INTEGRAL EQUALITIES FOR FUNCTIONS OF UNBOUNDED SPECTRAL OPERATORS IN BANACH SPACES

by

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DECLARATION

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by giving explicit references. A bibliography is appended.

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STATEMENT 2

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Contents

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to my brother ANDREA

Description of the main results

The thesis is dedicated to investigating a limiting procedure for extending "local" integral operator equalities to the "global " ones in the sense explained below, and to applying it to obtaining generalizations of the Newton-Leibnitz formula for operator-valued functions for a wide class of unbounded operators.

The integral equalities considered in the thesis have the following form

$$g(R_F)\int f_x(R_F)\,d\,\mu(x)=h(R_F). \tag{0.0.1}$$

They involve functions of the kind

$$X\ni x\mapsto f_x(R_F)\in B(F),$$

where X is a general locally compact space, F is a suitable Banach subspace of a fixed complex Banach space G, for example $F=G$. The integrals are with respect to a general complex Radon measure on X and with respect to the $\sigma(B(F),\mathcal{N}_F)-$ topology [1] on $B(F)$. R_F is a possibly unbounded scalar type spectral operator in F such that $\sigma(R_F)\subseteq\sigma(R_G)$, and for all $x\in X$ f_x and g,h are complex-valued Borelian maps on the spectrum $\sigma(R_G)$ of R_G.

If $F\neq G$ we call the integral equalities (0.0.1) "local", while if $F=G$ we call them "global".

Let G be a complex Banach space and $B(G)$ the Banach algebra of all bounded linear operators on G. *Scalar type spectral operators* in G were defined in [**DS**] Definition 18.2.12. [2] (see Section 2.1), and were created for providing a general Banach space with a class of unbounded linear operators for which it is possible to establish a Borel functional calculus similar to the well-known one for unbounded self-adjoint operators in a Hilbert space.

[1]Here $\mathcal{N}_F$ is a suitable subset of $B(F)^*$, the topological dual of $B(F)$, associated with the resolution of the identity of R_F.

[2]For the special case of bounded spectral operators on G see [**Dow**].

We start with the following useful formula [3] for the resolvent of T

$$(T - \lambda \mathbf{1})^{-1} = i \int_{-\infty}^{0} e^{-it\lambda} e^{itT}\, d\,t. \tag{0.0.2}$$

Here $\lambda \in \mathbb{C}$ is such that $Im(\lambda) > 0$ and the integral is with respect to the Lebesgue measure and with respect to the strong operator topology on $B(G)$ [4].

It is known that this formula holds for

(1) any bounded operator $T \in B(G)$ on a complex Banach space G with real spectrum $\sigma(T)$, see for example [**LN**];
(2) any infinitesimal generator T of a strongly continuous semi-group in a Banach space, see Corollary $8.1.16.$ of [**DS**], in particular for any unbounded self-adjoint operator $T : \mathbf{D}(T) \subset H \to H$ in a complex Hilbert space H.

Next we consider a more general case. Let S be an entire function and $L > 0$, then the Newton-Leibnitz formula

$$\boxed{R \int_{u_1}^{u_2} \frac{d\,S}{d\,\lambda}(tR)\, d\,t = S(u_2 R) - S(u_1 R),} \tag{0.0.3}$$

$\forall u_1, u_2 \in [-L, L]$ was known for any element R in a Banach algebra $\mathcal{A}$, where $S(tR)$ and $\frac{dS}{d\lambda}(tR)$ are understood in the standard framework of analytic functional calculus on Banach algebras, while the integral is with respect to the Lebesgue measure in the norm topology on $\mathcal{A}$ see for example [**Rud, Dieu, Schw**].

If E is the resolution of the identity of R then $\forall U \in \mathcal{B}(\mathbb{C})$ [5]

$$\mathfrak{L}_E^{\infty}(U) \doteqdot \left\{ f : \mathbb{C} \to \mathbb{C} \mid \|f \chi_U\|_{\infty}^{E} < \infty \right\}.$$

[3] An important application of this formula is made in proving the well-known Stone theorem for strongly continuous semigroups of unitary operators in Hilbert space, see Theorem 12.6.1. of [**DS**]. In [**Dav**] it has been used for showing the equivalence of uniform convergence in strong operator topology of a one-parameter semigroup depending on a parameter and the convergence in strong operator topology of the resolvents of the corresponding generators, Theorem 3.17.

[4] Notice that if $\zeta \doteqdot -i\lambda$ and $Q \doteqdot iT$, then the equality (0.0.2) turns into

$$(Q + \zeta \mathbf{1})^{-1} = \int_0^{\infty} e^{-t\zeta} e^{-Qt}\, d\,t,$$

which is referred in $IX.1.3.$ of [**Kat**] as the fact that the resolvent of Q is the *Laplace* trasform of the semigroup e^{-Qt}. Applications of this formula to perturbation theory are in $IX.2.$ of [**Kat**].

[5] $\mathcal{B}(\mathbb{C})$ is the class of all Borelian sets of $\mathbb{C}$.

Here $\chi_U : \mathbb{C} \to \mathbb{C}$ is equal to 1 in U and 0 in $\complement U$ and for all maps $F : \mathbb{C} \to \mathbb{C}$

$$\|F\|_\infty^E \doteqdot E - ess\sup_{\lambda\in\mathbb{C}} |F(\lambda)| \doteqdot \inf_{\{\delta\in\mathcal{B}(\mathbb{C}) | E(\delta)=\mathbf{1}\}} \sup_{\lambda\in\delta} |F(\lambda)|.$$

See [**DS**].

We say (see Definition 3.3.5) that $\mathcal{N}$ is an $E-$**appropriate set** if

(1) $\mathcal{N} \subseteq B(G)^*$ linear subspace;

(2) $\mathcal{N}$ separates the points of $B(G)$, namely

$$(\forall T \in B(G) - \{\mathbf{0}\})(\exists\,\omega \in \mathcal{N})(\omega(T) \neq 0);$$

(3) $(\forall \omega \in \mathcal{N})(\forall \sigma \in \mathcal{B}(\mathbb{C}))$ we have

$$\boxed{\omega \circ \mathcal{R}(E(\sigma)) \in \mathcal{N} \text{ and } \omega \circ \mathcal{L}(E(\sigma)) \in \mathcal{N}.} \tag{0.0.4}$$

Moreover, we say that $\mathcal{N}$ is an $E-$**appropriate set with the duality property** if in addition

$$\boxed{\mathcal{N}^* \subseteq B(G).} \tag{0.0.5}$$

Here for any Banach algebra $\mathcal{A}$, so in particular for $\mathcal{A} = B(G)$, we set $\mathcal{R} : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ and $\mathcal{L} : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ defined by

$$\begin{cases} \mathcal{R}(T) : \mathcal{A} \ni h \mapsto Th \in \mathcal{A} \\ \mathcal{L}(T) : \mathcal{A} \ni h \mapsto hT \in \mathcal{A}, \end{cases} \tag{0.0.6}$$

for all $T \in \mathcal{A}$. In (0.0.5) we mean

$$(\exists\, Y_0 \subseteq B(G))(\mathcal{N}^* = \{\hat{A} \upharpoonright \mathcal{N} \mid A \in Y_0\}),$$

where $\hat{(\cdot)} : B(G) \to (B(G)^*)^*$ is the canonical isometric embedding of $B(G)$ into its bidual.

In the thesis the following generalizations of (0.0.3) are proved for the case when $R : \mathbf{D} \subset G \to G$ is an unbounded scalar type spectral operator in a complex Banach space G, in particular when $R : \mathbf{D} \subset H \to H$ is an unbounded self-adjoint operator in a complex Hilbert space H. Under the assumptions that $S : U \to \mathbb{C}$ is an analytic map on an open neighbourhood U of the spectrum $\sigma(R)$ of R such that $(\exists\, L > 0)(]-L, L[\cdot U \subseteq U)$ and

$$\widetilde{S}_t \in \mathfrak{L}_E^\infty(\sigma(R)),\ \widetilde{\left(\frac{d\,S}{d\,\lambda}\right)_t} \in \mathfrak{L}_E^\infty(\sigma(R))$$

for all $t \in]-L,L[$, where $(S)_t(\lambda) \doteqdot S(t\lambda)$ and $(\frac{dS}{d\lambda})_t(\lambda) \doteqdot \frac{dS}{d\lambda}(t\lambda)$ for all $t \in]-L,L[$ and $\lambda \in U$, while for any map $F : U \to \mathbb{C}$ we set $\widetilde{F}$ the $\mathbf{0}-$extension of F to $\mathbb{C}$. The following statements are proved.

(1) If

$$\int^{*} \left\| \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \right\|_{\infty}^{E} dt < \infty \tag{0.0.7}$$

and $\forall \omega \in \mathcal{N}$ the map $]-L,L[\ni t \mapsto \omega\left(\frac{dS}{d\lambda}(tR)\right) \in \mathbb{C}$ is Lebesgue measurable, then in Corollary 3.5.1 it is proved that formula (0.0.3) holds where the integral is the weak-integral [6] with respect to the Lebesgue measure and with respect to the $\sigma(B(G),\mathcal{N})-$topology for any $E-$appropriate set $\mathcal{N}$ with the duality property. Moreover in Corollary 3.5.2 it is proved that formula (0.0.3) also holds when $\widetilde{\left(\frac{dS}{d\lambda}\right)_t} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ almost everywhere on $]-L,L[$ with respect to the Lebesgue measure.

(2) In particular it is proved that formula (0.0.3) holds where the integral is the weak-integral with respect to the Lebesgue measure and with respect to the sigma-weak operator topology, when G is a Hilbert space (Corollary 3.5.3).

(3) If in addition to (0.0.7), G is a reflexive complex Banach space then in Corollary 3.5.4 it is proved that formula (0.0.3) holds where the integral is the weak-integral with respect to the Lebesgue measure and with respect to the weak operator topology.

(4) If

$$\sup_{t\in]-L,L[} \left\| \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \right\|_{\infty}^{E} < \infty, \tag{0.0.8}$$

then in Theorem 2.3.6 it is proved that formula (0.0.3) holds where the integral is with respect to the Lebesgue measure and with respect to the strong operator topology.

(5) In Theorem 2.3.4 it is proved that if in addition to the (0.0.8)

$$\sup_{t\in]-L,L[} \|\widetilde{(S)_t}\|_{\infty}^{E} < \infty,$$

[6] See formula (0.0.17) below.

then $\forall v \in \mathbf{D}$ the mapping $]-L, L[\ni t \mapsto S(tR)v \in G$ is differentiable, and $(\forall v \in \mathbf{D})(\forall t \in]-L, L[)$

$$\boxed{\frac{dS(tR)v}{dt} = R\frac{dS}{d\lambda}(tR)v.} \tag{0.0.9}$$

(6) In Corollary 2.4.1 formula (0.0.2) is deduced from formula (0.0.3) for any unbounded scalar type spectral operator $T : \mathbf{D}(T) \subset G \to G$ in a complex Banach space G with real spectrum.

In these statements $\frac{dS}{d\lambda}(tR)$ and $S(tR)$ are understood in the framework of the Borel functional calculus for unbounded scalar type spectral operators in G. See definition $18.2.10.$ in Vol 3 of the Dunford-Schwartz monograph [**DS**] (also see Section 2.1 of the thesis).

In order to prove equality (0.0.3) when R is an unbounded scalar type spectral operator in G, we procede in two steps. First of all we consider the Banach spaces $G_{\sigma_n} \doteqdot E(\sigma_n)G$ where $\sigma_n \doteqdot B_n(\mathbf{0}) \subset \mathbb{C}$, with $n \in \mathbb{N}$, the bounded operators $R_{\sigma_n} \doteqdot RE(\sigma_n)$, and their restrictions $(R_{\sigma_n} \upharpoonright G_{\sigma_n})$ to G_{σ_n}. Then by Key Lemma 2.1.7 the operators $R_{\sigma_n} \upharpoonright G_{\sigma_n}$ are bounded *scalar type spectral* operators on G_{σ_n}, and for all $x \in G$

$$S(R)x = \lim_{n\in\mathbb{N}} S(R_{\sigma_n} \upharpoonright G_{\sigma_n})E(\sigma_n)x, \tag{0.0.10}$$

in G and

(0.0.11)
$$(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \int_{u_1}^{u_2} \frac{d\,S}{d\,\lambda}(t(R_{\sigma_n} \upharpoonright G_{\sigma_n}))\,d\,t = S(u_2(R_{\sigma_n} \upharpoonright G_{\sigma_n})) - S(u_1(R_{\sigma_n} \upharpoonright G_{\sigma_n})).$$

The second and most important step it is to set up a *limiting* procedure, which allows by using the convergence (0.0.10) to extend the "local" equality (0.0.11) to the "global" one (0.0.3).

As we shall see below such a limiting procedure can be set up in a very general context. First we wish to point out that the following equalities $\forall n \in \mathbb{N}$ and $\forall t \in]-L, L[$, which follow from Key Lemma 2.1.7 are essential for making this limiting procedure possible

$$\begin{cases} \frac{d\,S}{d\,\lambda}(tR)E(\sigma_n) = \frac{d\,S}{d\,\lambda}(t(R_{\sigma_n} \upharpoonright G_{\sigma_n}))E(\sigma_n), \\ S(tR)E(\sigma_n) = S(t(R_{\sigma_n} \upharpoonright G_{\sigma_n}))E(\sigma_n). \end{cases} \tag{0.0.12}$$

We note that one cannot replace in (0.0.11) $R_{\sigma_n} \upharpoonright G_{\sigma_n}$ with the simpler operator R_{σ_n} for the following reason. Although R_{σ_n} is a bounded operator on G for $n \in \mathbb{N}$ and $Rx = \lim_{n\in\mathbb{N}} R_{\sigma_n} x$ in G, in general R_{σ_n} is not a scalar type spectral operator, hence the expression $\frac{dS}{d\lambda}(tR_{\sigma_n})$ is not defined in the Dunford-Schwartz Functional Calculus for scalar type spectral operators, which turns to be mandatory in the sequel when using general Borelian maps not necessarily analytic.

Next we formulate a rather general statement allowing, by using a limiting procedure, to pass from "local" equalities similar to (0.0.11) to "global" ones similar to (0.0.3).

We generalize (0.0.3) in several directions. We replace

- the operator R to the left of the integral by a function $g(R)$, where g is a general Borelian map on $\sigma(R)$ [7],
- the compact set $[u_1, u_2]$ and the Lebesgue measure on it by a general locally compact space X and a complex Radon measure on it respectively,
- the map $[u_1, u_2] \ni t \to \left(\frac{dS}{d\lambda}\right)_t \in Bor(\sigma(R))$ by the map $X \ni x \to f_x \in Bor(\sigma(R))$ such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ where $\widetilde{f_x}$ is the $\mathbf{0}-$extension to $\mathbb{C}$ of f_x, and the map $X \ni x \to f_x(R) \in B(G)$ is strongly integrable with respect to the measure μ; [8]
- the map $S_{u_2} - S_{u_1}$ by a Borelian map h on $\sigma(R)$ such that $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$.

One of the main results of the thesis is Theorem 2.2.10 where we prove that if $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence [9], and [10] $\forall n \in \mathbb{N}$

$$R_{\sigma_n} \upharpoonright G_{\sigma_n} \doteqdot RE(\sigma_n) \upharpoonright (G_{\sigma_n} \cap Dom(R)), \tag{0.0.13}$$

and $\forall n \in \mathbb{N}$ the following *local* inclusion

$$g(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \int f_x(R_{\sigma_n} \upharpoonright G_{\sigma_n})\, d\,\mu(x) \subseteq h(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \tag{0.0.14}$$

[7] The most interesting case is when the operator $g(R)$ is unbounded.

[8] This means that $X \ni x \to f_x(R)v \in G$ is integrable with respect to the measure μ for all $v \in G$, in the sense of Ch 4, §4 of Bourbaki [**INT**], and the map $G \ni v \mapsto \int f_x(R)v \in G$ is a (linear) bounded operator on G.

[9] By definition this means that $(\forall n \in \mathbb{N})(\sigma_n \in \mathcal{B}(\mathbb{C}))$, $(\forall n, m \in \mathbb{N})(n > m \Rightarrow \sigma_n \supseteq \sigma_m)$; $supp(E) \subseteq \bigcup_{n\in\mathbb{N}} \sigma_n$; hence we have $\lim_{n\in\mathbb{N}} E(\sigma_n) = \mathbf{1}$ strongly.

[10] By Key Lemma 2.1.7 $R_{\sigma_n} \upharpoonright G_{\sigma_n}$ is a scalar type spectral operator in the complex Banach space G_{σ_n}, but in contrast to the previous case where $\sigma_n \doteqdot B_n(\mathbf{0})$ was bounded, here σ_n could be unbounded so it may happen that $G_{\sigma_n} \nsubseteq Dom(R)$ hence the restriction $R_{\sigma_n} \upharpoonright G_{\sigma_n}$ of R_{σ_n} to G_{σ_n} has to be defined on the set $G_{\sigma_n} \cap Dom(R)$, and it could be an unbounded operator in G_{σ_n}

holds, then $h(R) \in B(G)$ and the *global* equality holds, i.e.

$$g(R) \int f_x(R)\, d\,\mu(x) = h(R). \tag{0.0.15}$$

Here all the integrals are with respect to the strong operator topology.

Now we can describe Extension Theorem and the Newton-Leibnitz formula for the integration with respect to the $\sigma(B(G), \mathcal{N})-$ topology, where $\mathcal{N}$ is a suitable subset of $B(G)^*$, which, roughly speaking, is the weakest among reasonable locally convex topologies on $B(G)$, for which the aforementioned limiting procedure can be performed.

In Section 3.2 we recall the definition of scalar essential $\mu-$integrability and the weak-integral of maps defined on X and with values in a Hausdorff locally convex spaces, where μ is a Radon measure on a locally compact space X.

Here we need just to apply these definitions to the case of $\sigma(B(G), \mathcal{N})$, i.e. the weak topology on $B(G)$ defined by the standard duality between $B(G)$ and $\mathcal{N}$ where $\mathcal{N}$ is a subset of the (topological) dual $B(G)^*$ of $B(G)$ such that it separates the points of $B(G)$.

Thus $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is by definition scalarly essentially $\mu-$integrable or equivalently $f : X \to B(G)$ is scalarly essentially $\mu-$integrable with respect to the measure μ and with respect to the $\sigma(B(G), \mathcal{N})$ topology on $B(G)$ if for all $\omega \in \mathcal{N}$ the map $\omega \circ f : X \to \mathbb{C}$ is essentially $\mu-$integrable [11], so we can define its *integral* as the following linear operator

$$\mathcal{N} \ni \omega \mapsto \int \omega(f(x)) d\,\mu(x) \in \mathbb{C}.$$

Let $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ be scalarly essentially $\mu-$integrable and assume that

$$(\exists\, B \in B(G))(\forall \omega \in \mathcal{N}) \left(\omega(B) = \int \omega(f(x)) d\,\mu(x)\right). \tag{0.0.16}$$

Notice that the operator B is defined by this condition uniquely. In this case, by definition $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ *is scalarly essentially* $(\mu, B(G))-$*integrable* or $f : X \to B(G)$ *is scalarly essentially* $(\mu, B(G))-$*integrable with respect to the* $\sigma(B(G), \mathcal{N})-$ *topology* and its *weak-integral with respect to the measure* μ *and with respect to the* $\sigma(B(G), \mathcal{N})-$ *topology* or simply its **weak-integral**, is defined by

$$\int f(x) d\,\mu(x) \doteqdot B. \tag{0.0.17}$$

[11] See for the definition Ch. 5, §1, $n°3$, of [**INT**]

Next we can state **Theorem 3.4.2** , the main result of the thesis.

THEOREM 0.0.1 ($\sigma(B(G),\mathcal{N})-$ **Extension Theorem**). *Let G be a complex Banach space, X a locally compact space and μ a complex Radon measure on it. In addition let R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum, E its resolution of the identity and $\mathcal{N}$ an $E-$appropriate set. Let the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ $\mu-$ locally almost everywhere on X and the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ be scalarly essentially $(\mu, B(G))-$integrable. Finally let $g, h \in Bor(\sigma(R))$ and $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$.*

If $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence and $\forall n \in \mathbb{N}$

$$g(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \int f_x(R_{\sigma_n} \upharpoonright G_{\sigma_n})\, d\,\mu(x) \subseteq h(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \tag{0.0.18}$$

then $h(R) \in B(G)$ and

$$\boxed{g(R) \int f_x(R)\, d\,\mu(x) = h(R).} \tag{0.0.19}$$

In (0.0.18) *the weak-integral is with respect to the measure μ and with respect to the $\sigma(B(G_{\sigma_n}),\mathcal{N}_{\sigma_n})-$ topology* [12]*, while in* (0.0.19) *the weak-integral is with respect to the measure μ and with respect to the $\sigma(B(G),\mathcal{N})-$ topology.*

Notice that $g(R)$ is a possibly **unbounded** operator in G.

We list the most important results that allow to prove Theorem 3.4.2:

(1) Key Lemma 2.1.7;

(2) "Commutation" property (Theorem 3.3.7):

$$\forall \sigma \in \mathcal{B}(\mathbb{C}) \left[\int f_x(R) d\,\mu(x),\, E(\sigma)\right] = \mathbf{0}; \tag{0.0.20}$$

(3) "Restriction" property (Theorem 3.3.16): $\forall \sigma \in \mathcal{B}(\mathbb{C})$ we have that $f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)$, $\mu-$ locally almost everywhere on X, $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma),\mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable, and

$$\boxed{\int f_x(R_\sigma \upharpoonright G_\sigma)\, d\mu(x) = \int f_x(R)\, d\mu(x) \upharpoonright G_\sigma;} \tag{0.0.21}$$

[12] $\mathcal{N}_{\sigma_n}$ is, roughly speaking, the set of the restrictions to $B(G_{\sigma_n})$ of all the functionals belonging to $\mathcal{N}$. For the exact definition and properties see Definition 3.3.14 and Lemma 3.3.11.

(4) finally the fact that

$$Dom\left(g(R)\int f_x(R)\,d\,\mu(x)\right) \text{ is dense in } G.$$

We remark that the reason for introducing the concept of an $E-$appropriate set is primarily for obtaining the commutation and restriction properties.

Now we define

$$\mathcal{N}_{st}(G) \doteqdot \langle B(G), \tau_{st}(G)\rangle^* = \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\phi,v)} \mid (\phi,v) \in G^* \times G\}). \tag{0.0.22}$$

Here $\langle B(G), \tau_{st}(G)\rangle^*$ is the topological dual of $B(G)$ with respect to the strong operator topology, $\psi_{(\phi,v)} : B(G) \ni T \mapsto \phi(Tv) \in \mathbb{C}$, while $\mathfrak{L}_{\mathbb{C}}(J)$ is the complex linear space generated by the set $J \subseteq B(G)^*$. Then $\sigma(B(G), \mathcal{N}_{st}(G))$ is the weak operator topology on $B(G)$ and $\mathcal{N}_{st}(G)$ is an $E-$appropriate set for any spectral measure E.

Moreover we set in the case in which G is a complex Hilbert space

$$\mathcal{N}_{pd}(G) \doteqdot \text{ predual of } B(G),$$

which is by definition the linear space of all sigma-weakly continuous linear functionals on $B(G)$.

Note that

$$\mathcal{N}_{pd}(G)^* = B(G). \tag{0.0.23}$$

(See statement (iii) of Theorem 2.6., Ch. 2 of [**Tak**], or Proposition $2.4.18$ of [**BR**]). Here we mean that the normed subspace $\mathcal{N}_{pd}(G)^*$ of the bidual $(B(G)^*)^*$ is isometric to $B(G)$, through the canonical embedding of $B(G)$ into $(B(G)^*)^*$.

Hence we can apply the Extension Theorem 3.4.2 to the case $\mathcal{N} \doteqdot \mathcal{N}_{st}(G)$, or $\mathcal{N} \doteqdot \mathcal{N}_{pd}(G)$ and use the following additional property which is proved in Proposition 3.3.17

$$(\mathcal{N}_{st}(G))_\sigma = \mathcal{N}_{st}(G_\sigma), \text{ and } (\mathcal{N}_{pd}(G))_\sigma = \mathcal{N}_{pd}(G_\sigma). \tag{0.0.24}$$

The reason of introducing the concept of duality property for $E-$appropriate set is primarly for assuring that a map $f : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ scalarly essentially $\mu-$integrable is also $(\mu, B(G))-$integrable.

As an application of this fact and of the Extension theorem we obtain the Newton-Leibnitz formula in (0.0.3) by replacing $\mathcal{A}$ with $B(G)$, R with an unbounded scalar type spectral operator in a complex Banach space G, by considering S analytic in an open neighbourhood U of $\sigma(R)$ such that $]-L, L[\cdot U \subseteq U$, and the integral with respect to the $\sigma(B(G),\mathcal{N})-$topology, where $\mathcal{N}$ is an $E-$appropriate set with the duality property (**Corollary 3.5.1**).

Finally in a similar way we obtain the corresponding results for the cases of the sigma-weak operator topology (Corollary 3.5.3), and for the cases of weak operator topology (Corollary 3.5.4). The last result is a complement to Theorem 2.3.6.

Equality (0.0.3) where R is an element in a Banach algebra $\mathcal{A}$ under the assumptions on S formulated above follows from the appropriate differentiation formula

$$\frac{d\, g^{\mathbb{R}} \circ T}{dt}(t) = R\sum_{n=1}^{\infty} \alpha_n n t^{n-1} R^{n-1} = R\frac{d\, g}{d\lambda} \circ T(t) \tag{0.0.25}$$

$\forall t \in \left]-\frac{Q}{\|R\|}, \frac{Q}{\|R\|}\right[$, where $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$ is a complex power series with positive radius of convergence Q, $T(t) \doteqdot tR$, and $g^{\mathbb{R}} : B_Q(\mathbf{0}) \subset \mathcal{A}_{\mathbb{R}} \ni R \mapsto \sum_{n=0}^{\infty} \alpha_n R^n \in \mathcal{A}_{\mathbb{R}}$, where $\mathcal{A}_{\mathbb{R}}$ the real Banach algebra corresponding to $\mathcal{A}$.

In Chapter 1 we consider a general formula for the Fréchet differential of a power series in $\mathcal{A}$. In Theorem 1.0.11 it is proved that the Fréchet differential of the map $\mathcal{A} \ni T \mapsto \sum_{n=0}^{\infty} \alpha_n T^n \in \mathcal{A}$, can be expressed in terms of absolutely convergent series involving the commutant $C(T) : \mathcal{A} \ni h \mapsto hT - Th \in \mathcal{A}$ with $T \in \mathcal{A}$, in three different forms containing $C(T)$, $C(T^n)$ or $C(T)^n$. Explicitly if $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$, the radius of convergence is $r > 0$ and $B_r(\mathbf{0})$ is the ball of radius r in the Banach algebra $\mathcal{A}$, then we have

(1) for all $T \in B_r(\mathbf{0})$

(0.0.26)

$$\boxed{g^{[1]}(T) = \sum_{n=1}^{\infty} n\alpha_n \mathcal{L}(T)^{n-1} - \left\{\sum_{p=0}^{\infty}\left\{\sum_{n=p+2}^{\infty}(n-p-1)\alpha_n \mathcal{L}(T)^{n-(2+p)}\right\}\mathcal{R}(T)^p\right\}C(T)}$$

(here all the series converge absolutely uniformly on $B_s(\mathbf{0})$ for all $0 < s < r$),

(2) for all $T \in B_r(\mathbf{0})$

$$\boxed{g^{[1]}(T) = \sum_{n=1}^{\infty} n\alpha_n \mathcal{L}(T)^{n-1} - \sum_{k=2}^{\infty}\left\{\sum_{n=k}^{\infty} \alpha_n \mathcal{L}(T)^{n-k}\right\} C(T^{k-1}).} \tag{0.0.27}$$

(here all the series converge absolutely uniformly on $B_s(\mathbf{0})$ for all $0 < s < r$),

(3) $\forall T \in B_{\frac{r}{3}}(\mathbf{0})$

$$g^{[1]}(T) = \sum_{p=1}^{\infty} \frac{1}{p!}(g)^{(p)}(\mathcal{R}(T))C(T)^{p-1}. \tag{0.0.28}$$

(here the series converges absolutely uniformly on $B_s(\mathbf{0})$ for all $0 < s < \frac{r}{3}$ and $g^{(p)} : \mathbb{K} \to \mathbb{K}$ is the $p-$th derivative of the function g).

The proof of this result needs a general statement about the Fr$\acute{e}$chet differentiability *term by term* of a power series in a Banach algebra. Although this last result was known - indeed for complex Banach spaces was proved by [**Mar**] whereas the proof for real Banach spaces, given for the first time in [**Mic**] - the proof in Lemma 1.0.9 has the advantage of giving for the particular case of Banach algebras a unified approach for both the cases real and complex.

Summary of the main results

Let G be a complex Banach space, R an unbounded scalar type spectral operator in G, for example an unbounded self-adjoint operator in a Hilbert space, $\sigma(R)$ its spectrum and E its resolution of identity. The **main results** of the thesis are the following ones.

(1) Extension procedure leading from local equality (0.0.18) to global equality (0.0.19) for integration with respect to the $\sigma(B(G), \mathcal{N})-$topology (Theorem 3.4.2 if $\mathcal{N}$ is an $E-$appropriate set and Corollary 3.4.3 if $\mathcal{N}$ is an $E-$appropriate set with the duality property).

(2) Extension procedure leading from local equality (0.0.18) to global equality (0.0.19) for integration with respect to the sigma-weak topology (Corollary 3.4.5 and Theorem 3.4.6) and for integration with respect to the weak operator topology (Corollary 3.4.4 and Theorem 3.4.7 or Theorem 2.2.10 and Corollary 2.2.11).

(3) Newton-Leibnitz formula (0.0.3) for a suitable analytic map S for integration with respect to the $\sigma(B(G), \mathcal{N})-$ topology, where $\mathcal{N}$ is an $E-$appropriate set with the duality property (Corollary 3.5.1 and Corollary 3.5.2); for integration with respect to the sigma-weak topology (Corollary 3.5.3) and for integration with respect to the weak operator topology (Corollary 3.5.4 and Theorem 2.3.6).

(4) Differentiation formula (0.0.9) for a suitable analytic map S (Theorem 2.3.2 and Theorem 2.3.4).

(5) Formulas (0.0.26), (0.0.27) and (0.0.28) for the Fréchet differential of a power series in a Banach algebra in terms of commutants (Theorem 1.0.11).

(6) A new proof for the resolvent formula (0.0.2) via formula (0.0.3) (Corollary 2.4.1).

CHAPTER 1

Fr$\acute{e}$chet differentiable functions in Banach algebras

1.0.1. Fréchet differential of a power series of differentiable functions.

NOTATIONS 1.0.1. *We denote by $\mathbb{N}$ the set of all natural numbers $\{0, 1, 2, ...\}$. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\langle G, \|\cdot\|_G\rangle$, or simply G, be a Banach space over $\mathbb{K}$, then $(\forall a \in G)(\forall r > 0)$ we define the open ball centered in a of radius r, to be the following set $B_r(a) \doteqdot \{v \in G \mid \|v - a\|_G < r\}$, hence its closure in G is $\overline{B}_r(a) \doteqdot \overline{B_r(a)} = \{v \in G \mid \|v - a\|_G \leq r\}$.*

Let F, G be two Banach spaces over $\mathbb{K}$, briefly $\mathbb{K}-$Banach spaces, then $\left\langle B(F, G), \|\cdot\|_{B(F,G)}\right\rangle$, will denote the $\mathbb{K}-$Banach space of all linear continuous mappings of F to G and $\|U\|_{B(F,G)} \doteqdot \sup_{\|v\|_F \leq 1} \|U(v)\|_G$, we also set $\left\langle B(G), \|\cdot\|_{B(G)}\right\rangle \doteqdot \left\langle B(G, G), \|\cdot\|_{B(G,G)}\right\rangle$.

Let $\{G_1, ..., G_n\}$ be a finite set of $\mathbb{K}-$Banach spaces, then $\left\langle \prod_{k=1}^n G_k, \|\cdot\|_{\prod_{k=1}^n G_k}\right\rangle$ is the Banach space where $\prod_{k=1}^n G_k$ is the product of the vector spaces $\{G_1, ..., G_n\}$, and $\|(v_1, ..., v_n)\|_{\prod_{k=1}^n G_k} \doteqdot \max_{k \in \{1,...,n\}} \|v_k\|_{G_k}$.

If $(\forall k = 1, ..., n)(G_k = G)$, then we will use the following notation $\langle G^n, \|\cdot\|_{G^n}\rangle \doteqdot \left\langle \prod_{k=1}^n G_k, \|\cdot\|_{\prod_{k=1}^n G_k}\right\rangle$. Let $\{F_1, ..., F_n, G\}$ be a finite set of $\mathbb{K}-$Banach spaces, then $B_n(\prod_{k=1}^n F_k; G)$ is the $\mathbb{K}-$vector space of all $n-$multilinear continuous mappings defined on $\prod_{k=1}^n F_k$ with values in G. If $(\forall k \in \{1, ..., n\})(F_k = F)$ then we set $B_n(F^n; G) \doteqdot B_n(\prod_{k=1}^n F_k; G)$.

In the sequel we shall deal with Fréchet differentiable functions

$$f : U \subseteq F \to G$$

defined on an open set U of a $\mathbb{K}-$Banach space F and with values in a $\mathbb{K}-$Banach space G. Its Fréchet differential function will be denoted by

$$f^{[1]} : U \subseteq F \to B(F, G).$$

Recall that a map $f : U \subseteq F \to G$ is Fréchet differentiable at $x_0 \in U$ if there exists a $T \in B(F,G)$ such that

$$\lim_{\substack{h \to \mathbf{0} \\ h \neq \mathbf{0}}} \frac{\|f(x_0+h) - f(x_0) - T(h)\|_G}{\|h\|_F} = 0.$$

T is called the Fréchet differential of f at x_0 and is denoted by $f^{[1]}(x_0)$. f is Fréchet differentiable on U if f is Fréchet differentiable at each $x \in U$, and in this case the map $f^{[1]} : U \to B(F,G)$ is called the Fréchet differential function of f. For the properties of Fréchet differentials see Ch 8 of the Dieudonne book [**Dieu**].

Let $\mathcal{A}$ be an associative algebra over $\mathbb{K}$ (or briefly associative algebra) then the standard Lie product on $\mathcal{A}$ is the following map

$$[\cdot,\cdot] : \mathcal{A} \times \mathcal{A} \ni (A,B) \mapsto [A,B] \doteqdot AB - BA \in \mathcal{A}$$

the commutator of A, B. By denoting by $\mathcal{A}^{\mathcal{A}}$ the set of all maps from $\mathcal{A}$ to $\mathcal{A}$, let $\mathcal{R} : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ and $\mathcal{L} : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ be defined by

$$\begin{cases} \mathcal{R}(T) : \mathcal{A} \ni h \mapsto Th \in \mathcal{A} \\ \mathcal{L}(T) : \mathcal{A} \ni h \mapsto hT \in \mathcal{A} \end{cases} \tag{1.0.1}$$

for all $T \in \mathcal{A}$. We also define the map $C : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ by

$$C \doteqdot -ad = \mathcal{L} - \mathcal{R}.$$

We consider $\forall n \in \mathbb{N}$ the following mapping

$$u_n : \mathcal{A} \ni T \mapsto T^n \in \mathcal{A}.$$

A Banach algebra over $\mathbb{K}$ (or briefly Banach algebra), see for example [**Dal**] *or* [**Pal**]*, is an associative algebra $\mathcal{A}$ over $\mathbb{K}$ with a norm $\|\cdot\|$ on it such that $\langle \mathcal{A}, \|\cdot\| \rangle$ is a Banach space and $\forall A, B \in \mathcal{A}$ we have*

$$\|AB\| \leq \|A\| \|B\|.$$

If $\mathcal{A}$ contains the unit element then it is called unitary Banach algebra.

It is easy to verify directly that

$$(\forall T_1, T_2 \in \mathcal{A})([\mathcal{R}(T_1), \mathcal{L}(T_2)] = \mathbf{0}). \tag{1.0.2}$$

By recalling definition (1.0.1) *we have* $\forall T, h \in \mathcal{A}$ *that* $\|\mathcal{R}(T)(h)\|_{\mathcal{A}} \leq \|T\|_{\mathcal{A}}\|h\|_{\mathcal{A}}$, *and* $\|\mathcal{L}(T)(h)\|_{\mathcal{A}} \leq \|T\|_{\mathcal{A}}\|h\|_{\mathcal{A}}$, *therefore*

$$\mathcal{R}(T), \mathcal{L}(T) \in B(\mathcal{A}) \tag{1.0.3}$$

with

$$\|\mathcal{R}(T)\|_{B(\mathcal{A})} \leq \|T\|_{\mathcal{A}}, \|\mathcal{L}(T)\|_{B(\mathcal{A})} \leq \|T\|_{\mathcal{A}}, \|C(T)\|_{B(\mathcal{A})} \leq 2\|T\|_{\mathcal{A}}. \tag{1.0.4}$$

Since $\mathcal{L}$ *and* $\mathcal{R}$ *are linear mappings we can conclude that*

$$\begin{cases} \mathcal{L}, \mathcal{R} \in B(\mathcal{A}, B(\mathcal{A})) \\ \|\mathcal{R}\|_{B(\mathcal{A},B(\mathcal{A}))}, \|\mathcal{L}\|_{B(\mathcal{A},B(\mathcal{A}))} \leq 1. \end{cases} \tag{1.0.5}$$

We now present a simple formula which will be used later to decompose the commutator $C(T^n)$ in terms of $C(T)$.

LEMMA 1.0.2. *Let* $\mathcal{A}$ *be an associative algebra, then* $(\forall n \in \mathbb{N} - \{0\})(\forall A_1, ..., A_{n+1}, B \in \mathcal{A})$ *we have*

$$\left[\prod_{k=1}^{n+1} A_k, B\right] = \sum_{s=0}^{n} \left(\prod_{k=1}^{s} A_k\right) [A_{s+1}, B] \prod_{j=s+2}^{n+1} A_j. \tag{1.0.6}$$

If $s = 0$ *then the first factor of the summand should be omitted, if* $s = n$ *then the last one should be omitted.*

PROOF. For $n = 1$ it is easy to see that $[A_1A_2, B] = A_1[A_2, B] + [A_1, B]A_2$. We shall prove the statement by induction. Let $n \in \mathbb{N} - \{0, 1\}$, and (1.0.6) be true for $n-1$, then $(\forall A_1, ..., A_{n+1}, B \in \mathcal{A})$ we have

$$\left[\prod_{k=1}^{n+1} A_k, B\right] = \left[\left(\prod_{k=1}^{n} A_k\right) A_{n+1}, B\right]$$

by $[A_1A_2, B] = A_1[A_2, B] + [A_1, B]A_2$

$$= \left(\prod_{k=1}^{n} A_k\right) [A_{n+1}, B] + \left[\prod_{k=1}^{n} A_k, B\right] A_{n+1}$$

by hypothesis

$$= \left(\prod_{k=1}^{n} A_k\right)[A_{n+1}, B] + \sum_{s=0}^{n-1}\left(\prod_{k=1}^{s} A_k\right)[A_{s+1}, B]\left(\prod_{j=s+2}^{n} A_j\right)A_{n+1}$$
$$= \sum_{s=0}^{n}\left(\prod_{k=1}^{s} A_k\right)[A_{s+1}, B]\prod_{j=s+2}^{n+1} A_j.$$

□

COROLLARY 1.0.3. *Let $\mathcal{A}$ be a unitary associative algebra, then* $(\forall n \in \mathbb{N})(\forall T, h \in \mathcal{A})$ *we have*

$$C(T^{n+1}) = \sum_{s=0}^{n} \mathcal{R}(T)^s C(T)\mathcal{L}(T)^{n-s} = \sum_{s=0}^{n} \mathcal{R}(T)^s \mathcal{L}(T)^{n-s} C(T).$$

PROOF. The second equality follows by Lemma 1.0.2 where $A_1 = A_2 = ... = A_{n+1} = T$, the first one by the second and (1.0.2). □

The following equality is stated without proof in the exercise 19, $\S 1$, Ch. 1 of [**LIE**]. For the sake of completeness we give a proof.

LEMMA 1.0.4. *Let $\mathcal{A}$ be a unitary associative algebra, then we have $\forall T \in \mathcal{A}$ and $\forall n \in \mathbb{N}$ that*

$$C(T)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathcal{R}(T)^k \mathcal{L}(T)^{n-k}.$$

PROOF. Since by (1.0.2) $\mathcal{L}(T)$ and $\mathcal{R}(T)$ commute it follows the statement. □

LEMMA 1.0.5. *Let $\mathcal{A}$ be a unitary associative algebra. Then $\forall T \in \mathcal{A}$ and $\forall n \in \mathbb{N} - \{0\}$ we have*

$$\sum_{p=1}^{n} \binom{n}{p} \mathcal{R}(T)^{n-p} C(T)^{p-1} = \sum_{s=1}^{n} \mathcal{R}(T)^{n-s} \mathcal{L}(T)^{s-1}.$$

PROOF. Since $\mathcal{L} = C + \mathcal{R}$ and by (1.0.2) $C(T)$ and $\mathcal{R}(T)$ commute we have

$$
\begin{aligned}
\sum_{p=1}^{n} \mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1} &= \sum_{p=1}^{n} \mathcal{R}(T)^{n-p}(C(T) + \mathcal{R}(T))^{p-1} \\
&= \sum_{p=1}^{n} \mathcal{R}(T)^{n-p} \sum_{k=0}^{p-1} \binom{p-1}{k} \mathcal{R}(T)^{p-1-k} C(T)^{k} \\
&= \sum_{k=0}^{n-1} \left(\sum_{p=k+1}^{n} \binom{p-1}{k} \right) \mathcal{R}(T)^{n-1-k} C(T)^{k} \\
&= \sum_{s=1}^{n} \left(\sum_{p=s}^{n} \binom{p-1}{s-1} \right) \mathcal{R}(T)^{n-s} C(T)^{s-1} \\
&= \sum_{s=1}^{n} \binom{n}{s} \mathcal{R}(T)^{n-s} C(T)^{s-1}. \qquad (1.0.7)
\end{aligned}
$$

□

DEFINITION 1.0.6. *Let $\mathcal{A}$ be a unitary Banach algebra, $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$, where the series defined in $\mathbb{K}$ is with the coefficients $a_n \in \mathbb{K}$ and has the radius of convergence $R > 0$. Then $\forall T \in \mathcal{A}$ such that $\|T\|_{\mathcal{A}} < R$ we can define*

$$f(T) \doteqdot \sum_{n=0}^{\infty} \alpha_n T^n \in \mathcal{A}.$$

It is well-known that T^n is Fréchet differentiable. For the sake of completeness we give a direct proof of the Fréchet differential function of T^n in several forms which will be used in the sequel.

