



Interfaces with Other Disciplines

Optimal dividend and scale of business strategies with reinsurance and premium pricing for insurance company

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ABSTRACT

This paper investigates the optimal dividend and business scale strategies aimed at maximizing the value of an insurance company. While prior studies typically assume that insurers can only adjust their business scale through reinsurance, this study extends the framework by allowing the insurer to control the premium rate. Under more realistic market assumptions, we examine the joint optimization problem for two common types of reinsurance — proportional and excess-of-loss — across both arbitrage and non-arbitrage scenarios. We derive the optimal strategies for dividends and premium pricing, along with their corresponding value functions. The results show that the insurer should decrease the premium rate and reduce reinsurance coverage as the surplus increases. The optimal dividend policy follows a barrier strategy. Economic interpretations and numerical examples are provided to illustrate the findings.

1. Introduction

Safety and profitability are key considerations in a company's operations, with risk management and profit pursuit at the core. Though seemingly contradictory, an imbalance between these factors can be counterproductive—excessive profit-seeking increases short-term risks and insolvency potential, while overemphasis on safety reduces profitability. A successful corporate strategy requires balancing these elements to maximize company value. In corporate finance, this balance is often measured by maximizing expected discounted dividends before bankruptcy, inherently accounting for both profitability and risk. For insurance companies, this involves strategies related to dividends, reinsurance, and premium pricing.

Dividends play a crucial role in determining a company's value and act as a key mechanism for profit distribution. A well-structured dividend policy not only safeguards shareholder interests but also builds investor confidence and supports the company's long-term investment and growth objectives (Smith & Watts, 1992). The issue of when to declare dividends and how much to distribute has long attracted scholarly interest, dating back to the seminal work of De Finetti (1957). Since then, a substantial body of research has emerged on the optimal dividend problem from various perspectives, producing significant theoretical and practical insights (see, Asmussen & Taksar, 1997; Schmidli,

2007; Yao et al., 2011; Zhao et al., 2017; Chen et al., 2018; Yang et al., 2020; Zhu et al., 2020; Avanzi et al., 2021).

Reinsurance is an essential tool for insurance companies to manage business scale and transfer risk. Among the various types, proportional and excess-of-loss reinsurance are widely studied due to their operational simplicity and practical applicability (Cao et al., 2023, Amini et al., 2024 and Aboagye et al., 2025). The joint optimization of dividend and reinsurance strategies has attracted considerable academic attention, with notable contributions from Højgaard and Taksar (1999), Asmussen et al. (2000), Bai et al. (2010), Liang and Huang (2011), Zhou and Yuen (2012), Yao et al. (2016), Li et al. (2021), Azcue and Muler (2005), and Meng et al. (2016). However, these studies focus on the no-arbitrage setting. In contrast, this paper extends the analysis by exploring the optimal dividend, reinsurance, and pricing strategies of insurance companies under both no-arbitrage and arbitrage scenarios.

Premium pricing is one of the key determinants of a company's business scale and a fundamental pillar alongside dividends and reinsurance in insurance company management. Setting appropriate premiums requires careful consideration of consumer affordability, company profitability, risk coverage, and long-term sustainability. Prior research has made significant progress in these areas. For example, Martin-Löf (1983), Asmussen et al. (2019a, 2019b), Steffensen and Thøgersen

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(2019) and Thøgersen (2016) proposed effective methods for premium control. Zhou et al. (2017), Liu et al. (2020), and Jiang et al. (2020) investigated optimal investment and premium strategies aimed at maximizing the expected utility of terminal wealth. Højgaard (2002) analyzed the optimal dividend problem under premium control in a diffusion risk model, while Christensen et al. (2021) studied investment and premium strategies to maximize the expected discounted utility of dividends. However, to the best of our knowledge, the joint optimization of dividend, reinsurance, and premium pricing strategies remains underexplored.

Dividends, reinsurance, and premium pricing are crucial tools for managing an insurance company, and their interaction plays a significant role in maintaining financial stability and promoting sustainable growth. In this paper, we aim to bridge this gap in the literature by investigating the joint optimization of these three strategic levers in a continuous-time setting. We assume that both the insurer and the reinsurer follow the expected value principle for premium pricing. The insurer has the ability to influence the premium through the safety loading parameter γ . The arrival of claims is influenced by the premium rate and is modeled by a non-homogeneous Poisson process $\{N(t), t \geq 0\}$ with a time-dependent intensity λ_t . Following Højgaard (2002), we define the intensity as a function of the safety loading, i.e., $\lambda = h(\gamma)$.

To reflect practical considerations, we introduce an upper bound α on the premium rate, above which no customers are willing to purchase insurance, resulting in a claim arrival rate of zero. Based on the relative cost of reinsurance, we classify the problem into two cases: one where reinsurance is expensive and no arbitrage opportunity exists, and another where reinsurance is cheap, allowing for arbitrage. Within this framework, we explore how an insurance company can optimally balance its dividend policy, reinsurance strategy, and premium pricing to maximize its expected discounted dividend payouts. We consider both proportional reinsurance and excess-of-loss reinsurance models and derive closed-form solutions for the value function and the corresponding optimal strategies, which are shown to depend on the structural parameters of the risk model.

Our analysis reveals that, as the insurer's surplus increases, the premium rate tends to decline, thereby facilitating business expansion. In the absence of arbitrage opportunities, the company may optimally choose to exit the market and accept bankruptcy when its surplus reaches zero. In contrast, when arbitrage is possible, the company prefers to continue operating even at zero surplus, as the premium rate exceeds the cost of reinsurance. If reinsurance becomes prohibitively expensive, the insurer is inclined to forgo reinsurance entirely. Interestingly, when the surplus reaches a level sufficient to pay dividends, the premium strategies under both reinsurance types converge. By jointly modeling and analyzing dividends, reinsurance, and premium pricing, this study provides a unified framework that enhances our understanding of insurance company decision-making and offers valuable implications for strategic management under dynamic risk environments.

The remainder of the paper is organized as follows: Section 2 introduces the framework of risk model and raises the optimization problem. Optimal control techniques are employed to solve the optimization problems under proportional reinsurance and excess-of-loss reinsurance, presented in Sections 3 and 4, respectively. Closed-form solutions to the value function and the optimal strategies are presented. Section 5 provides numerical examples to illustrate the main results. Section 6 concludes the paper.

2. Model formulation and the optimal control problem

We start with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, where the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions, that is, $\{\mathcal{F}_t\}_{t \geq 0}$ is right continuous and P -completed. Throughout the paper, it assumes that all stochastic processes and random variables are well-defined and

adapted to the probability space. Similar to references such as Asmussen et al. (2000), Azcue and Muler (2005), we use the following classical risk model to simulate the uncontrolled surplus process:

$$R(t) = x + c(t) - \sum_{i=1}^{N(t)} X_i, \quad (2.1)$$

where $x \geq 0$ is the initial surplus; The claim times process, denoted by $\{N(t), t \geq 0\}$, is described as a non-homogeneous Poisson process with a time-dependent intensity function given by λ_t ; $\{X_i, i = 1, 2, \dots\}$ denote a sequence of positive, identically and independently distributed (i.i.d.) random variables representing the claim sizes, with common distribution function $F(x)$, the finite mean value $\mu > 0$, and the second moment $\sigma^2 > 0$; $c(t)$ denotes the accumulated premiums up to time t , which is determined by the expected value principle with safety loading $\gamma_t > 0$. Then we have

$$c(t) = \mu \int_0^t (1 + \gamma_s) \lambda_s ds. \quad (2.2)$$

Under a reinsurance policy, the positive risk X_i is decomposed into two parts: $S(X_i) \geq 0$ and $X_i - S(X_i) \geq 0$, where $S(X_i)$ is retained by the insurer and $X_i - S(X_i)$ is ceded to the reinsurer. Assume the reinsurance premium rate is also calculated by the expected value principle with a constant safety loading $\eta > 0$, then the surplus process of the insurance portfolio with reinsurance policy is given by

$$U(t) = x + C(t) - \sum_{i=1}^{N(t)} S(X_i), \quad (2.3)$$

where $C(t)$ represents the net premium for the insurer up to time t , calculated as

$$C(t) = \mu \int_0^t (\gamma_s - \eta) \lambda_s ds + (1 + \eta) E(S(X_1)) \int_0^t \lambda_s ds, \quad (2.4)$$

with $\mathbb{E}[S(X_1)] < \infty$ representing the expected claim size under the reinsurance policy, and $\mathbb{E}[S(X_1)^2] < \infty$ denoting its second moment.¹

Following Højgaard (2002) and Christensen et al. (2021), we assume that the insurer determines the premium rate by choosing the safety loading $\gamma_t > 0$ at time t . The claim times process $\{N(t), t \geq 0\}$ has an intensity λ_t that depends on the safety loading, expressed as $\lambda_t = h(\gamma_t)$. This implies that both the customer base and claim frequency are influenced by the premium rate, with λ_t decreasing as γ_t increases. It is reasonable to assume that $h(\gamma_t)$ is a strictly decreasing concave function, reflecting the observation that higher safety loadings reduce portfolio size and claim rates. To incorporate practical considerations, we assume a finite customer base, implying that $h(0) = \lambda_{\max} < \infty$. Furthermore, to ensure the model remains realistic, we assume the existence of a threshold α such that $h(\alpha) = 0$, signifying that no claims occur when the safety loading reaches or exceeds this level.

