

Families of G -constellations over Resolutions of Quotient Singularities

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Abstract. Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. A study is made of the ways in which resolutions of the quotient space \mathbb{C}^n/G can parametrise G -constellations, that is, G -regular finite length sheaves. These generalise G -clusters, which are used in the McKay correspondence to construct resolutions of orbifold singularities.

A complete classification theorem is achieved, in which all the natural families of G -constellations are shown to correspond to certain finite sets of G -Weil divisors, which are a special sort of rational Weil divisor, introduced in this paper. Moreover, it is shown that the number of equivalence classes of such families is always finite. Explicit examples are computed throughout using toric geometry.

1. Introduction

Let $G \subseteq \mathrm{SL}_3(\mathbb{C})$ be a finite subgroup and let X be the quotient space \mathbb{C}^3/G . Nakamura made a study of G -clusters, the G -invariant subschemes of dimension 0 whose coordinate ring, with the induced G -action, is the regular representation V_{reg} of G . He introduced the scheme $G\text{-Hilb } \mathbb{C}^3$, which parametrises all G -clusters and showed [13] that, in the case of G being abelian, it is a crepant resolution of \mathbb{C}^3/G , conjecturing that the same holds for the non-abelian case.

Craw and Reid [5] introduced an alternative way of explicit calculation of $G\text{-Hilb } \mathbb{C}^3$ and in his thesis [3] Craw introduced the concept of G -constellation as a generalisation of G -cluster. A G -constellation is a G -equivariant coherent sheaf whose global sections form the regular representation of G . In particular, the structure sheaf of any G -cluster is a G -constellation.

G -constellations can be interpreted in terms of representations of the McKay quiver of G . This allows for the use of an earlier result of King [10] on GIT construction of moduli spaces of quiver representations to introduce the stability conditions known as θ -stability on G -constellations and to construct their moduli spaces M_θ . In a quiver-theoretic context, Kronheimer [11] and Sardo Infirri [15, 16] have already considered these moduli spaces and have studied the chamber structure in the space Π of stability parameters θ , where all values of θ in the same chamber yield the same M_θ . Bridgeland, King and Reid [2] use

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derived category methods to show, in case of arbitrary $G \subseteq \mathrm{SL}_3(\mathbb{C})$ that $G\text{-Hilb } \mathbb{C}^3$ is a crepant resolution of X . Their method can be used to show that, for any chamber in Π , M_θ is a crepant resolution, however it yields little information about either the structure of the chamber space or the geometry of M_θ s.

Craw in his thesis conjectured that every projective crepant resolution of X can be realised as a moduli space M_θ of θ -stable G -constellations for some chamber in Π . A recent paper by Craw and Ishii [4] proves this for all abelian $G \subset \mathrm{SL}_3(\mathbb{C})$.

In this paper, we take a different approach to this issue. Rather than constructing a resolution as a moduli space of G -constellations, we shall take an arbitrary (not necessarily projective or crepant) resolution of X and study what families of G -constellations it can parametrise.

To start with let G be any finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$ and Y any scheme birational to the quotient space $X = \mathbb{C}^n/G$:

$$\begin{array}{ccc} Y & & \mathbb{C}^n \\ & \searrow \pi & \swarrow q \\ & X & \end{array}$$

Here q is the quotient map and π is the birational morphism.

Let R denote the coordinate ring $\mathbb{C}[x_1, \dots, x_n]$ of \mathbb{C}^n . A (G, R) -module is a G -representation V together with a G -equivariant action of R . The categories of finite-length G -equivariant coherent sheaves on \mathbb{C}^n and of (G, R) -modules are equivalent and in this paper we work in the latter category.

We would like the families of G -constellations which we study to be related, geometrically, to the space Y which parametrises them. That is, we would like to single out a set of ‘natural’ families of G -constellations on Y . For instance, for any point $y \in Y$ we have its image $\pi(y)$ in X and hence an orbit $q^{-1}\pi(y)$ of G in \mathbb{C}^n . On the other hand, a G -constellation is a G -equivariant finite-length sheaf and hence is supported on a finite union of G -orbits in \mathbb{C}^n . It seems reasonable to ask for the G -constellation parametrised by $y \in Y$ to be supported, set theoretically, precisely on $q^{-1}\pi(y)$.

Observe now that, due to dimension considerations, there is only one G -constellation supported at any free orbit of G in \mathbb{C}^n , up to an isomorphism. This G -constellation is precisely the structure sheaf \mathcal{O}_Z of G -cluster Z given by that orbit. Thus $q_*\mathcal{O}_{\mathbb{C}^n}$, over any subset U of X such that G acts freely on $q^{-1}(U)$, is a unique (up to a twist by a line bundle) family of G -constellations satisfying the wanted property on supports. Observe that its fiber at the generic point of X is the G -constellation $K(\mathbb{C}^n) \simeq V_{\mathrm{reg}} \otimes K(X)$, which we can think of as corresponding to the generic orbit of G . As any scheme birational to X shares its generic point p_X , the very least any natural family should do is to have p_X parametrise a G -constellation isomorphic to $K(\mathbb{C}^n)$. We call such families *deformations of*

the generic orbit of G across Y . We then show (see Proposition 2.5) that this requirement on the fiber of the family at the generic point implies much stronger naturality properties: for any point $y \in Y$, the support of the G -constellation it parametrises is indeed $q^{-1}\pi(y)$, set-theoretically. Moreover, any such family can be G and R equivariantly embedded into the constant sheaf $K(\mathbb{C}^n)$ on Y .

Now for G abelian, any family of G -constellations is a direct sum of invertible G -eigensheaves. On any scheme S , to consider invertible \mathcal{O}_S -submodules of $K(S)$ is to consider Cartier divisors on S . Therefore in Section 3 we extend the construction of Cartier divisors on Y , as global sections of $K^*(Y)/\mathcal{O}_Y^*$, by defining a G -Cartier divisor to be a global section of $K_G^*(\mathbb{C}^n)/\mathcal{O}_Y^*$, where $K_G^*(\mathbb{C}^n)$ is the group of all non-zero G -homogeneous rational functions on \mathbb{C}^n .

To make a link with Weil divisors, we make the natural extension of the concept of the valuation at a prime divisor from $K(Y)$ to $K_G^*(\mathbb{C}^n)$. We then define G -Weil divisors (see Definition 3.5) as a subset of \mathbb{Q} -Weil divisors on Y , in such a way as to have the correspondence between G -Weil and G -Cartier divisors in place when Y is smooth.

Now as any deformation \mathcal{F} of the generic orbit embeds into $K(\mathbb{C}^n)$ as a (G, R) -submodule, each of its eigensheaves \mathcal{F}_χ , together with its embedding into $K(\mathbb{C}^n)$ defines a G -Cartier divisor and consequently a G -Weil divisor D_χ . Conversely, any set $\{D_\chi\}$, where for each $\chi \in G^\vee$ we have one χ -Weil divisor D_χ , defines an \mathcal{O}_Y -submodule $\bigoplus \mathcal{L}(-D_\chi)$ of $K(\mathbb{C}^n)$. For it to be a (G, R) -submodule, and hence a deformation of the generic orbit, we need the R -action on $K(\mathbb{C}^n)$ to restrict down to it. We show that this is precisely equivalent to the condition that for (f) the principal divisor of any G -homogeneous $f \in R$,

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0$$

where $\rho(f)$ is the weight of f . Now it is clearly sufficient for this to be true just for $f = x_1, \dots, x_n$, the basic monomials. Thus we establish a one-to-one correspondence between deformations of the generic orbit and sets $\{D_\chi\}_{\chi \in G^\vee}$ of G -Weil divisors satisfying a finite number of inequalities.

It is usual in moduli problems to consider the families up to equivalence, that is twisting by a line bundle. We show that any equivalence class of deformations of the generic orbits contains a unique family with $D_{\chi_0} = 0$ in the corresponding divisor set. We call such deformations of the generic orbit *normalized*. On the other hand, the requirement for the subsheaf $\bigoplus \mathcal{L}(-D_\chi)$ of $K(\mathbb{C}^n)$ to be closed under R -action can be seen to imply that all the eigensheaves $\mathcal{L}(-D_\chi)$ must be, in a certain sense, close to each other inside $K(\mathbb{C}^n)$. When $D_{\chi_0} = 0$, this allows us to put a precise bound on how far from 0, numerically, all the other divisors D_χ can be. Explicitly, we define the set $\{M_\chi\}$ by

$$M_\chi = \sum_P \left(\min_{f \in R_\chi} v_P(f) \right) P,$$

where P ranges over all prime Weil divisors on Y . We show that $\bigoplus \mathcal{L}(-M_\chi)$ is a deformation of the generic orbit, and in case of Y being G -Hilb \mathbb{C}^n it is the tautological family of G -clusters parametrised by Y . Then we prove that for any normalized deformation of the generic orbit, the corresponding divisor set $\{D_\chi\}$ satisfies

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}}.$$

In particular, this implies that the number of equivalence classes is finite as we show that the only non-zero summands of M_χ are the exceptional divisors and the proper transforms in Y of images in X of coordinate hyperplanes of \mathbb{C}^n .

Thus our main result (see Theorem 5.25) is

Theorem 1.1 (Classification). *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X be the quotient of \mathbb{C}^n by the action of G and Y be a resolution of X . Then all deformations of the generic orbit across Y , up to isomorphism, are of form $\bigoplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$, where each D_χ is a χ -Weil divisor and the set $\{D_\chi\}$ satisfies the inequalities*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0$$

for all $\chi \in G^\vee$ and all G -homogeneous $f \in R$. Here $\rho(f)$ is the homogeneous weight of f . Conversely for any such set $\{D_\chi\}$, $\bigoplus \mathcal{L}(-D_\chi)$ is a deformation of the generic orbit.

Moreover, each equivalence class of families has precisely one family with $D_{\chi_0} = 0$. The divisor set $\{D_\chi\}$ corresponding to such a family satisfies inequalities

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}},$$

where $\{M_\chi\}$ is a fixed divisor set depending only on G and Y . In particular, the number of equivalence classes of families is finite.

Throughout the paper we illustrate the proceedings with examples from toric geometry, which allows for explicit calculations on Y whenever G is abelian. A brief summary of the toric setup as applied to our problem is given in Section 4. Then we introduce Y on which all of the examples will be calculated: a single toric flop of G -Hilb \mathbb{C}^3 with G being the cyclic subgroup of $\mathrm{GL}_3(\mathbb{C})$ of order 8 traditionally denoted $\frac{1}{8}(1, 2, 5)$.

2. Deformations of the generic orbit

2.1. G -constellations and families

Let G be a finite abelian group and let V_{giv} be an n -dimensional faithful representation of G . We identify the symmetric algebra $S(V_{\mathrm{giv}}^\vee)$ with the coordinate ring R of \mathbb{C}^n via a

choice of such an isomorphism that the induced action of G on \mathbb{C}^n is diagonal. By the dual action of G on R we shall mean the left action given by

$$(2.1) \quad g \cdot f(\mathbf{v}) = f(g^{-1} \cdot \mathbf{v}), \quad \forall \mathbf{v} \in \mathbb{C}^n.$$

Corresponding to the inclusion $R^G \subset R$ of the subring of G -invariant functions we have the quotient map $q: \mathbb{C}^n \rightarrow X$, where $X = \operatorname{Spec} R^G$ is the quotient space. This space is generally singular. So we are typically interested in taking resolutions $\pi: Y \rightarrow X$ of it:

$$\begin{array}{ccc} Y & & \mathbb{C}^n \\ & \searrow \pi & \swarrow q \\ & X & \end{array}$$

The purpose of this paper is to study the way in which Y can parametrise families of G -constellations.