LEMMA 1.0.7. *Let $\mathcal{A}$ be a unitary Banach algebra. Then $\forall n \in \mathbb{N}$ the map $u_n : \mathcal{A} \ni T \mapsto T^n \in \mathcal{A}$ is Fréchet differentiable and its Fréchet differential map $u_n^{[1]} : \mathcal{A} \to B(\mathcal{A})$*

is such that $u_0^{[1]}(T)=\mathbf{0}, u_1^{[1]}(T)=\mathbf{1}\in B(\mathcal{A})$, $\forall T\in\mathcal{A}$ *and* $\forall n\in\mathbb{N}-\{0,1\}$

$$
\begin{aligned}
u_n^{[1]}(T) &= \sum_{p=1}^{n}\mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1}\\
&= n\mathcal{L}(T)^{n-1}-\sum_{k=2}^{n}\mathcal{L}(T)^{n-k}C(T^{k-1})\\
&= \sum_{p=1}^{n}\binom{n}{p}\mathcal{R}(T)^{n-p}C(T)^{p-1}\\
&= n\mathcal{L}(T)^{n-1}-\sum_{s=0}^{n-2}(n-s-1)\mathcal{L}(T)^{n-(s+2)}\mathcal{R}(T)^{s}C(T). \qquad (1.0.8)
\end{aligned}
$$

Finally $\forall n\in\mathbb{N}-\{0\}$ *and* $\forall T\in\mathcal{A}$

$$
\|u_n^{[1]}(T)\|_{B(\mathcal{A})}\le n\|T\|_{\mathcal{A}}^{n-1}. \qquad (1.0.9)
$$

PROOF. For brevity in this proof we write $\|\cdot\|$ for $\|\cdot\|_{\mathcal{A}}$. The cases $n=0,1$ are trivial. Assume that $n\in\mathbb{N}-\{0,1\}$ and $T,h\in\mathcal{A}$

$$
\sum_{p=1}^{n}\mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1}(h)=hT^{n-1}+ThT^{n-2}+...+T^{k-1}hT^{n-k}+...+T^{n-1}h
$$

so

$$
(T+h)^n=T^n+\sum_{p=1}^{n}\mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1}(h)+\mathfrak{T}(h;T;2).
$$

Here $\mathfrak{T}(h;T;2)$ is a polynomial in the two variables T and h each monomial of which is at least of degree 2 with respect to the variable h. Hence

(1.0.10)

$$
\begin{aligned}
\lim_{h\to\mathbf{0}}\frac{\|(T+h)^n-T^n-\sum_{p=1}^{n}\mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1}(h)\|}{\|h\|} &= \lim_{h\to\mathbf{0}}\frac{\|\mathfrak{T}(h;T;2)\|}{\|h\|}\\
&\le \lim_{h\to\mathbf{0}}\frac{\mathfrak{T}(\|h\|;\|T\|;2)}{\|h\|}=0.
\end{aligned}
$$

Here $\mathfrak{T}(\|h\|;\|T\|;2)$ is the polynomial in the variables $\|h\|$ and $\|T\|$ obtained by replacing in $\mathfrak{T}(h;T;2)$ the variable h with $\|h\|$ and T with $\|T\|$. Hence

$$
u_n^{[1]}(T)=\sum_{p=1}^{n}\mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1} \qquad (1.0.11)
$$

and the first of equalities (1.0.8) follows. Therefore we have $(\forall T \in \mathcal{A})(\forall h \in \mathcal{A})$
(1.0.12)

$$\begin{aligned} u_n^{[1]}(T)(h) &= hT^{n-1} + ThT^{n-2} + ... + T^{k-1}hT^{n-k} + ... + T^{n-1}h \\ &= hT^{n-1} + [T,h]T^{n-2} + hT^{n-1} + ... + [T^{k-1},h]T^{n-k} + hT^{n-1} + ... \\ &+ [T^{n-1},h] + hT^{n-1} \\ &= nhT^{n-1} + \sum_{k=2}^{n}[T^{k-1},h]T^{n-k}. \end{aligned}$$

This is the second equality in (1.0.8). The fourth equality in (1.0.8) follows by the second one, by the commutativity property in (1.0.2) and by Corollary 1.0.3. By the first equality in (1.0.8) and Lemma 1.0.5 we obtain the third equality in (1.0.8). Finally $\forall T, h \in \mathcal{A}$ by (1.0.11) and the (1.0.5) we obtain (1.0.9)

$$\begin{aligned} \|u_n^{[1]}(T)(h)\| &= \|\sum_{p=1}^{n}\mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1}(h)\| \\ &\leq \sum_{p=1}^{n}\|\mathcal{R}(T)\|_{B(\mathcal{A})}^{n-p} \quad \|\mathcal{L}(T)\|_{B(\mathcal{A})}^{p-1} \quad \|h\| \\ &\leq \sum_{p=1}^{n}\|T\|^{n-p}\|T\|^{p-1}\|h\| \\ &= n\|T\|^{n-1}\|h\|. \end{aligned}$$

$\square$

REMARK 1.0.8. *Let $S \neq \emptyset$ and X be a Banach space over $\mathbb{K}$, then we define the space of bounded functions*

$$\mathbf{B}(S,X) \doteqdot \left\{ F : S \to X \mid \|F\|_{\mathbf{B}(S,X)} \doteqdot \sup_{T\in S}\|F(T)\|_X < \infty \right\}. \tag{1.0.13}$$

Then $\left\langle \mathbf{B}(S,X), \|\cdot\|_{\mathbf{B}(S,X)} \right\rangle$ is a Banach space over $\mathbb{K}$ and the convergence in it is called the uniform convergence on S in $\|\cdot\|_X-$topology *, or simply when this does not cause confusion, the* uniform convergence on S *, see Ch.*10 *of* **[GT]**.

Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathbf{B}(S,X)$ then the series $\sum_{n=0}^{\infty} f_n$ converges uniformly on S *if there exists $W \in \mathbf{B}(S,X)$ such that*

$$\lim_{n\in\mathbb{N}} \sup_{T\in S} \left\| W(T) - \sum_{k=1}^{n} f_k(T) \right\|_X = 0. \tag{1.0.14}$$

The series $\sum_{n=0}^{\infty} f_n$ converges absolutely uniformly on S *or* converges absolutely uniformly for $T \in S$ *if*

$$\sum_{n=0}^{\infty} \sup_{T \in S} \|f_n(T)\|_X < \infty.$$

Since $\mathbf{B}(S, X)$ *is a Banach space, the absolute uniform convergence implies uniform convergence.*

Now we shall show that a power series $g(T) \doteqdot \sum_{n=0}^{\infty} \alpha_n T^n$ in a Banach algebra $\mathcal{A}$ is Fréchet differentiable term by term, the corresponding power series of its Fréchet differential $g^{[1]}$ is uniformly convergent on $B_s(\mathbf{0})$ in the norm topology of $B(\mathcal{A})$ for all $0 < s < R$, and finally that $g^{[1]}$ is continuous, where the radius of convergence R of $\sum_{n=0}^{\infty} \alpha_n \lambda^n$ is different to zero. The proof is based on the well-known results stating that uniform convergence in Banach spaces, preserves Fréchet differentiability and continuity, see Theorem $8.6.3.$ of the [**Dieu**] for the first and Theorem (2), $\S 1.6.$, Ch. 10 of the [**GT**] for the second one.

The Fréchet differentiability of g can be seen as a particular case of the Fréchet differentiability of a power series of polynomials between two Banach spaces. The first time for complex Banach spaces was proved in [**Mar**]. Whereas the proof for real Banach spaces, given for the first time in [**Mic**], used a weak form of Markoff's inequality for the derivative of a polynomial, see [**Scha**].

Our proof has the advantage of giving for the particular case of Banach algebras a unified approach for both the cases real and complex.

LEMMA 1.0.9 (Fréchet differentiability of a power series in a Banach algebra). *Let* $\mathcal{A}$ *be a unitary Banach algebra,* $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{K}$ *be such that the radius of convergence of the series* $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *is* $R > 0$*.*

(1) *The series*

$$\sum_{n=0}^{\infty} \alpha_n u_n$$

converges absolutely uniformly on $B_s(\mathbf{0})$ *for all* $0 < s < R$.[1] *Hence we can define the map* $g : B_R(\mathbf{0}) \to \mathcal{A}$ *as* $g(T) \doteqdot \sum_{n=0}^{\infty} \alpha_n u_n(T)$*.*

[1]By Remark 1.0.8,

$$\sum_{n=0}^{\infty} \sup_{T \in B_s(\mathbf{0})} \|\alpha_n T^n\|_{\mathcal{A}} < \infty$$

for all $0 < s < R$.

(2) *g is Fréchet differentiable on $B_R(\mathbf{0})$ and*

$$\boxed{g^{[1]} = \sum_{n=1}^{\infty} \alpha_n u_n^{[1]}.} \tag{1.0.15}$$

Here the series converges absolutely uniformly on $B_s(\mathbf{0})$, for all $0 < s < R$ [2] *and $g^{[1]}$ is continuous.*

PROOF. $(\forall s \in (0,R))(\forall T \in B_s(\mathbf{0}))(\forall n \in \mathbb{N})$ we have $\|\alpha_n T^n\|_{\mathcal{A}} \leq |\alpha_n| \|T\|_{\mathcal{A}}^n \leq |\alpha_n| s^n$, so

$$\sum_{n=0}^{\infty} \sup_{T \in B_s(\mathbf{0})} \|\alpha_n T^n\|_{\mathcal{A}} \leq \sum_{n=0}^{\infty} |\alpha_n| s^n < \infty.$$

Which is statement (1).

By (1.0.9) for all $0 < s < R$

$$\sum_{n=0}^{\infty} \sup_{T \in B_r(\mathbf{0})} \|\alpha_n u_n^{[1]}(T)\|_{B(\mathcal{A})} \leq \sum_{n=0}^{\infty} |\alpha_n| n s^{n-1} < \infty.$$

Hence the series $\sum_{n=0}^{\infty} \alpha_n u_n^{[1]}$ converges absolutely uniformly on $B_s(\mathbf{0})$ for all $0 < s < R$. Thus the mapping

$$T \ni B_R(\mathbf{0}) \subset \mathcal{A} \mapsto \sum_{n=0}^{\infty} \alpha_n u_n^{[1]}(T) \in B(\mathcal{A}) \tag{1.0.16}$$

is well defined on $B_R(\mathbf{0})$ and the series converges uniformly for $T \in B_s(\mathbf{0})$ for all $0 < s < R$. Hence we can apply Theorem 8.6.3. of the [**Dieu**] and then deduce (1.0.15).

Now it remains to show the last part of the statement (2), i.e. the continuity of the differential function $g^{[1]}$. By the first part of Lemma 1.0.7 applied to the unitary Banach algebra $B(\mathcal{A})$ and by (1.0.5) $\forall n \in \mathbb{N}$ the maps

$$\mathcal{A} \ni T \mapsto \mathcal{L}(T)^n \in B(\mathcal{A}),\ \mathcal{A} \ni T \mapsto \mathcal{R}(T)^n \in B(\mathcal{A}) \tag{1.0.17}$$

are continuous. Moreover the product is continuous on $B(\mathcal{A}) \times B(\mathcal{A})$ so by (1.0.17) and the first equality in (1.0.8) $\forall n \in \mathbb{N}$

$$u_n^{[1]} : \mathcal{A} \to B(\mathcal{A}) \text{ is continuous.} \tag{1.0.18}$$

[2]By Remark 1.0.8

$$\sum_{n=1}^{\infty} \sup_{T \in B_s(\mathbf{0})} \|\alpha_n u_n^{[1]}(T)\|_{B(\mathcal{A})} < \infty$$

for all $0 < s < R$.

By (1.0.18), the uniform convergence of which in the first part of statement (2), and finally by the fact that the set of all continuous maps is closed with respect to the topology of uniform convergence, see for example Theorem (2), $\S 1.6.$, Ch. 10 of the [**GT**], we conclude that $\forall 0 < s < R$ the mapping $g^{[1]} \upharpoonright B_s(\mathbf{0}) : B_s(\mathbf{0}) \subset \mathcal{A} \to B(\mathcal{A})$ is continuous. This ends the proof of statement (2). $\square$

REMARK 1.0.10. *By statement* 2 *of Lemma 1.0.9 we have*

$$g^{[1]}(T)(h) = \sum_{n=1}^{\infty} \alpha_n u_n^{[1]}(T)(h).$$

Here the series converges absolutedly uniformly for $(T, h) \in B_s(\mathbf{0}) \times B_L(\mathbf{0})$*, for all* $L > 0$ *and* $0 < s < R$*.*

THEOREM 1.0.11 (**Fréchet differential of a power series**). *Let* $\mathcal{A}$ *be a unitary Banach algebra,* $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{K}$ *be such that the radius of convergence of the series* $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *is* $R > 0$*. Then*

(1) *for all* $T \in B_R(\mathbf{0})$

(1.0.19)
$$\boxed{g^{[1]}(T) = \sum_{n=1}^{\infty} n \alpha_n \mathcal{L}(T)^{n-1} - \left\{ \sum_{p=0}^{\infty} \left\{ \sum_{n=p+2}^{\infty} (n-p-1) \alpha_n \mathcal{L}(T)^{n-(2+p)} \right\} \mathcal{R}(T)^p \right\} C(T).}$$

Here all the series converge absolutely uniformly on $B_s(\mathbf{0})$ *for all* $0 < s < R$*.*

(2) *for all* $T \in B_R(\mathbf{0})$

$$\boxed{g^{[1]}(T) = \sum_{n=1}^{\infty} n \alpha_n \mathcal{L}(T)^{n-1} - \sum_{k=2}^{\infty} \left\{ \sum_{n=k}^{\infty} \alpha_n \mathcal{L}(T)^{n-k} \right\} C(T^{k-1}).} \tag{1.0.20}$$

Here all the series converge absolutely uniformly on $B_s(\mathbf{0})$ *for all* $0 < s < R$*.*

(3) $\forall T \in B_{\frac{R}{3}}(\mathbf{0})$

$$\boxed{g^{[1]}(T) = \sum_{p=1}^{\infty} \frac{1}{p!} (g)^{(p)}(\mathcal{R}(T)) C(T)^{p-1}.} \tag{1.0.21}$$

Here the series converges absolutely uniformly on $B_s(\mathbf{0})$ *for all* $0 < s < \frac{R}{3}$ *and* $g^{(p)} : \mathbb{K} \to \mathbb{K}$ *is the* $p-th$ *derivative of the function* g*.*

REMARK 1.0.12. *If* $R/3 \leq s < R$ *then in general the series in* (1.0.21) *may not converge, see for a counterexample the* [**Bur**].

REMARK 1.0.13. *Clearly both* (1.0.19) *and* (1.0.20) *immediately imply that if* $T, h \in \mathcal{A}$ *are such that* $[T, h] = \mathbf{0}$*, then*

$$g^{[1]}(T)(h) = \sum_{n=1}^{\infty} n\alpha_n h T^{n-1}.$$

PROOF OF THEOREM 1.0.11. By Lemma 1.0.9 and Lemma 1.0.7

$$\begin{aligned} g^{[1]}(T) &= \sum_{n=1}^{\infty} \alpha_n u_n^{[1]}(T) \\ &= \alpha_1 \mathbf{1} + \sum_{n=2}^{\infty} \alpha_n \left(n\mathcal{L}(T)^{n-1} - \sum_{s=0}^{n-2} (n-s-1)\mathcal{L}(T)^{n-(s+2)}\mathcal{R}(T)^s C(T) \right). \end{aligned}$$

By (1.0.4) for all $0 < r < R$

$$\sum_{n=2}^{\infty} \sup_{T \in B_r(\mathbf{0})} \|\alpha_n n\mathcal{L}(T)^{n-1}\|_{B(\mathcal{A})} \leq \sum_{n=2}^{\infty} n|\alpha_n| r^{n-1} < \infty$$

and

$$\sum_{n=2}^{\infty} \sum_{s=0}^{n-2} \sup_{T \in B_r(\mathbf{0})} \|\alpha_n (n-s-1)\mathcal{L}(T)^{n-(s+2)}\mathcal{R}(T)^s C(T)\|_{B(\mathcal{A})} \leq$$

$$\sum_{n=2}^{\infty} |\alpha_n| \sum_{s=0}^{n-2} (n-s-1) r^{n-2}(2r) = \sum_{n=2}^{\infty} |\alpha_n|(n-1)n r^{n-1} < \infty. \tag{1.0.22}$$

Therefore

$$g^{[1]}(T) = \sum_{n=1}^{\infty} \alpha_n n\mathcal{L}(T)^{n-1} - \sum_{n=2}^{\infty} \alpha_n \sum_{s=0}^{n-2} (n-s-1)\mathcal{L}(T)^{n-(s+2)}\mathcal{R}(T)^s C(T). \tag{1.0.23}$$

Here each series converges absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < R$. Inequality 1.0.22 also implies that

$$\begin{aligned} &\sum_{n=2}^{\infty} \alpha_n \sum_{s=0}^{n-2} (n-s-1)\mathcal{L}(T)^{n-(s+2)}\mathcal{R}(T)^s C(T) \\ &= \sum_{s=0}^{\infty} \sum_{n=s+2}^{\infty} (n-s-1)\alpha_n \mathcal{L}(T)^{n-(2+s)}\mathcal{R}(T)^s C(T) \\ &= \left\{ \sum_{s=0}^{\infty} \left\{ \sum_{n=s+2}^{\infty} (n-s-1)\alpha_n \mathcal{L}(T)^{n-(2+s)} \right\} \mathcal{R}(T)^s \right\} C(T). \end{aligned}$$

Here each series converging absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < R$ and statement (1) follows.

$$\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{k=2}^{n}\sup_{T\in B_r(\mathbf{0})}\left\|\sum_{s=0}^{k-2}\alpha_n\mathcal{L}(T)^{n-k}\mathcal{R}(T)^s\mathcal{L}(T)^{k-(2+s)}C(T)\right\|_{B(\mathcal{A})}\\
&=\sum_{n=2}^{\infty}\sum_{k=2}^{n}\sup_{T\in B_r(\mathbf{0})}\left\|\sum_{s=0}^{k-2}\alpha_n\mathcal{R}(T)^s\mathcal{L}(T)^{n-(2+s)}C(T)\right\|_{B(\mathcal{A})}\\
&\le\sum_{n=2}^{\infty}\sum_{k=2}^{n}\sum_{s=0}^{k-2}|\alpha_n|\sup_{T\in B_r(\mathbf{0})}\|\mathcal{R}(T)\|^s_{B(\mathcal{A})}\|\mathcal{L}(T)\|^{n-(2+s)}_{B(\mathcal{A})}\|C(T)\|_{B(\mathcal{A})}\\
&\le 2\sum_{n=2}^{\infty}\sum_{k=2}^{n}\sum_{s=0}^{k-2}|\alpha_n|\sup_{T\in B_r(\mathbf{0})}\|T\|^{n-1}_{\mathcal{A}}\\
&=\sum_{n=2}^{\infty}n(n-1)|\alpha_n|\sup_{T\in B_r(\mathbf{0})}\|T\|^{n-1}_{\mathcal{A}}\\
&=\sum_{n=2}^{\infty}n(n-1)|\alpha_n|r^{n-1}<\infty. \qquad (1.0.24)
\end{aligned}$$

Here in the second inequality we used (1.0.4). Therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{k=2}^{n}\sum_{s=0}^{k-2}\alpha_n\mathcal{L}(T)^{n-k}\mathcal{R}(T)^s\mathcal{L}(T)^{k-(2+s)}C(T)\\
&=\sum_{k=2}^{\infty}\sum_{n=k}^{\infty}\sum_{s=0}^{k-2}\alpha_n\mathcal{L}(T)^{n-k}\mathcal{R}(T)^s\mathcal{L}(T)^{k-(2+s)}C(T)\\
&=\sum_{k=2}^{\infty}\sum_{n=k}^{\infty}\alpha_n\mathcal{L}(T)^{n-k}\sum_{s=0}^{k-2}\mathcal{R}(T)^s\mathcal{L}(T)^{k-(2+s)}C(T)\\
&=\sum_{k=2}^{\infty}\left\{\sum_{n=k}^{\infty}\alpha_n\mathcal{L}(T)^{n-k}\right\}C(T^{k-1}). \qquad (1.0.25)
\end{aligned}$$

All the series uniformly converge for $T\in B_r(\mathbf{0})$. Here in the last equality we used Corollary 1.0.3 and the fact that $\mathcal{L}(C(T^{k-1}))\in B(B(\mathcal{A}))$. Moreover by (1.0.2)

$$\sum_{n=2}^{\infty}\sum_{k=2}^{n}\sum_{s=0}^{k-2}\alpha_n\mathcal{L}(T)^{n-k}\mathcal{R}(T)^s\mathcal{L}(T)^{k-(2+s)}C(T)=\sum_{n=2}^{\infty}\sum_{s=0}^{n-2}(n-s-1)\alpha_n\mathcal{L}(T)^{n-(2+s)}\mathcal{R}(T)^sC(T)$$

hence by (1.0.25) and (1.0.23) we obtain statement (2).

Finally we have $\forall s < \frac{R}{3}$

$$\begin{aligned}
\mathfrak{A} &\doteqdot \sum_{n=1}^{\infty}\sum_{p=1}^{n}\sup_{T\in B_s(\mathbf{0})}\left\|\alpha_n\binom{n}{p}\mathcal{R}(T)^{n-p}C(T)^{p-1}\right\|_{B(\mathcal{A})}\\
&\leq \sum_{n=1}^{\infty}\sum_{p=1}^{n}\binom{n}{p}|\alpha_n|\sup_{T\in B_s(\mathbf{0})}\|\mathcal{R}(T)\|_{B(\mathcal{A})}^{n-p}\|C(T)\|_{B(\mathcal{A})}^{p-1}\\
&\leq \sum_{n=1}^{\infty}\sum_{p=1}^{n}\binom{n}{p}|\alpha_n|\sup_{T\in B_s(\mathbf{0})}\|T\|_{\mathcal{A}}^{n-p}2^{p-1}\|T\|_{\mathcal{A}}^{p-1}\\
&= \sum_{n=1}^{\infty}\sum_{p=1}^{n}\binom{n}{p}|\alpha_n|s^{n-1}2^{p-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathfrak{A} &\leq \sum_{n=1}^{\infty}|\alpha_n|s^{n-1}\sum_{p=1}^{n}\binom{n}{p}2^{p-1}\\
&< \sum_{n=1}^{\infty}|\alpha_n|s^{n-1}\sum_{p=0}^{n}\binom{n}{p}2^{p}\\
&= \sum_{n=1}^{\infty}|\alpha_n|s^{n-1}3^{n}\\
&= s^{-1}\sum_{n=1}^{\infty}|\alpha_n|(3s)^{n} < \infty.
\end{aligned}$$

Thus by the third equality in Lemma 1.0.7 and Lemma 1.0.9 we obtain statement (3).

□

The previous Theorem 1.0.11 is the main result of the present chapter. Let $\mathcal{A}$ be a unitary $\mathbb{K}-$Banach algebra and $\sum_{n=0}^{\infty}\alpha_n\lambda^n$ a series at coefficients in $\mathbb{K}$ having radius of convergence $R > 0$. We give for the first time the Fréchet differential $g^{[1]}$ of the $\mathcal{A}-$valued function $g(T) = \sum_{n=0}^{\infty}\alpha_n T^n$, in a $C(T)-$ *depending uniformly convergent series on* $B_s(\mathbf{0})$*, for all* $0 < s < R$*, in statement* (1)*; and in a* $C(T^k)-$ *depending uniformly convergent series on* $B_s(\mathbf{0})$*, for all* $0 < s < R$ *and with* $k \geq 1$*, in statement* (2). This allows us to give immediately a simplified formula for the value $g^{[1]}(T)(h)$ in case of the commutativity $[T,h] = \mathbf{0}$, with $T \in B_R(\mathbf{0})$ and $h \in \mathcal{A}$, see Remark 1.0.13.

Finally we give a different proof respect to [**Rud**] and in such a way generalizing that in [**Bur**], of the known formula in statement (3), in case $0 < s < \frac{R}{3}$, see Remark 1.0.15 and Remark 1.0.20.

REMARK 1.0.14. *We note that formula* (1.0.19) *explicitly contains* $C(T)$ *as a factor, formula* (1.0.20) *gives an expansion in terms of* $C(T^k)$ *and finally formula* (1.0.21) *gives an expansion in terms of* $C(T)^k$.

REMARK 1.0.15. *For all* T *such that* $\|T\| < \frac{R}{3}$ *and* $\forall h \in \mathcal{A}$ *we have*

$$g^{[1]}(T)(h) = \sum_{p=1}^{\infty} \frac{1}{p!} g^{(p)}(T) C(T)^{p-1}(h). \tag{1.0.26}$$

Here the series is uniformly convergent for $(T, h) \in B_s(\mathbf{0}) \times B_L(\mathbf{0})$ *for all* $0 < s < \frac{R}{3}$ *and* $L > 0$.

COROLLARY 1.0.16 (Fréchet differential of a power series of differentiable functions defined on an open set of a $\mathbb{K}-$Banach Space and at value in a $\mathbb{K}-$Banach algebra $\mathcal{A}$). *Let* $\mathcal{A}$ *be a unitary Banach algebra, and* $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{K}$ *be such that the radius of convergence of the series* $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *is* $R > 0$ *and* $0 < s < R$. *Finally let* X *be a Banach space over* $\mathbb{K}$, $D \subseteq X$ *an open set in* X *and* $T : D \to \mathcal{A}$ *a Fréchet differentiable mapping such that* $T(D) \subseteq B_s(\mathbf{0})$ *or alternatively* D *is convex and bounded and* $\sup_{x \in D} \|T^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$. *If we set* $\widetilde{s} \doteqdot \sup_{x \in D} \|T(x)\|_{\mathcal{A}}$, *then*

(1) $\widetilde{s} < \infty$ *and if* $\widetilde{s} < R$ *then*

$$g \circ T = \sum_{n=0}^{\infty} \alpha_n T^n.$$

Here the series is uniformly convergent on D, *while* $T^n : D \ni x \mapsto T(x)^n$.

(2) *If* $0 < \widetilde{s} < R$ *then the function* $g \circ T$ *is Fréchet differentiable and*

$$\boxed{[g \circ T]^{[1]}(x) = \sum_{n=0}^{\infty} \alpha_n u_n^{[1]}(T(x)) T^{[1]}(x), \quad \forall x \in D.} \tag{1.0.27}$$

Here the series converges in $B(X, \mathcal{A})$. *Moreover*

(a) *If* $T^{[1]} : D \to B(X, \mathcal{A})$ *is continuous then the function* $[g \circ T]^{[1]} : D \to B(X, \mathcal{A})$, *is also continuous.*

(b) *If* $\sup_{x \in D} \|T^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$, *then the series in* (1.0.27) *absolutely uniformly converges on* D.

PROOF. Let us consider the case in which D is convex and bounded, and $\sup_{x \in D} \|T^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$. Let $a, b \in D$ and $S_{a,b}$ the segment jointing a, b. D is convex so $S_{a,b} \subset D$. By an application of the Mean Value Theorem, see Theorem 8.6.2.

of [**Dieu**], we have for any $x_0 \in D$

$$\|T(b) - T(a) - T^{[1]}(x_0)(b-a)\|_{\mathcal{A}} \leq \|b-a\|_X \cdot \sup_{x \in S_{a,b}} \|T^{[1]}(x) - T^{[1]}(x_0)\|_{B(X,\mathcal{A})}.$$

Thus by $\|A\| - \|B\| \leq \|A - B\|$ in any normed space, we have fixed $b \in D$ and $x_0 \in D$, that $\forall a \in D$

(1.0.28)

$$\begin{aligned}
&\sup_{a \in D} \|T(a)\|_{\mathcal{A}} \\
&\leq \sup_{a \in D} \|T(b) - T^{[1]}(x_0)(b-a)\|_{\mathcal{A}} + \sup_{a \in D} \|b-a\|_X \cdot \sup_{a \in D} \sup_{x \in S_{a,b}} \|T^{[1]}(x) - T^{[1]}(x_0)\|_{B(X,\mathcal{A})} \\
&\leq \|T(b)\|_{\mathcal{A}} + \|T^{[1]}(x_0)\|_{B(X,\mathcal{A})} \sup_{a \in D} \|b-a\|_X + \sup_{a \in D} \|b-a\|_X \cdot \sup_{x \in D} \|T^{[1]}(x) - T^{[1]}(x_0)\|_{B(X,\mathcal{A})} \\
&\leq \|T(b)\|_{\mathcal{A}} + \sup_{a \in D} \|b-a\|_X \cdot \left(2\|T^{[1]}(x_0)\|_{B(X,\mathcal{A})} + \sup_{x \in D} \|T^{[1]}(x)\|_{B(X,\mathcal{A})} \right) < \infty.
\end{aligned}$$

Here we considered that D is bounded and $\sup_{x \in D} \|T^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$ by hypothesis. So by (1.0.28) $\widetilde{s} < \infty$ which is the first part of statement (1).

Let $D \subseteq X$ be the open set of which in the hypotheses. By $\widetilde{s} < \infty$ we can assume that $0 < \widetilde{s} < R$, then the second part of statement (1) follows by statement (1) of Lemma 1.0.9.

In the sequel of the proof we assume that $0 < \widetilde{s} < R$. By statement (2) of Lemma 1.0.9 and by the Chain Theorem, see $8.2.1.$ of the [**Dieu**], $g \circ T$ is Fréchet differentiable and its differential map is

(1.0.29)

$$[g \circ T]^{[1]} : D \ni x \mapsto g^{[1]}(T(x)) \circ T^{[1]}(x) = \left\{ \sum_{n=0}^{\infty} \alpha_n u_n^{[1]}(T(x)) \right\} \circ T^{[1]}(x) \in B(X, \mathcal{A}),$$

where in the second equality we considered that uniform convergence implies puntual convergence. By statement (2) of Lemma 1.0.9 and $\widetilde{s} < R$ the previous series converges in $B(\mathcal{A})$, moreover if we set

$$\langle \cdot, \cdot \rangle : B(\mathcal{A}) \times B(X, \mathcal{A}) \ni (\phi, \psi) \mapsto \phi \circ \psi \in B(X, \mathcal{A})$$

then it is bilinear and continuous i.e.

$$\langle \cdot, \cdot \rangle \in B_2(B(\mathcal{A}) \times B(X, \mathcal{A}); B(X, \mathcal{A})), \tag{1.0.30}$$

since $\|\phi \circ \psi\|_{B(X,\mathcal{A})} \leq \|\phi\|_{B(\mathcal{A})} \cdot \|\psi\|_{B(X,\mathcal{A})}$. Therefore by (1.0.29) follows (1.0.27).

Set
$$\Gamma : D \ni x \mapsto (g^{[1]} \circ T(x), T^{[1]}(x)) \in B(\mathcal{A}) \times B(X, \mathcal{A}).$$
By (1.0.29)
$$[g \circ T]^{[1]} = \langle \cdot, \cdot \rangle \circ \Gamma. \tag{1.0.31}$$
$T^{[1]}$ is continuous by hypothesis, and $g^{[1]}$ is continuous by statement (2) of Lemma 1.0.9, while T is continuous being differentiable by hypothesis, so $g^{[1]} \circ T$ is continuous. Therefore by Proposition 1, §4.1., Ch 1, of the [**GT**] the map Γ is continuous. Thus by (1.0.31) and (1.0.30) $[g \circ T]^{[1]}$ is continuous and statement (2a) follows.

By (1.0.9)
$$\begin{aligned}
&\sum_{n=0}^{\infty} \sup_{x \in D} \| \alpha_n u_n^{[1]}(T(x)) \circ T^{[1]}(x) \|_{B(X,\mathcal{A})} \\
&\leq \sum_{n=0}^{\infty} \sup_{x \in D} |\alpha_n| \| u_n^{[1]}(T(x)) \|_{B(\mathcal{A})} \cdot \| T^{[1]}(x) \|_{B(X,\mathcal{A})} \\
&\leq \sum_{n=0}^{\infty} \sup_{x \in D} |\alpha_n| n \| T(x) \|_{\mathcal{A}}^{n-1} \cdot \| T^{[1]}(x) \|_{B(X,\mathcal{A})} \\
&\leq \sum_{n=0}^{\infty} \sup_{x \in D} |\alpha_n| n \| T(x) \|_{\mathcal{A}}^{n-1} \cdot \sup_{x \in D} \| T^{[1]}(x) \|_{B(X,\mathcal{A})} \\
&\leq J \sum_{n=0}^{\infty} |\alpha_n| n \widetilde{s}^{n-1} < \infty,
\end{aligned}$$
where $J \doteqdot \sup_{x \in D} \| T^{[1]}(x) \|_{B(X,\mathcal{A})}$ and statement (2b) follows. $\square$

REMARK 1.0.17. *By* (1.0.29)*, statement* (3) *of Theorem 1.0.11 and* (1.0.30)*, if* $0 < \widetilde{s} < \frac{R}{3}$*, we have* $\forall x \in D$
$$[g \circ T]^{[1]}(x) = \sum_{p=1}^{\infty} \frac{1}{p!} g^{(p)}(\mathcal{R}(T(x))) C(T(x))^{p-1} T^{[1]}(x). \tag{1.0.32}$$
In addition if $\sup_{x \in D} \| T^{[1]}(x) \|_{B(X,\mathcal{A})} < \infty$*, then the series in the* (1.0.32) *is absolutely uniformly convergent on* D*. If* $0 < \widetilde{s} < \frac{R}{3}$ *by* (1.0.32) *we have* $\forall h \in X$
$$[g \circ T]^{[1]}(x)(h) = \sum_{p=1}^{\infty} \frac{1}{p!} g^{(p)}(T(x)) C(T(x))^{p-1} (T^{[1]}(x)(h)). \tag{1.0.33}$$

In addition if $\sup_{x\in D}\|T^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$*, then the series in the* (1.0.33) *is absolutely uniformly convergent for* $(x,h) \in D \times B_L(\mathbf{0})$*, for all* $L > 0$*.*

REMARK 1.0.18. *In a similar way of the proof of statement 2b of Corollary 1.0.16 we have*

(1) *if* $0 < \widetilde{s} < R$

(1.0.34)
$$\begin{aligned}
&[g\circ T]^{[1]}(x)(h) = \sum_{n=1}^{\infty} n\alpha_n T^{[1]}(x)(h)T(x)^{n-1} + \\
&\sum_{n=2}^{\infty}\sum_{p=0}^{n-2}(n-p-1)\alpha_n T(x)^p[T(x),T^{[1]}(x)(h)]T(x)^{n-(2+p)} \\
&= \sum_{n=1}^{\infty} n\alpha_n T^{[1]}(x)(h)T(x)^{n-1} + \\
&\sum_{p=0}^{\infty}\sum_{n=p+2}^{\infty}(n-p-1)\alpha_n T(x)^p[T(x),T^{[1]}(x)(h)]T(x)^{n-(2+p)}.
\end{aligned}$$

If in addition $\sup_{x\in D}\|T^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$ *then all the series in* (1.0.34) *are absolutely uniformly convergent for* $(x,h) \in D \times B_L(\mathbf{0})$*, for all* $L > 0$*.*

(2) *If* $0 < \widetilde{s} < R$ *we have*

(1.0.35)
$$\begin{aligned}
&[g\circ T]^{[1]}(x) \\
&= \sum_{n=1}^{\infty} n\alpha_n \mathcal{L}(T(x))^{n-1}T^{[1]}(x) - \sum_{k=2}^{\infty}\left\{\sum_{n=k}^{\infty}\alpha_n\mathcal{L}(T(x))^{n-k}\right\}C(T(x)^{k-1})T^{[1]}(x).
\end{aligned}$$

If in addition $\sup_{x\in D}\|T^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$ *then all the series in* (1.0.35) *are absolutely uniformly convergent on* D*.*

DEFINITION 1.0.19. *Let* $\langle G, \|\cdot\|_G\rangle$ *be a* $\mathbb{C}-$*Banach space, then we denote by* $G_{\mathbb{R}}$ *the vector space* G *over* $\mathbb{R}$ *whose operation of summation is the same of that of the* $\mathbb{C}-$*vector space* G*, and whose multiplication by scalars is the restriction to* $\mathbb{R}\times G$ *of the multiplication by scalars on* $\mathbb{C}\times G$*, finally we set* $\|\cdot\|_{G_{\mathbb{R}}} \doteqdot \|\cdot\|_G$*. Then* $\langle G_{\mathbb{R}}, \|\cdot\|_{G_{\mathbb{R}}}\rangle$ *is a Banach space over* $\mathbb{R}$ *and will be called the* $\mathbb{R}-$*Banach space associated to* $\langle G, \|\cdot\|_G\rangle$*.*

Let F, G *be two* $\mathbb{C}-$*Banach spaces then of course* $B(F,G) \subset B(F_{\mathbb{R}}, G_{\mathbb{R}})$*, where the inclusion is to be intended only as a set inclusion. Let* $A \subseteq F$ *then if* A *is open in* F *it is open also in* $F_{\mathbb{R}}$*. For a mapping* $f : A \subseteq F \to G$*, we will denote with the symbol* $f^{\mathbb{R}} : A \subseteq F_{\mathbb{R}} \to G_{\mathbb{R}}$ *the same mapping but considered defined in the subset* A *of the*

$\mathbb{R}-$Banach space associated to F and at values in the $\mathbb{R}-$Banach space associated to G.

REMARK 1.0.20. *Let Y, Z be two $\mathbb{C}-$Banach spaces, then by considering that $B(Y,Z) \subset B(Y_{\mathbb{R}}, Z_{\mathbb{R}})$, we have that for each Fréchet differential function $f : A \subseteq Y \to Z$ the same function $f^{\mathbb{R}} : A \subseteq Y_{\mathbb{R}} \to Z_{\mathbb{R}}$ considered in the corresponding real Banach spaces, is differentiable, in addition $f^{[1]} = (f^{\mathbb{R}})^{[1]}$. Therefore if we get a real Banach space X, we shall obtain the same statements of Corollary 1.0.16, Remark 1.0.17 and Remark 1.0.18 by replacing $\mathcal{A}$ with $\mathcal{A}_{\mathbb{R}}$. In particular by taking $X \doteqdot \mathbb{R}$ we obtain by* (1.0.33) *that if $0 < \widetilde{s} < \frac{R}{3}$, and by denoting $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$ we have $\forall t \in D$*

$$\boxed{\frac{d\, g^{\mathbb{R}} \circ T}{dt}(t) = \sum_{p=1}^{\infty} \frac{1}{p!} g^{(p)}(T(t)) C(T(t))^{p-1} \left(\frac{dT}{dt}(t) \right).} \tag{1.0.36}$$

In addition if $\sup_{t \in D} \|\frac{dT}{dt}(t)\|_{\mathcal{A}} < \infty$, then the series in the (1.0.36) *is absolutely uniformly convergent on D. This formula has been shown for the first time by Victor I. Burenkov in* [**Bur**].

Notice that $C(T(t))^0 = \mathbf{1}$ and $\forall n \in \mathbb{N} - \{0\}$

$$C(T(t))^n \left(\frac{dT}{dt}(t) \right) = \left[\cdots \left[\left[\frac{dT}{dt}(t), T(t) \right], T(t) \right], \cdots \right].$$

In particular if $\left[\frac{dT}{dt}(t), T(t)\right] = \mathbf{0}$, then

$$\frac{d\, g^{\mathbb{R}} \circ T}{dt}(t) = g^{(1)}(T(t)) \frac{dT}{dt}(t). \tag{1.0.37}$$

If $\left[\left[\frac{dT}{dt}(t), T(t)\right], T(t)\right] = \mathbf{0}$ then

$$\frac{d\, g^{\mathbb{R}} \circ T}{dt}(t) = g^{(1)}(T(t)) \frac{dT}{dt}(t) + \frac{1}{2} g^{(2)}(T(t)) \left[\frac{dT}{dt}(t), T(t) \right]$$

and so on.

COROLLARY 1.0.21. *Let $\mathcal{A}$ be a unitary Banach algebra, $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{K}$ be such that the radius of convergence of the series $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$ is $R > 0$. Finally let $W \in \mathcal{A} - \{\mathbf{0}\}$, $0 < s < R$, $D_{(s,W)} \doteqdot \left] -\frac{s}{\|W\|}, \frac{s}{\|W\|} \right[$ and*

$$T(t) = tW$$

$\forall t \in D_{(s,W)}$. Then with the notations adopted in the statements of Lemma 1.0.9, we have

(1)
$$g^{\mathbb{R}} \circ T(t) = \sum_{n=0}^{\infty} \alpha_n t^n W^n$$
and the series is absolutely uniformly convergent for $t \in D_{(s,W)}$.