It is well known that jump models pose significant challenges in optimal control problems. Therefore, we adopt a classical approximation approach to tackle this challenge. Namely, we approximate the model (2.3) by a pure diffusion model $\{U_t, t \geq 0\}$ with the same drift and volatility. Specifically, the approximation process U_t satisfies the following stochastic differential equation

$$dU_t = [\mu(\gamma_t - \eta) \lambda_t + \lambda_t \eta E(S(X_1))] dt + \sqrt{\lambda_t E(S(X_1)^2)} dB_t, \quad (2.5)$$

with $U_0 = x \geq 0$, where $\{B_t, t \geq 0\}$ is a standard Brownian motion adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. $\mathcal{F}_t^B := \sigma\{B_s; 0 \leq s \leq t\}$ is the smallest complete information filtration generated by the Brownian motion process. Motivations and relevant references for this approach, along with more complex examples of diffusion approximations in risk

¹ Since $\{X_i\}_{i=1,2,3,\dots}$ are i.i.d. random variables. Thus, $\mathbb{E}(S(X_1)) = \mathbb{E}(S(X_2)) = \dots$, and we denote the common expectation as $\mathbb{E}(S(X_1))$ without loss of generality.

theory, can be found in Grandell (1991), Meng et al. (2016), Zhou and Yuen (2012).

Let L_t denote the cumulative dividends paid to shareholders from time 0 to time t . In a broader context, an insurance company manages multiple insurance and reinsurance contracts, each with its own signing time and terms. This indicates that the retention strategy for the entire portfolio is fluid and adjustable. This approach more accurately reflects the operational strategies of insurance companies, as they continuously adjust and optimize various contracts. We assume the insurer can dynamically modify retention levels $\{S_t(\cdot), t \geq 0\}$, premium rates $\{\gamma_t, t \geq 0\}$, and dividends $\{L_t, t \geq 0\}$. Given a strategy $\pi = \{\gamma, S(\cdot), L\}$, the controlled surplus process follows:

$$dU_t^\pi = [\mu(\gamma_t - \eta)h(\gamma_t) + \eta h(\gamma_t)E(S_t(X_t))]dt + \sqrt{h(\gamma_t)E(S_t(X_t)^2)}dB_t - dL_t. \quad (2.6)$$

Definition 2.1. A strategy $\pi = \{\gamma, S(\cdot), L\}$ is said to be admissible if (i) The process $\gamma = \{\gamma_t, t \geq 0\}$ is an $\mathcal{F}_t^B$ -predictable process and satisfies that $0 \leq \gamma_t \leq \eta$. (ii) $S(\cdot) = \{S_t(\cdot), t \geq 0\}$ satisfies the following conditions: (1) $S(0) = 0$, and (2) $0 \leq S'(X_t) \leq 1$ and for all $X_t \geq 0$. (iii) $L = \{L_t, 0 \leq t \leq \tau^\pi\}$ is an increasing, $\mathcal{F}_t^B$ -adapted càdlàg process with $L_{0-} = 0$ and $\Delta L_t = L_t - L_{t-} \leq U_{t-}^\pi$ for all $t \geq 0$.

Denote the set of all admissible strategies by Π .

Problem 2.1. Under the strategy $\pi \in \Pi$, the time of bankruptcy associated with U_t^π is defined as

$$\tau^\pi = \inf\{t \geq 0 : U_t^\pi < 0\}. \quad (2.7)$$

Let E_x denote the expectation conditional on $U_{0-}^\pi = x$, and $\delta > 0$ denote the discounted rate. In the view of corporation finance, we use the following performance function to measure the value of insurance company:

$$V(x, \pi) = E_x\left(\int_0^{\tau^\pi} e^{-\delta t} dL_t\right), \quad (2.8)$$

which is the expected present value of the future dividend payments until the time of bankruptcy. We aim at finding the value function

$$V(x) = \sup_{\pi \in \Pi} V(x, \pi), \quad (2.9)$$

and the optimal strategy $\pi^* = \{\gamma^*, S^*(\cdot), L^*\}$, such that $V(x) = V(x, \pi^*)$.

By defining an operator

$$\mathcal{L}^{\gamma, S(\cdot)}v(x) = \left[\frac{1}{2}h(\gamma)E(S(X_1)^2)v''(x) + (\mu(\gamma - \eta) + \eta E(S(X_1)))h(\gamma)v'(x) - \delta v(x)\right], \quad (2.10)$$

we obtain the HJB equation satisfied by the value function:

$$\max\left\{\sup_{S(\cdot), \gamma \in [0, \eta]} \mathcal{L}^{\gamma, S(\cdot)}v(x), 1 - V'(x)\right\} = 0, \quad (2.11)$$

$$\max\left\{-V(0), \sup_{\gamma \in [0, \alpha]} \{\mu(\gamma - \eta)h(\gamma)V'(0) - \delta V(0)\}\right\} = 0. \quad (2.12)$$

Eq. (2.12) represents the boundary condition, indicating that when the surplus reaches zero, the insurer faces two options: declaring bankruptcy or sustaining the company through arbitrage.

Theorem 2.1. Suppose $f(x)$ is an increasing, concave, and twice continuously differentiable solution of (2.11) and (2.12). Then, the following statements hold:

1. For each $\pi \in \Pi$, $f(x) \geq V(x, \pi)$, implying $f(x) \geq V(x)$ for all $x \geq 0$.
2. If there exists a strategy $\pi^* = \{\gamma^*, S^*(\cdot), L^*\} \in \Pi$ such that $f(x) = V(x, \pi^*)$, then $f(x) = V(x)$, and π^* is the associated optimal strategy.

Proof. The proof of this theorem is standard. We refer the readers to Schmidli (2007). $\square$

Next, we analyze the optimization problem under two specific reinsurance arrangements: proportional reinsurance and excess-of-loss reinsurance.

3. The optimal solution in the case of proportional reinsurance

Proportional reinsurance is one of the simplest and most popular forms of reinsurance. In this case, $S(X_t) = aX_t$ denotes a proportional reinsurance contract, where $a \in [0, 1]$ is the risk retention ratio. The reinsurance strategy $a = \{a_t, t \geq 0\}$ is said to be admissible, if it is $\mathcal{F}_t^B$ -adapted process with $0 \leq a_t \leq 1$ for all $t \geq 0$. Then under the strategy $\pi = \{\gamma, a, L\}$, the controlled surplus process (2.6) simplifies to

$$dU_t^\pi = [\mu(\gamma_t - \eta)h(\gamma_t) + \eta\mu a_t h(\gamma_t)]dt + a_t \sigma \sqrt{h(\gamma_t)}dB_t - dL_t. \quad (3.1)$$

As a special case of (2.8) and (2.9), let the performance function be denoted as $V_a(x, \pi) = E_x\left(\int_0^{\tau^\pi} e^{-\delta t} dL_t\right)$, and the value function as $V_a(x) = \sup_{\pi \in \Pi} V_a(x, \pi)$. Combined with Eq. (2.11), the corresponding HJB equation for the value function is:

$$\max\left\{\sup_{a \in [0, 1], \gamma \in [0, \eta]} \left\{\frac{1}{2}h(\gamma)(a\sigma)^2 v_a''(x) + \mu(\gamma - \eta + a\eta)h(\gamma)v_a'(x) - \delta v_a(x)\right\}, 1 - v_a'(x)\right\} = 0, \quad (3.2)$$

with the boundary condition (2.12). From Theorem 2.1, we conclude that $V_a(x) = v_a(x)$.

