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A new family of orthogonal polynomials related to Mellin transforms of Gegenbauer polynomials

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ABSTRACT

The polynomial family $p_n(\lambda, s)$, obtained from Mellin transforms of Gegenbauer polynomials $C_n^{\lambda}(x)$, mimics Riemann's zeta function $\zeta(s)$. They have zeros only on the critical line $\Re s = \frac{1}{2}$ and obey a reflective functional equation. We show the zeros of $p_n(\lambda, s)$ and $p_{n+2}(\lambda, s)$ interlace on the critical line. To do this, we construct a new family of orthogonal polynomials $Q_n(\lambda, s)$ which embed suitably transformed versions of $p_n(\lambda, s)$ and $p_{n+2}(\lambda, s)$. In the final section, we identify a related polynomial system $r_n(\lambda, s)$ which is an orthogonal system for certain values of λ. Taking the limit as $\lambda \to \infty$, we obtain a further orthogonal polynomial system which belongs to the Meixner family of orthogonal polynomials.

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1. Introduction

Many questions in mathematics require understanding the deeper interplay between addition and multiplication, a famous example being the enigma of the Riemann zeta function $\zeta(s)$ [1]. It can be written as a power sum and an Euler product, yet does not satisfy a second-order differential equation, whilst the distribution of its non-trivial zeros inside the critical strip is linked to the distribution of the prime numbers.

Progress with peering inside the critical strip has historically been elusive, and so more lateral approaches have been developed. One approach is to study functions that have similar properties to $\zeta(s)$, such as the rational approximations to $\zeta(s)$ considered by Ball [2].

In [3], the authors examined families of polynomials $p_n(\lambda, s)$, whose zeros all lie on the critical line $\Re s = 1/2$. They obey a similar reflective functional equation to that for the Bernoulli and Euler polynomials [4], satisfying $p_n(\lambda, s) = (-1)^{[n/2]} p_n(\lambda, 1-s)$, a signed version of the reflection formula for the Riemann xi function $\xi(s) = \xi(1-s)$. The polynomials families $p_n(\lambda, s)$ arise in finite Mellin transforms of Gegenbauer polynomials, similar to results of Bump et al. [5–7], where Mellin transforms of the orthogonal family of Hermite polynomials are identified in consideration of Riemann's analytic continuation and second proof for the functional equation for $\zeta(s)$.

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It was remarked in [3] that 'the zeros of $p_n(\lambda, s)$ and $p_{n+2}(\lambda, s)$ appear to obey an interlacing property with regard to their positions on the critical line $\Re s = 1/2$'. This is similar to the real zeros of orthogonal polynomials [8–10], which also interlace, partially motivating the results of this paper, although the polynomials $p_n(\lambda, s)$ do not form an orthogonal polynomial system. Another motivating factor is to further understand the effect of the finite Mellin transform on the orthogonality of the polynomials undergoing the transformation.

We establish that the zeros of the transformed polynomials $p_n(\lambda, s)$ and $p_{n+2}(\lambda, s)$ do interlace, by applying an embedding technique to the shifted and rotated polynomials $q_n^{\perp}(\lambda, x) = (-i)^n p_{2n}(\lambda, ix + 1/2)$. This approach yields in Theorem 3.2 new families of orthogonal polynomials $Q_n^{(l)}(\lambda, x)$, for $l \in \mathbb{N}$ and $n = 0, 1, 2, 3, \ldots$, where the interlacing properties from the zeros of $Q_n^{(l)}(\lambda, x)$ transfer to the zeros of $p_n(\lambda, s)$.

In Theorem 4.1, we deduce that for $\lambda \in (-\infty, 1/2) \cup (1/2, 1)$ the polynomial families $r_n(\lambda, x) = q_n^{\perp}(\lambda - 2n, x)$, form an orthogonal system, related to Mellin transforms of the Gegenbauer family of orthogonal polynomials. By taking the limit as $\lambda \to \infty$, in both families either $q_n^{\perp}(\lambda, x)$ or $r_n(\lambda, x)$ (normalized to be monic), we obtain an orthogonal family of monic Meixner polynomials of the second kind $m_n(x)$, as detailed in Theorem 4.2.

Our starting point is the generalized (finite) Mellin transform [11,12] considered by Coffey and the second author [5,13], where $f \mapsto \mathcal{M}(f)$, with

$$\mathcal{M}(f)(\lambda, s) = \int_0^1 \frac{x^{s-1}}{(1-x^2)^{3/4-\lambda/2}} f(x)\, \mathrm{d}x.$$

They applied (see [13], Equation 1.2) this transformation to the Gegenbauer family of orthogonal polynomials $\{C_n^{(\lambda)}(x)\}_{n=0}^{\infty}$, defined for $\lambda > -1/2$, $\lambda \neq 0$, [12,14–17] by the generating function

$$\frac{1}{(1-2xt+t^2)^{\lambda}} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x) t^n,$$

yielding the sequence of functions $\{M_n(\lambda, s)\}_{n=0}^{\infty}$, where

$$M_n(\lambda, s) = \mathcal{M}(C_n^{(\lambda)})(\lambda, s) = \int_0^1 \frac{x^{s-1}}{(1-x^2)^{3/4-\lambda/2}} C_n^{(\lambda)}(x)\, \mathrm{d}x. \tag{1.1}$$

The Gegenbauer family of polynomials satisfy respectively the *raising* and *lowering* differential difference relations [18–21]

$$(n+1)C_{n+1}^{(\lambda)}(x) = (n+2\lambda)xC_n^{(\lambda)}(x) - (1-x^2)\frac{d}{dx}C_n^{(\lambda)}(x), \tag{1.2}$$

$$(n+2\lambda-1)C_{n-1}^{(\lambda)}(x) = nxC_n^{(\lambda)}(x) + (1-x^2)\frac{d}{dx}C_n^{(\lambda)}(x), \tag{1.3}$$

which on application of the generalized Mellin transform to the raising relation (1.2) leads to the recurrence relation

$$(n+1)M_{n+1}(\lambda, s) = \left(n + \lambda - s + \frac{1}{2}\right) M_n(\lambda, s+1) + (s-1)M_n(\lambda, s-1). \tag{1.4}$$