(2) $g^{\mathbb{R}} \circ T$ *is derivable, the following map*
$$\frac{d\, g^{\mathbb{R}} \circ T}{dt}(t) = W \sum_{n=1}^{\infty} \alpha_n n t^{n-1} W^{n-1} = W \frac{d\, g}{d\lambda} \circ T(t) \tag{1.0.38}$$
$\forall t \in D_{(s,W)}$ *is the derivative function of* $g^{\mathbb{R}} \circ T$*, is continuous and the series in the* (1.0.38) *is absolutely uniformly convergent.*

PROOF. Statement (1) is trivial. The map T is derivable with constant derivative equal to $W \in \mathcal{A}$, hence we have statement (2) by Remark 1.0.20 and (1.0.34). $\square$

1.0.2. Application to the analytic functional calculus in a $\mathbb{C}-$Banach space. In this section G is a complex Banach space, we denote by $\sigma(T)$ the spectrum of T for all $T \in B(G)$ and $(\forall U \in Open(\mathbb{C}) \mid \sigma(T) \subset U)(\forall g : U \to \mathbb{C}$ analytic$)$ by $g(T)$ the operator belonging to $B(G)$ as defined in the analytic functional calculus framework given in Definition 7.3.9. of the [**DS**], that is
$$g(T) \doteqdot \frac{1}{2\pi i} \int_B g(\lambda) R(\lambda; T) d\,\lambda.$$
Here $R(\lambda; T) \doteqdot (\lambda \mathbf{1} - T)^{-1}$ is the resolvent of T, while $B \subset U$ is the boundary of an open set containing $\sigma(T)$ and consisting of a finite number of rectifiable Jordan curves. If U is an open neighborhood of 0 and $(\forall \lambda \in U)(g(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n)$ then by Theorem 7.3.10. of the [**DS**] $g(T) = \sum_{n=0}^{\infty} \alpha_n T^n$ converging in $B(G)$. Therefore for this case we can apply all the results in Section 1.0.1.

COROLLARY 1.0.22 (Fréchet differential of an operator valued analytic function defined on an open set of a $\mathbb{R}-$Banach Space). *Let* U_0 *be an open neighborhood of* $0 \in \mathbb{C}$, $g : U_0 \to \mathbb{C}$ *an analytic function such that* $(\forall \lambda \in U_0)(g(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n)$. *Let* $R > 0$ *be the radius of convergence of the series* $\sum_{n=0}^{\infty} \alpha_n \lambda^n$. *Finally let* X *be a Banach space over* $\mathbb{R}$, $D \subseteq X$ *an open set of* X *and* $T : D \to B(G)_{\mathbb{R}}$ *a Fréchet differentiable mapping such that* $(\exists\, s \in \mathbb{R}^+ \mid 0 < s < R)$

(1) $T(D) \subseteq B_s(\mathbf{0})$

(2) $(\forall x \in D)(\sigma(T(x)) \subseteq U_0)$

Then

(1)
$$g^{\mathbb{R}} \circ T = \sum_{n=0}^{\infty} \alpha_n T^n.$$
Here the series absolutely uniformly converges on D.

(2) *The statements of Corollary 1.0.16, Remark 1.0.17 and Remark 1.0.18 hold with $\mathcal{A}$ replaced by $B(G)_{\mathbb{R}}$, while Remark 1.0.20 holds with $\mathcal{A}$ replaced by $B(G)$.*

PROOF. The map $g^{\mathbb{R}} \circ T$ is well defined by the condition $(\forall x \in D)(\sigma(T(x)) \subseteq U_0)$, while the power series expansion follows by Theorem 7.3.10. of the [**DS**]. Therefore statement 1 follows by the hypothesis $T(D) \subseteq B_s(\mathbf{0})$, with $0 < s < R$ and Remark 1.0.20. Statement 2 is by Corollary 1.0.16 and Remark 1.0.20. $\square$

REMARK 1.0.23. *If we assume that G is a complex Hilbert space and $T(x)$ is a normal operator $\forall x \in D$, then the condition $T(D) \subseteq B_s(\mathbf{0})$ is equivalent to the following one $(\forall x \in D)(\sigma(T(x)) \subseteq B_s(\mathbf{0}))$.*

Although the following is a well-known result, for the sake of completeness we shall give a proof by using Corollary 1.0.21.

COROLLARY 1.0.24. *Let $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ be such that the radius of convergence of the series $g(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ is $R > 0$. Moreover let $W \in B(G) - \{\mathbf{0}\}$, $0 < s < R$, and $D_{(s,W)} \doteqdot \left]-\frac{s}{\|W\|}, \frac{s}{\|W\|}\right[$. Then the operator $g(tW)$ is well defined $\forall t \in D_{(s,W)}$ and*
$$g(tW) = \sum_{n=0}^{\infty} \alpha_n t^n W^n.$$
Here the series converges absolutely uniformly on $D_{(s,W)}$. Moreover the map $D_{(s,W)} \ni t \mapsto \frac{dg}{d\lambda}(tW)$ is Lebesgue integrable in $B(G)$ in the sense defined in [**INT**]*, Definition 2 Ch. IV, §3, $n^\circ 4$, and $\forall u_1, u_2 \in D_{(s,W)}$*
$$W \oint_{u_1}^{u_2} \frac{dg}{d\lambda}(tW)\, dt = g(u_2 W) - g(u_1 W).$$
Here $\oint_{u_1}^{u_2} \frac{dg}{d\lambda}(tW)\, dt$ is the Lebesgue integral of the map $D_{(s,W)} \ni t \mapsto \frac{dg}{d\lambda}(tW)$ as defined in Definition 1 Ch. IV, §4, $n^\circ 1$ of [**INT**].

PROOF. By (1.0.3)

(1.0.39) $$\mathcal{R}(W) \in B(B(G)).$$

Set $T(t) \doteqdot tW$ for all $t \in D_{(s,W)}$. Then $\frac{d\,g}{d\,\lambda} \circ T(t) = \sum_{n=1}^{\infty} \alpha_n n t^{n-1} W^{n-1}$ and the map $D_{(s,W)} \ni t \mapsto \frac{d\,g}{d\lambda} \circ T(t)$ is continuous in $B(G)$, as a corollary of (1.0.38), by replacing the map g with $\frac{dg}{d\lambda}$, hence it is Lebesgue-measurable in $B(G)$. Finally let $u_1, u_2 \in D_{(s,W)}$

$$\begin{aligned}
\int_{[u_1,u_2]}^{*} \|\frac{d\,g}{d\,\lambda}(tW)\| d\,t &= \int_{[u_1,u_2]}^{*} \|\sum_{n=1}^{\infty} \alpha_n n t^{n-1} W^{n-1}\| d\,t \\
&\leq \int_{[u_1,u_2]}^{*} \sum_{n=1}^{\infty} |\alpha_n| n s^{n-1} d\,t \\
&= |u_2 - u_1| \sum_{n=1}^{\infty} |\alpha_n| n s^{n-1} < \infty.
\end{aligned}$$

Here $\int_{[u_1,u_2]}^{*}$ is the upper integral of the Lebesgue measure on $[u_1, u_2]$. By this boundedness and by its Lebesgue- measurability we conclude by Theorem 5, $IV.71$ of [**INT**] that $[u_1, u_2] \ni t \mapsto \frac{d\,g}{d\lambda} \circ T(t) \in B(G)$ is Lebesgue-integrable in $B(G)$, so in particular by Definition 1, $IV.33$ of [**INT**]

$$\exists \oint_{u_1}^{u_2} \frac{d\,g}{d\,\lambda} \circ T(t) d\,t \in B(G). \tag{1.0.40}$$

Therefore by (1.0.39), (1.0.40), Theorem 1, $IV.35$ of [**INT**] and (1.0.38)

$$W \oint_{u_1}^{u_2} \frac{d\,g}{d\,\lambda} \circ T(t) d\,t = \oint_{u_1}^{u_2} W \frac{d\,g}{d\,\lambda} \circ T(t) d\,t = \oint_{u_1}^{u_2} \frac{d\,g^{\mathbb{R}}(T(t))}{dt} d\,t. \tag{1.0.41}$$

Furthermore by the continuity of the map $D_{(s,W)} \ni t \mapsto \frac{d\,g}{d\lambda} \circ T(t)$ in $B(G)$ and by (1.0.38), $D_{(s,W)} \ni t \mapsto \frac{d\,g^{\mathbb{R}}(T(t))}{dt}$ is continuous in $B(G)$ and it is the derivative of the map $D_{(s,W)} \ni t \mapsto g^{\mathbb{R}} \circ T$. Therefore it is Lebesgue integrable in $B(G)$, where the integral has to be understood as defined in Ch II of [**FVR**], see Proposition 3, $n°3$, §1, Ch II of [**FVR**]. Finally the Lebesgue integral for functions with values in a Banach space as defined in Ch II of [**FVR**], turns to be the integral with respect to the Lebesgue measure as defined in Ch. IV, §4, $n°1$ of [**INT**] (see Ch III, §3, $n°3$ and example in Ch IV, §4, $n°4$ of [**INT**]). Thus the statement follows by (1.0.41). □

One of the main aims of the sequel is proving this formula for a certain class of unbounded operators in a Banach space and by considering the integral in weaker topologies than that induced by the norm in $B(G)$.

CHAPTER 2

Extension theorem. The case of the strong operator topology

2.1. Key lemma

PRELIMINARIES 2.1.1. ***Integrals of bounded Borelian functions with respect to a vector valued measure.*** In the sequel $G \doteqdot \langle G, \|\cdot\|_G\rangle$ will be a complex Banach space. Denote by $\mathrm{Pr}(G)$ the class of all projectors on G, that is the class of $P \in B(G)$ such that $P^2 = P$. Consider a Boolean algebra $\mathcal{B}_X$, see Sec. 1.12 of [**DS**], of subsets of a set X, with respect to the order relation defined by $\sigma \geq \delta \Leftrightarrow \sigma \supseteq \delta$ and complemented by the operation $\sigma' \doteqdot \complement\sigma$. In particular $\mathcal{B}_X$ contains $\emptyset$ and X and is closed under finite intersection and finite union.

The map $E : \mathcal{B}_X \to B(G)$ is called a spectral measure in G on $\mathcal{B}_X$, or simply on X if X is a topological space and $\mathcal{B}_X$ is the Boolean algebra of its Borelian subsets, if

(1) $E(\mathcal{B}_X) \subseteq \mathrm{Pr}(G)$;
(2) $(\forall\sigma_1, \sigma_2 \in \mathcal{B}_X)(E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2))$;
(3) $(\forall\sigma_1, \sigma_2 \in \mathcal{B}_X)(E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2) - E(\sigma_1)E(\sigma_2))$;
(4) $E(X) = \mathbf{1}$;
(5) $E(\emptyset) = \mathbf{0}$.

(See Definition $15.2.1$ of [**DS**]).

If condition (3) is replaced by condition

$$(3')(\forall\sigma_1, \sigma_2 \in \mathcal{B}_X \mid \sigma_1 \cap \sigma_2 = \emptyset)(E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2)),$$

we obtain an equivalent definition.

Notice that if E is a spectral measure in G on $\mathcal{B}_X$, then it is a Boolean homomorphism onto the Boolean algebra $E(\mathcal{B}_X)$ with respect to the order relation induced by that defined in $\mathrm{Pr}(G)$ by $P \geq Z \Leftrightarrow Z = ZP$ and complemented by the operation $P' \doteqdot (\mathbf{1} - P)$. Indeed for all $\sigma, \delta \in \mathcal{B}_X$ we have $\delta \subseteq \sigma \Rightarrow E(\delta) = E(\delta \cap \sigma) \doteq E(\delta)E(\sigma) \Leftrightarrow E(\delta) \leq E(\sigma)$, while $\mathbf{1} = E(\sigma \cup \complement\sigma) = E(\sigma) + E(\complement\sigma)$.

A spectral measure E is called *(weakly) countable additive* if for all sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_X$ of disjoint sets, $\forall x \in G$ and $\forall \phi \in G^*$ we have

$$\phi\left(E(\bigcup_{n\in\mathbb{N}} \varepsilon_n)x\right) = \sum_{n=1}^{\infty} \phi\left(E(\varepsilon_n)x\right).$$

If $\mathcal{B}_X$ is a σ-field, i.e. a Boolean algebra closed under the operation of forming countable unions, we have by Corollary $15.2.4.$ of the [**DS**] that E is countably additive with respect to the strong operator topology, i.e. for all sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{C})$ of disjoint sets and for all $x \in G$ we have [1]

$$E(\bigcup_{n\in\mathbb{N}} \varepsilon_n)x = \sum_{n=1}^{\infty} E(\varepsilon_n)x = \sum_{n\in\mathbb{N}} E(\varepsilon_n)x. \tag{2.1.1}$$

Since $E(\bigcup_{n\in\mathbb{N}} \varepsilon_n) = E(\bigcup_{n\in\mathbb{N}} \varepsilon_{\rho(n)})$, for any permutation ρ of $\mathbb{N}$, hence $\sum_{n=1}^{\infty} E(\varepsilon_n)x = \sum_{n=1}^{\infty} E(\varepsilon_{\rho(n)})x$ for all $x \in G$, therefore by Proposition 9,§$5.7.$, Ch. 3 of [**GT**] we obtain the second equality in (2.1.1).

By $\mathcal{B}(\mathbb{C})$ we denote the set of the Borelian subsets of $\mathbb{C}$, and by $Bor(U)$ the complex linear space of all Borelian complex maps defined on a Borelian subset U of $\mathbb{C}$.

We denote with $\mathbf{TM}$ the space of the totally $\mathcal{B}(\mathbb{C})-$ measurable maps [2], which is the closure in the Banach space $\langle \mathbf{B}(\mathbb{C}), \|\cdot\|_{\sup}\rangle$ of all complex bounded functions on $\mathbb{C}$ with respect to the norm $\|g\|_{\sup} \doteqdot \sup_{\lambda\in\mathbb{C}} |g(\lambda)|$, of the linear space generated by the set $\{\chi_\sigma \mid \sigma \in \mathcal{B}(\mathbb{C})\}$, where χ_σ is the characteristic function of the set σ.

$\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$ is a Banach space, and the space of all bounded Borelian complex functions is in $\mathbf{TM}$ so dense in it. Finally $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$ is a C^*-subalgebra, in particular a Banach subalgebra, of $\langle \mathbf{B}(\mathbb{C}), \|\cdot\|_{\sup}\rangle$ if we define the pointwise operations of product and involution on $\mathbf{B}(\mathbb{C})$.

Let X be a complex Banach space and $F : \mathcal{B}(\mathbb{C}) \to X$ a weakly countably finite additive vector valued measure, see Section $4.10.$ of [**DS**], then we can define the integral with respect to F, see Section 10.1 of [**DS**], which will be denoted by $\int_{\mathbb{C}} f\, dF$. The operator

$$\mathbf{I}_{\mathbb{C}}^F : \mathbf{TM} \ni f \mapsto \int_{\mathbb{C}} f\, dF \in X \tag{2.1.2}$$

[1] By definition, see Ch.3 of [**GT**], $v = \sum_{n\in\mathbb{N}} E(\varepsilon_n)x$ if $v = \lim_{J\in\mathcal{P}_\omega(\mathbb{N})} \sum_{n\in J} E(\varepsilon_n)x$, where $\mathcal{P}_\omega(\mathbb{N})$ is the direct ordered set of all finite subsets of $\mathbb{N}$ ordered by inclusion.

[2] In [**DS**] denoted by $B(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, while by using the notations of [**Din2**] and considering $\mathbb{C}$ as a real Banach space we have $\mathbf{TM} = \mathbf{TM}_{\mathbb{R}}(\mathcal{B}(\mathbb{C}))$.

is linear and norm-continuous [3]. We have the following useful property if Y is a $\mathbb{C}-$Banach space and $Q \in B(X,Y)$, then

$$Q \circ \mathbf{I}_{\mathbb{C}}^{F} = \mathbf{I}_{\mathbb{C}}^{Q \circ F}, \tag{2.1.3}$$

see statement (f) of Theorem $4.10.8.$ of the [**DS**].

If $X \doteqdot B(G)$, the case we are mostly interested in, we have, as an immediate result of this property and the fact that the map $Q_x : B(G) \ni A \mapsto Ax \in G$ is linear and continuous $\forall x \in G$, that

$$(\forall x \in G)(\forall f \in \mathbf{TM})(\mathbf{I}_{\mathbb{C}}^{F}(f)x = \mathbf{I}_{\mathbb{C}}^{F^x}(f)). \tag{2.1.4}$$

Here $F^x : \mathcal{B}(\mathbb{C}) \ni \sigma \mapsto F(\sigma)x$. Finally if E is a spectral measure on $\mathbb{C}$, then $\mathbf{I}_{\mathbb{C}}^{E}$ is a continuous unital homomorphism between the two Banach algebras $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$, and $\left\langle B(G), \|\cdot\|_{B(G)}\right\rangle$ and $\mathbf{I}_{\mathbb{C}}^{E}(\chi_{supp\, E}) = \mathbf{1}$, see (2.1.6) and Section (2), Ch 15 of [**DS**].

Borel functional calculus for possibly unbounded scalar type spectral operators in G. If $T : \mathcal{D}(T) \subseteq G \to G$ is a possibly unbounded linear operator then we denote by $\sigma(T)$ its standard spectrum. A possibly unbounded linear operator $T : \mathcal{D}(T) \subseteq G \to G$ is called a **spectral operator in** G if it is closed and there exists a countably additive spectral measure $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ such that

i: for all bounded sets $\delta \in \mathcal{B}(\mathbb{C})$

$$E(\delta)G \subseteq \mathcal{D}(T);$$

ii: $(\forall \delta \in \mathcal{B}(\mathbb{C}))(\forall x \in \mathcal{D}(T))$ we have
(1) $(E(\delta)\mathcal{D}(T) \subseteq \mathcal{D}(T))$,
(2) $TE(\delta)x = E(\delta)Tx$;

iii: for all $\delta \in \mathcal{B}(\mathbb{C})$ we have

$$\sigma\left(T \restriction (\mathcal{D}(T) \cap E(\delta)G)\right) \subseteq \overline{\delta}.$$

Here $\sigma\left(T \restriction (\mathcal{D}(T) \cap E(\delta)G)\right)$ is the spectrum of the restriction of T to the domain $\mathcal{D}(T) \cap E(\delta)G$.

(See Definition $18.2.1.$ of the [**DS**]). We call any E with the above properties a **resolution of the identity of** T. Theorem $18.2.5.$ of [**DS**] states that the resolution of the identity of a spectral operator is unique.

[3]Notice that if we identify $B(G)$ with $B(\mathbb{R}, B(G))$ and recall that $\mathbf{TM} = \mathbf{TM}_{\mathbb{R}}(\mathcal{B}(\mathbb{C}))$, then with the notations of Definition 24, §1, Ch. 1 of [**Din2**] we have that $\mathbf{I}_{\mathbb{C}}^{E}$ is the immediate integral with respect to the vector valued measure $E : \mathcal{B}(\mathbb{C}) \to B(\mathbb{R}, B(G))$.

Finally we call support of a spectral measure E on $\mathcal{B}_X$, the following set

$$supp\, E \doteqdot \bigcap_{\{\sigma\in\mathcal{B}_X \mid E(\sigma)=\mathbf{1}\}} \overline{\sigma}.$$

It is easy to see [4] that

$$E(supp\, E) = \mathbf{1}. \tag{2.1.6}$$

Notice that an unbounded spectral operator T is closed by definition. Now we will show that T is also densely defined. In fact if E is the resolution of the identity of T and if $\{\sigma_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{C})$ is a non decreasing sequence of Borelian sets such that $\sigma(T) \subseteq \bigcup_{n\in\mathbb{N}} \sigma_n$, then by the strong countable additivity of E, the fact that $E(\sigma(T)) = \mathbf{1}$ we can deduce $\mathbf{1} = \lim_{n\in\mathbb{N}} E(\sigma_n)$ in the strong operator topology of $B(G)$, see (2.2.1).

Now we can choose $\{\sigma_n\}_{n\in\mathbb{N}}$ such that $\sigma_n \doteqdot B_n(\mathbf{0}) \doteqdot \{\lambda \in \mathbb{C} \mid |\lambda| < n\}$, or $\sigma_n \doteqdot W(\mathbf{0}, 2n) \doteqdot \{\lambda \in \mathbb{C} \mid |Re(\lambda)| < n, |Im(\lambda)| < n,\}$. But by the property (i) of the Definition $18.2.1.$ of [**DS**], we know that for all bounded sets $\sigma \in \mathcal{B}(\mathbb{C})$ we have $E(\sigma)G \subseteq Dom(T)$. Therefore we conclude that $(\forall v \in G)(v = \lim_{n\in\mathbb{N}} E(\sigma_n)v)$ and $(\forall n \in \mathbb{N})(E(\sigma_n)v \in Dom(T))$, so $Dom(T)$ is dense in G.

We want to remark that for each possibly unbounded spectral operator T in G by denoting with $\sigma(T)$ its spectrum and with $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ its resolution of the identity, we deduce by Lemma $18.2.25.$ of [**DS**] that $\sigma(T)$ is closed, that $supp\, E = \sigma(T)$ so by (2.1.6)

$$E(\sigma(T)) = \mathbf{1}.$$

Now we will give the definition of the Borel functional calculus for unbounded spectral operators in a complex Banach space G, that is essentially the same as in Definition $18.2.10.$ of the [**DS**].

DEFINITION 2.1.2. *Let X be a set, $S \subset X$, V a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $f : S \to V$. Then we define $\widetilde{f}^X$, or simply $\widetilde{f}$ when it doesn't cause confusion, to be the*

[4] Indeed let $S \doteqdot supp\, E$ then

$$\complement S = \bigcup_{\{\sigma\in\mathcal{B}_X \mid E(\sigma)=\mathbf{1}\}} \complement\overline{\sigma}. \tag{2.1.5}$$

Moreover E is order-preserving so for all $\sigma \in \mathcal{B}_X$ such that $E(\sigma) = \mathbf{1}$ we have $E(\complement\overline{\sigma}) \leq E(\complement\sigma) = \mathbf{1} - E(\sigma) = \mathbf{0}$. Hence by the definition of the order $E(\complement\overline{\sigma}) = E(\complement\overline{\sigma})\mathbf{0} = \mathbf{0}$. Therefore by the Principle of localization (Corollary, Ch 3, §2, $n°1$ of [**INT**]) which holds also for vector measures (footnote in Ch 6, §2, $n°1$ of [**INT**]) we deduce by (2.1.5) that $E(\complement S) = \mathbf{0}$. Finally

$$E(S) = \mathbf{1} - E(\complement S) = \mathbf{1}.$$

$\mathbf{0}-$*extension of* f *to* X*, i.e.* $\widetilde{f} : X \to V$ *such that* $\widetilde{f} \upharpoonright S = f$ *and* $\widetilde{f}(x) = \mathbf{0}$ *for all* $x \in (X - S)$*, where* $\mathbf{0}$ *is the zero vector of* V*.*

DEFINITION 2.1.3. *[Borel Functional Calculus of* E*] Assume that*

(1) $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ *is a countably additive spectral measure and* S *its support;*
(2) $f \in Bor(S)$*;*
(3) $\forall \sigma \subseteq \mathbb{C}$ *we set* $f_\sigma : \mathbb{C} \to \mathbb{C}$ *such that* $f_\sigma \doteqdot \widetilde{f} \cdot \chi_\sigma$*;*
(4) $\delta_n \doteqdot [-n, +n]$ *and*

$$f_n \doteqdot f_{\overset{-1}{|f|}(\delta_n)}.$$

Here $(\forall \sigma \subseteq \mathbb{C})(\forall g : \mathcal{D} \to \mathbb{C})(\overset{-1}{g}(\sigma) \doteqdot \{\lambda \in \mathcal{D} \mid g(\lambda) \in \sigma\})$.

Of course $f_n \in \mathbf{TM}$ *for all* $n \in \mathbb{N}$ *so we can define the following operator in* G

$$\begin{cases} Dom(f(E)) \doteqdot \{x \in G \mid \exists \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E}(f_n)x\} \\ (\forall x \in Dom(f(E)))(f(E)x \doteqdot \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E}(f_n)x). \end{cases} \tag{2.1.7}$$

Here all limits are considered in the space G*. We call the map* $f \mapsto f(E)$ *the* ***Borel functional calculus of the spectral measure*** E*.*

In the case in which E is the resolution of the identity of a possibly unbounded spectral operator T, recalling Lemma 18.2.25. of **[DS]** stating that $\sigma(T)$ is the support of E, we can define $f(T) \doteqdot f(E)$ for any map $f \in Bor(\sigma(T))$ and call the map

$$Bor(\sigma(T)) \ni f \mapsto f(T)$$

the **Borel functional calculus of the operator** T.

DEFINITION 2.1.4. *[*18.2.12. *of* **[DS]***] A* spectral operator of scalar type in G *or a* ***scalar type spectral operator in*** G *is a possibly unbounded linear operator* R *in* G *such that there exists a countably additive spectral measure* $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ *with support* S *and the property*

$$R = \imath(E).$$

Here $\imath : S \ni \lambda \mapsto \lambda \in \mathbb{C}$*, and* $\imath(E)$ *is relative to the Borel functional calculus of the spectral measure* E*. We call* E *a* ***resolution of the identity of*** R*.*

Let R be a scalar type spectral operator in G and E a resolution of the identity of R, then we have the following statements by **[DS]**:

- T is a spectral operator in G;

- E is the resolution of the identity of T as spectral operator;
- E is unique.

DEFINITION 2.1.5 ([**DS**]). *Let $E : \mathcal{B}(\mathbb{C}) \to Pr(G)$ be a countably additive spectral measure and $U \in \mathcal{B}(\mathbb{C})$, then the space of all $E-$*essentially bounded *maps is the following linear space*

$$\mathfrak{L}_E^\infty(U) \doteqdot \left\{f : \mathbb{C} \to \mathbb{C} \mid \|f\chi_U\|_\infty^E < \infty\right\}.$$

Here $\chi_U : \mathbb{C} \to \mathbb{C}$ is the characteristic map of U which is by definition equal to 1 in U and 0 in $\complement U$, and for each map $F : \mathbb{C} \to \mathbb{C}$

$$\|F\|_\infty^E \doteqdot E - ess \sup_{\lambda\in\mathbb{C}} |F(\lambda)| \doteqdot \inf_{\{\delta\in\mathcal{B}(\mathbb{C})|E(\delta)=\mathbf{1}\}} \sup_{\lambda\in\delta} |F(\lambda)|.$$

For a Borelian map $f : U \supset \sigma(R) \to \mathbb{C}$, with $U \in \mathcal{B}(\mathbb{C})$, we define $f(R)$ to be the operator $(f \upharpoonright \sigma(R))(R)$.

Let $g : U \subseteq \mathbb{C} \to \mathbb{C}$ be a Borelian map. Then g is $E-$essentially bounded if

$$E - ess \sup_{\lambda\in U} |g(\lambda)| \doteqdot \|\widetilde{g}\|_\infty^E < \infty.$$

See Definition 17.2.6. of [**DS**]. One formula arising by statement (i) of the Spectral Theorem 18.2.11. of the [**DS**], which will be used many times in the thesis is the following: for all Borelian complex function $f : \sigma(R) \to \mathbb{C}$ and for all $\phi \in G^*$ and $y \in Dom(f(R))$

$$\phi\left(f(R)y\right) = \int_{\mathbb{C}} \widetilde{f}\, d\, E_{(\phi,y)}. \tag{2.1.8}$$

Here G^* is the topological dual of G, that is the normed space of all $\mathbb{C}-$linear and continuous functionals on G with the $\sup-$norm, and $\forall\phi \in G^*, \forall y \in G$ we define $E_{(\phi,y)} : \mathcal{B}(\mathbb{C}) \ni \sigma \mapsto \phi(E(\sigma)y) \in \mathbb{C}$.

Finally if $P \in \Pr(G)$ then $\left\langle P(G), \|\cdot\|_{P(G)}\right\rangle$, with $\|\cdot\|_{P(G)} \doteqdot \|\cdot\|_G \upharpoonright P(G)$, is a Banach space. In fact let $\{v_n\}_{n\in\mathbb{N}} \subset G$ be such that $v = \lim_{n\in\mathbb{N}} Pv_n$, in $\|\cdot\|_G$, so $P = P^2$ being continuous we have that $Pv = \lim_{n\in\mathbb{N}} P^2 v_n = \lim_{n\in\mathbb{N}} Pv_n \doteqdot v$, so $v \in P(G)$, then $P(G)$ is closed in $\langle G, \|\cdot\|_G\rangle$, hence $\left\langle P(G), \|\cdot\|_{P(G)}\right\rangle$ is a Banach space.

If $E : \mathcal{B}_Y \to \Pr(G)$ is a spectral measure in G on $\mathcal{B}_Y$ and $\sigma \in \mathcal{B}_Y$, then we shall denote by G_σ^E or simply G_σ the complex Banach space $E(\sigma)G$, without expressing its dependence by E whenever it does not cause confusion. In addition for any Q possibly unbounded operator in G we define $\forall\sigma \in \mathcal{B}_Y$ the following possibly unbounded operator

operator in G

$$Q_\sigma \doteqdot QE(\sigma).$$

Finally we shall denote by $\mathcal{B}_b(\mathbb{C})$ the subclass of all bounded subsets of $\mathcal{B}(\mathbb{C})$.

DEFINITION 2.1.6. *Let F be a $\mathbb{C}-$Banach space, $P \in \Pr(F)$ and $S : Dom(S) \subseteq F \to F$, then we define*

$$SP \upharpoonright P(F) \doteqdot SP \upharpoonright (P(F) \cap Dom(SP)). \tag{2.1.9}$$

Notice that by the property $P^2 = P$ we have $P(F) \cap Dom(S) = P(F) \cap Dom(SP)$, and that

$$SP \upharpoonright P(F) = S \upharpoonright (P(F) \cap Dom(S)).$$

Moreover in the case in which $PS \subseteq SP$ then

$$SP \upharpoonright P(F) : P(F) \cap Dom(S) \to P(F).$$

That is $SP \upharpoonright P(F)$ is a linear operator in the Banach space $P(F)$. Let $E : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$, $\sigma \in \mathcal{B}_Y$ and Q a possibly unbounded operator in G such that $E(\sigma)Q \subseteq QE(\sigma)$, then

$$Q_\sigma \upharpoonright G_\sigma : G_\sigma \cap Dom(Q) \to G_\sigma.$$

In particular if R is a possibly unbounded scalar type spectral operator in G, E its resolution of the identity and $f \in Bor(\sigma(R))$, then by statement (g) of Theorem 18.2.11 *of* **[DS]**, *we have that for all $\sigma \in \mathcal{B}(\mathbb{C})$*

$$E(\sigma)f(R) \subseteq f(R)E(\sigma).$$

Hence for all $\sigma \in \mathcal{B}(\mathbb{C})$

$$\begin{cases} R_\sigma \upharpoonright G_\sigma = R_\sigma \upharpoonright (G_\sigma \cap Dom(R)) = R \upharpoonright (G_\sigma \cap Dom(R)) \\ f(R)_\sigma \upharpoonright G_\sigma = f(R)_\sigma \upharpoonright (G_\sigma \cap Dom(f(R))) = f(R) \upharpoonright (G_\sigma \cap Dom(f(R))) \end{cases} \tag{2.1.10}$$

are linear operators in G_σ.

Finally $E(\sigma(R)) = \mathbf{1}$ implies $E(\sigma) = E(\sigma \cap \sigma(R))$ for all $\sigma \in \mathcal{B}(\mathbb{C})$ so by (2.1.10)

$$\begin{cases} R_\sigma \upharpoonright G_\sigma = R_{\sigma\cap\sigma(R)} \upharpoonright G_{\sigma\cap\sigma(R)} \\ f(R)_\sigma \upharpoonright G_\sigma = f(R)_{\sigma\cap\sigma(R)} \upharpoonright G_{\sigma\cap\sigma(R)}. \end{cases} \tag{2.1.11}$$

LEMMA 2.1.7 (Key Lemma). *Let R be a possibly unbounded scalar type spectral operator in G, E its resolution of the identity, $\sigma(R)$ its spectrum and $f \in Bor(\sigma(R))$. Then $\forall \sigma \in \mathcal{B}(\mathbb{C})$*

(1) *$R_\sigma \upharpoonright G_\sigma$ is a scalar type spectral operator in G_σ whose resolution of the identity $\widetilde{E}_\sigma$ is such that for all $\delta \in \mathcal{B}(\mathbb{C})$*

$$\widetilde{E}_\sigma(\delta) = E(\delta) \upharpoonright G_\sigma \in B(G_\sigma),$$

(2)

$$\boxed{f(R)_\sigma \upharpoonright G_\sigma = f\left(R_\sigma \upharpoonright G_\sigma\right),}$$

(3) *$\forall g \in Bor(\sigma(R))$ such that $g(\sigma \cap \sigma(R))$ is bounded, we have that*

$$g(R)E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_\sigma) \in B(G).$$

PROOF. Let $\sigma \in \mathcal{B}(\mathbb{C})$. By the fact that $E(\sigma \cap \delta) = E(\delta)E(\sigma) = E(\sigma)E(\delta)$ for all $\delta \in \mathcal{B}(\mathbb{C})$ and $E(\sigma) \upharpoonright G_\sigma = \mathbf{1}_\sigma$ the unity operator on G_σ, we have for all $\delta \in \mathcal{B}(\mathbb{C})$

$$\widetilde{E}_\sigma(\delta) = E(\sigma \cap \delta) \upharpoonright G_\sigma \in B(G_\sigma). \tag{2.1.12}$$

In particular $\widetilde{E}_\sigma : \mathcal{B}(\mathbb{C}) \to B(G_\sigma)$, moreover E is a countably additive spectral measure in G, so

(2.1.13) $\widetilde{E}_\sigma$ is a countably additive spectral measure in G_σ.

By Lemma 18.2.2. of [**DS**] $\widetilde{E}_\sigma$ is the resolution of identity of the spectral operator $R_\sigma \upharpoonright G_\sigma$ so by Lemma 18.2.25. of [**DS**] applied to $R_\sigma \upharpoonright G_\sigma$

$$supp\, \widetilde{E}_\sigma = \sigma(R_\sigma \upharpoonright G_\sigma). \tag{2.1.14}$$

Furthermore by (2.1.11) and (iii) of Definition 18.2.1. of [**DS**] we have $\sigma(R_\sigma \upharpoonright G_\sigma) \subseteq \overline{\sigma \cap \sigma(R)}$, then by the equality $\overline{\overline{\sigma} \cap \sigma(R)} = \overline{\sigma} \cap \sigma(R)$, we deduce

$$\sigma(R_\sigma \upharpoonright G_\sigma) \subseteq \overline{\sigma} \cap \sigma(R) \subseteq \sigma(R). \tag{2.1.15}$$

Hence (2.1.14) and (2.1.15) imply that the operator function $f(\widetilde{E}_\sigma)$ is well defined.

For all $x \in Dom(f(R)_\sigma \restriction G_\sigma)$

$$\begin{aligned}
(f(R)_\sigma \restriction G_\sigma)x &= f(R)x && \text{by (2.1.10)}\\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E^x}(\widetilde{f}\cdot \chi_{\overset{-1}{|f|}(\delta_n)}) && \text{by (2.1.7), (2.1.4)}\\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{\widetilde{E}_\sigma^x}(\widetilde{f}\cdot \chi_{\overset{-1}{|f|}(\delta_n)}) && \text{by } x\in G_\sigma, \text{(2.1.13)}\\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{\widetilde{E}_\sigma}(\widetilde{f}\cdot \chi_{\overset{-1}{|f|}(\delta_n)})x && \text{by (2.1.4)}\\
&= f(\widetilde{E}_\sigma)x. && \text{by (2.1.7)}
\end{aligned}$$

So $f(R)_\sigma \restriction G_\sigma \subseteq f(\widetilde{E}_\sigma)$.
For all $x \in Dom(f(\widetilde{E}_\sigma))$

$$\begin{aligned}
f(\widetilde{E}_\sigma)x &= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{\widetilde{E}_\sigma^x}(\widetilde{f}\cdot \chi_{\overset{-1}{|f|}(\delta_n)}) && \text{by (2.1.7), (2.1.4)}\\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E^x}(\widetilde{f}\cdot \chi_{\overset{-1}{|f|}(\delta_n)})\\
&= \lim_{n\in\mathbb{N}} \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{f}\cdot \chi_{\overset{-1}{|f|}(\delta_n)})x && \text{by (2.1.4)}\\
&= (f(R)_\sigma \restriction G_\sigma)x. && \text{by (2.1.7), (2.1.10)}
\end{aligned}$$

So $f(\widetilde{E}_\sigma) \subseteq f(R)_\sigma \restriction G_\sigma$, then

$$f(R)_\sigma \restriction G_\sigma = f(\widetilde{E}_\sigma) \tag{2.1.16}$$

Therefore statement (1) follows by setting $f = \imath$, while statement (2) follows by statement (1) and (2.1.16).

Let $g \in Bor(\sigma(R))$ such that $g(\sigma\cap\sigma(R))$ is bounded, then

$$(\exists\, n\in\mathbb{N})(\forall m>n)(\sigma\cap \overset{-1}{|g|}(\delta_m) = \sigma\cap\sigma(R)).$$

Next $E(\sigma(R)) = \mathbf{1}$, so $E(\sigma) = E(\sigma)E(\sigma(R)) = E(\sigma(R) \cap \sigma)$. Since $\mathbf{I}_{\mathbb{C}}^{E}$ is an algebra homomorphism, $\forall m \in \mathbb{N}$

$$\begin{aligned}
\mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)})E(\sigma) &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)})E(\sigma \cap \sigma(R)) \\
&= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)})\mathbf{I}_{\mathbb{C}}^{E}(\chi_{\sigma\cap\sigma(R)}) \\
&= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)} \cdot \chi_{\sigma\cap\sigma(R)}) \\
&= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)\cap\sigma\cap\sigma(R)}) \\
&= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)\cap\sigma}).
\end{aligned}$$

This equality implies that

$$(\exists\, n \in \mathbb{N})(\forall m > n)(\mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)})E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{\sigma\cap\sigma(R)})). \tag{2.1.17}$$

Furthermore

$$\begin{aligned}
\mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{\sigma\cap\sigma(R)}) &= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\chi_{\sigma}\chi_{\sigma(R)}) \\
&= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\chi_{\sigma})\mathbf{I}_{\mathbb{C}}^{E}(\chi_{\sigma(R)}) \\
&= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\chi_{\sigma})E(\sigma(R)) \\
&= \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g}\chi_{\sigma}).
\end{aligned}$$

Therefore by (2.1.17)

$$(\exists\, n \in \mathbb{N})(\forall m > n)(\mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_m)})E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{\sigma})). \tag{2.1.18}$$

Moreover by definition in (2.1.7) we have $\forall x \in Dom(g(R))$ that

$$g(R)x \doteqdot \lim_{n\to\infty} \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{|g|^{-1}(\delta_n)})x$$

and $Dom(g(R))$ is the set of $x \in G$ such that such a limit exists; thus by (2.1.18) we can conclude that $E(\sigma)G \subseteq Dom(g(R))$ and $g(R)E(\sigma) = \mathbf{I}_{\mathbb{C}}^{E}(\widetilde{g} \cdot \chi_{\sigma}) \in B(G)$, which is statement (3). $\square$

COROLLARY 2.1.8. *Let R be a possibly unbounded scalar type spectral operator in G, and $f \in Bor(\sigma(R))$. Then $\forall \sigma \in \mathcal{B}(\mathbb{C})$*

$$f(R)E(\sigma) = f\left(R_{\sigma} \upharpoonright G_{\sigma}\right) E(\sigma).$$

Moreover if $f(\sigma \cap \sigma(R))$ is bounded then

$$f\left(R_\sigma \restriction G_\sigma\right) E(\sigma) \in B(G).$$

PROOF. Let $y \in Dom(f(R)E(\sigma))$ then $E(\sigma)y \in G_\sigma \cap Dom(f(R))$ hence by (2.1.10), Lemma 2.1.7

$$f(R)E(\sigma)y = (f(R)_\sigma \restriction G_\sigma)E(\sigma)y = f(R_\sigma \restriction G_\sigma)E(\sigma)y.$$

So $f(R)E(\sigma) \subseteq f(R_\sigma \restriction G_\sigma)E(\sigma)$.

Next let $y \in Dom(f(R_\sigma \restriction G_\sigma)E(\sigma))$, then $E(\sigma)y \in Dom(f(R_\sigma \restriction G_\sigma))$, hence by Lemma 2.1.7 and (2.1.10)

$$f(R_\sigma \restriction G_\sigma)(E(\sigma)y) = f(R)E(\sigma)E(\sigma)y = f(R)E(\sigma)y.$$

So $f(R_\sigma \restriction G_\sigma)E(\sigma) \subseteq f(R)E(\sigma)$. Thus we obtain statement (1). Statement (2) follows by statement (1) and statement (3) of Lemma 2.1.7. $\square$

2.2. Extension theorem for strong operator integral equalities

NOTATIONS 2.2.1. Let X be a locally compact space and μ a measure on X in the sense of the Bourbaki text **[INT]** see $III.7$, Definition 2, that is a continuous linear $\mathbb{C}-$functional on the $\mathbb{C}-$locally convex space $H(X)$ of all compactly supported complex continuous functions on X, with the direct limit topology (or inductive limit) of the spaces $H(X;K)$ with K running in the class of all compact subsets of X, where $H(X;K)$ is the space of all complex continuous functions $f : X \to \mathbb{C}$ such that $supp(f) \doteqdot \overline{\{x \in X \mid f(x) \neq 0\}} \subseteq K$ with the norm topology of uniform convergence [5]. In the thesis any measure μ on X in the sense of **[INT]** will be called complex Radon measure on X.

For the definition of *$\mu-$integrable functions defined on X and with values in a $\mathbb{C}-$Banach space G* see $IV.23$. Definition 2 of **[INT]**, while the *integral with respect to μ of a $\mu-$integrable function $f : X \to G$*, which will be denoted with $\int f(x)\, d\,\mu(x) \in G$, is defined in Definition 1, $III.33$ and Definition 1, $IV.33$ of **[INT]**.

For the definition of the total variation $|\mu|$, and definition and properties of the upper integral $\int^* g\, d|\mu|(x)$ see Ch. $3-4$ of **[INT]**. We denote by $Comp(X)$ the class of the compact subsets of X and by $\mathfrak{F}_1(X;\mu)$ the seminormed space, with seminorm $\|\cdot\|_{\mathfrak{F}_1(X;\mu)}$,

[5] $H(X;K)$ is isometric to the Banach space of all continuous maps $g : K \to \mathbb{C}$ equal to 0 on ∂K, with the norm topology of uniform convergence

of all maps $F : X \to \mathbb{C}$ such that

$$\|F\|_{\mathfrak{F}_1(X;\mu)} \doteqdot \int^* |F(x)|\, d\,|\mu|(x) < \infty.$$

In this section it will be assumed, unless the contrary is stated, that X is a locally compact space and μ is a complex Radon measure over X.