Denote $b_a = \inf\{x \geq 0 : v_a'(x) \leq 1\}$. Then, by the concavity of $v_a(x)$, it has $v_a'(x) \geq 1$ for $x \leq b_a$. So, for $0 \leq x \leq b_a$, (3.2) reduces to

$$\sup_{a \in [0, 1], \gamma \in [0, \eta]} \left\{\frac{1}{2}h(\gamma)(a\sigma)^2 v_a''(x) + \mu(\gamma - \eta + a\eta)h(\gamma)v_a'(x) - \delta v_a(x)\right\} = 0. \quad (3.3)$$

Differentiating with respect to a and γ respectively, and setting the derivative to zero, we get

$$a\sigma^2 v_a''(x) + \eta\mu v_a'(x) = 0, \quad (3.4)$$

$$(a\sigma)^2 h'(\gamma)v_a''(x) + 2\mu[h(\gamma) + (\gamma - \eta + a\eta)h'(\gamma)]v_a'(x) = 0. \quad (3.5)$$

Solving (3.4) and (3.5) yields

$$\frac{v_a'(x)}{v_a''(x)} = -\frac{a\sigma^2}{\eta\mu} = -\frac{(a\sigma)^2 h'(\gamma)}{2\mu[h(\gamma) + (\gamma - \eta + a\eta)h'(\gamma)]}. \quad (3.6)$$

Moreover, (3.6) gives

$$a = G(\gamma) := -\frac{2[h(\gamma) + (\gamma - \eta)h'(\gamma)]}{\eta h'(\gamma)}. \quad (3.7)$$

The property $G'(\gamma) = -\frac{2}{\eta} + \frac{2h(\gamma)h''(\gamma)}{\eta(h'(\gamma))^2} < 0$ establishes that the function $G(\gamma)$ is decreasing with respect to γ . Now we obtain the relationship between a^* and γ^* . Next, we will discuss the scenario based on whether arbitrage exists or not.²

3.1. No arbitrage: $\eta \geq \alpha$

In this section, we examine the scenario where arbitrage is absent, i.e., reinsurance becomes prohibitively expensive ($\eta \geq \alpha$). This may result from heightened risk uncertainty, constraints in market supply, or reinsurers increasing premiums to mitigate risk and adhere to regulatory requirements. Then boundary condition satisfied by the value function are:

$$v_a(0) = 0, \quad \text{and} \quad \sup_{\gamma \in [0, \alpha]} \{\mu(\gamma - \eta)h(\gamma)v_a'(0) - \delta v_a(0)\} \leq 0. \quad (3.8)$$

² The condition $\eta \geq \alpha$ eliminates any arbitrage potential for the insurer. Conversely, when $\eta < \alpha$, the insurer can achieve arbitrage by transferring all risks to the reinsurer while retaining the profits.

Combined with (3.3) and (3.4), we get

$$(\gamma - \eta + \frac{1}{2}a\eta)\mu h(\gamma)v'_a(x) - \delta v_a(x) = 0. \quad (3.9)$$

Similarly, (3.3) and (3.5) result in

$$-\frac{\mu h^2(\gamma)}{h'(\gamma)}v'_a(x) - \delta v_a(x) = 0. \quad (3.10)$$

Due to $v_a(0) = 0$, it is easy to get that $\gamma^*(0) = \alpha$, and

$$a(0) = \frac{2(\eta - \alpha)}{\eta}. \quad (3.11)$$

Since $a(0) \in [0, 1]$, we conclude that $a^*(0) = 1$ if $a(0) \geq 1$, indicating that reinsurance is too expensive and insurers forgo risk transfer. On the other hand, if $a(0) < 1$, then $a^*(0) = a(0)$.

3.1.1. $\eta \geq 2\alpha$

If $\eta \geq 2\alpha$, which means that reinsurance is too expensive, insurers forgo risk transfer. Then the HJB equation satisfied by the value function is

$$\max \left\{ \sup_{\gamma \in [0, \eta]} \left\{ \frac{1}{2}h(\gamma)\sigma^2 v''_a(x) + \mu\gamma h(\gamma)v'_a(x) - \delta v_a(x) \right\}, 1 - v'_a(x) \right\} = 0, \quad (3.12)$$

Repeating step (3.5), we can get (3.10). For ease of derivation, we first give a lemma.

Lemma 3.1. *There exists a unique solution $\gamma_2 \in (0, \alpha)$ to the equation $h(s) + sh'(s) = 0$.*

Proof. Consider the function $h(s) + sh'(s)$. Its derivative with respect to s is given by

$$\frac{d}{ds} (h(s) + sh'(s)) = h'(s) + h'(s) + sh''(s) = 2h'(s) + sh''(s) < 0,$$

which implies that the function $h(s) + sh'(s)$ is strictly decreasing.

Moreover, we are given that

$$h(\alpha) + \alpha h'(\alpha) < 0 \quad \text{and} \quad h(0) = \lambda_{\max} > 0.$$

Since $h(s) + sh'(s)$ is continuous and strictly decreasing, by the Intermediate Value Theorem, there exists a unique point $\gamma_2 \in (0, \alpha) \subset (0, \eta)$ such that $h(\gamma_2) + \gamma_2 h'(\gamma_2) = 0$. $\square$

Define the function $H(s) := \frac{\mu h^2(s)}{h'(s)} < 0$. Taking the derivative of $H(s)$ with respect to s yields

$$H'(s) = 2\mu h(s) - \mu \frac{h^2(s)h''(s)}{(h'(s))^2} \geq 0, \quad \text{for } s \in [0, \alpha]. \quad (3.13)$$

Combined with Lemma 3.1, $h(s) + sh'(s) < 0$ for $s \in (\gamma_2, \alpha)$. Next, define the function $Q_1(\gamma)$ as

$$x = Q_1(\gamma) := \int_{\gamma}^{\alpha} \frac{H'(s)}{\delta - \frac{2\mu(h(s)+sh'(s))}{h'(s)\sigma^2} H(s)} ds, \quad \gamma \in [\gamma_2, \alpha] \quad (3.14)$$

Combined with (3.13) and $H(s) < 0$ for $s \leq \alpha$, the integrand in (3.14) is positive, which implies that the inverse function exists.

Theorem 3.1. *If $\eta \geq 2\alpha$, then the value function is*

$$V_a(x) = v_a(x) = \begin{cases} \int_{x_a}^x e^{\int_y^{b_a} \frac{2\mu[h(Q_1^{-1}(s))+Q_1^{-1}(s)]h'(Q_1^{-1}(s))}{h'(Q_1^{-1}(s))\sigma^2} dy} dy, & 0 \leq x \leq b_a, \\ x - b_a + v_a(b_a), & x > b_a, \end{cases} \quad (3.15)$$

which is an increasing, concave and twice continuously differentiable solution of (3.2). Here $b_a = Q_1(\gamma_2)$, where $Q_1(\cdot)$ is defined by (3.14). Then the associated optimal premium strategy is described by the safety loading as

$$\gamma^*(x) = \begin{cases} Q_1^{-1}(x), & 0 \leq x \leq b_a, \\ \gamma_2, & x > b_a. \end{cases} \quad (3.16)$$

It is a decreasing function, which means that the insurer should lower the premium rate as the surplus increases. And the optimal risk-taking ratio is to take full risk, that is $a^*(x) = 1$. The optimal dividend strategy is of barrier style with parameter $b_a > 0$, namely,

$$L_t^{\pi^*} = (x - b_a)_+ 1_{\{t \geq 0\}} + \int_0^t 1_{\{X_s^{\pi^*} = b_a\}} dL_s^{\pi^*}. \quad (3.17)$$

It means that whenever the (modified) surplus exceeds the barrier b_a , the excess is paid out immediately as dividends.

Proof. The proof of this theorem is standard. We refer the reader to Højgaard (2002). $\square$

3.1.2. $2\alpha > \eta \geq \alpha$

When $2\alpha > \eta \geq \alpha$, although reinsurance is still expensive, insurers still choose to buy reinsurance to transfer risk when their surplus is low. Combined with (3.9) and (3.10), it is easy to get $\gamma^*(0) = \alpha$ and $a^*(0) = \frac{2(\eta - \alpha)}{\eta}$.

Lemma 3.2. *There exists a unique $\gamma_1 \in (\gamma_2, \alpha)$ such that $G(\gamma_1) = 1$.*

Proof. Since $G(\gamma_2) = -\frac{2[h(\gamma_2) + (\gamma_2 - \eta)h'(\gamma_2)]}{\eta h'(\gamma_2)}$, simplifying gives:

$$G(\gamma_2) = \frac{2\eta h'(\gamma_2)}{\eta h'(\gamma_2)} = 2$$

Additionally, $G(\alpha) = -\frac{2(\alpha - \eta)}{\eta} < 1$ and $G(\gamma)$ is decreasing in γ . By the Intermediate Value Theorem, there exists a unique $\gamma_1 \in (\gamma_2, \alpha)$ such that $G(\gamma_1) = 1$. $\square$

Next, define the function $Q_2(\gamma)$ as

$$x = Q_2(\gamma) := \int_{\gamma}^{\alpha} \frac{H'(s)}{\delta - \frac{\eta\mu}{G(s)\sigma^2} H(s)} ds, \quad \gamma \in [\gamma_1, \alpha] \quad (3.18)$$

Since $\frac{H'(s)}{\delta - \frac{\eta\mu}{G(s)\sigma^2} H(s)} > 0$, the inverse function $\gamma = Q_2^{-1}(x)$ exists and is unique for $\gamma \in [\gamma_1, \alpha]$. Therefore, we have $a(x) = G(Q_2^{-1}(x))$. Define x_a as the point at which the entire risk is assumed, then $x_a := Q_2(G^{-1}(1)) = Q_2(\gamma_1)$.

