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ALMOST SURE CENTRAL LIMIT THEOREM FOR THE HYPERBOLIC ANDERSON MODEL WITH LÉVY WHITE NOISE

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Abstract. In this paper, we present an almost sure central limit theorem (ASCLT) for the hyperbolic Anderson model (HAM) with a Lévy white noise in a finite-variance setting, complementing a recent work by Balan and Zheng (*Trans. Amer. Math. Soc.*, 2024) on the (quantitative) central limit theorems for the solution to the HAM. We provide two different proofs: one uses the Clark-Ocone formula and takes advantage of the martingale structure of the white-in-time noise, while the other is obtained by combining the second-order Gaussian Poincaré inequality with Ibragimov and Lifshits' method of characteristic functions. Both approaches are different from the one developed in the PhD thesis of C. Zheng (2011), allowing us to establish the ASCLT without lengthy computations of star contractions. Moreover, the second approach is expected to be useful for similar studies on SPDEs with colored-in-time noises, when the former approach, based on Itô calculus, is not applicable.

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1. Introduction

This short note is devoted to the study of a stochastic (linear) wave equation

$$\begin{cases} (\partial_{tt}^2 - \partial_{xx}^2)u = u\dot{L} \\ \big(u(0,\bullet), \partial_t u(0,\bullet)\big) = (1,0) \end{cases} \quad \text{on} \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1.1}$$

driven by a space-time *pure-jump Lévy white noise*; see Section 1.3 for a precise description of the noise and the solution.

1.1. **Central limit theorems for SPDEs.**

In a recent work [24], Huang, Nualart, and Viitasaari established a central limit theorem (CLT) for the spatial integral of the solution to a stochastic nonlinear heat equation with space-time Gaussian white noise and constant initial condition on $\mathbb{R}_+ \times \mathbb{R}$:

$$(\partial_t - \tfrac{1}{2}\partial_{xx}^2)u = \sigma(u)\dot{W} \quad \text{and} \quad u(0,x) = 1, \tag{1.2}$$

where $\dot{W}$ denotes the space-time Gaussian white noise on $\mathbb{R}_+ \times \mathbb{R}$ and the nonlinearity $\sigma(u)$ is described by a Lipschitz function $\sigma : \mathbb{R} \to \mathbb{R}$. More precisely, letting

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$u = \{u(t,x)\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}}$ be the solution to (1.2), and denoting (with $t_0 > 0$ fixed)

$$F_\theta := \int_{-\theta}^{\theta} \big[u(t_0,x) - 1\big]dx \quad \text{and} \quad \sigma_\theta := \sqrt{\mathrm{Var}(F_\theta)}, \tag{1.3}$$

for any $\theta > 0$, the authors of [24] established that F_θ/σ_θ admits Gaussian fluctuation as $\theta \to \infty$. This result was partially motivated by the study of mathematical intermittency of the random field solution. In retrospect, (a) it is not hard to expect that the solution u is spatially stationary due to the constant initial condition and homogeneous noise; (b) it is also natural to investigate furthermore the spatial ergodicity of u that will lead to the first-order result, a law of large number: as $\theta \to +\infty$, $\frac{1}{2\theta}\int_{-\theta}^{\theta} u(t,x)dx \to 1$ in $L^2(\mathbb{P})$ and almost surely; (c) therefore, the central limit theorem (the second-order fluctuation) would naturally come into the picture.

Since the appearance of the work [24], there have been a growing body of literature on similar CLT results for heat equations with various Gaussian noises; see, for example, [25, 40, 37, 15, 16, 17, 39, 42, 45, 36]. Meanwhile, such a program was carried out by Nualart, Zheng, and their collaborators to investigate the stochastic (nonlinear) wave equations driven by Gaussian noises; see [19, 13, 42, 41, 7] and see also the recent works [21, 22] by Ebina on three and higher-dimensional stochastic wave equations. In a recent paper [8] by Balan and Zheng, a similar program for the SPDEs with Lévy noises has been initiated; see Section 1.3 for an overview.

1.2. **Almost sure central limit theorems.** The almost sure central limit theorem (ASCLT) was first formulated by Paul Lévy in his book [32, page 270] without a proof. It had not gained much attention until being rediscovered by various authors in the 1980's ([23, 14, 47, 29]). The simplest form of ASCLT can be stated as follows. Let $\{X_n\}_{n\geq 1}$ be an independent and identically distributed sequence of centered random variables with the unit variance. Then, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k}\delta_{\frac{S_k}{\sqrt{k}}(\omega)} \Longrightarrow \gamma \quad \text{as } n \to +\infty, \tag{1.4}$$

where δ_x denotes the Dirac mass at x, $S_k = X_1 + \dots + X_k$, γ stands for the standard normal distribution, and "$\Longrightarrow$" represents the weak convergence of finite measures. In other words, the Gaussian asymptotic behavior can be observed along a generic trajectory via this logarithmic average. It was explored in several papers [29, 11, 2, 31] that the above hypothesis of i.i.d. sequence of random variables with mean zero and unit variance is optimal for ASCLT. See also [10, 27] for more details. Let us first state a few definitions, one of which generalizes (1.4).

Definition 1.1. (i) [Discrete version] A sequence $\{F_k\}_{k\in\mathbb{N}}$ of real random variables is said to satisfy the ASCLT if for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\mu_N^\omega := \frac{1}{\log N}\sum_{k=1}^{N} \frac{1}{k}\delta_{F_k(\omega)} \Longrightarrow \gamma \quad \text{as } N \to \infty. \tag{1.5}$$

(ii) [Continuum version] A family $\{F_y\}_{y\geq 1}$ of real random variables is said to satisfy the ASCLT if for $\mathbb{P}$-almost every $\omega \in \Omega$, the map $y \mapsto F_y(\omega)$ is almost surely measurable and [1]

[1] The (random) measure ν_T^ω can be understood by first testing it with $f \in C_c(\mathbb{R})$ (continuous function with compact support), i.e., $(\star)$ $\langle \nu_T^\omega, f\rangle = \frac{1}{\log T}\int_1^T f\big(F_y(\omega)\big)\frac{dy}{y}$, which defines a positive

$$\nu_T^\omega := \frac{1}{\log T}\int_1^T \delta_{F_y(\omega)}\frac{dy}{y} \Longrightarrow \gamma \quad \text{as } T\to+\infty. \tag{1.6}$$

In this note, we aim to establish an ASCLT for the spatial integrals of the solution to an SPDE. Then, it is more natural for us to proceed with the continuum version of Definition 1.1, while the discrete analogue can be dealt with in the same way.

Remark 1.2. (i) For simplicity, we take the logarithmic average in (1.5) and (1.6). In fact, one can consider more general weights, for example, using (d_k, D_N) in place of $(\frac{1}{k}, \log N)$, where $d_k > 0$, $D_N = d_1 + \cdots + d_N$ with $D_{N+1}/D_N \to 1$ as $N\to+\infty$; see also the discussion in [29, page 204].
(ii) By the separability of $\mathbb{R}$, there exists a *countable* family $\{\phi_n\}_{n\in\mathbb{N}}$ of bounded Lipschitz functions on $\mathbb{R}$ such that $\{\phi_n\}_{n\in\mathbb{N}}$ forms a *separating class* for the weak convergent probability measures on $\mathbb{R}$; see [27, Proposition 2.2]. Thus, the validity of (1.6) (for $\mathbb{P}$-almost every $\omega\in\Omega$) is equivalent to each of the following statements:[2]

(a) $d_{\rm FM}(\nu_T^\omega,\gamma) \xrightarrow[T\to+\infty]{\text{almost surely}} 0$, with $d_{\rm FM}$ defined as in (2.1);
(b) $\forall f\in C_b(\mathbb{R})$, almost surely, $\frac{1}{\log T}\int_1^T \frac{1}{\theta} f(F_\theta)\,d\theta \xrightarrow{T\to+\infty} \int_{\mathbb{R}} f(x)\gamma(dx)$; here $C_b(\mathbb{R})$ is the set of real bounded continuous functions on $\mathbb{R}$. In view of [38, Proposition C.3.2], the above statements are also equivalent to the following statements;
(c) almost surely, $\forall t\in\mathbb{R}$, $\frac{1}{\log T}\int_1^T \frac{1}{\theta}\mathbf{1}_{\{F_\theta\le t\}}\,d\theta \xrightarrow{T\to+\infty} \gamma((-\infty,t])$;
(d) $d_{\rm Kol}(\nu_T^\omega,\gamma) \xrightarrow[T\to+\infty]{\text{almost surely}} 0$, with $d_{\rm Kol}$ defined as in (2.3).

