}^{\otimes}$ of $\mathcal{M}$ contains a $\mathcal{C}$-generator (and is thus in particular not zero). If in addition $\mathcal{C}$ and $\mathcal{M}$ are finite abelian, then the objects in $m \in \mathcal{M}$ for which* mod-$\underline{\mathrm{Hom}}(m,m)$ *is equivalent to $\mathcal{M}$ as a $(\mathcal{C}, \otimes)$-module are precisely the $\mathcal{C}$-generators in $\widehat{\mathcal{M}}^{\otimes}$, while the objects $n \in \mathcal{M}$ for which* comod-$\underline{\mathrm{coHom}}(n,n)$ *is equivalent to $\mathcal{M}$ as a $(\mathcal{C}, \circledast)$-module are precisely the $\mathcal{C}$-cogenerators in $\widehat{\mathcal{M}}^{\circledast}$.*

Working with algebraic GV-module categories, one can generalize the statement in Corollary 4.30 to a sufficient criterion for objects in $\widehat{\mathcal{M}}^{\otimes}$:

Lemma 4.33. *Let $\mathcal{C}$ be a GV-category and $\mathcal{M}$ a GV-module category over $\mathcal{C}$. In case $\mathcal{C}$ and $\mathcal{M}$ are additive, the subcategory $\widehat{\mathcal{M}}^{\otimes}$ is closed under direct sums and under direct summands. Moreover, if $\mathcal{M} =$* mod-A *for an algebra A in $\mathcal{C}$, then the following holds: If for $m \in \mathcal{M}$ there exist objects $m' \in \mathcal{M}$ and $x \in \widehat{\mathcal{C}}^{\otimes}$ with an isomorphism*

$$m \oplus m' \cong \mathrm{Ind}_A(x) = x \otimes A \tag{4.47}$$

in $\mathcal{M}$, then we have $m \in \widehat{\mathcal{M}}^{\otimes}$.

Proof. As is well known (compare e.g. Lemmas 12.17.2 and 12.17.3 of [20]), the direct sum $F \oplus G$ of functors F and G has a right adjoint if and only if both F and G have a right adjoint. Moreover, if F and G are weak module functors, then $F \oplus G$ is a strong module functor if and only if both F and G are strong module functors. The first claim follows by applying this result to the functor $\underline{\mathrm{Hom}}(m \oplus m', -)$. The second claim follows from the first and Corollary 4.30. □

Note that for $\mathcal{C} = \mathrm{vect}_{\Bbbk}$, this result reproduces the familiar condition for m to be projective.

In the remainder of this section we treat the case of the GV-category of finite-dimensional right modules over a commutative algebra A from Example 2.5 in detail and classify the algebraic module categories over mod-A.

Proposition 4.34. *Let A be a commutative $\Bbbk$-algebra. For any abelian module category $\mathcal{M}$ over the category $\mathcal{C} = \text{mod-}A$ of finite-dimensional A-modules we have:*

(i) *Any $m \in \widehat{\mathcal{M}}^{\otimes}$ is projective in $\mathcal{M}$.*
(ii) *Any $m \in \widehat{\mathcal{M}}^{\boldsymbol{\otimes}}$ is injective in $\mathcal{M}$.*

Proof. (i) The object $1 = A_A$ is free and thus projective in $\mathcal{C} = \text{mod-}A$. It thus follows from Proposition 4.24.3 that $m \cong 1 \triangleright m$ is projective for $m \in \widehat{\mathcal{M}}^{\otimes}$.

(ii) The statement follows from Proposition 4.24.3 as well, since $K \cong G(1)$ is injective by Remark 2.6. □

Remark 4.35. The argument would fail for $\mathcal{C} = A\text{-Bimod}$, since the monoidal unit ${}_AA_A$ is, in general, not projective as a bimodule.

Proposition 4.36. *Let A be a finite-dimensional commutative $\Bbbk$-algebra. The algebraic module categories over the category* mod-A *of finite-dimensional A-modules are, up to equivalence, given by* mod-B, *where B is an algebra extension of A: B is a $\Bbbk$-algebra together with a unital algebra morphism $\iota\colon A \longrightarrow Z(B)$ from A to the center of B.*

Proof. Recall that for a commutative ring R, an algebra object in the category mod-R is the same as an R-algebra [16, Ch. VII.3]. Analogously, denoting the forgetful functor from $\mathcal{C}$ to vect by U, an algebra object B in $\mathcal{C}$ is the same as a $\Bbbk$-algebra $U(B)$ together with a unital algebra morphism $\iota\colon A \longrightarrow Z(U(B))$. Given such an algebra B, the category $\text{mod}_{\mathcal{C}}\text{-}B$ of right B-modules in $\mathcal{C} = \text{mod-}A$ is equivalent to the category mod-$U(B)$ of B-modules in vect. Indeed, every $m \in \text{mod}_{\mathcal{C}}\text{-}B$ defines an object $U(m) \in \text{mod-}U(B)$ as follows. As a vector space, $U(m) = m$, and the action of B is the composite $U(m) \otimes B \longrightarrow m \otimes_A B \longrightarrow m$ of the canonical projection and the B-action on $m \in \text{mod}_{\mathcal{C}}\text{-}B$. Conversely, given an object $n \in \text{mod-}U(B)$, by restriction n is also an A-module, hence $n \in \mathcal{C}$. Moreover, the action $n \otimes B \longrightarrow n$ is A-balanced so that n naturally has the structure of an object in $\text{mod}_{\mathcal{C}}\text{-}B$. The two constructions are readily seen to be functorial and to provide an equivalence of categories. □

In the case at hand we have the following description of the objects in $\widehat{\mathcal{M}}^{\otimes}$.