Definition 2.1. [4] A G -constellation is a G -equivariant coherent sheaf \mathcal{F} on \mathbb{C}^n such that $H^0(\mathcal{F})$ is isomorphic, as a $\mathbb{C}[G]$ -module, to the regular representation V_{reg} .

Of course as \mathcal{F} is coherent, it is uniquely determined by $H^0(\mathcal{F})$ via the $\tilde{\cdot}$ construction [8, p. 110]. The actions of G and R on \mathcal{F} are entirely determined by their restrictions to $H^0(\mathcal{F})$. In this paper we shall adopt this more algebraic point of view, and consider a following class of objects.

Definition 2.2. A (G, R) -module is a $\mathbb{C}[G]$ -module V together with an equivariant R -action, that is,

$$(2.2) \quad g \cdot (f \cdot \mathbf{v}) = (g \cdot f) \cdot (g \cdot \mathbf{v})$$

must hold for all $\mathbf{v} \in V$, $g \in G$ and all $f \in R$.

A morphism of (G, R) -modules is a G and R equivariant linear map of the underlying vector spaces.

The functors $\tilde{\cdot}$ and $H^0(\cdot)$ provide an equivalence between the categories of finite-length coherent G -equivariant sheaves on \mathbb{C}^n and of (G, R) -modules, thus we can use both concepts interchangeably.

Any R -action on V is defined by an element of $\operatorname{Hom}_{\mathbb{C}}(R \otimes_{\mathbb{C}} V, V)$. As $R = S(V_{\text{giv}}^{\vee})$ it is sufficient to consider restrictions to $\operatorname{Hom}_{\mathbb{C}}(V_{\text{giv}}^{\vee} \otimes V, V)$. The condition (2.2) is precisely equivalent to asking for this homomorphism to be G -equivariant.

Conversely, $\alpha \in \operatorname{Hom}_G(V_{\text{giv}}^{\vee} \otimes V, V)$ defines an R -action on V if and only if it satisfies

$$(2.3) \quad \alpha(v_1 \otimes \alpha(v_2 \otimes v)) = \alpha(v_2 \otimes \alpha(v_1 \otimes v)).$$

Thus we see that there exists a one-to-one correspondence between all the (G, R) -modules with an underlying $\mathbb{C}[G]$ -module V and the elements of $Z_{R,G} \subseteq \text{Hom}_G(V_{\text{giv}}^\vee \otimes V, V)$ satisfying the commutator conditions (2.3).

Further, it can be seen that the R -structures of two isomorphic (G, R) -modules on V differ by conjugation by an element of $\text{Aut}_G(V)$. Therefore we have a one-to-one correspondence between isomorphism classes of (G, R) -modules with underlying $\mathbb{C}[G]$ -module V and the orbits of $\text{Aut}_G(V)$ in $Z_{R,G}$.

Definition 2.3. A *family of (G, R) -modules parametrised by a scheme S* is a locally free sheaf \mathcal{F} of \mathcal{O}_S -modules with G and R acting by \mathcal{O}_S -linear endomorphisms, so that

$$g \cdot (f \cdot s) = (g \cdot f) \cdot (g \cdot s)$$

for all $g \in G$, $f \in R$ and any local section s of \mathcal{F} .

We shall say that two families \mathcal{F} and \mathcal{F}' are equivalent if there exists an invertible sheaf \mathcal{L} on S such that \mathcal{F} is (G, R) -equivariantly isomorphic to $\mathcal{F}' \otimes \mathcal{L}$.

We shall call \mathcal{F} a *family of G -constellations* if its fiber $\mathcal{F}|_p$ at any point $p \in Y$ is a G -constellation.

Any sheaf \mathcal{F} with a G -action must split into G -eigensheaves, which are locally free if \mathcal{F} is. In particular, we see that for an abelian G any family of G -constellations must split as

$$\bigoplus_{\chi \in G^\vee} \mathcal{L}_\chi$$

where G acts on each invertible sheaf \mathcal{L}_χ by the character χ .

Any free G -orbit $Z \subset \mathbb{C}^n$ is a G -cluster, its structure sheaf \mathcal{O}_Z a G -constellation. Considering $H^0(\mathcal{O}_Z)$ as the fibre of $q_*\mathcal{O}_{\mathbb{C}^n}$ at $x = q(Z) \in X$, we see that over any $U \subset X$ such that G acts freely on $q^{-1}(U)$, we have a natural family of G -constellations $\mathcal{F} = q_*\mathcal{O}_{\mathbb{C}^n}$.

Now consider the generic point p_X of X . Its pre-image in \mathbb{C}^n is the generic point $p_{\mathbb{C}^n}$, which can be viewed as the generic orbit of G . The fibre of $\mathcal{O}_{\mathbb{C}^n}$ at $p_{\mathbb{C}^n}$ is the function field $K(\mathbb{C}^n)$ and that of \mathcal{O}_X at p_X is $K(X) = K(\mathbb{C}^n)^G$. The extension $K(\mathbb{C}^n) : K(\mathbb{C}^n)^G$ is Galois, so the Normal Basis Theorem from Galois theory [7, Theorem 19.6] implies that $K(\mathbb{C}^n) = V_{\text{reg}} \otimes_{\mathbb{C}} K(X)$. Thus $K(\mathbb{C}^n)$ is a family of G -constellations parametrised by a single point-scheme p_X . Moreover it is natural, in the sense that it is precisely the fiber of the natural family $q_*(\mathcal{O}_{\mathbb{C}^n})$ at p_X . We now proceed to single out a class of families whose fiber at the generic point is isomorphic to the natural one.

Definition 2.4. Let Y be a scheme birational to X and let p_Y denote the generic point of Y . A *deformation of the generic orbit of G across Y* is a family of G -constellations parametrised by Y equipped with a (G, R) -equivariant isomorphism

$$\iota: \mathcal{F}|_{p_Y} \xrightarrow{\sim} K(\mathbb{C}^n) = (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})|_{p_Y}.$$

We now show that, in fact, any family which agrees with the natural one at the generic point must agree with it wherever G acts freely.

Proposition 2.5. *Let $\pi: Y \rightarrow X$ be a birational morphism and let \mathcal{F} be a family of G -constellations on Y . Then the following are equivalent:*

- (1) *There exists an isomorphism*

$$(2.4) \quad \mathcal{F}|_{p_Y} \simeq K(\mathbb{C}^n)$$

which makes \mathcal{F} into a deformation of the generic orbit of G across Y .

- (2) *There exists a (G, R) -equivariant embedding $\iota': \mathcal{F} \hookrightarrow K(\mathbb{C}^n)$, where $K(\mathbb{C}^n)$ is considered as a constant sheaf of (G, R) -modules on Y .*

- (3) *For any open $U \subseteq Y$, $s \in \mathcal{F}(U)$ and $f \in R^G$, we have*

$$(2.5) \quad f \cdot s = fs,$$

where on the left-hand side f acts as an element of R and on the right-hand side as a section of \mathcal{O}_Y , via the inclusion $\mathcal{O}_X \hookrightarrow \pi_ \mathcal{O}_Y$.*

- (4) *For any open $U \subset X$ such that G acts freely on $q^{-1}U$,*

$$(2.6) \quad \mathcal{F}|_{\pi^{-1}U} \simeq \pi^* q_* \mathcal{O}_{\mathbb{C}^n}|_{\pi^{-1}U} \otimes \mathcal{L}$$

for some invertible sheaf \mathcal{L} on $\pi^{-1}U$.

Before tackling this proposition, we prove a useful lemma, which provides a nice geometrical interpretation of the condition (2.5).

Lemma 2.6. *Let \mathcal{F} be a family of G -constellations on Y satisfying (2.5). Then for any $p \in Y$ we have a scheme-theoretic inclusion*

$$(2.7) \quad \text{Supp } \mathcal{F}|_p \subseteq q^{-1}\pi(p).$$

Moreover, set-theoretically we have equality. Further, if G acts freely on $q^{-1}(p)$, we have

$$\mathcal{F}|_p \simeq (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})|_p$$

as G -constellations.

Proof. Given an arbitrary G -constellation V , the support of V is the vanishing set of the ideal $\text{Ann}_R V \subset R$. On the other hand, $q^{-1}\pi(p)$ is the vanishing of the ideal in R generated by $\mathfrak{m}_{\pi p} \in R^G$. So scheme-theoretically (2.7) is equivalent to

$$\text{Ann}_{R^G} k(\pi p) \subset \text{Ann}_R \mathcal{F} \otimes_{\mathcal{O}_Y} k(p),$$

which follows immediately from (2.5).

To show the set-theoretic equality, we observe from (2.2) that the ideal $\text{Ann}_R \mathcal{F}_p$ is G -invariant, and so, set-theoretically $\text{Supp } \mathcal{F}_p$ is a union of G -orbits in \mathbb{C}^n . But (2.7) now implies that it is contained in a single orbit: the closed points of $q^{-1}\pi(p)$. Therefore we have equality.

For the last bit, we observe that \mathcal{F}_p is a finite length sheaf on \mathbb{C}^n and so splits as a direct sum

$$\bigoplus_{x \in \text{Supp } \mathcal{F}_p} (\mathcal{F}_p)_{|x}$$

of its fibers at each closed point in its support. But as G acts freely on $q^{-1}\pi(p)$, the size of the orbit is $|G|$. Since this is also the dimension of \mathcal{F}_p , each $(\mathcal{F}_p)_{|x}$ must be 1-dimensional and hence

$$\mathcal{F}_p = \bigoplus_{x \in q^{-1}\pi(p)} (\mathcal{O}_{\mathbb{C}^n})_{|x} \simeq (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{|p}. \quad \square$$

Proof of Proposition 2.5. (4) \Rightarrow (1) is obtained by considering the restriction of the isomorphism (2.6) to stalks at p_Y .

(1) \Leftrightarrow (2). Consider the sheaf $\mathcal{F} \otimes_{\mathcal{O}_Y} K(Y)$. On any open U where \mathcal{F} is a free \mathcal{O}_Y -module, $\mathcal{F} \otimes_{\mathcal{O}_Y} K(Y)$ is the constant sheaf \mathcal{F}_{p_Y} for which we have the (G, R) -equivariant isomorphism (2.4) to the constant sheaf $K(\mathbb{C}^n)$. A sheaf constant on an open cover must be constant as Y is irreducible. Now the natural map $\mathcal{F} \hookrightarrow \mathcal{F} \otimes K(Y)$ becomes the requisite embedding.

(2) \Rightarrow (3) is immediate because $K(\mathbb{C}^n)$, as a (G, R) -module clearly satisfies (2.5).

So we are left with proving (3) \Rightarrow (4). We begin with a local version: if $p \in \pi^{-1}(U) \subset Y$, then $\mathcal{F}_p \simeq (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_p$, that is, the stalks at p are (G, R) -equivariantly isomorphic.

Now $(\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_p$ (which we can write as $R \otimes_{R^G} \mathcal{O}_{Y,p}$) is a free $\mathcal{O}_{Y,p}$ -module of rank $|G|$. This is because G acting freely on $q^{-1}\pi(p)$ implies that the quotient map q is flat and $|G|$ -to-one at $\pi(p)$. \mathcal{F}_p is also a free $\mathcal{O}_{Y,p}$ -module of rank $|G|$, because \mathcal{F} is a family of G -constellations. Therefore we can consider the determinant of any (G, R) -equivariant $\mathcal{O}_{Y,p}$ -morphism between the two, and it would suffice to find a morphism whose determinant is invertible.

Consider the map $\theta: (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_p \rightarrow \mathcal{F}_p$ defined by

$$m \otimes f \rightarrow m \cdot (fs_0), \quad m \in R, \quad f \in \mathcal{O}_{Y,p},$$

where s_0 is a fixed choice of any $\mathcal{O}_{Y,p}$ -generator of the χ_0 -eigenspace of \mathcal{F}_p .