Let $B \subseteq X$ be a $\mu-$measurable set, then by $\mu-a.e.(B)$ we mean " almost everywhere in B with respect to the measure μ ".

Let $f : X \to B(G)$ be a map $\mu-$integrable in the normed space $B(G)$ (Definition 2 Ch. IV, §3, $n°4$ of [**INT**]) then we convene to denote with the symbol

$$\oint f(x)\, d\,\mu(x) \in B(G)$$

its integral in $B(G)$ (Definition 1 Ch. IV, §4, $n°1$ of [**INT**]), which is uniquely determined by the following property $\forall \phi \in B(G)^*$

$$\phi(\oint f(x)\, d\,\mu(x)) = \int \phi(f(x))\, d\,\mu(x).$$

For any scalar type spectral operator S in a complex Banach space G and for any Borelian map $f : U \supseteq \sigma(S) \to \mathbb{C}$ we assume that $f(S)$ is the closed operator defined in (2.1.7) and recall that we denote by $\widetilde{f}$ the $\mathbf{0}-$extension of f to $\mathbb{C}$, see Definition 2.1.2.

DEFINITION 2.2.2 ($E-$sequence). *Let $E : \mathcal{B}_Y \to Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$ then we say that $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$*sequence *if there exists an $S \in \mathcal{B}_Y$ such that $E(S) = \mathbf{1}$ and*

- $(\forall n \in \mathbb{N})(\sigma_n \in \mathcal{B}_Y)$;
- $(\forall n, m \in \mathbb{N})(n > m \Rightarrow \sigma_n \supseteq \sigma_m)$;
- $S \subseteq \bigcup_{n\in\mathbb{N}} \sigma_n$.

PROPOSITION 2.2.3. *Let $E : \mathcal{B}_Y \to Pr(G)$ be a countably additive spectral measure in G on a $\sigma-$field $\mathcal{B}_Y$, and $\{\sigma_n\}_{n\in\mathbb{N}}$ an $E-$sequence. Then*

$$\lim_{n\to\infty} E(\sigma_n) = \mathbf{1} \quad \textit{in strong operator topology.} \tag{2.2.1}$$

PROOF. Let $S \in \mathcal{B}_Y$ of which in Definition 2.2.2 associated to the $E-$sequence $\{\sigma_n\}_{n\in\mathbb{N}}$. So $E(S) = \mathbf{1}$ and E is an order-preserving map, then $E(\bigcup_{n\in\mathbb{N}} \sigma_n) \geq E(S) = \mathbf{1}$. Since $\mathbf{1}$ is a maximal element in $\langle E(\mathcal{B}_Y), \geq\rangle$

$$E(\bigcup_{n\in\mathbb{N}} \sigma_n) = \mathbf{1}.$$

Let us define $\eta_1 \doteqdot \sigma_1$, and $(\forall n \geq 2)(\eta_n \doteqdot \sigma_n \cap \complement \sigma_{n-1})$, so $(\forall n \in \mathbb{N})(\sigma_n = \bigcup_{k=1}^{n} \eta_k)$ and $(\forall n \neq m \in \mathbb{N})(\eta_n \cap \eta_m = \emptyset)$, finally $\bigcup_{n\in\mathbb{N}} \eta_n = \bigcup_{n\in\mathbb{N}} (\bigcup_{k=1}^{n} \eta_k) = \bigcup_{n\in\mathbb{N}} \sigma_n$. Therefore by the countable additivity of E with respect to the strong operator topology

$$
\begin{aligned}
E(\bigcup_{n\in\mathbb{N}} \sigma_n) = E(\bigcup_{n\in\mathbb{N}} \eta_n) &= \sum_{n=1}^{\infty} E(\eta_n) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} E(\eta_k) = \lim_{n\to\infty} E(\bigcup_{k=1}^{n} \eta_k) \\
&= \lim_{n\to\infty} E(\sigma_n).
\end{aligned}
$$

Here all limits are with respect to the strong operator topology, hence the statement. □

DEFINITION 2.2.4 (Integration in the Strong Operator Topology). *Let G_1, G_2 be two complex Banach spaces, and $f : X \to B(G_1, G_2)$. Then we say that* f is $\mu-$ integrable with respect to the strong operator topology *if*

(1) *$\forall v \in G_1$ the map $X \ni x \mapsto f(x)v \in G_2$ is $\mu-$integrable;*
(2) *if we set*

$$F : G_1 \ni v \mapsto \int f(x)(v)\, d\,\mu(x) \in G_2$$

then $F \in B(G_1, G_2)$.

In such a case we set $\int f(x)\, d\,\mu(x) \doteqdot F$, in other words the integral $\int f(x)\, d\,\mu(x)$ of f with respect to the measure μ and the strong operator topology is a bounded linear operator from G_1 to G_2 such that for all $v \in G_1$

$$\left(\int f(x)\, d\,\mu(x)\right)(v) = \int f(x)(v)\, d\,\mu(x).$$

We shall need the following version of the Minkowski inequality

PROPOSITION 2.2.5. *Let G_1, G_2 be two complex Banach spaces, and a map $f : X \to B(G_1, G_2)$ such that*

(1) *$(\forall v \in G_1)(\forall \phi \in G_2^*)$ the complex map $X \ni x \mapsto \phi(f(x)v) \in \mathbb{C}$ is $\mu-$measurable;*
(2) *$(\forall v \in G_1)(\forall K \in Comp(X))(\exists H \subset G_2)$ such that H is countable and $f(x)v \in \overline{H}$ $\mu - a.e.(K)$;*
(3) *$(X \ni x \mapsto \|f(x)\|_{B(G_1,G_2)}) \in \mathfrak{F}_1(X; \mu)$,*

Then f is $\mu-$integrable with respect to the strong operator topology and we have

$$\left\| \int f(x)\, d\,\mu(x) \right\|_{B(G_1,G_2)} \leq \int^* \|f(x)\|_{B(G_1,G_2)}\, d\,|\mu|(x).$$

PROOF. By hypothesis (3) we have $\forall v \in G_1$

$$\int^* \|f(x)v\|_{G_2}\, d\,|\mu|(x) \leq \|v\|_{G_1} \int^* \|f(x)\|_{B(G_1,G_2)}\, d\,|\mu|(x) < \infty. \tag{2.2.2}$$

By hypothesis $(1-2)$ and Corollary 1, $IV.70$ of [**INT**], we have $\forall v \in G_1$ that the map $X \mapsto f(x)v \in G_2$ is $\mu-$measurable. Therefore by (2.2.2) and by Theorem 5, $IV.71$ of [**INT**] we deduce $\forall v \in G_1$ that $X \mapsto f(x)v \in G_2$ is $\mu-$integrable. So in particular by Definition 1, $IV.33$ of [**INT**] $(\forall v \in G_1)(\exists \int f(x)v\, d\,\mu(x) \in G_2)$ while by Proposition 2, $IV.35$ of [**INT**] and the (2.2.2) we obtain $\forall v \in G_1$

$$\left\| \int f(x)v\, d\,\mu(x) \right\|_{G_2} \leq \|v\|_{G_1} \int^* \|f(x)\|_{B(G_1,G_2)}\, d\,|\mu|(x)$$

Hence the statement follows. □

REMARK 2.2.6. *As it follows by the above proof Proposition 2.2.5 is also valid if we replace the hypotheses $(1-2)$ with the following one*

(1') $\quad \forall v \in G_1 \quad$ *the map* $X \ni x \mapsto f(x)v \in G_2 \quad$ *is $\mu-$measurable.*

LEMMA 2.2.7. *Let X, Y, Z be three normed spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $R : Dom(R) \subseteq Y \to Z$ a possibly unbounded closed linear operator and $A \in B(X, Y)$. Then $R \circ A : \mathsf{D} \to Z$ is a closed operator, where $\mathsf{D} \doteqdot Dom(R \circ A)$*

PROOF. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{D} \doteqdot \{x \in X \mid A(x) \in Dom(R)\}$, and $(x, z) \in X \times Z$ such that $x = \lim_{n\to\infty} x_n$, and $z = \lim_{n\to\infty} R \circ A(x_n)$. A being continuous we have $A(x) = \lim_{n\to\infty} A(x_n)$, but $z = \lim_{n\to\infty} R(Ax_n)$, and R is closed, so $z = R(A(x)) \doteqdot R \circ A(x)$, hence $(x, z) \in Graph(R \circ A)$, which is just the statement. □

LEMMA 2.2.8. *Let X be a normed space and Y a Banach space over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, finally $U : \mathsf{D} \subseteq X \to Y$ be a linear operator. If U is continuous and closed then D is closed.*

PROOF. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathsf{D}$ and $x \in X$ such that $x = \lim_{n\to\infty} x_n$. So by the continuity of U we have $\forall n, m \in \mathbb{N}$ that $\|U(x_n) - U(x_m)\| = \|U(x_n - x_m)\| \leq \|U\| \|x_n - x_m\|$, hence $\lim_{(n,m)\in\mathbb{N}^2} \|U(x_n) - U(x_m)\| = 0$, thus Y being a Banach space we have that $(\exists\, y \in Y)(y = \lim_{n\to\infty} U(x_n))$. But U is closed, therefore $y = U(x)$, so $x \in \mathsf{D}$, which is the statement. □

THEOREM 2.2.9. *Let R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum and E its resolution of the identity. Let the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $(\forall x \in X)(\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R)))$ where $X \ni x \mapsto f_x(R) \in B(G)$ is $\mu-$integrable with respect to the strong operator topology.*

Then

(1) $\forall \sigma \in \mathcal{B}(\mathbb{C})$ *the map* $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)$ *is* $\mu-$*integrable with respect to the strong operator topology and*

$$\left\| \int f_x(R_\sigma \upharpoonright G_\sigma)\, d\,\mu(x) \right\|_{B(G_\sigma)} \leq \left\| \int f_x(R)\, d\,\mu(x) \right\|_{B(G)}.$$

(2) *If* $g, h \in Bor(\sigma(R))$, $\{\sigma_n\}_{n \in \mathbb{N}}$ *is an* $E-$*sequence, and* $\forall n \in \mathbb{N}$

$$g(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \int f_x(R_{\sigma_n} \upharpoonright G_{\sigma_n})\, d\,\mu(x) \subseteq h(R_{\sigma_n} \upharpoonright G_{\sigma_n}). \tag{2.2.3}$$

then

$$g(R) \int f_x(R)\, d\,\mu(x) \upharpoonright \Theta = h(R) \upharpoonright \Theta, \tag{2.2.4}$$

where $\Theta \doteqdot Dom\left(g(R) \int f_x(R)\, d\,\mu(x)\right) \cap Dom(h(R))$ *and all the integrals are with respect to the strong operator topologies.*

Notice that $g(R)$ is possibly an **unbounded** operator in G.

PROOF. Let $\sigma \in \mathcal{B}(\mathbb{C})$ then by (2.1.15)

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\sigma(R_\sigma \upharpoonright G_\sigma) \subseteq \overline{\sigma} \cap \sigma(R) \subseteq \sigma(R)).$$

which implies that all the following operator functions $g(R_\sigma \upharpoonright G_\sigma)$, $h(R_\sigma \upharpoonright G_\sigma)$ and $\forall x \in X$ the $f_x(R_\sigma \upharpoonright G_\sigma)$, are well defined.

By the fact that $\{\delta \in \mathcal{B}(\mathbb{C}) \mid E(\delta) = \mathbf{1}\} \subseteq \{\delta \in \mathcal{B}(\mathbb{C}) \mid \widetilde{E}_\sigma(\delta) = \mathbf{1}_\sigma\}$ which follows by statement (1) of Lemma 2.1.7, we deduce $\forall x \in X$

$$\|\widetilde{f_x}\|_\infty^{\widetilde{E}_\sigma} \leq \|\widetilde{f_x}\|_\infty^E = \|\widetilde{f_x}\chi_{\sigma(R)}\|_\infty^E < \infty,$$

where the last equality came by $\widetilde{f_x}\chi_{\sigma(R)} = \widetilde{f_x}$, while the boundedness by the hypothesis $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$. Thus $\widetilde{f_x} \in \mathfrak{L}_{\widetilde{E}_\sigma}^\infty(\mathbb{C})$ hence by statement (c) of Theorem 18.2.11. of [**DS**] applied to the scalar type spectral operator $R_\sigma \upharpoonright G_\sigma$

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)). \tag{2.2.5}$$

A more direct way for obtaining (2.2.5) is to use statement (2) of Lemma 2.1.7 and the fact that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ implies $f_x(R) \in B(G)$.

$\forall \sigma \in \mathcal{B}(\mathbb{C})$ we claim that $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)$ is μ-integrable with respect to the strong operator topology. By Lemma 2.1.7 we have $(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\forall v \in G_\sigma)$

$$\int^* \|f_x(R_\sigma \upharpoonright G_\sigma)v\|_{G_\sigma}\, d\,|\mu|(x) = \int^* \|f_x(R)v\|_G\, d\,|\mu|(x) < \infty. \tag{2.2.6}$$

Here the boundedness comes by Theorem 5, $IV.71$ of [**INT**] applied to the $\mu-$integrable map $X \ni x \mapsto f_x(R)v \in G$. By Corollary 1, $IV.70$ and Theorem 5, $IV.71$ of [**INT**] applied, for any $v \in G$, to the $\mu-$integrable map $X \ni x \mapsto f_x(R)v \in G$, we have $(\forall v \in G)(\forall K \in Comp(X))(\exists\, H^v \subseteq G \text{ countable })(f_x(R)v \in \overline{H^v},\, \mu - a.e.(K))$. But by statement (g) of Theorem 18.2.11. of [**DS**] and $f_x(R) \in B(G)$, we have $(\forall \sigma \in \mathcal{B}(\mathbb{C}))([f_x(R), E(\sigma)] = \mathbf{0})$, hence by the previous equation and by the fact that $E(\sigma) \in B(G)$, so is continuous, we obtain $(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\forall v \in G)(\forall K \in Comp(X))$

$$(\exists\, H^v \subseteq G \text{ countable })(f_x(R)E(\sigma)v = E(\sigma)f_x(R)v \in \overline{H^v_\sigma},\, \mu - a.e.(K)).$$

Here $H^v_\sigma \doteqdot E(\sigma)H^v$. Therefore by Lemma 2.1.7 we state that $(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\forall v \in G_\sigma)(\forall K \in Comp(X))$

$$(\exists\, H^v_\sigma \subset G_\sigma \text{ countable })(f_x(R_\sigma \upharpoonright G_\sigma)v \in \overline{H^v_\sigma} \subseteq G_\sigma,\, \mu - a.e.(K)). \tag{2.2.7}$$

That $\overline{H^v_\sigma} \subseteq G_\sigma$ follows by the fact that G_σ is closed in G. Therefore we can consider the closure $\overline{H^v_\sigma}$ as the closure in the Banach space G_σ. By the Hahn-Banach Theorem, see Corollary 3, $II.23$ of the [**TVS**], $\forall \sigma \in \mathcal{B}(\mathbb{C})$

$$\{\phi \upharpoonright G_\sigma \mid \phi \in G^*\} = (G_\sigma)^*. \tag{2.2.8}$$

Moreover by Corollary 1, $IV.70$ and Theorem 5, $IV.71$ of [**INT**] applied, for any $v \in G$, to the $\mu-$integrable map $X \ni x \mapsto f_x(R)E(\sigma)v \in G$, we have $\forall \phi \in G^*$

$$X \ni x \mapsto \phi(f_x(R)E(\sigma)v) \in \mathbb{C} \text{ is } \mu-\text{measurable.}$$

Thus by Lemma 2.1.7 we have $(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\forall v \in G_\sigma)(\forall \phi \in G^*))$

$$X \ni x \mapsto \phi(f_x(R_\sigma \upharpoonright G_\sigma)v) \in \mathbb{C} \text{ is } \mu-\text{measurable.}$$

Hence by (2.2.8) we can state $(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\forall v \in G_\sigma)$ that

$$(\forall \phi_\sigma \in (G_\sigma)^*)(X \ni x \mapsto \phi_\sigma(f_x(R_\sigma \upharpoonright G_\sigma)v) \in \mathbb{C} \text{ is } \mu-\text{measurable.}) \tag{2.2.9}$$

Now by collecting (2.2.9), (2.2.6) and (2.2.7), where the closure $\overline{H_\sigma^v}$ is to be intended how the closure in the Banach space G_σ, we can apply Corollary 1, $IV.70$ and Theorem 5, $IV.71$ of [**INT**] to the map $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma)v \in G_\sigma$, in order to state that

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\forall v \in G_\sigma)(X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma)v \in G_\sigma \text{ is } \mu-\text{integrable.}) \tag{2.2.10}$$

This means in particular that there exists its integral, so $(\forall \sigma \in \mathcal{B}(\mathbb{C}))(\forall v \in G_\sigma)$

$$\begin{aligned} \left\| \int f_x(R_\sigma \upharpoonright G_\sigma)v \, d\,\mu(x) \right\|_{G_\sigma} &= \left\| \int f_x(R)v \, d\,\mu(x) \right\|_G && \text{by Lemma 2.1.7} \\ &\leq \left\| \int f_x(R) \, d\,\mu(x) \right\|_{B(G)} \|v\|_{G_\sigma}. && (2.2.11) \end{aligned}$$

Here the inequality follows by the hypothesis that $X \ni x \mapsto f_x(R) \in B(G)$ is $\mu-$integrable in the strong operator topology. Therefore by Definition 2.2.4 and (2.2.5), (2.2.10), (2.2.11) we can conclude that

(2.2.12)

$$\begin{cases} (\forall \sigma \in \mathcal{B}(\mathbb{C}))(X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma) \text{ is } \mu-\text{integrable in the strong operator topology}) \\ \left\| \int f_x(R_\sigma \upharpoonright G_\sigma) \, d\,\mu(x) \right\|_{B(G_\sigma)} \leq \left\| \int f_x(R) \, d\,\mu(x) \right\|_{B(G)}. \end{cases}$$

Which is the claim we wanted to show, then statement (1) follows.

Statement (1) proves that the assumption (2.2.3) is well set, so we are able to start the proof of the statement (2).

$\forall y \in \Theta$

$$
\begin{aligned}
&= g(R)\int f_x(R)\,d\,\mu(x)y \\
&= \lim_{n\in\mathbb{N}} E(\sigma_n)g(R)\int f_x(R)\,d\,\mu(x)y && \text{by (2.2.1)} \\
&= \lim_{n\in\mathbb{N}} g(R)E(\sigma_n)\int f_x(R)\,d\,\mu(x)y && \text{by } (g) \text{ of Theorem } 18.2.11 \text{ of } \textbf{[DS]} \\
&= \lim_{n\in\mathbb{N}} g(R)E(\sigma_n)\int f_x(R)y\,d\,\mu(x) && \text{by Definition 2.2.4} \\
&= \lim_{n\in\mathbb{N}} g(R)E(\sigma_n)\int E(\sigma_n)f_x(R)y\,d\,\mu(x) && \text{by Theorem } 1,\ IV.35 \text{ of } \textbf{[INT]} \\
&= \lim_{n\in\mathbb{N}} g(R)E(\sigma_n)\int f_x(R)E(\sigma_n)y\,d\,\mu(x) && \text{by } (g) \text{ of Theorem } 18.2.11 \text{ of } \textbf{[DS]} \\
&= \lim_{n\in\mathbb{N}} g(R_{\sigma_n}\upharpoonright G_{\sigma_n})\int f_x(R_{\sigma_n}\upharpoonright G_{\sigma_n})E(\sigma_n)y\,d\,\mu(x) && \text{by } \textbf{Lemma 2.1.7} \\
&= \lim_{n\in\mathbb{N}} g(R_{\sigma_n}\upharpoonright G_{\sigma_n})\int f_x(R_{\sigma_n}\upharpoonright G_{\sigma_n})\,d\,\mu(x)E(\sigma_n)y && \text{by statement } (1) \text{ and Definition 2.2.4} \\
&= \lim_{n\in\mathbb{N}} h(R_{\sigma_n}\upharpoonright G_{\sigma_n})E(\sigma_n)y && \text{by hypothesis (2.2.3)} \\
&= \lim_{n\in\mathbb{N}} h(R)E(\sigma_n)y && \text{by Lemma 2.1.7} \\
&= \lim_{n\in\mathbb{N}} E(\sigma_n)h(R)y && \text{by } (g) \text{ of Theorem } 18.2.11 \text{ of } \textbf{[DS]} \\
&= h(R)y && \text{by (2.2.1).}
\end{aligned}
$$

Therefore

$$g(R)\int f_x(R)\,d\,\mu(x)\upharpoonright\Theta = h(R)\upharpoonright\Theta.$$

□

THEOREM 2.2.10 (**Strong Extension Theorem**). *Let X be a locally compact space, μ a complex Radon measure on X, R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum and E its resolution of the identity. Let the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $(\forall x \in X)(\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R)))$, where the map $X \ni x \mapsto f_x(R) \in B(G)$ be $\mu-$integrable with respect to the strong operator topology. Finally let $g, h \in Bor(\sigma(R))$ and $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$.*

If $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence and $\forall n \in \mathbb{N}$

$$g(R_{\sigma_n}\upharpoonright G_{\sigma_n})\int f_x(R_{\sigma_n}\upharpoonright G_{\sigma_n})\,d\,\mu(x) \subseteq h(R_{\sigma_n}\upharpoonright G_{\sigma_n}) \tag{2.2.13}$$

then $h(R) \in B(G)$ *and*

$$\boxed{g(R) \int f_x(R)\, d\,\mu(x) = h(R).}$$

Here all the integrals are with respect to the strong operator topologies.

Notice that $g(R)$ is possibly an **unbounded** operator on G.

PROOF. $h(R) \in B(G)$ by Theorem $18.2.11.$ of [**DS**] and the hypothesis $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$, so by (2.2.4)

$$g(R) \int f_x(R)\, d\,\mu(x) \subseteq h(R). \tag{2.2.14}$$

Let us set

$$(\forall n \in \mathbb{N})(\delta_n \doteqdot \overset{-1}{|g|}([0,n])). \tag{2.2.15}$$

We claim that

$$\begin{cases} \bigcup_{n\in\mathbb{N}} \delta_n = \sigma(R) \\ n \geq m \Rightarrow \delta_n \supseteq \delta_m \\ (\forall n \in \mathbb{N})(g(\delta_n) \text{ is bounded. }) \end{cases} \tag{2.2.16}$$

In addition being $|g| \in Bor(\sigma(R))$ we have $\delta_n \in \mathcal{B}(\mathbb{C})$ for all $n \in \mathbb{N}$, so $\{\delta_n\}_{n\in\mathbb{N}}$ is an $E-$sequence, hence by (2.2.1)

$$\lim_{n\in\mathbb{N}} E(\delta_n) = \mathbf{1} \tag{2.2.17}$$

with respect to the strong operator topology on $B(G)$. Indeed the first equality follows since $\bigcup_{n\in\mathbb{N}} \delta_n \doteq \bigcup_{n\in\mathbb{N}} \overset{-1}{|g|}([0,n]) = \overset{-1}{|g|}\left(\bigcup_{n\in\mathbb{N}}[0,n]\right) = \overset{-1}{|g|}(\mathbb{R}^+) = Dom(g) \doteqdot \sigma(R)$, the second by the fact that $\overset{-1}{|g|}$ preserves the inclusion, the third since $|g|(\delta_n) \subseteq [0,n]$. Hence our claim.

By the third statement of (2.2.16), $\delta_n \in \mathcal{B}(\mathbb{C})$ and statement 3 of Lemma 2.1.7

$$(\forall n \in \mathbb{N})(E(\delta_n)G \subseteq Dom(g(R))). \tag{2.2.18}$$

$f_x(R)E(\delta_n) = E(\delta_n)f_x(R)$, by statement (g) of Theorem $18.2.11$ of [**DS**], so $\forall v \in G$

$$\begin{aligned} \int f_x(R)\, d\,\mu(x) E(\delta_n) v &\doteq \int f_x(R) E(\delta_n) v\, d\,\mu(x) \\ &= \int E(\delta_n) f_x(R) v\, d\,\mu(x) = E(\delta_n) \int f_x(R) v\, d\,\mu(x), \end{aligned}$$

where the last equality follows by applying Theorem $1, IV.35.$ of [**INT**]. Hence $\forall n \in \mathbb{N}$

$$\int f_x(R)\, d\,\mu(x) E(\delta_n) G \subseteq E(\delta_n) G \subseteq Dom(g(R)),$$

where the last inclusion is by (2.2.18). Therefore

$$(\forall n \in \mathbb{N})(\forall v \in G)\left(E(\delta_n)v \in Dom\left(g(R)\int f_x(R)\, d\,\mu(x)\right)\right).$$

Hence by (2.2.17)

$$\mathbf{D} \doteqdot Dom\left(g(R)\int f_x(R)\, d\,\mu(x)\right) \text{ is dense in } G. \tag{2.2.19}$$

Next $\int f_x(R)\, d\,\mu(x) \in B(G)$ and $g(R)$ is closed by Theorem $18.2.11.$ of [**DS**], so by Lemma 2.2.7

$$g(R)\int f_x(R)\, d\,\mu(x) \text{ is closed.} \tag{2.2.20}$$

Moreover $h(R) \in B(G)$ hence by (2.2.14)

$$g(R)\int f_x(R)\, d\,\mu(x) \in B(\mathbf{D}, G). \tag{2.2.21}$$

(2.2.20), (2.2.21) and Lemma 2.2.8 allow us to state that $\mathbf{D}$ is closed in G, thus by (2.2.19) we have

$$\mathbf{D} = G.$$

Therefore by (2.2.14) the statement follows. □

Now we shall prove a corollary of the previous result in which conditions are given ensuring the strong operator integrability of the map $f_x(R)$.

COROLLARY 2.2.11. *Let R be a possibly unbounded scalar type spectral operator in G. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be an $E-$sequence and $(\forall x \in X)(f_x \in Bor(\sigma(R)))$ such that*

$$(X \ni x \mapsto \|\widetilde{f_x}\|_\infty^E) \in \mathfrak{F}_1(X;\mu)$$

and $X \ni x \mapsto f_x(R) \in B(G)$ satisfies the conditions $(1-2)$ of Proposition 2.2.5, (respectively $\forall v \in G$ the map $X \ni x \mapsto f_x(R)v \in G$ is $\mu-$measurable). Finally let $g, h \in Bor(\sigma(R))$. If we assume that $\forall n \in \mathbb{N}$ holds (2.2.13) *and that $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$, then the same conclusions of Thm. 2.2.10 hold.*

PROOF. By statement (c) of Theorem 18.2.11 of [**DS**] and Proposition 2.2.5, (respectively Remark 2.2.6) the map $X \ni x \mapsto f_x(R) \in B(G)$ is $\mu-$integrable with respect to the strong operator topology and

$$\left\| \int f_x(R) \, d\,\mu(x) \right\|_{B(G)} \leq 4M \int^* \|\widetilde{f}_x\|_\infty^E \, d\,|\mu|(x)$$

Here $M \doteqdot \sup_{\sigma \in \mathcal{B}(\mathbb{C})} \|E(\sigma)\|_{B(G)}$. Therefore the statement follows by Theorem 2.2.10.

$\square$

2.3. Generalization of the Newton-Leibnitz formula

The main result of this section is Theorem 2.3.6 which generalizes the Newton-Leibnitz formula to the case of unbounded scalar type spectral operators in G.

For proving Theorem 2.3.6 we need two preliminary results, the first is Theorem 2.3.2, concerning the Newton-Leibnitz formula for any bounded scalar type spectral operator on G and any analytic map on an open neighbourhood of its spectrum, so generalizing Corollary 1.0.24. The second, Theorem 2.3.4, concerns strong operator continuity, and under additional conditions also differentiability, for operator maps of the type $K \ni t \mapsto S(tR) \in B(G)$, where K is an open interval of $\mathbb{R}$, $S(tR)$ arises by the Borel functional calculus for the unbounded scalar type spectral operator R in G and S is any analytic map on an open neighbourhood U of $\sigma(R)$ such that $K \cdot U \subseteq U$.

Let Z be a non empty set, Y a $\mathbb{K}-$linear space ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), $U \subseteq Y$, $K \subseteq \mathbb{K}$ such that $K \cdot U \subseteq U$ and $F : U \to Z$. Then we set $F_t : U \to Z$ such that $F_t(\lambda) \doteqdot F(t\lambda)$, for all $t \in K$ and $\lambda \in U$.

If F, G are two $\mathbb{C}-$Banach spaces, $A \subseteq F$ open and $f : A \subseteq F \to G$ a map, we convene to denote $F_{\mathbb{R}}$ and $G_{\mathbb{R}}$ by F and G respectively and with the symbol f the map $f^{\mathbb{R}} : A \subseteq F_{\mathbb{R}} \to G_{\mathbb{R}}$. See Definition 1.0.19.

LEMMA 2.3.1. *Let $\langle Y, d \rangle$ be a metric space, U an open of Y and σ a compact such that $\sigma \subseteq U$. Then $\exists\, Q > 0$*

$$K \doteqdot \overline{\bigcup_{\{y \in \sigma\}} \overline{B}_Q(y)} \subseteq U, \tag{2.3.1}$$

moreover if σ is of finite diameter then K is of finite diameter.

PROOF. By Remark §2.2., Ch. 9 of [**GT**] we deduce

$$P \doteqdot dist(\sigma, \complement U) \neq 0,$$

where $dist(A, B) \doteqdot \inf_{\{x \in A, y \in B\}} d(x, y)$, for all $A, B \subseteq Y$.

Set
$$Q \doteqdot \frac{P}{2}$$
then $(\forall y \in \sigma)(\forall x \in \overline{B}_Q(y))(\forall z \in \complement U)$ we have

$$d(x,z) \geq d(z,y) - d(y,x) \geq \frac{P}{2} \neq 0. \tag{2.3.2}$$

Thus by applying Proposition 2, §2.2., Ch. 9 of **[GT]** $\overline{B}_Q(y) \cap \complement U = \emptyset$, i.e. $\overline{B}_Q(y) \subseteq U$, then
$$A \doteqdot \bigcup_{\{y \in \sigma\}} \overline{B}_Q(y) \subseteq U.$$
Moreover by Proposition 3, §2.2., Ch. 9 of **[GT]** the map $x \mapsto d(x, \complement U)$ is continuous on Y, hence by (2.3.2) $\forall x \in \overline{A}$
$$d(x, \complement U) = \lim_{n \in \mathbb{N}} d(x_n, \complement U) \geq \frac{P}{2} \neq 0,$$
$\forall \{x_n\}_{n \in \mathbb{N}} \subset A$ such that $x = \lim_{n \in \mathbb{N}} x_n$. Therefore by Proposition 2, §2.2., Ch. 9 of **[GT]** (2.3.1) follows.

Let $B \subset Y$ be of finite diameter then by the continuity of the map $d : Y \times Y \to \mathbb{R}^+$ it is of finite diameter also $\overline{B}$. Indeed let $diam(B) \doteqdot \sup_{\{x,y \in B\}} d(x,y)$, if by absurdum $\sup_{\{x,y \in \overline{B}\}} d(x,y) = \infty$ then

$$(\exists\, x_0, y_0 \in \overline{B})(d(x_0,y_0) > diam(B) + 1). \tag{2.3.3}$$

Let $\{(x_\alpha, y_\alpha)\}_{\alpha \in D} \subset B \times B$ be a net such that $\lim_{\alpha \in D}(x_\alpha, y_\alpha) = (x_0, y_0)$ limit in $\langle Y, d\rangle \times \langle Y, d\rangle$. Thus by the continuity of d
$$d(x_0, y_0) = \lim_{\alpha \in D} d(x_\alpha, y_\alpha) \leq diam(B)$$
which contradicts (2.3.3), so $\sup_{\{x,y \in \overline{B}\}} d(x,y) < \infty$.

Therefore if A is of finite diameter it is so K. Let $z_1, z_2 \in A$ then there exist $y_1, y_2 \in \sigma$ such that $z_k \in \overline{B}_Q(y_k)$, for $k \in \{1,2\}$. Then
$$d(z_1,z_2) \leq d(z_1,y_1) + d(y_1,y_2) + d(y_2,z_2) \leq 2Q + diam(\sigma) < \infty,$$
where $diam(\sigma) \doteqdot \sup_{\{x,y \in \sigma\}} d(x,y)$. Hence A is of finite diameter. □

THEOREM 2.3.2. *Let $T \in B(G)$ be a scalar type spectral operator, $\sigma(T)$ its spectrum. Assume that $0 < L \leq \infty$, U is an open neighbourhood of $\sigma(T)$ such that $]-L, L[\cdot U \subseteq U$ and $F : U \to \mathbb{C}$ is an analytic map. Then $\forall t \in]-L, L[$*

(1)

$$F(tT) = F_t(T); \tag{2.3.4}$$

(2)

$$\boxed{\frac{d\,F(tT)}{d\,t} = T\frac{d\,F}{d\,\lambda}(tT);} \tag{2.3.5}$$

(3) $\forall u_1, u_2 \in]-L, L[$

$$\boxed{T \oint_{u_1}^{u_2} \frac{d\,F}{d\,\lambda}(tT)d\,t = F(u_2T) - F(u_1T).} \tag{2.3.6}$$

Here $F_t(T)$, (respectively $\frac{d\,F}{d\,\lambda}(tT)$ and $F(tT)$) are the operators arising by the Borelian functional calculus of the operator T (respectively tT) $\forall t \in]-L, L[$.

PROOF. T is a bounded operator on G so $\sigma(T)$ is compact. Let us denote by $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle$ the Banach algebra of all continuous complex valued maps defined on $\sigma(T)$ with the norm $\|g\|_{\sup} \doteqdot \sup_{\lambda\in\sigma(T)} |g(\lambda)|$. Set

$$\begin{cases} \widetilde{\mathcal{C}}(\sigma(T)) \doteqdot \left\{ f : \mathbb{C} \to \mathbb{C} \mid f \upharpoonright \sigma(T) \in \mathcal{C}(\sigma(T)),\ f \upharpoonright \complement\sigma(T) = \mathbf{0} \right\}, \\ J : \mathcal{C}(\sigma(T)) \ni g \mapsto \widetilde{g} \in \widetilde{\mathcal{C}}(\sigma(T)). \end{cases} \tag{2.3.7}$$

Notice that $\widetilde{\mathcal{C}}(\sigma(T))$ is an algebra moreover J is a surjective morphism of algebras and $\sup_{\lambda\in\mathbb{C}} |J(g)(\lambda)| = \|g\|_{\sup}$ for all $g \in \mathcal{C}(\sigma(T))$ furthermore $J(g) \in Bor(\mathbb{C})$ since $g \in Bor(\sigma(T))$ and $\sigma(T) \in \mathcal{B}(\mathbb{C})$. Hence $\widetilde{\mathcal{C}}(\sigma(T))$ is a subalgebra of $\mathbf{TM}$, moreover J is an isometry between $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle$ and $\left\langle \widetilde{\mathcal{C}}(\sigma(T)), \|\cdot\|_{\sup}\right\rangle$.

Thus $\left\langle \widetilde{\mathcal{C}}(\sigma(T)), \|\cdot\|_{\sup}\right\rangle$ is a Banach subalgebra of the Banach algebra $\langle \mathbf{TM}, \|\cdot\|_{\sup}\rangle$ and J is an isometric isomorphism of algebras.

Therefore by denoting with E the resolution of the identity of T, by (2.1.2) we have that $\mathbf{I}_{\mathbb{C}}^{E} \circ J$ is a unital [6] morphism of algebras such that $\mathbf{I}_{\mathbb{C}}^{E} \circ J \in B(\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle, B(G))$.

[6]indeed by setting $\mathbf{1} : \mathbb{C} \ni \lambda \mapsto 1 \in \mathbb{C}$ the unity element in $\mathbf{TM}$ then $\mathbf{I}_{\mathbb{C}}^{E} \circ J(\mathbf{1} \upharpoonright \sigma(T)) = \mathbf{I}_{\mathbb{C}}^{E}(\mathbf{1} \cdot \chi_{\sigma(T)}) = \mathbf{I}_{\mathbb{C}}^{E}(\mathbf{1})\mathbf{I}_{\mathbb{C}}^{E}(\chi_{\sigma(T)}) = \mathbf{1}$.

In the sequel we convene to denote for brevity with the symbol $\mathbf{I}_{\mathbb{C}}^{E}$ the operator $\mathbf{I}_{\mathbb{C}}^{E} \circ J$ so

$$\begin{cases} \mathbf{I}_{\mathbb{C}}^{E} \in B(\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle, B(G)), \\ \mathbf{I}_{\mathbb{C}}^{E} \text{ is a unital morphism of algebras} \\ (\forall g \in \mathcal{C}(\sigma(T)))(g(T) = \mathbf{I}_{\mathbb{C}}^{E}(g)). \end{cases} \tag{2.3.8}$$

In particular $\mathbf{I}_{\mathbb{C}}^{E}$ is Fréchet differentiable with constant differential map equal to $\mathbf{I}_{\mathbb{C}}^{E}$. In the sequel we shall denote with $\mathbf{0}$ the zero element of the Banach space $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle$.

Let $t \in]-L, L[-\{0\}$, and $\imath_t \doteqdot t \cdot \imath$, where $\imath : \sigma(T) \ni \lambda \mapsto \lambda$. So $\imath_t(T) = \mathbf{I}_{\mathbb{C}}^{E}(t \cdot \imath) = t\mathbf{I}_{\mathbb{C}}^{E}(\imath) = tT$. Hence by the general spectral mapping theorem $18.2.21.$ of [**DS**] applied to the map $\imath_t$, the fact that $\sigma(T)$ is closed and the product by no zero scalars in $\mathbb{C}$ is a homeomorphism, we deduce that tT is a scalar type spectral operator and $E_t : \mathcal{B}(\mathbb{C}) \ni \delta \mapsto E(t^{-1}\delta)$ its resolution of the identity. Finally

$$(\forall t \in]-L, L[)(\sigma(tT) = t\sigma(T) \subseteq U),$$

the inclusion is by hypothesis. So $F(tT)$ arising by the Borel functional calculus of the operator tT is well defined and by (2.3.8)

$$\begin{aligned} F(tT) = \mathbf{I}_{\mathbb{C}}^{E_t}(F \upharpoonright \sigma(tT)) &\doteq \mathbf{I}_{\mathbb{C}}^{E \circ \imath_{t^{-1}}}(F \upharpoonright \sigma(tT)) \\ &= \mathbf{I}_{\mathbb{C}}^{E}(F \circ \imath_t) \\ &= \mathbf{I}_{\mathbb{C}}^{E}(F_t \upharpoonright \sigma(T)) = F_t(T). \end{aligned} \tag{2.3.9}$$

Thus (2.3.4).

Set

$$\Delta :]-L, L[\ni t \mapsto (F \circ \imath_t) \in \langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup}\rangle,$$

by the third equality in (2.3.9)

$$(\forall t \in]-L, L[)(F(tT) = \mathbf{I}_{\mathbb{C}}^{E} \circ \Delta(t)). \tag{2.3.10}$$

We claim that Δ is derivable (i.e. Fréchet differentiable) and $\forall t \in]-L, L[$

$$\frac{d\,\Delta}{dt}(t) = \imath \cdot \left(\frac{d\,F}{d\lambda}\right)_t \upharpoonright \sigma(T). \tag{2.3.11}$$

Set

$$\begin{cases} \mathcal{C}_U(\sigma(T)) \doteqdot \{f \in \mathcal{C}(\sigma(T)) \mid f(\sigma(T)) \subseteq U\}, \\ \zeta :]-L, L[\ni t \mapsto \imath_t \in \mathcal{C}_U(\sigma(T)) \subset \langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup} \rangle \\ \Upsilon : \mathcal{C}_U(\sigma(T)) \ni f \mapsto F \circ f \in \langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup} \rangle . \end{cases}$$

Notice

$$\Delta = \Upsilon \circ \zeta, \tag{2.3.12}$$

moreover ζ is Fréchet differentiable and $\forall t \in]-L, L[$

$$\frac{d\,\zeta}{dt}(t) = \imath. \tag{2.3.13}$$

Next $\forall f \in \mathcal{C}_U(\sigma(T))$ by Lemma 2.3.1 applied to the compact $f(\sigma(T))$, $\exists\, Q_f > 0$

$$K_f \doteqdot \overline{\bigcup_{\{\lambda \in \sigma(T)\}} \overline{B}_{Q_f}(f(\lambda))} \subseteq U, \tag{2.3.14}$$

in particular

$$\overline{B}_{Q_f}(f) \subseteq \mathcal{C}_U(\sigma(T)). \tag{2.3.15}$$

Thus $\mathcal{C}_U(\sigma(T))$ is an open set of the space $\langle \mathcal{C}(\sigma(T)), \|\cdot\|_{\sup} \rangle$, therefore we can claim that Υ is Fréchet differentiable and its differential map $\Upsilon^{[1]} : \mathcal{C}_U(\sigma(T)) \to B(\mathcal{C}(\sigma(T)))$ is such that $(\forall f \in \mathcal{C}_U(\sigma(T)))(\forall h \in \mathcal{C}(\sigma(T)))(\forall \lambda \in \sigma(T))$

$$\begin{cases} \Upsilon^{[1]}(f)(h)(\lambda) = \frac{dF}{d\lambda}(f(\lambda))h(\lambda), \\ \|\Upsilon^{[1]}(f)\|_{B(\mathcal{C}(\sigma(T)))} \leq \|\frac{dF}{d\lambda} \circ f\|_{\sup} \end{cases} \tag{2.3.16}$$

Let us fix $f \in \mathcal{C}_U(\sigma(T))$ and K_f as in (2.3.14), so by the boundedness of $f(\sigma(T))$ and Lemma 2.3.1 K_f is compact. Morever $\frac{dF}{d\lambda}$ is continuous on U therefore uniformly continuous on the compact K_f, hence $(\forall \varepsilon > 0)(\exists\, \delta > 0)(\forall h \in \overline{B}_{Q_f}(\mathbf{0}) \cap \overline{B}_\delta(\mathbf{0}))$

$$\sup_{t \in [0,1]} \sup_{\lambda \in \sigma(T)} \left| \frac{dF}{d\lambda}(f(\lambda) + th(\lambda)) - \frac{dF}{d\lambda}(f(\lambda)) \right| \leq \varepsilon, \tag{2.3.17}$$

indeed $f(\lambda) + th(\lambda) \in K_f$ and $|f(\lambda) + th(\lambda) - f(\lambda)| \leq |h(\lambda)| \leq \delta$, for all $\lambda \in \sigma(T)$ and $t \in [0, 1]$.

