Strong Admissibility for Infinite Argumentation Frameworks

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Abstract

Strong admissibility plays an important role in formal argumentation under the grounded semantics, especially when explaining the acceptance of an argument. However, strong admissibility has so far only been defined in the context of finite argumentation frameworks. In the current paper, we examine the case of infinite argumentation frameworks. In particular, we assess what the challenges are when moving from finite to infinite argumentation frameworks and we show that despite these challenges, strong admissibility can be meaningfully defined and applied in the context of infinite argumentation frameworks.

Keywords

Abstract Argumentation, Strong Admissibility, Infinite Argumentation Frameworks

1. Introduction

Formal argumentation has become one of the key approaches for symbolic reasoning under uncertainty [1]. Within formal argumentation, strong admissibility [2, 3, 4] plays a key role, especially in the context of grounded semantics. In essence, strong admissibility relates to grounded semantics in a similar way as admissibility relates to preferred semantics, especially when it comes to proof procedures. In order to show that an argument is in a preferred extension, it is not necessary to construct the entire preferred extension. Instead, it is sufficient to show that the argument is in an admissible set. Similarly, in order to show that an argument is in the grounded extension, it is not necessary to construct the entire grounded extension. Instead, it is sufficient to show that the argument is in a strongly admissible set [4]. Such a strongly admissible set can then either be presented in its original form, or be the basis for an interactive explanation in the form of a discussion game [5].

Strong admissibility was originally only defined for finite argumentation frameworks [2, 3, 4, 6, 7]. This can be a limitation, especially when applying strong admissibility in the context of instantiated argumentation. For instance, when applying ASPIC$^+$ [8] with domain independent strict rules (that is, with strict rules based on classical logic entailment) the mere fact that there exist an infinite number of tautologies implies that there will be an infinite number of arguments. As such, it is worthwhile to explore how the concept of strong admissibility can be applied to infinite argumentation frameworks as well.

In the current paper, we examine the challenges when it comes to applying strong admissibility in the context of infinite argumentation frameworks. We provide a novel definition of strong admissibility that can be applied in a meaningful way for both finite and infinite argumentation frameworks. Moreover, we show that our new definition is backwards compatible with existing definitions of strong admissibility that were restricted to finite argumentation frameworks only. That is, we show that for finite argumentation frameworks, our new definition coincides with the existing definitions of strong admissibility. In addition, we show that (even in the context of infinite argumentation frameworks) our new definition satisfies properties very similar to what is satisfied by the existing definitions in the context of finite argumentation frameworks.

ArgU'25: First European Workshop on Argumentation and Uncertainty, September 23, 2025, Hagen, Germany
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The current work is closely related to a paper that has recently been accepted to the ECSQARU 2025 conference [9]. However, where the ECSQARU paper restricts itself to *finitary* argumentation frameworks,[1] the current work reports on subsequent research where we are able to lift this restriction and define strong admissibility for infinite argumentation frameworks in general (finitary or not).

The current paper is structured as follows. First, in Section 2, we provide some basic definitions and formal preliminaries. Then, in Section 3 we present some of the existing definitions of strong admissibility and examine why these are problematic in the context of infinite argumentation frameworks.[2] Then, in Section 4 we introduce a new definition of strong admissibility, one that is well-defined for both finite and infinite argumentation frameworks, is backwards compatible with previous definitions of strong admissibility and satisfies similar properties. We round off in Section 5 with a discussion of the obtained results.

2. Preliminaries

In the current section, we briefly restate some of the key concepts of abstract argumentation theory, in its extension-based form.

Definition 1. *An* argumentation framework *is a pair* (Ar, att) *where* Ar *is a set of entities, called arguments, whose internal structure can be left unspecified, and* att *is a binary relation on* Ar*. For any* $A, B \in Ar$ *we say that* A attacks B *iff* $(A, B) \in att$*. An argumentation framework is called* finite *iff* Ar *is finite, and is called* finitary *iff for each* $A \in Ar$*, the set* $\{B \mid (B, A) \in att\}$ *is finite.*

Definition 2. *Let* $AF = (Ar, att)$ *be an argumentation framework,* $A \in Ar$ *and* $\mathcal{A}rgs \subseteq Ar$*. We define* A^+ *as* $\{B \in Ar \mid A \text{ attacks } B\}$*,* A^- *as* $\{B \in Ar \mid B \text{ attacks } A\}$*,* $\mathcal{A}rgs^+$ *as* $\cup\{A^+ \mid A \in \mathcal{A}rgs\}$*, and* $\mathcal{A}rgs^-$ *as* $\cup\{A^- \mid A \in \mathcal{A}rgs\}$*.* $\mathcal{A}rgs$ *is said to be* conflict-free *iff* $\mathcal{A}rgs \cap \mathcal{A}rgs^+ = \emptyset$*.* $\mathcal{A}rgs$ *is said to* defend A *iff* $A^- \subseteq \mathcal{A}rgs^+$*. The characteristic function* $F_{AF} : 2^{Ar} \rightarrow 2^{Ar}$ *is defined as* $F_{AF}(\mathcal{A}rgs) = \{A \mid \mathcal{A}rgs \text{ defends } A\}$*.*

Definition 3. *Let* $AF = (Ar, att)$ *be an argumentation framework.* $\mathcal{A}rgs \subseteq Ar$ *is said to be:*

- *an* admissible set *of* AF *iff* $\mathcal{A}rgs$ *is conflict-free and* $\mathcal{A}rgs \subseteq F_{AF}(\mathcal{A}rgs)$
- *a* complete extension *of* AF *iff* $\mathcal{A}rgs$ *is conflict-free and* $\mathcal{A}rgs = F_{AF}(\mathcal{A}rgs)$
- *a* grounded extension *of* AF *iff* $\mathcal{A}rgs$ *is the (unique) smallest (w.r.t.* $\subseteq$*) complete extension*
- *a* preferred extension *of* AF *iff* $\mathcal{A}rgs$ *is a maximal (w.r.t.* $\subseteq$*) complete extension*

3. Strong Admissibility and Infinite Argumentation Frameworks

In the current section, we provide a brief overview of strong admissibility in its different forms,[3] as well as of the challenges one encounters when trying to apply this concept in the context of infinite argumentation frameworks. Due to space limitations, we are unable to provide a general discussion of how strong admissibility is applied for finite argumentation frameworks. For this, we refer the reader to [4].

The concept of strong admissibility was first introduced by Baroni and Giacomin [2], using the notion of *strong defence*.

Definition 4 ([2])**.** *Let* (Ar, att) *be an argumentation framework,* $A \in Ar$ *and* $\mathcal{A}rgs \subseteq Ar$*.* A *is* strongly defended *by* $\mathcal{A}rgs$ *iff each attacker* $B \in Ar$ *of* A *is attacked by some* $C \in \mathcal{A}rgs \setminus \{A\}$ *such that* C *is strongly defended by* $\mathcal{A}rgs \setminus \{A\}$*.*

[1] An argumentation framework is finitary iff each argument has a finite number of attackers [10].
[2] Most of the contents of Section 3 is also contained in [9]. The reason for including it here as well is to make the paper self-contained.
[3] Please notice that we restrict ourselves to set-based (instead of labelling-based) definitions of strong admissibility.

Baroni and Giacomin say that a set $\mathcal{A}rgs$ satisfies the strong admissibility property iff it strongly defends each of its arguments [2]. However, it is also possible to define strong admissibility in an equivalent way without having to refer to strong defence [4].