Proposition 4.37. *Let $\mathcal{M} = \text{mod-}B$ be a module category over $\mathcal{C} = \text{mod-}A$ as in Proposition 4.36. The objects in $\widehat{\mathcal{M}}^{\otimes}$ are precisely the projective B-modules.*

Proof. If m is projective, then there are objects $n \in \mathcal{M}$ and $V \in \text{vect}$ with an isomorphism $m \oplus n \cong V \otimes_{\Bbbk} B \cong (V \otimes_{\Bbbk} A) \otimes_A B$. Since we have $V \otimes_{\Bbbk} A \in \widehat{\mathcal{C}}^{\otimes}$, it follows from Lemma 4.33 that $m \in \widehat{\mathcal{M}}^{\otimes}$. Conversely, if $m \in \widehat{\mathcal{M}}^{\otimes}$, then by Proposition 4.34, m is projective. □

5. Relative Serre functors

The prototypical idea behind relative Serre functors is the following. Let $\mathcal{C}$ be a monoidal category with a duality functor $D\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{opp}}$ – where D could for instance arise from a rigid duality or from a GV-duality on $\mathcal{C}$. Let furthermore $H\colon \mathcal{M}^{\mathrm{opp}} \times \mathcal{M} \longrightarrow \mathcal{C}$ be a functor, such as a Hom functor or an internal Hom. Then a *relative Serre functor* $S\colon \mathcal{M} \longrightarrow \mathcal{M}$ (for $\mathcal{C}$ with respect to the functor H) is an endofunctor together with a natural family

$$H(n, S(m)) \cong D(H(m, n)) \tag{5.1}$$

of isomorphisms for all $m, n \in \mathcal{M}$. In the situation of our interest we will encounter the weaker variant [17, Def. 3.2] of a *partially defined relative Serre functor*

$$S: \quad \widehat{\mathcal{M}} \longrightarrow \mathcal{M} \tag{5.2}$$

with an isomorphism (5.1) for $n \in \mathcal{M}$ and $m \in \widehat{\mathcal{M}}$, for $\widehat{\mathcal{M}} \subset \mathcal{M}$ a subcategory. As we will see, the subcategory $\widehat{\mathcal{M}}$ relevant to us is the subcategory $\widehat{\mathcal{M}}^{\otimes}$ of a GV-module category $\mathcal{M}$ as introduced in Definition 4.23. Note that $\widehat{\mathcal{M}}^{\otimes}$ was only designed for objects whose internal Ends provide algebras representing the module category $\mathcal{M}$. It is therefore quite remarkable that the same subcategories appear naturally in the definition of relative Serre functors.

5.1. Internal Homs and representable functors

The internal (co)Homs can be used to discuss (co)representability of module functors. Recall for a GV-module category $\mathcal{M}$ over $\mathcal{C}$ the module functors $\mathrm{R}_m^{\triangleright}\colon {}_{\mathcal{C}}\mathcal{C} \longrightarrow {}_{\mathcal{C}}\mathcal{M}$ and $\mathrm{R}_m^{\blacktriangleright}$ from (3.36) and (3.37).

Lemma 5.1. *Let $\mathcal{C}$ be a GV-category and let $\mathcal{M}$ be a left GV-module category over $\mathcal{C}$.*

1. *Let $F\colon \mathcal{C} \to \mathcal{M}$ be a strong $\triangleright$-module functor. Then there exists an object $m \in \mathcal{M}$ with a module natural isomorphism $F \xrightarrow{\cong} R_m^{\triangleright}$. The object m is unique up to unique isomorphism. Furthermore, F has a right adjoint.*
2. *Let $J\colon \mathcal{C} \to \mathcal{M}$ be a strong $\blacktriangleright$-module functor. Then there exists an object $m \in \mathcal{M}$ with a module natural isomorphism $J \xrightarrow{\cong} R_m^{\blacktriangleright}$. The object m is unique up to unique isomorphism. Furthermore, J has a left adjoint.*
3. *Let $H\colon \mathcal{M} \to \mathcal{C}$ be a strong $\blacktriangleright$-module functor that admits a left adjoint. Then there exists a (unique up to unique isomorphism) object $m \in \mathcal{M}$ with a module natural isomorphism $H \xrightarrow{\cong} \underline{\mathrm{Hom}}(m, -)$.*
4. *Let $H\colon \mathcal{M} \to \mathcal{C}$ be a strong $\triangleright$-module functor admitting a right adjoint. Then there exists a (unique up to unique isomorphism) object $m \in \mathcal{M}$ with a module natural isomorphism $H \xrightarrow{\cong} \underline{\mathrm{coHom}}(m, -)$.*

Proof. To obtain the first statement, set $m := F(1)$. From the module constraint f of F we obtain a natural isomorphism $f_{c,1}\colon F(c) = F(c \otimes 1) \xrightarrow{\cong} c \triangleright F(1) = c \triangleright m$ for every $c \in \mathcal{C}$. The pentagon axiom for the module constraint implies that this family of isomorphisms constitutes a module natural isomorphism $F \xrightarrow{\cong} R_m^{\triangleright}$. Since $R_m^{\triangleright}$ has a right adjoint, we conclude that F has a right adjoint as well. The second statement follows analogously.

To show the third statement, we invoke Lemma 3.20 to conclude that $H^{\text{l.a.}}\colon \mathcal{C} \longrightarrow \mathcal{M}$ is a strong $\triangleright$-module functor, so that by the first statement there is a module natural isomorphism $H^{\text{l.a.}} \xrightarrow{\cong} R_m^{\triangleright}$ for some $m \in \mathcal{M}$. But then from the isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(c, H(n)) \cong \mathrm{Hom}_{\mathcal{M}}(H^{\text{l.a.}}(c), n) \cong \mathrm{Hom}_{\mathcal{M}}(c \triangleright m, n) \cong \mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(m, n)) \quad (5.3)$$

we conclude with the Yoneda Lemma that there is a natural isomorphism $H \xrightarrow{\cong} \underline{\mathrm{Hom}}(m, -)$. Since all isomorphisms in Equation (5.3) are module natural isomorphisms, we thus have $H \cong \underline{\mathrm{Hom}}(m, -)$ as module functors. The last statement follows analogously. $\square$

In view of Lemma 5.1 the following terminology makes sense, for $\mathcal{C}$ a finite GV-category and $\mathcal{M}$ a GV-module category:

Definition 5.2. A left exact ($\blacktriangleright$-module) functor $F\colon \mathcal{M} \longrightarrow \mathcal{C}$ is called *internally representable* if it is isomorphic (as a $\blacktriangleright$-module functor) to a functor of the form

$$\begin{aligned} \mathcal{M} &\longrightarrow \mathcal{C}\,, \\ m &\longmapsto \underline{\mathrm{Hom}}(m_0, m) \end{aligned} \quad (5.4)$$

for some $m_0 \in \mathcal{M}$.

We call a right exact ($\triangleright$-module) functor $F\colon \mathcal{M} \longrightarrow \mathcal{C}$ *internally representable* if it is isomorphic (as a $\triangleright$-module functor) to a functor of the form

$$\begin{aligned} \mathcal{M} &\longrightarrow \mathcal{C}\,, \\ m &\longmapsto \underline{\mathrm{coHom}}(m_0, m) \end{aligned} \quad (5.5)$$

for some $m_0 \in \mathcal{M}$.