The usual CLT does not require the random variables to be defined on a common probability space, while the ASCLT does; and furthermore, the convergence to normality at a fast rate does not necessarily imply the ASCLT. This can be readily confirmed by, for example, setting $F_k \equiv Y\sim\gamma$ for all k, leading to a clear violation of the ASCLT (1.5). In other words, a finer understanding of the whole family of random variables is necessary to investigate the ASCLT.

1.3. SPDEs driven with Lévy noises.

Stochastic differential equations driven by Lévy processes have been studied intensively in the literature since 1970, often using semi-martingale techniques. There are already several monographs dedicated to this topic (e.g., [12, 44, 1]). The study of SPDEs driven by Lévy noise is a relatively new area in stochastic analysis, which extends these techniques to problems that incorporate a spatially-dependent component for the noise. These equations can be studied using either the variational approach (developed at length in the mongraph [43]), or the random field approach (see, e.g., [5, 6]). One needs to distinguish between the finite-variance case (in which many techniques are similar to the Gaussian case) and the infinite-variance case (see, e.g., [4, 28]).

The present paper constitues a new contribution to this area, and focuses on the hyperbolic Anderson model in dimension 1, driven by a finite-variance Lévy noise. This model can be used for describing the evolution of a wave perturbed by random

linear functional on $C_c(\mathbb{R})$, and hence, as a result of Riesz's representation theorem, ν_T^ω defined in (1.6) makes sense as the unique Radon measure satisfying $(\star)$

[2](b) is equivalent to the seemingly stronger statement: (b') almost surely, for any bounded continuous f, $\frac{1}{\log T}\int_1^T \frac{1}{\theta} f(F_\theta)\,d\theta \xrightarrow{T\to+\infty} \int_{\mathbb{R}} f(x)\gamma(dx)$.

forces, which are characterized by a sequence of impulses in space-time (described rigorously by a Poisson random measure). Models with this type of noise could be useful in a variety of situations, when the Gaussian space-time white noise is not a correct model for describing the sources of randomness perturbing the system.

Consider the equation (1.1) driven by a space-time Lévy white noise $\dot{L}$. Throughout this paper, we make the following assumptions:

(i) $\dot{L}$ is the space-time (pure-jump) Lévy noise on $\mathbb{R}_+ \times \mathbb{R}$ with finite variance:

- (i-a) let $\mathbf{Z} := \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_0$, with $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ equipped with the distance $d(x,y) = |x^{-1} - y^{-1}|$, and let $\mathcal{Z}$ be the Borel σ-algebra on $\mathbf{Z}$, and let $m = \text{Leb} \times \nu$, with Leb the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}$ and ν a σ-finite measure on $\mathbb{R}_0$ satisfying the *Lévy-measure condition* $\int_{\mathbb{R}_0} \min\{1, |z|^2\}\nu(dz) < \infty$ and the *finite-variance condition* $m_2 := \int_{\mathbb{R}_0} |z|^2 \nu(dz) < \infty$;
- (i-b) let N be a Poisson random measure on the space $(\mathbf{Z}, \mathcal{Z})$ with intensity m, and let $\widehat{N} = N - m$ be the compensated version of N, and we put $L(A) = \int_{A \times \mathbb{R}_0} z \widehat{N}(ds, dy, dz)$ for any Borel set $A \subset \mathbb{R}_+ \times \mathbb{R}_0$ with $\text{Leb}(A) < \infty$, where the above integral $L(A)$ is defined in the Itô sense, which is an *infinitely divisible* random variable with Lévy-Khintchine formula $\mathbb{E}\big[e^{i\lambda L(A)}\big] = \exp\big(\text{Leb}(A) \int_{\mathbb{R}_0} (e^{i\lambda z} - 1 - i\lambda z)\nu(dz)\big)$, $\lambda \in \mathbb{R}$;
- (i-c) $\dot{L}(s,y) = L(ds, dy)$ is the formal derivative $\partial_s \partial_y L$;

(ii) the product of the unknown u and the Lévy noise $\dot{L}$ is intetpreted in the Itô sense; and in Duhamel formulation, we rewrite (1.1) in the following integral form:

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) u(s,y) L(ds, dy), \tag{1.7}$$

and $G_t(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}$ denotes the wave kernel.[3]

It is known that equation (1.1) admits a unique solution $u = \{u(t,x)\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}}$ that is an $\mathbb{F}$-predictable process satisfying (1.7) and $\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathbb{E}[|u(t,x)|^2] < \infty$ for any finite $T > 0$; see [6, Theorem 1.1]. Here, $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ denotes the natural filtration generated by the Lévy noise $\dot{L}$.[4]

In a recent work [8], Balan and Zheng established the following result on spatial ergodicity and CLT results for (1.1).

Theorem 1.3 ([8, Theorem 1.1]). *Let $u = \{u(t,x)\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}}$ be the solution to* (1.1)*. Fix $t_0 > 0$, and let F_θ and σ_θ be defined as in* (1.3) *for $\theta > 0$. Then,*

- (i) *$\{u(t_0, x)\}_{x\in\mathbb{R}}$ is strictly stationary and ergodic.*
- (ii) *$\sigma_\theta = \sqrt{\text{Var}(F_\theta)} \asymp \sqrt{\theta}$ as $\theta \to \infty$.*
- (iii) *Assume additionally that $m_{2+2\alpha} + m_{1+\alpha} < \infty$ for some $\alpha \in (0,1]$,*[5] *where*

$$m_p := \int_{\mathbb{R}_0} |z|^p \nu(dz), \quad p \in [1, \infty). \tag{1.8}$$

[3] The integral in (1.7) is understood as $\int_{\mathbb{R}\times\mathbb{R}} X(s,y) L(ds,dy) = \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}_0} X(s,y) z \widehat{N}(ds,dy,dz)$, whenever the right side exists.

[4] More precisely, let $\mathcal{F}_t^0$ be the σ-algebra generated by the random variables $N([0,s]\times A \times B)$ with $s \in [0,t]$ and $\text{Leb}([0,s]\times A) + \nu(B) < \infty$. And let $\mathcal{F}_t = \sigma\big(\mathcal{F}_t^0 \cup \mathcal{N}\big)$ be the σ-algebra generated by $\mathcal{F}_t^0$ and the set $\mathcal{N}$ of $\mathbb{P}$-null sets. This gives us a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$.

[5] In particular, this forces m_2 to be finite.