Definition 5 ([4]). *Let* (Ar, att) *be an argumentation framework.* $\mathcal{A}rgs \subseteq Ar$ *is* strongly admissible *iff every* $A \in \mathcal{A}rgs$ *is defended by some* $\mathcal{A}rgs' \subseteq \mathcal{A}rgs \setminus \{A\}$ *which in its turn is again strongly admissible.*

It is important to note that Definition 4 and Definition 5 have so far only been applied in the context of finite argumentation frameworks (that is, argumentation frameworks in which the number of arguments is finite). Unfortunately, these definitions cannot easily be applied in the context where the argumentation framework is infinite. To see why, consider the infinite argumentation framework $AF_1 = (Ar, att)$ where $Ar = \{A_1, A_2, A_3, \ldots\}$ and $att = \{(A_{i+1}, A_i) \mid i \geq 1\}$. This argumentation framework is shown in Figure 1.

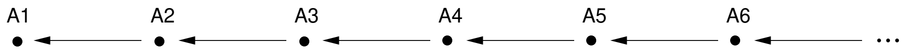

Figure 1: AF_1: each argument is attacked by its successor

In argumentation framework AF_1 there exist precisely three admissible sets: $\emptyset$, $\{A_i \mid i \text{ is odd }\}$ and $\{A_i \mid i \text{ is even }\}$. The first set is the grounded extension. The second and third set are the preferred extensions. However, when trying to apply either Definition 4 or Definition 5 to assess whether the latter two sets are strongly admissible, one stumbles upon a problem. Take for instance the set $\{A_i \mid i \text{ is odd }\}$. When applying Definition 4 to assess whether A_1 is strongly defended by $\{A_i \mid i \text{ is odd }\}$, we observe that A_1's attacker A_2 is attacked by $A_3 \in \{A_i \mid i \text{ is odd }\} \setminus \{A_1\}$. So we need to assess whether A_3 is strongly defended by $\{A_i \mid i \text{ is odd }\} \setminus \{A_1\}$. For this, we need to assess whether A_5 is strongly defended by $\{A_i \mid i \text{ is odd }\} \setminus \{A_1, A_3\}$, etc. The point here is that Definition 4 has a recursive nature, and for the argumentation framework AF_1 the recursion does not end. As such, one could either assume that for each odd j, A_j is strongly defended by $\{A_i \mid i \text{ is odd }\} \setminus \{A_k \mid k \text{ is odd and } k < j\}$, or that for each odd j, A_j is *not* strongly defended by $\{A_i \mid i \text{ is odd }\} \setminus \{A_k \mid k \text{ is odd and } k < j\}$. Both assumptions are consistent with Definition 4, yet only one of them can hold.

A similar problem occurs in the context of Definition 5. Here, in order to determine whether $\{A_i \mid i \text{ is odd }\}$ is a strongly admissible set, we have to determine whether A_1 is defended by some subset of $\{A_i \mid i \text{ is odd }\} \setminus \{A_1\}$ which in its turn is strongly admissible. In essence, Definition 5 is another example of a recursive definition of which the recursion does not end for argumentation framework AF_1.

A third definition of strong admissibility was provided in [4, Lemma 2, Theorem 1].[4]

Definition 6. *Let* $AF = (Ar, att)$ *be an argumentation framework and let* $\mathcal{A}rgs \subseteq Ar$. *Let* $H^0_{\mathcal{A}rgs} = \emptyset$ *and* $H^{i+1}_{\mathcal{A}rgs} = F_{AF}(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$ $(i \geq 0)$. $\mathcal{A}rgs$ *is* strongly admissible *iff* $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs} = \mathcal{A}rgs$.

Definition 6 is not recursive. As such, it avoids the problem of potential infinite recursion. In particular, for AF_1 it can be observed that for any set $\mathcal{A}rgs$, $H^0_{\mathcal{A}rgs} = \emptyset$, $H^1_{\mathcal{A}rgs} = F(H^0_{\mathcal{A}rgs}) \cap \mathcal{A}rgs = \emptyset$, $H^2_{\mathcal{A}rgs} = F(H^1_{\mathcal{A}rgs}) \cap \mathcal{A}rgs = \emptyset$, etc. As such, the only set that is strongly admissible is the empty set, which as we observed before, is also the grounded extension.

Although Definition 6 allows one to unambiguously assess, even for infininite argumentation frameworks, whether a particular set is strongly admissible or not, it still has some issues. Consider the argumentation framework $AF_2 = (Ar, att)$ with $Ar = \{A_i \mid i \geq 1\} \cup \{B\}$ and $att = \{(A_i, A_{i+1}) \mid i \geq 1\} \cup \{(A_j, B) \mid j \text{ is even }\}$. This argumentation framework is shown in Figure 2.

AF_2 only has one complete extension: $\{A_j \mid j \text{ is odd }\} \cup \{B\}$, which is also the grounded extension. Yet, this grounded extension is not strongly admissible, at least not according to Definition 6. This

[4]It has been shown that Definition 4, Definition 5 and Definition 6 are equivalent to each other in the context of finite argumentation frameworks [4].

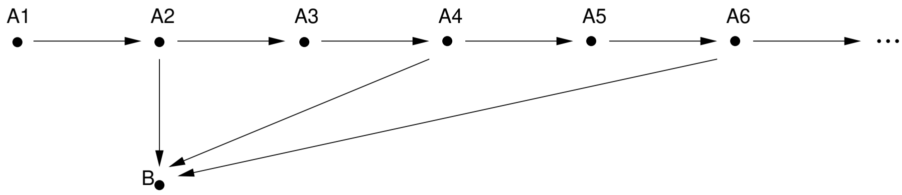

Figure 2: AF_2: an argumentation framework that is not finitary in the sense of [10]

is because (when taking $\mathcal{A}rgs$ as $\{A_j \mid j \text{ is odd }\} \cup \{B\}$) $\cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$ is $\{A_j \mid j \text{ is odd }\}$ instead of $\{A_j \mid j \text{ is odd }\} \cup \{B\}$.[5] More seriously, even though B is in the grounded extension, there is no strongly admissible set that contains B, at least not according to Definition 6. This is a problem, as the whole idea of strong admissibility is to show that an argument is in the grounded extension by showing that it is in a strongly admissible set [6].[6] For finite argumentation frameworks, this property actually holds; in particular, it also holds that the grounded extension is always strongly admissible. For infinite argumentation frameworks, the property unfortunately does not always hold, as shown by the counter example of AF_2.

In [9] the notion of strong admissibility is broadened from finite argumentation frameworks to finitary argumentation frameworks. It is shown that, when restricted to finitary argumentation frameworks, strong admissibility as defined by Definition 6 satisfies the following properties:[7]

- each strongly admissible set is an admissible set
- the empty set is the smallest strongly admissible set (w.r.t. $\subseteq$)
- the grounded extension is the biggest strongly admissible set (w.r.t. $\subseteq$)
- the strongly admissible sets form a lattice (w.r.t. $\subseteq$)

4. A New Definition of Strong Admissibility

In the current section we introduce a new definition of strong admissibility and examine its properties. However, before doing so, we first need to formally define the restriction of an argumentation framework to a set of arguments.

Definition 7. *Let* $AF = (Ar, att)$ *be a (possibly non-finitary) argumentation framework and let* $\mathcal{A}rgs \subseteq Ar$*. We define* $AF_{|\mathcal{A}rgs}$ *as* $(\mathcal{A}rgs, att \cap (\mathcal{A}rgs \times \mathcal{A}rgs))$*.*

Our new definition of strong admissibility is as follows.