Note that a necessary condition for a lax module functor to be internally representable is that the module structure is strong.

Remark 5.3. Dually to Definition 5.2 we call a left exact ($\blacktriangleright$-module) functor $F\colon \mathcal{M}^{\text{opp}} \longrightarrow \mathcal{C}$ *internally corepresentable* if it is isomorphic (as a $\blacktriangleright$-module functor) to a functor from $\mathcal{M}^{\text{opp}}$ to $\mathcal{C}$ that maps objects as $m \longmapsto \underline{\mathrm{Hom}}(m, m_0)$ for some $m_0 \in \mathcal{M}$. Analogously we call a right exact ($\triangleright$-module) functor $F\colon \mathcal{M}^{\text{opp}} \longrightarrow \mathcal{C}$ internally corepresentable if it is isomorphic (as a $\triangleright$-module functor) to a functor that maps objects as $m \longmapsto \underline{\mathrm{coHom}}(m, m_0)$

for some $m_0 \in \mathcal{M}$. The statements about internal representability shown below have obvious analogues for corepresentable functors.

We next examine a suitable subcategory of objects, on which the internal Hom functor is internally representable. Recall from Definition 4.23 the subcategories $\widehat{\mathcal{M}}^{\otimes}$ and $\widehat{\mathcal{M}}^{\circledast}$ of $\otimes$- and $\circledast$-admissible objects of a GV-module category $\mathcal{M}$.

Lemma 5.4. *Let $m \in \mathcal{M}$, and let Y_m and W_m be the internal Hom and coHom functors as defined in* (4.34)*.*

1. *The object m is $\otimes$-admissible if and only if the $\triangleright$-module functor $\underline{\mathrm{Hom}}(m,-)$ is internally representable in the form of* (5.5)*, i.e. if and only if there exist an object $S(m) \in \mathcal{M}$ and an isomorphism $\phi_m \colon \underline{\mathrm{Hom}}(m,-) \xrightarrow{\cong} \underline{\mathrm{coHom}}(Sm,-)$ of $\triangleright$-module functors. If this is the case, then we can choose*

$$S(m) = Y_m^{\mathrm{r.a.}}(K)\,. \tag{5.6}$$

2. *The object m is $\circledast$-admissible if and only if the $\blacktriangleright$-module functor $\underline{\mathrm{coHom}}(m,-)$ is internally representable as in* (5.4)*, i.e. if and only if there exist an object $\widetilde{S}(m) \in \mathcal{M}$ and an isomorphism $\psi_m \colon \underline{\mathrm{coHom}}(m,-) \xrightarrow{\cong} \underline{\mathrm{Hom}}(\widetilde{S}m,-)$ of $\blacktriangleright$-module functors. If this is the case, then we can choose*

$$\widetilde{S}(m) = W_m^{\mathrm{l.a.}}(1)\,. \tag{5.7}$$

The objects $S(m)$ and $\widetilde{S}(m)$ are unique up to unique isomorphism.

Proof. This is a direct consequence of Lemma 5.1. □

In particular we obtain natural isomorphisms

$$\rho_m(n): \quad \mathrm{Hom}(n, S(m)) \cong \mathrm{Hom}(\underline{\mathrm{Hom}}(m,n), K) \tag{5.8}$$

for $m \in \widehat{\mathcal{M}}^{\otimes}$ and $n \in \mathcal{M}$. Since the objects $S(m)$ for $m \in \widehat{\mathcal{M}}^{\otimes}$ internally represent functors, the assignment $m \mapsto S(m)$ extends to a functor

$$S: \quad \widehat{\mathcal{M}}^{\otimes} \longrightarrow \mathcal{M}\,. \tag{5.9}$$

Analogously we obtain a functor

$$\widetilde{S}: \quad \widehat{\mathcal{M}}^{\circledast} \longrightarrow \mathcal{M}\,. \tag{5.10}$$

The functors S and $\widetilde{S}$ are indeed relative Serre functors:

Corollary 5.5. *For every $m \in \widehat{\mathcal{M}}^{\otimes}$ and every $n \in \mathcal{M}$ there is a natural isomorphism*

$$\widetilde{\phi}_m(n): \quad \underline{\mathrm{Hom}}(n, Sm) \xrightarrow{\;\cong\;} G(\underline{\mathrm{Hom}}(m,n))\,. \tag{5.11}$$

Analogously, for every $m \in \widehat{\mathcal{M}}^{\boldsymbol{\otimes}}$ and every $n \in \mathcal{M}$ there is a natural isomorphism

$$\underline{\mathrm{Hom}}(\widetilde{S}m, n) \xrightarrow{\;\cong\;} G(\underline{\mathrm{Hom}}(n,m))\,. \tag{5.12}$$

Proof. By Lemma 3.41 we have an isomorphism $\underline{\mathrm{coHom}}(Sm,n) \cong G^{-1}\underline{\mathrm{Hom}}(n,Sm)$. The isomorphisms (5.11) and (5.12) thus follow directly by invoking the isomorphisms from the definitions of S and $\widetilde{S}$. □

Definition 5.6. The functors $S\colon \widehat{\mathcal{M}}^{\otimes} \longrightarrow \mathcal{M}$ and $\widetilde{S}\colon \widehat{\mathcal{M}}^{\boldsymbol{\otimes}} \longrightarrow \mathcal{M}$ defined by the formulas (5.6) and (5.7) are called the *relative Serre functor S of $\mathcal{M}$* and the *inverse relative Serre functor $\widetilde{S}$*, respectively. Considering $\mathcal{C}$ as a left $\mathcal{C}$-module category defines the relative Serre functor S of $\mathcal{C}$.

There is also a variant of a relative Serre functor for $\mathcal{C}$ when considering $\mathcal{C}$ as right $\mathcal{C}$-module category.

Theorem 5.7. *The relative Serre functors provide an equivalence of categories between the subcategories $\widehat{\mathcal{M}}^{\otimes}$ and $\widehat{\mathcal{M}}^{\boldsymbol{\otimes}}$ of $\mathcal{M}$, i.e., slightly abusing notation by keeping the symbols S and $\widetilde{S}$, we have*

$$S: \quad \widehat{\mathcal{M}}^{\otimes} \xrightarrow{\;\simeq\;} \widehat{\mathcal{M}}^{\boldsymbol{\otimes}} \qquad \text{and} \qquad \widetilde{S}: \quad \widehat{\mathcal{M}}^{\boldsymbol{\otimes}} \xrightarrow{\;\simeq\;} \widehat{\mathcal{M}}^{\otimes}. \tag{5.13}$$

In particular, for every $m \in \widehat{\mathcal{M}}^{\otimes}$ the object $S(m)$ is in the subcategory $\widehat{\mathcal{M}}^{\boldsymbol{\otimes}}$ of $\mathcal{M}$, and analogously, $\widetilde{S}(n) \in \widehat{\mathcal{M}}^{\otimes}$ for every $n \in \widehat{\mathcal{M}}^{\boldsymbol{\otimes}}$.