Then, $\operatorname{dist}\big(\frac{F_\theta}{\sigma_\theta}, \mathcal{N}(0,1)\big) \lesssim \theta^{-\frac{\alpha}{1+\alpha}}$*, where the implicit constant does not depend on* θ*, and one can choose the distributional metric* dist *to be one of the following: Fortet-Mourier, 1-Wasserstein, and Kolmogorov distances* (*see* (2.1)-(2.3)).

Note that $\sigma_\theta > 0$ for $\theta > 0$; see [8, Remark 4.3]. One of the key ingredients for part (iii) is a second-order Poincaré inequality that goes back to Last, Peccati, and Schulte's paper [30], and has been recently improved by Trauthwein [48].

The goal of the current note is to provide the following ASCLT.

Theorem 1.4. *With the notation as in Theorem 1.3 and let* $\widetilde{F}_\theta = F_\theta/\sigma_\theta$ *for* $\theta > 0$*. Assume* $m_{1+\alpha}+m_{2+2\alpha} < \infty$ *for some* $\alpha \in (0,1]$*. Then,* $\{\widetilde{F}_\theta\}_{\theta\geq 1}$ *satisfy the* ASCLT.

The paper is organized as follows. In Section 2, we collect some preliminaries for the proofs of Theorem 1.4 presented in Section 3. Proofs of technical lemmas will be given in the Appendix A.

2. Preliminaries

2.1. **Notations.** In this paper, all random objects are defined on a rich common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{E}$, $\operatorname{Var}, \operatorname{Cov}$ stand for the associated expectation, variance, covariance operators respectively. We use $\|X\|_p := \|X\|_{L^p(\Omega)}$ for any $p \in [1,\infty)$ and real-valued random variable X, and we write $\|f\|_\infty$ for the essential-sup norm for any measurable $f : \mathbb{R} \to \mathbb{R}$. We write $Y \sim \gamma$ to mean that Y is a standard normal random variable. For any two (probability) measures μ_1 and μ_2 on $\mathbb{R}$, we define the Fortet-Mourier distance

$$d_{\mathrm{FM}}(\mu_1,\mu_2) := \sup_{\|\phi\|_\infty + \|\phi\|_\infty < 1} \Big| \int_{\mathbb{R}} \phi(x)\mu_1(dx) - \int_{\mathbb{R}} \phi(x)\mu_2(dx)\Big|. \tag{2.1}$$

It is well-known that d_{FM} characterizes the weak convergence of probability measures on $\mathbb{R}$; see [20, Theorem 11.3.3]. Another two stronger metrics, the Wasserstein distance d_{Wass} and the Kolmogorov distance d_{Kol}, are defined, respectively, by

$$d_{\mathrm{Wass}}(\mu_1,\mu_2) := \sup_{\|\phi'\|_\infty \leq 1} \Big| \int_{\mathbb{R}} \phi(x)\mu_1(dx) - \int_{\mathbb{R}} \phi(x)\mu_2(dx)\Big|, \tag{2.2}$$

$$d_{\mathrm{Kol}}(\mu_1,\mu_2) := \sup_{t\in\mathbb{R}} |\mu_1((-\infty,t]) - \mu_2((-\infty,t])|. \tag{2.3}$$

For two random variables X and Y, we also write $d_{\mathrm{Wass}}(X,Y)$ for the Wasserstein distance between their distributions. On the L^2 probability space generated by the Poisson random measure N, one can develop the Malliavin calculus; see [8, Section 2] for more details and for any unexplained notation. In particular, we recall that for $F \in \mathbb{D}^{1,2}$, $D_\xi F$ denotes the Malliavin derivative $DF \in L^2(\Omega; L^2(\mathbf{Z}, m))$ valued at $\xi \in \mathbf{Z}$, where for convenience, ξ and $m(d\xi)$ are short for $\xi = (r,y,z) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_0$ and $m(d\xi) = dr \times dy \times \nu(dz)$ respectively, whenever no ambiguity arises. For two functions $g(t)$ and $h(t)$, $g(t) \asymp h(t)$ means that $0 < c_1 \leq \liminf_{t\to+\infty} g(t)/h(t) \leq \limsup_{t\to+\infty} g(t)/h(t) \leq c_2 < \infty$ for some constants c_1 and c_2.

2.2. **Preliminary results.**

Proposition 2.1 ([8, Proposition 3.2])**.** *Let* m_p *be defined as in* (1.8) *for* $p \geq 1$*. Assume* $m_2 + m_{2+2\alpha} < \infty$ *for some* $\alpha \in (0,1]$*. Then, for any* $0 < r_1 < r_2 < t_0$*,*

$$\|D_{r_1,y,z}u(t_0,x)\|_{2+2\alpha} \lesssim G_{t_0-r_1}(x-y)|z|,$$

and

$$\|D_{r_2,y_2,z_2}D_{r_1,y_1,z_1}u(t_0,x)\|_{2+2\alpha} \lesssim |z_1z_2|G_{t_0-r_2}(x-y_2)G_{r_2-r_1}(y_2-y_1). \tag{2.4}$$

The following proposition is a variant of [48, Theorem 3.4] for random variables whose variances are not necessarily one. The modification is standard within the Stein's method ([46, 18, 38]). For the completeness, we provide a sketchy proof.

First, we recall the framework used in [48]. Let $(\mathbb{X},\mathcal{X},\lambda)$ be a σ-finite measure space, and $\mathbf{N}_{\mathbb{X}}$ be the set of $\mathbb{N}_0\cup\{\infty\}$-valued measures on $\mathbb{X}$. Let $\mathcal{N}_{\mathbb{X}}$ be the smallest σ-field on $\mathbf{N}_{\mathbb{X}}$ for which all maps $\mathbf{N}_{\mathbb{X}}\ni\xi\mapsto\xi(B)$ are measurable for $B\in\mathcal{X}$.

Let χ be a Poisson random measure (PRM) on $\mathbb{X}$ of intensity λ, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. For any χ-measurable random variable F, there exists a measurable function $f:\mathbf{N}_{\mathbb{X}}\to\mathbb{R}$ such that $F=f(\chi)$ a.s. For such a variable, we define the add-one cost operator $D_xF=f(\chi+\delta_x)-f(\chi)$ for all $x\in\mathbb{X}$. It is known that for $F\in\mathbb{D}^{1,2}$, DF coincides with the add-one cost operator and hence we will simply use the same notation; see [8, Remark 2.7]. In our framework, $(\mathbb{X},\mathcal{X},\lambda)=(\mathbf{Z},\mathcal{Z},m)$. If η is a PRM on $\mathbb{Y}=\mathbb{X}\times[0,1]$ of intensity $\overline{\lambda}=\lambda\otimes dt$, then $\chi=\eta(\cdot\times[0,1])$ is a PRM on $\mathbb{X}$ of intensity λ, and for any χ-measurable random variable F, $D_{(x,s)}F=D_xF$ for all $(x,s)\in\mathbb{Y}$.