Let us identify for the moment $\mathbb{C}$ as the $\mathbb{R}-$normed space $\mathbb{R}^2$, then the usual product $(\cdot) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}-$bilinear, therefore the map $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is Fréchet differentiable and $(\forall x \in U)(\forall h \in \mathbb{R}^2)$

$$F^{[1]}(x)(h) = \frac{d\,F}{d\lambda}(x) \cdot h. \tag{2.3.18}$$

$\forall h \in \overline{B}_{Q_f}(\mathbf{0})$

$$\begin{aligned} &\sup_{\lambda \in \sigma(T)} \left|(F(f(\lambda)+h(\lambda)) - F(f(\lambda)) - \frac{d\,F}{d\lambda}(f(\lambda))h(\lambda)\right| = \\ &\sup_{\lambda \in \sigma(T)} \left|(F(f(\lambda)+h(\lambda)) - F(f(\lambda)) - F^{[1]}(f(\lambda))(h(\lambda)))\right| \leq \\ &\sup_{t \in [0,1]} \sup_{\lambda \in \sigma(T)} \|F^{[1]}(f(\lambda)+th(\lambda)) - F^{[1]}(f(\lambda))\|_{B(\mathbb{R}^2)} \sup_{\lambda \in \sigma(T)} |h(\lambda)| = \\ &\sup_{t \in [0,1]} \sup_{\lambda \in \sigma(T)} \left|\frac{d\,F}{d\lambda}(f(\lambda)+th(\lambda)) - \frac{d\,F}{d\lambda}(f(\lambda))\right| \|h\|_{\sup}. \end{aligned} \tag{2.3.19}$$

Here in the first equality we use (2.3.18), in the first inequality an application of the Mean value theorem applied to the Fréchet differentiable map $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, in the second equality we use a corollary of (2.3.18).

Finally by (2.3.19) and (2.3.17) $(\forall \varepsilon > 0)(\exists\, \delta > 0)(\forall h \in \overline{B}_{Q_f}(\mathbf{0}) \cap \overline{B}_\delta(\mathbf{0}) - \{\mathbf{0}\})$

$$\frac{\sup_{\lambda \in \sigma(T)} \left|(F(f(\lambda)+h(\lambda)) - F(f(\lambda)) - \frac{d\,F}{d\lambda}(f(\lambda))h(\lambda)\right|}{\|h\|_{\sup}} \leq \varepsilon.$$

Equivalently

$$\lim_{\substack{h \to \mathbf{0} \\ h \neq \mathbf{0}}} \frac{\|\Upsilon(f+h) - \Upsilon(f) - \Upsilon^{[1]}(f)(h)\|_{\sup}}{\|h\|_{\sup}} = 0 \tag{2.3.20}$$

Moreover

$$\|\Upsilon^{[1]}(f)(h)\|_{\sup} \leq \|\frac{d\,F}{d\lambda} \circ f\|_{\sup} \|h\|_{\sup}$$

then by (2.3.20) we proved the claimed (2.3.16).

By (2.3.12), (2.3.13) and (2.3.16) we deduce that Δ is derivable in addition $\forall t \in]-L, L[$ and $\forall \lambda \in \sigma(T)$

$$\begin{aligned} \frac{d\,\Delta}{dt}(t)(\lambda) &= \Upsilon^{[1]}(\zeta(t))(\imath)(\lambda) \\ &= \frac{d\,F}{d\lambda}(\zeta_t(\lambda))\imath(\lambda) = \imath\left(\frac{d\,F}{d\lambda}\right)_t(\lambda). \end{aligned}$$

Thus the claimed (2.3.11).

In conclusion by the fact that $\mathbf{I}_{\mathbb{C}}^{E}$ is a morphism of algebras, (2.3.10), (2.3.8) and (2.3.11) $\forall t \in]-L, L[$

$$\begin{aligned}\frac{d\,F(tT)}{dt} &= \frac{d}{dt}\left(\mathbf{I}_{\mathbb{C}}^{E} \circ \Delta\right)(t) = \mathbf{I}_{\mathbb{C}}^{E}\left(\frac{d\,\Delta}{dt}(t)\right) \\ &= \mathbf{I}_{\mathbb{C}}^{E}\left(\imath \cdot \left(\frac{d\,F}{d\lambda}\right)_{t} \upharpoonright \sigma(T)\right) \\ &= \mathbf{I}_{\mathbb{C}}^{E}(\imath)\,\mathbf{I}_{\mathbb{C}}^{E}\left(\left(\frac{d\,F}{d\lambda}\right)_{t} \upharpoonright \sigma(T)\right) = T\left(\frac{d\,F}{d\lambda}\right)_{t}(T).\end{aligned}$$

Therefore statement (2) by statement (1) applied to the map $\frac{d\,F}{d\lambda}$.

The map $]-L, L[\ni t \mapsto \frac{d\,F}{d\lambda}(tT) \in B(G)$ is continuous by (2.3.5) (by replacing the map F with $\frac{dF}{d\lambda}$) hence it is Lebesgue-measurable in $B(G)$. Let $u_1, u_2 \in]-L, L[$, by statement (1) and Theorem $18.2.11.$ of [**DS**]

$$\begin{aligned}\int_{[u_1,u_2]}^{*}\left\|\frac{d\,F}{d\,\lambda}(tT)\right\| d\,t &= \int_{[u_1,u_2]}^{*}\left\|\left(\frac{d\,F}{d\,\lambda}\right)_{t}(T)\right\| d\,t \\ &\leq 4M\int_{[u_1,u_2]}^{*}\left\|\left(\frac{d\,F}{d\,\lambda}\right)_{t} \upharpoonright \sigma(T)\right\|_{\sup} d\,t \\ &\leq 4MD|u_2 - u_1| < \infty,\end{aligned}$$

where $M \doteqdot \sup_{\delta \in \mathcal{B}(\mathbb{C})}\|E(\delta)\|$, and

$$D \doteqdot \sup_{t \in [u_1,u_2]}\left\|\left(\frac{d\,F}{d\,\lambda}\right)_{t} \upharpoonright \sigma(T)\right\|_{\sup} = \sup_{(t,\lambda) \in [u_1,u_2] \times \sigma(T)}\left|\frac{d\,F}{d\,\lambda}(t\lambda)\right| < \infty,$$

indeed $[u_1, u_2] \times \sigma(T)$ is compact and the map $(t, \lambda) \mapsto \frac{d\,F}{d\,\lambda}(t\lambda)$ is continuous on $]-L, L[\times U$.

Therefore by Theorem 5, $IV.71$ of [**INT**] $]-L, L[\ni t \mapsto \frac{d\,F}{d\lambda}(tT)$ is Lebesgue-integrable with respect to the norm topology on $B(G)$, so in particular by Definition 1, $IV.33$ of [**INT**]

$$\exists \oint_{u_1}^{u_2} \frac{d\,F}{d\,\lambda}(tT)\,d\,t \in B(G). \tag{2.3.21}$$

Therefore by (1.0.3), (2.3.21), Theorem 1, $IV.35$ of [**INT**] and (2.3.5)

$$T\oint_{u_1}^{u_2} \frac{d\,F}{d\,\lambda}(tT)\,d\,t = \oint_{u_1}^{u_2} T\frac{d\,F}{d\,\lambda}(tT)\,d\,t = \oint_{u_1}^{u_2} \frac{d\,F(tT)}{d\,t}\,dt. \tag{2.3.22}$$

By (2.3.5) the map $]-L, L[\ni t \mapsto F(tT)$, is derivable moreover its derivative $]-L, L[\ni t \mapsto \frac{d\,F(tT)}{d\,t}$ is continuous in $B(G)$ by (2.3.5) and the continuity of the map

$]-L, L[\ni t \mapsto \frac{dF}{d\lambda}(tT)$ in $B(G)$. Therefore $[u_1, u_2] \ni t \mapsto \frac{dF(tT)}{dt}$ is Lebesgue integrable in $B(G)$, where the integral has to be understood as defined in Ch II of [**FVR**], see Proposition 3, $n°3$, $\S 1$, Ch II of [**FVR**].

Finally the Lebesgue integral for functions with values in a Banach space as defined in Ch II of [**FVR**], turns to be the integral with respect to the Lebesgue measure as defined in Ch. IV, $\S 4$, $n°1$ of [**INT**] (see Ch III, $\S 3$, $n°3$ and example in Ch IV, $\S 4$, $n°4$ of [**INT**]). Thus statement (3) follows by (2.3.22). $\square$

LEMMA 2.3.3. *Let R be a possibly unbounded scalar type spectral operator in G, $\sigma(R)$ its spectrum, E its resolution of the identity, $K \neq \emptyset$ and $\forall t \in K$ be $f_t \in Bor(\sigma(R))$ such that*

$$N \doteqdot \sup_{t \in K} \|\widetilde{f}_t\|_\infty^E < \infty. \tag{2.3.23}$$

If $g \in Bor(\sigma(R))$ and $\{\sigma_n\}_{n \in \mathbb{N}}$ is an $E-$sequence then $\forall v \in Dom(g(R))$

$$\lim_{n \in \mathbb{N}} \sup_{t \in K} \|f_t(R)g(R)v - f_t(R)g(R)E(\sigma_n)v\| = 0.$$

PROOF. By statement (g) of Theorem $18.2.11.$ of [**DS**] the statement is well set. Let $M \doteqdot \sup_{\sigma \in \mathcal{B}(\mathbb{C})} \|E(\sigma)\|_{B(G)}$ then $M < \infty$ by Corollary 15.2.4. of [**DS**]. Hypothesis (2.3.23) together statement (c) of Theorem $18.2.11.$ of [**DS**], imply that $(\forall t \in K)(f_t(R) \in B(G))$ and

$$\sup_{t \in K} \|f_t(R)\|_{B(G)} \leq 4MN.$$

Therefore $\forall v \in Dom(g(R))$ we have

$$\begin{aligned}
&\lim_{n \in \mathbb{N}} \sup_{t \in K} \|f_t(R)g(R)v - f_t(R)g(R)E(\sigma_n)v\| \\
&\leq \lim_{n \in \mathbb{N}} \sup_{t \in K} \|f_t(R)\| \cdot \|g(R)v - g(R)E(\sigma_n)v\| \\
&\leq 4MN \lim_{n \in \mathbb{N}} \|g(R)v - g(R)E(\sigma_n)v\| \\
&= 4MN \lim_{n \in \mathbb{N}} \|g(R)v - E(\sigma_n)g(R)v\| && \text{by } (g) \text{ of Theorem } 18.2.11. \text{ of } [\mathbf{DS}] \\
&= 0 && \text{by (2.2.1).}
\end{aligned}$$

$\square$

THEOREM 2.3.4 (**Strong operator derivability**). *Let R be a possibly unbounded scalar type spectral operator in G, $K \subseteq \mathbb{R}$ an open interval of $\mathbb{R}$ and U an open neighbourhood of $\sigma(R)$ such that $K \cdot U \subseteq U$. Assume that $f : U \to \mathbb{C}$ is an analytic map* *and*

$$\sup_{t\in K}\|\widetilde{f}_t\|_{\infty}^{E}<\infty.$$

Then

(1) *the map* $K \ni t \mapsto f(tR) \in B(G)$ *is continuous in the strong operator topology,*

(2) *if*

$$\sup_{t\in K}\left\|\widetilde{\left(\frac{d\,f}{d\,\lambda}\right)}_t\right\|_{\infty}^{E}<\infty, \tag{2.3.24}$$

then $(\forall v \in Dom(R))(\forall t \in K)$

$$\boxed{\frac{df(tR)v}{dt}=R\frac{df}{d\lambda}(tR)v\in G.}$$

PROOF. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be an $E-$sequence of compact sets, then by Lemma 2.3.3 applied for the Borelian map $g:\sigma(R)\ni\lambda\to 1\in\mathbb{C}$, so $g(R)=\mathbf{1}$, and by (2.3.4) we have $\forall v\in G$

$$\lim_{n\in\mathbb{N}}\sup_{t\in K}\|f(tR)v-f(tR)E(\sigma_n)v\|=0. \tag{2.3.25}$$

By (2.3.4) and Key Lemma 2.1.7 $\forall n\in\mathbb{N}$

$$\begin{aligned} f(tR)E(\sigma_n)&=f_t(R)E(\sigma_n)=f_t(R_{\sigma_n}\restriction G_{\sigma_n})E(\sigma_n)\\ &=f(t(R_{\sigma_n}\restriction G_{\sigma_n}))E(\sigma_n).\end{aligned} \tag{2.3.26}$$

σ_n is bounded so by Key Lemma 2.1.7 $R_{\sigma_n}\restriction G_{\sigma_n}$ is a scalar type spectral operator such that $R_{\sigma_n}\restriction G_{\sigma_n}\in B(G_{\sigma_n})$, moreover by (2.1.15) U is an open neighbourhood of $\sigma(R_{\sigma_n}\restriction G_{\sigma_n})$. Thus by statement (2) of Theorem 2.3.2 the map

$$K\ni t\mapsto f(t(R_{\sigma_n}\restriction G_{\sigma_n}))\in B(G_{\sigma_n})$$

is derivable, so in particular $\|\cdot\|_{B(G_{\sigma_n})}-$continuous.

Now $(\forall n\in\mathbb{N})(\forall v_n\in G_{\sigma_n})$ define $\xi_{v_n}:B(G_{\sigma_n})\ni A\mapsto Av_n\in G$, then $\xi_{v_n}\in B(B(G_{\sigma_n}),G)$, thus as a composition of two continuous maps also the following map

$$K\ni t\mapsto \xi_{E(\sigma_n)v}\left(f(t(R_{\sigma_n}\restriction G_{\sigma_n}))\right)\in G \tag{2.3.27}$$

is $\|\cdot\|_G-$continuous, $\forall n\in\mathbb{N}$ and $\forall v\in G$. Hence by (2.3.26) we have $\forall n\in\mathbb{N}$

$$K\ni t\mapsto f(tR)E(\sigma_n)\in B(G)\text{ is strongly continuous.} \tag{2.3.28}$$

Finally by (2.3.28) and (2.3.25) we can apply Theorem 2, §$1.6.$, Ch. 10 of [**GT**] to the uniform space $B(G)_{st}$ whose uniformity is generated by the set of seminorms defining the strong operator topology on $B(G)$. Thus we conclude that $K \ni t \mapsto f(tR) \in B(G)$ is strongly continuous, and statement (1) follows.

Let $n \in \mathbb{N}$ and $v_n \in G_{\sigma_n}$ so $\xi_{v_n} \in B(B(G_{\sigma_n}), G)$ thus ξ_{v_n} is Fréchet differentiable with constant differential map $\xi_{v_n}^{[1]} : B(G_{\sigma_n}) \ni A \mapsto \xi_{v_n} \in B(B(G_{\sigma_n}), G)$. Therefore by statement (2) of Theorem 2.3.2 $(\forall n \in \mathbb{N})(\forall v \in G)$ the map in (2.3.27) is Fréchet differentiable as composition of two Fréchet differentiable maps, and its derivative is $\forall t \in K$

$$
\begin{aligned}
\frac{d}{dt}\left(f(t(R_{\sigma_n} \restriction G_{\sigma_n}))E(\sigma_n)v\right) &= \xi_{E(\sigma_n)v}\left(\frac{d}{dt}(f(t(R_{\sigma_n} \restriction G_{\sigma_n})))\right) \\
&= \frac{d}{dt}(f(t(R_{\sigma_n} \restriction G_{\sigma_n})))E(\sigma_n)v \\
&= (R_{\sigma_n} \restriction G_{\sigma_n})\frac{df}{d\lambda}(t(R_{\sigma_n} \restriction G_{\sigma_n}))E(\sigma_n)v, && \text{by (2.3.5)} \\
&= \frac{df}{d\lambda}(t(R_{\sigma_n} \restriction G_{\sigma_n}))(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v, && \text{by 18.2.11., [DS]} \\
&= \left(\frac{df}{d\lambda}\right)_t (R_{\sigma_n} \restriction G_{\sigma_n})(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v && \text{by (2.3.4)} \\
&= \left(\frac{df}{d\lambda}\right)_t (R)(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v && \text{by Key Lemma 2.1.7} \\
&= \frac{df}{d\lambda}(tR)(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v. && \text{by (2.3.4)}
\end{aligned}
\tag{2.3.29}
$$

Thus by (2.3.26) $(\forall n \in \mathbb{N})(\forall v \in G)$

$$
\begin{cases}
K \ni t \mapsto f(tR)E(\sigma_n)v \in G \text{ is differentiable and} \\
K \ni t \mapsto \frac{df}{d\lambda}(tR)(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v \in G \text{ is its derivative.}
\end{cases}
\tag{2.3.30}
$$

By (2.3.24) we can apply Lemma 2.3.3 to the maps $\left(\frac{df}{d\lambda}\right)_t \restriction \sigma(R)$ and $g = \imath : \sigma(R) \ni \lambda \mapsto \lambda \in \mathbb{C}$, so $g(R) = R$, hence by (2.3.4) $\forall v \in Dom(R)$

$$
\lim_{n\in\mathbb{N}} \sup_{t\in K} \left\| \frac{df}{d\lambda}(tR)Rv - \frac{df}{d\lambda}(tR)(R_{\sigma_n} \restriction G_{\sigma_n})E(\sigma_n)v \right\| = 0.
\tag{2.3.31}
$$

Moreover $\forall a \in K$ let $r_a \in \mathbb{R}^+$ be such that $B_{r_a}(a) \subset K$ which exists K being open, then the equations (2.3.31), (2.3.30) and (2.3.25) hold again if we replace K by $B_{r_a}(a)$. Hence we can apply Theorem $8.6.3.$ of [**Dieu**] and deduce $\forall v \in Dom(R)$ that the map

$K \ni t \mapsto f(tR)v \in G$ is derivable, and its derivative map is

$$K \ni t \mapsto \frac{df}{d\lambda}(tR)Rv \in G.$$

Finally $(\forall v \in Dom(R))(R\frac{df}{d\lambda}(tR)v = \frac{df}{d\lambda}(tR)Rv)$, by $Dom(\frac{df}{d\lambda}(tR)) = G$ and the commutativity property of the Borel functional calculus expressed in statement (f) of Theorem 18.2.11. of [**DS**]. Hence the statement follows. □

COROLLARY 2.3.5. *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighbourhood of $\sigma(R)$ and $S : U \to \mathbb{C}$ an analytic map. Assume that $\exists\, L > 0$ such that $]-L, L[\cdot U \subseteq U$ and*

(1) *$\widetilde{S}_t \in \mathfrak{L}_E^{\infty}(\sigma(R))$ and $\widetilde{\left(\frac{dS}{d\lambda}\right)_t} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ for all $t \in]-L, L[$;*

(2)

$$\int^{*} \left\| \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \right\|_{\infty}^{E} dt < \infty$$

(here the upper integral is with respect to the Lebesgue measure on $]-L, L[$);

(3) *$\forall v \in G$ the map $]-L, L[\ni t \mapsto \frac{dS}{d\lambda}(tR)v \in G$ is Lebesgue measurable.*

Then $\forall u_1, u_2 \in]-L, L[$

$$R \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt = S(u_2 R) - S(u_1 R) \in B(G).$$

Here the integral is with respect to the Lebesgue measure on $[u_1, u_2]$ and with respect to the strong operator topology on $B(G)$, see Definition 2.2.4.

PROOF. Let $M \doteqdot \sup_{\sigma \in \mathcal{B}(\mathbb{C})} \|E(\sigma)\|_G$ and μ the Lebesgue measure on $[u_1, u_2]$, then by (2.3.4), hypotheses, and statement (c) of Theorem 18.2.11 of [**DS**] we have

a: $(\forall t \in [u_1, u_2])(S(tR) \in B(G))$;

b: $(\forall t \in [u_1, u_2])(\frac{dS}{d\lambda}(tR) \in B(G))$;

c: $([u_1, u_2] \ni t \mapsto \|\frac{dS}{d\lambda}(tR)\|_{B(G)}) \in \mathfrak{F}_1([u_1, u_2]; \mu)$.

So by hypothesis (3), the (c) and Remark 2.2.6 we have that the map

$$[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$$

is Lebesgue integrable with respect to the strong operator topology. This means that, except for (2.2.13), the hypotheses of Theorem 2.2.10 hold for $X \doteqdot [u_1, u_2]$, $h \doteqdot (S_{u_2} - S_{u_1}) \upharpoonright \sigma(R)$, $g : \sigma(R) \ni \lambda \mapsto \lambda \in \mathbb{C}$ and finally for the maps $f_t \doteqdot \left(\frac{dS}{d\lambda}\right)_t \upharpoonright \sigma(R)$, for all $t \in [u_1, u_2]$.

Next let $\sigma \in \mathcal{B}(\mathbb{C})$ be bounded, so by Key Lemma 2.1.7 $R_\sigma \upharpoonright G_\sigma$ is a scalar type spectral operator such that $R_\sigma \upharpoonright G_\sigma \in B(G_\sigma)$, moreover by (2.1.15) U is an open neighbourhood of $\sigma(R_\sigma \upharpoonright G_\sigma)$. Thus we can apply statement (3) of Theorem 2.3.2 to the Banach space G_σ, the analytic map S and to the operator $R_\sigma \upharpoonright G_\sigma$.

In particular the map $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma)) \in B(G_\sigma)$ is Lebesgue integrable in $\|\cdot\|_{B(G_\sigma)}-$topology, that is in the meaning of Definition 2, $IV.23$ of [**INT**]. Next we consider $\forall v \in G_\sigma$, the following map

$$T \in B(G_\sigma) \mapsto Tv \in G_\sigma$$

which is linear and continuous in the norm topologies. Thus by Theorem 1, $IV.35$ of the [**INT**], $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))v \in G_\sigma$ is Lebesgue integrable $\forall v \in G_\sigma$ and

$$\int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))v\,dt = \left(\oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,dt\right) v.$$

Therefore by Definition 2.2.4 we can state that $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma)) \in B(G_\sigma)$ is Lebesgue integrable with respect to the strong operator topology on $B(G_\sigma)$ and

$$\int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,dt = \oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,dt. \tag{2.3.32}$$

Here $\int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))dt$ is the integral of $\frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))$ with respect to the Lebesgue measure on $[u_1, u_2]$ and the strong operator topology on $B(G_\sigma)$. Furthermore by statement (3) of Theorem 2.3.2

$$(R_\sigma \upharpoonright G_\sigma) \oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,dt = S(u_2(R_\sigma \upharpoonright G_\sigma)) - S(u_1(R_\sigma \upharpoonright G_\sigma)).$$

Thus by (2.3.32)

$$(R_\sigma \upharpoonright G_\sigma) \int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\,dt = S(u_2(R_\sigma \upharpoonright G_\sigma)) - S(u_1(R_\sigma \upharpoonright G_\sigma)). \tag{2.3.33}$$

Which implies (2.2.13), by choosing for example $\sigma_n \doteqdot B_n(\mathbf{0})$, for all $n \in \mathbb{N}$. Therefore by Theorem 2.2.10 we obtain the statement. $\square$

THEOREM 2.3.6 (**Strong operator Newton-Leibnitz formula**). *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighbourhood of $\sigma(R)$ and $S : U \to \mathbb{C}$ an analytic map. Assume that $\exists\, L > 0$ such that $]-L, L[\cdot U \subseteq U$ and*

(1) *$\widetilde{S}_t \in \mathfrak{L}_E^\infty(\sigma(R))$ for all $t \in]-L, L[$;*

(2)
$$\sup_{t\in]-L,L[}\left\|\widetilde{\left(\frac{d\,S}{d\,\lambda}\right)}_t\right\|_{\infty}^{E}<\infty.$$

Then

(1) $\forall u_1,u_2\in]-L,L[$

$$\boxed{R\int_{u_1}^{u_2}\frac{d\,S}{d\,\lambda}(tR)\,d\,t=S(u_2R)-S(u_1R)\in B(G).}$$

Here the integral is with respect to the Lebesgue measure on $[u_1,u_2]$ *and with respect to the strong operator topology on* $B(G)$*.*

(2) *If also* $\sup_{t\in]-L,L[}\left\|\widetilde{S}_t\right\|_{\infty}^{E}<\infty$*, then* $(\forall v\in Dom(R))(\forall t\in]-L,L[)$

$$\boxed{\frac{dS(tR)v}{dt}=R\frac{dS}{d\lambda}(tR)v.}$$

PROOF. By hypothesis (2) and statement (1) of Theorem 2.3.4 $\forall v\in G$ the map $]-L,L[\ni t\mapsto\frac{d\,S}{d\,\lambda}(tR)v\in G$ is continuous. Thus statement (1) by Corollary 2.3.5 and the fact that continuity implies measurability. Statement (2) follows by statement (2) of Theorem 2.3.4. □

REMARK 2.3.7. *We end this section by remarking that* $f:X\to B(G)$ *is* $\mu-$*integrable with respect to the strong operator topology as defined in Definition 2.2.4, if and only if* $f:X\to B(G)$ *is scalarly* $(\mu,B(G))-$*integrable with respect to the weak operator topology in the sense explained in Notations 3.2.1.*

In Chapter 3 we shall extend the results of Chapter 2 to the case of integration with respect to the measure μ *and with respect to the* $\sigma(B(G),\mathcal{N})-$*topology, where* $\mathcal{N}\subset B(G)^*$ *is a suitable linear subspace of the topological dual of* $B(G)$*.*

2.4. Application to resolvents of unbounded scalar type spectral operators in a Banach space G

COROLLARY 2.4.1. *Let* T *be a possibly unbounded scalar type spectral operator in* G *with real spectrum* $\sigma(T)$*. Then*

(1) $\forall\lambda\in\mathbb{C}\mid Im(\lambda)>0$

$$\boxed{(T-\lambda\mathbf{1})^{-1}=i\int_{-\infty}^{0}e^{-it\lambda}e^{itT}\,d\,t\in B(G).}\tag{2.4.1}$$

(2) $(\forall v \in Dom(T))(\forall t \in \mathbb{R})$

$$\frac{d\, e^{it(T-\lambda\mathbf{1})}v}{d\, t} = i(T-\lambda\mathbf{1})e^{i(T-\lambda\mathbf{1})t}v.$$

REMARK 2.4.2. *If we set the map* $(\forall \lambda \in \mathbb{C})(S(\lambda) \doteqdot \exp(i\lambda))$ *then the operator functions in Corollary 2.4.1 are so defined* $e^{itT} \doteqdot S_t(T)$ *and* $e^{it(T-\lambda\mathbf{1})} \doteqdot S_t(T-\lambda\mathbf{1})$*, in the sense of the Borelian functional calculus for the scalar type spectral operators* T *and* $(T-\lambda\mathbf{1})$*, respectively, as defined in Definition 2.1.3.*
The integral in Corollary 2.4.1 is with respect to the Lebesgue measure and with respect to the strong operator topology on $B(G)$*. Meaning by definition that*

$$\int_{-\infty}^{0} e^{-it\lambda}e^{itT}\, d\, t \in B(G)$$

such that $\forall v \in G$

$$\left(\int_{-\infty}^{0} e^{-it\lambda}e^{itT}\, d\, t\right)v \doteqdot \lim_{u\to-\infty}\left(\int_{u}^{0} e^{-it\lambda}e^{itT}\, d\, t\right)v = \lim_{u\to-\infty}\int_{u}^{0} e^{-it\lambda}e^{itT}v\, d\, t.$$

Here the integral in the right side of the first equality is with respect to the Lebesgue measure on $[u,0]$ *and with respect to the strong operator topology on* $B(G)$*.*

PROOF. Let $\lambda \in \mathbb{C}$ and set $R \doteqdot (T-\mathbf{1}\lambda)$, then R is a scalar type spectral operator, see Theorem 18.2.17. of the [**DS**]. Let $\lambda \in \mathbb{C} \mid Im(\lambda) \neq 0$ and E be the resolution of the identity of R, then $\sigma(R) = \sigma(T) - \lambda$, as as corollary of the well-known spectral mapping theorem. Then $\forall t \in \mathbb{R}$

$$\begin{aligned} E - ess \sup_{\nu\in\sigma(R)} \left|\frac{dS}{d\lambda}(t\nu)\right| &= E - ess \sup_{\nu\in\sigma(R)} |S(t\nu)| \leq \\ &\leq \sup_{\nu\in\sigma(R)} |S(t\nu)| \\ &= \sup_{\mu\in\sigma(T)} |e^{i(\mu-\lambda)t}| \\ &= e^{Im(\lambda)t}. \end{aligned}$$

Therefore are verified the hypotheses of Corollary 2.3.6 with the position $R \doteqdot (T-\lambda\mathbf{1})$, then we can state $\forall v \in G$ and $\forall u \in \mathbb{R}$ that

$$i(T-\lambda\mathbf{1})\int_{u}^{0} e^{it(T-\lambda\mathbf{1})}v\, d\, t = v - e^{iu(T-\lambda\mathbf{1})}v. \tag{2.4.2}$$

Here $e^{it(T-\lambda\mathbf{1})} \doteqdot S_t(R)$. One should note an apparent ambiguity about the symbol $e^{it(T-\lambda\mathbf{1})}$, standing here for the operator $S_t(R) = S(tR)$, which could be seen also as a Borelian function of the operator T. By setting $g^{[\lambda]}(\mu) \doteqdot \mu - \lambda$, so $g^{[\lambda]} = \imath - \lambda \cdot \mathbf{1}$ with $\mathbf{1} : \mathbb{C} \ni \lambda \mapsto 1$, considering that by the composition rule, see Theorem 18.2.24 of [**DS**], we have $S_t \circ g^{[\lambda]}(T) = S_t(g^{[\lambda]}(T))$, finally $R = \imath(T) - \lambda \mathbf{1}(T) = (\imath - \lambda \cdot \mathbf{1})(T) = g^{[\lambda]}(T)$, we can assert

$$\begin{cases} T - \lambda \mathbf{1} = g^{[\lambda]}(T) \doteqdot T - \lambda \\ e^{it(T-\lambda \mathbf{1})} \doteqdot S_t(T - \lambda \mathbf{1}) = S_t \circ g^{[\lambda]}(T) = e^{it(T-\lambda)}. \end{cases} \tag{2.4.3}$$

Therefore we can consider the operator $e^{it(T-\lambda\mathbf{1})}$ as an operator function of R or of T. Now $(\forall t \in \mathbb{R})(\sup_{\mu \in \sigma(T)} |\exp(i\mu t)| = 1)$, therefore we can deduce by statement (c) of Theorem 18.2.11. of [**DS**]

$$\sup_{t \in \mathbb{R}} \| \exp(iTt) \|_{B(G)} \leq 4M. \tag{2.4.4}$$

Here $M \doteqdot \sup_{\sigma \in \mathcal{B}(\mathbb{C})} \|E(\sigma)\|_G$. But with the notations before adopted we have $\forall \mu \in \mathbb{C}$ that $S_t \circ g^{[\lambda]}(\mu) = \exp(it(\mu - \lambda)) = \exp(-it\lambda) S_t(\mu)$, therefore by considering that $S_t(T) = S(tT)$, see (2.3.4), we have $S_t \circ g^{[\lambda]}(T) = \exp(-it\lambda) S_t(T) = \exp(-it\lambda) S(tT)$. Thus by (2.4.3) we have $\forall t \in \mathbb{R}$ and $\forall \lambda \in \mathbb{C} \mid Im(\lambda) > 0$

$$e^{it(T-\lambda\mathbf{1})} = \exp(-it\lambda) S(tT) \doteq \exp(-it\lambda) e^{itT}. \tag{2.4.5}$$

We have by (2.4.5) and (2.4.4)

$$\lim_{u \to -\infty} \|e^{iu(T-\lambda\mathbf{1})}\|_{B(G)} \leq 4M \lim_{u \to -\infty} \exp(Im(\lambda) u) = 0$$

or equivalently $\lim_{u \to -\infty} e^{iu(T-\lambda\mathbf{1})} = \mathbf{0}$ in $\| \cdot \|_{B(G)}-$topology. Hence by (2.4.2) $\forall v \in G$

$$v = i \lim_{u \to -\infty} (T - \lambda \mathbf{1}) \int_u^0 e^{it(T-\lambda\mathbf{1})} v \, dt \text{ in } \| \cdot \|_G. \tag{2.4.6}$$

By considering that $Im(\lambda) \neq 0$ we have $\{\mu \in \mathbb{C} \mid g^{[\lambda]}(\mu) = 0\} \cap \sigma(T) = \emptyset$, therefore if we denote with F the resolution of the identity of the spectral operator T, we have $F(\sigma(T)) = \mathbf{1}$ so $F(\{\mu \in \mathbb{C} \mid g^{[\lambda]}(\mu) = 0\}) = F(\{\mu \in \mathbb{C} \mid g^{[\lambda]}(\mu) = 0\} \cap \sigma(T)) = F(\emptyset) \doteqdot \mathbf{0}$. Thus by applying statement (h) of Theorem 18.2.11. of [**DS**], we can assert that

$$\exists\, (T - \lambda)^{-1} = \frac{1}{g^{[\lambda]}}(T) \doteq \frac{1}{T - \lambda}.$$

Finally $F - ess\sup_{\mu\in\sigma(T)}|\frac{1}{g^{[\lambda]}(\mu)}| \leq \sup_{\mu\in\sigma(T)}|\frac{1}{g^{[\lambda]}(\mu)}| = \sup_{\mu\in\sigma(T)}|\frac{1}{\mu-\lambda}| = \frac{1}{\inf_{\mu\in\sigma(T)}|(\mu-\lambda)|} \leq \frac{1}{|Im(\lambda)|} < \infty$, so

$$\frac{1}{g^{[\lambda]}}(T) \in B(G).$$

Hence by the previous equation and the fact $T - \lambda = T - \lambda\mathbf{1}$, see (2.4.3), we can state

$$(T - \lambda\mathbf{1})^{-1} \in B(G).$$

Finally by following a standard argument, see for example [**LN**], by this one and (2.4.6) we can deduce $\forall v \in G$ that

$$\begin{aligned}(T - \lambda\mathbf{1})^{-1}v &= i \lim_{u\to-\infty}(T - \lambda\mathbf{1})^{-1}(T - \lambda\mathbf{1})\int_u^0 e^{it(T-\lambda\mathbf{1})}v\,dt \\ &= i \lim_{u\to-\infty}\int_u^0 e^{it(T-\lambda\mathbf{1})}v\,dt.\end{aligned}$$

So statement (1) by (2.4.5).

By (2.4.5), the fact that $S_t(T) = S(tT)$ and statement (2) of Theorem 2.3.4 applied to the operator T and to the map $S : \mathbb{C} \ni \mu \mapsto e^{i\mu}$, we obtain statement (2). □

REMARK 2.4.3. *An important application of this formula is made in proving the well-known Stone theorem for strongly continuous semigroups of unitary operators in Hilbert space, see Theorem* 12.6.1. *of* [**DS**]. *In* [**Dav**] *it has been used for showing the equivalence of uniform convergence in strong operator topology of a one-parameter semigroup depending on a parameter and the convergence in strong operator topology of the resolvents of the corresponding generators, Theorem* 3.17..
Notice that if $\zeta \doteqdot -i\lambda$ *and* $Q \doteqdot iT$, *then the equality* (2.4.1) *turns into*

$$(Q + \zeta\mathbf{1})^{-1} = \int_0^\infty e^{-t\zeta}e^{-Qt}\,dt,$$

which is referred in $IX.1.3.$ *of* [**Kat**] *as the fact that the resolvent of* Q *is the* Laplace *transform of the semigroup* e^{-Qt}. *Applications of this formula to perturbation theory are in* $IX.2.$ *of* [**Kat**].

CHAPTER 3

Extension theorem. The case of the topology $\sigma(B(G), \mathcal{N})$

3.1. Introduction

Let R be an unbounded scalar type spectral operator R in a complex Banach space G and E its resolution of identity. The main results of this chapter and of the thesis are of two types.

The results of the first type are Extension Theorems for integration with respect to the $\sigma(B(G), \mathcal{N})-$topology, when $\mathcal{N}$ is an $E-$appropriate set: Theorems 3.4.2 and when $\mathcal{N}$ is an $E-$appropriate set with the duality property: Corollary 3.4.3.

As an application we will prove, by using (3.3.19), the Extension theorems for the integration with respect to the sigma-weak topology: Corollary 3.4.5 and Corollary 3.4.6, and for integration with respect to the weak operator topology: Corollary 3.4.4, and Corollary 3.4.7.

The results of the second type are Newton-Leibnitz formulas for integration with respect to the $\sigma(B(G), \mathcal{N})-$topology, when $\mathcal{N}$ is an $E-$appropriate set with the duality property: Corollary 3.5.1 and Corollary 3.5.2; for integration with respect to the sigma-weak topology: Corollary 3.5.3; for integration with respect to the weak operator topology: Corollary 3.5.4

For obtaining the Extension Theorem 3.4.2 we need to introduce the concept of $E-$appropriate set, Definition 3.3.5, which allows us to establish two important properties for the proof of Theorem 3.4.2, namely the "Commutation" property, Theorem 3.3.7, and the "Restriction" property, Theorem 3.3.16.

Finally for obtaining Corollary 3.4.3 and the Newton-Leibnitz formula in Corollary 3.5.1 we have to introduce the concept of an $E-$appropriate set $\mathcal{N}$ with the duality property, Definition 3.3.5, which allows us to establish conditions ensuring that a map is scalarly essentially $(\mu, B(G))-$integrable with respect to the $\sigma(B(G), \mathcal{N})-$topology, Theorem 3.2.2. Similar results for the weak operator topology are contained in Theorem 3.2.5 and Corollary 3.2.6.

3.2. Existence of the weak-integral with respect to the $\sigma(B(G),\mathcal{N})-$ topology

In this section we shall obtain a general result, Theorem 3.2.2 about conditions ensuring that a map is scalarly essentially $(\mu, B(G))-$integrable with respect to the $\sigma(B(G),\mathcal{N})-$topology, where $\mathcal{N}$ is a suitable subset of $B(G)^*$.

NOTATION WITH COMMENTS 3.2.1. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, Z a linear space over $\mathbb{K}$ and τ a locally convex topology on Z, then we indicate with $\langle Z, \tau\rangle$ the associated locally convex space over $\mathbb{K}$. We denote with $LCS(\mathbb{K})$ the class of all the locally convex spaces over $\mathbb{K}$ and for any $\langle Z, \tau\rangle \in LCS(\mathbb{K})$ we set $\langle Z, \tau\rangle^*$ for its topological dual, that is the $\mathbb{K}-$linear space of all $\mathbb{K}-$linear continuous functionals on Z.

Let Y be a linear space over $\mathbb{K}$ and U a subspace of $Hom(Y, \mathbb{K})$, then we indicate with the symbol $\sigma(Y, U)$ the weakest (locally convex) topology on Y such that $U \subseteq \langle Y, \sigma(Y, U)\rangle^*$, Def. 2, $II.42$ of [**TVS**], which coincides with the locally convex topology on Y generated by the set of seminorms $\Gamma(U)$ associated to U where $\Gamma(U) \doteqdot \{q_\phi : Y \ni y \mapsto |\phi(y)| \mid \phi \in U\}$.

It is not hard to see that $\sigma(Y, U)$ is the topology generated by the set of seminorms $\Gamma(S)$ for any S such that $U = \mathfrak{L}_{\mathbb{K}}(S)$, where $\mathfrak{L}_{\mathbb{K}}(S)$ is the $\mathbb{K}-$linear space generated by the set S.

By Proposition 2, $II.43$ of [**TVS**], $\sigma(Y, U)$ is a Hausdorff topology if and only if U separates the points of Y, i.e.

$$(\forall T \in Y)(T \neq \mathbf{0} \Rightarrow (\exists\, \phi \in U)(\phi(T) \neq 0)). \tag{3.2.1}$$

Also by Proposition 3, $II.43$ of [**TVS**]

$$\langle Y, \sigma(Y, U)\rangle^* = U. \tag{3.2.2}$$

Let X be a locally compact space and μ a $\mathbb{K}-$ Radon measure on X, Definition 2, §1, n°3, Ch. 3, of [**INT**] where it is called just measure. We denote with $|\mu|$ the total variation of μ, §1, n°6, Ch. 3, of [**INT**], and with $\int^*$ the upper integral with respect to a positive measure, as for example $|\mu|$, Definition 1, §1, n°1, Ch. 4, of [**INT**], With $\int^\bullet$ we denote the essential upper integral with respect to a positive measure, Definition 1, §1, n°1, Ch. 5, of [**INT**]. [1] We readdress for the definition of essentially $\mu-$integrable map $f : X \to \mathbb{K}$, to Ch. 5, §1, n°3, of [**INT**].

Let $\langle Y, \tau\rangle \in LCS(\mathbb{K})$ of Hausdorff then $f : X \to \langle Y, \tau\rangle$ is *scalarly essentially $\mu-$integrable* or equivalently $f : X \to Y$ is *scalarly essentially $\mu-$integrable with*

[1] In general $\int^\bullet \leq \int^*$, however if X is $\sigma-$compact, in particular compact, then $\int^\bullet = \int^*$.

respect to the measure μ and with respect to the $\tau-$ topology on Y if for all $\omega \in \langle Y, \tau \rangle^*$ the map $\omega \circ f : X \to \mathbb{K}$ is essentially $\mu-$integrable, so we can define its *integral* as the following linear operator

$$\langle Y, \tau \rangle^* \ni \omega \mapsto \int \omega(f(x)) \, d\mu(x) \in \mathbb{K}.$$

See Ch. 6, §1, $n°1$ for $\mathbb{K} = \mathbb{R}$, and for the extension to the case $\mathbb{K} = \mathbb{C}$ see the end of §2, $n°10$, of [**INT**].