It was shown in [13] that the Mellin transformed functions can be written as a product of Gamma factors and polynomials in s. Here we write them such that

$$M_n(\lambda, s) = \begin{cases} \dfrac{p_n(\lambda, s)\Gamma((s+1)/2)}{\Phi(n, \lambda, s)}, & \text{for } n \text{ odd}, \\ \dfrac{p_n(\lambda, s)\Gamma(s/2)}{\Phi(n, \lambda, s)}, & \text{for } n \text{ even}, \end{cases} \tag{1.5}$$

where

$$\Phi(n, \lambda, s) = \frac{2\Gamma((s+n+\lambda)/2 + 1/4)}{\Gamma(\lambda/2 + 1/4)}. \tag{1.6}$$

The trivial zeros for $\zeta(s)$ lie at the negative even integers, and from (1.6), analogous zeros arise when $\Gamma((s+n+\lambda)/2+1/4) \to \pm\infty$. This happens when $(s+n+\lambda)/2+1/4$ is a non-positive integer $-m$, say with $m \geq 0$. In the case $\lambda = 1/2$ for the Legendre polynomials, the 'trivial' zeros occur when $s = n-m-1$, with integer $m \geq n-1$.

Carrying out the integration in (1.1) for the cases $0 \leq n \leq 4$ we find the $p_n(\lambda, s)$, $0 \leq n \leq 4$ are polynomials in s given by

$$\begin{aligned} p_0(\lambda, s) &= 1, \quad p_1(\lambda, s) = 2\lambda, \quad p_2(\lambda, s) = \frac{1}{4}(2s-1)\lambda(2\lambda+1), \\ p_3(\lambda, s) &= \frac{1}{6}(2s-1)\lambda(\lambda+1)(2\lambda+1), \\ p_4(\lambda, s) &= \frac{1}{96}\lambda(\lambda+1)(2\lambda+1)(6\lambda+15+4s(s-1)(2\lambda+3)). \end{aligned}$$

It can be seen that $p_2(\lambda, s)$ and $p_3(\lambda, s)$ both have one zero at $1/2$ and $p_4(\lambda, s)$ has zeros at $1/2 \pm i\sqrt{(\lambda+3)/(2\lambda+3)}$. These zeros lie on the critical line $\Re(s) = 1/2$ and it was shown in Theorem 5.1 of [3] that this is true for all $p_n(\lambda, s)$. The approach employed the polynomials $q_n(\lambda, z) = p_n(\lambda, z+1/2)$, and establishing that the zeros of the $q_n(\lambda, z)$ lie on the imaginary axis. This result will be re-derived at the end of Section 3 using an alternative argument.

Although the zeros of $q_{n+2}(\lambda, z)$ and $q_n(\lambda, z)$ appeared to interlace on the imaginary axis, no proof was forthcoming. In what follows, we will establish that this is true for the case n being an even non-negative integer. A similar argument can be used to establish that the zeros of $q_{n+2}(\lambda, z)$ and $q_n(\lambda, z)$ also interlace when n is an odd non-negative integer.

We begin with an example. Consider the zeros of $q_8(\lambda, z)$ and $q_{10}(\lambda, z)$, with $\lambda = 1$. The five zeros of $q_{10}(1, z)$ are shown, to two decimal places, in bold below and between them are the four zeros of $q_8(1, z)$.

$$\mathbf{-9.15}\,i, \quad -5.61\,i, \quad \mathbf{-2.05}\,i, \quad -0.71\,i, \quad \mathbf{0}, \quad 0.71\,i, \quad \mathbf{2.05}\,i, \quad 5.61\,i, \quad \mathbf{9.15}\,i.$$

These zeros can be seen to interlace on the imaginary axis.

To aide our calculations, we now make a further transformation to introduce the polynomials $q_n^{\perp}(\lambda, x)$ defined by $q_n^{\perp}(\lambda, x) = (-i)^n q_{2n}(\lambda, ix)$. The first six of these polynomials are

$$
\begin{aligned}
q_0^{\perp}(\lambda,x) &= 1,\\
q_1^{\perp}(\lambda,x) &= \frac{1}{2!}x\lambda(2\lambda+1),\\
q_2^{\perp}(\lambda,x) &= -\frac{1}{4!}\lambda(\lambda+1)(2\lambda+1)(\lambda+3-x^2(2\lambda+3)),\\
q_3^{\perp}(\lambda,x) &= -\frac{1}{6!}x\lambda(\lambda+1)(\lambda+2)(2\lambda+1)(2\lambda+5)\left(7\lambda+33-x^2(2\lambda+3)\right),\\
q_4^{\perp}(\lambda,x) &= \frac{1}{8!}\lambda(\lambda+1)(\lambda+2)(\lambda+3)(2\lambda+1)(2\lambda+5)\\
&\quad\times\left(15(\lambda+5)(\lambda+7)-2x^2(2\lambda+7)(11\lambda+69)\right.\\
&\quad\left.+x^4(2\lambda+3)(2\lambda+7)\right),\\
q_5^{\perp}(\lambda,x) &= \frac{1}{10!}x\lambda(\lambda+1)((\lambda+2)(\lambda+3)(\lambda+4)(2\lambda+1)(2\lambda+5)(2\lambda+9)\\
&\quad\times\left(12369+\lambda(3260+211\lambda)-10x^2(2\lambda+7)(5\lambda+39)\right.\\
&\quad\left.+x^4(2\lambda+3)(2\lambda+7)\right).
\end{aligned}
\tag{1.7}
$$

The above transformation rotates all of the zeros of $q_{2n}(\lambda,x)$ about the origin, from the imaginary axis onto the real axis. Hence the zeros of the polynomials $q_n^{\perp}(\lambda,x)$ all lie on the real axis and by construction the zeros of $q_n^{\perp}(\lambda,x)$ interlace on the real axis, if and only if the zeros of $q_{2n}(\lambda,x)$ interlace on the imaginary axis.

Remark 1.1: We note in passing that the polynomials listed above are comprised entirely from either even powers of x or odd powers of x. As such they alternate between even and odd functions, for consecutive values of n.