Proposition 2.2. *Let χ be a Poisson random measure on $\mathbb{X}$ of intensity λ. Let $F\in\mathbb{D}^{1,2}$ be such that $\mathbb{E}(F)=0$ and $\mathbb{E}(F^2)=\sigma^2$. Then,*

$$d_{\rm Wass}(F,\mathcal{N}(0,1))\le\frac{2}{\sqrt{\pi}}|1-\sigma^2|+\gamma_1+\gamma_2+\gamma_3, \tag{2.5}$$

where, for $p,q\in(1,2]$,

$$\begin{aligned}
\gamma_1 &:= \frac{2^{\frac{2}{p}+\frac12}}{\sqrt{\pi}}\bigg(\int_{\mathbb{X}}\bigg[\int_{\mathbb{X}}\|D_{x_2}F\|_{2p}\|D_{x_1}D_{x_2}F\|_{2p}\,\lambda(dx_2)\bigg]^p\lambda(dx_1)\bigg)^{\frac1p},\\
\gamma_2 &:= \frac{2^{\frac{2}{p}-\frac12}}{\sqrt{\pi}}\bigg(\int_{\mathbb{X}}\bigg[\int_{\mathbb{X}}\|D_{x_1}D_{x_2}F\|_{2p}^2\,\lambda(d\xi_2)\bigg]^p\lambda(dx_1)\bigg)^{\frac1p},\\
\gamma_3 &:= 2\int_{\mathbb{X}}\mathbb{E}|D_xF|^{q+1}\,\lambda(dx).
\end{aligned} \tag{2.6}$$

Proof. Let η be given as above, then we can view F as a η-measurable random variable F with $D_yF=D_xF$ for all $y=(x,s)\in\mathbb{Y}$.

We use [48, (3.3)] (whose proof does not rely on the fact that $\mathbb{E}(F^2)=1$):

$$d_{\rm Wass}(F,\mathcal{N}(0,1))\le\sqrt{2/\pi}\,\mathbb{E}\left|1-\sigma^2-\mathbf{G}\right|+2\int_{\mathbb{Y}}|E[D_yF|\eta]|\cdot|D_yF|^q\overline{\lambda}(dy), \tag{2.7}$$

where $\mathbf{G}:=\int_{\mathbb{Y}}D_yF\,\mathbb{E}[D_yF|\eta]\overline{\lambda}(dy)-\sigma^2$ has mean zero in view of [48, (D.20)], and the second term in (2.7) is bounded by γ_3 in view of [48, (D.47)]. The same argument as for [48, (D.18)] shows that $\mathbb{E}|\mathbf{G}|\le\beta_1'+\beta_2'$, where β_1' and β_2' have the same expressions as β_1,β_2 in [48, Theorem 3.2], *but without* σ^{-2}, and without $\sqrt{2/\pi}$ (which is a typo in [48]). Hence, the above Wasserstein distance (2.7) is bounded by $\sqrt{2/\pi}|1-\sigma^2|+\sqrt{2/\pi}\mathbb{E}|\mathbf{G}|+\gamma_3$. Then $\sqrt{2/\pi}(\beta_1'+\beta_2')=\beta_1+\beta_2$, where β_1 and β_2 are given as in [48, Theorem 3.2] without the factor σ^{-2}. The conclusion follows since $\beta_1+\beta_2\le\gamma_1+\gamma_2$, by [48, D.46]. □

Remark 2.3. Proposition 2.2 will be combined with Ibragimov-Lifshits' method of characteristic functions (Proposition 3.3) to provide the second proof of Theorem

1.4 in Section 3.2. It is important to note that F in (2.5) may not have the unit variance, whereas the original statement of [48, Theorem 3.4] gives an estimate for $d_{\rm Wass}(\widehat{F}, Y)$ with $\widehat{F} = (F - \mathbb{E}(F))/\sqrt{\mathrm{Var}(F)}$. This seemingly minor modification is crucial for obtaining (3.14): applying directly [48, Theorem 3.4] would require a uniform (positive) lower bound for the variance $V_{\theta,\rm w}$ in (3.15), but $V_{\theta,\theta} = 0$.

3. Two proofs of Theorem 1.4

3.1. **Proof of Theorem 1.4 via the Clark-Ocone formula.** In view of Remark 1.2-(ii), it suffices to show

$$\frac{1}{\log T}\int_1^T \frac{1}{\theta} f(\widetilde{F}_\theta)d\theta \xrightarrow[T\to+\infty]{a.s.} \int_{\mathbb{R}} f(x)\gamma(dx) \tag{3.1}$$

for any bounded Lipschitz function f on $\mathbb{R}$ with the Lipschitz constant $\mathrm{Lip}(f)$. By Theorem 1.3-(iii), $\lim_{\theta\to\infty}\mathbb{E}[f(\widetilde{F}_\theta)] = \int_{\mathbb{R}} f(x)\gamma(dx)$. Then, (3.1) is equivalent to

$$\frac{1}{\log T}\int_1^T \frac{1}{\theta}\big(f(\widetilde{F}_\theta) - \mathbb{E}[f(\widetilde{F}_\theta)]\big)d\theta \xrightarrow[T\to+\infty]{a.s.} 0. \tag{3.2}$$

Put

$$H_\theta \coloneqq f(\widetilde{F}_\theta) - \mathbb{E}[f(\widetilde{F}_\theta)]. \tag{3.3}$$

Then, one can deduce from the Clark-Ocone formula (see, e.g., [8, Lemma 2.5]) that

$$H_\theta = \int_0^{t_0}\int_{\mathbb{R}}\int_{\mathbb{R}_0} \mathbb{E}\big[D_{r,y,z}H_\theta|\mathcal{F}_r\big]\widehat{N}(dr,dy,dz), \tag{3.4}$$

which is an Itô integral with respect to the compensated Poisson random measure $\widehat{N}$. Then, we deduce from the Itô isometry, the chain rule (e.g. [8, (2.42)]), the Cauchy-Schwarz, and Jensen's inequalities that for $\theta < \rm w$,

$$\big|\mathbb{E}[H_\theta H_{\rm w}]\big| \le \frac{\mathrm{Lip}^2(f)}{\sigma_\theta\sigma_{\rm w}}\int_0^{t_0}\int_{\mathbb{R}}\int_{\mathbb{R}_0}\|D_{r,y,z}F_\theta\|_2\cdot\|D_{r,y,z}F_{\rm w}\|_2\, drdy\nu(dz); \tag{3.5}$$

while Proposition 2.1, together with Minkowski's inequality, implies that

$$\|D_{r,y,z}F_\theta\|_2 \le \int_{-\theta}^{\theta}\|D_{r,y,z}u(t_0,x)\|_2\,dx \lesssim |z|\int_{-\theta}^{\theta}G_{t_0-r}(x-y)dx. \tag{3.6}$$

Thus, it follows from (3.5), (3.6), Theorem 1.3-(ii), and the following inequalities

$$G_{t_0-r}(\bullet) \le G_{t_0}(\bullet), \quad G_{t_0}(x_1-y)G_{t_0}(x_2-y) \le G_{t_0}(x_1-y)G_{2t_0}(x_1-x_2)$$

that

$$\begin{aligned}\big|\mathbb{E}[H_\theta H_{\rm w}]\big| &\lesssim \frac{1}{\sqrt{\theta \rm w}}\int_0^{t_0}\int_{\mathbb{R}}\int_{\mathbb{R}_0}|z|^2\left(\int_{-\theta}^{\theta}G_{t_0-r}(x_1-y)dx_1\right)\\ &\qquad\cdot\left(\int_{-\rm w}^{\rm w}G_{t_0-r}(x_2-y)dx_2\right)drdy\nu(dz)\\ &\le \frac{m_2t_0}{\sqrt{\theta\rm w}}\int_{\mathbb{R}}\left(\int_{-\theta}^{\theta}G_{t_0}(x_1-y)dx_1\right)\cdot\left(\int_{-\rm w}^{\rm w}G_{2t_0}(x_2-x_1)dx_2\right)dy\\ &\le 4m_2t_0^3\big(\theta/\rm w\big)^{\frac12} \qquad \text{for } \theta < \rm w,\end{aligned} \tag{3.7}$$

where the last step is obtained by integrating in the order of dx_2, dy, then dx_1. Hence, the proof of (3.2) is done by invoking the following Lemma 3.1. ☐

Lemma 3.1. *Let $\{H_\theta\}_{\theta>0}$ be a family uniformly bounded random variables with*

$$\big|\mathbb{E}[H_\theta H_{\mathrm{w}}]\big| \le C_\beta \big(\theta/\mathrm{w}\big)^\beta$$

for any $0<\theta<\mathrm{w}$, where the exponent $\beta>0$ and the constant C_β do not depend on (θ,w). Assume also that $\theta\mapsto H_\theta$ is a measurable function almost surely. Then,

$$L_T := \frac{1}{\log T}\int_1^T \frac{1}{\theta} H_\theta d\theta \xrightarrow[T\to+\infty]{a.s.} 0. \tag{3.8}$$

A discrete analogue of Lemma 3.1 can be formulated in a straightforward manner. Lemma 3.1 follows essentially from a Borel-Cantelli argument, and for the sake of completeness, we present a short proof in the Appendix A.