Notice that the previous definitions depend only on the dual space $\langle Y, \tau \rangle^*$, hence both the concepts of scalar essential $\mu-$integrability and integral will be invariant if we replace τ with any other Hausdorff locally convex topology τ_2 on Y compatible with the duality $(Y, \langle Y, \tau \rangle^*)$, i.e. such that $\langle Y, \tau \rangle^* = \langle Y, \tau_2 \rangle^*$.

Therefore as a corollary of the well-known Mackey-Arens Theorem, see Theorem 1, $IV.2$ of [**TVS**] or Theorem 5 §8.5. of [**Jar**], fixed a locally convex space $\langle Y, \tau \rangle$ and denoted by $\mathcal{N} \doteqdot \langle Y, \tau \rangle^*$ its topological dual, we have that scalar essential $\mu-$integrability (respectively integral) is an invariant property (respectively functional) under the variation of any Hausdorff locally convex topology τ_1 on Y such that

$$\sigma(Y, \mathcal{N}) \leq \tau_1 \leq \tau(Y, \mathcal{N}).$$

Here $a \leq b$ means a is weaker than b and $\tau(Y, \mathcal{N})$ is the Mackey topology associated to the canonical duality $(Y, \mathcal{N})$.

Let $f : X \to \langle Y, \tau \rangle$ be scalarly essentially $\mu-$integrable and assume that

$$\boxed{(\exists\, B \in Y)(\forall \omega \in \langle Y, \tau \rangle^*) \left(\omega(B) = \int \omega(f(x)) \, d\mu(x) \right).} \tag{3.2.3}$$

Notice that by the Hahn-Banach theorem $\langle Y, \tau \rangle^*$ separates the points of Y, so the element B is defined by this condition uniquely. In this case, by definition $f : X \to \langle Y, \tau \rangle$ **is scalarly essentially** $(\mu, Y)-$**integrable** (or $f : X \to Y$ **is scalarly essentially** $(\mu, Y)-$**integrable with respect to the** $\tau-$ **topology on** Y) and its **weak-integral with respect to the measure** μ **and with respect to the** $\tau-$**topology**, or briefly its **weak-integral**, is defined by

$$\int f(x) \, d\mu(x) \doteqdot B. \tag{3.2.4}$$

In the thesis we shall use this integral for the case $\langle Y, \tau \rangle \doteqdot \langle B(G), \sigma(B(G), \mathcal{N}) \rangle$, where $\mathcal{N}$ is a linear subspace of $B(G)^*$ which separates the points of $B(G)$. Notice that by (3.2.2) $\langle B(G), \sigma(B(G), \mathcal{N}) \rangle^* = \mathcal{N}$.

Let G be a $\mathbb{K}-$normed space, then the strong operator topology $\tau_{st}(G)$ on $B(G)$ is defined to be the locally convex topology generated by the following set of seminorms $\{q_v : B(G) \ni A \mapsto \|Av\|_G \mid v \in G\}$. Hence $\tau_{st}(G)$ is a Hausdorff topology, in fact a base of the neighbourhoods of $A \in B(G)$ is the class of the sets $U_{\overline{v},\varepsilon}(A) \doteqdot \{B \in B(G) \mid \sup_{k=1,\dots,n} \|(A-B)\overline{v}_k\|_G < \varepsilon\}$, with $\overline{v}$ running in $\bigcup_{n\in\mathbb{N}} G^n$ and ϵ in $\mathbb{R}^+ - \{0\}$. So $B \in \overline{\{\mathbf{0}\}}$, the closure of $\{\mathbf{0}\}$ in the strong operator topology, if and only if $(\forall\varepsilon \in \mathbb{R}^+ - \{0\})(\forall v \in G)(\|Bv\|_G < \varepsilon)$, that is $B = \mathbf{0}$. Hence $\overline{\{\mathbf{0}\}} = \{\mathbf{0}\}$ and then $\tau_{st}(G)$ is of Hausdorff. By Ch. 6, §1, $n°3$, of [**INT**]

$$\mathcal{N}_{st}(G) \doteqdot \langle B(G), \tau_{st}(G)\rangle^* = \mathfrak{L}_{\mathbb{K}}(\{\psi_{(\phi,v)} \mid (\phi, v) \in G^* \times G\}). \tag{3.2.5}$$

Here

$$\psi_{(\phi,v)} : B(G) \ni T \mapsto \phi(Tv) \in \mathbb{K}.$$

Here if Z is a $\mathbb{K}-$linear space and $S \subseteq Z$ then $\mathfrak{L}_{\mathbb{K}}(S)$ is the space of all $\mathbb{K}-$linear combinations of elements in S.

The first locally convex space in which are mainly interested, is $\langle B(G), \sigma(B(G), \mathcal{N}_{st}(G))\rangle$, for which by (3.2.2) we have

$$\langle B(G), \sigma(B(G), \mathcal{N}_{st}(G))\rangle^* = \mathcal{N}_{st}(G). \tag{3.2.6}$$

Notice that by what said $\sigma(B(G), \mathcal{N}_{st}(G))$ is the topology on $B(G)$ generated by the set of seminorms associated to the set $\{\psi_{(\phi,v)} \mid (\phi, v) \in G^* \times G\}$, hence $\sigma(B(G), \mathcal{N}_{st}(G))$ is nothing but the usual weak operator topology on $B(G)$.

Notice that by (3.2.1), and the Hahn-Banach theorem applied to G we have that $\sigma(B(G), \mathcal{N}_{st}(G))$ is a topology of Hausdorff.

Let G be a complex Hilbert space. We define

$$\mathcal{N}_{pd}(G) \doteqdot B(G)_*.$$

Here $B(G)_*$ is the "predual" of the von Neumann algebra $B(G)$, see for example Definition $2.4.17.$ of [**BR**], or Definition $2.13.$, Ch. 2 of [**Tak**], So every $\omega \in \mathcal{N}_{pd}(G)$ has the following form, see Proposition $2.4.6$ of [**BR**] or statement $(ii.4)$ of Theorem $2.6.$, Ch. 2 of [**Tak**]

$$\omega : B(G) \ni B \mapsto \sum_{n=0}^{\infty} \langle u_n, Bw_n\rangle \in \mathbb{C}. \tag{3.2.7}$$

Here $\{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \subset G$ are such that $\sum_{n=0}^{\infty} \|u_n\|^2 < \infty$ and $\sum_{n=0}^{\infty} \|w_n\|^2 < \infty$.

We say that ω *is determined by* $\{u_n\}_{n\in\mathbb{N}}$, $\{w_n\}_{n\in\mathbb{N}}$ if (3.2.7) holds. Notice that ω is well-defined, indeed $\forall B \in B(G)$ we have $\sum_{n=0}^{\infty} |\langle u_n, Bw_n\rangle|^2 \leq \|B\|^2 \left(\sum_{n=0}^{\infty}\|u_n\|^2\right)\left(\sum_{n=0}^{\infty}\|w_n\|^2\right) < \infty$, hence $\exists\,\omega(B)$ and $\omega \in B(G)^*$, so

$$\mathcal{N}_{pd}(G) \subseteq B(G)^*. \tag{3.2.8}$$

The second locally convex space in which are mainly interested is $\langle B(G), \sigma(B(G), \mathcal{N}_{pd}(G))\rangle$, for which by (3.2.2) we have

$$\langle B(G), \sigma(B(G), \mathcal{N}_{pd}(G))\rangle^* = \mathcal{N}_{pd}(G). \tag{3.2.9}$$

By the fact that every $\omega \in \mathcal{N}_{st}(G)$ is determined by the $\{u_n\}_{n=1}^{N}$, $\{w_n\}_{n=1}^{N}$, for some $N \in \mathbb{N}$, we have that $\mathcal{N}_{st}(G) \subset \mathcal{N}_{pd}(G)$. Hence being $\sigma(B(G), \mathcal{N}_{st}(G))$ a topology of Hausdorff we can conclude by (3.2.1) that it is so also the $\sigma(B(G), \mathcal{N}_{pd}(G))-$ topology.

Notice that by what said $\sigma(B(G), \mathcal{N}_{pd}(G))$ is the topology on $B(G)$ generated by the set of seminorms associated to the set $\mathcal{N}_{pd}(G)$, hence is nothing but the usual sigma-weak operator topology on $B(G)$, see for example for its definition Section $2.4.1$ of [**BR**], so often we shall refer to it just as the sigma-weak operator topology on $B(G)$.

We want just to remark that as a corollary of the beforementioned invariance property for the weak-integration, when we change the topology τ on Y with any other Hausdorff topology compatible with it, we deduce by (3.2.5) that $f : X \to B(G)$ is scalarly essentially $(\mu, B(G))-$integrable with respect to the measure μ and with respect to the $\sigma(B(G), \mathcal{N}_{st}(G))$ topology on $B(G)$, if and only if it is so with respect to the strong topology $\tau_{st}(G)$ on $B(G)$, and in this case their weak-integrals coincide.

Let $\mathcal{A}$ be a $\mathbb{K}-$Banach algebra then $(\forall A, B \in \mathcal{A})([A, B] \doteqdot AB - BA)$, while the map $\mathcal{R} : \mathcal{A} \to B(\mathcal{A})$ and $\mathcal{L} : \mathcal{A} \to B(\mathcal{A})$, as previously defined in (1.0.1), see also (1.0.3) and (1.0.5), are such that for all $T \in \mathcal{A}$

$$\begin{cases} \mathcal{R}(T) : \mathcal{A} \ni h \mapsto Th \in \mathcal{A} \\ \mathcal{L}(T) : \mathcal{A} \ni h \mapsto hT \in \mathcal{A}. \end{cases} \tag{3.2.10}$$

Let G be a $\mathbb{K}-$Banach space and $\mathcal{N} \subseteq B(G)^*$ a linear subspace of the normed space $B(G)^*$, then we introduce the following notations

$$\begin{cases} \mathcal{N}^* \subseteq B(G) \overset{def}{\Leftrightarrow} (\exists\, Y_0 \subseteq B(G))(\mathcal{N}^* = \{\hat{A} \upharpoonright \mathcal{N} \mid A \in Y_0\}); \\ \mathcal{N}^* \overset{\|\cdot\|}{\subseteq} B(G) \overset{def}{\Leftrightarrow} \\ (\exists\, Y_0 \subseteq B(G))(\forall \phi \in \mathcal{N}^*)(\exists\, A \in Y_0)((\phi = \hat{A} \upharpoonright \mathcal{N}) \wedge (\|\phi\|_{\mathcal{N}^*} = \|A\|_{B(G)})). \end{cases}$$

Here $(\hat{\cdot}) : B(G) \to (B(G)^*)^*$ is the canonical isometric embedding of $B(G)$ into its bidual.

By statement (iii) of Theorem $2.6.$, Ch. 2 of [**Tak**], or Proposition $2.4.18$ of [**BR**]

$$\mathcal{N}_{pd}(G)^* \overset{\|\cdot\|}{\subseteq} B(G). \tag{3.2.11}$$

Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$ then we continue to follow the notation

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))(G_\sigma \doteqdot \mathbf{H}(\sigma)G),$$

without expressing the dependence on $\mathbf{H}$ everywhere it does not cause confusion.

In this Chapter we assume to be fixed a complex Banach space G, a locally compact space X a complex Radon measure μ on X, a possibly unbounded scalar type spectral operator R with spectrum $\sigma(R)$ and resolution of the the identity E.

For each map $f : U \subset \mathbb{C} \to \mathbb{C}$ we denote by $\widetilde{f}$ the $\mathbf{0}-$extension of f to $\mathbb{C}$.

Finally we shall denote with $\mathfrak{F}_{ess}(X;\mu)$ the seminormed space, with the seminorm $\|\cdot\|_{\mathfrak{F}_{ess}(X;\mu)}$, of all maps $H : X \to \mathbb{C}$ such that

$$\|H\|_{\mathfrak{F}_{ess}(X;\mu)} \doteqdot \int^\bullet |H(x)|\ d|\mu|(x) < \infty.$$

By $\mu - l.a.e.(X)$ we shall mean "locally almost everywhere on X with respect to μ". Moreover if $f : X_0 \to \mathbb{C}$ is a map defined $\mu - l.a.e.(X)$, then we convene to say that $f \in \mathfrak{F}_{ess}(X;\mu)$ if there exists a map $F : X \to \mathbb{C}$ such that $F \upharpoonright X_0 = f$ and $F \in \mathfrak{F}_{ess}(X;\mu)$. In such a case we set

$$\|f\|_{\mathfrak{F}_{ess}(X;\mu)} \doteqdot \|F\|_{\mathfrak{F}_{ess}(X;\mu)}. \tag{3.2.12}$$

(3.2.12) is well-defined since the definition is independent of which map $F \in \mathfrak{F}_{ess}(X;\mu)$ extends f, as an application of statement (a) of Proposition 1, $n°1$, $\S1$, Ch. V of [**INT**].

Moreover let $\langle Y, \tau \rangle$ be a locally convex space and $f : X_0 \to Y$ a map defined $\mu - l.a.e.(X)$, then we for brevity say that the map $f : X \to \langle Y, \tau \rangle$ is scalarly essentially

$(\mu, Y)-$integrable if there exists a map $F : X \to Y$ such that $F \upharpoonright X_0 = f$ and $F : X \to \langle Y, \tau \rangle$ is scalarly essentially $(\mu, Y)-$integrable. In this case we define

$$\boxed{\int f(x)\, d\mu(x) \doteqdot \int F(x)\, d\mu(x).} \tag{3.2.13}$$

This definition is well-defined since it does not depend by the scalarly essentially $(\mu, Y)-$integrable map F which extends f.

Indeed let for all $k \in \{1, 2\}$ the map $F_k : X \to Y$ be such that $F_k \upharpoonright X_0 = f$ and $F_k : X \to \langle Y, \tau \rangle$ be scalarly essentially $(\mu, Y)-$integrable, then $(\forall \omega \in \langle Y, \tau \rangle^*)(\forall k \in \{1, 2\})$

$$\omega\left(\int F_k(x)\, d\mu(x)\right) = \int \omega(F_k(x))\, d\mu(x) = \int \chi_{X_0}(x)\omega(F_k(x))\, d\mu(x).$$

Next $(\forall x \in X)(\chi_{X_0}(x)\omega(F_1(x)) = \chi_{X_0}(x)\omega(f(x)) = \chi_{X_0}(x)\omega(F_2(x)))$, so $\forall \omega \in \langle Y, \tau \rangle^*$

$$\omega\left(\int F_1(x)\, d\mu(x)\right) = \omega\left(\int F_2(x)\, d\mu(x)\right)$$

then by (3.2.1) follows $\int F_1(x)\, d\mu(x) = \int F_2(x)\, d\mu(x)$.

Now we will show some result about which functions are scalarly essentially $(\mu, B(G))-$integrable with respect to the $\sigma(B(G), \mathcal{N})-$topology. Here $\mathcal{N} \subseteq B(G)^*$, such that separates the points of $B(G)$ and $\mathcal{N}^* \subseteq B(G)$. Then we apply these results to the case when G is a Hilbert space and $\mathcal{N} = \mathcal{N}_{pd}(G)$.

THEOREM 3.2.2. *Let G be a complex Banach space, a subspace $\mathcal{N} \subseteq B(G)^*$ be such that $\mathcal{N}$ separates the points of $B(G)$ and*

$$\mathcal{N}^* \subseteq B(G).$$

Let $F : X \to B(G)$ be a map such that $\forall \omega \in \mathcal{N}$ the map $\omega \circ F : X \to \mathbb{C}$ is $\mu-$measurable and

$$(X \ni x \mapsto \|F(x)\|_{B(G)}) \in \mathfrak{F}_{ess}(X; \mu). \tag{3.2.14}$$

Then the map $F : X \to \langle B(G), \sigma(B(G), \mathcal{N}) \rangle$ is scalarly essentially $(\mu, B(G))-$integrable, if in addition $\mathcal{N}^* \overset{\|\cdot\|}{\subseteq} B(G)$ then its weak-integral is such that

$$\left\| \int F(x)\, d\mu(x) \right\|_{B(G)} \leq \int^{\bullet} \|F(x)\|_{B(G)}\, d|\mu|(x). \tag{3.2.15}$$

PROOF. For all $\omega \in \mathcal{N}$ we have $|\omega(F(x))| \leq \|\omega\| \|F(x)\|_{B(G)}$, hence $\forall \omega \in \mathcal{N}$

$$\int^{\bullet} |\omega(F(x))| \, d|\mu|(x) \leq \|\omega\| \int^{\bullet} \|F(x)\|_{B(G)} \, d|\mu|(x). \tag{3.2.16}$$

Moreover the map $\omega \circ F$ is $\mu-$measurable by hypothesis, therefore by (3.2.16) and Proposition 9, $\S 1$, $n^{\circ}3$, Ch. 5 of [**INT**] we have that $\omega \circ F$ is essentially $\mu-$integrable.

By this fact we can define the following map

$$\Psi : \mathcal{N} \ni \omega \mapsto \int \omega(F(x)) \, d\mu(x) \in \mathbb{C}$$

which is linear. Moreover for any essentially $\mu-$integrable map $H : X \to \mathbb{C}$

$$\left| \int H(x) \, d\mu(x) \right| \leq \int^{\bullet} |H(x)| \, d|\mu|(x), \tag{3.2.17}$$

hence by (3.2.16)

$$\Psi \in \mathcal{N}^*. \tag{3.2.18}$$

Finally by the duality property $\mathcal{N}^* \subseteq B(G)$ in hypothesis the statement follows by (3.2.18) and (3.2.16). $\square$

REMARK 3.2.3. *Let G be a complex Hilbert space, then the statement of Theorem 3.2.2 holds if we set $\mathcal{N} \doteqdot \mathcal{N}_{pd}(G)$. Indeed we have the duality property* (3.2.11)*.*

Now we give similar results for $\mathcal{N} = \mathcal{N}_{st}(G)$.

LEMMA 3.2.4. *Let G be reflexive, that is $(G^*)^*$ is isometric to G through the natural injective embedding of any normed space into its bidual. In addition let $B : G^* \times G \to \mathbb{C}$ be a bounded bilinear form, that is*

$$(\exists \, C > 0)(\forall (\phi, v) \in G^* \times G)(|B(\phi, v)| \leq C \|\phi\|_{G^*} \|v\|_G).$$

Then

$$(\exists \, ! L \in B(G))(\forall \phi \in G^*)(\forall v \in G)(B(\phi, v) = \phi(L(v))$$

and $\|L\|_{B(G)} \leq \|B\|$, where $\|B\| \doteqdot \sup_{\{(\phi,v) \mid \|\phi\|_{G^*}, \|v\|_G \leq 1\}} |B(\phi, v)|$.

PROOF. $\forall v \in G$ let $T(v) : G^* \ni \phi \mapsto B(\phi, v) \in \mathbb{C}$ so $T(v) \in (G^*)^*$ such that $\|T(v)\|_{(G^*)^*} \leq \|B\| \cdot \|v\|_G$. G is reflexive, hence $(\forall v \in G)(\exists \, ! L(v) \in G)(\forall \phi \in G^*)(\phi(L(v)) = T(v)(\phi))$, in addition $\|L(v)\|_G = \|T(v)\|_{(G^*)^*} \leq \|B\| \cdot \|v\|_G$. L is linear by the linearity of T and by the fact that G^* separates the points of G by the Hahn-Banach theorem. Thus L is linear and bounded and $\|L\|_{B(G)} \leq \|B\|$. This implies the existence of L. Let now $L' \in B(G)$ be another operator with the same property, so $(\forall \phi \in G^*)(\forall v \in G)(\phi(L(v)) = \phi(L'(v)))$, thus by the Hahn-Banach theorem $(\forall v \in G)(L(v) = L'(v))$, which shows the uniqueness. $\square$

THEOREM 3.2.5. *Let G be reflexive, $F : X \to B(G)$ be a map such that for all $(\phi, v) \in G^* \times G$ the map $X \ni x \mapsto \phi(F(x)v) \in \mathbb{C}$ is $\mu-$measurable, finally assume that* (3.2.14) *holds. Then the map $F : X \to \langle B(G), \sigma(B(G), \mathcal{N}_{st}(G))\rangle$ is scalarly essentially $(\mu, B(G))-$integrable and its weak-integral satisfies* (3.2.15).

PROOF. We have $(\forall \phi \in G^*)(\forall v \in G)$ and $\forall x \in X$ that $|\phi(F(x)v)| \leq \|\phi\|\|v\|\|F(x)\|_{B(G)}$, hence

$$\int^{\bullet} |\phi(F(x)v)| \, d|\mu|(x) \leq \|\phi\|\|v\| \int^{\bullet} \|F(x)\|_{B(G)} \, d|\mu|(x) \tag{3.2.19}$$

Furthermore the map $X \ni x \mapsto \phi(F(x)v)$ is $\mu-$measurable by hypothesis, therefore by (3.2.19) and Proposition 9, $\S 1$, $n^{\circ}3$, Ch. 5 of [**INT**] we have that $X \ni x \mapsto \phi(F(x)v)$ is essentially $\mu-$integrable.

By this fact we can define the following map

$$B : G^* \times G \ni (\phi, v) \mapsto \int \phi(F(x)v) \, d\mu(x) \in \mathbb{C},$$

which is bilinear. So by (3.2.17) and (3.2.19)

$$|B(\phi, v)| \leq \|\phi\|\|v\| \int^{\bullet} \|F(x)\|_{B(G)} \, d|\mu|(x).$$

Hence B is a bounded bilinear form whose norm is such that $\|B\| \leq \int^{\bullet} \|F(x)\|_{B(G)} \, d|\mu|(x)$, then the statement by Lemma 3.2.4. $\square$

COROLLARY 3.2.6. *Let G be reflexive, $F : X \to B(G)$ a map $\sigma(B(G), \mathcal{N}_{st}(G))-$continuous, i.e. for all $(\phi, v) \in G^* \times G$ the map $X \ni x \mapsto \phi(F(x)v) \in \mathbb{C}$ is continuous, finally assume that* (3.2.14) *holds.*

Then the map $F : X \to \langle B(G), \sigma(B(G), \mathcal{N}_{st}(G))\rangle$ is scalarly essentially $(\mu, B(G))-$integrable and its weak-integral satisfies (3.2.15).

PROOF. By definition of $\mu-$measurability we have that the continuity condition implies that $\forall(\phi, v) \in G^* \times G$ the map $X \ni x \mapsto \phi(F(x)v) \in \mathbb{C}$ is $\mu-$measurable, hence the statement by Theorem 3.2.5. $\square$

3.3. Commutation and restriction properties

Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$, then in the sequel we shall introduce a special class of subspaces of $B(G)^*$, the class of all "$\mathbf{H}-$appropriate sets", which allows one to show two important properties for proving the main Extension Theorem 3.4.2. These are

(1) "Commutation" property: Theorem 3.3.7, for a general $E-$appropriate set $\mathcal{N}$, and Corollary 3.3.8 for $\mathcal{N} = \mathcal{N}_{pd}(G)$ or $\mathcal{N} = \mathcal{N}_{st}(G)$;
(2) "Restriction" property: Theorem 3.3.16 for a general $E-$appropriate set $\mathcal{N}$.

LEMMA 3.3.1. *Let $A \in B(G)$ such that $AR \subseteq RA$ and $f \in Bor(\sigma(R))$. Then*

$$Af(R) \subseteq f(R)A.$$

PROOF. By Corollary 18.2.4. of **[DS]**

$$(\forall \sigma \in \mathcal{B}(\mathbb{C}))([A, E(\sigma)] = \mathbf{0}). \tag{3.3.1}$$

By (1.0.3) $(\forall T \in B(G))(\mathcal{R}(T), \mathcal{L}(T) \in B(B(G)))$, so by using the notations in Preliminaries 2.1.1, we have $\forall n \in \mathbb{N}$

$$\begin{aligned}
\mathbf{I}_{\mathbb{C}}^{E}(f_n)A &= \left(\mathcal{L}(A) \circ \mathbf{I}_{\mathbb{C}}^{E}\right)(f_n) \\
&= \mathbf{I}_{\mathbb{C}}^{\mathcal{L}(A)\circ E}(f_n) && \text{by (2.1.3), } \mathcal{L}(A) \in B(B(G)) \\
&= \mathbf{I}_{\mathbb{C}}^{\mathcal{R}(A)\circ E}(f_n) && \text{by (3.3.1)} \\
&= \left(\mathcal{R}(A) \circ \mathbf{I}_{\mathbb{C}}^{E}\right)(f_n) && \text{by (2.1.3), } \mathcal{R}(A) \in B(B(G)) \\
&= A\,\mathbf{I}_{\mathbb{C}}^{E}(f_n). && (3.3.2)
\end{aligned}$$

Let $x \in Dom(f(R))$ then by (2.1.7), the fact that $A \in B(G)$ and (3.3.2)

$$Af(R)x = \lim_{n\to\infty} \mathbf{I}_{\mathbb{C}}^{E}(f_n)Ax.$$

Hence (2.1.7) implies $Ax \in Dom(f(R))$ and

$$f(R)Ax = \lim_{n\to\infty} \mathbf{I}_{\mathbb{C}}^{E}(f_n)Ax = Af(R)x.$$

□

LEMMA 3.3.2. *Let $\mathcal{N} \subseteq B(G)^*$ be such that $\sigma(B(G), \mathcal{N})$ is a Hausdorff topology, $A \in B(G)$, and the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ $\mu - l.a.e.(X)$. Assume that*

(1) *the map* $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ *is scalarly essentially* $(\mu, B(G))-$*integrable;*
(2) $\phi \circ \mathcal{R}(A) \in \mathcal{N}$ *and* $\phi \circ \mathcal{L}(A) \in \mathcal{N}$, $\forall \phi \in \mathcal{N}$*;*
(3) $AR \subseteq RA$.

Then

$$\boxed{\left[\int f_x(R)\, d\mu(x),\ A\right] = \mathbf{0}.}$$

PROOF. By the hypothesis $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R)),\ \mu - l.a.e.(X)$ and statement (c) of Theorem 18.2.11. of **[DS]** applied to the scalar type spectral operator R, we have $f_x(R) \in B(G),\ \mu - l.a.e.(X)$.

Let us set $X_0 \doteqdot \{x \in X \mid f_x(R) \in B(G)\}$. By the hypothesis (1) we deduce that $\exists\, F : X \to B(G)$ such that

- $(\forall x \in X_0)(F(x) = f_x(R))$;
- $F : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable.

Thus by definition

$$\int f_x(R)\, d\mu(x) \doteqdot \int F(x)\, d\mu(x) \tag{3.3.3}$$

Notice that $(\forall x \in X)(\forall \phi \in \mathcal{N})$

$$\chi_{X_0}(x)\, \phi \circ \mathcal{L}(A)(F(x)) = \chi_{X_0}(x)\, \phi \circ \mathcal{R}(A)(F(x)), \tag{3.3.4}$$

since by Lemma 3.3.1 $\forall x \in X_0$

$$F(x)A = f_x(R)A = Af_x(R) = AF(x).$$

Moreover $\forall \phi \in \mathcal{N}$

$$\begin{cases} \int \phi \circ \mathcal{L}(A)\,(F(x))\ d\mu(x) = \int \chi_{X_0}(x)\, \phi \circ \mathcal{L}(A)\,(F(x))\ d\mu(x), \\ \int \phi \circ \mathcal{R}(A)\,(F(x))\ d\mu(x) = \int \chi_{X_0}(x)\, \phi \circ \mathcal{R}(A)\,(F(x))\ d\mu(x). \end{cases} \tag{3.3.5}$$

Indeed $\phi \circ \mathcal{L}(A) \in \mathcal{N}$ hence $X \ni x \mapsto \phi \circ \mathcal{L}(A)\,(F(x))$ is essentially $\mu-$integrable so by Proposition 9 $n^\circ 3$ $\S 1$ Ch 5 of **[INT]**

$$\int^\bullet \left|\chi_{X_0}(x)\, \phi \circ \mathcal{L}(A)\,(F(x))\right| d\,|\mu|(x) \leq \int^\bullet \left|\phi \circ \mathcal{L}(A)\,(F(x))\right| d\,|\mu|(x) < \infty.$$

Furthermore by Proposition 6 $n^\circ 2$ $\S 5$ Ch 4 of **[INT]** $X \ni x \mapsto \chi_{X_0}(x)\, \phi \circ \mathcal{L}(A)\,(F(x))$ is $\mu-$measurable. Thus by Proposition 9 $n^\circ 3$ $\S 1$ Ch 5 of **[INT]** the map $X \ni x \mapsto \chi_{X_0}(x)\, \phi \circ \mathcal{L}(A)\,(F(x))$ is essentially $\mu-$integrable and we obtain the first statement of

(3.3.5) by the fact that two essentially $\mu-$integrable maps that are equal $\mu - l.a.e.(X)$ have the same integral. In the same way it is possible to show also the second statement of (3.3.5).

Therefore $\forall\phi \in \mathcal{N}$

$$
\begin{aligned}
\phi\left(\int f_x(R)\,d\mu(x) A\right) &= \phi \circ \mathcal{L}(A)\left(\int f_x(R)\,d\mu(x)\right) \\
&= \phi \circ \mathcal{L}(A)\left(\int F(x)\,d\mu(x)\right) && \text{by (3.3.3)} \\
&= \int \phi \circ \mathcal{L}(A)\,(F(x))\ d\mu(x) && \text{by } \phi \circ \mathcal{L}(A) \in \mathcal{N} \\
&= \int \phi \circ \mathcal{R}(A)\,(F(x))\ d\mu(x) && \text{by (3.3.5), (3.3.4)} \\
&= \phi \circ \mathcal{R}(A)\left(\int F(x)\,d\mu(x)\right) && \text{by } \phi \circ \mathcal{R}(A) \in \mathcal{N} \\
&= \phi\left(A \int f_x(R)\,d\mu(x)\right). && \text{by (3.3.3)}
\end{aligned}
$$

Then the statement by (3.2.1) $\square$

REMARK 3.3.3. *By definition of* $\mathcal{N}_{st}(G)$*, see* (3.2.5)*, the hypothesis* (2) *of Lemma 3.3.2 holds* $\forall A \in B(G)$ *and for* $\mathcal{N} = \mathcal{N}_{st}(G)$*. Moreover* $\sigma(B(G), \mathcal{N}_{st}(G))$ *is a Hausdorff topology on* $B(G)$*.*

Let G *be a Hilbert space, by* (3.2.7) *we note that for all* $A \in B(G)$ *we have* $\omega \circ \mathcal{L}(A) \in \mathcal{N}_{pd}(G)$*, and* $\omega \circ \mathcal{R}(A) \in \mathcal{N}_{pd}(G)$*, indeed if* ω *is determined by* $\{u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}}$*, then* $\omega \circ \mathcal{L}(A)$*, (respectively* $\omega \circ \mathcal{R}(A)$*), is determined by* $\{u_n\}_{n\in\mathbb{N}}, \{Aw_n\}_{n\in\mathbb{N}}$*, (respectively* $\{A^* u_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}}$*). Hence the hypothesis* (2) *of Lemma 3.3.2 holds* $\forall A \in B(G)$ *and for* $\mathcal{N} = \mathcal{N}_{pd}(G)$*. Furthermore* $\sigma(B(G), \mathcal{N}_{pd}(G))$ *is a Hausdorff topology on* $B(G)$*.*

REMARK 3.3.4. *By Definition* 18.2.1 *of* [**DS**] $(\forall\sigma \in \mathcal{B}(\mathbb{C}))(E(\sigma)R \subseteq RE(\sigma))$*, thus hypothesis* (3) *of Lemma 3.3.2 holds for* $A \doteqdot E(\sigma)$*.*

DEFINITION 3.3.5 ($\mathbf{H}-$**appropriate set**). *Let* $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ *be a spectral measure in* G *on* $\mathcal{B}_Y$*, see Preliminaries 2.1.1. Then we define* $\mathcal{N}$ *to be an* $\mathbf{H}-$***appropriate set*** *, if*

(1) $\mathcal{N} \subseteq B(G)^*$ *linear subspace;*

(2) $\mathcal{N}$ *separates the points of* $B(G)$*, namely*

$$(\forall T \in B(G))(T \neq \mathbf{0} \Rightarrow (\exists\,\phi \in \mathcal{N})(\phi(T) \neq 0));$$

(3) $(\forall \phi \in \mathcal{N})(\forall \sigma \in \mathcal{B}_Y)$

$$\boxed{\phi \circ \mathcal{R}(\mathbf{H}(\sigma)) \in \mathcal{N} \quad \phi \circ \mathcal{L}(\mathbf{H}(\sigma)) \in \mathcal{N}.} \tag{3.3.6}$$

Furthermore $\mathcal{N}$ is an $\mathbf{H}-$***appropriate set with the duality property*** *if $\mathcal{N}$ is an* $\mathbf{H}-$*appropriate set such that*

$$\boxed{\mathcal{N}^* \subseteq B(G).}$$

Finally $\mathcal{N}$ is an $\mathbf{H}-$appropriate set with the isometric duality property if $\mathcal{N}$ is an $\mathbf{H}-$appropriate set such that

$$\mathcal{N}^* \overset{\|\cdot\|}{\subseteq} B(G).$$

REMARK 3.3.6. *Some comments about the previous definition. The separation property is equivalent to require that $\sigma(B(G), \mathcal{N})$ is a Hausdorff topology on $B(G)$, while* (3.3.6) *is equivalent to require that for all $\sigma \in \mathcal{B}_Y$ the maps on $B(G)$, $\mathcal{R}(\mathbf{H}(\sigma))$ and $\mathcal{L}(\mathbf{H}(\sigma))$ are continuous with respect to the $\sigma(B(G), \mathcal{N})-$ topology. Moreover the duality property $\mathcal{N}^* \subseteq B(G)$ ensures that suitable scalarly essentially $\mu-$integrable maps with respect to the $\sigma(B(G), \mathcal{N})-$ topology, are $(\mu, B(G))-$integrable, see Theorem 3.2.2.*

Finally by Remark 3.3.3 $\mathcal{N}_{st}(G)$ and $\mathcal{N}_{pd}(G)$, in case in which G is a Hilbert space, are $\mathbf{H}-$appropriate sets for any spectral measure $\mathbf{H}$, furthermore by (3.2.11)*, $\mathcal{N}_{pd}(G)$ is an $\mathbf{H}-$appropriate set with the isometric duality property.*

THEOREM 3.3.7 (**Commutation 1**). *Let $\mathcal{N}$ be an $E-$appropriate set, the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^{\infty}(\sigma(R)) \quad \mu - l.a.e.(X)$. Assume that the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable. Then $\forall \sigma \in \mathcal{B}(\mathbb{C})$*

$$\boxed{\left[\int f_x(R)\, d\mu(x),\ E(\sigma)\right] = \mathbf{0}.} \tag{3.3.7}$$

PROOF. $\mathcal{N}$ being an $E-$appropriate set ensures that hypothesis (2) of Lemma 3.3.2 is satisfied for $A \doteqdot E(\sigma)$ for all $\sigma \in \mathcal{B}(\mathbb{C})$, so the statement by Remark 3.3.4 and Lemma 3.3.2. □

COROLLARY 3.3.8 (**Commutation 2**). (3.3.7) *holds if we replace $\mathcal{N}$ in Theorem 3.3.7 with $\mathcal{N}_{st}(G)$ or with $\mathcal{N}_{pd}(G)$ and assume that G is a Hilbert space.*

PROOF. By Remark 3.3.6 and Theorem 3.3.7. □

Now we are going to present some results necessary for showing the Restriction property in Theorem 3.3.16, namely that the map $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in$

$\langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable, where $\mathcal{N}$ is a $E-$appropriate set, and $\mathcal{N}_\sigma$ could be thought as the "restriction" of $\mathcal{N}$ to $B(G_\sigma)$ for all $\sigma \in \mathcal{B}(\mathbb{C})$.

In particular when $\mathcal{N} = \mathcal{N}_{st}(G)$, respectively $\mathcal{N} = \mathcal{N}_{pd}(G)$, we can replace $\mathcal{N}_\sigma$ with $\mathcal{N}_{st}(G_\sigma)$, respectively $\mathcal{N}_{pd}(G_\sigma)$, Proposition 3.3.17.

LEMMA 3.3.9. *Let $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$, see Preliminaries 2.1.1. Then $(\forall \sigma \in \mathcal{B}_Y)(G = G_\sigma \bigoplus G_{\sigma'})$, where $\sigma' \doteqdot \complement\sigma$.*

PROOF. $\mathbf{H}(\sigma) + \mathbf{H}(\sigma') = \mathbf{H}(\sigma \cup \sigma') = \mathbf{1}$ so $\mathbf{H}(\sigma') = \mathbf{1} - \mathbf{H}(\sigma)$ and $\mathbf{H}(\sigma)\mathbf{H}(\sigma') = \mathbf{H}(\sigma')\mathbf{H}(\sigma) = \mathbf{0}$. Hence $(\forall v \in G)(v = \mathbf{H}(\sigma)v + \mathbf{H}(\sigma')v)$, or $G = G_\sigma + G_{\sigma'}$. But for any $\delta \in \mathcal{B}_Y$ we have $G_\delta = \{y \in G \mid y = \mathbf{H}(\delta)y\}$ then $G_\sigma \cap G_{\sigma'} = \{y \in G \mid y = \mathbf{H}(\sigma)\mathbf{H}(\sigma')y\} = \{\mathbf{0}\}$. Thus $G_\sigma + G_{\sigma'} = G_\sigma \bigoplus G_{\sigma'}$. $\square$

DEFINITION 3.3.10. *Let $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$, $\sigma \in \mathcal{B}_Y$ and $\sigma' \doteqdot \complement\sigma$. Then Lemma 3.3.9 allows us to define the operator $\xi_\sigma^{\mathbf{H}} : B(G_\sigma) \to B(G)$, such that $\forall T_\sigma \in B(G_\sigma)$*

$$\xi_\sigma^{\mathbf{H}}(T_\sigma) \doteqdot T_\sigma \oplus \mathbf{0}_{\sigma'} \in B(G). \tag{3.3.8}$$

Whenever it does not cause confusion we shall denote $\xi_\sigma^{\mathbf{H}}$ simply by ξ_σ. Here $\mathbf{0}_{\sigma'} \in B(G_{\sigma'})$ is the null element of the space $B(G_{\sigma'})$, while the direct sum of two operators $T_\sigma \in B(G_\sigma)$ and $T_{\sigma'} \in B(G_{\sigma'})$ is the following standard definition

$$(T_\sigma \oplus T_{\sigma'}) : G_\sigma \bigoplus G_{\sigma'} \ni (v_\sigma \oplus v_{\sigma'}) \mapsto T_\sigma v_\sigma \oplus T_{\sigma'} v_{\sigma'} \in G_\sigma \bigoplus G_{\sigma'}.$$

LEMMA 3.3.11. *Let $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ be a spectral measure in G on $\mathcal{B}_Y$, then $\forall \sigma \in \mathcal{B}_Y$ and $\forall T_\sigma \in B(G_\sigma)$ we have that*

$$\xi_\sigma^{\mathbf{H}}(T_\sigma) = T_\sigma \mathbf{H}(\sigma). \tag{3.3.9}$$

Hence $\xi_\sigma^{\mathbf{H}}$ is well-defined, injective, $\xi_\sigma^{\mathbf{H}} \in B(B(G_\sigma), B(G))$ and $\|\xi_\sigma^{\mathbf{H}}\|_{B(B(G_\sigma),B(G))} \leq \|\mathbf{H}(\sigma)\|_{B(G)}$.

PROOF. Let $\sigma \in \mathcal{B}_Y$ then $\forall v \in G$ we have

$$(T_\sigma \oplus \mathbf{0}_{\sigma'})v = (T_\sigma \oplus \mathbf{0}_{\sigma'})(\mathbf{H}(\sigma)v \oplus \mathbf{H}(\sigma')v) = (T_\sigma \mathbf{H}(\sigma)v \oplus \mathbf{0}) = T_\sigma \mathbf{H}(\sigma)v,$$

then the first part. Let $T_\sigma \in B(G_\sigma)$ such that $\xi_\sigma(T_\sigma) = \mathbf{0}$, then $T_\sigma \mathbf{H}(\sigma) = \mathbf{0}$, which implies that for all $v_\sigma \in G_\sigma$ we have $T_\sigma v_\sigma = T_\sigma \mathbf{H}(\sigma) v_\sigma = \mathbf{0}$. So $T_\sigma = \mathbf{0}_\sigma$.

Let us consider $\mathbf{H}(\sigma) \in B(G, G_\sigma)$, and $T_\sigma \in B(G_\sigma, G)$, so $T_\sigma \mathbf{H}(\sigma) \in B(G)$ and $\|T_\sigma \mathbf{H}(\sigma)\|_{B(G)} \leq \|T_\sigma\|_{B(G_\sigma,G)} \cdot \|\mathbf{H}(\sigma)\|_{B(G,G_\sigma)} = \|T_\sigma\|_{B(G_\sigma)} \cdot \|\mathbf{H}(\sigma)\|_{B(G)}$. $\square$

Notice that by (3.3.9) and the fact that $B(G_\sigma)$ is a Banach space, it is possible to show that $\xi_\sigma(B(G_\sigma))$ is a Banach subspace of $B(G)$, thus ξ_σ has a continuous inverse.

REMARK 3.3.12. *Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$, and $\sigma \in \mathcal{B}_Y$. If we consider the product space $G_\sigma \times G_{\sigma'}$ with the standard linearization and define*

$$\begin{cases} \|(x_\sigma, x_{\sigma'})\|_\oplus \doteqdot \|x_\sigma + x_{\sigma'}\|_G, \\ I : G_\sigma \times G_{\sigma'} \ni (x_\sigma, x_{\sigma'}) \mapsto x_\sigma + x_{\sigma'} \in G, \end{cases} \tag{3.3.10}$$

then by $G = G_\sigma \bigoplus G_{\sigma'}$, see Lemma 3.3.9, the two spaces $\langle G_\sigma \times G_{\sigma'}, \|\cdot\|_\oplus \rangle$ and $\langle G, \|\cdot\|_G \rangle$ are isomorphics, thus isometric and I is an isometry between them.