Proof. Let m be $\otimes$-admissible. According to Proposition 3.45 the two functors $\underline{\mathrm{Hom}}(m,-)$ and $\underline{\mathrm{coHom}}(Sm,-)$ are GV-module functors. Hence it follows from Theorem 3.13 that the $\triangleright$-module natural isomorphism $\phi_m\colon \underline{\mathrm{Hom}}(m,-) \xrightarrow{\cong} \underline{\mathrm{coHom}}(Sm,-)$ is also a natural isomorphism of $\blacktriangleright$-module functors. As a consequence, for $m \in \widehat{\mathcal{M}}^{\otimes}$ one has $S(m) \in \widehat{\mathcal{M}}^{\boldsymbol{\otimes}}$, using that $\underline{\mathrm{coHom}}(Sm,-) \cong \underline{\mathrm{Hom}}(m,-)$ is a strong $\blacktriangleright$-module functor, and by the same isomorphism we see that $\widetilde{S}(S(m)) \cong m$. Analogously it follows that $S(\widetilde{S}(m)) \cong m$. Thus S and $\widetilde{S}$ define an equivalence of categories. □

Proposition 5.8. *Let $\mathcal{C}$ be a GV-category and $\mathcal{M}$ a left GV-module category over $\mathcal{C}$.*

1. *The relative Serre functor of $\mathcal{C}$ is canonically a monoidal equivalence*

$$S_{\mathcal{C}}: \quad \widehat{\mathcal{C}}^{\otimes} \longrightarrow \widehat{\mathcal{C}}^{\boldsymbol{\otimes}} \tag{5.14}$$

with inverse monoidal functor $\widetilde{S}_{\mathcal{C}}$.

2. *The relative Serre functor $S_{\mathcal{M}}$ of $\mathcal{M}$ is a twisted module functor, in the sense that there are natural isomorphisms*

$$S_{\mathcal{M}}(c \triangleright m) \xrightarrow{\cong} S_{\mathcal{C}}(c) \blacktriangleright S_{\mathcal{M}}(m) \tag{5.15}$$

for $c \in \widehat{\mathcal{C}}^{\otimes}$ and $m \in \widehat{\mathcal{M}}^{\otimes}$.
Analogously we have $\widetilde{S}_{\mathcal{M}}(d \blacktriangleright n) \cong \widetilde{S}_{\mathcal{C}}(d) \triangleright \widetilde{S}_{\mathcal{M}}(n)$ for $d \in \widehat{\mathcal{C}}^{\otimes\!\!\!\bullet}$ and $n \in \widehat{\mathcal{M}}^{\otimes\!\!\!\bullet}$.

Proof. To obtain the first statement, first note that the isomorphisms (4.35) imply that $S_{\mathcal{C}}(1) \cong K$. The isomorphism $S_{\mathcal{C}}(x \otimes y) \cong S_{\mathcal{C}}(x) \circledast S_{\mathcal{C}}(y)$ for $x, y \in \widehat{\mathcal{C}}^{\otimes}$ follows by applying the Yoneda Lemma to the composite of the isomorphisms

$$\begin{aligned} \underline{\mathrm{Hom}}(x \otimes y, -) \cong \underline{\mathrm{Hom}}(x, \underline{\mathrm{Hom}}(y, -)) &\cong \underline{\mathrm{coHom}}(S_{\mathcal{C}}x, \underline{\mathrm{coHom}}(S_{\mathcal{C}}y, -)) \\ &\cong \underline{\mathrm{coHom}}(S_{\mathcal{C}}x \circledast S_{\mathcal{C}}y, -) \end{aligned} \tag{5.16}$$

of module functors from Proposition 3.45 together with the isomorphism $\underline{\mathrm{Hom}}(x \otimes y, -) \cong \underline{\mathrm{Hom}}\ (S_{\mathcal{C}}(x \otimes y), -)$. All isomorphisms in the sequence (5.16) are coherent, hence $S_{\mathcal{C}}$ is a monoidal equivalence.

The natural isomorphisms follow from the analogous computation for $\underline{\mathrm{Hom}}(x \triangleright m, -)$ with $x \in \widehat{\mathcal{C}}^{\otimes}$ and $m \in \widehat{\mathcal{M}}^{\otimes}$. The statements for $\widetilde{S}$ are shown analogously. □

For the case that the module category is algebraic we will obtain a stronger statement in Proposition 5.15.

5.2. The relative Serre functor of $\mathcal{C}$

We now restrict our attention to the important regular case, already considered in Proposition 5.8, of $\mathcal{C}$ as a left GV-module category over itself. Since the internal Homs of $\mathcal{C}$ can now be expressed in terms of the monoidal structures, we can compute S as follows.

Proposition 5.9. *Let $\mathcal{C}$ be a GV-category and $c \in \mathcal{C}$. For any $c \in \mathcal{C}$ the following statements are equivalent:*

1. *$c \in \mathcal{C}$ is $\otimes$-admissible.*
2. *For all $x, y \in \mathcal{C}$ the distributors $c \otimes (x \circledast y) \longrightarrow (c \otimes x) \circledast y$ are isomorphisms.*
3. *c has a right $\otimes$-rigid dual object.*

Proof. By applying G to the distributors in statement 2 and invoking Proposition 3.30 we see that these distributors are all isomorphisms if and only if the oplax $\blacktriangleright$-module functor Y_c is strong. Statement 3 means that there exists an object $c^{\vee}$ together with morphisms $c^{\vee} \otimes c \longrightarrow 1$ and $1 \longrightarrow c \otimes c^{\vee}$ that obey the snake identities. The equivalence of this statement with statement 1 is shown in [13, Prop. 5.2]. Thus all three statements are equivalent. □

By analogously working with the right module category $\mathcal{C}_\mathcal{C}$ and/or the left exact tensor product $\circledast$ one finds that the statement in Proposition 5.9 has in total four incarnations: For instance, for $\mathcal{C}_\mathcal{C}$ with the monoidal structure $\otimes$ we obtain the statement that the objects c that have a left $\otimes$-rigid dual are those whose distributors $(x \circledast y) \otimes c \longrightarrow x \circledast (y \otimes c)$ are isomorphisms. In the case of ${}_\mathcal{C}\mathcal{C}$ and $\circledast$, we see that the following statements are equivalent for every $c \in \mathcal{C}$:

1. $c \in \mathcal{C}$ is $\circledast$-admissible.
2. For all $x, y \in \mathcal{C}$ the distributors $(c \circledast x) \otimes y \longrightarrow c \circledast (x \otimes y)$ are isomorphisms.
3. c has a left $\circledast$-rigid dual object.