Our argument will utilize the property that consecutive terms in families of orthogonal polynomials have interlacing zeros. This would be straightforward to establish for all even n if it could be shown that the $q_n^{\perp}(\lambda,x)$ satisfied a three term recurrence relation of the form required for alternatively even and odd function orthogonal polynomials, that is

$$
q_{n+1}^{\perp}(\lambda,x) = a_n x q_n^{\perp}(\lambda,x) + c_n q_{n-1}^{\perp}(\lambda,x),
\tag{1.8}
$$

where a_n and c_n do not depend on x. This is not the case, as may be seen by attempting to equate coefficients of powers of x, where it is not possible to choose a_4 and c_4 so that

$$
q_5^{\perp}(\lambda,x) = a_4 x q_4^{\perp}(\lambda,x) + c_4 q_3^{\perp}(\lambda,x).
$$

However there does exist a three-term recurrence relation between $q_5^{\perp}(\lambda,x)$, $q_4^{\perp}(\lambda,x)$ and $q_3^{\perp}(\lambda,x)$ given by

$$
\begin{aligned}
q_5^{\perp}(\lambda,x) = {}& \frac{2(\lambda+8)\,(63+\lambda(15+2\lambda))}{180(\lambda+7)} x q_4^{\perp}(\lambda,x)\\
& - \frac{(\lambda+3)(\lambda+9)(2\lambda+7)\left(x^2+(7+\lambda)^2\right)}{180(\lambda+7)} q_3^{\perp}(\lambda,x),
\end{aligned}
\tag{1.9}
$$

as may be verified using the list of polynomials given in (1.7). In Section 3, we will show that this holds for all n as a consequence of Lemma 2.1. Note that the above coefficient of $q_3^{\perp}(\lambda, x)$ contains x^2 and so this is not of the form (1.8). There is a relation of the form (1.8) but it has $q_3^{\perp}(\lambda+2, x)$ instead of $q_3^{\perp}(\lambda, x)$. This relation is

$$\begin{aligned} q_5^{\perp}(\lambda, x) &= \frac{1}{90}(\lambda+4)(2\lambda+9)xq_4^{\perp}(\lambda, x) \\ &\quad - \frac{1}{180}\lambda(\lambda+1)(\lambda+9)(2\lambda+1)q_3^{\perp}(\lambda+2, x). \end{aligned} \tag{1.10}$$

We also find that

$$\begin{aligned} q_4^{\perp}(\lambda, x) &= \frac{\lambda(\lambda+1)(2\lambda+1)(2\lambda+3)}{56(\lambda+4)(2\lambda+9)}xq_3^{\perp}(\lambda+2, x) \\ &\quad - \frac{\lambda(\lambda+1)(\lambda+2)(\lambda+3)(2\lambda+1)(2\lambda+5)}{112(\lambda+4)(2\lambda+9)}q_2^{\perp}(\lambda+4, x), \\ q_3^{\perp}(\lambda+2, x) &= \frac{(\lambda+2)(\lambda+3)(2\lambda+5)(2\lambda+7)}{30(\lambda+5)(2\lambda+11)}xq_2^{\perp}(\lambda+4, x) \\ &\quad - \frac{(\lambda+2)(\lambda+3)(\lambda+4)(2\lambda+5)(2\lambda+9)}{60(2\lambda+11)}q_1^{\perp}(\lambda+6, x), \\ q_2^{\perp}(\lambda+4, x) &= \frac{(\lambda+4)(\lambda+5)(2\lambda+9)(2\lambda+11)}{12(\lambda+6)(2\lambda+13)}xq_1^{\perp}(\lambda+6, x) \\ &\quad - \frac{(\lambda+4)(\lambda+5)(\lambda+7)(2\lambda+9)}{24}q_0^{\perp}(\lambda+8, x). \end{aligned} \tag{1.11}$$

These may be verified using (1.7) and in Section 3 these λ shifted recurrence relations will be extended to all index values n.

We now use an idea inspired by García-Ardila, Marcellán and Marriaga, in [22], with which we define in generality the embedding of $q_{n+1}^{\perp}(\lambda, x)$ and $q_n^{\perp}(\lambda, x)$ in a new family of polynomial functions $Q_n^{(l)}(\lambda, x)$. We begin by focusing on the specific case when $l = 4$. This embedding of $q_5^{\perp}(\lambda, x)$ and $q_4^{\perp}(\lambda, x)$ allows to establish a new family of orthogonal polynomials $Q_n^{(4)}(\lambda, x)$.

A proof for the generalization of this technique is given in Theorem 3.2, once further results regarding the recurrence relations arising from the Mellin transforms have been established.

Definition 1.1: For integer $l \geq 1$, we define the polynomials $Q_n^{(l)}(\lambda, x)$ in terms of the $q_n^{\perp}(\lambda, x)$ by

$$\begin{aligned} Q_n^{(l)}(\lambda, x) &= q_n^{\perp}(\lambda+2l-2n, x) \quad \text{for } 0 \leq n \leq l-1, \\ Q_n^{(l)}(\lambda, x) &= q_n^{\perp}(\lambda, x) \quad \text{for } l \leq n \leq l+1, \\ Q_n^{(l)}(\lambda, x) &= x\,Q_{n-1}^{(l)}(\lambda, x) - Q_{n-2}^{(l)}(\lambda, x) \quad \text{for } l+2 \leq n. \end{aligned}$$

Example 1.1: From the above definition when $l = 4$, we obtain the polynomials $Q_n^{(4)}(\lambda, x)$ in terms of the $q_n^{\perp}(\lambda, x)$ such that

$$\begin{aligned}
Q_n^{(4)}(\lambda, x) &= q_n^{\perp}(\lambda + 8 - 2n, x) \quad \text{for } 0 \le n \le 3,\\
Q_4^{(4)}(\lambda, x) &= q_4^{\perp}(\lambda, x),\\
Q_5^{(4)}(\lambda, x) &= q_5^{\perp}(\lambda, x),\\
Q_n^{(4)}(\lambda, x) &= x\, Q_{n-1}^{(4)}(\lambda, x) - Q_{n-2}^{(4)}(\lambda, x), \quad \text{for } 6 \le n.
\end{aligned}$$

We now have the following theorem.

Theorem 1.1: *Let $\lambda > -\frac{1}{2}$ and $\lambda \neq 0$. Then the system of polynomials $Q_n^{(4)}(\lambda, x)$, $n = 0, 1, 2, \ldots$, given in the above example, satisfy a three term recurrence relation of the form*

$$Q_{n+1}^{(4)}(\lambda, x) = A_n x\, Q_n^{(4)}(\lambda, x) + C_n Q_{n-1}^{(4)}(\lambda, x),$$

with $A_n A_{n-1} C_n < 0$ for $n \ge 0$. They form an orthogonal polynomial system whose zeros are real and interlace on the real line.