Remark 3.2. The aforementioned Lemma 3.1 for $H_\theta = f(\widetilde{F}_\theta) - \mathbb{E}\big[f(\widetilde{F}_\theta)\big]$ with bounded Lipschitz f is enough to prove Theorem 1.4. A crucial point is the usage of the Clark-Ocone representation H_θ as an Itô integral (3.4), and applying the moment estimate of the Malliavin derivatives (Proposition 2.1). This idea has been used in papers [33, 34, 35] for similar studies on SPDEs driven by Gaussian noises. However when the noise is not white in time (e.g., [7, 39]), the strategy, based on Itô's calculus, is not applicable any more. This motivates us to provide another proof using a combination of the Ibragimov-Lifshits' criteria for ASCLT and the second-order Gaussian Poincaré inequalities; see also Remark 3.4 for a further discussion.

3.2. **Proof of Theorem 1.4 based on the Ibragimov-Lifshits' criterion.** Let us first state a variant of the Ibragimov-Lifshits' result.

Proposition 3.3. ([26, Ibragimov-Lifshits' criterion]) *A family of random variables $\{F_\theta\}_{\theta\ge1}$ satisfies the* ASCLT *if $\theta\mapsto F_\theta$ is measurable almost surely, and*

$$\sup_{|s|\le T}\int_2^\infty \frac{\mathbb{E}\big[|\mathbf{K}_t(s)|^2\big]}{t\log t}dt<\infty, \tag{3.9}$$

for any finite $T>0$, where

$$\mathbf{K}_t(s) := \frac{1}{\log t}\int_1^t \frac{1}{\theta}\big(e^{isF_\theta} - e^{-s^2/2}\big)d\theta, \quad t\in(1,\infty). \tag{3.10}$$

In Ibragimov-Lifshits' original paper [26], the criterion is proved for the discrete version (see Definition 1.1-(i)). For readers' convenience, we provide a sketchy proof of Proposition 3.3 in the Appendix A though it is almost identical to that in [26].

Let F_θ be defined as in (1.3) and $\mathbf{K}_t(s)$ be given by (3.10) with F_θ replaced by $\widetilde{F}_\theta = \frac{1}{\sigma_\theta}F_\theta$. By expanding $|\mathbf{K}_t(s)|^2$, we write

$$\begin{aligned}|\mathbf{K}_t(s)|^2 &= \frac{1}{(\log t)^2}\int_{[1,t]^2}\frac{1}{\theta\mathrm{w}}\big(e^{is\widetilde{F}_\theta} - e^{-\frac{s^2}{2}}\big)\big(e^{-is\widetilde{F}_\mathrm{w}} - e^{-\frac{s^2}{2}}\big)d\theta d\mathrm{w}\\ &= \frac{1}{(\log t)^2}\int_{[1,t]^2}\frac{1}{\theta\mathrm{w}}\big(e^{is(\widetilde{F}_\theta-\widetilde{F}_\mathrm{w})} + e^{-s^2} - e^{is\widetilde{F}_\theta}e^{-\frac{s^2}{2}} - e^{-is\widetilde{F}_\mathrm{w}}e^{-\frac{s^2}{2}}\big)d\theta d\mathrm{w}\\ &= \mathbb{I}_t(s) - e^{-\frac{s^2}{2}}\mathbb{II}_t(s),\end{aligned}$$

where

$$\begin{aligned}
\mathbb{I}_t(s) &:= \frac{1}{(\log t)^2}\int_{[1,t]^2}\frac{1}{\theta \mathrm{w}}\Big(e^{is(\widetilde{F}_\theta-\widetilde{F}_{\mathrm{w}})}-e^{-s^2}\Big)\,d\theta d\mathrm{w}, \\
\mathbb{II}_t(s) &:= \frac{1}{\log t}\int_1^t \frac{1}{\theta}\big(e^{is\widetilde{F}_\theta}+e^{-is\widetilde{F}_\theta}-2e^{-\frac{s^2}{2}}\big)\,d\theta.
\end{aligned} \tag{3.11}$$

Therefore, thanks to Proposition 3.3, it suffices to show that

$$A_1(s) := \int_2^\infty \frac{\mathbb{E}\big[\mathbb{I}_t(s)\big]}{t\log t}dt \quad \text{and} \quad A_2(s) := \int_2^\infty \frac{\mathbb{E}\big[\mathbb{II}_t(s)\big]}{t\log t}dt, \quad s\in[-T,T]$$

are both uniformly bounded for any given $T>0$.

⋆ *Estimation of the A_2 term.* Applying Euler's formula $e^{iz}+e^{-iz}=2\cos(z)$, $z\in\mathbb{R}$, and $\mathbb{E}[e^{isY}]=e^{-\frac{s^2}{2}}$ for $Y\sim\gamma$, one can write, with $\phi_s(z)=2\cos(sz)$, that

$$\Big|\mathbb{E}\big(e^{is\widetilde{F}_\theta}+e^{-is\widetilde{F}_\theta}-2e^{-\frac{s^2}{2}}\big)\Big| = \Big|\mathbb{E}\big(\phi_s(\widetilde{F}_\theta)-\phi_s(Y)\big)\Big| \le 2|s| d_{\mathrm{Wass}}(\widetilde{F}_\theta, Y), \tag{3.12}$$

where we used (2.2) and the fact that ϕ_s is a Lipschitz function with $\mathrm{Lip}(\phi_s)\le 2|s|\le 2T$. Then, it follows from (3.11), (3.12), and Theorem 1.3-(iii) that

$$\sup_{|s|\le T}\big|\mathbb{E}[\mathbb{II}_t(s)]\big| \lesssim_T \frac{1}{\log t}\int_1^\infty \frac{d\theta}{\theta^{1+\frac{\alpha}{1+\alpha}}} \lesssim_T \frac{1}{\log t},$$

where $\lesssim_T$ suggests the implicit constant in the inequality only depends on T. Therefore, it holds that $\sup\{|A_2(s)| : s\in[-T,T]\}<\infty$ for any finite $T>0$.