In view of the third characterization of admissibility in Proposition 5.9, by applying G or G^{-1} to the evaluation and coevaluation one sees that for any $c \in \widehat{\mathcal{C}}^{\otimes}$ the object $G(c^\vee)$ is a right $\circledast$-rigid dual of $G(c)$. Analogous statements hold for the other three incarnations of Proposition 5.9.

Corollary 5.10. *The functor G^2 preserves the $\otimes$- as well as the $\circledast$-admissible objects; that is, the functors*

$$G^2: \quad \widehat{\mathcal{C}}^{\otimes} \longrightarrow \widehat{\mathcal{C}}^{\otimes} \qquad \text{and} \qquad G^2: \quad \widehat{\mathcal{C}}^{\circledast} \longrightarrow \widehat{\mathcal{C}}^{\circledast} \tag{5.17}$$

are monoidal equivalences.

Proof. Since G^2 is a monoidal equivalence for $\otimes$ as well as of $\circledast$, it preserves the class of objects that have a right $\otimes$-rigid dual, as well as the class of objects that have a left $\circledast$-rigid dual. □

Lemma 5.11. *Let $\mathcal{C}$ be a GV-category and $c \in \mathcal{C}$ be an object for which the three equivalent conditions in Proposition 5.9 are satisfied. Then Y_c has*

$$Y_c^{\mathrm{r.a.}} = - \circledast (G^2(c) \otimes K) \tag{5.18}$$

as a right adjoint functor. Moreover, the right $\otimes$-rigid dual of c is given by

$$c^\vee \cong 1 \circledast Gc \tag{5.19}$$

and we have

$$y \circledast Gc \cong y \otimes c^\vee \tag{5.20}$$

as well as

$$c \circledast y \cong c \otimes (1 \circledast y) \tag{5.21}$$

for every $y \in \mathcal{C}$*.*

Proof. The isomorphism $Y_c(y) = y \circledast Gc \cong (y \otimes 1) \circledast Gc \cong y \otimes (1 \circledast Gc)$ implies that Y_c has the functor (5.18) as a right adjoint. The expression (5.19) for $c^\vee$ is shown in [13, Prop. 5.2]. The isomorphism (5.20) follows, for instance, from the fact that the two functors $- \circledast Gc$ and $- \otimes c^\vee$ are both right adjoint to $- \otimes c$. Finally, (5.21) is the special case $x = 1$ of the isomorphism in the second statement of Proposition 5.9. $\square$

Next we describe the relative Serre functor of $\mathcal{C}$ explicitly:

Proposition 5.12. *The relative Serre functor* S *of* $\mathcal{C}$ *is given by*

$$S(c) \cong G^2(c) \otimes K \tag{5.22}$$

for $c \in \widehat{\mathcal{C}}^{\otimes}$*. This natural isomorphism is monoidal in the sense that the diagram*

$$\begin{tikzcd}
S(c \otimes d) \arrow[r, "\cong"] \arrow[ddd, "\cong"'] & G^2(c \otimes d) \otimes K \arrow[dr, "\cong"] & \\
& & G^2(c) \otimes G^2(d) \otimes K \arrow[d, "\cong"] \\
& & G^2(c) \otimes (K \circledast (G^2(d) \otimes K)) \arrow[dl, "\delta"] \\
S(c) \circledast S(d) \arrow[r, "\cong"] & (G^2(c) \otimes K) \circledast (G^2(d) \otimes K) &
\end{tikzcd} \tag{5.23}$$

with $\delta = \delta^{\mathrm{l}}_{G^2c, K, G^2d \otimes K}$ *(which is an isomorphism) commutes for all* $c, d \in \widehat{\mathcal{C}}^{\otimes}$*. The inverse of* S *is given by*

$$\widetilde{S}(c) \cong G^2(c) \circledast 1 \tag{5.24}$$

for $c \in \widehat{\mathcal{C}}^{\otimes}$*.*

Proof. Note that because of $G^2(c) \in \widehat{\mathcal{C}}^{\otimes}$ the distributor $\delta^{\mathrm{l}}_{G^2c, K, G^2d \otimes K}$ is an isomorphism. The expression (5.22) for the relative Serre functor follows from $S(c) = Y_c^{\text{r.a.}}(K)$. The monoidality follows from (5.16), which expresses the monoidal structure of S; this is seen as follows. For $c \in \widehat{\mathcal{C}}^{\otimes}$ we have by Equation (5.11) the isomorphism

$$G(\underline{\mathrm{Hom}}(c, x)) \cong \underline{\mathrm{Hom}}(x, Sc) \tag{5.25}$$

for every $x \in \mathcal{C}$. If we insert $Sc \cong G^2 c \otimes K$, the isomorphism (5.25) computes as

$$G(x \circledast c) \cong Gc \otimes Gx \cong (G^2c \otimes K) \circledast Gx \cong Sc \circledast Gx\,, \tag{5.26}$$

where the second isomorphism is a distributor. For the isomorphisms (5.26) the coherence diagram (5.16) translates to the diagram (5.23). The expression (5.24) for $\widetilde{S}$ follows analogously. □

Remark 5.13. Also, upon a change of dualizing object from K to $\widetilde{K} = g \otimes K$ with invertible $g \in \mathcal{C}$, combining the formula (5.22) for the relative Serre functor with the resulting change (2.6) in G^2 shows that S changes to

$$S'(c) = \widetilde{G}^2(c) \otimes \widetilde{K} = g \otimes G^2c \otimes g^{-1} \otimes g \otimes K \cong g \otimes G^2c \otimes K \cong g \otimes S(c)\,. \tag{5.27}$$

It follows e.g. that if $S(c) \cong c$, then there exists an isomorphism $S'(c) \cong c$ if and only if $g \otimes c \cong c$.