Proof: Using the expressions in (1.10), (1.11), and the above definitions of $Q_n^{(4)}(\lambda, x)$, we find that the recurrence relation is satisfied. We need to verify the condition $A_n A_{n-1} C_n < 0$ for $n \ge 0$. This is clear when $\lambda > 0$ as then $A_n > 0$ and $C_n < 0$ for $n \ge 0$. When $-1/2 < \lambda < 0$ all A_n and C_n have the same sign as before except for A_3, C_3, C_4 all of which change sign which means we still have $A_4 A_3 C_4 < 0$ and $A_3 A_2 C_3 < 0$ as required.

The result follows as the family of polynomials $Q_n^{(4)}(\lambda, x)$, $n = 0, 1, 2, \ldots \infty$ satisfy the conditions of Favard's Theorem (see Theorem 4.4 of [16]), and therefore are orthogonal polynomials with a positive definite moment function and so have interlacing zeros on the real line. ■

In particular as the zeros of $q_{n+1}^{\perp}(\lambda, x) = Q_{n+1}^{(4)}(\lambda, x)$ and $q_n^{\perp}(\lambda, x) = Q_n^{(4)}(\lambda, x)$ lie on the real axis and interlace. In turn this shows the zeros of $q_{2n+2}(\lambda, x)$ and $q_{2n}(\lambda, x)$ lie on the imaginary axis and so the zeros of $p_{2n+2}(\lambda, x)$ and $p_{2n}(\lambda, x)$ lie on the critical line where they also interlace. This will be shown to hold for all n in Theorem 3.2.

2. Three lemmas

In this section, we derive three lemmas, each of which determines a three-term recurrence relation for our finite Mellin transform function $M_n(\lambda, s)$ in terms of $M_n(\lambda, s), M_{n+2}(\lambda, s), M_{n+4}(\lambda, s)$. The index step of $+2$ in these three-term recurrences allows them to be applied to the polynomials $q_n^{\perp}(\lambda, x)$, via the $p_n(\lambda, s)$, which enables a clearance of common Gamma factors. The three lemmas are sequential in that the third lemma relies on the second lemma, which itself relies on the first lemma.

We begin by returning to the generalized Mellin transform of Equation (1.2), which is obtained by multiplying both sides by

$$\frac{x^{s-1}}{(1-x^2)^{3/4-\lambda/2}}$$

and integrating, so that

$$\begin{aligned}\int_0^1 &\frac{x^{s-1}}{(1-x^2)^{3/4-\lambda/2}}(n+1)C_{n+1}^{(\lambda)}(x)\,\mathrm{d}x\\ &= \int_0^1 \frac{x^{s-1}}{(1-x^2)^{3/4-\lambda/2}}(n+2\lambda)xC_n^{(\lambda)}(x)\,\mathrm{d}x\\ &\quad - \int_0^1 \frac{x^{s-1}}{(1-x^2)^{3/4-\lambda/2}}(1-x^2)\frac{d}{dx}C_n^{(\lambda)}(x)\,\mathrm{d}x.\end{aligned}$$

This gives us

$$(n+1)M_{n+1}(\lambda,s) = (n-s+\lambda+1/2)M_n(\lambda,s+1) + (s-1)M_n(\lambda,s-1). \tag{2.1}$$

In the same way, (1.3) becomes

$$(n+2\lambda-1)M_{n-1}(\lambda,s) = (n+s+\lambda-1/2)M_n(\lambda,s+1) - (s-1)M_n(\lambda,s-1). \tag{2.2}$$

Adding (2.1) and (2.2), we obtain

$$(n+1)M_{n+1}(\lambda,s) = (-2\lambda-n+1)M_{n-1}(\lambda,s) + 2(\lambda+n)M_n(\lambda,s+1), \tag{2.3}$$

and eliminating $M_n(\lambda,s+1)$ gives us

$$\begin{aligned}(n+1)(2(\lambda+n+s)-1)M_{n+1}(\lambda,s) &= (2\lambda+n-1)(2(\lambda+n-s)+1)M_{n-1}(\lambda,s)\\ &\quad + 4(s-1)(\lambda+n)M_n(\lambda,s-1).\end{aligned} \tag{2.4}$$

Lemma 2.1: *The $M_n(\lambda,s)$ satisfy*

$$\begin{aligned}&(n+3)(n+4)(\lambda+n+1)(2(\lambda+n+s)+5)M_{n+4}(\lambda,s)\\ &\quad - 2(2s-1)(\lambda+n+2)\left(\lambda(2\lambda+3)+n^2+2(\lambda+2)n+3\right)M_{n+2}(\lambda,s)\\ &\quad - (\lambda+n+3)(2\lambda+n)(2\lambda+n+1)(2(\lambda+n-s)+3)M_n(\lambda,s)) = 0.\end{aligned}$$

Proof: The case $n=0$ is easily verified to hold. From (2.3), putting $n=k+2$ then $n=k+4$, we get

$$M_{k+2}(\lambda,s+1) = \frac{(1+k+2\lambda)M_{k+1}(\lambda,s)+(k+3)M_{k+3}(\lambda,s)}{2(2+k+\lambda)}, \tag{2.5}$$

$$M_{k+4}(\lambda,s+1) = \frac{(3+k+2\lambda)M_{k+3}(\lambda,s)+(k+5)M_{k+5}(\lambda,s)}{2(4+k+\lambda)}, \tag{2.6}$$

and from (2.4), putting $n = k+1$ and replacing s by $s+1$, we get

$$M_{k+1}(\lambda, s) = -\frac{(k+2\lambda)(2(k+\lambda-s)+1)M_k(\lambda, s+1)}{4s(k+\lambda+1)} + \frac{(k+2)(2(k+\lambda+s)+3)M_{k+2}(\lambda, s+1)}{4s(k+\lambda+1)}. \tag{2.7}$$