This map is a well-defined $\mathcal{O}_{Y,p}$ -module map, that is, it descends from the set-theoretic product $R \times \mathcal{O}_{Y,p}$ to the tensor product, precisely because both \mathcal{F}_p and $R \otimes \mathcal{O}_{Y,p}$ satisfy (2.5). It is G -equivariant because $1 \mapsto s_0$ ensures that χ_0 -eigenspace maps to χ_0 -eigenspace

and (2.2) forces the rest. Finally not only θ is defined to be R -action equivariant, but the reader can verify that it is the unique element of $\mathrm{Hom}_{(G,R)}(R \otimes \mathcal{O}_{Y,p}, \mathcal{F}_p)$ which maps 1 to s_0 . Note that in particular, this shows that

$$(2.8) \quad \mathrm{Hom}_{(G,R)}(R \otimes \mathcal{O}_{Y,p}, \mathcal{F}_p) \simeq (\mathcal{F}_p)_{\chi_0} \simeq \mathcal{O}_{Y,p},$$

where θ is a (G, R) -equivariant morphism. It descends to the (G, R) -equivariant morphism

$$\bar{\theta}: (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})|_p \rightarrow \mathcal{F}|_p$$

on fibers. Similar to (2.8),

$$\mathrm{Hom}_{(G,R)}((\pi^* q_* \mathcal{O}_{\mathbb{C}^n})|_p, \mathcal{F}_p) \simeq \mathbb{C},$$

i.e., all (G, R) -equivariant morphisms between the two are scalar multiples of each other. Since by Lemma 2.6, the two fibers are (G, R) -equivariantly isomorphic, we have that unless $\bar{\theta}$ is a zero map, it is an isomorphism. But it maps $[1]$ to $[s_0]$, and the latter can not be 0 by the choice of s_0 . So $\det \bar{\theta} \neq 0$ implying that $\det \theta \in \mathcal{O}_{Y,p}^*$, as required.

The isomorphisms on stalks give isomorphisms $\theta_i: R \otimes_{RG} \mathcal{O}_{U_i} \rightarrow \mathcal{F}|_{U_i}$ on an open cover $\{U_i\}$ of U , as both sheaves are locally free and of finite rank. Then on each intersection $U_i \cap U_j$, $\theta_i \circ \theta_j^{-1}$ is a (G, R) -automorphism of $R \otimes_{RG} \mathcal{O}_{U_i \cap U_j}$. Any such, by an argument identical to (2.8), is a multiplication by an element of $\mathcal{O}_{U_i \cap U_j}^*$, which concludes the proof. \square

For the rest of this paper, we shall concern ourselves only with those families of G -constellations which are deformations of the generic orbit.

Observe that the map $\iota: \mathcal{F}|_{p_Y} \xrightarrow{\sim} K(\mathbb{C}^n) = (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})|_{p_Y}$ uniquely determines the embedding $\iota': \mathcal{F} \hookrightarrow K(\mathbb{C}^n)$. The notion of the isomorphism of deformations demands for the (G, R) -equivariant sheaf isomorphism $\theta: \mathcal{F} \rightarrow \mathcal{F}'$ to have its restriction to stalks at p_Y form a commutative triangle with maps $\iota_{\mathcal{F}}$ and $\iota_{\mathcal{F}'}$ for \mathcal{F} and \mathcal{F}' to be isomorphic as deformations of the generic orbit. Consequently θ itself must form a commutative triangle with $\iota'_{\mathcal{F}}$ and $\iota'_{\mathcal{F}'}$, in particular images of \mathcal{F} and \mathcal{F}' in $K(\mathbb{C}^n)$ must coincide. Thus isomorphism classes of deformations of the generic orbit are precisely in one-to-one correspondence with deformations of the generic orbit which are subsheaves of $K(\mathbb{C}^n)$.

3. Line bundles and G -Cartier divisors

As we deal with families of G -constellations which are subsheaves of $K(\mathbb{C}^n)$, it would be useful to have a language similar to that of the Cartier divisors to describe the invertible sub- \mathcal{O}_Y -modules of $K(\mathbb{C}^n)$ with non-trivial G -action. In this section we shall extend the familiar construction of Cartier divisors using the larger group of non-zero G -homogeneous rational functions, which we shall denote by $K_G^*(\mathbb{C}^n)$, instead of the group of non-zero invariant rational functions $K^*(Y)$.

Definition 3.1. We shall say that a rational function $f \in K(\mathbb{C}^n)$ is *G-homogeneous of weight* $\chi \in G^\vee$ if

$$(3.1) \quad g \cdot f = \chi(g^{-1})f, \quad \forall g \in G.$$

We shall denote by $K_\chi(\mathbb{C}^n)$ the subset of $K(\mathbb{C}^n)$ of G -homogeneous elements of a specific weight χ and by the $K_G(\mathbb{C}^n)$ the subset of $K(\mathbb{C}^n)$ of all the G -homogeneous elements. We shall use R_χ and R_G to mean $R \cap K_\chi(\mathbb{C}^n)$ and $R \cap K_G(\mathbb{C}^n)$ respectively.

The choice of a sign in this definition is motivated as follows: we want a function $p \in R$ to be G -homogeneous of weight $\chi \in G^\vee$ if $p(g \cdot v) = \chi(g)p(v)$ for any $g \in G$ and $v \in \mathbb{C}^n$. E.g., usual concept of a homogeneous polynomial, whose degree, an integer number, is precisely its weight as a character of \mathbb{C}^* acting diagonally on \mathbb{C}^n . In view of (2.1), this means we must have $\chi(g^{-1})$ instead of $\chi(g)$ in (3.1).

Now consider $K_G^*(\mathbb{C}^n)$, the invertible elements of $K_G(\mathbb{C}^n)$. Using the fact that $K(Y) = K(X) = K(\mathbb{C}^n)^G$, we have a short exact sequence of multiplicative groups:

$$1 \rightarrow K^*(Y) \rightarrow K_G^*(\mathbb{C}^n) \rightarrow G^\vee \rightarrow 1.$$

What makes this enlargement of $K^*(Y)$ useful is that we can still define a valuation of a G -homogeneous rational function at a prime Weil divisor.

Definition 3.2. Let $D \subset Y$ be a prime Weil divisor on Y . Given any $f \in K_G^*(\mathbb{C}^n)$, we choose any $n \in \mathbb{Z}$ such that f^n is invariant, i.e., $f^n \in K(Y)$. For instance, $n = |G|$. Then we define

$$v_D(f) = \frac{1}{n}v_D(f^n) \in \mathbb{Q},$$

where $v_D(f^n)$ is the ordinary valuation of f^n in the local ring $\mathcal{O}_{D,Y}$ of the generic point of D . This is well-defined since for any $g \in K(Y)$, we have $v_D(g^k) = kv_D(g)$.

In what follows, we shall write

$$\{n\} = n - \lfloor n \rfloor$$

for the fractional part of $n \in \mathbb{Q}$. Generally, the valuations defined above are \mathbb{Q} -valued. However, if f and g in $K_G^*(\mathbb{C}^n)$ are both χ -homogeneous, then f/g is G -invariant and hence for any Weil divisor D on Y , $v_D(f) - v_D(g) \in \mathbb{Z}$. Therefore the fractional part of $v_D(f)$ is independent of the choice of f in $K_\chi^*(\mathbb{C}^n)$.

Definition 3.3. We define $v(D, \chi)$ to be the number $\{v_D(f)\} \in \mathbb{Q}$, where f is any element of $K_\chi^*(\mathbb{C}^n)$.

We can now replicate, almost word-for-word, the definitions in [8, pp. 140–141].

Definition 3.4. A G -Cartier divisor on Y is a global section of the sheaf of multiplicative groups $K_G^*(\mathbb{C}^n)/\mathcal{O}_Y^*$, i.e., the quotient of the constant sheaf $K_G^*(\mathbb{C}^n)$ on Y by the sheaf \mathcal{O}_Y^* of invertible regular functions.

As usual, such a section can be described by a choice of an open cover $\{U_i\}$ of Y and functions $\{f_i\} \subseteq K_G^*(\mathbb{C}^n)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_Y^*)$. Observe that, as their ratios are invariant, the f_i must all be homogenous of the same weight $\chi \in G^\vee$. In such a case, we further say that the divisor is χ -Cartier.

As with ordinary Cartier divisors, a G -Cartier divisor is said to be principal if it lies in the image of the natural map $K_G^*(\mathbb{C}^n) \rightarrow K_G^*(\mathbb{C}^n)/\mathcal{O}_Y^*$ and two divisors are said to be linearly equivalent if their difference is principal.

However when defining a corresponding enlargement of the group of Weil divisors, we have to be a little bit careful.

Definition 3.5. A χ -Weil divisor on Y is a finite sum $\sum q_i D_i$ (where $q_i \in \mathbb{Q}$) of prime Weil divisors on Y , such that

$$(3.2) \quad q_i - v(D_i, \chi) \in \mathbb{Z}$$

for all i .

We shall further use the term G -Weil divisor to refer to all χ -divisors for any $\chi \in G^\vee$.

Definition 3.6. For any $f \in K_G^*(\mathbb{C}^n)$, we define the principal G -Weil divisor of f to be

$$(f) = \sum v_P(f) P$$

with the sum taken over all prime Weil divisors P on Y . This sum is finite as $f^{|G|}$ is a regular function on Y and hence has non-zero valuations only on finitely many prime divisors.

Given any $\chi, \chi' \in G^\vee$, we can see that, for any prime divisor D ,

$$v(D, \chi) + v(D, \chi') - v(D, \chi\chi') \in \mathbb{Z}$$

as it is equal to the valuation at D of an invariant function. Hence G -Weil divisors form an additive group. We define two G -Weil divisors to be linearly equivalent if their difference is principal and a divisor $\sum q_i D_i$ to be effective if all $q_i \geq 0$.

Recall that from [8, Proposition 6.11] there is an injective homomorphism from the group of Cartier divisors to the group of Weil divisors which is an isomorphism when Y is smooth. The definition extends naturally to an injective homomorphism from the group of G -Cartier divisors to the group of G -Weil divisors, but some care needs to be taken to show that it is surjective when Y is smooth.

Definition 3.7. Define the map ϕ from the group of G -Cartier divisors to the group of G -Weil divisors on Y by

$$\{(f_i, U_i)\} \mapsto \sum k_D D,$$

where the sum is taken over all prime Weil divisors D on Y and $k_D = v_D(f_i)$ for any f_i such that $U_i \cap D$ is not empty. Once again the sum is finite, as each f_i has non-zero valuation only on finitely many prime Weil divisors.

Proposition 3.8. *Let ϕ be the injective homomorphism defined above. If Y is smooth, then ϕ is an isomorphism.*

Proof. We need surjectivity. So suppose we have a χ -Weil divisor D on Y . Take any $f \in K_\chi^*(\mathbb{C}^n)$. Then $D - (f)$ is an ordinary Weil divisor and as Y is smooth, it has a Cartier divisor $\{(U_i, g_i)\}$ corresponding to it as before. Then $\{(U_i, g_i f)\}$ is the χ -Cartier divisor which ϕ maps to D . \square

The point of introducing G -Cartier divisors is that they correspond to invertible sheaves which carry a G -action in the same way that ordinary Cartier divisors correspond to the ordinary invertible sheaves.

Indeed consider D , the χ -Cartier divisor on Y specified by a collection $\{(U_i, f_i)\}$ where U_i form an open cover of Y and $f_i \in K_\chi^*(\mathbb{C}^n)$. We define an invertible sheaf $\mathcal{L}(D)$ on Y as the sub- \mathcal{O}_Y -module of $K(\mathbb{C}^n)$ generated by f_i^{-1} on U_i . Observe that we have an action of G on $\mathcal{L}(D)$, restricted from the one on $K(\mathbb{C}^n)$, and it acts on every section by the character χ .