Definition 8. *Let* $AF = (Ar, att)$ *be a (possibly non-finitary) argumentation framework and let* $\mathcal{A}rgs \subseteq Ar$*.* $\mathcal{A}rgs$ *is called a* strongly admissible *set iff* $\mathcal{A}rgs$ *is the grounded extension of* $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$*.*

It should be mentioned that Definition 8 is not in any way restricted to finite or even finitary argumentation frameworks. It is designed to be applied to *any* argumentation framework (finite or infinite, finitary or non-finitary).

As an example of how Definition 8 is applied, consider the argumentation framework AF_3, depicted in Figure 3. Here, the strongly admissible sets are $\emptyset$, $\{A\}$, $\{A, C\}$, $\{A, C, F\}$, $\{D\}$, $\{A, D\}$, $\{A, C, D\}$, $\{D, F\}$, $\{A, D, F\}$ and $\{A, C, D, F\}$, the latter also being the grounded extension of AF_3. As an example, the set $\{A, C, F\}$ is strongly admissible because it is the grounded extension of $AF_{3|\{A,C,F\} \cup \{A,C,F\}^-} = AF_{3|\{A,B,C,E,F\}} = (\{A, B, C, E, F\}, \{(A, B), (B, C), (C, E), (E, F)\})$. As another example, the set $\{F\}$, although admissible, is *not* strongly admissible, because it is not the

[5] A similar problem was observed in [10] w.r.t. the inductive proof procedure for grounded semantics.

[6] In a similar way, one shows that an argument is in a preferred extension by showing that it is in an admissible set.

[7] The same properties have previously been proven in [4] in the more restricted context of finite (instead of finitary) argumentation frameworks.

grounded extension of $AF_{3|\{F\}\cup\{F\}^-} = AF_{3|\{E,F\}} = (\{E,F\},\{(E,F),(F,E)\})$, as this grounded extension is $\emptyset$, not $\{F\}$. As for argumentation framework AF_2 (Figure 2), we observe that $\{A_i \mid i$ is odd $\} \cup \{B\}$ is indeed a strongly admissible set. This is because $(\{A_i \mid i$ is odd $\} \cup \{B\}) \cup (\{A_i \mid i$ is odd $\} \cup \{B\})^- = Ar$, $AF_{2|Ar} = AF_2$ and $\{A_i \mid i$ is odd $\} \cup \{B\}$ is indeed the grounded extension of AF_2. As for argumentation framework AF_1 (Figure 1), although there are three admissible sets ($\emptyset$, $\{A_i \mid i$ is odd $\}$ and $\{A_i \mid i$ is even $\}$), only one of them ($\emptyset$) is strongly admissible. To see why for instance $\{A_i \mid i$ is odd $\}$ is *not* strongly admissible, we observe that $\{A_i \mid i$ is odd $\} \cup \{A_i \mid i$ is even $\}^-$ $= \{A_i \mid i$ is odd or even $\} = Ar$, that $AF_{1|Ar} = AF_1$, and that $\{A_i \mid i$ is odd $\}$ is *not* the grounded extension of AF_1 ($\emptyset$ is).

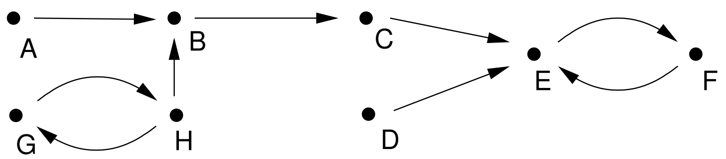

Figure 3: An example of a finite argumentation framework.

It can be observed that our new definition of strong admissibility (Definition 8) is backwards compatible with one of the previous definitions of strong admissibility (Definition 6). In particular, we show that for finitary argumentation frameworks, Definition 8 and Definition 6 coincide.

Theorem 1. *Let $AF = (Ar, att)$ be a finitary argumentation framework and let $\mathcal{A}rgs \subseteq Ar$. $\mathcal{A}rgs$ is a strongly admissible set (in the sense of Definition 8) iff $\mathcal{A}rgs$ is a strongly admissible set (in the sense of Definition 6).*

Proof.

"$\Rightarrow$" Let $\mathcal{A}rgs$ be a strongly admissible set in the sense of Definition 8. That is, $\mathcal{A}rgs$ is the grounded extension of $AF' = AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$. This means that $\mathcal{A}rgs = \cup_{i=0}^{\infty} F^i$ with $F^0 = \emptyset$ and $F^{i+1} = F_{AF'}(F^i)$. We need to prove that $\mathcal{A}rgs = \cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$, with $H^0_{\mathcal{A}rgs} = \emptyset$ and $H^{i+1}_{\mathcal{A}rgs} = F_{AF}(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$. We proceed to show by induction that for each $i \geq 0$, $F^i = H^i_{\mathcal{A}rgs}$.

BASIS ($i = 0$) $F^0 = \emptyset = H^0_{\mathcal{A}rgs}$.

STEP Suppose that for some i it holds that $F^i = H^i_{\mathcal{A}rgs}$. We proceed to show that $F^{i+1} = H^{i+1}_{\mathcal{A}rgs}$.

"$\subseteq$" Suppose $A \in F^{i+1}$. That is, $A \in F_{AF'}(F^i)$. We need to prove that $A \in F_{AF}(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$. From the induction hypothesis, it follows that it is sufficient to show that $A \in F_{AF}(F^i) \cap \mathcal{A}rgs$. We first show that $A \in \mathcal{A}rgs$. This follows from the fact that $\mathcal{A}rgs = \cup_{i=0}^{\infty} F^i$ and that $A \in F^{i+1}$. We proceed to show that $A \in F_{AF}(F^i)$. Let B be an attacker of A in AF. From the fact that $A \in \mathcal{A}rgs$, it follows that $B \in \mathcal{A}rgs^-$. Therefore, B is also an attacker of A in AF'. As $A \in F_{AF'}(F^i)$ it follows that F^i contains a C that attacks B in AF'. But then the same C also attacks B in AF. That is, $A \in F_{AF}(F^i)$.

"$\supseteq$" Suppose $A \in H^{i+1}$. That is, $A \in F_{AF}(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$. We need to prove that $A \in F_{AF'}(F^i)$. From the induction hypothesis, it follows that it is sufficient to show that $A \in F_{AF'}(H^i_{\mathcal{A}rgs})$. That is, we need to prove that A is defended by $H^i_{\mathcal{A}rgs}$ in AF'. The fact that A is defended by $H^i_{\mathcal{A}rgs}$ in AF means that for every attacker B of A in AF, $H^i_{\mathcal{A}rgs}$ contains a C that attacks B in AF. Every attacker of A in AF' is also an attacker of A in AF, so for every attacker B of A in AF', $H^i_{\mathcal{A}rgs}$ contains a C that attacks B in AF. The same C also attacks B in AF'. That is, A is defended by $H^i_{\mathcal{A}rgs}$ in AF'. That is, $A \in F_{AF'}(H^i_{\mathcal{A}rgs})$.

From the thus proved fact that for each $i \geq 0$, $F^i = H^i_{\mathcal{A}rgs}$, it follows that $\cup_{i=0}^{\infty} F^i = \cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$, so from $\mathcal{A}rgs = \cup_{i=0}^{\infty} F^i$ it follows that $\mathcal{A}rgs = \cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$.