Assume the statement of Lemma 2.1 holds for $n \leq k$ then, putting $n = k$ and replacing s by $s+1$, we find

$$\begin{aligned} M_k(\lambda, s+1) &= -\frac{2(2s+1)(k+\lambda+2)\left(k^2+2\lambda^2+2k(\lambda+2)+3\lambda+3\right)M_{k+2}(\lambda, s+1)}{(k+\lambda+3)(k+2\lambda)(k+2\lambda+1)(2k+2\lambda-2s+1)} \\ &\quad + \frac{(k+3)(k+4)(k+\lambda+1)(2k+2\lambda+2s+7)M_{k+4}(\lambda, s+1)}{(k+\lambda+3)(k+2\lambda)(k+2\lambda+1)(2k+2\lambda-2s+1)}. \end{aligned} \tag{2.8}$$

Eliminating $M_k(\lambda, s+1)$, $M_{k+2}(\lambda, s+1)$ and $M_{k+4}(\lambda, s+1)$ from (2.5) to (2.8) we obtain, after cancelling a non-zero factor, the statement of Lemma 2.1 for the case $n = k+1$ thereby establishing the inductive step. ■

Lemma 2.2: *The $M_n(\lambda, s)$ satisfy*

$$\begin{aligned} &(n+3)(n+4)(2\lambda+2n+2s+5)M_{n+4}(\lambda, s) \\ &\quad + (1-2s)(2\lambda+n+2)(2\lambda+n+3)M_{n+2}(\lambda, s) \\ &\quad - 8\lambda(\lambda+1)(\lambda+n+3)M_n(\lambda+2, s)) = 0. \end{aligned}$$

Proof: The case $n = 0$ is easily verified to hold. From Lemma 2.1, putting $n = k-3$ and replacing λ by $\lambda + 2$ we get

$$\begin{aligned} &-(k+\lambda+2)(k+2\lambda+1)(k+2\lambda+2)(2k+2\lambda-2s+1)M_{k-3}(\lambda+2, s) \\ &\quad - 2(2s-1)(k+\lambda+1)\left(k^2+2\lambda^2+2(\lambda+1)k+5\lambda+2\right)M_{k-1}(\lambda+2, s) \\ &\quad + k(k+1)(k+\lambda)(2(k+\lambda+s)+3)M_{k+1}(\lambda+2, s) = 0. \end{aligned} \tag{2.9}$$

Again from Lemma 2.1 putting $n = k-1$ then $n = k+1$ we get

$$\begin{aligned} &-(k+\lambda+2)(k+2\lambda-1)(k+2\lambda)(2k+2\lambda-2s+1)M_{k-1}(\lambda, s) \\ &\quad - 2(2s-1)(k+\lambda+1)\left(k^2+2(\lambda+1)k+\lambda(2\lambda+1)\right)M_{k+1}(\lambda, s) \\ &\quad \times (k+2)(k+3)(k+\lambda)(2k+2\lambda+2s+3)M_{k+3}(\lambda, s) = 0 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &-(k+\lambda+4)(k+2\lambda+1)(k+2\lambda+2)(2k+2\lambda-2s+5)M_{k+1}(\lambda, s) \\ &\quad - 2(2s-1)(k+\lambda+3)\left(k^2+2\lambda^2+2(\lambda+3)k+5\lambda+8\right)M_{k+3}(\lambda, s) \\ &\quad + (k+4)(k+5)(k+\lambda+2)(2k+2\lambda+2s+7)M_{k+5}(\lambda, s) = 0. \end{aligned} \tag{2.11}$$

Now assume that Lemma 2.2 holds for $n \leq k$. Putting $n = k-1$ and $n = k-3$, we get

$$\begin{aligned}
&- 8\lambda(\lambda+1)(k+\lambda+2)M_{k-1}(\lambda+2,s) \\
&\quad + (1-2s)(k+2\lambda+1)(k+2\lambda+2)M_{k+1}(\lambda,s) \\
&\quad + (k+2)(k+3)(2k+2\lambda+2s+3)M_{k+3}(\lambda,s) = 0
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&- 8\lambda(\lambda+1)(k+\lambda)M_{k-3}(\lambda+2,s) \\
&\quad + (1-2s)(k+2\lambda-1)(k+2\lambda)M_{k-1}(\lambda,s) \\
&\quad + k(k+1)(2k+2\lambda+2s-1)M_{k+1}(\lambda,s) = 0.
\end{aligned} \tag{2.13}$$

Eliminating $M_{k-3}(\lambda+2,s)$, $M_{k-1}(\lambda,s)$, $M_{k-1}(\lambda+2,s)$ and $M_{k+1}(\lambda,s)$ from (2.9)–(2.13) we obtain, after cancelling a non-zero factor, the statement of Lemma 2.2 for the case $n = k+1$ thereby establishing the inductive step. ■

Lemma 2.3: *The $M_n(\lambda,s)$ satisfy*

$$\begin{aligned}
&(n+3)(n+4)(2\lambda+n+4)(2\lambda+n+5)M_{n+4}(\lambda,s) \\
&\quad + 4\lambda(\lambda+1)(2\lambda+3)(1-2s)M_{n+2}(\lambda+2,s) \\
&\quad - 16\lambda(\lambda+1)(\lambda+2)(\lambda+3)M_n(\lambda+4,s) = 0.
\end{aligned}$$

Proof: From Lemma 2.2, replacing λ by $\lambda+2$, we get

$$\begin{aligned}
&- 8(\lambda+2)(\lambda+3)(\lambda+n+5)M_n(\lambda+4,s) \\
&\quad + (1-2s)(2(\lambda+2)+n+2)(2(\lambda+2)+n+3)M_{n+2}(\lambda+2,s) \\
&\quad + (n+3)(n+4)(2(\lambda+2)+2n+2s+5)M_{n+4}(\lambda+2,s) = 0,
\end{aligned} \tag{2.14}$$

replacing n by $n+2$ we get

$$\begin{aligned}
&- 8\lambda(\lambda+1)(\lambda+n+5)M_{n+2}(\lambda+2,s) \\
&\quad + (1-2s)(2\lambda+n+4)(2\lambda+n+5)M_{n+4}(\lambda,s) \\
&\quad + (n+5)(n+6)(2\lambda+2(n+2)+2s+5)M_{n+6}(\lambda,s) = 0,
\end{aligned} \tag{2.15}$$