Proposition 3.9. *The map $D \rightarrow \mathcal{L}(D)$ gives an isomorphism between the group $G\text{-Cl}(Y)$ of G -Cartier divisors up to linear equivalence and the group $G\text{-Pic}(Y)$ of invertible G -sheaves on Y .*

Proof. A standard argument from [8, Corollary 6.15] shows that it is an injective homomorphism. To show that it is an isomorphism, we need to be able to embed any invertible G -sheaf \mathcal{L} , with G acting by some $\chi \in G^\vee$, as a sub- \mathcal{O}_Y -module into $K(\mathbb{C}^n)$.

Given such \mathcal{L} , we consider the sheaf $\mathcal{L} \otimes_{\mathcal{O}_Y} K(Y)$. On every open set U_i where \mathcal{L} is trivial, it is G -equivariantly isomorphic to the constant sheaf $K_\chi(\mathbb{C}^n)$. On an irreducible scheme a sheaf constant on an open cover is constant itself, so as Y is irreducible we have $\mathcal{L} \otimes_{\mathcal{O}_Y} K(Y) \simeq K_\chi(\mathbb{C}^n)$ and a particular choice of this isomorphism gives the necessary embedding as

$$\mathcal{L} \rightarrow \mathcal{L} \otimes_{\mathcal{O}_Y} K(Y) \simeq K_\chi(\mathbb{C}^n) \subset K(\mathbb{C}^n). \quad \square$$

A curious thing about G -divisors and valuations of G -homogeneous functions is the fact that on the quotient space X every prime Weil divisor is a principal divisor of some G -homogeneous function. In particular, every G -Weil divisor is G -Cartier.

Proposition 3.10. *Let P be a prime Weil divisor on X . Then there exists an $f \in R_G^*$ such that $P = (f)$, that is,*

$$v_D(f) = \begin{cases} 1 & \text{when } D = P, \\ 0 & \text{when } D \neq P \end{cases}$$

for any prime divisor D on Y .

Proof. Let $I_P \subset R^G$ be the prime ideal of height 1 corresponding to P . Consider the ring extension $R^G \subseteq R$. By a classical result of Emmy Noether [1, Theorem 1.3.1], this extension is integral. This then implies from [12, Theorem 9.3] that there exists a prime ideal I' of height 1 in R lying over I_P , that is $I_P = I' \cap R^G$ and that every other prime ideal lying over I_P is conjugate to I' by an element of G . As R is a UFD, every prime ideal of height one is principal and so there exists some $y' \in R$ such that $I_P = (y') \cap R^G$.

So take $g_0 = 1, g_1, \dots, g_k \in G$ to be such that the principal ideals $(y'), (g_1 \cdot y'), \dots, (g_k \cdot y')$ are all the distinct prime ideals lying over I_P . Then we claim that $y = \prod g_i \cdot y'$ is a G -homogeneous function and that $I_P = (y) \cap R^G$. Indeed, $(h \cdot y) = \bigcap ((hg_i) \cdot y')$. The ideals $((hg_i) \cdot y')$ are all distinct prime ideals lying over I_P and therefore

$$(h \cdot y) = \bigcap ((hg_i) \cdot y') = \bigcap (g_i \cdot y') = (y),$$

which implies $h \cdot y \in \mathbb{C}^* y$. For the second claim, observe that $I_P = g_i \cdot I_P = (g_i \cdot y') \cap R^G$ for all i . Consequently $I_P = \left(\bigcap (g_i \cdot y') \right) \cap R^G = (y) \cap R^G$.

Thus we have $I_P = (y) \cap R^G$. Note that (y) is precisely the vanishing ideal of the pre-image of P in \mathbb{C}^n . Now let k be the ramification index of the valuation ring extension $R_{I_P}^G \subset R_{(y)}$. Then for any $w \in K(\mathbb{C}^n)^G$ we have $v_P(w) = \frac{1}{k} v_{(y)}(w)$, which immediately extends to the \mathbb{Q} -valued valuation $v_P(w)$ of any G -homogeneous $w \in K_G^*(\mathbb{C}^n)$. In particular, we see that $v_P(y) = \frac{1}{k}$. Now take any other prime divisor D on Y . We have $I_D = (u) \cap R^G$ for some prime $u \in R$. If now $v_D(y) \neq 0$, then as y is regular we have $y \in (u)$ and so $g_i \cdot y' \in (u)$ for some i . Then $(u) = (g_i \cdot y')$ and $D = P$.

Now taking $f = y^k$ finishes the proof. \square

In the course of the proof of Proposition 3.10, we see that the valuations of G -homogeneous functions are actually non-integer only at ramification divisors of q . We now contemplate along which actual divisors the ramification can occur.

Proposition 3.11. *There are only finitely many prime divisors P on X with ramification index greater than 1. More precisely, if we write the ideal of each such P as $(y) \cap R^G$ for $y \in R_G^*$ as per Proposition 3.10, then we will have at most one y of weight χ for each character $\chi \in G^\vee$.*

Explicitly, the ramification can only occur along the images of coordinate hyperplanes $(x_1), \dots, (x_n)$ of \mathbb{C}^n and in the case of $G \subset \mathrm{SL}_n(\mathbb{C})$ ramification never occurs at all.

Proof. For each character $\chi \in G^\vee$ fix a G -homogeneous function $f_\chi \in R$ of weight χ . We further demand that it is minimal such, in a sense that no element of R^G other than 1 divides it. We shall now show that ramification could only occur along one of the $(f_\chi) \cap R^G$ and only when f_χ is the unique function satisfying these conditions.

To see it, take any prime divisor P on X . Write $I_P = (y) \cap R^G$ for $y \in R_G^*$ as per Proposition 3.10. Unless $f_\chi \in (y)$, $v_{(y)}(f_\chi) = 0$ and hence $v_{(y)}(\frac{y}{f_\chi}) = 1$ and so there is no ramification along P . But if $f_\chi \in (y)$ then minimality condition forces $f_\chi = y$.

Explicitly, when G is abelian we know that the character map $\rho: \mathbb{Z}^n \rightarrow G^\vee$ is surjective (see (4.2)). Given a character $\chi \in G$, there exists $m \in \mathbb{Z}^n$ such that $x^m = \prod x_i^{m_i}$ is G -homogeneous of weight χ . Then above implies that ramification can only occur along $(y) \cap R^G$ if y is monomial. But recalling proof of Proposition 3.10, $y = \prod g_i \cdot y'$ where y' is prime. This implies y' must be one of the basic monomials x_i .

In case when $G \subseteq \mathrm{SL}_n(\mathbb{C})$, we know that $x_1 \cdots x_n$ is invariant. As $v_{(x_i)}(x_1 \cdots x_n) = 1$, there is no ramification along any of $(x_i) \cap R^G$ either. \square

Propositions 3.10 and 3.11 have an immediate corollary in terms of the numbers $v(P, \chi)$ on X .

Corollary 3.12. *For any P , a prime Weil divisor on X which is not a ramification divisor of q , and $\chi \in G^\vee$, there exists a monomial $m \in R_\chi$ such that $v_P(m) = 0$. Consequently*

$$v(P, \chi) = 0.$$

Proof. Unless $P = (x_i) \cap R^G$, one can take m to be any monomial in R of weight χ . If $P = (x_i) \cap R^G$, then, unless there is ramification at P , there exists a $p \in R^G$ whose valuation at (x_i) in \mathbb{C}^n is 1. Note that we can take p to be monomial by considering its monomial summands. Then $\frac{p}{x_i} \in R_{\chi^{-1}}$ and $v_P(\frac{p}{x_i}) = 0$, so we can take $m = \frac{p}{x_i} |G|^{-1}$. \square

Let us look at some concrete examples of the ramification occurring and not occurring.

Example 3.13. First consider $G = \frac{1}{3}(1, 2)$, the group of 3rd complex roots of unity embedded into $\mathrm{SL}_2(\mathbb{C})$ by

$$\xi \mapsto \begin{pmatrix} \xi^1 & \\ & \xi^2 \end{pmatrix}.$$

If we write χ_k for the character of G given by $\xi \mapsto \xi^k$, then x is of weight χ_1 and y of weight χ_2 .

Let P be the image in X of the hyperplane $x = 0$. It is a prime Weil divisor (but not a Cartier one) given by $(x^3, xy) = (x) \cap R^G$. $v_{(x)}(xy) = 1$, so there is no ramification. And consequently, $v_P(x) = v_{(x)}(x) = 1$ as $x^3 = (xy)^3 y^{-3}$.

Now take $G = \frac{1}{4}(1, 2)$. Then the divisor P is given by $(x^4, x^2 y)$. So we see that index of ramification is $v_{(x)}(x^2 y) = 2$ and correspondingly $v_P(x) = \frac{1}{2} v_{(x)}(x) = \frac{1}{2}$.

Corollary 3.14. *Let $\pi: Y \rightarrow X$ be a resolution and P a prime Weil divisor on Y , which is neither exceptional nor a proper transform of a ramification divisor of q in X . Then for any $\chi \in G^\vee$ there exists $m \in R_\chi$ such that $v_P(m) = 0$, implying*

$$v(P, \chi) = 0.$$

Proof. This is a straightforward consequence of Corollary 3.12. Consider $P' = \pi(P)$, the image of P in X . Unless P is exceptional, P' is a prime Weil divisor on X . Its generic point lies in the open set on which the resolution map is an isomorphism, which implies that for any $f \in K(\mathbb{C}^n)$, $v_P(f) = v_{P'}(f)$. Now Corollary 3.12 gives the result. \square

4. Toric picture

4.1. Basics

In this section we give a brief exposition of the necessary toric background and then translate some of the results of Section 3 into the toric language.

A more thorough exposition of toric geometry in general can be found in [6] and of toric geometry as related to quotient singularities in [9].

Consider the maximal torus $(\mathbb{C}^*)^n \subset \mathrm{GL}_n(\mathbb{C})$ containing G . We have an exact sequence of abelian groups:

$$(4.1) \quad 0 \longrightarrow G \longrightarrow (\mathbb{C}^*)^n \longrightarrow T \longrightarrow 0$$

where T is the quotient torus which acts on the quotient space X .

By applying $\mathrm{Hom}(\cdot, \mathbb{C}^*)$ to (4.1) we obtain an exact sequence

$$(4.2) \quad 0 \longrightarrow M \longrightarrow \mathbb{Z}^n \xrightarrow{\rho} G^\vee \longrightarrow 0$$

where \mathbb{Z}^n is thought of as the lattice of exponents of Laurent monomials. Thus given $m = (k_1, \dots, k_n) \in \mathbb{Z}^n$ we shall write x^m for $x_1^{k_1} \cdots x_n^{k_n}$. M is the sublattice in \mathbb{Z}^n of (exponents of) G -invariant Laurent monomials.

Note that each Laurent monomial is a G -homogeneous function and ρ is precisely the weight map, that is, $x^m(g \cdot \mathbf{v}) = \rho(m)(g) x^m(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{C}^n$.

Applying $\mathrm{Hom}(\cdot, \mathbb{Z})$ to (4.2) we obtain

$$0 \longrightarrow (\mathbb{Z}^n)^\vee \longrightarrow L \longrightarrow \mathrm{Ext}^1(G^\vee, \mathbb{Z}) \longrightarrow 0$$

where we write $(\mathbb{Z}^n)^\vee$ for the dual lattice of \mathbb{Z}^n , L for the dual of M and note that $\mathrm{Hom}(G^\vee, \mathbb{Z}) = 0$ as G^\vee is finite and $\mathrm{Ext}^1(\mathbb{Z}^n, \mathbb{Z}) = 0$ as \mathbb{Z}^n is free.