"⇐" Let $\mathcal{A}rgs$ be a strongly admissible set in the sense of Definition 6. That is, $\mathcal{A}rgs = \cup_{i=0}^{\infty} H^i_{\mathcal{A}rgs}$ with $H^0_{\mathcal{A}rgs} = \emptyset$ and $H^{i+1}_{\mathcal{A}rgs} = F_{AF}(H^i_{\mathcal{A}rgs}) \cap \mathcal{A}rgs$. We need to prove that $\mathcal{A}rgs$ is the grounded extension of $AF' = AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$. That is, we need to prove that $\mathcal{A}rgs = \cup_{i=0}^{\infty} F^i$ with $F^0 = \emptyset$ and $F^{i+1} = F_{AF'}(F^i)$ This follows from the induction proof above (at "⇒") where it was shown that for each $i \geq 0$, $F^i = H^i_{\mathcal{A}rgs}$.

□

It has previously been shown that for finite argumentation frameworks, Definition 4, Definition 5 and Definition 6 coincide with each other [4]. As such, it follows from Theorem 1 that Definition 8 is backwards compatible not just with Definition 6 but also with Definition 4 and Definition 5 in the context of finite argumentation frameworks.

Now that we have shown our new definition of strong admissibility (Definition 8) to be backwards compatible with the previous definitions of strong admissibility, the next step is to show that it satisfies similar properties. That is, we aim to show that strong admissibility in the sense of Definition 8 satisfies the following properties:

- each strongly admissible set is an admissible set
- the empty set is the smallest strongly admissible set (w.r.t. $\subseteq$)
- the grounded extension is the biggest strongly admissible set (w.r.t. $\subseteq$)
- the strongly admissible sets form a lattice (w.r.t. $\subseteq$)

These properties are to be shown for arbitrary argumentation frameworks (finite or infinite, finitary or non-finitary) instead of just for finite argumentation frameworks [4] or finitary argumentation frameworks [9].

We start with showing that under our new definition of strong admissibility (Definition 8), each strongly admissible set is also an admissible set.

Theorem 2. *Let $AF = (Ar, att)$ be a (possibly non-finitary) argumentation framework and let $\mathcal{A}rgs \subseteq Ar$ be a strongly admissible set (in the sense of Definition 8). It holds that $\mathcal{A}rgs$ is also an admissible set of AF.*

Proof. The fact that $\mathcal{A}rgs$ is a strongly admissible set in the sense of Definition 8 means that $\mathcal{A}rgs$ is the grounded extension of $AF' = AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$. As the grounded extension is admissible, it follows that $\mathcal{A}rgs$ is an admissible set of AF'. As such, $\mathcal{A}rgs$ is conflict-free in AF', from which it follows that $\mathcal{A}rgs$ is also conflict-free in AF. We proceed to show that $\mathcal{A}rgs$ also defends all its elements in AF. Let $A \in \mathcal{A}rgs$. Then for each B that attacks A in AF', there exists a $C \in \mathcal{A}rgs$ that attacks B in AF'. Let B be an attacker of A in AF. As $A \in \mathcal{A}rgs$, it follows that $B \in \mathcal{A}rgs^-$, so B also attacks A in AF'. It then follows that there is a $C \in \mathcal{A}rgs$ that attacks B in AF'. The same C also attacks B in AF. That is, $\mathcal{A}rgs$ defends A in AF. The thus derived facts that $\mathcal{A}rgs$ is conflict-free in AF and defends all of its elements in AF mean that $\mathcal{A}rgs$ is an admissible set in AF. □

The next thing to observe is that the empty set is always strongly admissible.

Proposition 1. *Let $AF = (Ar, att)$ be a (possibly non-finitary) argumentation framework. It holds that the empty set ($\emptyset$) is strongly admissible in the sense of Definition 8.*

Proof. It holds that $AF_{|\emptyset \cup \emptyset^-} = AF_{|\emptyset}$ is the empty argumentation framework, which has the empty set as its grounded extension. □

As no set can be smaller than the empty set, it trivially follows that the empty set is the *smallest* strongly admissible set in the sense of Definition 8.

We proceed to show that the grounded extension is the biggest strongly admissible set. For this, we first show that the grounded extension is strongly admissible.

Theorem 3. *Let $AF = (Ar, att)$ be a (possibly non-finitary) argumentation framework. The grounded extension of AF is strongly admissible (in the sense of Definition 8).*

Proof. Let GE be the grounded extension of AF. That is, GE is the smallest fixpoint of F_{AF}. We have to prove that GE is also the grounded extension of $AF_{|GE \cup GE^-}$. For this, we have to show that GE is the smallest fixpoint of $F_{AF_{|GE \cup GE^-}}$ in $AF_{|GE \cup GE^-}$. We start with showing that GE is a fixpoint of $F_{AF_{|GE \cup GE^-}}$.

1. $GE \subseteq F_{AF_{|GE \cup GE^-}}(GE)$
   Let $A \in GE$. Then from the fact that GE is a fixpoint of F_{AF} it follows that $A \in F_{AF}(GE)$. That is, for each B that attacks A in AF, there is a $C \in GE$ that attacks B in AF. Let B be an argument that attacks A in $AF_{|GE \cup GE^-}$. Then B also attacks A in AF (because every attack in $AF_{|GE \cup GE^-}$ is also an attack in AF), so there is a $C \in GE$ that attacks B in AF. The same C also attacks B in $AF_{|GE \cup GE^-}$ (this is because every attack in AF between arguments in $GE \cup GE^-$ is also an attack in $AF_{|GE \cup GE^-}$, together with the fact that $A \in GE$ and $B \in GE^-$) so $A \in F_{AF_{|GE \cup GE^-}}(GE)$.
2. $F_{AF_{GE \cup GE^-}}(GE) \subseteq GE$.
   Let $A \in F_{AF_{GE \cup GE^-}}(GE)$. Then for each B that attacks A in $AF_{|GE \cup GE^-}$ there is a $C \in GE$ that attacks B in $AF_{|GE \cup GE^-}$. Let B be an argument that attacks A in AF. Then B also attacks A in $AF_{|GE \cup GE^-}$ (this is because B attacks $A \in GE$, so $B \in GE^-$, so both A and B are in $GE \cup GE^-$) so there exists a $C \in GE$ that attacks B in $AF_{|GE \cup GE^-}$. The same C also attacks B in AF (because every attack in $AF_{|GE \cup GE^-}$ is also an attack in AF) so $A \in F_{AF}(GE)$. From the fact that GE is a fixpoint of F_{AF} it follows that $A \in GE$.

Now that we have proved that GE is a fixpoint of $AF_{|GE \cup GE^-}$, the next thing to prove is that GE is also the *smallest* fixpoint of $AF_{|GE \cup GE^-}$. Let $GE' \subseteq GE$ be a fixpoint in $AF_{|GE \cup GE^-}$. That is, $GE' = F_{AF_{|GE \cup GE^-}}(GE')$. We proceed to show that GE' is also a fixpoint in AF.