and replacing n by $n+4$ we get

$$\begin{aligned}
&- 8\lambda(\lambda+1)(\lambda+n+7)M_{n+4}(\lambda+2,s) \\
&\quad + (1-2s)(2\lambda+n+6)(2\lambda+n+7)M_{n+6}(\lambda,s) \\
&\quad + (n+7)(n+8)(2\lambda+2(n+4)+2s+5)M_{n+8}(\lambda,s) = 0,
\end{aligned} \tag{2.16}$$

and from Lemma 2.1, replacing n by $n+4$, we have that

$$\begin{aligned}
&- (\lambda+n+7)(2\lambda+n+4)(2\lambda+n+5)(2\lambda+2(n+4)-2s+3)M_{n+4}(\lambda,s) \\
&\quad - 2(2s-1)(\lambda+n+6)\left(\lambda(2\lambda+3)+2(\lambda+2)(n+4)+(n+4)^2+3\right)M_{n+6}(\lambda,s) \\
&\quad + (n+7)(n+8)(\lambda+n+5)(2\lambda+2(n+4)+2s+5)M_{n+8}(\lambda,s) = 0.
\end{aligned} \tag{2.17}$$

Eliminating $M_{n+8}(\lambda,s)$, $M_{n+6}(\lambda,s)$ and $M_{n+4}(\lambda+2,s)$ from (2.14)–(2.17) we obtain, after cancelling a non-zero factor, the statement of Lemma 2.3. ■

3. The Mellin polynomials

In this section, we relate the lemmas of Section 2 to the $p_n(\lambda, s)$ polynomials, obtaining the corresponding three-term recurrence relations for the $q_n^{\perp}(\lambda, x)$ polynomials.

Theorem 3.1: *The functions $p_n(\lambda, s)$ satisfy the three-term recurrence relation*

$$\begin{aligned}
&16(n+3)(n+4)(\lambda+n+1)p_{n+4}(\lambda, s)\\
&\quad - 8(2s-1)(\lambda+n+2)\left(\lambda(2\lambda+3)+n^2+2(\lambda+2)n+3\right)p_{n+2}(\lambda, s)\\
&\quad - (\lambda+n+3)(2\lambda+n)(2\lambda+n+1)(2(\lambda+n-s)+3)(2(\lambda+n+s)+1)p_n(\lambda, s) = 0.
\end{aligned} \tag{3.1}$$

Corollary 3.1: *The sequence $\{p_n(\lambda, s)\}_{n=0}^{\infty}$ is a sequence of polynomials in s.*

Proof: If we substitute (1.5) into the statement of Lemma 2.1, we can adjust the Gamma function factors so that all terms have the same factors. These may then be cancelled to obtain the recurrence relation stated.

To see the corollary, because $p_0(\lambda, s), p_1(\lambda, s), p_2(\lambda, s), p_3(\lambda, s)$ and $p_4(\lambda, s)$ are all polynomials in s it follows from the recurrence relation (3.1), that $p_n(\lambda, s)$ is a polynomial in s for all $n \geq 0$. ■

Now the polynomials $q_n(\lambda, s)$ are defined by $q_n(\lambda, s) = p_n(\lambda, s+1/2)$, so in turn they satisfy

$$\begin{aligned}
&4(n+3)(n+4)(\lambda+n+1)q_{n+4}(\lambda, s)\\
&\quad - 4s(\lambda+n+2)\left(\lambda(2\lambda+3)+n^2+2(\lambda+2)n+3\right)q_{n+2}(\lambda, s)\\
&\quad - (\lambda+n+3)(2\lambda+n)(2\lambda+n+1)(\lambda+n-s+1)(\lambda+n+s+1)q_n(\lambda, s) = 0.
\end{aligned}$$

Furthermore, the polynomials $q_n^{\perp}(\lambda, x)$ are defined by $q_n^{\perp}(\lambda, x) = (-i)^n q_{2n}(\lambda, ix)$, so they satisfy

$$\begin{aligned}
&4(n+2)(2n+3)(\lambda+2n+1)q_{n+2}^{\perp}(\lambda, x)\\
&\qquad - 2(\lambda+2(n+1))\left(\lambda(2\lambda+3)+4n(\lambda+n+2)+3\right)xq_{n+1}^{\perp}(\lambda, x)\\
&\qquad + (\lambda+n)(\lambda+2n+3)(2(\lambda+n)+1)\left((\lambda+1)^2+4n(\lambda+n+1)+x^2\right)q_n^{\perp}(\lambda, x)\\
&\quad = 0.
\end{aligned} \tag{3.2}$$

Setting $n = 3$ in Equation (3.2), we obtain the required expression given in (1.9).

In the same way from Lemma 2.2, we find that the polynomials $q_n^{\perp}(\lambda, x)$ satisfy

$$\begin{aligned}
&4(n+2)(2n+3)q_{n+2}^{\perp}(\lambda, x) - 2(\lambda+n+1)(2(\lambda+n)+3)xq_{n+1}^{\perp}(\lambda, x)\\
&\quad + \lambda(\lambda+1)(2\lambda+1)(\lambda+2n+3)q_n^{\perp}(\lambda+2, x) = 0.
\end{aligned} \tag{3.3}$$

Similarly, from Lemma 2.3, the polynomials $q_n^{\perp}(\lambda,x)$ satisfy

$$\begin{aligned}&4(n+2)(2n+3)(\lambda+n+2)(2(\lambda+n)+5)q_{n+2}^{\perp}(\lambda,x)\\&\quad-2\lambda(\lambda+1)(2\lambda+1)(2\lambda+3)xq_{n+1}^{\perp}(\lambda+2,x)\\&\quad+\lambda(\lambda+1)(\lambda+2)(\lambda+3)(2\lambda+1)(2\lambda+5)q_n^{\perp}(\lambda+4,x)=0.\end{aligned}\tag{3.4}$$

We are now in a position to generalize Theorem 1.1 and show that the zeros of $p_{2l+2}(\lambda,s)$ interlace on the critical line with those of $p_{2l}(\lambda,s)$. The proof follows the discussion in Section 1. We first note that from (3.3) we have

$$\begin{aligned}q_{n+2}^{\perp}(\lambda,x)=&\frac{2(\lambda+n+1)(2(\lambda+n)+3)}{4(n+2)(2n+3)}xq_{n+1}^{\perp}(\lambda,x)\\&-\frac{\lambda(\lambda+1)(2\lambda+1)(\lambda+2n+3)}{4(n+2)(2n+3)}q_n^{\perp}(\lambda+2,x).\end{aligned}\tag{3.5}$$