Thus we see that $L/(\mathbb{Z}^n)^\vee \simeq \mathrm{Ext}^1(G^\vee, \mathbb{Z})$. Taking an injective resolution of \mathbb{Z} :

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

we see that $\text{Ext}^1(G^\vee, \mathbb{Z}) \simeq \text{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$ as $\text{Hom}(G^\vee, \mathbb{Q}) = 0$. Now a choice of a map $\mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{C}^*$ which is equivalent to a simultaneous choice of a primitive n th complex root of unity for all $n \in \mathbb{N}$, would give us

$$L/(\mathbb{Z}^n)^\vee \simeq \text{Hom}(G^\vee, \mathbb{C}^*) = G,$$

allowing us to identify points in $L/(\mathbb{Z}^n)^\vee$ with the elements of the group.

Tautologically, we have a \mathbb{Z} -valued pairing between M and L . This pairing extends naturally to a \mathbb{Q} -valued pairing between \mathbb{Z}^n and L . For the purposes of the exposition to follow, it will be convenient to think of elements of L as functions on the monomial lattices $M \hookrightarrow \mathbb{Z}^n$. Henceforth, given $l \in L$ and $m \in \mathbb{Z}^n$, we shall write $l(m)$ to denote the pairing above.

For any cone $\tau \subset \mathbb{Z}^n \otimes \mathbb{R}$, $\tau \cap M$ and $\tau \cap \mathbb{Z}^n$ are abelian semigroups. We shall write $\mathbb{C}[\tau \cap M]$ and $\mathbb{C}[\tau \cap \mathbb{Z}^n]$ for the \mathbb{C} -algebras generated by the corresponding Laurent monomials. Whenever we omit the lattice, writing $\mathbb{C}[\tau]$, it should be assumed that the lattice is M .

The fan of X in L consists of the single cone L_+ , the dual of the cone M_+ of regular Laurent monomials in M (similarly, we shall use \mathbb{Z}_+^n and $(\mathbb{Z}^n)_+^\vee$). The fan of any toric resolution of X is given by a subdivision of L_+ into basic cones.

Fix such a toric resolution Y . Write \mathfrak{F} for the set of basic cones which make up the fan of Y . We shall denote by A_σ the toric variety $\text{Spec } \mathbb{C}[\sigma^\vee]$ corresponding to any cone σ in $L \otimes \mathbb{R}$. Then Y is constructed in toric geometry by gluing together $\{A_\sigma\}_{\sigma \in \mathfrak{F}}$: A_{σ_1} and A_{σ_2} are glued along $A_{\sigma_1 \cap \sigma_2} = \text{Spec } \mathbb{C}[(\sigma_1 \cap \sigma_2)^\vee]$. Thus $\{A_\sigma\}_{\sigma \in \mathfrak{F}}$ is an open affine cover of Y .

Now write $\mathfrak{E} \subset L$ for the set of all generators of these basic cones. In the toric geometry each element of \mathfrak{E} corresponds to either an exceptional divisor on Y or the proper transform of one of the coordinate hyperplanes in X . For $e_i \in \mathfrak{E}$, write E_i for the divisor on Y corresponding to it.

It is often important whether the resolution is crepant or not. The discrepancy of each E_i depends only on e_i and not on the choice of Y . If $e_i = (k_1, \dots, k_n) \in L$, then (see [9, 1.4] and [14, Proposition 4.8] for technicalities) the discrepancy of E_i is $(\sum k_i) - 1$, so the crepant divisors correspond to the elements of L which lie in the junior simplex:

$$\Delta = \left\{ (k_1, \dots, k_n) \in L \otimes \mathbb{R} \mid k_i > 0 \text{ and } \sum k_i = 1 \right\}.$$

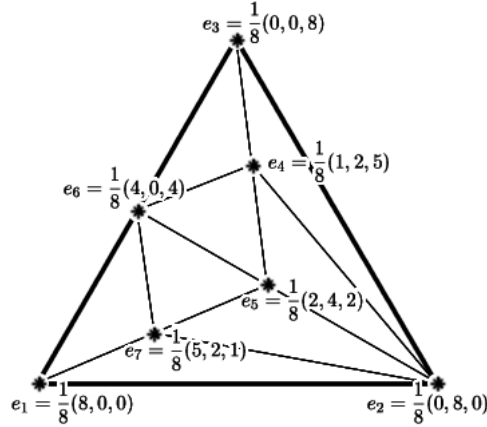
Note that if a basic cone contains $e \in \Delta \cap L$, then e must be one of its generators. So, for any resolution, $\Delta \cap L$ is a subset of \mathfrak{E} and the crepant ones are precisely those for which this inclusion is an equality.

Example 4.1. Consider the group G being $\frac{1}{8}(1, 2, 5)$, the group of 8th complex roots of unity embedded into $\mathrm{SL}_3(\mathbb{C})$ by

$$\xi \mapsto \begin{pmatrix} \xi^1 & & \\ & \xi^2 & \\ & & \xi^5 \end{pmatrix}.$$

We shall write χ_k for the character of G given by $\xi \mapsto \xi^k$. So x has weight χ_1 , y weight χ_2 and z weight χ_5 .

The lattice L is generated in $(\mathbb{Z}^3)^\vee \otimes \mathbb{Q}$ by elements of $(\mathbb{Z}^3)^\vee$ and $\frac{1}{8}(1, 2, 5)$. The cone L_+ , the positive octant, is the fan of X . A crepant resolution of Y is given by a triangulation of the junior simplex Δ into basic triangles. For the subsequent examples, we choose the following triangulation:



So $\mathfrak{E} = \Delta \cap L = \{e_1, \dots, e_7\}$. And the basic cones of the fan \mathfrak{F} of Y are

$$\mathfrak{F} = \{ \langle e_1, e_2, e_7 \rangle, \langle e_7, e_2, e_5 \rangle, \langle e_4, e_2, e_5 \rangle, \langle e_4, e_3, e_2 \rangle, \\ \langle e_3, e_4, e_6 \rangle, \langle e_4, e_6, e_5 \rangle, \langle e_6, e_5, e_7 \rangle, \langle e_1, e_6, e_7 \rangle \}.$$

This shall be the setup for all the subsequent examples.

4.2. Valuations

We now establish two simple results which translate the notions defined in Section 3 into toric language.

Proposition 4.2. *Let Y be a toric resolution of X , \mathfrak{F} its fan and \mathfrak{E} the set in L of the generators of \mathfrak{F} . For any $e_i \in \mathfrak{E}$ and $m \in \mathbb{Z}^n$,*

$$v_{E_i}(x^m) = e_i(m) \in \mathbb{Q}.$$

Proof. Take any basic cone $\sigma \in \mathfrak{F}$ such that $e_i \in \sigma$. Without loss of generality $i = 1$ and $\sigma = \langle e_1, \dots, e_n \rangle$. Let $\check{e}_1, \dots, \check{e}_n$ be the dual basis in M .

For any $m \in \mathbb{Z}^n$, $|G|m \in M$. Using the dual basis

$$|G|m = \sum |G|e_i(m) \check{e}_i,$$

therefore

$$x^{|G|m} = (x^{\check{e}_1})^{|G|e_1(m)} \dots (x^{\check{e}_n})^{|G|e_n(m)}.$$

The restriction of the exceptional divisor E_1 to A_σ is given by the principal Weil divisor $(x^{\check{e}_1})$. Thus the local ring of E_i is the coordinate ring of A_σ localised at the ideal $(x^{\check{e}_1})$, and so the valuation of $x^{|G|m} \in \mathcal{O}_Y$ is $|G|e_1(m)$. By definition, $v_{E_1}(x^m) = \frac{1}{|G|}v_{E_1}(x^{|G|m}) = e_1(m)$. \square

The second result establishes which compatibility conditions a set of monomials $\{x^{m_\sigma}\}_{\sigma \in \mathfrak{F}}$ must satisfy for it to define a G -Cartier divisor. When the conditions are satisfied, we further establish the form which the corresponding G -Weil divisor must take.

Proposition 4.3. *A set $\{x^{m_\sigma}\}_{\sigma \in \mathfrak{F}} \subset \mathbb{C}[\mathbb{Z}^n]$ of Laurent monomials defines a G -Cartier divisor $\{(A_\sigma, x^{m_\sigma})\}_{\sigma \in \mathfrak{F}}$ on Y if and only if for any $e_i \in \mathfrak{E}$,*

$$(4.3) \quad e_i(m_\sigma) = e_i(m_\tau) \quad \text{for all } \sigma, \tau \ni e_i.$$

When (4.3) holds, denote by q_i the value of $e_i(m_\sigma)$ for any $\sigma \ni e_i$. Then, under the isomorphism ϕ from Proposition 3.8, $\{(A_\sigma, x^{m_\sigma})\}_{\sigma \in \mathfrak{F}}$ corresponds to the G -Weil divisor

$$\sum_{e_i \in \mathfrak{E}} q_i E_i.$$

Proof. Observe that if $\sigma, \tau \in \mathfrak{E}$ are such that e_i belongs to both, then the generic point p_{E_i} of E_i lies in $A_\sigma \cap A_\tau$. If $\{(A_\sigma, x^{m_\sigma})\}$ is a G -Cartier divisor, then $x^{m_\sigma}/x^{m_\tau} \in \mathcal{O}^*(A_\sigma \cap A_\tau)$, so we have $v_{E_i}(x^{m_\sigma}/x^{m_\tau}) = 0$ and hence

$$e_i(m_\sigma) = v_{E_i}(x^{m_\sigma}) = v_{E_i}(x^{m_\tau}) = e_i(m_\tau).$$

Conversely suppose we have $e_i(m_\sigma) = e_i(m_\tau)$ for all $e_i \in \sigma \cap \tau$. Then $m_\sigma - m_\tau \in (\sigma \cap \tau)^\perp$, and hence x^{m_σ}/x^{m_τ} is invertible in $\mathbb{C}[(\sigma \cap \tau)^\vee] = \mathcal{O}_Y(A_\sigma \cap A_\tau)$ as required.

For the last part, recall that $\phi(\{(A_\sigma, x^{m_\sigma})\})$ is defined as the sum $\sum n_D D$ over all prime divisors on Y where $n_D = v_D(x^{m_\sigma})$ for any σ such that $D \cap A_\sigma \neq \emptyset$. So it suffices to prove that, for all $\sigma \in \mathfrak{F}$, the restrictions of the principal divisor (x^{m_σ}) and $\sum_{i \in \mathfrak{E}} q_i E_i$ to A_σ are identical.

Without loss of generality, we can take $\sigma = \langle e_1, \dots, e_n \rangle$. Then $\mathcal{O}_{A_\sigma} = \mathbb{C}[t_1, \dots, t_n]$ where $t_i = x^{\check{e}_i}$. We have $x^{m_\sigma} = \prod_{e_i \in \sigma} t_i^{q_i}$ and recall (proof of Proposition 4.2) that $E_i|_{A_\sigma} = (t_i)$. Therefore

$$(x^{m_\sigma})|_{A_\sigma} = \sum_{e_i \in \sigma} q_i(t_i) = \left(\sum_{e_i \in \sigma} q_i E_i \right)|_{A_\sigma}$$

and the result follows. \square

Remark 4.4. (1) Observe that the ‘only if’ part of the proof is completely general and doesn’t rely on the toric technology. It is the standard argument used to show that the morphism ϕ taking Cartier divisors to Weil divisors is well-defined.

On the other hand the ‘if’ argument is toric-specific and relies heavily on the fact that the invertible functions on $A_\sigma \cap A_\tau$ are precisely the monomials in $(\sigma \cap \tau)^\vee$.