1. $GE' \subseteq F_{AF}(GE')$
   Let $A \in GE'$. Then from $GE' = F_{AF_{|GE \cup GE^-}}(GE')$ it follows that each B that attacks A in $AF_{|GE \cup GE^-}$ there is a $C \in GE'$ that attacks B in $AF_{|GE \cup GE^-}$. Let B be an argument that attacks A in AF. From the fact that B attacks $A \in GE'$, it follows that $B \in GE'^-$. From the fact that $GE' \subseteq GE$ it follows that $GE'^- \subseteq GE^-$, so $B \in GE^-$. Similarly, from the fact that $A \in GE'$ and $GE' \subseteq GE$ it follows that $A \in GE$. So $A, B \in GE \cup GE^-$. This means that B attacks A not only in AF but also in $AF_{|GE \cup GE^-}$. From the fact that $A \in F_{AF_{GE \cup GE^-}}$ (as $A \in GE' = F_{AF_{GE \cup GE^-}}$) it then follows that there is a $C \in GE'$ that attacks B in $AF_{GE \cup GE^-}$. The same C also attacks B in AF (as every attack in $AF_{|GE \cup GE^-}$ is also an attack in AF). So $A \in F_{AF}(GE')$.
2. $F_{AF}(GE') \subseteq GE'$
   Let $A \in F_{AF}(GE')$. We first observe that as F_{AF} is a monotonic function, $GE' \subseteq GE$ implies that $F_{AF}(GE') \subseteq F_{AF}(GE)$. So the fact that $A \in F_{AF}(GE')$ implies that $A \in F_{AF}(GE)$, so $A \in GE$, which means that A is an argument in $AF_{|GE \cup GE^-}$. The fact that $A \in F_{AF}(GE')$ implies that for every B that attacks A in AF there exists a $C \in GE'$ that attacks B in AF. Let B be an argument that attacks A in $AF_{|GE \cup GE^-}$. Then B also attacks A in AF (as every attack in $AF_{|GE \cup GE^-}$ is also an attack in AF) so there is a $C \in GE'$ that attacks B in AF. The same C also attacks B in $AF_{|GE \cup GE^-}$. So $A \in F_{AF_{|GE \cup GE^-}}(GE')$. As $GE' = F_{AF_{|GE \cup GE^-}}(GE')$ it follows that $A \in GE'$.

From the thus obtained fact that GE' is a fixpoint of F_{AF}, together with the fact that GE is the *smallest* fixpoint of F_{AF} (as GE is the grounded extension of AF) it follows that $GE \subseteq GE'$. This, together with our initial assumption that $GE' \subseteq GE$ implies that $GE = GE'$. Hence, GE is the *smallest* fixpoint of $F_{AF_{|GE \cup GE^-}}$. □

Now that we have established that the grounded extension is a strongly admissible set, we proceed to show that it is also the *biggest* strongly admissible set.

Theorem 4. *Let $AF = (Ar, att)$ be a (possibly non-finitary) argumentation framework. The grounded extension of AF is the* biggest *strongly admissible set of AF in the sense of Definition 8.*

Proof. Let GE be the grounded extension of AF and let $\mathcal{A}rgs \subseteq Ar$ be a strongly admissible set of AF (in the sense of Definition 8) such that $GE \subseteq \mathcal{A}rgs$. We need to prove that $\mathcal{A}rgs \subseteq GE$.
We start with proving that GE is a complete extension of $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$. First of all, from the fact that GE is conflict-free in AF, it follows that GE is also conflict-free in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$ (as each attack in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$ is also an attack in AF). We proceed to prove that $GE = F_{AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}}(GE)$.

"$\subseteq$" Let $A \in GE$. From the fact that $GE \subseteq \mathcal{A}rgs$ it follows that $A \in \mathcal{A}rgs$, so A is an argument in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$. Let B be an argument that attacks A in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$. Then B also attacks A in AF (as every attack in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$ is also an attack in AF). From the fact that $GE = F_{AF}(GE)$ it follows that there is a $C \in GE$ that attacks B in AF. The same C also attacks B in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$, as $C \in GE$ and $GE \subseteq \mathcal{A}rgs$, so $C \in \mathcal{A}rgs \cup \mathcal{A}rgs^-$. Therefore, $A \in F_{AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}}(GE)$.

"$\supseteq$" Let $A \in F_{AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}}(GE)$. From the fact that $F_{AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}}$ is a monotonic function, the fact that $GE \subseteq \mathcal{A}rgs$ implies $F_{AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}}(GE) \subseteq F_{AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}}(\mathcal{A}rgs)$, so $A \in F_{AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}}(\mathcal{A}rgs)$. As $\mathcal{A}rgs$ is a strongly admissible set of AF, it holds by definition that $\mathcal{A}rgs$ is the grounded extension of $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$, so $F_{AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}}(\mathcal{A}rgs) = \mathcal{A}rgs$, so $A \in \mathcal{A}rgs$. The fact that $A \in F_{AF_{\mathcal{A}rgs \cup \mathcal{A}rgs^-}}(GE)$ means that for each B that attacks A in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$ there is a $C \in GE$ that attacks B in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$. Let B be an argument that attacks A in AF. Then the fact that $A \in \mathcal{A}rgs$ implies that B attacks $\mathcal{A}rgs$, so $B \in \mathcal{A}rgs^-$. Therefore, B also attacks A in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$, so there exists a $C \in GE$ that attacks B in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$. The same C also attacks B in AF (as every attack in $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$ is also an attack in AF). Therefore, $A \in F_{AF}(GE)$. As $F_{AF}(GE) = GE$, it directly follows that $A \in GE$.

Now that we obtained that GE is a complete extension of $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$, we can infer that this is a superset of the grounded extension of $AF_{|\mathcal{A}rgs \cup \mathcal{A}rgs^-}$. That is, $GE \supseteq \mathcal{A}rgs$, which is precisely what we needed to prove. □

The next step is to show that the union of two strongly admissible sets is strongly admissible. To do so, we need the following proposition.

Proposition 2. *Let $AF = (Ar, att)$ be a (possibly non-finitary) argumentation framework and let $\mathcal{A}rgs \subseteq Ar$. If $\mathcal{A}rgs$ is conflict-free then $F_{AF}(\mathcal{A}rgs)$ is also conflict-free.*

Proof. Suppose $F_{AF}(\mathcal{A}rgs)$ is not conflict-free. Then $F_{AF}(\mathcal{A}rgs)$ contains arguments A and B such that A attacks B. From the fact that B is defended by $\mathcal{A}rgs$ it follows that there must be a $C \in \mathcal{A}rgs$ that attacks A. From the fact that A is defended by $\mathcal{A}rgs$ it follows that there must be some $D \in \mathcal{A}rgs$ that attacks C. Hence, $\mathcal{A}rgs$ is not conflict-free. □

Theorem 5. *Let $AF = (Ar, att)$ be a (possibly non-finitary) argumentation framework and let $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ be strongly admissible sets of AF (in the sense of Defintion 8). It holds that $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is also a strongly admissible set of AF (in the sense of Definition 8).*

Proof. The fact that $\mathcal{A}rgs_1$ is a strongly admissible set of AF means that $\mathcal{A}rgs_1$ is the grounded extension of $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}$. Similarly, the fact that $\mathcal{A}rgs_2$ is a strongly admissible set of AF means that $\mathcal{A}rgs_2$ is the grounded extension of $AF_{|\mathcal{A}rgs_2 \cup \mathcal{A}rgs_2^-}$. We need to show that $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is the grounded extension of $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$. For this, we first show that $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is a fixpoint of $F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}$. That is, we show that $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ = $F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}(\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2)$.