⋆ *Estimation of the A_1 term.* We first write

$$\begin{aligned}
\sup_{|s|\le T} A_1(s) &\le 2\sqrt{2}T\int_2^\infty \frac{1}{t(\log t)^3}\bigg(\int_{[1,t]^2}\frac{1}{\theta\mathrm{w}}\,d_{\mathrm{Wass}}\big(\tfrac{\widetilde{F}_\theta-\widetilde{F}_{\mathrm{w}}}{\sqrt{2}},Y\big)\,d\theta d\mathrm{w}\bigg)dt \\
&= 4\sqrt{2}T\int_2^\infty \frac{1}{t(\log t)^3}\bigg(\int_{1<\theta<\mathrm{w}<t}\frac{1}{\theta\mathrm{w}}\,d_{\mathrm{Wass}}\big(\tfrac{\widetilde{F}_\theta-\widetilde{F}_{\mathrm{w}}}{\sqrt{2}},Y\big)\,d\theta d\mathrm{w}\bigg)dt,
\end{aligned} \tag{3.13}$$

where we used $d_{\mathrm{Wass}}(-X,Y)=d_{\mathrm{Wass}}(X,Y)$ for $Y\sim\gamma$ and the following bound

$$\big|\mathbb{E}\big[e^{is(\widetilde{F}_\theta-\widetilde{F}_{\mathrm{w}})}-e^{-s^2}\big]\big| = \big|\mathbb{E}\big[e^{i\sqrt{2}s(\frac{\widetilde{F}_\theta-\widetilde{F}_{\mathrm{w}}}{\sqrt{2}})}-e^{i\sqrt{2}sY}\big]\big| \le 2\sqrt{2}|s| d_{\mathrm{Wass}}\big(\tfrac{\widetilde{F}_\theta-\widetilde{F}_{\mathrm{w}}}{\sqrt{2}},Y\big).$$

We **claim** that there exist positive real numbers β_1,β_2, and β_3 such that

$$d_{\mathrm{Wass}}\big(\tfrac{\widetilde{F}_\theta-\widetilde{F}_{\mathrm{w}}}{\sqrt{2}},Y\big) \lesssim \theta^{-\beta_1}+\mathrm{w}^{-\beta_2}+(\theta/\mathrm{w})^{\beta_3} \tag{3.14}$$

for $1<\theta<\mathrm{w}<\infty$. It is then easy to deduce the finiteness of (3.13) from the claim (3.14). The rest of the proof is then devoted to verifying the claim (3.14).

Observe that $\widetilde{F}_\theta$ and $\widetilde{F}_{\mathrm{w}}$ are centered with unit variance, and thus,

$$V_{\theta,\mathrm{w}} := \mathbb{E}\Big[\big(\tfrac{\widetilde{F}_\theta-\widetilde{F}_{\mathrm{w}}}{\sqrt{2}}\big)^2\Big] = 1-\mathrm{Cov}(\widetilde{F}_\theta,\widetilde{F}_{\mathrm{w}}). \tag{3.15}$$

It follows from Proposition 2.2 with (3.15) that

$$d_{\mathrm{Wass}}\big(\tfrac{\widetilde{F}_\theta-\widetilde{F}_{\mathrm{w}}}{\sqrt{2}},Y\big) \le |\mathrm{Cov}(\widetilde{F}_\theta,\widetilde{F}_{\mathrm{w}})| + \sum_{j=1}^3 \gamma_j(\theta,\mathrm{w}), \tag{3.16}$$

where $\gamma_j(\theta,\mathrm{w})$, $j=1,2,3$, are defined as in (2.6) with $F=\frac{\widetilde{F}_\theta-\widetilde{F}_{\mathrm{w}}}{\sqrt{2}}$ and $p=1+\alpha$, $q=\min\{2,1+2\alpha\}$ for some $\alpha\in(0,1]$; namely, ignoring the constants

$$\gamma_1 := \left[\int_{\mathbf{Z}}\left(\int_{\mathbf{Z}}\big\|D_{\xi_2}(\widetilde{F}_\theta-\widetilde{F}_{\rm w})\big\|_{2p}\big\|D_{\xi_1}D_{\xi_2}(\widetilde{F}_\theta-\widetilde{F}_{\rm w})\big\|_{2p}\,m(d\xi_2)\right)^{1+\alpha} m(d\xi_1)\right]^{\frac{1}{1+\alpha}},$$

$$\gamma_2 := \left[\int_{\mathbf{Z}}\left(\int_{\mathbf{Z}}\big\|D_{\xi_1}D_{\xi_2}(\widetilde{F}_\theta-\widetilde{F}_{\rm w})\big\|_{2p}^2\,m(d\xi_2)\right)^{1+\alpha} m(d\xi_1)\right]^{\frac{1}{1+\alpha}},$$

and

$$\gamma_3 := \int_{\mathbf{Z}}\big\|D_\xi(\widetilde{F}_\theta-\widetilde{F}_{\rm w})\big\|_{q+1}^{q+1}\,m(d\xi).$$

Step (i). The estimate for $\mathrm{Cov}(\widetilde{F}_\theta,\widetilde{F}_{\rm w})$ is already there, if one notices that $\mathrm{Cov}(\widetilde{F}_\theta,\widetilde{F}_{\rm w}) = \mathbb{E}[H_\theta H_{\rm w}]$, where H is defined as in (3.3) with $f(x)=x$ for all $x\in\mathbb{R}$. As a consequence of (3.7), for all $0<\theta<{\rm w}$,[6]

$$|\mathrm{Cov}(\widetilde{F}_\theta,\widetilde{F}_{\rm w})| \lesssim (\theta/{\rm w})^{\frac{1}{2}}. \tag{3.17}$$

Step (ii). In this step we provide the estimate for γ_1. It is clear that

$$\gamma_1^{1+\alpha}(\theta,{\rm w}) \lesssim \mathbf{T}_1(\theta,\theta)+\mathbf{T}_1(\theta,{\rm w})+\mathbf{T}_1({\rm w},\theta)+\mathbf{T}_1({\rm w},{\rm w}), \tag{3.18}$$

where

$$\mathbf{T}_1(\theta,{\rm w}) := \int_{\mathbf{Z}}\left[\int_{\mathbf{Z}}\|D_{\xi_2}\widetilde{F}_\theta\|_{2p}\|D_{\xi_1}D_{\xi_2}\widetilde{F}_{\rm w}\|_{2p}\,m(d\xi_2)\right]^{1+\alpha} m(d\xi_1).$$

Note that $\mathbf{T}_1(\theta,\theta)$ and $\mathbf{T}_1({\rm w},{\rm w})$ have been already dealt with in [8], and we have

$$\mathbf{T}_1(\theta,\theta)\lesssim\theta^{-\alpha}\quad\text{and}\quad\mathbf{T}_1({\rm w},{\rm w})\lesssim{\rm w}^{-\alpha}; \tag{3.19}$$

see equations (4.21)–(4.24) therein. The other two terms in (3.18) can be addressed similarly. Using Proposition 2.1, we can write next inequities analogous to (3.6),

$$\begin{aligned}\|D_{r,y,z}F_\theta\|_{2p} &\lesssim |z|\cdot\int_{-\theta}^{\theta}G_{t_0}(x-y)dx,\\ \|D_{r_1,y_1,z_1}D_{r_2,y_2,z_2}F_{\rm w}\|_{2p} &\lesssim |z_1z_2|\cdot\int_{-{\rm w}}^{{\rm w}}G_{t_0}(x-y_1)G_{t_0}(y_2-y_1)dx,\end{aligned} \tag{3.20}$$

where t_0 is fixed as in Proposition 2.1; see also equations (4.18)–(4.20) in [8].[7]

Now, we are ready to estimate $\mathbf{T}_1(\theta,{\rm w})$ by utilizing (3.20) and Theorem 1.3-(ii):

$$\begin{aligned}\mathbf{T}_1(\theta,{\rm w}) &\le \frac{1}{(\sigma_\theta\sigma_{\rm w})^{1+\alpha}}\int_0^{t_0}\int_{\mathbb{R}\times\mathbb{R}_0}m(d\xi_1)\,|z_1|^{1+\alpha}\left[\int_0^{t_0}\int_{\mathbb{R}\times\mathbb{R}_0}m(d\xi_2)\,|z_2|^2\right.\\ &\qquad\left.\cdot\left(\int_{-\theta}^{\theta}G_{t_0}(x_2-y_2)dx_2\right)\left(\int_{-{\rm w}}^{{\rm w}}G_{t_0}(x_1-y_1)G_{t_0}(y_2-y_1)dx_1\right)\right]^{1+\alpha}\\ &\lesssim \frac{1}{(\theta{\rm w})^{\frac{1+\alpha}{2}}}\int_{\mathbb{R}}dy_1\widetilde{\mathbf{T}}_1(\theta,{\rm w},y_1)^{1+\alpha},\end{aligned}$$

---

[6] To limit the length of this note, here we take a shortcut by using directly (3.7) although this does not really fit the spirit of the second proof. Instead of an application of (3.7), the explicit chaos expansion of $\widetilde{F}_\theta$ can be used to verify the bound (3.17); we leave this for interested readers. It is worth pointing out that this approach based on the chaos expansion requires $f(x)\equiv x$ for all $x\in\mathbb{R}$ in (3.3), and the obtention of (3.7) for general f crucially relies on the Clark-Ocone formula for $f(\widetilde{F}_\theta)$, which is not available in the colored-in-time setting.