(2) Note that, in particular, we have proved that for any $m \in \mathbb{Z}^n$, the sum

$$\sum_{i \in \mathfrak{E}} v(E_i, x^m) E_i$$

is a valid G -Weil divisor on Y . Recalling the definition of G -Weil divisors, this provides an independent proof that for any prime divisor D which is not E_i for some $i \in \mathfrak{E}$, we have

$$v(D, \chi) = 0$$

for all $\chi \in G^\vee$, since $v(D, \chi)$ is defined as the fractional part of the valuation of any homogeneous rational function of weight χ on D .

Example 4.5. To illustrate the above, in the context of Example 4.1, we shall calculate explicitly the χ_6 -Cartier divisor corresponding to the χ_6 -Weil divisor

$$D = \frac{7}{4}E_4 + \frac{1}{2}E_5 - \frac{1}{4}E_7.$$

Consider the cone $\sigma = \langle e_4, e_5, e_6 \rangle$. Calculating the dual basis which generates the abelian semigroup $\check{\sigma} \cap M$, we get

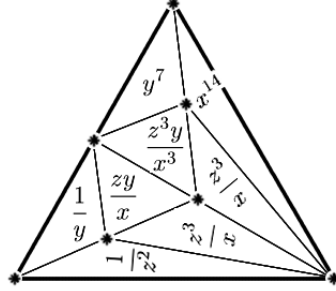
$$\check{e}_4 = (-2, 0, 2), \quad \check{e}_5 = (1, 2, -1), \quad \check{e}_6 = (2, -1, 0).$$

So $A_\sigma = \text{Spec } \mathbb{C}\left[\frac{z^2}{x^2}, \frac{xy^2}{z}, \frac{x^2}{y}\right]$ and the restrictions of E_4 , E_5 and E_6 to A_σ are given by $\left(\frac{z^2}{x^2}\right)$, $\left(\frac{xy^2}{x^2}\right)$ and $\left(\frac{x^2}{y}\right)$, respectively. To specify D on A_σ we need $f \in K_{\chi_6}(\mathbb{C}^3)$ such that $v_{E_4}(f) = \frac{7}{4}$, $v_{E_5}(f) = \frac{1}{2}$ and $v_{E_6}(f) = 0$, so we take

$$\left(\frac{z^2}{x^2}\right)^{7/4} \left(\frac{xy^2}{z}\right)^{1/2} \left(\frac{x^2}{y}\right)^0 = \frac{z^3 y}{x^3}$$

to be f .

Repeating the same calculations for the remaining cones in the fan \mathfrak{F} we get the χ_6 -Cartier divisor given by



and we can indeed see that, as all the monomials representing the divisor have weight χ_6 , their ratios are all invariant and the sub- \mathcal{O}_Y -module of $K(\mathbb{C}^n)$ they generate is an invertible sheaf on Y with the natural action of G by χ_2 .

4.3. Representations of the McKay quiver

We now introduce a useful way to visualise the mechanics of a family of G -constellations over a particular toric affine piece of Y . Suppose we have a family \mathcal{F} of G -constellations on Y and a cone σ in the fan \mathfrak{F} . In this section, we are interested in looking up close at the structure of \mathcal{F} restricted to the corresponding affine piece A_σ .

Over A_σ the sheaf \mathcal{F} is trivialised and we have

$$\mathcal{F}(A_\sigma) \simeq \mathbb{C}[\sigma^\vee] \otimes_{\mathbb{C}} V_{\text{reg}} \simeq \bigoplus_{\chi} F_{\chi},$$

where each F_{χ} is isomorphic to $\mathbb{C}[\sigma^\vee]$ and G acts on it by χ . Evidently, the whole structure of \mathcal{F} as a family of G -constellations on A_σ is contained in the way that R acts on F_{χ} 's. An effective method to visualise the mechanics of this is to consider the representations of the *McKay quiver of G* . We shall briefly summarize the necessary background. For a more detailed exposition of the following material, see [3].

Definition 4.6. A *quiver* consists of a vertex set Q_0 , an arrow set Q_1 and two maps $h: Q_1 \rightarrow Q_0$ and $t: Q_1 \rightarrow Q_0$ giving the head $hq \in Q_0$ and the tail $tq \in Q_0$ of each arrow $q \in Q_1$.

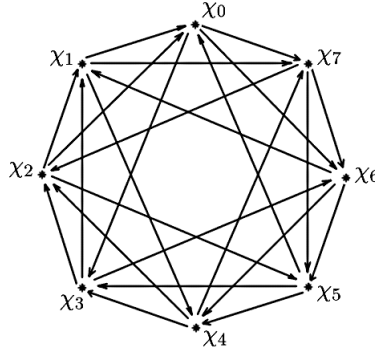
Definition 4.7. Let G be a finite subgroup of $\text{GL}(V_{\text{giv}})$. Then the *McKay quiver of G* is the quiver with the vertex set Q_0 labelled by the irreducible representations ρ of G and the arrow set Q_1 which has precisely $\dim \text{Hom}_G(\rho_i, \rho_j \otimes V_{\text{giv}})$ arrows going from the vertex ρ_i to the vertex ρ_j .

Example 4.8. (1) In our case of G being abelian and V_{giv} identified with \mathbb{C}^n , we have a decomposition of V_{giv}^\vee into irreducible representations as $\bigoplus \mathbb{C}x_i$, where x_i s are the basic monomials. Then, writing U_χ for the representation corresponding to $\chi \in G^\vee$,

$$\text{Hom}_G(U_{\chi_i}, U_{\chi_j} \otimes \mathbb{C}^n) = \bigoplus_{x_k | \chi_i \rho^{-1}(x_k) = \chi_j} \text{Hom}_G(x_k \otimes U_{\chi_i}, U_{\chi_j}),$$

where by $x_k \otimes U_{\chi_i}$, we denote the space $\mathbb{C}x_k \otimes_{\mathbb{C}} U_{\chi_i}$. Each of the spaces $\text{Hom}_G(x_k \otimes U_{\chi_i}, U_{\chi_j})$ is one-dimensional and so has one arrow from χ_i to χ_j corresponding to it. Thus the quiver consists of $|G|$ vertices labelled by characters $\chi \in G^\vee$ and out of each vertex χ emerge n arrows, each corresponding to one of the one-dimensional spaces $\text{Hom}_G(x_k \otimes U_\chi, U_{\chi \rho(x_k)})$. We shall write $(\chi, x_k) \in Q_1$ to denote such an arrow.

(2) For a concrete example, the reader can verify that the McKay quiver for $G = \frac{1}{8}(1, 2, 5)$ (see Example 4.1) looks like



A good reason for contemplating the McKay quiver of G is that it is possible to establish a one-to-one correspondence between a subset of its *representations* and (G, R) -modules.

Definition 4.9. A *representation* of a quiver is a graded vector space $\bigoplus_{i \in Q_0} V_i$ and a collection $\{\alpha_q: V_{tq} \rightarrow V_{hq}\}_{q \in Q_1}$ of linear maps indexed by the arrow set of the quiver. A morphism from $(\bigoplus V_i, \{\alpha_q\})$ to $(\bigoplus V'_i, \{\alpha'_q\})$ is a collection of linear maps $\{\theta_i: V_i \rightarrow V'_i\}_{i \in Q_0}$ forming commutative squares with α_q s and α'_q s.

Given a G -representation V , it is traditional, in case of G being a general finite subgroup of GL_n , to consider representations of the McKay quiver on a graded vector space $\bigoplus V_\rho$ where $V_\rho = \text{Hom}_G(\rho, V)$. It is then possible (see [16]) to establish a one-to-one correspondence between such representations and elements of $\text{Hom}_G(V_{\text{giv}}^\vee \otimes V, V)$. And, in the light of the remarks after Definition 2.2, there is a one-to-one correspondence between all the (G, R) -module structures on V and the elements of $\text{Hom}_G(V_{\text{giv}}^\vee \otimes V, V)$ which satisfy the commutator relations (2.3).

However, in the case when the group G is abelian, a considerable shortcut can be taken by considering the representations directly into graded vector space $\bigoplus V_\chi$, where V_χ is the χ -eigenspace of V . We again have the correspondence between representations of McKay quiver on $\bigoplus V_\chi$ and elements of $\text{Hom}_G(V_{\text{giv}}^\vee \otimes V, V)$ and consequently the correspondence with G -constellations. Explicitly, if we have a (G, R) -structure on V , then the action map $V \rightarrow V$ for each basic monomial x_i is G -equivariant and so splits into maps $V_\chi \rightarrow V_{\chi/\rho(x_i)}$. Each such map gives precisely the map $\alpha_{\chi, x_i} \in \text{Hom}(V_\chi, V_{\chi/\rho(x_i)})$ in the corresponding representation of the quiver.

In case of $V = V_{\text{reg}}$, if we make an explicit choice of a basis vector e_χ for each V_χ , this gives us bases for all $\text{Hom}_G(x_i \otimes V_\chi, V_{\chi/\rho(x_i)})$. Then every McKay quiver representation on $\bigoplus V_\chi$ gains a unique map $\xi: Q_1 \rightarrow \mathbb{C}$ associated with it, defined by

$$\alpha_{\chi, x_i}(e_\chi) = \xi(\chi, x_i)e_{\chi/\rho(x_i)}.$$

Considering a family of G -constellations \mathcal{F} parametrised by an affine piece A_σ of Y , we have, as outlined in the beginning of the section,

$$\mathcal{F}(A_\sigma) \simeq \mathbb{C}[\sigma^\vee] \otimes_{\mathbb{C}} V_{\text{reg}}.$$

We then write the χ -eigenspace decomposition $\mathcal{F}(A_\sigma) = \bigoplus F_\chi$, and all the correspondences above work just as well with $\mathbb{C}[\sigma^\vee]$ -modules as they did with complex vector spaces.

This technology presents us with a compact way to write down the R -module structure on $\mathcal{F}|_{A_\sigma}$. After a choice of bases, a representation of the McKay quiver becomes a map $\xi: Q_1 \rightarrow \mathbb{C}[\sigma^\vee]$ readily pictured as a McKay quiver of G with $\xi(\chi, x_i)$ written above each arrow $(\chi, x_i) \in Q_1$. In this way it is also easy to calculate explicitly the G -constellation in \mathcal{F} parametrised by any point of A_σ . If a point $p \in A_\sigma$ is defined by a map $\text{ev}_p: \mathbb{C}[\sigma^\vee] \rightarrow \mathbb{C}$, then the corresponding quiver representation is given by the map $\xi_p = \text{ev}_p \circ \xi: Q_1 \rightarrow \mathbb{C}$.

Finally, let us consider deformations of the generic orbit. If \mathcal{F} is one such, then it comes with an embedding $\iota: \mathcal{F} \rightarrow K(\mathbb{C}^n)$. Its image $\iota(\mathcal{F})$ splits into χ -eigenspaces, which are invertible sheaves, so we can take a set $\{f_\chi\} \in K(\mathbb{C}^n)$, where each f_χ is homogeneous of weight χ and a generator of χ^{-1} -eigenspace of \mathcal{F} over A_σ . The R -module structure comes for free with the embedding into $K(\mathbb{C}^n)$ and the corresponding quiver representation is given by the map $\xi: Q_1 \rightarrow \mathbb{C}[\sigma^\vee]$ defined by

$$(\chi^{-1}, x_i) \mapsto \frac{x_i f_\chi}{f_{\rho(x_i)\chi}}$$

with respect to the choice of generators f_χ .

Example 4.10. Let us work through an actual example. Let $G = \frac{1}{8}(1, 2, 5)$ and $\sigma = \langle e_4, e_5, e_6 \rangle$. Recall from Example 4.5 that the calculation of the dual basis in M gives us the local coordinates on $A_\sigma = \text{Spec } \mathbb{C}[\sigma^\vee]$ as $\mathbb{C}[\sigma^\vee] = \mathbb{C}\left[\frac{z^2}{x^2}, \frac{xy^2}{z}, \frac{x^2}{y}\right]$.