Next, define the function $Q_3(\gamma)$ as

$$x = Q_3(\gamma) := \int_{\gamma}^{\gamma_1} \frac{H'(s)}{\delta - \frac{2\mu(h(s)+sh'(s))}{h'(s)\sigma^2} H(s)} ds + x_a, \quad \gamma \in [\gamma_2, \gamma_1] \quad (3.19)$$

From (3.7), it is easy to get $h'(\gamma_1)\eta = 2(h(\gamma_1) + \gamma_1 h'(\gamma_1)) < 0$. Combined with Lemmas 3.1 and 3.2, $h(s) + sh'(s) < 0$ for $s \in (\gamma_2, \gamma_1)$. The inverse function $\gamma = Q_3^{-1}(x)$ exists and is unique for $\gamma \in [\gamma_2, \gamma_1]$. In this case, $b_a = Q_3(\gamma_2)$.

Theorem 3.2. *Under proportional reinsurance contract, when $2\alpha > \eta \geq \alpha$, the value function has the following form*

$$V_a(x) = v_a(x) = \begin{cases} n_1 \int_0^x e^{\int_y^{x_a} \frac{\eta\mu}{G(Q_2^{-1}(s))\sigma^2} dy} dy, & 0 \leq x \leq x_a, \\ \int_{x_a}^x e^{\int_y^{b_a} \frac{2\mu[h(Q_3^{-1}(s))+Q_3^{-1}(s)]h'(Q_3^{-1}(s))}{h'(Q_3^{-1}(s))\sigma^2} dy} dy + v_a(x_a), & x_a < x \leq b_a, \\ x - b_a + v_a(b_a), & x > b_a, \end{cases} \quad (3.20)$$

which is an increasing, concave and twice continuously differentiable solution of (3.2). Here $n_1 = e^{\int_{x_a}^{b_a} \frac{2\mu[h(Q_3^{-1}(s))+Q_3^{-1}(s)]h'(Q_3^{-1}(s))}{h'(Q_3^{-1}(s))\sigma^2} ds}$, where $G(\cdot)$, $Q_2(\cdot)$, $Q_3(\cdot)$ are defined by (3.7), (3.18), (3.19), respectively. Then the associated optimal premium strategy is described by the safety loading as

$$\gamma^*(x) = \begin{cases} Q_2^{-1}(x), & 0 \leq x \leq x_a, \\ Q_3^{-1}(x), & x_a < x \leq b_a, \\ \gamma_2, & x > b_a, \end{cases} \quad (3.21)$$

which is a decreasing function. The optimal reinsurance strategy is characterized by the following risk retention ratio

$$a^*(x) = \begin{cases} G(Q_2^{-1}(x)), & 0 \leq x \leq x_a, \\ 1, & x > x_a. \end{cases} \quad (3.22)$$

It is an increasing function, which means that the insurer should reduce reinsurance as the surplus increases. The optimal dividend strategy follows a barrier style, similar to that described in (3.17).

Proof. We will prove that the value function is a solution of the HJB Eq. (3.2). First, we demonstrate that $v'_a(x) \leq 1$ holds for $0 \leq x \leq b_a$. By straightforward calculation, we obtain $v'_a(b_a-) = 1$. The second derivative is explicitly given by

$$v''_a(x) = \begin{cases} -n_1 \frac{\mu\eta}{G(Q_2^{-1}(x))\sigma^2} e^{\int_x^{x_a} \frac{\mu\eta}{G(Q_2^{-1}(s))\sigma^2} ds}, & 0 \leq x \leq x_a, \\ -\frac{2\mu[h(Q_3^{-1}(x)) + Q_3^{-1}(x)h'(Q_3^{-1}(x))]}{h'(Q_3^{-1}(x))\sigma^2} e^{\int_x^{b_a} \frac{2\mu[h(Q_3^{-1}(s)) + Q_3^{-1}(s)h'(Q_3^{-1}(s))]}{h'(Q_3^{-1}(s))\sigma^2} ds}, & x_a < x \leq b_a. \end{cases} \quad (3.23)$$

Combining this result with Lemma 3.1, we deduce $v''_a(x) \leq 0$, which implies $v'_a(x) \geq 1$ for $0 \leq x \leq b_a$.

We now prove that Eq. (3.20) is the solution to (3.2). We rewrite (3.10) as

$$H(\gamma(x))v'_a(x) + \delta v_a(x) = 0. \quad (3.24)$$

Then, taking derivative respect to x yields

$$H(\gamma(x))v''_a(x) + H'(\gamma(x))\gamma'(x)v'_a(x) + \delta v'_a(x) = 0. \quad (3.25)$$

And using (3.4), we obtain the following ordinary differential equation (ODE)

$$\gamma'(x) = \frac{\frac{\eta\mu}{G(\gamma(x))\sigma^2} H(\gamma(x)) - \delta}{H'(\gamma(x))} < 0, \quad (3.26)$$

which means the relationship between γ and x satisfies (3.10). And use (3.6), we can obtain that (3.20) is the solution to (3.2) for $0 \leq x \leq x_a$. Similarly, we can prove that (3.20) is the solution to (3.2) for $x_a < x \leq b_a$.

For $x > b_a$, $v_a(x)$ satisfies that $1 - v'_a(x) = 0$, so $v_a(x) = x - b_a + v_a(b_a)$. And we can get

$$\begin{aligned} & \frac{1}{2} h(\gamma_2)\sigma^2 v''_a(x) + \mu\gamma_2 h(\gamma_2) v'_a(x) - \delta v_a(x) \\ &= \mu\gamma_2 h(\gamma_2) - \delta v_a(x) \\ &< \mu\gamma_2 h(\gamma_2) - \delta v_a(b_a) \\ &= 0. \end{aligned} \quad (3.27)$$

which means that $v_a(x)$ satisfies HJB Eq. (3.2).

It is straightforward to verify that $v_a(x)$ satisfies the boundary condition (2.12). Therefore, we omit the details here. In summary, $v_a(x)$ is an increasing, concave, and twice continuously differentiable solution to the HJB Eq. (3.2). $\square$

3.2. Arbitrage exists: $\alpha > \eta$

When $\alpha > \eta$, reinsurance is relatively cheap and the possibility of arbitrage exists. Combined with (3.9) and (3.10), we can get $\eta(0) = \alpha$ and $a(0) = \frac{2(\eta-\alpha)}{\eta} < 0$ under the boundary condition $v_a(0) = 0$, which is inconsistent with the definition of a . Then the boundary condition becomes

$$-v_a(0) \leq 0, \quad \text{and} \quad \sup_{\gamma \in [0, \alpha]} \{ \mu(\gamma - \eta)h(\gamma)v'_a(0) - \delta v_a(0) \} = 0. \quad (3.28)$$

Similarly to the above, we take the first-order optimality condition,

$$[\mu h(\gamma) + \mu(\gamma - \eta)h'(\gamma)] v'_a(0) = 0. \quad (3.29)$$

Now, define $\zeta(\gamma) := \mu h(\gamma) + \mu(\gamma - \eta)h'(\gamma)$.

Lemma 3.3. There exists a unique $\tilde{\gamma} \in [\eta, \alpha]$ such that $\zeta(\tilde{\gamma}) = 0$.

Proof. Consider the function $\zeta(\gamma)$ defined on the interval $[\eta, \alpha]$. From the derivative

$$\zeta'(\gamma) = 2\mu h'(\gamma) + \mu(\gamma - \eta)h''(\gamma) < 0.$$

Hence, $\zeta(\gamma)$ is strictly decreasing on $[\eta, \alpha]$.