It is not difficult to see that the topology induced by the norm $\|\cdot\|_\oplus$ is the product topology on $G_\sigma \times G_{\sigma'}$ [2]*, which implies the following property that in any case we prefer to show directly.*

PROPOSITION 3.3.13. *Let $\mathbf{H} : \mathcal{B}_Y \to \Pr(G)$ be a spectral measure in G on $\mathcal{B}_Y$ and assume the notations in* (3.3.10) *and Definition 3.3.10. For all $T_\sigma \in B(G_\sigma)$ and $T_{\sigma'} \in B(G_{\sigma'})$ set*

$$T_\sigma \times T_{\sigma'} : G_\sigma \times G_{\sigma'} \ni (x_\sigma, x_{\sigma'}) \mapsto (T_\sigma x_\sigma, T_{\sigma'} x_{\sigma'}) \in G_\sigma \times G_{\sigma'}$$

Then

$$\begin{cases} T_\sigma \oplus T_{\sigma'} = I(T_\sigma \times T_{\sigma'})I^{-1} = T_\sigma \mathbf{H}(\sigma) + T_{\sigma'} \mathbf{H}(\sigma') \in B(G) \\ T_\sigma \times T_{\sigma'} = I^{-1}(T_\sigma \mathbf{H}(\sigma) + T_{\sigma'} \mathbf{H}(\sigma'))I \in B(G_\sigma \times G_{\sigma'}). \end{cases} \tag{3.3.11}$$

PROOF. $I(T_\sigma \times T_{\sigma'})I^{-1}(x_\sigma \oplus x_{\sigma'}) = I(T_\sigma x_\sigma, T_{\sigma'} x_{\sigma'}) = T_\sigma x_\sigma \oplus T_{\sigma'} x_{\sigma'}$, for all $x_\sigma \in G_\sigma$ and $x_{\sigma'} \in G_{\sigma'}$, so the first equality. $\forall x \in G$

$$\begin{aligned} I(T_\sigma \times T_{\sigma'})I^{-1}(x) &= I(T_\sigma \times T_{\sigma'})I^{-1}(\mathbf{H}(\sigma)x + \mathbf{H}(\sigma')x) \\ &= I(T_\sigma \mathbf{H}(\sigma)x, T_{\sigma'} \mathbf{H}(\sigma')x) \\ &= T_\sigma \mathbf{H}(\sigma)x + T_{\sigma'} \mathbf{H}(\sigma')x. \end{aligned}$$

Then the second equality. The third equality is by the second and the fact that I is an isometry. □

Notice that by the first statement in (3.3.11) we obtain (3.3.9).

[2] Indeed let $\sigma \in \mathcal{B}_Y$ such that $\mathbf{H}(\sigma) \neq \mathbf{0}$, set $M \doteqdot \max\{\|\mathbf{H}(\sigma)\|, \|\mathbf{H}(\sigma')\|\}$ and $\forall r > 0$ define $B_r^\oplus(\mathbf{0}) \doteqdot \{(x_\sigma, x_{\sigma'}) \in G_\sigma \times G_{\sigma'} \mid \|(x_\sigma, x_{\sigma'})\|_\oplus < r\}$. Thus $\forall \varepsilon > 0$ by setting $\eta \doteqdot \frac{\varepsilon}{2}$ we have $B_\eta(\mathbf{0}_\sigma) \times B_\eta(\mathbf{0}'_\sigma) \subset B_\varepsilon^\oplus(\mathbf{0})$, while $\forall \varepsilon_1, \varepsilon_2 > 0$ by setting $\zeta \doteqdot \frac{\min\{\varepsilon_1, \varepsilon_2\}}{M}$ we have $B_\zeta^\oplus(\mathbf{0}) \subset B_{\varepsilon_1}(\mathbf{0}_\sigma) \times B_{\varepsilon_2}(\mathbf{0}'_\sigma)$.

DEFINITION 3.3.14. *Let* $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ *be a spectral measure in* G *on* $\mathcal{B}_Y$ *and* $\mathcal{N} \subseteq B(G)^*$. *We define* $(\forall \sigma \in \mathcal{B}_Y)(\forall \psi \in \mathcal{N})$

$$\begin{cases} \psi_\sigma^{\mathbf{H}} \doteqdot \psi \circ \xi_\sigma^{\mathbf{H}} \in B(G_\sigma)^* \\ \mathcal{N}_\sigma^{\mathbf{H}} \doteqdot \{\psi_\sigma^{\mathbf{H}} \mid \psi \in \mathcal{N}\}, \end{cases} \tag{3.3.12}$$

where $\xi_\sigma^{\mathbf{H}}$ *has been defined in* (3.3.8). *We shall express* $\psi_\sigma^{\mathbf{H}}$ *and* $\mathcal{N}_\sigma^{\mathbf{H}}$ *simply by* ψ_σ *and* $\mathcal{N}_\sigma$ *respectively, whenever it does not cause confusion.*

PROPOSITION 3.3.15. *Let* $\mathbf{H} : \mathcal{B}_Y \to \mathrm{Pr}(G)$ *be a spectral measure in* G *on* $\mathcal{B}_Y$, $\mathcal{N} \subseteq B(G)^*$ *such that* $\mathcal{N}$ *separates the points of* $B(G)$ *and* $\sigma \in \mathcal{B}_Y$. *Then* $\mathcal{N}_\sigma$ *separates the points of* $B(G_\sigma)$.

PROOF. Let $T_\sigma \in B(G_\sigma) - \{\mathbf{0}_\sigma\}$, by Lemma 3.3.11 ξ_σ is injective so $\xi_\sigma(T_\sigma) \neq \mathbf{0}$. But $\mathcal{N}$ separates the points of $B(G)$, so $(\exists\, \psi \in \mathcal{N})(\psi(\xi_\sigma(T_\sigma)) \neq 0)$. □

THEOREM 3.3.16 (**Restriction**). *Let* $\mathcal{N}$ *be an* $E-$*appropriate set, the map* $X \ni x \mapsto f_x \in Bor(\sigma(R))$ *be such that* $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R)) \quad \mu - l.a.e.(X)$ *Assume that the map* $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ *is scalarly essentially* $(\mu, B(G))-$*integrable.*

Then $\forall \sigma \in \mathcal{B}(\mathbb{C})$ *the map* $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ *is scalarly essentially* $(\mu, B(G_\sigma))-$*integrable and*

$$\boxed{\int f_x(R_\sigma \upharpoonright G_\sigma)\, d\mu(x) = \int f_x(R)\, d\mu(x) \upharpoonright G_\sigma.} \tag{3.3.13}$$

PROOF. Let $\sigma \in \mathcal{B}(\mathbb{C})$ then (2.1.15) implies that $\forall x \in X$ the operator $f_x(R_\sigma \upharpoonright G_\sigma)$ is well-defined.

By the hypothesis $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$, $\mu - l.a.e.(X)$ and statement (c) of Theorem 18.2.11. of [**DS**] applied to the scalar type spectral operator R, we have $f_x(R) \in B(G)$, $\mu - l.a.e.(X)$. Let us set

$$X_0 \doteqdot \{x \in X \mid f_x(R) \in B(G)\},$$

thus by statement (2) of Lemma 2.1.7 we obtain

$$(\forall x \in X_0)(f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)). \tag{3.3.14}$$

Hence $f_x(R_\sigma \upharpoonright G_\sigma) \in B(G_\sigma)$, $\mu - l.a.e.(X)$. So by Proposition 3.3.15 and (3.2.1) it is well-defined the statement that $X \ni x \mapsto f_x(R_\sigma \upharpoonright G_\sigma) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable.

By hypothesis we deduce that $\exists\, F : X \to B(G)$ such that

- $(\forall x \in X_0)(F(x) = f_x(R))$;
- $F : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable.

Thus by (3.2.13)

$$\int f_x(R)\, d\mu(x) \doteqdot \int F(x)\, d\mu(x) \tag{3.3.15}$$

Now $\forall \sigma \in \mathcal{B}(\mathbb{C})$ let us define the map $F^\sigma : X \to B(G_\sigma)$ such that $\forall x \in X$

$$F^\sigma(x) \doteqdot E(\sigma)F(x) \upharpoonright G_\sigma.$$

By (3.3.14) we can claim that

(1) $(\forall x \in X_0)(F^\sigma(x) = f_x(R_\sigma \upharpoonright G_\sigma))$;
(2) the map $F^\sigma : X \to \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable, moreover

$$\int F^\sigma(x)\, d\mu(x) = \int f_x(R)\, d\mu(x) \upharpoonright G_\sigma. \tag{3.3.16}$$

Then the statement will follow by setting according (3.2.13)

$$\int f_x(R_\sigma \upharpoonright G_\sigma)\, d\mu(x) \doteqdot \int F^\sigma(x)\, d\mu(x).$$

$\forall x \in X_0$

$$\begin{aligned} F^\sigma(x) &= E(\sigma)f_x(R) \upharpoonright G_\sigma \\ &= f_x(R)E(\sigma) \upharpoonright G_\sigma \text{ by } [f_x(R), E(\sigma)] = \mathbf{0} \\ &= f_x(R_\sigma \upharpoonright G_\sigma) \qquad \text{by Key Lemma 2.1.7.} \end{aligned}$$

Hence (1) of our claim follows.

$(\forall \psi \in \mathcal{N})(\forall x \in X)$

$$\begin{aligned} \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma))\,(F(x)) &\doteq \psi\,(E(\sigma)F(x)E(\sigma)) \\ &= \psi_\sigma\,(E(\sigma)F(x) \upharpoonright G_\sigma) \\ &\doteq \psi_\sigma\,(F^\sigma(x))\,. \end{aligned} \tag{3.3.17}$$

Here in the second equality we deduce by Lemma 3.3.11 that for all $T \in B(G)$ we have $\xi_\sigma\,(E(\sigma)T \upharpoonright G_\sigma) = E(\sigma)TE(\sigma)$.

$F : X \to \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially $\mu-$integrable, and $(\forall \psi \in \mathcal{N})(\psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma)) \in \mathcal{N})$, hence by (3.3.17) the map

$$F^\sigma : X \to \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle \text{ is scalarly essentially } \mu-\text{integrable.}$$

Now by (3.3.7) we have $\forall v \in G_\sigma$

$$\int f_x(R)\,d\mu(x)v = \int f_x(R)\,d\mu(x)E(\sigma)v = E(\sigma)\int f_x(R)\,d\mu(x)v \in G_\sigma, \tag{3.3.18}$$

moreover $\int f_x(R)\,d\mu(x) \in B(G)$ so

$$\int f_x(R)\,d\mu(x) \upharpoonright G_\sigma \in B(G_\sigma).$$

Therefore $\forall \psi \in \mathcal{N}$

$$\begin{aligned}
&\psi_\sigma\left(\int f_x(R)\,d\mu(x) \upharpoonright G_\sigma\right)\\
&= \psi_\sigma\left(E(\sigma)\int f_x(R)\,d\mu(x) \upharpoonright G_\sigma\right) \quad \text{by (3.3.18)}\\
&= \psi\left(E(\sigma)\int f_x(R)\,d\mu(x)\,E(\sigma)\right) \quad \text{by Lemma 3.3.11}\\
&\doteq \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma))\left(\int f_x(R)\,d\mu(x)\right)\\
&\doteq \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma))\left(\int F(x)\,d\mu(x)\right) \quad \text{by (3.3.15)}\\
&= \int \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma))\,(F(x))\;d\mu(x) \;\; \text{by } \psi \circ \mathcal{L}(E(\sigma)) \circ \mathcal{R}(E(\sigma)) \in \mathcal{N}\\
&= \int \psi_\sigma\left(F^\sigma(x)\right)\,d\mu(x) \;\; \text{by (3.3.17).}
\end{aligned}$$

Hence (3.3.16) by (3.2.3) and (3.2.4) and the statement follows. □

PROPOSITION 3.3.17. $\forall \sigma \in \mathcal{B}(\mathbb{C})$

$$\boxed{(\mathcal{N}_{st}(G))_\sigma = \mathcal{N}_{st}(G_\sigma) \text{ and } (\mathcal{N}_{pd}(G))_\sigma = \mathcal{N}_{pd}(G_\sigma);} \tag{3.3.19}$$

PROOF. By the Hahn-Banach theorem

$$(G_\sigma)^* = \{\phi \upharpoonright G_\sigma \mid \phi \in G^*\}. \tag{3.3.20}$$

Then we have

$$\begin{aligned}
(\mathcal{N}_{st}(G))_\sigma &\doteq \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\phi,v)} \circ \xi_\sigma \mid (\phi, v) \in G^* \times G\})\\
&= \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\phi \upharpoonright G_\sigma, w)} \mid (\phi, w) \in G^* \times G_\sigma\})\\
&= \mathfrak{L}_{\mathbb{C}}(\{\psi_{(\rho, w)} \mid (\rho, w) \in (G_\sigma)^* \times G_\sigma\})\\
&\doteq \mathcal{N}_{st}(G_\sigma).
\end{aligned}$$

Here in the third equality we used (3.3.20), while in the second equality we considered that $\forall(\phi, v) \in G^* \times G$ and $\forall T_\sigma \in B(G_\sigma)$

$$
\begin{aligned}
\psi_{(\phi,v)} \circ \xi_\sigma(T_\sigma) &= \phi(T_\sigma E(\sigma) v) \qquad\qquad \text{by (3.3.9)} \\
&= (\phi \upharpoonright G_\sigma)\, (T_\sigma E(\sigma) v) \\
&= \psi_{(\phi \upharpoonright G_\sigma, E(\sigma) v)}(T_\sigma).
\end{aligned} \tag{3.3.21}
$$

Let G be a complex Hilbert space then

$$
\begin{aligned}
(\mathcal{N}_{pd}(G))_\sigma &= \left\{ \left(\sum_{n=0}^{\infty} \psi_{(u_n^\dagger, w_n)} \right) \circ \xi_\sigma \middle| \{u_n\}_{n\in\mathbb{N}},\ \{w_n\}_{n\in\mathbb{N}} \subset G, \sum_{n=0}^{\infty} \|u_n\|_G^2 < \infty, \sum_{n=0}^{\infty} \|w_n\|_G^2 < \infty \right\} \\
&= \left\{ \sum_{n=0}^{\infty} (\psi_{(u_n^\dagger, w_n)} \circ \xi_\sigma) \middle| \{u_n\}_{n\in\mathbb{N}},\ \{w_n\}_{n\in\mathbb{N}} \subset G, \sum_{n=0}^{\infty} \|u_n\|_G^2 < \infty, \sum_{n=0}^{\infty} \|w_n\|_G^2 < \infty \right\} \\
&= \left\{ \sum_{n=0}^{\infty} \psi_{(u_n^\dagger \upharpoonright G_\sigma, E(\sigma) w_n)} \middle| \{u_n\}_{n\in\mathbb{N}},\ \{w_n\}_{n\in\mathbb{N}} \subset G, \sum_{n=0}^{\infty} \|u_n\|_G^2 < \infty, \sum_{n=0}^{\infty} \|w_n\|_G^2 < \infty \right\} \\
&= \left\{ \sum_{n=0}^{\infty} \psi_{((E(\sigma)^* u_n)^\dagger \upharpoonright G_\sigma, E(\sigma) w_n)} \middle| \{u_n\}_{n\in\mathbb{N}},\ \{w_n\}_{n\in\mathbb{N}} \subset G, \sum_{n=0}^{\infty} \|u_n\|_G^2 < \infty, \sum_{n=0}^{\infty} \|w_n\|_G^2 < \infty \right\} \\
&= \left\{ \sum_{n=0}^{\infty} \psi_{(a_n^\dagger, b_n)} \middle| \{a_n\}_{n\in\mathbb{N}},\ \{b_n\}_{n\in\mathbb{N}} \subset G_\sigma, \sum_{n=0}^{\infty} \|a_n\|_{G_\sigma}^2 < \infty, \sum_{n=0}^{\infty} \|b_n\|_{G_\sigma}^2 < \infty \right\} \\
&= \mathcal{N}_{pd}(G_\sigma).
\end{aligned}
$$

Here for any Hilbert space F we set $u^\dagger \in F^*$ such that $u^\dagger(v) \doteqdot \langle u, v \rangle$ for all $u, v \in F$, and the series in the first equality is converging with respect to the strong operator topology on $B(G)^*$, while all the others are converging with respect to the strong operator topology on $B(G_\sigma)^*$.

The first equality follows by 3.2.7, the third is by (3.3.21), the forth by the fact that $E(\sigma) \upharpoonright G_\sigma = \mathbf{1}_\sigma$ the identity operator on G_σ.

Now we shall show the fifth equality. Notice that

$$
\sum_{n=0}^{\infty} \|E(\sigma) w_n\|_{G_\sigma}^2 \doteq \sum_{n=0}^{\infty} \|E(\sigma) w_n\|_G^2 \leq \|E(\sigma)\|^2 \sum_{n=0}^{\infty} \|w_n\|_G^2 < \infty.
$$

While the fact that $\dagger : H \to H^*$ is an antisometry, we have for all $n \in \mathbb{N}$ that exists only one $a_n \in G_\sigma$ such that $a_n^\dagger = (E(\sigma)^* u_n)^\dagger \upharpoonright G_\sigma$ moreover

$$
\|a_n\|_{G_\sigma} = \|\, (E(\sigma)^* u_n)^\dagger \upharpoonright G_\sigma\|_{G_\sigma^*}.
$$

Next

$$\begin{aligned}\|\,(E(\sigma)^* u_n)^\dagger \upharpoonright G_\sigma\|_{G_\sigma^*} &= \sup_{\{v\in G_\sigma \mid \|v\|_{G_\sigma}\leq 1\}} |\,\langle E(\sigma)^* u_n, v\rangle\,| \\ &= \sup_{\{v\in G_\sigma \mid \|v\|_{G_\sigma}\leq 1\}} |\,\langle u_n, v\rangle\,| \leq \sup_{\{v\in G \mid \|v\|_{G}\leq 1\}} |\,\langle u_n, v\rangle\,| \\ &= \|u_n^\dagger\|_{G^*} = \|u_n\|_G.\end{aligned}$$

Hence $\sum_{n=0}^\infty \|a_n\|_{G_\sigma}^2 \leq \sum_{n=0}^\infty \|u_n\|_G^2 < \infty$ and the fifth equality follows. □

3.4. Extension theorem for integral equalities with respect to the $\sigma(B(G),\mathcal{N})-$topology

In the present section wil shall prove the Extension Theorems for integration with respect to the $\sigma(B(G),\mathcal{N})-$topology, when $\mathcal{N}$ is an $E-$appropriate set: Theorems 3.4.2 and when $\mathcal{N}$ is an $E-$appropriate set with the duality property: Corollary 3.4.3.

As an application we shall consider the cases of the sigma-weak topology: Corollary 3.4.5 and Corollary 3.4.6; and weak operator topology: Corollary 3.4.4, and Corollary 3.4.7.

In this section it will be adopted all the notations defined in Section 3.2.

THEOREM 3.4.1. *Let $\mathcal{N}$ be an $E-$appropriate set and $\{\sigma_n\}_{n\in\mathbb{N}}$ be an $E-$sequence (see Definition 2.2.2) and the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ $\mu - l.a.e.(X)$. Let $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle$ be scalarly essentially $(\mu, B(G))-$integrable and $g, h \in Bor(\sigma(R))$.*

If $\forall n \in \mathbb{N}$

$$g(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \int f_x(R_{\sigma_n} \upharpoonright G_{\sigma_n})\, d\,\mu(x) \subseteq h(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \tag{3.4.1}$$

then

$$g(R) \int f_x(R)\, d\,\mu(x) \upharpoonright \Theta = h(R) \upharpoonright \Theta. \tag{3.4.2}$$

In (3.4.1) *the weak-integral is with respect to the measure μ and with respect to the $\sigma(B(G_{\sigma_n}),\mathcal{N}_{\sigma_n})-$ topology, while in* (3.4.2)

$$\Theta \doteqdot Dom\left(g(R) \int f_x(R)\, d\,\mu(x)\right) \cap Dom(h(R))$$

and the weak-integral is with respect to the measure μ and with respect to the $\sigma(B(G),\mathcal{N})-$ topology.

PROOF. (3.4.1) is well set since Theorem 3.3.16.

By (2.2.1) for all $y \in \Theta$

$$g(R) \int f_x(R)\, d\,\mu(x)\, y = \lim_{n \in \mathbb{N}} E(\sigma_n) g(R) \int f_x(R)\, d\,\mu(x)\, y$$

by statement (g) of Theorem $18.2.11$ of [**DS**] and (3.3.7)

$$= \lim_{n \in \mathbb{N}} g(R) \int f_x(R)\, d\,\mu(x)\, E(\sigma_n) y$$

by (3.3.13) and Lemma 2.1.7 applied to $g(R)$

$$= \lim_{n \in \mathbb{N}} g(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \int f_x(R_{\sigma_n} \upharpoonright G_{\sigma_n})\, d\,\mu(x)\, E(\sigma_n) y$$

by hypothesis (3.4.1)

$$= \lim_{n \in \mathbb{N}} h(R_{\sigma_n} \upharpoonright G_{\sigma_n})\, E(\sigma_n) y$$

by Lemma 2.1.7 and statement (g) of Theorem $18.2.11$ of [**DS**]

$$= \lim_{n \in \mathbb{N}} E(\sigma_n) h(R)\, y$$

$$= h(R)\, y. \tag{3.4.3}$$

In the last equality we considered (2.2.1). $\square$

THEOREM 3.4.2 ($\sigma(B(G), \mathcal{N})-$ **Extension Theorem**). *Let R be a possibly unbounded scalar type spectral operator in G, E its resolution of the identity, $\mathcal{N}$ an $E-$appropriate set. Let the map $X \ni x \mapsto f_x \in Bor(\sigma(R))$ be such that $\widetilde{f_x} \in \mathfrak{L}_E^\infty(\sigma(R))$ $\mu - l.a.e.(X)$ and the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ be scalarly essentially $(\mu, B(G))-$integrable. Finally let $g, h \in Bor(\sigma(R))$ and $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$.*

If $\{\sigma_n\}_{n \in \mathbb{N}}$ is an $E-$sequence and $\forall n \in \mathbb{N}$

$$g(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \int f_x(R_{\sigma_n} \upharpoonright G_{\sigma_n})\, d\,\mu(x) \subseteq h(R_{\sigma_n} \upharpoonright G_{\sigma_n}) \tag{3.4.4}$$

then $h(R) \in B(G)$ and

$$\boxed{g(R) \int f_x(R)\, d\,\mu(x) = h(R).} \tag{3.4.5}$$

In (3.4.4) *the weak-integral is with respect to the measure* μ *and with respect to the* $\sigma(B(G_{\sigma_n}), \mathcal{N}_{\sigma_n})-$ *topology, while in* (3.4.5) *the weak-integral is with respect to the measure* μ *and with respect to the* $\sigma(B(G), \mathcal{N})-$ *topology.*

Notice that $g(R)$ is a possibly **unbounded** operator in G.

PROOF. Theorem 18.2.11. of **[DS]** and hypothesis $\widetilde{h} \in \mathfrak{L}_E^\infty(\sigma(R))$ imply that $h(R) \in B(G)$, so by (3.4.2) we can deduce

$$g(R) \int f_x(R)\, d\,\mu(x) \subseteq h(R). \tag{3.4.6}$$

Let us set

$$(\forall n \in \mathbb{N})(\delta_n \doteqdot \overset{-1}{|g|}([0, n])). \tag{3.4.7}$$

We claim that

$$\begin{cases} \bigcup_{n\in\mathbb{N}} \delta_n = \sigma(R) \\ n \geq m \Rightarrow \delta_n \supseteq \delta_m \\ (\forall n \in \mathbb{N})(g(\delta_n) \text{ is bounded. }) \end{cases} \tag{3.4.8}$$

Since $|g| \in Bor(\sigma(R))$ we have $\delta_n \in \mathcal{B}(\mathbb{C})$ for all $n \in \mathbb{N}$, so $\{\delta_n\}_{n\in\mathbb{N}}$ is an $E-$sequence, hence by (2.2.1)

$$\lim_{n\in\mathbb{N}} E(\delta_n) = \mathbf{1} \tag{3.4.9}$$

with respect to the strong operator topology on $B(G)$.

Indeed the first equality follows by $\bigcup_{n\in\mathbb{N}} \delta_n \doteq \bigcup_{n\in\mathbb{N}} \overset{-1}{|g|}([0, n]) = \overset{-1}{|g|}\left(\bigcup_{n\in\mathbb{N}}[0, n]\right) = \overset{-1}{|g|}(\mathbb{R}^+) = Dom(g) \doteqdot \sigma(R)$, the second by the fact that $\overset{-1}{|g|}$ preserves the inclusion, the third by the inclusion $|g|(\delta_n) \subseteq [0, n]$. Hence our claim.

By the third statement of (3.4.8), $\delta_n \in \mathcal{B}(\mathbb{C})$ and statement 3 of Lemma 2.1.7

$$(\forall n \in \mathbb{N})(E(\delta_n)G \subseteq Dom(g(R))). \tag{3.4.10}$$

By (3.3.7) and (3.4.10) for all $n \in \mathbb{N}$

$$\int f_x(R)\, d\,\mu(x) E(\delta_n)G \subseteq E(\delta_n)G \subseteq Dom(g(R)).$$

Therefore

$$(\forall n \in \mathbb{N})(\forall v \in G)\left(E(\delta_n)v \in Dom\left(g(R) \int f_x(R)\, d\,\mu(x)\right)\right).$$

Hence by (3.4.9)

$$\mathbf{D} \doteqdot Dom\left(g(R)\int f_x(R)\,d\,\mu(x)\right) \text{ is dense in } G. \tag{3.4.11}$$

But $\int f_x(R)\,d\,\mu(x) \in B(G)$ and $g(R)$ is closed by Theorem 18.2.11. of [**DS**], so by Lemma 2.2.7 we have

$$g(R)\int f_x(R)\,d\,\mu(x) \text{ is closed.} \tag{3.4.12}$$

But we know that $h(R) \in B(G)$ so by (3.4.6) we deduce

$$g(R)\int f_x(R)\,d\,\mu(x) \in B(\mathbf{D}, G). \tag{3.4.13}$$

The (3.4.12), (3.4.13) and Lemma 2.2.8 allow us to state that $\mathbf{D}$ is closed in G, therefore by (3.4.11)

$$\mathbf{D} = G.$$

Hence by (3.4.6) we can conclude that the statement holds. $\square$

Now we shall show a corollary of the previous theorem, in which we give conditions on the maps f_x ensuring that $f_x(R) \in B(G)$, and that $X \ni x \mapsto f_x(R) \in B(G)$ is scalarly essentially $(\mu, B(G))-$integrable with respect to the $\sigma(B(G), \mathcal{N})-$ topology.

COROLLARY 3.4.3 ($\sigma(B(G), \mathcal{N})-$ **Extension Theorem. Duality case.**). *Let $\mathcal{N}$ be an $E-$appropriate set with the duality property and $X \ni x \mapsto f_x \in Bor(\sigma(R))$. Assume that $\exists\, X_0 \subseteq X$ such that $\complement X_0$ is $\mu-$locally negligible and $\widetilde{f_x} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ for all $x \in X_0$, moreover let there exists $F : X \to B(G)$ extending $X_0 \ni x \mapsto f_x(R) \in B(G)$ such that $\forall \omega \in \mathcal{N}$ the map $X \ni x \mapsto \omega(F(x)) \in \mathbb{C}$ is $\mu-$measurable and*

$$(X \ni x \mapsto \|F(x)\|_{B(G)}) \in \mathfrak{F}_{ess}(X;\mu). \tag{3.4.14}$$

If $g, h \in Bor(\sigma(R))$ is such that $\widetilde{h} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ is an $E-$sequence such that $\forall n \in \mathbb{N}$ holds (3.4.4) *then the statement of Theorem 3.4.2 holds. Moreover if $\mathcal{N}$ is an $E-$appropriate set with the isometric duality property*

$$\left\|\int f_x(R)\,d\mu(x)\right\|_{B(G)} \leq \int^{\bullet} \|f_x(R)\|_{B(G)}\,d|\mu|(x).$$

PROOF. By the duality property of hypothesis, and Theorem 3.2.2 the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N})\rangle$ is scalarly essentially $(\mu, B(G))-$integrable. Hence the first part of the statement by Theorem 3.4.2. The inequality follows by (3.2.15), (3.2.13) and (3.2.12). $\square$

Now we will give the corollaries of the previous two results in the cases in which $\mathcal{N} = \mathcal{N}_{st}(G)$ or $\mathcal{N} = \mathcal{N}_{pd}(G)$ and G be a Hilbert space.

COROLLARY 3.4.4. *The statement of Theorem 3.4.2 holds if $\mathcal{N}$ is replaced by $\mathcal{N}_{st}(G)$ and $\mathcal{N}_{\sigma_n}$ is replaced by $\mathcal{N}_{st}(G_{\sigma_n})$, for all $n \in \mathbb{N}$.*

PROOF. By Remark 3.3.6 we know that $\mathcal{N}_{st}(G)$ is an $E-$appropriate set, therefore the statement by (3.3.19) and Theorem 3.4.2. $\square$

COROLLARY 3.4.5. *The statement of Theorem 3.4.2 holds if G is a complex Hilbert space, $\mathcal{N}$ is replaced by $\mathcal{N}_{pd}(G)$ and $\mathcal{N}_{\sigma_n}$ is replaced by $\mathcal{N}_{pd}(G_{\sigma_n})$, for all $n \in \mathbb{N}$.*

PROOF. By Remark 3.3.6 we know that $\mathcal{N}_{pd}(G)$ is in particular an $E-$appropriate set, therefore the statement by (3.3.19) and Theorem 3.4.2. $\square$

THEOREM 3.4.6 (**Sigma-weak Extension Theorem**). *Let G be a Hilbert space, then the statement of Corollary 3.4.3 holds if we set $\mathcal{N} \doteqdot \mathcal{N}_{pd}(G)$ and $\mathcal{N}_{\sigma_n} \doteqdot \mathcal{N}_{pd}(G_{\sigma_n})$ for all $n \in \mathbb{N}$.*

PROOF. By Remark 3.3.6 $\mathcal{N}_{pd}(G)$ is an $E-$appropriate set with the isometric duality property, so we obtain the statement by Corollary 3.4.3 and by (3.3.19). $\square$

COROLLARY 3.4.7 (**Weak Extension Theorem**). *Let G be reflexive, then the statement of Corollary 3.4.3 holds if we set $\mathcal{N} \doteqdot \mathcal{N}_{st}(G)$ and $\mathcal{N}_{\sigma_n} \doteqdot \mathcal{N}_{st}(G_{\sigma_n})$ for all $n \in \mathbb{N}$.*

PROOF. By Theorem 3.2.5 we have that the map $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), \mathcal{N}_{st}(G))\rangle$ is scalarly essentially $(\mu, B(G))-$integrable. Hence the first part of the statement by Corollary 3.4.4. While the inequality follows by (3.2.15), (3.2.13) and (3.2.12). $\square$

REMARK 3.4.8. *In the case in which G is an Hilbert space we can obtain Corollary 3.4.7 as an application of the duality property of the predual $\mathcal{N}_{pd}(G)$. Indeed as we know $\mathcal{N}_{st}(G) \subset \mathcal{N}_{pd}(G)$, hence by the Hahn-Banach theorem $(\forall \Psi_0 \in \mathcal{N}_{st}(G)^*)(\exists\, \Psi \in \mathcal{N}_{pd}(G)^*)(\Psi \upharpoonright \mathcal{N}_{st}(G) = \Psi_0)$, thus by the duality property $\mathcal{N}_{pd}(G)^* = B(G)$ we obtain $(\forall \Psi_0 \in \mathcal{N}_{st}(G)^*)(\exists\, B \in B(G))(\forall \omega \in \mathcal{N}_{st}(G))(\Psi_0(\omega) = \omega(B))$, which ensures that the weak-integral with respect to the measure μ and with respect to the weak operator topology of the map $X \ni x \mapsto f_x(R) \in B(G)$ belongs to $B(G)$.*

REMARK 3.4.9. *Let* $\mathsf{D} \subset G$ *be a linear subspace of* G *and* $E : \mathcal{B}(\mathbb{C}) \to \Pr(G)$ *be a countably additive spectral measure, then by* (2.1.3) *for all* $f \in \mathbf{TM}$, $\phi \in G^*$ *and* $v \in \mathsf{D}$ *that*

$$|\phi\left(\mathbf{I}_{\mathbb{C}}^{E}(f)v\right)| = \left|\int f(\lambda)\, d\left(\psi_{\phi,v} \circ E\right)(\lambda)\right| \leq 4M\|f\|_{\sup}\|\phi\|\|v\|, \tag{3.4.15}$$

where $M \doteqdot \sup_{\delta\in\mathcal{B}(\mathbb{C})}\|E(\delta)\|$, $\psi_{\phi,v} : B(G) \ni A \mapsto \phi(Av) \in \mathbb{C}$ *and* $\mathbf{TM}$ *is the space of all totally* $\mathcal{B}(\mathbb{C})-$*measurable complex maps on* $\mathbb{C}$.

Next we know that

$$H(\mathbb{C}) \subset \mathbf{TM}. \tag{3.4.16}$$

Here $H(\mathbb{C})$ *is the space of all compactly supported complex continuous functions on* $\mathbb{C}$, *with the direct limit topology, of the spaces* $H(\mathbb{C};K)$ *with* K *running in the class of all compact subsets of* $\mathbb{C}$; *where* $H(\mathbb{C};K)$ *is the space of all complex continuous functions* $f : \mathbb{C} \to \mathbb{C}$ *such that* $supp(f) \subset K$ *with the topology of uniform convergence.*

Let us set

$$F_w^{\mathsf{D}} \doteqdot \overline{B(\mathsf{D}, G)} \text{ in } \mathfrak{L}_w(\mathsf{D}, G),$$

where $\mathfrak{L}_w(\mathsf{D}, G)$ *is the Hausdorff locally convex space of all linear operators on* D *at values in* G *with the topology generated by the following set of seminorms*

$$\{\mathfrak{L}_w(\mathsf{D}, G) \ni B \mapsto |q_{\phi,v}(B)| \mid (\phi, v) \in G^* \times \mathsf{D}\},$$

where $q_{\phi,v}(B) \doteqdot \phi(Bv)$ *for all* $(\phi, v) \in G^* \times \mathsf{D}$ *and* $B \in \mathfrak{L}_w(\mathsf{D}, G)$, *while* $B(\mathsf{D}, G)$ *is the space of all bounded operators belonging to* $\mathfrak{L}_w(\mathsf{D}, G)$.

By (3.4.16) *we can define*

$$\mathbf{m}_E : H(\mathbb{C}) \ni f \mapsto \left(\mathbf{I}_{\mathbb{C}}^{E}(f) \upharpoonright \mathsf{D}\right) \in F_w^{\mathsf{D}}$$

Moreover by (3.4.15) *we have, with the notations in 2.2.1, that for all compact* K *the operator* $\mathbf{m}_E \upharpoonright H(\mathbb{C};K)$ *is continuous.*

Therefore as a corollary of the general result in statement (ii) *Proposition* 5, $n°4$, §4, *Ch* 2 *of* [**TVS**] *about locally convex final topologies, so in particular for the inductive limit topology, we deduce that* $\mathbf{m}_E$ *is continuous on* $H(\mathbb{C})$ *i.e.*

$$\mathbf{m}_E \text{ is a vector measure on } \mathbb{C} \text{ with vales in } F_w^{\mathsf{D}}.$$

Here, by generalizing to the complex case the definition 1, $n°1$, §2, *Ch* 6 *of* [**INT**], *we call a vector measure on a locally compact space* X *with values in a complex Hausdorff locally convex space* Y *any* $\mathbb{C}-$*linear continuous map* $\mathbf{m} : H(X) \to Y$. *Furthermore*

$\forall(\phi, v) \in G^* \times \mathsf{D}$

$$\begin{aligned} q_{\phi,v} \circ \mathbf{m}_E &= \psi_{\phi,v} \circ \mathbf{I}_{\mathbb{C}}^{E} \restriction H(\mathbb{C}) \\ &= \mathbf{I}_{\mathbb{C}}^{\psi_{\phi,v}\circ E} \restriction H(\mathbb{C}). \quad \textit{by } (2.1.3) \end{aligned}$$

Hence

$$\mathfrak{L}_1(\mathbb{C}; q_{\phi,v} \circ \mathbf{m}_E) = \mathfrak{L}_1(\mathbb{C}; \psi_{\phi,v} \circ E),$$

where the left hand side it is intended in the sense of Ch 4 *of* [**INT**]*, while the right hand side it is intended in the standard sense, and for all* $f \in \mathfrak{L}_1(\mathbb{C}; q_{\phi,v} \circ \mathbf{m}_E)$

$$\int f(\lambda)\, d\,(q_{\phi,v} \circ \mathbf{m}_E)(\lambda) = \int f(\lambda)\, d\,(\psi_{\phi,v} \circ E)(\lambda) \tag{3.4.17}$$

Finally let us assume that D *is dense, then for all* $f \in Bor(supp\, E)$ *such that* $Dom(f(E)) = \mathsf{D}$ *by* (2.1.7) *we have*

$$f(E) \in F_w^{\mathsf{D}},$$

and by Theorem $18.2.11$ *of* [**DS**] *for all* $(\phi, v) \in G^* \times \mathsf{D}$ *we have* $f \in \mathfrak{L}_1(\mathbb{C}; \psi_{\phi,v} \circ E)$ *and*

$$q_{\phi,v}(f(E)) = \int f(\lambda)\, d\,(\psi_{\phi,v} \circ E)(\lambda). \tag{3.4.18}$$

Therefore by adopting the definitions in $n^\circ 2$*,* $\S 2$*, Ch* 6 *of* [**INT**]*, we deduce by* (3.4.17) *that each* $f \in Bor(supp\, E)$ *such that* $Dom(f(E)) = \mathsf{D}$ *is* essentially integrable for $\mathbf{m}_E$ *and*

$$f(E) = \int f(\lambda)\, d\,\mathbf{m}_E(\lambda).$$

Here $\int f(\lambda)\, d\,\mathbf{m}_E(\lambda)$ *is the* integral of f with respect to $\mathbf{m}_E$.

Thus if R *is an unbounded scalar type spectral operator in* G*, then for all* $f \in Bor(\sigma(R))$ *such that* $Dom(f(R)) = \mathsf{D}$ f is essentially integrable for $\mathbf{m}_E$ *and*

$$f(R) = \int f(\lambda)\, d\,\mathbf{m}_E(\lambda).$$

3.5. Generalization of the Newton-Leibnitz formula

In this section we shall apply the results of the previous one in order to prove Newton-Leibnitz formulas for integration with respect to the $\sigma(B(G), \mathcal{N})-$topology, when $\mathcal{N}$ is an $E-$appropriate set with the duality property, for integration with respect to the sigma-weak topology, and for integration with respect to the weak operator topology.

COROLLARY 3.5.1 ($\sigma(B(G),\mathcal{N})-$ **Newton-Leibnitz formula 1**). *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighborhood of $\sigma(R)$, $S: U \to \mathbb{C}$ an analytic map and $\mathcal{N}$ an $E-$appropriate set with the duality property. Assume that $S : U \to \mathbb{C}$ is an analytic map and $\exists\, L > 0$ such that $]-L, L[\,\cdot U \subseteq U$ and*

(1) $\widetilde{S}_t \in \mathfrak{L}_E^{\infty}(\sigma(R))$, $\widetilde{\left(\frac{dS}{d\lambda}\right)_t} \in \mathfrak{L}_E^{\infty}(\sigma(R))$ *for all* $t \in]-L, L[$;

(2)

$$\int^{*} \left\| \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \right\|_{\infty}^{E} dt < \infty$$

(here the upper integral is with respect to the Lebesgue measure on $]-L, L[$);

(3) $\forall \omega \in \mathcal{N}$ *the map* $]-L, L[\ni t \mapsto \omega\left(\frac{dS}{d\lambda}(tR)\right) \in \mathbb{C}$ *is Lebesgue measurable.*

Then $\forall u_1, u_2 \in]-L, L[$

$$\boxed{R\int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt = S(u_2 R) - S(u_1 R) \in B(G).}$$

Here the integral is the weak-integral of the map $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$ with respect to the Lebesgue measure on $[u_1, u_2]$ and with respect to the $\sigma(B(G),\mathcal{N})-$topology.

Moreover if $\mathcal{N}$ is an $E-$appropriate set with the isometric duality property and $M \doteqdot \sup_{\sigma \in \mathcal{B}(\mathbb{C})} \|E(\sigma)\|_{B(G)}$ then

$$\left\| \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt \right\|_{B(G)} \leq 4M \int_{[u_1,u_2]}^{*} \left\| \widetilde{\left(\frac{dS}{d\lambda}\right)_t} \right\|_{\infty}^{E} dt. \tag{3.5.1}$$

PROOF. Let μ the Lebesgue measure on $[u_1, u_2]$. By (2.3.4), the hypotheses, and statement (c) of Theorem 18.2.11 of **[DS]** we have

a: $(\forall t \in [u_1, u_2])(S(tR) \in B(G))$;

b: $(\forall t \in [u_1, u_2])(\frac{dS}{d\lambda}(tR) \in B(G))$;

c: $([u_1, u_2] \ni t \mapsto \|\frac{dS}{d\lambda}(tR)\|_{B(G)}) \in \mathfrak{F}_1([u_1, u_2]; \mu)$,

So by hypothesis (3), by (c) and Theorem 3.2.2 we have that the map

$$[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in \langle B(G), \sigma(B(G),\mathcal{N})\rangle \tag{3.5.2}$$

is scalarly essentially $(\mu, B(G))-$ integrable and if $\mathcal{N}$ is an $E-$appropriate set with the isometric duality property then its weak-integral satisfies (3.5.1).

This means that, made exception for (3.4.4), all the hypotheses of Theorem 3.4.2 hold for $X \doteqdot [u_1, u_2]$, $h \doteqdot (S_{u_2} - S_{u_1}) \upharpoonright \sigma(R)$, $g : \sigma(R) \ni \lambda \mapsto \lambda \in \mathbb{C}$ and finally for the map $[u_1, u_2] \ni t \mapsto f_t \doteqdot \left(\frac{dS}{d\lambda}\right)_t \upharpoonright \sigma(R)$.