"$\subseteq$" Let $A \in \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$. Without loss of generality, assume that $A \in \mathcal{A}rgs_1$ (the case of $A \in \mathcal{A}rgs_2$ is similar). Let B be an argument that attacks A in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$. Then by definition, $B \in \mathcal{A}rgs_1^-$. So the fact that $A \in \mathcal{A}rgs_1 = F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}}$ implies that there is a $C \in \mathcal{A}rgs_1$ that attacks B in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}$. The same $C \in \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ also attacks B in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$ (this is because every attack in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}$ is also an attack in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$). So A is defended by $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$. That is, $A \in F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}(\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2)$.

"$\supseteq$" Suppose $A \in F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}(\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2)$. That is, each B that attacks A in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$ is attacked by some $C \in \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$. We need to prove that $A \in \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$. From the fact that A is an argument in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$ it follows that $A \in \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-$. We distinguish two cases:

1. $A \in \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$.
   In that case, we're done.
2. $A \in \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-$.
   Assume without loss of generality that $A \in \mathcal{A}rgs_1^-$ (the case of $A \in \mathcal{A}rgs_2^-$ is similar).
   We first show that $\mathcal{A}rgs_1 \subseteq F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}(\mathcal{A}rgs_1)$. Let $A' \in \mathcal{A}rgs_1$. Let B be an attacker of A' in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$. As $A' \in \mathcal{A}rgs_1$, it follows that $B \in \mathcal{A}rgs_1^-$. Therefore, B also attacks A' in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}$. so from $A' \in \mathcal{A}rgs_1 = F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}}(\mathcal{A}rgs_1)$ it follows that A' is defended by $\mathcal{A}rgs_1$ in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}$, so there is a $C \in \mathcal{A}rgs_1$ that attacks B in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}$. The same C also attacks B in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$. Therefore, A' is defended by $\mathcal{A}rgs_1$ in $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$. That is, $A' \in F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}(\mathcal{A}rgs_1)$.
   From the trivial fact that $\mathcal{A}rgs_1 \subseteq \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ and the fact that $F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}$ is a monotonic function, it follows that $F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}(\mathcal{A}rgs_1) \subseteq F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}(\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2)$. This, together with the above derived fact that $\mathcal{A}rgs_1 \subseteq F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}(\mathcal{A}rgs_1)$ implies that $\mathcal{A}rgs_1 \subseteq F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}(\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2)$. The fact that $A \in \mathcal{A}rgs_1^-$ means A attacks $\mathcal{A}rgs_1$. Hence, $F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}(\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2)$ is not conflict-free. From Lemma 2 it then follows that $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is not conflict-free. However, as both $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ are strongly admissible sets of AF, it follows that the grounded extension of AF contains both $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ (as it follows from Theorem 4 that the grounded extension is the biggest strongly admissible set of AF). That is, the grounded extension of AF contains $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$. Therefore, the grounded extension of AF is not conflict-free. Contradiction.

Now that we have proved that $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is a fixpoint of $F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}$, we proceed to prove that it is also the *smallest* fixpoint. For this, we first observe that $(AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-})_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-} = AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}$. Hence, the fact that $\mathcal{A}rgs_1$ is the grounded extension of $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}$ (as it is an admissible set of AF) trivially implies that it is the grounded extension of $(AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-})_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-} = AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_1^-}$. Hence, $\mathcal{A}rgs_1$ is a strongly admissible set of $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$. This implies that $\mathcal{A}rgs_1$ is a subset of the grounded extension of $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$ (Theorem 4). For similar reasons, $\mathcal{A}rgs_2$ is also a subset of the grounded extension of $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$. So overall, we obtain that $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is a subset of the grounded extension of $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$. The fact that the grounded extension of $AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}$ contains $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ means that *every* fixpoint of $F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}$ contains $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$. Hence, there cannot be a smaller fixpoint of $F_{AF_{|\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2 \cup \mathcal{A}rgs_1^- \cup \mathcal{A}rgs_2^-}}$ than $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$. □

Theorem 5 allows us to infer that $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is a *least* upper bound of $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ as any upper bound of $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ has to be a superset of $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$.

Now that we established that any two strongly admissible sets have a least upper bound, we proceed to examine whether they also have a greatest upper bound. For this, we need the following lemma.[8]

Lemma 1. *Let $AF = (Ar, att)$ be a (possibly non-finitary) argumentation framework and let $\mathcal{A}rgs \subseteq Ar$. $\mathcal{A}rgs$ has a unique biggest (w.r.t. $\subseteq$) strongly admissible subset (in the sense of Definition 8).*

Proof. We first observe that there is always at least one strongly admissible subset of $\mathcal{A}rgs$ (the empty set). We also observe that every increasing sequence $\mathcal{A}rgs_1, \mathcal{A}rgs_2, \mathcal{A}rgs_3, \ldots$ of strongly admissible subsets of $\mathcal{A}rgs$ has an upper bound ($\cup_{i=1}^{\infty} \mathcal{A}rgs_i$ which is again strongly admissible; this follows from Theorem 5; also, it is still a subset of $\mathcal{A}rgs$). This allows us to apply Zorn's lemma and obtain that there is at least one maximal strongly admissible subset of $\mathcal{A}rgs$.[9] We now proceed to show that this maximal strongly admissible subset is unique. Let $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ be maximal strongly admissible subsets of $\mathcal{A}rgs$. Now consider $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$. From Theorem 5 it follows that this is again a strongly admissible set. From the fact that $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ are *maximal* strongly admissible subsets, it follows that if $\mathcal{A}rgs_1 \subseteq \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ then $\mathcal{A}rgs_1 = \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$, and that if $\mathcal{A}rgs_2 \subseteq \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ then $\mathcal{A}rgs_2 = \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$, so we obtain that $\mathcal{A}rgs_1 = \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ and $\mathcal{A}rgs_2 = \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ so $\mathcal{A}rgs_1 = \mathcal{A}rgs_2$. □

We are now ready to introduce one of our main results: the fact that the strongly admissible sets (in the sense of Definition 8) form a lattice.[10]

Theorem 6. *Let $AF = (Ar, att)$ be a (possibly non-finitary) argumentation framework. The strongly admissible sets of AF (in the sense of Definition 8) form a lattice (w.r.t. $\subseteq$).*

Proof. We need to prove that each two strongly admissible sets have a supremum (a least upper bound) and a infimum (a greatest lower bound).

supremum Let $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ be strongly admissible sets. From Theorem 5 it follows that $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is also a strongly admissible set. Since, by definition, $\mathcal{A}rgs_1 \subseteq \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ and $\mathcal{A}rgs_2 \subseteq \mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$, it follows that $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ is an upper bound. Moreover, it is also a *least* upper bound, since any proper subset of $\mathcal{A}rgs_1 \cup \mathcal{A}rgs_2$ will not be a superset of $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$.

infimum Let $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$ be strongly admissible sets. Let $\mathcal{A}rgs_3$ be $\mathcal{A}rgs_1 \cap \mathcal{A}rgs_2$. $\mathcal{A}rgs_3$ has (unique) biggest strongly admissible subset (Lemma 1) which we will refer to as $\mathcal{A}rgs_3'$. We proceed to show that $\mathcal{A}rgs_3'$ is the infimum of $\mathcal{A}rgs_1$ and $\mathcal{A}rgs_2$.

lower bound From the fact that $\mathcal{A}rgs_3' \subseteq \mathcal{A}rgs_3 = \mathcal{A}rgs_1 \cap \mathcal{A}rgs_2 \subseteq \mathcal{A}rgs_1$ and the fact that $\mathcal{A}rgs_3' \subseteq \mathcal{A}rgs_3 = \mathcal{A}rgs_1 \cap \mathcal{A}rgs_2 \subseteq \mathcal{A}rgs_2$ it follows that $\mathcal{A}rgs_3' \subseteq \mathcal{A}rgs_1$ and $\mathcal{A}rgs_3' \subseteq \mathcal{A}rgs_2$.

greatest lower bound Let $\mathcal{A}rgs_3''$ be a strongly admissible set such that $\mathcal{A}rgs_3'' \subseteq \mathcal{A}rgs_1$ and $\mathcal{A}rgs_3'' \subseteq \mathcal{A}rgs_2$. This implies that $\mathcal{A}rgs_3'' \subseteq \mathcal{A}rgs_1 \cap \mathcal{A}rgs_2 = \mathcal{A}rgs_3$. As $\mathcal{A}rgs_3'$ is the *biggest* strongly admissible subset of $\mathcal{A}rgs_3$, it follows that $\mathcal{A}rgs_3'' \subseteq \mathcal{A}rgs_3'$.