Moreover, we have the boundary values $\zeta(\eta) > 0$ and $\zeta(\alpha) < 0$. By the Intermediate Value Theorem, since $\zeta(\gamma)$ is continuous on $[\eta, \alpha]$ and strictly decreasing, there exists a unique $\tilde{\gamma} \in [\eta, \alpha]$ such that $\zeta(\tilde{\gamma}) = 0$. $\square$

From Lemma 3.3, we obtain $a = G(\tilde{\gamma}) = 0$. This result implies that when $x = 0$, the optimal safety loading is $\tilde{\gamma}$, and the risk-taking ratio is 0. Next, define the function $Q_4(\gamma)$ as

$$x = Q_4(\gamma) := \int_{\gamma}^{\tilde{\gamma}} \frac{H'(s)}{\delta - \frac{\eta\mu}{G(s)\sigma^2} H(s)} ds, \quad \gamma \in [\gamma_1, \tilde{\gamma}]. \quad (3.30)$$

Similar to the (3.18), the inverse function $\gamma = Q_4^{-1}(x)$ exists and is unique for $\gamma \in [\gamma_1, \tilde{\gamma}]$. Therefore, we have $a(x) = G(Q_4^{-1}(x))$, and $x_a := Q_4(G^{-1}(1)) = Q_4(\gamma_1)$.

Next, define the function $Q_5(\gamma)$ as

$$x = Q_5(\gamma) := \int_{\gamma}^{\gamma_1} \frac{H'(s)}{\delta - \frac{2\mu(h(s) + s h'(s))}{h'(s)\sigma^2} H(s)} ds + x_a, \quad \gamma \in [\gamma_2, \gamma_1]. \quad (3.31)$$

Similar to the (3.19), the inverse function $\gamma = Q_5^{-1}(x)$ exists and is unique for $\gamma \in [\gamma_2, \gamma_1]$. In this case, $b_a = Q_5(\gamma_2)$.

Theorem 3.3. Under proportional reinsurance contract, when $\alpha > \eta$, the value function has the following form

$$V_a(x) = v_a(x) = \begin{cases} n_2 \int_0^x e^{\int_y^{x_a} \frac{\mu\eta}{G(Q_4^{-1}(s))\sigma^2} ds} dy + v_a(0), & 0 < x \leq x_a, \\ \int_{x_a}^x e^{\int_y^{b_a} \frac{2\mu(h(Q_5^{-1}(s)) + Q_5^{-1}(s)h'(Q_5^{-1}(s)))}{h'(Q_5^{-1}(s))\sigma^2} ds} dy + v_a(x_a), & x_a < x \leq b_a, \\ x - b_a + v_a(b_a), & x > b_a, \end{cases} \quad (3.32)$$

which is an increasing, concave and twice continuously differentiable solution of Eq. (3.2). Here $n_2 = e^{\int_0^{x_a} \frac{2\mu(h(Q_5^{-1}(s)) + Q_5^{-1}(s)h'(Q_5^{-1}(s)))}{h'(Q_5^{-1}(s))\sigma^2} ds}$, where $G(\cdot)$, $Q_4(\cdot)$, $Q_5(\cdot)$ are defined by Eqs. (3.7), (3.30), (3.31), respectively. And

$$v_a(0) = \frac{(\tilde{\gamma} - \eta)\mu n_2}{\delta} e^{\int_0^{x_a} \frac{\mu\eta}{G(Q_4^{-1}(s))\sigma^2} ds} \quad (3.33)$$

Then the associated optimal premium strategy is described by the safety loading as

$$\gamma^*(x) = \begin{cases} Q_4^{-1}(x), & 0 < x \leq x_a, \\ Q_5^{-1}(x), & x_a < x \leq b_a, \\ \gamma_2, & x > b_a, \end{cases} \quad (3.34)$$

which is a decreasing function. The optimal reinsurance strategy is characterized by the following risk retention ratio

$$a^*(x) = \begin{cases} G(Q_4^{-1}(x)), & 0 < x \leq x_a, \\ 1, & x > x_a, \end{cases} \quad (3.35)$$

which is an increasing function. The optimal dividend strategy follows a barrier style, similar to that described in (3.17).

Proof. The proof is analogous to that of Theorem 3.2 and is omitted for brevity. $\square$

4. The optimal solution in the case of excess-of-loss reinsurance

In this section, we shall discuss the problem in the case of excess-of-loss reinsurance, which is another popular reinsurance style. Let

$S(\cdot) = \min\{\cdot, d\}$ denote an excess-of-loss reinsurance contract, where d is the risk retention level. Denote $\bar{F}(s) = P(X_i > s) = 1 - F(s)$. Considering the insurable conditions, we assume that the support of claim distribution is limited, i.e., $\xi := \sup\{x : F(x) < 1\} < \infty$. It is easy to prove that $2\mu\xi > \sigma^2$, see Yao et al. (2016). Assume that the insurer can dynamically adjust the retention level d . The excess-of-loss reinsurance strategy is determined by the process $d = \{d_t, t \geq 0\}$, which is an $\mathcal{F}_t^B$ -adapted process and $0 \leq d_t \leq \xi$. Define the following functions:

$$g(d) = E(X_i \wedge d) = \int_0^d \bar{F}(s) ds, \quad (4.1)$$

$$p(d) = E((X_i \wedge d)^2) = \int_0^d 2s\bar{F}(s) ds. \quad (4.2)$$

Then under the strategy $\pi = \{\gamma, d, L\} \in \Pi$, the controlled surplus process follows

$$dU_t^\pi = [\mu(\gamma_t - \eta)h(\gamma_t) + \eta h(\gamma_t)g(d_t)]dt + \sqrt{h(\gamma_t)p(d_t)}dB_t - dL_t. \quad (4.3)$$

Similar to Section 3, in the case of excess-of-loss reinsurance, denote the performance function $V_d(x, \pi) = E_x\left(\int_0^\pi e^{-\delta t} dL_t\right)$ and the value function $V_d(x) = \sup_{\pi \in \Pi} V_d(x, \pi)$, respectively. Correspondingly, we give the following HJB equation associated with the value function:

$$\max \left\{ \sup_{d \in [0, \xi], \gamma \in [0, \eta]} \left\{ \frac{1}{2} h(\gamma_t) p(d) v_d''(x) + [\mu(\gamma_t - \eta)h(\gamma_t) + \eta h(\gamma_t)g(d)]v_d'(x) - \delta v_d(x) \right\}, 1 - v_d'(x) \right\} = 0, \quad (4.4)$$

with the boundary condition (2.12).

Assume that the switching point $b_d = \inf\{x \geq 0 : v_d'(x) \leq 1\}$ exists. Then, by the concavity, $v_d'(x) \geq 1$ for $x \leq b_d$. So, for $0 \leq x \leq b_d$, (4.3) reduces to

$$\sup_{d \in [0, \xi], \gamma \in [0, \eta]} \left\{ \frac{1}{2} h(\gamma) p(d) v_d''(x) + [\mu(\gamma - \eta)h(\gamma) + \eta h(\gamma)g(d)]v_d'(x) - \delta v_d(x) \right\} = 0. \quad (4.5)$$

Differentiating with respect to γ and d respectively, and setting the derivative to zero, we obtain

$$h'(\gamma)p(d)v_d''(x) + [2\mu(h(\gamma) + (\gamma - \eta)h'(\gamma)) + 2\eta g(d)h'(\gamma)]v_d'(x) = 0, \quad (4.6)$$

$$d v_d''(x) + \eta v_d'(x) = 0. \quad (4.7)$$

Suppose that the insurer can implement optimal reinsurance and premium strategies simultaneously. From (4.6) and (4.7), we can get the relationship between γ and d as follow

$$\frac{v_d'(x)}{v_d''(x)} = -\frac{d}{\eta} = \frac{-h'(\gamma)p(d)}{2\mu[h(\gamma) + (\gamma - \eta)h'(\gamma)] + 2\eta g(d)h'(\gamma)} \quad (4.8)$$

which is the relationship between optimal retention levels $d^*(x)$ and safety loading $\gamma^*(x)$. Next, we will discuss two scenarios: the presence of arbitrage and the absence of arbitrage.