Next let $\sigma \in \mathcal{B}(\mathbb{C})$ be bounded, so by Key Lemma 2.1.7 $R_\sigma \upharpoonright G_\sigma$ is a scalar type spectral operator such that $R_\sigma \upharpoonright G_\sigma \in B(G_\sigma)$, moreover by (2.1.15) U is an open neighborhood of $\sigma(R_\sigma \upharpoonright G_\sigma)$. Thus we can apply statement (3) of Theorem 2.3.2 to the Banach space G_σ, the analytic map S and to the operator $R_\sigma \upharpoonright G_\sigma$.

In particular the map $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma)) \in B(G_\sigma)$ is Lebesgue integrable in $\| \cdot \|_{B(G_\sigma)}-$topology, that is in the meaning of Definition 2, $n°4$, $\S 3$, Ch. IV of **[INT]**.

By Lemma 3.3.11 $\xi_\sigma \in B(B(G_\sigma), B(G))$, so

$$\mathcal{N}_\sigma \subset B(G_\sigma)^*.$$

Therefore we deduce by using Theorem 1, $IV.35$ of the **[INT]**, that $(\forall \omega_\sigma \in \mathcal{N}_\sigma)([u_1, u_2] \ni t \mapsto \omega_\sigma\left(\frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\right) \in \mathbb{C})$ is Lebesgue integrable, in addition $\forall \omega_\sigma \in \mathcal{N}_\sigma$

$$\int_{u_1}^{u_2} \omega_\sigma\left(\frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\right) dt = \omega_\sigma\left(\oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\, dt\right).$$

Thus we can state that $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma)) \in \langle B(G_\sigma), \sigma(B(G_\sigma), \mathcal{N}_\sigma)\rangle$ is scalarly essentially $(\mu, B(G_\sigma))-$integrable and its weak-integral is such that

$$\text{(3.5.3)} \qquad \int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\, dt = \oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\, dt.$$

Moreover by statement (3) of Theorem 2.3.2

$$(R_\sigma \upharpoonright G_\sigma) \oint_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\, dt = S(u_2(R_\sigma \upharpoonright G_\sigma)) - S(u_1(R_\sigma \upharpoonright G_\sigma)).$$

Thus by (3.5.3)

$$\text{(3.5.4)} \quad (R_\sigma \upharpoonright G_\sigma) \int_{u_1}^{u_2} \frac{dS}{d\lambda}(t(R_\sigma \upharpoonright G_\sigma))\, dt = S(u_2(R_\sigma \upharpoonright G_\sigma)) - S(u_1(R_\sigma \upharpoonright G_\sigma)).$$

This implies exactly the hypothesis (3.4.4) of Theorem 3.4.2, by choosing for example $\sigma_n \doteqdot B_n(\mathbf{0})$, for all $n \in \mathbb{N}$. Therefore by Theorem 3.4.2 we obtain the statement. □

COROLLARY 3.5.2 ($\sigma(B(G), \mathcal{N})-$ **Newton-Leibnitz formula 2**). *Let R be a possibly unbounded scalar type spectral operator in G, U an open neighborhood of $\sigma(R)$, $S : U \to \mathbb{C}$ an analytic map and $\mathcal{N}$ an $E-$appropriate set with the duality property. Assume* *that* $\exists\, L > 0$ *such that* $]-L, L[\,\cdot U \subseteq U$, $(\forall t \in]-L, L[)(\widetilde{S}_t \in \mathfrak{L}_E^{\infty}(\sigma(R)))$ *and* $\exists K_0 \subset]-L, L[$ *such that* $\complement K_0$ *is a Lebesgue negligible set and* $(\forall t \in K_0)\left(\widetilde{\left(\frac{dS}{d\lambda}\right)_t} \in \mathfrak{L}_E^{\infty}(\sigma(R))\right)$ *moreover*

(1) $\exists\, F :]-L, L[\to B(G)$ *extending* $K_0 \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$ *such that*

$$\int^* \|F(t)\|_{B(G)}\, dt < \infty$$

(here the upper integral is with respect to the Lebesgue measure on $]-L, L[$*),*

(2) $\forall \omega \in \mathcal{N}$ *the map* $]-L, L[\ni t \mapsto \omega(F(t)) \in \mathbb{C}$ *is Lebesgue measurable.*

Then $\forall u_1, u_2 \in]-L, L[$

$$\boxed{R \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt = S(u_2 R) - S(u_1 R) \in B(G).}$$

Here the integral is the weak-integral of the map $[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in B(G)$ *with respect to the Lebesgue measure on* $[u_1, u_2]$ *and with respect to the* $\sigma(B(G), \mathcal{N})-$*topology. Moreover if* $\mathcal{N}$ *is an* $E-$*appropriate set with the isometric duality property then*

$$\left\| \int_{u_1}^{u_2} \frac{dS}{d\lambda}(tR)\, dt \right\|_{B(G)} \leq \int^*_{[u_1, u_2]} \left\| \frac{dS}{d\lambda}(tR) \right\|_{B(G)} dt.$$

PROOF. By Theorem 3.2.2 and (3.2.13)

$$[u_1, u_2] \ni t \mapsto \frac{dS}{d\lambda}(tR) \in \langle B(G), \sigma(B(G), \mathcal{N}) \rangle$$

is scalarly essentially $(\mu, B(G))-$integrable and if $\mathcal{N}$ is an $E-$appropriate set with the isometric duality property its weak integral satisfies by (3.2.12) the inequality in the statement. Thus the proof goes on as that in Corollary 3.5.1. $\square$

COROLLARY 3.5.3 (Sigma-Weak Newton-Leibnitz formula). *The statement of Corollary 3.5.1 (respectively Corollary 3.5.2) holds if* G *is a complex Hilbert space and everywhere* $\mathcal{N}$ *is replaced by* $\mathcal{N}_{pd}(G)$.

PROOF. By Remark 3.3.6, $\mathcal{N}_{pd}(G)$ is an $E-$appropriate set with the isometric duality property, hence the statement by Corollary 3.5.1 (respectively Corollary 3.5.2). $\square$

COROLLARY 3.5.4 (Weak Newton-Leibnitz formula). *The statement of Corollary 3.5.1 (respectively Corollary 3.5.2) holds if* G *is a reflexive complex Banach space and everywhere* $\mathcal{N}$ *is replaced by* $\mathcal{N}_{st}(G)$.

PROOF. By using Corollary 3.2.6 instead of Theorem 3.2.2, we obtain (3.5.2) and (3.5.1) by replacing $\mathcal{N}$ with $\mathcal{N}_{st}(G)$. Then the proof procedes similarly to that of Corollary 3.5.1 (respectively Corollary 3.5.2). $\square$

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Appendix. Further developments

Although it will be not included in this thesis, we found some results on the possibility of generalizing the concept of spectral operator in Banach space; here we want to outline some of them.

Introduction

There could be many ways for trying to generalize the Extension Theorem 3.4.2, for example by considering integration with respect to a measure on general topological spaces, as defined in Ch. 9 of [**INT**], or in the other side checking other locally convex topologies in $B(G)$. Or by replacing $B(G)$ with a suitable locally convex space of operators defined in a subset of G and with values in G, case which we performed.

But in our opinion what is most important to do firstly it is to generalize the concept of spectral operator by recognizing that an unbounded spectral operator (not only of scalar type), as defined by using its resolution of the identity as in Ch 18 of [**DS**], it is a geometric object. Only after that we can perform the generalization of all previous results in a rather wide sense by defining a new kind of integration.

More exactly it is possible to show, by using Key Lemma 2.1.7, that to any unbounded spectral operator R in a complex Banach space, with E its resolution of the identity and $\sigma(R)$ its spectrum, and any $D \subset G$ such that $E(\mathcal{B}(\mathbb{C}))' D \subseteq D$ [3], corresponds a suitable presheaf $\Xi \doteqdot \left\langle \mathcal{M}(U), \hat{P}_{U,V} \right\rangle$ over $\sigma(R)$ [4] at values in the category **ST** of **Spectral**

[3] $E(\mathcal{B}(\mathbb{C}))'$ is the commutant in $B(G)$ of $E(\mathcal{B}(\mathbb{C}))$, i.e. the class of all operators $T \in B(G)$ such that $E(\sigma)T = TE(\sigma)$ for all $\sigma \in \mathcal{B}(\mathbb{C})$.

[4] Abridgement of $\left\{ \left\langle \mathcal{M}(U), \hat{P}_{U,V} \right\rangle \mid U, V \in Op(\sigma(R)) \mid U \supseteq V \right\}$

Structures [5] such that

$$
(3.5.5) \qquad \begin{cases} g(R_U \restriction G_U) \restriction D(U) \in \mathcal{M}(U) \\ \hat{P}_{U,V}\left[g(R_U \restriction G_U) \restriction D(U)\right] = \left[g(R_V \restriction G_V) \restriction D(V)\right] \end{cases}
$$

$\forall U, V \in Op(\sigma(R))$ such that $U \supseteq V$ and for all maps g in a suitable class $\mathfrak{F}^{R,D} \subset Bor(\sigma(R))$. Where $D(U) \doteqdot E(U)D$, and $\mathcal{M}(U)$ is the Hausdorff completion of the topological unital algebra $\mathcal{A}(U) \doteqdot E(U)E(\mathcal{B}(\mathbb{C}))'$ with the locally convex topology generated by the set of seminorms $\{q^U_{\phi,v} : \mathcal{A}(U) \ni A \mapsto |\phi(Av)| \mid (\phi, v) \in G^* \times D(U)\}$ [6]. Finally $\hat{P}_{U,V}$ is the extension by continuity to $\mathcal{M}(U)$ of the operator $P_{U,V} : \mathcal{A}(U) \ni a \mapsto E(V)a \in \mathcal{A}(V)$.

This presheaf characterization allowed me to generalize the definition of spectral operators in Banach spaces to a rather general categorical framework.

More exactly

DEFINITION 3.5.5. (Θ, Γ) *is a* sheaf couple *over* Y *at values in* $\mathcal{C}$ *if* Y *is a topological space,* $\mathcal{C}$ *is a small category* [7], $\Gamma \doteqdot \langle \mathcal{N}(U), Q_{U,V}\rangle$ *and* $\Theta \doteqdot \langle \mathcal{O}(U), S_{U,V}\rangle$ *are respectively a presheaf and a* ***sheaf*** *on* Y *at values in* $\mathcal{C}$ [8]*, moreover* $\forall U \in Op(Y)$ *we have* $\mathcal{N}(U) \subseteq \mathcal{O}(U)$ *and*

$$
(\forall V \in Op(Y) \mid U \supseteq V)(Q_{U,V} = S_{U,V} \restriction \mathcal{N}(U)).
$$

Then we can set

[5] Which is a small category whose class of objects is a subclass of that of all bimodules over topological algebras, bimodules which are also Hausdorff complete locally convex spaces.

By definition a presheaf over a topological space Y with values in a category $\mathfrak{H}$ is just a functor with values in the category $\mathfrak{H}$ and defined in the opposite category $Op(Y)^\circ$ of the the category $Op(Y)$ whose objects are open sets of Y and $Mor(V, U) = \{\imath_{V,U}\}$ if $U \supseteq V$, otherwise is the empty set, where $\imath_{V,U}$ is the identity map of V into U. See Ch. 17 of [**KS**] for the definition of presheaves and sheaves; while for the case $\mathfrak{H} = SET$ see [**Bor**] Vol.3, or [**McLM**]. Finally for category theory see [**McL**].

[6] The following consideration is the first step towards the proof that $[g(R_U \restriction G_U) \restriction D(U)] \in \mathcal{M}(U)$. If we set $\sigma_n \doteqdot \overset{-1}{|g|}([0, n])$, for $n \in \mathbb{N}$, then by the fact that $\lim_{n\in\mathbb{N}} E(\sigma_n) = \mathbf{1}$ strongly, by Corollary 2.1.8 and by the commutativity property in statement (g) of 18.2.11. of [**DS**], we have

$$
g(R_U \restriction G_U)y = \lim_{n\in\mathbb{N}} E(U)g(R)E(\sigma_n)y
$$

for all $y \in D(U)$, in addition $g(R)E(\sigma_n) \in E(\mathcal{B}(\mathbb{C}))'$.

[7] By definition any category whose objects are sets.

[8] By restricting our attention to the case of a small category as in our case $\mathcal{C}$, by definition $\langle \mathcal{O}(U), S_{U,V}\rangle$ is a sheaf on Y with values in $\mathcal{C}$ if it is a presheaf on Y with values in $\mathcal{C}$ such that the following *gluing property* holds: for each $U \in Op(Y)$, open cover $\{U_i\}_{i\in I}$ of U and family $\{f_i \mid f_i \in \mathcal{O}(U_i)\}_{i\in I}$ such that $(\forall i, j \in I)(S_{U_i,U_i\cap U_j}(f_i) = S_{U_j,U_i\cap U_j}(f_j))$ there exists only one element $f \in \mathcal{O}(U)$ such that $(\forall i \in I)(f_i = S_{U,U_i}(f))$, see pg 66 of [**McLM**] for the case $\mathcal{C} = SET$, the category of sets.

DEFINITION 3.5.6. *Let* (Θ,Γ) *be a sheaf couple over* Y *at values in* $\mathcal{C}$*, then by denoting* $\Gamma \doteqdot \langle \mathcal{N}(U), Q_{U,V}\rangle$ *and* $\Theta \doteqdot \langle \mathcal{O}(U), S_{U,V}\rangle$ *we define the class of all* ***G-spectral operators associated to*** (Θ,Γ) *to be the following class*

$$SO\,(\Theta,\Gamma) \doteqdot \{\widetilde{A} \in \mathcal{O}(Y) \mid (\forall U \in Op(Y))(S_{X,U}(\widetilde{A}) \in \mathcal{N}(U))\}.$$

Finally $\forall \widetilde{A} \in SO\,(\Theta,\Gamma)$*, we call the map*

$$Open(Y) \ni U \mapsto S_{X,U} \in Mor(\mathcal{C})$$

the ***G-resolution of identity of*** $\widetilde{A}$*.*

Notice that $\mathcal{N}(Y) \subseteq SO\,(\Theta,\Gamma) \subseteq \mathcal{O}(Y)$.

In case where $\mathcal{C}$ satisfies the request 17.4.1. of [**KS**] [9], the most natural example of a sheaf couple is $(\widetilde{\Gamma},\Gamma)$ where $\widetilde{\Gamma}$ is the sheaf generated by the presheaf Γ.

Hence if we denote with $\widetilde{\Xi} \doteqdot \left\langle \widetilde{\mathcal{M}}(U), \widetilde{\widehat{P}}_{U,V} \right\rangle$ the sheaf generated by Ξ, then $(\widetilde{\Xi},\Xi)$ is a sheaf couple over $\sigma(R)$ at values in the category of Spectral Structures. In addition by (3.5.5) $g(R) \upharpoonright D$ is a G-spectral operator associated to $(\widetilde{\Xi},\Xi)$ such that $g(R) \upharpoonright D \in \mathcal{M}(\sigma(R))$, and $Open(\sigma(R)) \ni U \mapsto \widetilde{\widehat{P}}_{\sigma(R),U} \in Mor(\mathbf{ST})$ is its G-resolution of identity.

The easier way for constructing G-spectral operators is that which uses the gluing property of the sheaf generated by a fixed presheaf.

More exactly if $\Gamma \doteqdot \langle \mathcal{N}(U), Q_{U,V}\rangle$ is a presheaf on a topological space Y, at values in a small category $\mathcal{C}$ satisfying the request 17.4.1. of [**KS**] and $\mathcal{B}$ is an open cover filter basis of Y [10] then we can define the class of *spectral elements of* Γ *relative to* $\mathcal{B}$ as the class of sets $\{f_U\}_{U\in\mathcal{B}}$ such that

$$\begin{cases}(\forall U \in \mathcal{B})((f_U \in \mathcal{N}(U))\\ (\forall V \in \mathcal{B} \mid U \supseteq V)(Q_{U,V}(f_U) = f_V)).\end{cases}$$

Being the presheaf Γ embedded canonically into the sheaf $\widetilde{\Gamma} = \left\langle \widetilde{\mathcal{N}}(U), \widetilde{Q}_{U,V} \right\rangle$ by it generated, we have that any spectral element $\{f_U\}_{U\in\mathcal{B}}$ of Γ could be considered as a spectral element of $\widetilde{\Gamma}$, therefore by using gluing property of any sheaf we can associate

[9] The request 17.4.1., which is satisfied for example by the small category of Spectral Structures, is intended to provide a tool for associating to any presheaf at values in $\mathcal{C}$ a sheaf valued in the same category, see Theorem 17.4.7. of [**KS**].

[10] By definition $\mathcal{B} \subseteq Op(Y)$ such that $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B})(W \subseteq U \cap V)$, and $\bigcup_{U\in\mathcal{B}} U = X$, see [**GT**] for the definition of a filter basis

to $\{f_U\}_{U\in\mathcal{B}}$ an unique element $f \in \widetilde{\mathcal{N}}(Y)$ such that $\widetilde{Q}_{X,U}(f) = f_U$ for all $U \in \mathcal{B}$. We will call such an element *global* associated to $\{f_U\}_{U\in\mathcal{B}}$.

Now it is not so difficult to see that $SO\left(\widetilde{\Gamma},\Gamma\right)$ is the class of the global elements associated to all spectral elements $\{f_U\}_{U\in\mathcal{B}}$ of Γ, where $\mathcal{B}$ is any open cover filter basis of Y.

We encountered a spectral element, in fact by (3.5.5) $\{g(R_U \upharpoonright G_U) \upharpoonright D(U) \mid U \in \mathcal{B}\}$ is a spectral element of Ξ relative to any open covering filter basis $\mathcal{B}$ of $\sigma(R)$, for all $g \in \mathfrak{F}^{R,D}$, in addition, by the uniqueness in the gluing property, its global element belongs to $\mathcal{M}(\sigma(R))$ and is $g(R)$.

But it is clear that the most interessant and new cases of G-spectral operators associated to $(\widetilde{\Xi},\Xi)$ to analyze, are those belonging to $\widetilde{\mathcal{M}}(\sigma(R)) \cap \complement\mathcal{M}(\sigma(R))$.

In any case the gluing property in the sheaf framework allows us to generalize the weak-integration, in the sense of Ch 6 of [**INT**], to suitable maps with values in $\mathcal{M}(\sigma(R))$ resp. in $\widetilde{\mathcal{M}}(\sigma(R))$, which might be "not" scalarly essentially $\mu-$integrable with respect to a Radon measure μ on a locally compact space X and with respect to the topology on $\mathcal{M}(\sigma(R))$ resp. on $\widetilde{\mathcal{M}}(\sigma(R))$.

The main idea is to use this property for constructing a functor $\mathcal{SH}$ defined on a suitable category and a values in the category of vector spaces, and then apply it to contruct the generalization

$$\mathcal{SH}\left(\int\right)$$

of the weak-integral $\int$ to a wide variety of maps possibly non-integrable in the standard sense.

Finally we apply this new integral to the case of maps of the kind $X \ni x \mapsto f_x(R) \in \widetilde{\mathcal{M}}(\sigma(R))$, where $f_x \in \mathfrak{F}^{R,D}$ locally $\mu-$ $a.e.(X)$, for generalizing the Extension Theorem 3.4.2. We add the Appendix "Further developments" for describing some result we found in this direction.

Description of some results

Some of the main results in this direction that we found are the following.

(1) Construction of a suitable category and a **functor** $\mathcal{SH}$ defined on it and a values in the category of vector spaces, such that the wanted generalization of the integral $\int$ is the linear operator $\mathcal{SH}\left(\int\right)$. More exactly let us fix two

categories $\mathcal{C}$ and $\mathcal{D}$ and a topological space Y [11]. Then we introduce a category - which we called category of of **Sh-Spectral Elements associated to** $\langle \mathcal{C}, \mathcal{D} \rangle$ - rougly speaking whose objects are couples $\langle \Gamma, \Theta \rangle$ where Γ is a presheaf over Y with values in a subcategory $\mathcal{C}_0$ of $\mathcal{C}$ depending by Γ, while $\Theta : \mathcal{C}_0 \to \mathcal{D}$ is a suitable functor such that $\Theta \circ \Gamma$ is a sheaf, finally the class of natural transformations between the two functors $\Theta_1 \circ \Gamma_1$ and $\Theta_2 \circ \Gamma_2$ is the class of morphisms $Mor(\langle \Gamma_1, \Theta_1 \rangle, \langle \Gamma_2, \Theta_2 \rangle)$. After that we construct a functor $\mathcal{SH}$ defined in the category of $Sh-$Specral Elements associated to $\langle \mathcal{C}, \mathcal{D} \rangle$ and a values in $\mathcal{D}$.

(2) Now we can apply the result before described to the case in which $\mathcal{C}$ is the category $\mathcal{HLCS}$ of Hausdorff locally convex spaces, $\mathcal{D}$ is the category $\mathcal{VS}$ of vector spaces. And the result we found is the possibility of generalizing the concept of integration [12] over X with respect to μ, of maps with values in a locally convex space. More exactly let us fix a presheaf $\langle \Gamma, Q_{U,V} \rangle$ over Y, shortly Γ, at values in the category $\mathcal{HLCS}$ of Hausdorff locally convex spaces and as usual let us indicate with $\left\langle \widetilde{\Gamma}, \widetilde{Q}_{U,V} \right\rangle$, the corresponding sheaf. Now the couple $\left\langle \widetilde{\Gamma}, F_2 \right\rangle$ with F_2 the forgetful functor is of course an object of the category of Spectral Elements associated to $\langle \mathcal{HLCS}, \mathcal{VS} \rangle$. In order to use the functor $\mathcal{SH}$, firstly we showed that also the couple $\left\langle \widetilde{\Gamma}^X, F_1 \right\rangle$ is an object of the same category. Where $\widetilde{\Gamma}^X$ is rougly speaking the presheaf over Y with values in a subcategory $\mathcal{C}_0 \subset \mathcal{HLCS}$ of space of scalarly essentially $\mu-$integrable maps, so that $\forall U \in Op(Y)$ the $\widetilde{\Gamma}^X(U)$ is the space of all scalarly essentially $\mu-$integrable functions $\widetilde{f}^U : X \to \widetilde{\Gamma}(U)$ whose integral belongs to $\widetilde{\Gamma}(U)$, with a locally convex topogy making the integral a continuous operator, and with structure maps $\widetilde{f}^U \mapsto \widetilde{Q}_{U,V} \circ \widetilde{f}^U$, where $U, V \in Op(Y)$. Finally $F_1 : \mathcal{C}_0 \ni S \mapsto Z^X \in \mathcal{D}$ if S the space of all scalarly essentially $\mu-$integrable functions defined on X and at values in a locally convex space Z and whose integral belongs to Z. Secondly we showed that the integral operator $\int$ is a morphism belonging to $Morph\left(\left\langle \widetilde{\Gamma}^X, F_1 \right\rangle, \left\langle \widetilde{\Gamma}, F_2 \right\rangle\right)$. Hence we were able to

[11] Although the general definition which we made consider presheaves over different topological spaces, here for simplifying the notations, it shall be assumed fixed a topological space Y.
[12] In the sense exposed in Ch 6 of [**INT**].

set the following definition

$$\boxed{Sh - \mathbf{I} \doteqdot \mathcal{SH}\left(\int\right) : \mathcal{SH}(\widetilde{\Gamma}^X) \subset \widetilde{\Gamma}(Y)^X \to \widetilde{\Gamma}(Y)} \tag{3.5.6}$$

as a linear map between the linear spaces $\mathcal{SH}(\widetilde{\Gamma}^X) \subset \widetilde{\Gamma}(Y)^X$ and $\mathcal{SH}(\widetilde{\Gamma}) \doteqdot \widetilde{\Gamma}(Y)$. The remarkable property of the *Sh-integral* $Sh - \mathbf{I}$ is to generalize the integral operator $\int$ to the space $\mathcal{SH}(\widetilde{\Gamma}^X)$ which contains all maps $\widetilde{f} : X \to \widetilde{\Gamma}(Y)$ such that the map $\widetilde{f^i} : X \to \widetilde{\Gamma}(U_i)$ is scalarly essentially $\mu-$integrable and its integral belongs to $\widetilde{\Gamma}(U_i)$, for all $i \in I$ where $\widetilde{f^i} \doteqdot \widetilde{Q}_{Y,U_i} \circ \widetilde{f}$ and $\{U_i\}_{i \in I}$ is an open cover of Y which for technical reasons we prefer to get closed under finite intersection. Now we can consider $\Gamma(U)$ as a subspace of $\widetilde{\Gamma}(U)$, while $\widetilde{Q}_{U,V}$ an extension of $Q_{U,V}$, so for example, for what said, the $Sh-$Integral applies to every map $f : X \to \Gamma(Y)$ possibly "not" scalarly essentially $\mu-$integrable or such that its integral does NOT belongs to $\Gamma(Y)$ such that the map $f^i : X \mapsto Q_{Y,U_i} \in \Gamma(U_i) \circ f^i$ is scalarly essentially $\mu-$integrable and its integral belongs to $\Gamma(U_i)$, for all $i \in I$.
And even if we get a family of maps $\{f^i : X \to \Gamma(U_i)\}_{i \in I}$ such that $(\forall U_i \supseteq U_j)(Q_{U_i,U_j} \circ f^i = f^j)$, and f_i is scalarly essentially $\mu-$integrable and its integral belongs to $\Gamma(U_i)$, for all $i \in I$; then $Sh-$Integral applies to the map $\widetilde{f} : X \to \widetilde{\Gamma}(Y)$ possibly not scalarly essentially $\mu-$integrable or such that its integral does not belongs to $\widetilde{\Gamma(Y)}$ such that $\widetilde{f}(x)$ is obtained by *gluing together* the family $\{f^i(x)\}_{i \in I}$ for all $x \in X$

(3) Finally as an application of the $Sh-$Integral to the choice

$$\Gamma \doteqdot \mathcal{M}$$

we were able to generalize the extension formula (0.0.19) to suitable possibly "not" scalarly essentially $\mu-$integrable maps $X \ni x \mapsto f_x(R) \in \mathcal{M}(\sigma(R))$ by replacing the standard integral $\int$ with the $Sh-$Integral $Sh - \mathbf{I}$. Similar generalization holds also for the integro-differential formula (0.0.3). See below for more details.

(4) In the beginning it was outlined that the real possibility of extending the concept of scalar type spectral operator to more general setting, is the construction of preshaves valued in the category of Spectral Structures. To this end, by using a result in locally convex quasi-algebras, see [**Schm**] or [**AIT**], we construct a presheaf Γ over $\mathbb{C}$ and with values in the category of Spectral Structures so

that the space of the global sections $\Gamma(\mathbb{C})$ is a suitable subspace of the locally convex space [13] of all continuous linear operators defined in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with its standard topology and at values in its conjugate topological dual $\mathcal{S}'(\mathbb{R}^n)$ with the topology of uniform convergence over bounded sets of $\mathcal{S}(\mathbb{R}^n)$.

Now we shall see a litle bit more in details in what consists and how to construct the presheaf $\mathcal{M}$ associated to an unbounded scalar type spectral operator.

First of all let $D \subset G$ a linear space so that

$$E(\mathcal{B}(\mathbb{C}))'D \subseteq D \tag{3.5.7}$$

for example the set of analytic elements $\bigcap_{n\in\mathbb{N}} Dom(R^n)$ of a self-adjoint operator in a Hilbert space. Let $\mathfrak{F}^{R,D}$ be the family of all Borelian functions g on $\sigma(R)$ such that $D \subseteq Dom(g(R))$, $g \in \mathfrak{F}^{R,D}$, and $\sigma_n \doteqdot \overset{-1}{|g|}([0,n])$, for $n \in \mathbb{N}$. Hence $\mathbf{1} = \lim_{n\in\mathbb{N}} E(\sigma_n)$, strongly and for all $y \in Dom(g(R_U \upharpoonright G_U))$

$$\begin{aligned} g(R_U \upharpoonright G_U)y &= \lim_{n\in\mathbb{N}} E(\sigma_n) g(R_U \upharpoonright G_U) E(U)y \\ &= \lim_{n\in\mathbb{N}} E(\sigma_n) g(R) E(U)y \\ &= \lim_{n\in\mathbb{N}} E(U) g(R) E(\sigma_n)y. \end{aligned} \tag{3.5.8}$$

Here the second equality comes by Corollary 2.1.8, while the third by the commutativity in statement (g) of $18.2.11.$ of [**DS**]

But $g(R)E(\sigma_n) \in B(G)$ [14] so $g(R)E(\sigma_n) \in E(\mathcal{B}(\mathbb{C}))'$ by (g) of $18.2.11.$ of [**DS**].

Therefore by the convergence (3.5.8), if we set $D(U) \doteqdot E(U)D$, the restriction $g(R_U \upharpoonright G_U) \upharpoonright D(U)$ [15] of the operator $g(R_U \upharpoonright G_U)$ to $D(U)$ could be considered, see below, as an element of the Hausdorff completion $\mathcal{M}(\langle \mathcal{A}_R(U), \tau_D(U)\rangle)$ of the topological unital algebra $\langle \mathcal{A}_R(U), \tau_D(U)\rangle$, where

$$\mathcal{A}_R(U) \doteqdot E(U)E(\mathcal{B}(\mathbb{C}))'$$

with S' is the commutant in the algebra $B(G)$ of $S \subset B(G)$, i.e. the set of all elements of $B(G)$ which commutate with every elements of S. While the locally convex topology

[13] with respect to a suitable locally convex topology

[14] In fact $g(\sigma_n \cap \sigma(R))$ is bounded by construction of σ_n so by Key Lemma 2.1.7 stat. (3) we have $g(R)E(\sigma_n) \in B(G)$.

[15] Well set indeed by (3.5.7) $E(U)D(U) = E(U)D \subseteq D \subseteq Dom(g(R))$, so $D(U) \subseteq Dom(g(R)E(U)) \cap G_U$, therefore by Corollary 2.1.8 we conclude $D(U) \subseteq Dom(g(R_U \upharpoonright G_U))$.

$\tau_D(U)$ is that induced by the following set of seminorms

$$\{q^U_{(\phi,y)} : \mathcal{A}_R(U) \ni A \mapsto |\phi(Ax)| \mid (\phi, x) \in G^* \times D(U)\}.$$

Here $D(U) \doteqdot E(U)D$. As previously said we note that $\mathcal{A}_R(U)$ is unital with unity $E(U)$. In addition the request (3.5.7) assures that $\langle \mathcal{A}_R(U), \tau_D(U)\rangle$ is a topological algebra, i.e. an algebra with a topological vector space structure making its product separatedly continuous. For making notations more readible, we convene $\forall U \in Op(\sigma(R))$ to denote $\langle \mathcal{A}_R(U), \tau_D(U)\rangle$ with $\mathcal{A}(U)$ and $\mathcal{M}(\langle \mathcal{A}_R(U), \tau_D(U)\rangle)$ with $\mathcal{M}(U)$. Infact we shown that $\forall U \in Op(\sigma(R))$ after suitable hypotheses

$$\boxed{\mathcal{M}(U) = \overline{(\mathcal{A}_R(U) \restriction D(U))} \text{ in } \mathcal{L}(D(U), G).} \tag{3.5.9}$$

Here $\mathcal{A}_R(U) \restriction D(U) \doteqdot \{A \restriction D(U) \mid A \in \mathcal{A}_R(U)\}$ and the closure is in $\mathcal{L}(D(U), G)$ the locally convex space of all maps $A \restriction D(U)$, with $A \in B(G)$, with the topology induced by the set of seminorms $\{A \ni\mapsto |\phi(Av)| \mid (\phi, v) \in G^* \times D(U)\}$ [16]. Therefore by (3.5.8) and the fact that $D(U) \subseteq Dom(g(R_U \restriction G_U))$

$$\boxed{(\forall U \in Op(\sigma(R)))(\forall g \in \mathfrak{F}^{R,D})([g(R_U \restriction G_U) \restriction D(U)] \in \mathcal{M}(U)).} \tag{3.5.10}$$

Notice that $\mathcal{A}(U)$ is Hausdorff iff $D(U)$ is dense and in general the product is not jointly continuous.

Let us set $\forall U, V$ Borelian so that $V \subseteq U$ the map

$$P_{U,V} : \mathcal{A}(U) \ni A \mapsto E(V)A$$

then of course is linear, in addition by the property $E(\delta_1 \cap \delta_2) = E(\delta_1)E(\delta_2)$ we have $P_{U,V} : \mathcal{A}(U) \to \mathcal{A}(V)$, and $P_{V,W}P_{U,V} = P_{U,W}$. Finally by the fact that $E(V)^2 = E(V)$ and $E(V)$ commutates with any element of $\mathcal{A}(U)$ we deduce that $P_{U,V}$ is a morphism of algebras, while by (3.5.7) we deduce that $\mathcal{A}(U)D(U) \subseteq D(U)$ hence together the fact that $E(U)$ is linear and continuous, we have that $P_{U,V}$ is continuous with respect the $(\tau_{D(U)}, \tau_{D(V)})-$ topologies. Therefore we have shown that

$$\langle \mathcal{A}(U), P_{U,V}\rangle \text{ is a presheaf over } \sigma(R) \text{ of Locally Convex Algebras.} \tag{3.5.11}$$

Finally it is a well known fact [17] that the completion $\mathcal{M}(\mathcal{A})$ of any topological algebra $\mathcal{A}$ has the structures of a topological vector space and of a bimodule over the algebra $\mathcal{A}$

[16] Notice that $\mathcal{M}(U) = \overline{(\mathcal{A}_R(U) \restriction D)}$ in $\mathcal{L}_{inv}(D(U), G)$ where $\mathcal{L}_{inv}(D(U), G)$ is the locally convex algebra of all elements $A \in \mathcal{L}(D(U), G)$ such that $AD(U) \subseteq D(U)$.

[17] Especially in the theory of partial algebras in which this type of bimodules are called quasi-algebras, see [**AIT**]

whose two external products $\mathcal{A} \times \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})$ are continuous in the right place. In addition to these we found other algebro-topological nice properties holding by $\mathcal{M}(\mathcal{A})$ which could be encoded in a better way in a categorical framework. For this reason we define the category of Spectral Structures respect to which $\mathcal{M}(\mathcal{A})$ is one of its objects.

What it is important, is that such a construction $\mathcal{M} : \mathcal{A} \mapsto \mathcal{M}(\mathcal{A})$ is a functor from the category of topological algebras to that of Spectral Structures. Therefore any presheaf over the space $\sigma(R)$ of topological algebras, being by definition a functor, can be extended,by componing it with $\mathcal{M}$, to a presheaf over $\sigma(R)$ at values in the category of Spectral Structures. Hence by (3.5.11) we have

(3.5.12) $\boxed{\begin{array}{l} \left\langle \mathcal{M}(U), \hat{P}_{U,V} \right\rangle \\ \text{is a presheaf over } \sigma(R) \\ \text{at values in the category of Spectral Structures.} \end{array}}$

Here the map $\hat{P}_{U,V} : \mathcal{M}(U) \to \mathcal{M}(V)$ is obtained by extending by continuity to the completion $\mathcal{M}(U)$ the linear and continuous map $P_{U,V} : \mathcal{A}(U) \to \mathcal{A}(V)$.

Therefore (3.5.10) and (3.5.12) make clear the connection between the field of Borelian functional calculus for unbounded scalar type spectral operators in a Banach space and the field of presheaf with values in the category of bimodules over topological algebras (or more specifically the category of Spectral Structures).

Why it is so important to translate the theory of Spectral Operators into that of presheaf valued in the category of Spectral Structures?

The reason is very simple: because to every presheaf with values in that category, we can associate a sheaf and therefore use the gluing property of any sheaf for exctracting by a problem concerning unbounded scalar type spectral operators, *global* informations - i.e. related to $\sigma(R)$ - from *local* informations - i.e. related to any open U_i where $\{U_i\}_{i \in I}$ is an open cover of $\sigma(R)$. As application of this general sheaf approach we mention some result which we obtained in this direction. Let X a locally compact space and μ a Radon measure on X, R an unbounded scalar type spectral operator in a Banach space G and $\Gamma \doteqdot \{U_i\}_{i \in I}$ an open cover of $\sigma(R)$ which is closed for finite intersection. Finally recall that for any locally convex algebra $\mathcal{A}$ the bimodule $\mathcal{M}(\mathcal{A})$ has by construction the standard locally convex topology of the Hausdorff completion of $\mathcal{A}$. Then

(1) let us set the class of the maps $\mathfrak{H}$

$$f : X \to \mathcal{M}(\sigma(R))$$

so that $\forall i \in I$ the map $f_i : X \ni x \mapsto \hat{P}_{\sigma(R),U_i}(f(x)) \in \mathcal{M}(U_i)$ is scalarly essentially $\mu-$integrable, in the meaning of $Ch.6$ of **[INT]**, and its integral belongs to $\mathcal{M}(U_i)$.

Let us indicate with $\left\langle \widetilde{\mathcal{M}}(U), \widetilde{\hat{P}}_{U,V} \right\rangle$ the sheaf associated to the presheaf $\left\langle \mathcal{M}(U), \hat{P}_{U,V} \right\rangle$ over $\sigma(R)$ [18] Let us set the space of maps $\widetilde{\mathfrak{H}}$ whose elements are maps $\widetilde{f} : X \to \widetilde{\mathcal{M}}(\sigma(R))$ so that there exists a map $f \in \mathfrak{H}$ such that $\forall x \in X$ the $\widetilde{f}(x)$ is the element in $\widetilde{\mathcal{M}}(\sigma(R))$ obtained by gluing togeter the class of elements $\{f_i(x)\}_{i\in I}$. Now by applying the general construction (3.5.6) we can define the operator

$$Sh - \mathbf{I} : \widetilde{\mathfrak{H}} \to \widetilde{\mathcal{M}}(\sigma(R))$$

called $Sh-$Integral so that $Sh - \mathbf{I}(\widetilde{f})$ is exactly the element in $\widetilde{\mathcal{M}}(\sigma(R))$ obtained by *gluing* the class of elements [19]

$$\left\{ \int f_i(x)\, d\mu \right\}_{i\in I}.$$

Now let us mention two remarkable facts

(a) If $\widetilde{f}(x) \subseteq \mathcal{M}(\sigma(R))\ \ \mu - l.a.e.(X)$, then by the uniqueness of the gluing property we have $\widetilde{f} = f,\ \mu - l.a.e.(X)$ therefore we can apply the $Sh-$Integral to f and calculate the value

$$Sh - \mathbf{I}(f) \in \mathcal{M}(\sigma(R)); \tag{3.5.13}$$

(b) If $\widetilde{f}(x) \subseteq \mathcal{M}(\sigma(R))\ \ \mu - l.a.e.(X)$, and f is scalarly essentially $\mu-$integrable, and its integral belongs to $\mathcal{M}(\sigma(R))$, then by using the continuity property of the integral and again the uniqueness of the gluing

[18] Roughly speaking we can see $\widetilde{\mathcal{M}}(U)$ as the space of all maps defined in U and at values in $\mathcal{M}(U)$, with the topology of pointwise convergence. Hence $\mathcal{M}(U)$ is canonically embedded in $\widetilde{\mathcal{M}}(U)$ as the space of costant maps.

[19] More exactly, in the notations used before, we have $Sh - \mathbf{I} = \mathcal{SH}\left(\int\right)$, where $\int$ is considered as a morphism $\int \in Morph\left(\left\langle \widetilde{\mathcal{M}}^X, F_1 \right\rangle \left\langle \widetilde{\mathcal{M}}, F_2 \right\rangle\right)$ in the category of $Sh-$Spectral Elemnets associated to $\langle \mathcal{HLCS}, \mathcal{VS} \rangle$, where the functors F_1, F_2 at values in the category of vector spaces are such that $(F_1 \circ \widetilde{\mathcal{M}}^X)(U) = \widetilde{M}(U)^X$, while $(F_2 \circ \widetilde{\mathcal{M}})(U) = \widetilde{M}(U)$, for all U open set of $\sigma(R)$.

property, we can deduce that

$$Sh - \mathbf{I}(f) = \int f \, d\mu. \tag{3.5.14}$$

Therefore by (3.5.13) and (3.5.14) the $Sh-$Integral $Sh - \mathbf{I}$ *extends* the usual locally convex vector valued integration also to possibly NOT-integrable maps.

(2) One should note that (3.5.5) allows us to construct maps in $\mathfrak{H}$ by using suitable maps in $\mathfrak{F}^{R,D}$.

Just by employing this fact we can shown the fundamental result that the extension formula in (0.0.19) still holds if we

(a) *replace the standard* $\int$ *with the* $Sh-$*Int* $Sh - \mathbf{I}$;

(b) replace the hypothesis $X \ni x \mapsto f_x(R)$ is scalarly essentially $\mu-$integrable with integral belonging to $B(G)$ with $f_x \in \mathfrak{F}^{R,D}$, $\mu - a.e.$ in X and the map $X \ni x \mapsto [f_x(R_n \upharpoonright G_{\sigma_n}) \upharpoonright D(\sigma_n)] \in \mathcal{M}(\sigma_n)$ is scalarly essentially $\mu-$integrable with integral belonging to $\mathcal{L}(D(\sigma_n), G)$ [20] for all $n \in \mathbb{N}$;

(c) replace $f_x(R)$ with $\widetilde{f_x}(R)$.

Now what is important to point out is that if $\widetilde{f_x}(R) \in \mathcal{M}(\sigma(R)),\ \mu - l.a.e.(X)$ then by what said $\widetilde{f_x}(R) = f_x(R)$ hence we have the remarkable fact that, *although* $X \ni x \mapsto f_x(R) \in \mathcal{M}(\sigma(R))$ *could be NOT scalarly essentially integrable or its integral is not belonging to* $\mathcal{M}(\sigma(R))$*; the extension formula* (0.0.19) *is still true by providing to replace* $\int$ *with* $Sh - \mathbf{I}$.

(3) we have shown a similar result also for the integro-differential formula (0.0.3).

[20]Hence belogs to $\mathcal{M}(\sigma_n)$ being it closed in $\mathcal{L}(D(\sigma_n), G)$