□

[8] Lemma 1 uses a similar structure as Lemma 2 of [9]. The main difference is that it uses Theorem 5, which is based on Definition 8.

[9] Although not explicitly mentioned in [10], a similar form of reasoning is needed to prove that maximal admissible sets (i.e. preferred extensions) always exist, even for an infinite argumentation framework with an infinite sequences of ever increasing admissible sets.

[10] Theorem 6 uses a similar structure as Theorem 5 of [4]. The main difference is that it uses Theorem 5 and Lemma 1, which are based on Definition 8.

5. Discussion

Grounded semantics plays an important role in computational argumentation because it is one of the few mainstream semantics that is computationally tractable[11] [11].

However, in many cases, the aim is not just to compute whether an argument is accepted, but also to *show* or *explain* why it is accepted. For preferred semantics, there is a well-known concept (that of an admissible set) that can be used to show that an argument is in a preferred extension. For grounded semantics, the concept of a strongly admissible set plays a similar role. Instead of having to construct (and show) the entire grounded extension, it suffices to show that the argument in question is in a strongly admissible set. As such, strong admissibility provides a local property that can be used to explain membership of the grounded extension.[12] Also, unlike an admissible set, a strongly admissible set can be constructed in polynomial time [6, 7].

Traditionally, an important limitation of strong admissibility was its restricted scope. Whereas the notion of an admissible set was applicable to each argumentation framework (finite or infinite) right from the start [10], the notion of a strongly admissible set was originally only defined for finite argumentation frameworks [2, 4]. The contribution of the current paper is that it breaks with this limitation, making the notion of strong admissibility applicable to *all* argumentation frameworks.

The road towards the current result can be seen as consisting of three steps. The first step was defining strong admissibility in the context of finite argumentation frameworks [2, 4]. This resulted in Definition 4 [2], Definition 5 [4] and Definition 6 [4]. It was proved that in the context of finite argumentation frameworks, these three definitions are equivalent to each other and satisfy the following properties:

1. each strongly admissible set is an admissible set
2. the empty set is the smallest strongly admissible set (w.r.t. $\subseteq$)
3. the grounded extension is the biggest strongly admissible set (w.r.t. $\subseteq$)
4. the strongly admissible sets form a lattice (w.r.t. $\subseteq$)

The second step was to observe that the scope of Definition 6 can be broadened from finite argumentation frameworks to finitary argumentation frameworks, while still satisfying the above four properties. The third step, as presented in the current paper, was to come up with a new definition of strong admissibility (Definition 8) that satisfies the above four properties for *any* arrgumentation framework (finite or infinite). Moreover, we observed that the new definition is backwards compatible with the previous definitions in their respective domains. That is, for finite argumentation frameworks Definition 8 coincides with Definition 4, Definition 5 and Definition 6, and for finitary argumentation frameworks Definition 8 coincides with 6.

On the Relevance of Infinite Argumentation Frameworks

Infinite argumentation frameworks play an important role in the context of instantiated argumentation, where the argumentation framework is constructed using an underlying knowledge base.[13] As an example of how such construction works, consider the case of logic programming based argumentation [14, 15, 16]. Let P_1 be the following logic program [15].

$$\begin{array}{ll} b \leftarrow c, \texttt{not}\ a & a \leftarrow \texttt{not}\ b \\ p \leftarrow c, d, \texttt{not}\ p & p \leftarrow \texttt{not}\ a \\ c \leftarrow d & d \leftarrow \end{array}$$

P_1 would yield the argumentation framework AF_{P_1} depicted in Figure 4 [15].

The basic idea of argument construction in [14, 15, 16] is that the logic programming rules are used to form what can be described as an "derivation tree". Each rule in such a derivation tree that contains

[11] together with conflict-free semantics and native semantics
[12] See [12] for an example of what such an explanation can look like.
[13] Such instantiated argumentation formalisms can be applied for reasoning under incomplete or uncertain information [13].

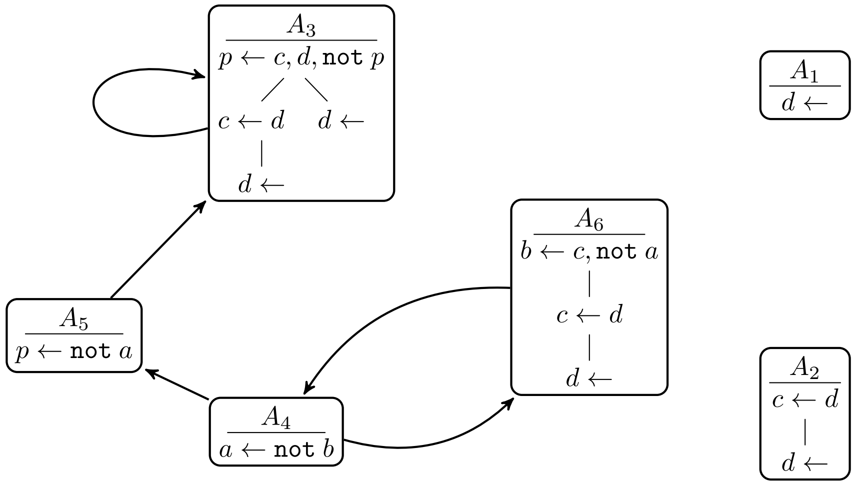

Figure 4: AF_{P_1}: the argumentation framework built using logic program P_1

a strong literal (that is, a literal not preceded by `not`) in its body has to have a child rule that has this literal in its head. Weak negation is ignored when constructing the arguments and is only used when it comes to defining the attacks. That is, an argument A attacks an argument B iff the conclusion of A (that is, the consequent of its top rule) is contained as a weakly negated literal in the body of one of the rules in B. The intuition behind this is that "$\texttt{not}\ c$" means that c is not derivable. So if there actually is a derivation (argument) for c, then this argument attacks everything that contains "$\texttt{not}\ c$".