4.1. No arbitrage: $\eta \geq \alpha$

Similar to the previous section, we first examine the scenario where reinsurance is more expensive and no arbitrage opportunities exist. That is to say, the boundary condition becomes:

$$v_a(0) = 0, \quad \text{and} \quad \sup_{\gamma \in [0, \alpha]} \{\mu(\gamma - \eta)h(\gamma)v_a'(0) - \delta v_a(0)\} \leq 0. \quad (4.9)$$

When $d = \xi$, (4.8) leads to

$$-\frac{\xi}{\eta} = -\frac{h'(\gamma)\sigma^2}{2\mu[h(\gamma) + \gamma h'(\gamma)]}.$$

Thus, define $v(\gamma) := 2\mu\xi[h(\gamma) + \gamma h'(\gamma)] - \eta h'(\gamma)\sigma^2$, and derive that

$$v'(\gamma) = 2\mu\xi[2h'(\gamma) + \gamma h''(\gamma)] - \eta h''(\gamma)\sigma^2 \quad (4.10)$$

$$\begin{aligned} &= 4\xi\mu h'(\gamma) + 2\mu\xi\gamma h''(\gamma) - \eta h''(\gamma)\sigma^2 \\ &< 4\xi\mu h'(\gamma) + 2\mu\xi\eta h''(\gamma) - \eta h''(\gamma)\sigma^2 \\ &= 4\xi\mu h'(\gamma) + \eta h''(\gamma)(2\mu\xi - \sigma^2) \\ &< 0. \end{aligned}$$

The first inequality holds because of $\eta > \gamma$ and the secondary inequality holds because of $2\mu\xi > \sigma^2$. Noting that $v(0) = 2\mu\xi h(0) - \eta h'(0)\sigma^2 > 0$ and $v(\alpha) = 2\mu\xi\alpha h'(\alpha) - \eta h'(\alpha)\sigma^2$. We can conclude the following:

1. When $v(\alpha) < 0$ (i.e., $\alpha < \eta < \frac{2\mu\xi}{\sigma^2}\alpha$), there exists $\gamma_4 \in (\gamma_2, \alpha)$ such that $v(\gamma_4) = 0$, and insurers adopt reinsurance strategies.
2. Conversely, when $v(\alpha) \geq 0$ (i.e., $\eta \geq \frac{2\mu\xi}{\sigma^2}\alpha$), insurers do not consider reinsurance strategies.

4.1.1. $\eta \geq \frac{2\mu\xi}{\sigma^2}\alpha$

When $\eta \geq \frac{2\mu\xi}{\sigma^2}\alpha$, the cost of maintaining an excess-of-loss reinsurance strategy becomes prohibitively high for the insurer, leading them to abandon the reinsurance strategy. In this case, the insurer assumes the entire risk and dynamically adjusts the safety loading of the premium to maximize the company's value. The resulting optimal value function and optimal strategy align with the findings of Theorem 3.1.

4.1.2. $\frac{2\mu\xi}{\sigma^2}\alpha > \eta \geq \alpha$

When $\frac{2\mu\xi}{\sigma^2}\alpha > \eta \geq \alpha$, although reinsurance is more expensive, insurers still choose reinsurance to transfer risk. Putting (4.6) back into (4.5) yields

$$-\frac{\mu h^2(\gamma)}{h'(\gamma)} v_d'(x) - \delta v_d(x) = 0. \quad (4.11)$$

Similar to (3.10), we can get $\gamma(0) = \alpha$.

Lemma 4.1. *There exists a function $q(\cdot)$ such that $d = q(\gamma)$, where $q(\cdot)$ characterizes the relationship between the optimal retention level and the safety loading.*

Proof. For clarity of presentation, we define

$$r_\gamma(d) := 2d\mu[h(\gamma) + (\gamma - \eta)h'(\gamma)] + 2d\eta g(d)h'(\gamma) - \eta h'(\gamma)p(d),$$

which follows from (4.8). Taking the derivative of the above function with respect to d yields

$$r_\gamma'(d) = 2\mu[h(\gamma) + (\gamma - \eta)h'(\gamma)] + 2\eta g(d)h'(\gamma),$$

and further differentiating gives

$$r_\gamma''(d) = 2\eta \bar{F}(d)h'(\gamma) < 0.$$

Thus, $r_\gamma(d)$ is a concave function of d . Combined with the conditions $r_\gamma'(0) = 2\mu[h(\gamma) + (\gamma - \eta)h'(\gamma)] > 0$ and $r_\gamma'(\xi) = 2\mu[h(\gamma) + \gamma h'(\gamma)] < 0$, we conclude that $r_\gamma(d)$ first increases and then decreases. Moreover, since $r_\gamma(0) = 0$ and

$$r_\gamma(\xi) = 2\mu\xi[h(\gamma) + \gamma h'(\gamma)] - \eta h'(\gamma)\sigma^2 < 0$$

for $\gamma \in (\gamma_4, \alpha)$, it follows by the Intermediate Value Theorem that there exists a unique solution $d \in (0, \xi]$ to $r_\gamma(d) = 0$. Therefore, there exists a unique value d_1 such that $r_a(d_1) = 0$, corresponding to the optimal retention level at $x = 0$. Additionally, there exists a function $q(\cdot)$ such that $d = q(\gamma)$ and $r_\gamma(q(\gamma)) = 0$, with $q(\alpha) = d_1$ and $q(\gamma_4) = \xi$. $\square$

Now, define the function:

$$x = I_1(\gamma) := \int_\gamma^\alpha -\frac{H'(s)}{\frac{\eta}{q(s)}H(s) - \delta} ds, \quad \alpha > \gamma \geq \gamma_4. \quad (4.12)$$

Additionally, define $x_d = I_1(\gamma_4)$ as the point at which all risk is assumed. Due to $H(\alpha) = 0$ and $H'(\gamma) \geq 0$, we can get $H(\gamma) < 0$ for $\gamma \in [0, \alpha)$. Then the integrand is greater than 0, and we have $\gamma = I_1^{-1}(x)$ for $0 < x \leq x_d$. In addition, define

$$x = I_2(\gamma) := \int_\gamma^{\gamma_4} -\frac{H'(s)}{\frac{2\mu(h(s) + s h'(s))}{h'(s)\sigma^2}H(s) - \delta} ds + x_d, \quad \gamma_4 > \gamma \geq \gamma_2, \quad (4.13)$$

similar to (3.19), the inverse function exists $\gamma = I_2^{-1}(x)$ for $x_d < x \leq I_2(\gamma_2)$. And

$$k_1 = \exp \left\{ \int_{x_d}^{b_d} \frac{2\mu[h(I_2^{-1}(s)) + I_2^{-1}(s)h'(I_2^{-1}(s))]}{h'(I_2^{-1}(s))\sigma^2} ds \right\} \quad (4.14)$$

Theorem 4.1. Under excess-of-loss reinsurance contract, when $\alpha \leq \eta < \frac{2\mu\xi}{\sigma^2}\alpha$, the value function is given by

$$V_d(x) = v_d(x) = \begin{cases} k_1 \int_0^x e^{\int_y^{x_d} \frac{\eta}{q(I_1^{-1}(s))} dy} dy, & 0 \leq x \leq x_d, \\ \int_{x_d}^x e^{\int_y^{b_d} \frac{2\mu[h(I_2^{-1}(s)) + I_2^{-1}(s)h'(I_2^{-1}(s))]}{h'(I_2^{-1}(s))\sigma^2} ds} dy + v_d(x_d), & x_d < x \leq b_d, \\ x - b_d + v_d(b_d), & x > b_d, \end{cases} \quad (4.15)$$

which is an increasing, concave and twice continuously differentiable solution to (4.4). Here $b_d = I_2(\gamma_2)$ and where k_1 , $I_1(\cdot)$ and $I_2(\cdot)$ are defined by (4.14), (4.12) and (4.13) respectively. The optimal premium control strategy is described by the safety loading as

$$\gamma^*(x) = \begin{cases} I_1^{-1}(x), & 0 \leq x \leq x_d, \\ I_2^{-1}(x), & x_d < x \leq b_d, \\ \gamma_2, & x > b_d. \end{cases} \quad (4.16)$$

It is a decreasing function, which means that the insurer should lower the premium rate as the surplus increases. The optimal reinsurance strategy is characterized by the following retention level

$$d^*(x) = \begin{cases} q(I_1^{-1}(x)), & 0 \leq x \leq x_d, \\ \xi, & x > x_d. \end{cases} \quad (4.17)$$

It is an increasing function, which means that the insurer should reduce reinsurance as the surplus increases. Barrier dividend strategy with parameter $b_d > 0$ is optimal, namely,

$$L_t^{\pi^*} = (x - b_d)_+ 1_{\{t \geq 0\}} + \int_0^t 1_{\{X_s^{\pi^*} = b_d\}} dL_s^{\pi^*}. \quad (4.18)$$

Proof. First, we demonstrate that $v_d'(x) \leq 1$ holds for $0 \leq x \leq b_d$. By straightforward calculation, we obtain $v_d'(b_d-) = 1$. The second derivative is explicitly given by

$$v_d''(x) = \begin{cases} -\frac{\eta}{q(I_1^{-1}(x))} e^{\int_x^{x_d} \frac{\eta}{q(I_1^{-1}(s))} ds}, & 0 \leq x \leq x_d, \\ -\frac{2\mu[h(I_2^{-1}(x)) + I_2^{-1}(x)h'(I_2^{-1}(x))]}{h'(I_2^{-1}(x))\sigma^2} e^{\int_x^{b_d} \frac{2\mu[h(I_2^{-1}(s)) + I_2^{-1}(s)h'(I_2^{-1}(s))]}{h'(I_2^{-1}(s))\sigma^2} ds}, & x_d < x \leq b_d. \end{cases} \quad (4.19)$$

Combining this result with Lemma 3.1, we deduce $v_d''(x) \leq 0$, which implies $v_d'(x) \geq 1$ for $0 \leq x \leq b_d$.