Since $G^2 \colon \mathcal{C} \longrightarrow \mathcal{C}$ is a monoidal equivalence both for $\otimes$ and for $\circledast$, by composing S with its inverse we obtain directly

Corollary 5.14. *Let $\mathcal{C}$ be a GV-category. The functor*

$$- \otimes K : \quad \widehat{\mathcal{C}}^{\otimes} \longrightarrow \widehat{\mathcal{C}}^{\circledast} \tag{5.28}$$

is a monoidal equivalence with inverse functor $- \circledast 1$.

We can now conclude that for algebraic module categories the relative Serre functor is twisted by G^2, a situation familiar from the rigid case [12, Lemma 4.23]:

Proposition 5.15. *Let $\mathcal{M}$ be an algebraic left $\mathcal{C}$-module category. Then $S_{\mathcal{M}} \colon \widehat{\mathcal{M}}^{\otimes} \longrightarrow \mathcal{M}$ is a twisted module functor in the sense that there are coherent natural isomorphisms*

$$S_{\mathcal{M}}(c \triangleright m) \cong G^2c \triangleright S_{\mathcal{M}}(m) \tag{5.29}$$

for all $c \in \widehat{\mathcal{C}}^{\otimes}$ and $m \in \widehat{\mathcal{M}}^{\otimes}$.

Proof. By Proposition 4.21 the module distributors of $\mathcal{M}$ can be obtained from the distributors of $\mathcal{C}$, which together with Proposition 5.9 implies that for all $c \in \widehat{\mathcal{C}}^{\otimes}$, all $d \in \mathcal{C}$ and all $m \in \mathcal{M}$ the distributor

$$c \triangleright (x \blacktriangleright m) \longrightarrow (c \otimes x) \blacktriangleright m \tag{5.30}$$

is an isomorphism. But then we obtain with Proposition 5.8 for all $c \in \widehat{\mathcal{C}}^{\otimes}$ and $m \in \widehat{\mathcal{M}}^{\otimes}$ an isomorphism

$$\begin{aligned} S_{\mathcal{M}}(c \triangleright m) &\cong S_{\mathcal{C}}(c) \blacktriangleright S_{\mathcal{M}}(m) \cong (G^2 c \otimes K) \blacktriangleright S_{\mathcal{M}}(m) \\ &\cong G^2 c \triangleright (K \blacktriangleright S_{\mathcal{M}}(m)) \cong G^2 c \triangleright S_{\mathcal{M}}(m) . \end{aligned} \tag{5.31}$$

It follows from Proposition 5.12 that the composite isomorphism is also coherent. □

The following example shows that the category of admissible objects is in general not abelian, even if $\mathcal{C}$ is:

Example 5.16. [13, Lemma 5.5] Let A be a finite-dimensional $\Bbbk$-algebra and $\mathcal{C} = A$-bimod the corresponding GV-category. The following statements are equivalent for ${}_AM_A \in \mathcal{C}$:

(i) M is admissible, i.e. $M \in \widehat{\mathcal{C}}^{\otimes}$.
(ii) M has a $\otimes_A$-right dual.
(iii) M_A is projective as a right A-module.

5.3. Relative Serre functors and Frobenius algebras

Finally, generalizing an argument from [19], we show that a trivialization of the relative Serre functor equips the internal End algebra with a Frobenius structure. Recall from Equation (5.11) the isomorphism $\widetilde{\phi}_m(n)\colon \underline{\mathrm{Hom}}(n, S(m)) \xrightarrow{\cong} G(\underline{\mathrm{Hom}}(m,n))$.

Lemma 5.17. *Let $m \in \widehat{\mathcal{M}}^{\otimes}$. For all objects $c \in \mathcal{C}$ and $m, n \in \mathcal{M}$ the diagram*

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(n, S(m))) \arrow[r] \arrow[dd, "\widetilde{\phi}_m(n)"'] & \mathrm{Hom}_{\mathcal{M}}(c \triangleright n, S(m)) \arrow[d, "\rho_m(c \triangleright n)"] \\ & \mathrm{Hom}(\underline{\mathrm{Hom}}(m, c \triangleright n), K) \arrow[d, "\cong"] \\ \mathrm{Hom}(c, G(\underline{\mathrm{Hom}}(m,n))) \arrow[r] & \mathrm{Hom}(c \otimes \underline{\mathrm{Hom}}(m,n), K) \end{tikzcd} \tag{5.32}$$

commutes.

Proof. From the proof of Lemma 5.1 we obtain a commuting diagram

$$\begin{tikzcd} \mathrm{Hom}(\underline{\mathrm{Hom}}(m,n), c) \arrow[r, "\cong"] \arrow[d, "(\phi_m(n))^*"'] & \mathrm{Hom}(n, Y_m^{\mathrm{r.a.}}(c)) \arrow[d, "\cong"] \\ \mathrm{Hom}(\underline{\mathrm{coHom}}(Y_m^{\mathrm{r.a.}}(K), n), c) \arrow[r, "\cong"] & \mathrm{Hom}(n, c \blacktriangleright Y_m^{\mathrm{r.a.}}(K)) \end{tikzcd} \tag{5.33}$$

Inserting the definition of the strong $\blacktriangleright$-module functor structure of $Y_m^{\mathrm{r.a.}}$, it follows that the outer hexagon in the diagram

$$\begin{tikzcd}
\mathrm{Hom}(\underline{\mathrm{coHom}}(Sm,n),c) \arrow[rr, "\cong"] \arrow[dr, "G^{-1}"] \arrow[dd, "\phi_m(n)^*"'] & & \mathrm{Hom}(n, c \blacktriangleright S(m)) \arrow[dd, "\cong"] \\
& \mathrm{Hom}(Gc, \underline{\mathrm{Hom}}(n,Sm)) \arrow[d, "\widetilde{\phi}"'] \arrow[dr, "\cong"] & \\
\mathrm{Hom}(\underline{\mathrm{Hom}}(m,n),c) \arrow[r, "G^{-1}"] \arrow[d, "\cong"'] & \mathrm{Hom}(Gc, G(\underline{\mathrm{Hom}}(m,n))) & \mathrm{Hom}(Gc \triangleright n, Sm) \arrow[d, "\cong"] \\
\mathrm{Hom}(Gc, \otimes \underline{\mathrm{Hom}}(m,n), K) \arrow[ur, "\cong"'] \arrow[rr, "\cong"] & & \mathrm{Hom}(\underline{\mathrm{Hom}}(m, Gc \triangleright n), K)
\end{tikzcd} \tag{5.34}$$

commutes. It is also readily seen that the inner triangle and the two inner quadrangles commute. Thus the pentagon in the lower right corner commutes as well; this yields the diagram (5.32). □

Following [19, Def. 3.7] we give

Definition 5.18. Let $m \in \widehat{\mathcal{M}}^{\otimes}$. The *trace of* m is the composite morphism

$$\mathrm{tr}_m : \quad \underline{\mathrm{Hom}}(m, S(m)) \xrightarrow{\widetilde{\phi}_m(m)} G(\underline{\mathrm{Hom}}(m,m)) \xrightarrow{G(u_m)} G(1) = K \tag{5.35}$$

with the unit $u_m : 1 \longrightarrow \underline{\mathrm{Hom}}(m,m)$.