[7] The bounds in (3.20) follow easily from those in [8, (4.18)–(4.20)] and triangle inequality with the fact $G_{t_0-r}\le G_{t_0}$.

where $m(d\xi_i) = dr_i dy_i \nu(dz_i)$ for $i = 1, 2$ (see Section 2);

$$\widetilde{\mathbf{T}}_1(\theta, \mathrm{w}, y_1) := \int_{\mathbb{R}} dy_2 \int_{-\theta}^{\theta} dx_2 \int_{-\mathrm{w}}^{\mathrm{w}} dx_1 G_{t_0}(x_2 - y_2) G_{t_0}(x_1 - y_1) G_{t_0}(y_2 - y_1) \leq t_0^3;$$

and in the last inequality we used the finiteness of m_2 and $m_{1+\alpha}$ (see Footnote 5). Therefore, an application of Jensen's inequality implies that

$$\mathbf{T}_1(\theta, \mathrm{w}) \lesssim \frac{t_0^{3\alpha}}{(\theta \mathrm{w})^{\frac{1+\alpha}{2}}} \int_{\mathbb{R}} dy_1 \widetilde{\mathbf{T}}_1(\theta, \mathrm{w}, y_1) \lesssim \frac{\theta}{(\theta \mathrm{w})^{\frac{1+\alpha}{2}}} \leq (\theta/\mathrm{w})^{\frac{1+\alpha}{2}} \tag{3.21}$$

for $1 \leq \theta \leq \mathrm{w}$, where in the second-to-last inequality, we have simply performed integration in the exact order of dx_1, dy_1, dy_2, dx_2. In the same way, we can obtain

$$\mathbf{T}_1(\mathrm{w}, \theta) \leq (\theta/\mathrm{w})^{\frac{1+\alpha}{2}} \quad \text{for } 1 \leq \theta \leq \mathrm{w}. \tag{3.22}$$

Therefore, in view of (3.18), (3.19), (3.21), and (3.22), we get

$$\gamma_1(\theta, \mathrm{w}) \lesssim \theta^{-\alpha} + \mathrm{w}^{-\alpha} + (\theta/\mathrm{w})^{\frac{1+\alpha}{2}}. \tag{3.23}$$

Step (iii). For the term γ_2, we can write

$$\begin{aligned} \gamma_2^{1+\alpha}(\theta, \mathrm{w}) &\lesssim \int_{\mathbf{Z}} \left[\int_{\mathbf{Z}} \| D_{\xi_1} D_{\xi_2} \widetilde{F}_\theta \|_{2p}^2 \, m(d\xi_2) \right]^{1+\alpha} m(d\xi_1) \\ &\quad + \int_{\mathbf{Z}} \left[\int_{\mathbf{Z}} \| D_{\xi_1} D_{\xi_2} \widetilde{F}_\mathrm{w} \|_{2p}^2 \, m(d\xi_2) \right]^{1+\alpha} m(d\xi_1) \lesssim \theta^{-\alpha} + \mathrm{w}^{-\alpha}, \end{aligned} \tag{3.24}$$

which is essentially done in [8, (4.25)–(4.27)].

Step (iv). Now we consider the last term $\gamma_3(\theta, \mathrm{w})$ with $q = \min\{2, 1+2\alpha\} \in (1, 2]$.

$$\gamma_3(\theta, \mathrm{w}) \lesssim \int_{\mathbf{Z}} \| D_\xi \widetilde{F}_\theta \|_{q+1}^{q+1} \, m(d\xi) + \int_{\mathbf{Z}} \| D_\xi \widetilde{F}_\mathrm{w} \|_{q+1}^{q+1} \, m(d\xi) \lesssim \theta^{-\frac{q-1}{2}} + \mathrm{w}^{-\frac{q-1}{2}}, \tag{3.25}$$

where the last step is essentially done in [8, (4.28)–(4.30)]. Claim (3.14) follows from (3.16), (3.17), (3.23), (3.24), and (3.25). Hence, the proof is complete. □

Remark 3.4. In this proof, we merged Ibragimov-Lifshits' criteria and the second-order Gaussian Poincaré inequalities. The strategy was largely motivated by earlier works [9, 50, 51, 3] based on a combination of Ibragimov-Lifshits' method and the Malliavin-Stein approach ([38]). With the chaos expansion, papers [9, 50, 51] have established sufficient conditions (in terms of contractions) for the ASCLT on Gaussian, Poisson, and Rademacher settings. Our strategy shares the same root as the aforementioned references but differs by incorporating the second-order Gaussian Poincaré inequalities. This novelty avoids lengthy computation of asymptotic negligibility of the (star-)contractions, and will be further illustrated in the work [49].

Appendix A. Proofs of Lemma 3.1 and Proposition 3.3

Proof of Lemma 3.1. Let us first compute the second moment of L_T in (3.8):

$$\begin{aligned} \mathbb{E}\big[L_T^2\big] &= \frac{1}{(\log T)^2} \int_1^T \int_1^T \frac{1}{\theta \mathrm{w}} \mathbb{E}[H_\theta H_\mathrm{w}] \, d\theta d\mathrm{w} \\ &\leq \frac{2}{(\log T)^2} \int_{1<\theta<\mathrm{w}\leq T} \frac{1}{\theta \mathrm{w}} C_\beta \big(\theta/\mathrm{w}\big)^\beta \, d\theta d\mathrm{w} \leq \frac{2C_\beta}{\beta \log T}. \end{aligned}$$

In particular, we get (by Fubini)

$$\mathbb{E}\sum_{k=1}^{\infty} L^2_{2^{k^2}} < +\infty, \quad \text{and therefore } L_{2^{k^2}} \xrightarrow{a.s.} 0 \text{ as } k \to +\infty.$$

Next, we will show the almost sure convergence along the continuum parameter (as $T \to +\infty$). By the uniform boundedness of $\{H_\theta\}_{\theta>0}$, one can find a constant $M > 0$ such that $|H_\theta| \le M$, $\forall \theta > 0$, with probability one. Then, for any $T > 1$, one can find some (unique) $k = k_T \in \mathbb{N}_{\ge 0}$ such that

$$2^{k_T^2} \le T < 2^{(k_T+1)^2}. \tag{A.1}$$

It follows that

$$\begin{aligned}|L_T| &= \Big|\frac{\log(2^{k^2})}{\log T} L_{2^{k^2}} + \frac{1}{\log T}\int_{2^{k^2}}^T \frac{1}{\theta} H_\theta \, d\theta\Big| \\ &\le \big|L_{2^{k^2}}\big| + \frac{M}{\log T}\big[\log T - \log(2^{k^2})\big] \quad \text{with } k = k_T \text{ as in (A.1)} \\ &\le \big|L_{2^{k^2}}\big| + \frac{M}{k^2}\big[(k+1)^2 - k^2\big],\end{aligned}$$

which goes to 0 as $T \to \infty$ ($k = k_T \to \infty$ as well). Hence the proof is complete. □