From (3.4), after replacing n by $n-1$, we have

$$\begin{aligned}q_{n+1}^{\perp}(\lambda,x)=&\frac{2\lambda(\lambda+1)(2\lambda+1)(2\lambda+3)}{4(n+1)(2n+1)(\lambda+n+1)(2(n+\lambda)+3)}xq_n^{\perp}(\lambda+2,x)\\&-\frac{\lambda(\lambda+1)(\lambda+2)(\lambda+3)(2\lambda+1)(2\lambda+5)}{4(n+1)(2n+1)(\lambda+n+1)(2(n+\lambda)+3)}q_{n-1}^{\perp}(\lambda+4,x).\end{aligned}\tag{3.6}$$

Again from (3.4), after replacing n by $n-2$ and λ by $\lambda+2$, we have

$$\begin{aligned}q_n^{\perp}(\lambda+2,x)=&\frac{2(\lambda+2)(\lambda+3)(2\lambda+5)(2\lambda+7)}{4n(2n-1)(\lambda+n+2)(2(n+\lambda)+5)}xq_{n-1}^{\perp}(\lambda+4,x)\\&-\frac{(\lambda+2)(\lambda+3)(\lambda+4)(\lambda+5)(2\lambda+5)(2\lambda+9)}{4n(2n-1)(\lambda+n+2)(2(n+\lambda)+5)}q_{n-2}^{\perp}(\lambda+6,x).\end{aligned}\tag{3.7}$$

This may be continued until we reach

$$\begin{aligned}&q_2^{\perp}(\lambda+2n-2,x)\\&\quad=\frac{2(\lambda+2n-2)(\lambda+2n-1)(2\lambda+4n-3)(2\lambda+4n-1)}{24(\lambda+2n)(2\lambda+4n+1)}xq_1^{\perp}(\lambda+2n,x)\\&\quad-\frac{(\lambda+2n-2)(\lambda+2n-1)(2\lambda+4n-3)(\lambda+2n+1)}{24}q_0^{\perp}(\lambda+2n+2,x).\end{aligned}\tag{3.8}$$

Applying the embedding construction in Definition 1.1 to $q_{l+1}^{\perp}(\lambda,x)$ and $q_l^{\perp}(\lambda,x)$, we now show that the $Q_n^{(l)}(\lambda,x),\ n=0,1,2,\dots$ are a new family of orthogonal polynomial functions.

Theorem 3.2: *Let $\lambda>-\frac{1}{2}$ and $\lambda\neq 0$. Then for integer $l\geq 1$, the system of polynomials $Q_n^{(l)}(\lambda,x),\ n=0,1,2,\dots$, satisfy a three-term recurrence relation of the form*

$$Q_{n+1}^{(l)}(\lambda,x)=A_nx\,Q_n^{(l)}(\lambda,x)+C_nQ_{n-1}^{(l)}(\lambda,x),$$

with $A_nA_{n-1}C_n<0$ for $n\geq 1$. They form an orthogonal polynomial system whose zeros are real and interlace on the real line.

Proof: Equations (3.5)–(3.8) and the definition of the $Q_n^{(l)}$ serve to define A_n and C_n and show the recursion relation above is satisfied. We also see the condition $A_nA_{n-1}C_n < 0$ for $n \geq 0$ is satisfied. This is clear when $\lambda > 0$ as then $A_n > 0$ and $C_n < 0$ for $n \geq 0$. When $-1/2 < \lambda < 0$ all A_n and C_n have the same sign as before except for A_l, C_l, C_{l+1} all of which change sign which means we still have $A_{l+1}\,A_l\,C_{l+1} < 0$ and $A_l\,A_{l-1}\,C_l < 0$ as required. We may use Favard's theorem (see Theorem 4.4 of [16]) to deduce that the polynomials $Q_n^{(l)}$ are orthogonal polynomials with a positive definite moment function and so have interlacing zeros. This shows that the zeros of $q_l^{\perp}(\lambda, x)$ and $q_{l+1}^{\perp}(\lambda, x)$ lie on the real axis and interlace. Because they lie on the real axis the zeros of $q_{2l}(\lambda, x)$ and $q_{2l+2}(\lambda, x)$ lie on the imaginary axis and interlace, and the zeros of $p_{2l}(\lambda, x)$ and $p_{2l+2}(\lambda, x)$ lie on the critical line and interlace. ∎

4. Related orthogonal polynomial systems

Applying Lemma 2.3 we found that the polynomials $q_n^{\perp}(\lambda, x)$ satisfy the relation (3.4). We now consider values of λ for which the shifted polynomial sequence $\{r_n(\lambda, x)\}_{n=0}^{\infty}$ defined by

$$r_n(\lambda, x) = q_n^{\perp}(\lambda - 2n, x)$$

forms an orthogonal polynomial system.

Theorem 4.1: *The polynomials $\{r_n(\lambda, x)\}_{n=0}^{\infty}$ form an orthogonal polynomial system with a positive definite moment function if and only if $\lambda \in (-\infty, 1/2) \cup (1/2, 1)$.*

Proof: These polynomials satisfy $a_n r_{n+1}(\lambda, x) = b_n x r_n(\lambda, x) - c_n r_{n-1}(\lambda, x)$, where

$$\begin{aligned} a_n &= 4(n+1)(2n+1)(-2\lambda+2n+1)(-\lambda+n+1), \\ b_n &= 2(-2\lambda+4n+1)(-2\lambda+4n+3)(-\lambda+2n+1)(-\lambda+2n+2), \\ c_n &= (-2\lambda+4n-1)(-2\lambda+4n+3)(-\lambda+2n-1)(2n-\lambda) \\ &\quad \times (-\lambda+2n+1)(-\lambda+2n+2). \end{aligned}$$

The sequence $\{r_n(\lambda, x)\}_{n=0}^{\infty}$ will be an orthogonal polynomial system with a positive definite moment function if and only if

$$\frac{a_{n-1}c_n}{b_n b_{n-1}} > 0 \quad \forall\, n \geq 1. \tag{4.1}$$

The monic version of this sequence $\{\hat{r}_n(\lambda, x)\}_{n=0}^{\infty}$ is given by

$$\hat{r}_n(\lambda, x) = \frac{a_{n-1}a_{n-1}\cdots a_0}{b_{n-1}b_{n-2}\cdots b_0} r_n(\lambda, x),$$

and satisfies

$$\hat{r}_{n+1}(\lambda, x) = x\hat{r}_n(\lambda, x) - \frac{a_{n-1}c_n}{b_n b_{n-1}}\hat{r}_{n-1}. \tag{4.2}$$