We then obtain the following analogue of [19, Lemma 3.8]:

Lemma 5.19. *Let* $m \in \widehat{\mathcal{M}}^{\otimes}$. *With respect to the multiplication* $\underline{\mu}_{m,n,k} : \underline{\mathrm{Hom}}(n,k) \otimes \underline{\mathrm{Hom}}(m,n) \longrightarrow \underline{\mathrm{Hom}}(m,k)$ *we have a commuting diagram*

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(n,Sm) \otimes \underline{\mathrm{Hom}}(m,n) \arrow[r, "\underline{\mu}_{m,n,Sm}"] \arrow[d, "\widetilde{\phi}_m(m)"'] & \underline{\mathrm{Hom}}(m,Sm) \arrow[r, "\mathrm{tr}_m"] & K \\
G(\underline{\mathrm{Hom}}(m,n)) \otimes \underline{\mathrm{Hom}}(m,n) \arrow[urr, "\mathrm{ev}_{\underline{\mathrm{Hom}}(m,n)}"'] & &
\end{tikzcd} \tag{5.36}$$

Proof. This follows, as in [19], by chasing the identity morphism in the upper left corner in (5.32) for $c = \underline{\mathrm{Hom}}(n,Sm)$ through the diagram. The downward-right-path yields the element $\mathrm{ev}_{\underline{\mathrm{Hom}}(m,n)} \circ \widetilde{\phi}_m(m) \in \mathrm{Hom}(\underline{\mathrm{Hom}}(n,Sm) \otimes \underline{\mathrm{Hom}}(m,n), K)$. Using Lemma 5.19, the other path in (5.32) gives the morphism $\underline{\mu}_{m,n,Sm} \in \mathrm{Hom}(\underline{\mathrm{Hom}}(n,Sm) \otimes \underline{\mathrm{Hom}}(m,n), K)$. □

This leads us to the following analogue of Theorem 3.14 of [19]:

Theorem 5.20. *Let $m \in \widehat{\mathcal{M}}^{\otimes}$ be such that the object $S(m)$ is isomorphic to m. Then for every choice of isomorphism $p\colon m \longrightarrow Sm$ in $\mathcal{M}$, $(\underline{\mathrm{Hom}}(m,m), \mu = \underline{\mu}_{m,m,m})$ is a GV-Frobenius algebra in $\mathcal{C}$ with Frobenius form*

$$\lambda : \quad \underline{\mathrm{Hom}}(m,m) \xrightarrow{\underline{\mathrm{Hom}}(m,p)} \underline{\mathrm{Hom}}(m,Sm) \xrightarrow{\mathrm{tr}_m} K\,. \tag{5.37}$$

Proof. The proof given in [19] generalizes: By Lemma 5.19 for $A = \underline{\mathrm{Hom}}(m,m)$, the composite morphism $\lambda \circ \mu\colon A \otimes A \longrightarrow K$ is equal to the composite

$$\begin{aligned} \underline{\mathrm{Hom}}(m,m) \otimes A &\xrightarrow{\underline{\mathrm{Hom}}(m,p)\otimes \mathrm{id}} \underline{\mathrm{Hom}}(m,Sm) \otimes A \\ &\xrightarrow{\widetilde{\phi}_m(m)\otimes \mathrm{id}} G(\underline{\mathrm{Hom}}(m,m)) \otimes \underline{\mathrm{Hom}}(m,m) \xrightarrow{\mathrm{ev}_{\underline{\mathrm{Hom}}(m,m)}} K\,. \end{aligned} \tag{5.38}$$

The first two morphisms in this composite are isomorphisms, and the last is the duality pairing, which is non-degenerate. Thus in total λ is a Frobenius form on $\underline{\mathrm{Hom}}(m,m)$. $\square$

Remark 5.21. Suppose that there is an isomorphism $p\colon m \longrightarrow S(m)$, i.e. that m is a fixed point under the action of the relative Serre functor. Then m is an object in both $\widehat{\mathcal{M}}^{\otimes}$ and $\widehat{\mathcal{M}}^{\color{gray}\otimes}$. (But we do better than just intersecting the classes of objects of these subcategories, because we also trivialize the relative Serre functor.) By Theorem 5.20, for such a Serre fixed point m the algebra $\underline{\mathrm{Hom}}(m,m)$ is Frobenius. In general it is not symmetric, though – there need not even exist a pivotal structure on $\mathcal{C}$.

In the proof of Theorem 5.20 no coherence requirement on the isomorphism p needs to be imposed. From Lemma 5.19 and the naturality of the multiplication $\underline{\mu}_{m,n,l}$ we can directly deduce a stronger non-degeneracy of the Frobenius pairing:

Corollary 5.22. *Let $m \in \widehat{\mathcal{M}}^{\otimes}$ be an object with an isomorphism $S(m) \cong m$ and a corresponding Frobenius form $\lambda\colon \underline{\mathrm{Hom}}(m,m) \longrightarrow K$. Then for every $n \in \mathcal{M}$ the pairing*

$$\underline{\mathrm{Hom}}(n,m) \otimes \underline{\mathrm{Hom}}(m,n) \xrightarrow{\underline{\mu}_{m,n,m}} \underline{\mathrm{Hom}}(m,m) \xrightarrow{\lambda} K \tag{5.39}$$

is non-degenerate.