Consider $\mathcal{F} = \bigoplus_{\chi_i \in G^\vee} \mathcal{O}_{A_\sigma} f_i \subset K(\mathbb{C}^n)$ where

$$f_0 = 1, \quad f_1 = x, \quad f_2 = y, \quad f_3 = xy, \quad f_4 = \frac{z}{x}, \quad f_5 = z, \quad f_6 = \frac{yz}{x}, \quad f_7 = yz.$$

Now for any choice of f_i , as long as each $f_i \in K_{\chi_i}(\mathbb{C}^n)$, the generic fiber $\bigoplus K(Y) f_i$ is the whole of $K(\mathbb{C}^n)$. The latter has a natural structure of a G -constellation, and so it has a corresponding quiver representation. Let $\xi': Q_1 \rightarrow K(Y)$ be the map specifying it with respect to $\{f_i\}$ s as the choice of eigenspace bases.

We claim that \mathcal{F} is closed under R -action in $K(\mathbb{C}^n)$ and hence defines a family of G -constellations parametrised by A_σ . We shall verify this statement in the course of calculating the map ξ' and seeing that it restricts to a map $Q_1 \rightarrow \mathbb{C}[\sigma^\vee]$, which defines the quiver representation corresponding to our family.

Consider the arrow (χ_0, x) . As described above, in the corresponding quiver representation the map $K(Y)f_0 \rightarrow K(Y)f_1$ is given by multiplication by x . Hence we get

$$f_0 \mapsto 1 f_1,$$

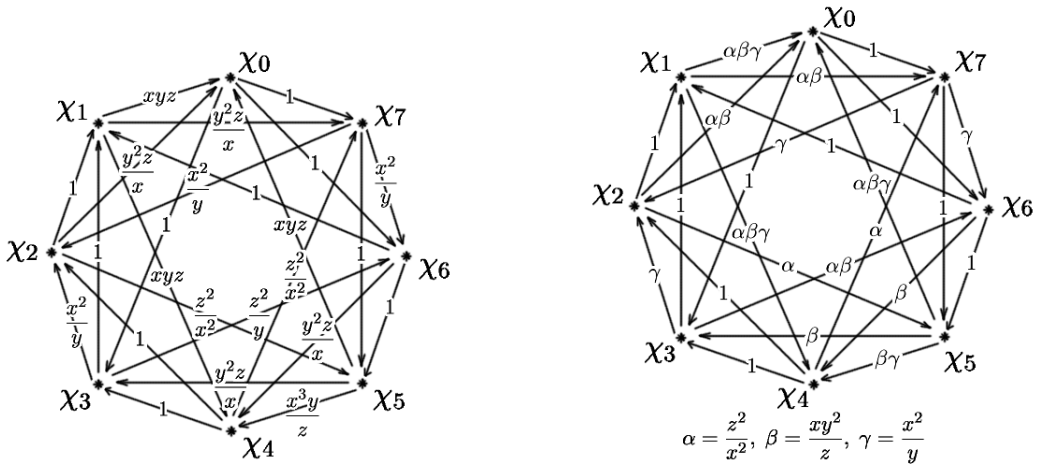
and so we label this arrow by

$$1 = \left(\frac{z^2}{x^2} \right)^0 \left(\frac{xy^2}{z} \right)^0 \left(\frac{x^2}{y} \right)^0.$$

Similarly the arrow (χ_5, z) corresponds to the map $f_3 \mapsto xyz f_0$ and so we label it by

$$xyz = \left(\frac{z^2}{x^2} \right)^1 \left(\frac{xy^2}{z} \right)^1 \left(\frac{x^2}{y} \right)^1.$$

Repeating this for all the arrows of the quiver we obtain



In the diagram on the right we have written all the functions marking the arrows in terms of positive powers of the local coordinates α, β, γ on A_σ . This demonstrates that

we indeed have a map

$$\xi: Q_1 \rightarrow \mathbb{C}[\sigma^\vee] = \mathbb{C}\left[\frac{z^2}{x^2}, \frac{xy^2}{z}, \frac{x^2}{y}\right],$$

so \mathcal{F} is indeed a family of G -constellations parametrised by $A_\sigma = \text{Spec}[\alpha, \beta, \gamma]$. The G -constellations parametrised by each point of A_σ are readily calculated by assigning specific values to α , β and γ in the diagram on the right.

5. Reductors

5.1. Reductor pieces

As in Section 4.3, let Y be a toric resolution, $\sigma \in \mathfrak{F}$ a cone in its fan and \mathcal{F} a deformation of the generic orbit across Y . If we have a set of generators $\{f_\chi \mid f_\chi \in K_\chi(\mathbb{C}^n)\}$ such that

$$\iota(\mathcal{F})(A_\sigma) = \bigoplus \mathbb{C}[\sigma^\vee]f_\chi,$$

then we must have

$$(5.1) \quad \frac{x_i f_\chi}{f_{\rho(x_i)\chi}} \in \mathbb{C}[\sigma^\vee]$$

for all basic monomials x_i and $\chi \in G^\vee$.

But observe that, conversely, for any set $\{f_\chi \mid f_\chi \in K_\chi(\mathbb{C}^n)\}$ for which (5.1) holds, the $\mathbb{C}[\sigma^\vee]$ -submodule of $K(\mathbb{C}^n)$ generated by f_χ is closed under the natural action of R on $K(\mathbb{C}^n)$ by multiplication. It is certainly closed under the G -action, so it is a (G, R) -submodule of $K(\mathbb{C}^n)$ and a family of G -constellations parametrised by A_σ .

This observation motivates the rest of this section. But first we make a useful definition

Definition 5.1. A *reductor piece* for a basic cone $\sigma \subset L$ of the fan \mathfrak{F} of the toric resolution Y is a set $\{f_\chi \mid f_\chi \in K_\chi(\mathbb{C}^n)\}$ such that for any basic monomial x_i and any $\chi \in G^\vee$ we have

$$\frac{x_i f_\chi}{f_{\rho(x_i)\chi}} \in \mathbb{C}[\sigma^\vee].$$

Thus, if we wanted to explicitly construct a family of G -constellations parametrised by Y , we could do it by producing a reductor piece for each cone σ in the fan \mathfrak{F} . Every such would give a family of G -constellations parametrised by open affine piece A_σ . However, we would need these families to ‘glue together’, i.e., the restrictions to $A_\sigma \cap A_{\sigma'}$ of the families generated on A_σ and $A_{\sigma'}$, respectively, must be isomorphic for any two cones $\sigma, \sigma' \in \mathfrak{F}$. The general way to guarantee this is independent of the toric technology altogether, taking us back to G -Weil divisors and to where Section 3 left off.

5.2. Reductor sets

From now on Y is once again an arbitrary, not necessarily toric, resolution of X .

Let \mathcal{F} be a deformation of the generic orbit. It comes with a choice of an embedding $\iota': \mathcal{F} \hookrightarrow K(\mathbb{C}^n)$. Then \mathcal{F} splits into G -eigensheaves as $\bigoplus \mathcal{F}_\chi$ and, as per Section 3, each \mathcal{F}_χ defines a linear equivalence class of χ -divisors embedding it into $K(\mathbb{C}^n)$, and $\iota'(\mathcal{F}_\chi)$ pinpoints a specific element of that class. Hence $\iota'(\mathcal{F}) = \bigoplus_\chi \mathcal{L}(-D_\chi)$ for some unique set of G -divisors $\{D_\chi\}_{\chi \in G^\vee}$. Note that it is important here that $\mathcal{L}(-D_\chi)$ is not merely an abstract line bundle corresponding to D_χ , but a specific sub- \mathcal{O}_Y -module of $K(\mathbb{C}^n)$ as per its definition.

Thus each subsheaf of the constant sheaf $K(\mathbb{C}^n)$ on Y , which is an image of an isomorphism class of deformations of the generic orbit, is of the form $\bigoplus \mathcal{L}(-D_\chi)$, where each D_χ is a χ -divisor on Y .

Lemma 5.2. *Let $\mathcal{F} = \bigoplus \mathcal{L}(-D_\chi)$ and $\mathcal{F}' = \bigoplus \mathcal{L}(-D'_\chi)$ be two deformations of the generic orbit across Y . Then they are isomorphic as sheaves of (G, R) -modules if and only if there exists $g \in K(Y)$ such that*

$$D'_\chi - D_\chi = (g)$$

for all $\chi \in G^\vee$.

Proof. The ‘if’ part is immediate, observe that we have a natural isomorphism $\mathcal{L}(A) \otimes \mathcal{L}(B) \rightarrow \mathcal{L}(A+B)$ given by multiplication in $K(\mathbb{C}^n)$. Applying this to $-D_\chi - (g) = -D'_\chi$ yields isomorphism $\mathcal{F} \rightarrow \mathcal{F}'$ given by $s \mapsto s/g$.

For the ‘only if’ part, let $\phi: \bigoplus \mathcal{L}(-D_\chi) \rightarrow \bigoplus \mathcal{L}(-D'_\chi)$ be a (G, R) -equivariant isomorphism. Then it restricts to $\phi_\chi: \mathcal{L}(-D_\chi) \xrightarrow{\sim} \mathcal{L}(-D'_\chi)$ for all $\chi \in G^\vee$. Then ϕ_χ induces a map $\mathcal{L}(0) \xrightarrow{\sim} \mathcal{L}(-D'_\chi + D_\chi)$, so let $g_\chi \in K(\mathbb{C}^n)^G$ be an image of 1 under this map. Then $D'_\chi - D_\chi = (g_\chi)$ and ϕ_χ is given by $s \mapsto g_\chi s$ for any $s \in \mathcal{L}(-D_\chi)$.

It remains to show that all g_χ are equal. Fix any $\chi \in G^\vee$ and consider any G -homogeneous $m \in R$ of weight χ . Take any $s \in \mathcal{L}(-D_{\chi_0}) \subset K(\mathbb{C}^n)$. Then $ms \in \mathcal{L}(-D_\chi)$ and using R -equivariance of ϕ :

$$\phi(ms) = m\phi(s) = g_{\chi_0}ms,$$

and hence $g_\chi = g_{\chi_0}$ for all $\chi \in G^\vee$. □

Corollary 5.3. *Let $\mathcal{F} = \bigoplus \mathcal{L}(-D_\chi)$ and $\mathcal{F}' = \bigoplus \mathcal{L}(-D'_\chi)$ be two deformations of the generic orbit across Y . Then they are equivalent if and only if there exists a χ_0 -divisor N such that*

$$D'_\chi - D_\chi = N$$

for all $\chi \in G^\vee$.

Proof. Once again, the ‘if’ direction is immediate: an isomorphism $\mathcal{F} \otimes \mathcal{L}(-N) \rightarrow \mathcal{F}'$ is given by multiplication in $K(\mathbb{C}^n)$.

Conversely, if the families are equivalent then let \mathcal{N} be an invertible sheaf on Y such that $\mathcal{F}' \simeq \mathcal{F} \otimes \mathcal{N}$. Choose any Weil divisor N' such that $\mathcal{N} = \mathcal{L}(-N')$. Then apply Lemma 5.2 to the isomorphic families $\bigoplus \mathcal{L}(-D_\chi - N')$ and $\mathcal{L}(-D'_\chi)$ to obtain $g \in K(\mathbb{C}^n)$ such that $D'_\chi - D_\chi - N' = (g)$ for all $\chi \in G^\vee$. Setting $N = N' + (g)$ finishes the proof. \square

Corollary 5.4. *In every equivalence class of deformations of the generic orbit there exists a unique family \mathcal{F} of the form $\bigoplus \mathcal{L}(-D_\chi)$ with $D_{\chi_0} = 0$.*

Proof. Given an arbitrary deformation of the generic orbit \mathcal{F} we can find an isomorphic family of the form $\bigoplus \mathcal{L}(-D_\chi)$. Then setting $D'_\chi = D_\chi - D_{\chi_0}$ we obtain an equivalent family $\mathcal{L}(-D'_\chi)$ with the required properties. Finally, Corollary 5.3 shows the uniqueness. \square

In the view of all of the above, we make the following definition.