Although in the example above, a finite logic program P_1 generates a finite argumentation framework AF_{P_1}, this is not always the case. Consider the following logic program P_2.

$$\begin{array}{ll} a \leftarrow & b \leftarrow a \\ a \leftarrow b & c \leftarrow \texttt{not}\ b \end{array}$$

Using P_2, we can construct an argument for c (consisting of the single rule $c \leftarrow \texttt{not}\ b$). We can also construct an infinite number of arguments for b, as there is no limit on how often one can apply the sequence of rules $b \leftarrow a$ and $a \leftarrow b$ in the construction of an argument. This means that the argument for c has an infinite number of attackers, making the argumentation framework non-finitary.[14]

Even when defining a formalism in such a way to prevent the same rule from occurring more than once in a root-originated path of the derivation tree, there are instantiated argumentation formalisms for which this is not sufficient. An example of this would be ASPIC+ [17]. In ASPIC+ there are two types of rules: strict and defeasible. The idea is that strict rules represent inferences that can be made in an underlying classical logic (e.g. propositional logic). In this way, ASPIC+ is able to encapsulate classical logic, in a similar way as for instance Default Logic [18] is able to encapsulate classical logic.[15] Argument construction is done in a comparable way as we showed above for logic programming, with an argument essentially consisting of a "derivation tree" of strict and defeasible rules.

ASPIC+ can lead to an infinite number of arguments because of the way it encapsulates classical logic. As an example, suppose there is an ASPIC+ argument (say A) of which the conclusion is $p \wedge q$. If we would apply the strict rule $p \wedge q \rightarrow p$ (as the strict rules are based on classical logic, and the fact that $p \wedge q \vdash p$ means there is a strict rule $p \wedge q \rightarrow p$) we obtain another argument that consists of A with the rule $p \wedge q \rightarrow p$ on top of it. The point, however, is that from $p \wedge q$ there is an infinite number of ways to derive q, especially if more than one rule is involved. For instance, instead of extending A with the single rule $p \wedge q \rightarrow p$, one could instead extend A with the pair of rules $p \wedge q \rightarrow p \wedge p$ and $p \wedge p \rightarrow p$, or with the pair of rules $p \wedge q \rightarrow p \wedge p \wedge p$ and $p \wedge p \wedge p \rightarrow p$, etc. As such, there is an infinite

[14] Although [14, 15, 16] prevent this situation by disallowing using a rule that has already been used further down the argument, other rule-based instantiated argumentation formalisms such as ASPIC+ [17] do not implement such preventive measures and can therefore generate an infinite argumentation framework even from a finite rule base.

[15] As such, it should not come as a surprise that ASPIC+ is able to model (prioritised) default logic [19].

number of arguments with conclusion p, even without any rules occurring more than once in any of these arguments. Moreover, if there is an argument (say B) that contains an assumption of which p is the contrary, such an argument will have an infinite number of attackers, making the argumentation framework non-finitary.

One last formalism that we would like to mention is Assumption-Based Argumentation (ABA) [20]. In ABA, a derivation is basically a tree-based structure, comparable to what is done in logic programming based argumentation [14, 15, 16] or ASPIC+ [17]. Arguments, however, abstract from the particular structure of such a derivation. This is done by representing arguments as pairs, written as $Asms \vdash c$, where $Asms$ is a set of assumptions such that there exists a derivation that uses these assumptions to infer conclusion c. As such, an ABA argument in essence represents an equivalence class of derivations, each of which uses the same assumptions and derives the same conclusion. Even if there is an infinite number of such derivations in the equivalence class, it will generate only one associated ABA argument. This allows ABA to deal with the kind of issues discussed above. However, in spite of this, ABA can still generate an infinite number of arguments when encapsulating classical logic. This is for instance the case because in classical logic a tautology can be inferred without needing any premises or assumptions, which leads to an ABA argument $\emptyset \vdash t$ for each tautology t. As classical logic supports an infinite number of tautologies, this translates to an infinite number of ABA arguments.

The point we want to make is that the current formalisms for instantiated argumentation tends to generate infinite argumentation frameworks, even when applying a finite knowledge base. In order for any theory of abstract argumentation to be useful for instantiated argumentation, it will need to be applicable for such infinite argumentation frameworks. Theories that are only defined for finite argumentation frameworks are not going to cut it.

Acknowledgments

This publication is supported in part by the Joint Research and Innovation Seed Grants Program between Cardiff University and the University of Illinois System.

References

[1] P. Baroni, D. Gabbay, M. Giacomin, L. van der Torre, Handbook of Formal Argumentation, volume 1, College Publications, 2018.

[2] P. Baroni, M. Giacomin, On principle-based evaluation of extension-based argumentation semantics, Artificial Intelligence 171 (2007) 675–700.

[3] M. Caminada, Strong admissibility revisited, in: S. Parsons, N. Oren, C. Reed, F. Cerutti (Eds.), Computational Models of Argument; Proceedings of COMMA 2014, IOS Press, 2014, pp. 197–208.

[4] M. Caminada, P. Dunne, Strong admissibility revisited: theory and applications, Argument & Computation 10 (2019) 277–300.

[5] M. Caminada, Argumentation semantics as formal discussion, in: Handbook of Formal Argumentation, volume 1, College Publications, 2018, pp. 487–518.

[6] M. Caminada, S. Harikrishnan, Tractable algorithms for strong admissibility, Argument & Computation 16 (2025) 212–235.

[7] M. Caminada, S. Harikrishnan, An evaluation of algorithms for strong admissibility, in: Proceedings of SAFA 2024, 2024, pp. 69–82.

[8] S. Modgil, H. Prakken, The aspic+ framework for structured argumentation: a tutorial, Argument & Computation 5 (2014) 31–62. Special Issue: Tutorials on Structured Argumentation.

[9] M. Caminada, Strong admissibility and infinite argumentation frameworks, in: The 18th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty, 2025. In print.

[10] P. Dung, On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games, Artificial Intelligence 77 (1995) 321–357.

[11] W. Dvořák, P. Dunne, Computational problems in formal argumentation and their complexity, in: Handbook of Formal Argumentation, volume 1, College Publications, 2018, pp. 631–687.
[12] M. Caminada, A discussion game for grounded semantics, in: E. Black, S. Modgil, N. Oren (Eds.), Theory and Applications of Formal Argumentation (proceedings TAFA 2015), Springer, 2015, pp. 59–73.
[13] D. Odekerken, T. Lehtonen, A. Borg, J. Wallner, M. Järvisalo, Argumentative reasoning in aspic+ under incomplete information, in: 20th International Conference on Principles of Knowledge Representation and Reasoning (KR 2023), 2023, pp. 531–541.
[14] Y. Wu, M. Caminada, D. Gabbay, Complete extensions in argumentation coincide with 3-valued stable models in logic programming, Studia Logica 93 (2009) 383–403. Special issue: new ideas in argumentation theory.
[15] M. Caminada, S. Sá, J. Alcântara, W. Dvořák, On the equivalence between logic programming semantics and argumentation semantics, International Journal of Approximate Reasoning 58 (2015) 87–111.
[16] M. Caminada, S. Harikrishnan, S. Sá, Comparing logic programming and formal argumentation; the case of ideal and eager semantics, Argument & Computation 13 (2022) 91–120.
[17] S. Modgil, H. Prakken, Abstract rule-based argumentation, in: Handbook of Formal Argumentation, volume 1, College Publications, 2018, pp. 287–364.
[18] R. Reiter, A logic for default reasoning, Artificial Intelligence 13 (1980) 81–132.
[19] A. Young, S. Modgil, O. Rodrigues, Prioritised default logic as rational argumentation, in: Proceedings of the 15th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2016), 2016, pp. 626–634.
[20] K. Čyras, X. Fan, C. Schulz, F. Toni, Assumption-based argumentation: Disputes, explanations, preferences, in: Handbook of Formal Argumentation, volume 1, College Publications, 2018, pp. 365–408.