This, in turn, characterizes the Frobenius structures on the algebra $\underline{\mathrm{Hom}}(m,m)$ that we obtain this way:

Proposition 5.23. *Let $m \in \widehat{\mathcal{M}}^{\otimes}$ and $\lambda\colon \underline{\mathrm{Hom}}(m,m) \longrightarrow K$ a morphism, such that the pairing (5.39) is non-degenerate for every $n \in \mathcal{M}$. Then there exists an isomorphism $p\colon m \xrightarrow{\cong} Sm$ such that λ is constructed from p as the composite (5.37).*

Proof. The statement follows directly from the Yoneda lemma applied to the composite isomorphism $\underline{\mathrm{Hom}}(n,m) \xrightarrow{\cong} G(\underline{\mathrm{Hom}}(m,n)) \xrightarrow{\cong} \underline{\mathrm{Hom}}(n,Sm)$, in which the first isomorphism comes from the non-degeneracy of (5.39) and the second from the definition of the relative Serre functor. □

Corollary 5.24. *If $x \in \mathcal{C}$ has a rigid right dual and there is an isomorphism $x \cong G^2(x) \otimes K$, then the internal End $\underline{\mathrm{Hom}}(x,x) \cong x \circledast Gx$ has the structure of a GV-Frobenius algebra.*

Let us finally turn our attention to the situation that the GV-category $\mathcal{C}$ admits a pivotal structure. Assume that a specific pivotal structure has been fixed.

Corollary 5.25. *Let $\mathcal{C}$ be a GV-category with a pivotal structure. Then for any $c \in \widehat{\mathcal{C}}^{\otimes}$ the rigid right dual obeys*

$$c^{\vee} \cong 1 \circledast Gc \cong G(c \otimes K)\,. \tag{5.40}$$

It follows that c has $G(c)$ as a rigid right dual if and only if there is an isomorphism $S(c) \cong c$. In this case the internal End $\underline{\mathrm{Hom}}(c,c)$ has the structure of a Frobenius algebra in $\mathcal{C}$.

Proof. The presence of a pivotal structure implies that the relative Serre functor (5.22) on $\widehat{\mathcal{C}}^{\otimes}$ is given by $S(c) \cong c \otimes K$. By Lemma 5.11 this provides the isomorphism (5.40). It follows that there is an isomorphism $S(c) \cong c$ if and only if $c \otimes K \cong c$ which, in turn, is the case if and only if $c^{\vee} \cong G(c)$. □

Analogously, by (5.24) in a pivotal GV-category we have $\widetilde{S}(c) \cong c \circledast 1$.

Example 5.26. These observations connect nicely with the conditions given in [15, Def. 3.8] for the subcategory relevant to the classification of boundary conditions in the so-called logarithmic triplet model of conformal field theory [15, Sect. 3]. There the case of a pivotal GV-category with monoidal isomorphism $\pi\colon \mathrm{id}_{\mathcal{C}} \longrightarrow G^2$, as in Section 4.2, is considered. Denote the image of π under the GV-adjunction ϖ of (2.1) by

$$f_c = \varpi_{c,Gc}(\pi_c): \quad c \otimes Gc \longrightarrow K\,. \tag{5.41}$$

The subcategory classifying boundary conditions is constructed in two stages. First there is the full subcategory $\mathcal{C}^r$ consisting of all objects admitting both a left and a right dual (this is a two-handed version of our subcategories of admissible objects). Within $\mathcal{C}^r$ there is then a further restriction to the full subcategory $\mathcal{C}^b$ of all objects $c \in \mathcal{C}^r$ for which Gc is both a left and a right dual for c (an additional non-degeneracy condition, which is not relevant here, is also imposed). It was then noted that for $c \in \mathcal{C}^b$ the internal End $\underline{\mathrm{Hom}}(c,c)$ is isomorphic to $c \otimes Gc$ and shown in [15, Thm. 3.10] that the composition of such an isomorphism with f_c is non-degenerate (the choice of isomorphism was left implicit in [15], but from Corollary 5.25 we know that such a choice is equivalent to a choice of isomorphism $S(c) \cong c$). This matches Corollary 5.24 asserting that such internal Ends admit the structure of a Frobenius algebra.

Example 5.27. Note that the subcategory $\widehat{\mathcal{C}}^{\otimes}$ of admissible objects of $\mathcal{C}$ does not depend on the choice of GV-structure, since it only involves the right exact tensor product $\otimes$. In the pivotal case the subcategory of objects $c \in \widehat{\mathcal{C}}^{\otimes}$ for which there exists an isomorphism $c \cong c \otimes K$ is of interest as well: the objects in this subcategory are precisely those for which $c \cong 1 \circledast c$; they also satisfy $c \cong S(c)$ and their internal End is a Frobenius algebra. This subcategory is obviously very sensitive to the choice of dualizing object K. For an abelian group A and choice of normalized abelian 3-cocycle (F, Ω), consider the braided monoidal category $\mathrm{vect}_{\Bbbk,A}^{(F,\Omega)}$: objects of $\mathrm{vect}_{\Bbbk,A}^{(F,\Omega)}$ are finite dimensional A-graded $\Bbbk$-vector spaces (so the isomorphism classes of simple objects are in bijection with the elements of A), morphisms are grade-preserving linear maps, the tensor functor is the standard tensor product of graded vector spaces, and the braiding and associativity on simple objects are given by Ω and F, respectively. As is well known, this category is rigid (hence $\mathrm{vect}_{\Bbbk,A}^{(F,\Omega)} = \widehat{\mathrm{vect}}_{\Bbbk,A}^{(F,\Omega),\otimes}$) and every simple object $\Bbbk_h$ is invertible. Thus any $\Bbbk_h$, $h \in A$, can be taken to be a dualizing object. If we choose $\Bbbk_0$ to be the dualizing object, then for $m \in \mathrm{vect}_{\Bbbk,A}^{(F,\Omega)}$ the condition $m \cong S_0(m) \cong m \otimes \Bbbk_0$ is empty. On the other hand, if we choose $\Bbbk_h$ with $h \neq 0$ to be the dualizing object, then an isomorphism $m \cong S_h(m) \cong m \otimes \Bbbk_h$ can only exist if m is a direct sum of the form $\bigoplus_{\tilde{h} \in \langle h \rangle} \Bbbk_{g+\tilde{h}}$, where $\langle h \rangle$ is the subgroup of A generated by h. Note that this sum is contained in $\mathrm{vect}_{\Bbbk,A}^{(F,\Omega)}$ only if h has finite order, otherwise completions are required. So objects m admitting isomorphisms $m \cong S_h(m)$ are sums of “cosets of $\langle h \rangle$ in A”.

Acknowledgments

We thank Max Demirdilek and Lukas Woike as well as the anonymous referees for valuable comments. J.F. is supported by the Swedish Research Council VR under project no. 2022-02931. S.W. is supported by the Engineering and Physical Sciences Research Council (EPSRC) EP/V053787/1 and by the Alexander von Humboldt Foundation. C.S. is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy - EXC 2121 “Quantum Universe” - 390833306 and the Collaborative Research Center - SFB 1624 “Higher structures, moduli spaces and integrability” - 506632645.

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