We now prove that Eq. (4.15) is the solution to (4.4). We rewrite (4.11) as

$$H(\gamma(x))v_d'(x) + \delta v_d(x) = 0. \quad (4.20)$$

Then, taking derivative respect to x yields

$$H(\gamma(x))v_d''(x) + H'(\gamma(x))\gamma'(x)v_d'(x) + \delta v_d'(x) = 0. \quad (4.21)$$

And using (4.8) and Lemma 4.1, we obtain the following ordinary differential equation(ODE)

$$\gamma'(x) = \frac{\frac{\eta}{q(x)}H(\gamma(x)) - \delta}{H'(\gamma(x))} < 0, \quad (4.22)$$

which means the relationship between γ and x satisfies (4.12). And use (4.8), we can obtain that (4.15) is the solution to (4.4) for $0 \leq x \leq x_d$. Similarly, we can prove that (4.15) is the solution to (4.4) for $x_d < x \leq b_d$; the details are omitted here for brevity.

Similar to the proof of Theorem 4.2, for $x > b_d$, $v_d(x)$ satisfies $1 - v_d'(x) = 0$, which implies $v_d(x) = x - b_d + v_d(b_d)$. Using (3.27), we can verify that it is indeed a solution to (4.4). It is also straightforward to check that $v_d(x)$ satisfies the boundary condition (2.12); thus, we omit the details for brevity. In summary, $v_d(x)$ is an increasing, concave, and twice continuously differentiable solution to the HJB equation. $\square$

4.2. Arbitrage exists: $\alpha > \eta$

When $\alpha > \eta$, there is room for arbitrage for the insurer. At this point, the boundary conditions are:

$$-v_d(0) \leq 0, \quad \text{and} \quad \sup_{\gamma \in [0, \alpha]} \{\mu(\gamma - \eta)h(\gamma)v_d'(0) - \delta v_d(0)\} = 0. \quad (4.23)$$

Similar to the proportional reinsurance scenario, the insurer adopts a safety loading of $\tilde{\gamma}$ and sets the retention level to 0 at $x = 0$. Combined with Lemma 3.3, we can get $v(\tilde{\gamma}) < 0$. Following an argument analogous to (4.10), we further show that $v'(\gamma) < 0$ for all $\gamma \in [\gamma_2, \tilde{\gamma}]$. The strict monotonicity of v guarantees the existence of a unique $\gamma_4 \in [\gamma_2, \tilde{\gamma}]$ satisfying $v(\gamma_4) = 0$.

Lemma 4.2. There exists a function $q_1(\cdot)$ such that $d = q_1(\gamma)$, with $q_1(\tilde{\gamma}) = 0$ and $q_1(\gamma_4) = \xi$, where $q_1(\cdot)$ characterizes the relationship between the optimal retention level and the safety loading.

Proof. The proof methodology mirrors that of Lemma 4.1, and is omitted here for brevity. $\square$

Next, define the function $I_3(\gamma)$ as

$$x = I_3(\gamma) := \int_{\gamma}^{\tilde{\gamma}} -\frac{H'(s)}{\frac{\eta}{q_1(s)}H(s) - \delta} ds, \quad \tilde{\gamma} > \gamma \geq \gamma_4. \quad (4.24)$$

Similar to the (3.18), the inverse function $\gamma = I_3^{-1}(x)$ exists and is unique for $\gamma \in [\gamma_4, \tilde{\gamma}]$. Therefore, we have $d(x) = q_1(I_3^{-1}(x))$, and $x_d := I_3(\gamma_4)$.

Theorem 4.2. Under excess-of-loss reinsurance contract, when $\alpha > \eta$, the value function has the following form

$$V_d(x) = v_d(x) = \begin{cases} k_2 \int_0^x e^{\int_y^{x_d} \frac{\eta}{q_1(I_3^{-1}(s))} dy} dy + v_d(0), & 0 < x \leq x_d, \\ \int_{x_d}^x e^{\int_y^{b_d} \frac{2\mu[h(I_2^{-1}(s)) + I_2^{-1}(s)h'(I_2^{-1}(s))]}{h'(I_2^{-1}(s))\sigma^2} ds} dy + v_d(x_d), & x_d < x \leq b_d, \\ x - b_d + v_d(b_d), & x > b_d, \end{cases} \quad (4.25)$$

which is an increasing, concave and twice continuously differentiable solution of (4.4). Here $k_2 = e^{\int_{x_d}^{b_d} \frac{2\mu[h(I_2^{-1}(s)) + I_2^{-1}(s)h'(I_2^{-1}(s))]}{h'(I_2^{-1}(s))\sigma^2} ds} dy$, where $I_2(\cdot)$, $I_3(\cdot)$ are defined by (4.13), (4.24), respectively. And

$$v_d(0) = \frac{(\tilde{\gamma} - \eta)\mu k_2}{\delta} e^{\int_0^{x_d} \frac{\eta}{q_1(I_3^{-1}(s))} ds} \quad (4.26)$$

Then the associated optimal premium strategy is described by the safety loading as

$$\gamma^*(x) = \begin{cases} I_3^{-1}(x), & 0 \leq x \leq x_d, \\ I_2^{-1}(x), & x_d < x \leq b_d, \\ \gamma_2, & x > b_d. \end{cases} \quad (4.27)$$

which is a decreasing function. The optimal reinsurance strategy is characterized by the following risk retention level

$$d^*(x) = \begin{cases} q_1(I_3^{-1}(x)), & 0 \leq x \leq x_d, \\ \xi, & x > x_d, \end{cases} \quad (4.28)$$

which is an increasing function. The optimal dividend strategy follows a barrier style, similar to that described in (4.18).

Proof. The proof is analogous to that of [Theorem 4.1](#) and is omitted for brevity. $\square$

Remark 1. When $\eta = 0$, we have $\gamma_1 = \gamma_2 = \gamma_4 = \tilde{\gamma}$. In this case, the insurer does not assume any risk and transfers the entire risk to the reinsurer. At this point, the value function is given by $v_d(x) = v_a(x) = \frac{(\tilde{\gamma}-\eta)\mu}{\delta} + x$.

Based on the discussion of the two reinsurance forms, we find that when $x = 0$, whether or not an insurer using proportional reinsurance adopts a reinsurance strategy and how much risk it takes on mainly considers the safety loading associated with reinsurance and primary premiums. In contrast, those opting for excess-of-loss reinsurance must also take into account the distribution of claims, as it significantly affects their risk exposure. When the insurer retains the entire risk, the optimal premium control strategy depends on three factors: the function $h(\gamma)$, the mean of the claims, and its variance.

5. Numerical examples

In the above two sections, we have presented the value functions and associated optimal strategies in the cases of two common reinsurance contracts, respectively. Next, we will use numerical examples to illustrate the results and make insightful comparisons.

In this section, as the safety loading η varies, we can observe the change of optimal strategies adopted by two companies who buy proportional reinsurance and excess-of-loss reinsurance, respectively. Due to $\sigma^2 < \xi\mu$, $\frac{2\mu\xi}{\sigma^2}\alpha > 2\alpha$. In other words, as the safety loading of reinsurance price η increases, the insurer who adopts proportional reinsurance will abandon reinsurance earlier.

Remark 2. When the safety loading on reinsurance premiums η continues to rise, insurance companies employing proportional reinsurance strategies will be the first to cease transferring risks through reinsurance as the associated costs become prohibitively high.

Now, let us compare the size of γ between Sections 3 and 4, i.e. compare γ_1 and γ_4 , when insurers do not buy reinsurance. (3.6) gives

$$\frac{h(\gamma_1) + \gamma_1 h'(\gamma_1)}{h'(\gamma_1)} = \frac{\eta}{2}, \quad (5.1)$$

and (4.8) gives

$$\frac{h(\gamma_4) + \gamma_4 h'(\gamma_4)}{h'(\gamma_4)} = \frac{\eta\sigma^2}{2\mu\xi}. \quad (5.2)$$

Now, let us define $\phi(x) := \frac{h(x) + xh'(x)}{h'(x)}$, then it has

$$\phi'(x) = \frac{2(h'(x))^2 - h(x)h''(x)}{(h'(x))^2} > 0. \quad (5.3)$$

Due to $\sigma^2 < \mu\xi$, we can get

$$\frac{\eta}{2} > \frac{\eta\sigma^2}{2\mu\xi}. \quad (5.4)$$

Combined with (5.3) and (5.4), we can get $\gamma_4 < \gamma_1$.

Remark 3. If $\eta < 2\alpha$, insurers will implement a reinsurance strategy. Under full risk retention, companies utilizing excess-of-loss reinsurance tend to offer lower premium pricing.