The condition (4.1) will be satisfied if, for a given λ,

$$\frac{n(2n-1)(-2\lambda+2n-1)(n-\lambda)}{(-2\lambda+4n-3)(-2\lambda+4n-1)} > 0 \quad \forall\, n \geq 1.$$

This will be the case if and only if $\lambda \in (-\infty, 1/2)$ or $\lambda \in (1/2, 1)$. Therefore the sequence of polynomials $\{r_n(\lambda, x)\}_{n=0}^{\infty}$ will be an orthogonal polynomial system with a positive definite moment function if and only if $\lambda \in (-\infty, 1/2) \cup (1/2, 1)$. ∎

We now consider the sequence $\{\hat{r}_n(\lambda, x)\}_{n=0}^{\infty}$ in the case $\lambda \to \infty$. We find that

$$\begin{aligned}
\hat{r}_0(\lambda, x) &= 1,\\
\hat{r}_1(\lambda, x) &= x,\\
\hat{r}_2(\lambda, x) &= \frac{-\lambda + (2\lambda-5)x^2 + 1}{2\lambda - 5},\\
\hat{r}_3(\lambda, x) &= \frac{x\left(-7\lambda + (2\lambda-9)x^2 + 9\right)}{2\lambda - 9},\\
\hat{r}_4(\lambda, x) &= \frac{15\left(\lambda^2 - 4\lambda + 3\right)}{4\lambda^2 - 44\lambda + 117} + x^4 + \frac{2(19-11\lambda)x^2}{2\lambda - 13}.
\end{aligned}$$

Taking the limit as $\lambda \to \infty$ for these polynomials and defining the polynomials $m_n(x)$ such that

$$m_n(x) = \hat{r}_n(\infty, x) = \lim_{\lambda\to\infty} \hat{r}_n(\lambda, x) \tag{4.3}$$

we find that first few terms are given by

$$\begin{aligned}
&m_0(x) = 1, \quad m_1(x) = x,\\
&m_2(x) = x^2 - \frac{1}{2}, \quad m_3(x) = x^3 - \frac{7x}{2},\\
&m_4(x) = x^4 - 11x^2 + \frac{15}{4}.
\end{aligned}$$

Our new polynomials $m_n(x)$ are in fact an orthogonal polynomial family, as detailed in the following theorem.

Theorem 4.2: *The polynomials $\{m_n(x)\}_{n=0}^{\infty}$, defined in* (4.3), *satisfy the three-term recurrence relation*

$$m_{n+1}(x) = x m_n(x) - n\left(n - \frac{1}{2}\right) m_{n-1}(x),$$

and belong the Meixner class of orthogonal polynomials of the second kind. They arise in the limit as $\lambda \to \infty$ for both the $\hat{r}_n(\lambda, x)$ and $q_n^{\perp}(\lambda, x)$ polynomial families, so that

$$m_n(x) = \lim_{\lambda\to\infty} q_n^{\perp}(\lambda, x) = \lim_{\lambda\to\infty} \hat{r}_n(\lambda, x),$$

with generating function

$$G(x,t) = \sum_{n=0}^{\infty} \frac{m_n(x)}{n!} t^n = \left(1+t^2\right)^{-\frac{1}{4}} e^{x \tan^{-1} t}$$
$$= \left(\prod_{j=0}^{\infty} \exp\left(\frac{x(-1)^j t^{2j+1}}{2j+1}\right)\right)\left(\sum_{k=0}^{\infty} (-1^k)\binom{-\frac{1}{4}}{k} t^{2k}\right).$$

Proof: We see that the limits exist for $\hat{r}_0(\lambda, x)$ and $\hat{r}_1(\lambda, x)$. We also have

$$\lim_{\lambda\to\infty} \frac{a_{n-1}c_n}{b_n b_{n-1}} = n\left(n - \frac{1}{2}\right).$$

Therefore, by induction, all limits exist and the recurrence relation follows from (4.2).

Regarding the generating function and that the polynomials $m_n(x)$ belong to the class of Meixner orthogonal polynomials of the second kind, we refer to [16, p. 179, equation 3.17] with $\delta = 0$ and $\eta = 1/2$. The alternative expressions for the generating function are obtained after the application of Gregory's series for $\arctan x$ and the negative binomial expansion.

For the $q_n^{\perp}(\lambda, x)$, they satisfy (3.2) which may be written as

$$a_n q_{n+1}^{\perp}(\lambda, x) = b_n q_n^{\perp}(\lambda, x) - c_n q_{n-1}^{\perp}(\lambda, x),$$

with

$$\begin{aligned}
a_n &= 4(n+1)(2n+1)(\lambda+2n-1),\\
b_n &= 2(\lambda+2n)(\lambda(2\lambda+3)+4(n-1)(\lambda+n+1)+3),\\
c_n &= (\lambda+n-1)(\lambda+2n+1)(2\lambda+2n-1),\\
&\quad \left((\lambda+1)^2+4(n-1)(\lambda+n)+x^2\right).
\end{aligned}$$

As the term c_n contains the term x^2, $q_n^{\perp}(\lambda, x)$ do not satisfy a three-term recurrence relation of the form required to be an orthogonal polynomial system.

We now repeat the above argument used for $\hat{r}_n(\lambda, x)$, obtaining the monic version of the sequence to establish the same recurrence relation holds for this polynomial family. Taking into account that in the limit as $\lambda \to \infty$, we have $q_0^{\perp}(\lambda, x) = 1$ and $q_1^{\perp}(\lambda, x) = x$, the same initial values for $\hat{r}_0(\lambda, x)$ and $\hat{r}_1(\lambda, x)$, it follows that both polynomials equate to $m_n(x)$, as required. ∎

5. Areas for further consideration

The orthogonal polynomials considered here can be viewed as bi-variate in the two variables λ and s. Initial investigations have been fruitful and this will be a topic for future research. Connections also exist between orthogonal polynomials and random matrix theory [23], including discrete and q-analogues which may lead to tangible mathematical applications of these results.

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