Proof of Proposition 3.3. The proof consists of two parts: (i) for any $s \in \mathbb{R}$, $\mathbf{K}_t(s) \to 0$ a.s. as $t \to \infty$; and (ii) for $\mathbb{P}$-almost every $\omega \in \Omega$ and for any sequence $t_n \uparrow \infty$ (as $n \to \infty$), the family of probability measures $\{\nu^\omega_{t_n} : n \ge 1\}$ is tight, where we have used the notation ν^ω_T from (1.6). Then, we can deduce from (i) and (ii) together that $\mathbb{P}(\{\omega \in \Omega : \nu^\omega_T \Longrightarrow \gamma \text{ as } T \to +\infty\}) = 1$. That is, ASCLT holds for $\{F_\theta\}_{\theta \ge 1}$.

To show part (i), we fix any $s \in \mathbb{R}$. Let $h > 1$ and define

$$I_j = [e^{h^j}, e^{h^{j+1}}], \quad j \in \mathbb{N}_{\ge 0}.$$

It is clear that $t \mapsto \mathbb{E}\big[|\mathbf{K}_t(s)|^2\big]$ is continuous, and therefore there exists some (deterministic) $s_j \in I_j$ such that

$$\begin{aligned}\mathbb{E}\big[|\mathbf{K}_{s_j}(s)|^2\big] = \min_{t \in I_j} \mathbb{E}\big[|\mathbf{K}_t(s)|^2\big] &\le \Big(\int_{I_j} \frac{1}{t\log t} dt\Big)^{-1} \int_{I_j} \frac{\mathbb{E}\big[|\mathbf{K}_t(s)|^2\big]}{t \log t} dt \\ &= \frac{1}{\log h} \int_{I_j} \frac{\mathbb{E}\big[|\mathbf{K}_t(s)|^2\big]}{t\log t} dt,\end{aligned} \tag{A.2}$$

which is summable in j by assumption (3.9). It follows from Fubini that

$$\text{almost surely, } \mathbf{K}_{s_j}(s) \to 0 \text{ as } j \to \infty. \tag{A.3}$$

Next, for $e^{h^j} \le a < b \le e^{h^{j+1}}$, we get

$$\begin{aligned}\big|\mathbf{K}_a(s) - \mathbf{K}_b(s)\big| &= \Big|\Big(\frac{1}{\log a} - \frac{1}{\log b}\Big)\int_1^a \frac{1}{\theta} e^{isF_\theta} \, d\theta - \frac{1}{\log b}\int_a^b \frac{1}{\theta} e^{isF_\theta} \, d\theta\Big| \\ &\le 2(1 - \tfrac{1}{h}).\end{aligned} \tag{A.4}$$

Thus, part (i) follows from (A.3) and (A.4): almost surely,

$$\limsup_{t\to\infty} |\mathbf{K}_t(s)| \le 2(1 - \tfrac{1}{h}) \quad \text{for all } h > 1. \tag{A.5}$$

Now, let us continue with part (ii). First, we deduce from the same continuity argument as in (A.2) that for $r > 0$, there exists some (deterministic) $s_j^* \in I_j$ (depending on r as well) such that

$$\mathbb{E}\int_{-r}^{r} |\mathbf{K}_{s_j^*}(s)|^2 ds = \min_{t\in I_j} \mathbb{E}\int_{-r}^{r} |\mathbf{K}_t(s)|^2 ds \le \frac{1}{\log h}\int_{I_j} \frac{dt}{t\log t}\bigg(\mathbb{E}\int_{-r}^{r} |\mathbf{K}_t(s)|^2 ds\bigg),$$

which, together with Fubini and the assumption (3.9), implies that

$$\mathbb{E}\sum_j \int_{-r}^{r} |\mathbf{K}_{s_j^*}(s)|^2 ds \le \frac{2r}{\log h}\sup_{|s|\le r}\int_2^\infty \frac{1}{t\log t}\mathbb{E}\big[|\mathbf{K}_t(s)|^2\big]dt < +\infty.$$

Thus, it holds almost surely that $\int_{-r}^{r} |\mathbf{K}_{s_j^*}(s)|^2 ds \to 0$ as $j \to +\infty$. Therefore, we can obtain by the same arguments as in (A.3), (A.4), and (A.5) that with probability one, $\int_{-r}^{r} |\mathbf{K}_t(s)|^2 ds \to 0$ as $t \to +\infty$. Then, we get from Hölder inequality that

$$\text{(A.6)} \qquad \text{almost surely,} \quad \int_{-r}^{r} \phi_{\nu_t^\omega}(s) ds \to \int_{-r}^{r} \phi_\gamma(s) ds \quad \text{as } t \to +\infty,$$

where $\phi_\mu(s) = \int_{\mathbb{R}} e^{isx}\mu(dx)$ denotes the characteristic function of a probability measure μ on $\mathbb{R}$, and γ denotes the standard Gaussian measure on $\mathbb{R}$ in this paper. Using the fact $|\sin(z)| \le |z|$, $\forall z \in \mathbb{R}$, we deduce that

$$\text{(A.7)} \qquad \begin{aligned} \mu\big(\{x \in \mathbb{R} : |x| \le \tfrac{1}{r^2}\}\big) &\ge \int_{|x|\le r^{-2}} \frac{\sin(rx)}{rx}\mu(dx) \\ &\ge \int_{\mathbb{R}} \frac{\sin(rx)}{rx}\mu(dx) - r = \frac{1}{2r}\int_{-r}^{r} \phi_\mu(s) ds - r. \end{aligned}$$

It follows from (A.7) and (A.6) that for $\mathbb{P}$-almost every $\omega \in \Omega$ and for every $\varepsilon > 0$,

$$\liminf_{t\to+\infty} \nu_t^\omega(|x| \le r^{-2}) \ge \frac{1}{2r}\int_{-r}^{r} \phi_\gamma(s) ds - r \ge 1 - \frac{\varepsilon}{2}$$

for small enough $r = r_\varepsilon$ that does not depend on ω. For any increasing divergent sequence $\{t_n\}$, the above bound indicates that there is some $N = N_\varepsilon$ such that

$$\text{(A.8)} \qquad \nu_{t_n}^\omega(|x| \le r_\varepsilon^{-2}) \ge 1 - \varepsilon, \quad \forall n \ge N_\varepsilon;$$

while by choosing another small enough $r'_\varepsilon > 0$, we get

$$\text{(A.9)} \qquad \nu_{t_n}^\omega(|x| \le \tfrac{1}{(r'_\varepsilon)^2}) \ge 1 - \varepsilon, \quad \forall n < N_\varepsilon.$$

Combining (A.8) and (A.9) yields the tightness of $\{\nu_{t_n}^\omega : n \ge 1\}$. Therefore, the result in part (i) implies that $\nu_{t_n}^\omega \Longrightarrow \gamma$ as $n \to \infty$, and such weak convergence holds for true for any increasing divergent sequence $\{t_n\}_{n\in\mathbb{N}}$. Hence, we proved $\mathbb{P}\{\omega \in \Omega : \nu_T^\omega \Longrightarrow \gamma \text{ as } T \to +\infty\} = 1$. □

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