During the premium rate adjustment process, the safety loading γ progressively decreases as the surplus x increases. When the surplus reaches the dividend payment barrier, the safety loading attains its minimum value, γ_2 , as demonstrated in Sections 3 and 4. Consequently, insurers employing two distinct reinsurance strategies will have identical premium rates at the point of dividend payment.

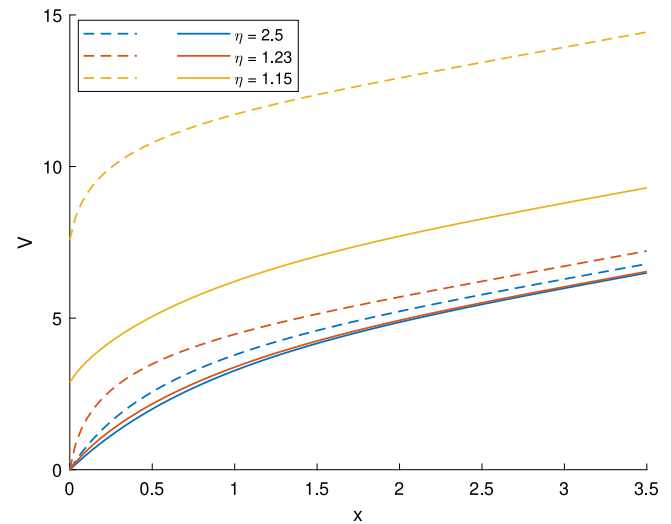


Fig. 1. The value functions with different η .

The solid lines correspond to the proportional reinsurance; The dotted lines correspond to the excess-of-loss reinsurance.

Table 1
Effect of η on optimal strategy.

η	Proportional reinsurance			Excess-of-loss reinsurance		
	1.15	1.23	2.5	1.15	1.23	2.5
$x_d(x_d)$	1.1033	1.0563	0	2.7631	2.8239	3.1670
$b_d(b_d)$	3.5531	3.6146	3.6750	2.8651	2.9327	3.3795

Remark 4. The safety loading γ decreases progressively with surplus growth, attaining its minimum value γ_2 at the dividend payment barrier where insurers employing two distinct reinsurance strategies share identical premium rates during dividend distribution.

Now let us use a numerical example to explore further. Assume that distribution of the claim size is heavy-tailed and follows that

$$F(x) = 1 - \frac{1}{(x+1)^3}, \quad x \leq 100, \quad (5.5)$$

with $\mu = 0.5$ and $\sigma^2 = 0.9803$. Let $\delta = 0.05$ in the following calculations. Our primary focus will be on investigating the impact of the reinsurance premium safety loading η . It should be noted that a higher η signifies greater costs for insurers in terms of risk transfer. Furthermore, we posit that $h(\gamma) = 1.44 - \gamma^2$.

Fig. 1 demonstrates an inverse relationship between η and the company's value. **Table 1** reveals that the dividend threshold b_d (b_d) increases with η , indicating that higher reinsurance costs require greater retention of reserves and delayed dividend distribution. Notably, insurers adopting proportional reinsurance exhibit later dividend payments than those using excess-of-loss reinsurance, as reflected by $b_d > b_d$.

The γ - η relationship is established through **Fig. 2** and **Table 1**. Higher reinsurance costs drive up insurers' premium pricing. Proportional reinsurance users maintain systematically higher premiums than excess-of-loss adopters. This discrepancy stems from fundamental risk transfer differences: excess-of-loss reinsurance shifts catastrophic tail risks to reinsurers, enabling primary insurers to reduce premiums for market expansion and profit maximization. **Remark 4** and **Fig. 2** further identify γ_2 as the unified minimum premium threshold across strategies. **Table 1** shows that increased η delays the attainment of γ_2 , with excess-of-loss insurers reaching this threshold earlier than their proportional reinsurance counterparts.

Fig. 3 and **Table 1** uncover distinct risk retention patterns. Proportional reinsurance insurers assume full risk exposure earlier than

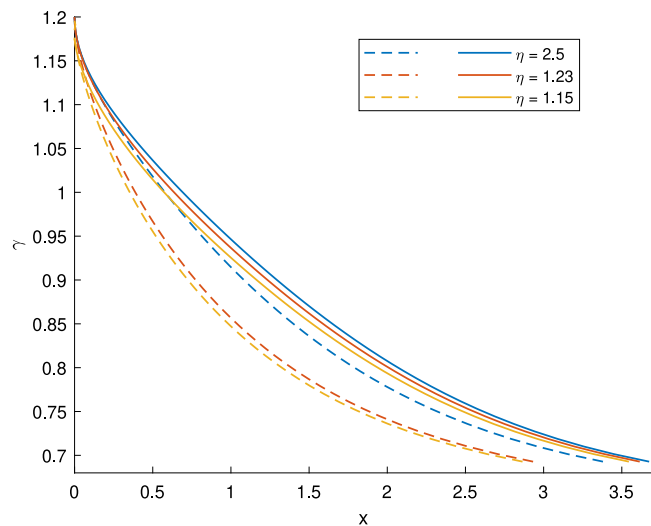


Fig. 2. The optimal premium rates with different η . The solid lines correspond to the proportional reinsurance; The dotted lines correspond to the excess-of-loss reinsurance.

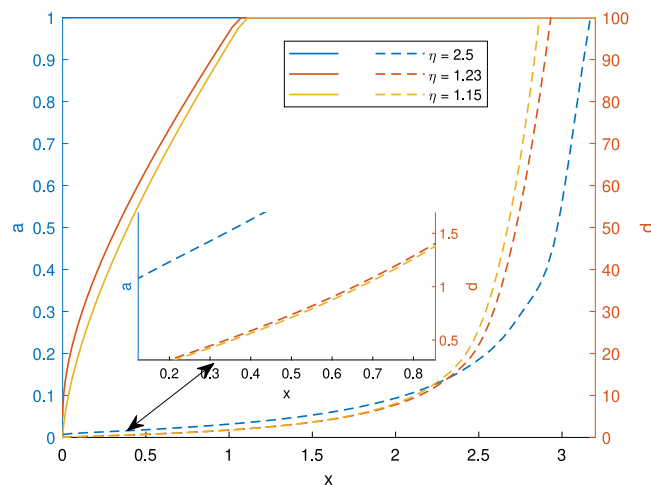


Fig. 3. The optimal retention ratios for reinsurance with different η . The solid lines correspond to the proportional reinsurance; The dotted lines correspond to the excess-of-loss reinsurance.

excess-of-loss users. Crucially, higher η accelerates complete risk retention for proportional reinsurers but delays it for excess-of-loss insurers. The latter exhibit η -dependent risk profiles: lower surplus corresponds to higher risk retention with increased η , while abundant surplus shows an inverse pattern.

6. Conclusion and summary

This paper examines the optimal dividend and business scaling strategy for an insurance company. It assumes that the insurer aims to maximize profits by paying dividends and managing its business scale and risks through reinsurance purchases and premium rate adjustments. Under the object of maximizing the value of company, the trade-off between return and risk is discussed. As a result, a continuous-time optimization problem is formulated, and two scenarios — proportional reinsurance and excess-of-loss reinsurance — are examined under both the presence and absence of arbitrage.

Using stochastic control techniques, closed-form expressions for the value function and optimal strategies are derived for all cases. The

results show that as the surplus increases, the insurer should lower the premium rate and reduce its reliance on reinsurance. The optimal dividend strategy follows a barrier structure. When arbitrage opportunities exist, the insurer chooses to remain in business even when the surplus falls to zero; in contrast, without arbitrage, the insurer opts to exit the market at zero surplus. Furthermore, the insurer will choose to pay dividends directly without assuming any risk only when reinsurance is zero. If reinsurance costs are present, the insurer prefers to retain the risk itself, provided the surplus is sufficient. When the insurer deems the surplus sufficient to support dividend payments, both scenarios converge to the same strategy: retaining all the risk and setting the premium rate at the lowest level γ_2 .

Future research could extend this work by considering more sophisticated dependence structures, such as the common shock model or the thinning-dependence model, to provide deeper insights into optimal insurance management strategies.

CRedit authorship contribution statement

Dingjun Yao: Writing – original draft, Supervision, Methodology, Investigation, Conceptualization. **Bo Yang:** Writing – original draft, Validation, Formal analysis, Data curation, Conceptualization. **Xin Xu:** Writing – original draft, Software, Resources, Formal analysis, Data curation. **Youwei Li:** Writing – review & editing, Supervision, Project administration, Investigation. **Yizhi Wang:** Writing – original draft, Writing – review & editing, Conceptualization, Methodology, Software, Investigation, Formal analysis, Data curation.

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