Definition 5.5. Let $\{D_\chi\}_{\chi \in G^\vee}$ be a set of G -divisors. We shall call it a *prereductor set* if each D_χ is a χ -Weil divisor. We shall call it a *reductor set* if $\bigoplus \mathcal{L}(-D_\chi)$ with the inclusion map into $K(\mathbb{C}^n)$ is a deformation of the generic orbit. We shall say the reductor set is *normalised* if $D_{\chi_0} = 0$.

5.3. Reductor condition

We have seen that a deformation of the generic orbit can be specified (up to an isomorphism) by a set of G -Weil divisors on Y which gives its embedding into $K(\mathbb{C}^n)$. Here we investigate an opposite question. For which prereductor sets $\{D_\chi\}$, is $\bigoplus \mathcal{L}(-D_\chi)$ a family of G -constellations?

We observe that $\bigoplus \mathcal{L}(-D_\chi)$ is always a sub- \mathcal{O}_Y -module of $K(\mathbb{C}^n)$ closed under the G -action. However, for a general choice of divisors D_χ , there is no guarantee that the $\bigoplus \mathcal{L}(-D_\chi)$ will be closed under the R -action on $K(\mathbb{C}^n)$.

Proposition 5.6 (Reductor condition). *Let $\{D_\chi\}$ be a prereductor set. Then it is a reductor set if and only if, for any $f \in R_G$, a G -homogeneous polynomial, the divisor*

$$(5.2) \quad D_\chi + (f) - D_{\chi\rho(f)} \geq 0,$$

i.e., it is effective.

Remark 5.7. (1) It is, of course, sufficient to check (5.2) only for f being one of the basic monomials x_1, \dots, x_n . This leaves us with a finite number of inequalities to check. Note also that the principal divisor (x_j) is very easy to compute in toric case. It follows

immediately from Proposition 4.3 that it is $\sum_{e_i \in \mathfrak{C}} e_i(x_j)E_i$. Observe that $e_i(x_j)$ is simply the j th coordinate of e_i in L .

(2) Numerically, if we write each D_χ as $\sum q_{\chi,P}P$, each inequality (5.2) becomes a set of inequalities

$$(5.3) \quad q_{\chi,P} + v_P(f) - q_{\chi\rho(f),P} \geq 0$$

for all prime divisors P on Y . The important thing to notice here is that the subsets of inequalities for each prime divisor P are all independent of each other. We can speak of $\{D_\chi\}$ satisfying or not satisfying the reductor condition at a given prime divisor P . Moreover, we can construct reductor sets $\{D_\chi\}$ by independently choosing for each prime divisor P any of the sets of numbers $\{q_{\chi,P}\}_{\chi \in G^\vee}$ which satisfy (5.3).

Proof of Proposition 5.6. Take an open cover U_i on which all $\mathcal{L}(-D_\chi)$ are trivialised and write $g_{\chi,i}$ for the generator of $\mathcal{L}(-D_\chi)$ on U_i . $\{D_\chi\}$ being a reductor set is equivalent to $\bigoplus \mathcal{L}(-D_\chi)$ being closed under R -action on $K(\mathbb{C}^n)$. As R is a direct sum of its G -homogeneous parts, it is sufficient to check the closure under the action of just the homogeneous functions. So on each U_i , we want

$$f g_{\chi,i} \in \mathcal{O}_Y(U_i) g_{\chi\rho(f),i}$$

to hold for all $f \in R_G$, $\chi \in G^\vee$.

On the other hand, with the notation above, G -Cartier divisor $D_\chi + (f) - D_{\chi\rho(f)}$ is given on U_i by $\frac{f g_{\chi,i}}{g_{\chi\rho(f),i}}$ and it being effective is equivalent to

$$\frac{f g_{\chi,i}}{g_{\chi\rho(f),i}} \in \mathcal{O}_Y(U_i)$$

for all U_i 's. The result now follows. \square

We now translate the reductor condition (5.2) into toric language and investigate what it implies for the reductor pieces of the family on the open toric charts A_σ of a toric resolution Y .

Example 5.8. Let G and Y be as in previous examples. Let $\{D_\chi\}$ be a prereducator set where each $D_\chi = \sum q_{\chi,i}E_i$ is given as follows:

$$\begin{aligned} D_{\chi_0} &= 0, & D_{\chi_1} &= \frac{1}{8}E_4 + \frac{2}{8}E_5 + \frac{4}{8}E_6 + \frac{5}{8}E_7, \\ D_{\chi_2} &= \frac{2}{8}E_4 + \frac{4}{8}E_5 + \frac{2}{8}E_7, & D_{\chi_3} &= \frac{3}{8}E_4 + \frac{6}{8}E_5 + \frac{4}{8}E_6 + \frac{7}{8}E_7, \\ D_{\chi_4} &= \frac{4}{8}E_4 + \frac{4}{8}E_7, & D_{\chi_5} &= \frac{5}{8}E_4 + \frac{2}{8}E_5 + \frac{4}{8}E_6 + \frac{1}{8}E_7, \\ D_{\chi_6} &= \frac{6}{8}E_4 + \frac{4}{8}E_5 + \frac{6}{8}E_7, & D_{\chi_7} &= \frac{7}{8}E_4 + \frac{6}{8}E_5 + \frac{4}{8}E_6 + \frac{3}{8}E_7. \end{aligned}$$

In the view of Proposition 4.2, the reductor condition (5.2) is equivalent to

$$(5.4) \quad q_{\chi,i} + e_i(m) - q_{\chi\rho(m),i} \geq 0$$

for all $\chi \in G^\vee$, $e_i \in \mathfrak{E}$ and $m \in \mathbb{Z}_+^n$.

The careful reader could now verify that (5.4) holds for $m = (1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and hence $\{D_\chi\}$ is a reductor set and $\bigoplus \mathcal{L}(-D_\chi)$ is a family of G -constellations.

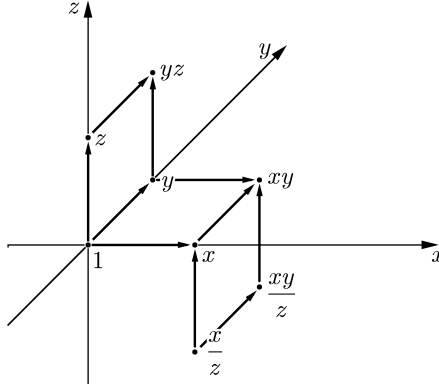
Recall now reductor pieces introduced in Definition 5.1. Let us calculate the reductor piece $\{x^{p_\chi}\}$ specified by the generators of $\mathcal{L}(-D_\chi)$ on the affine piece $A_{\langle e_5, e_6, e_7 \rangle}$. This is the same calculation of a generator of a G -Weil divisor on a given open toric chart that we saw in Example 4.5, e.g.,

$$p_{\chi_1} = q_{\chi_1,5} \check{e}_5 + q_{\chi_1,6} \check{e}_6 + q_{\chi_1,7} \check{e}_7,$$

and so

$$x^{p_{\chi_1}} = \left(\frac{y^2 z}{x}\right)^{2/8} \left(\frac{z^2}{y}\right)^{4/8} \left(\frac{x^2}{z^2}\right)^{5/8} = x.$$

Repeating this for each $\chi \in G^\vee$, we obtain $\{x^{p_\chi}\} = \{1, x, y, xy, \frac{x}{z}, z, \frac{xy}{z}, yz\}$, the reductor piece pictured below as a diagram in the monomial lattice \mathbb{Z}^n :



The inequalities (5.4) now translate into the following form

$$e_i(p_\chi + m - p_{\chi\rho(m)}) > 0, \quad i = 5, 6, 7,$$

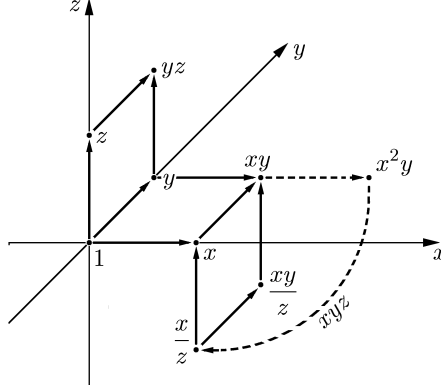
that is,

$$(5.5) \quad \frac{x^{p_\chi} x^m}{x^{p_{\chi\rho(m)}}} \in \mathbb{C}[\sigma^\vee]$$

for every $m \in \mathbb{Z}_+^n$. This agrees with the discussion in Section 5.1, where it is precisely the condition for $\bigoplus \mathcal{O}_{A_\sigma} x^{p_\chi}$ to be a family of G -constellations parametrised by A_σ .

The reader may find the diagrams set in the monomial lattice \mathbb{Z}^n convenient for checking if a given monomial set $\{x^{p_\chi}\}$ satisfies the reductor equations in the form (5.5). One

merely needs to check that when adding $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$ to any p_χ , the vector reducing the result to $p_{\chi'}$ (for appropriate χ') lies within the cone σ^\vee .



5.4. Existence and symmetries

So far we have seen no indication that, over an arbitrary resolution Y , there exist any deformations of the generic orbit in the first place. There is apriori, for an arbitrary Y , no reason why it could at all be able to parametrise a family of G -constellations with such a strong relation to the geometry of Y (see Proposition 2.5) as that of a deformation of the generic orbit. However, the following result shows that, for an absolutely any resolution Y , we always have at least one such family.

Proposition 5.9 (Canonical family). *For an arbitrary resolution Y of X the set of G -Weil divisors given by $D_\chi = \sum v(P, \chi)P$, where P runs over all prime Weil divisors on Y , satisfies the reductor condition.*

We shall call the family $\mathcal{F} = \bigoplus \mathcal{L}(-D_\chi)$ the canonical deformation of the generic orbit of G across Y .

Remark 5.10. For $D_\chi = \sum v(P, \chi)P$ to be a G -Weil divisor we need, in particular, for it to be a finite sum. This is implied by Corollary 3.12.

Proof of Proposition 5.9. We need to show that for any $\chi \in G^\vee$, any G -homogeneous $f \in R_G$ and any prime divisor P on Y we have

$$v(P, \chi) + v_P(f) - v(P, \chi\rho(f)) \geq 0.$$

First observe that the above expression must be integer valued. Also $v(P, \chi) \geq 0$ and $-v(P, \chi\rho(f)) > -1$ by definition, while $v_P(f) \geq 0$ since f^n is regular on all of Y . So we must have

$$v(P, \chi) + v_P(f) - v(P, \chi\rho(f)) > -1,$$

and the result follows. \square

Corollary 5.11. *Let Y be a toric resolution of X . Then the canonical family of G -constellations on Y is given by $\{D_\chi\}$ where*

$$D_\chi = \sum_{i \in \mathfrak{C}} v(E_i, \chi) E_i.$$

Moreover, on any affine open piece A_σ , we have

$$\mathcal{F}(A_\sigma) = \mathbb{C}[\sigma \cap \mathbb{Z}^n].$$

Proof. The first statement follows trivially from the definition of the canonical family and the fact that $v(P, \chi) = 0$ whenever P is not one of the divisors E_i (see Corollary 3.14).

For the second statement, without loss of generality let $\sigma = \langle e_1, \dots, e_n \rangle$. Write $\mathcal{F}(A_\sigma) = \bigoplus \mathbb{C}[\sigma^\vee \cap M] x^{p_\chi}$, where x^{p_χ} are the generators of $\mathcal{L}(-D_\chi)(A_\sigma)$. Proposition 4.3 implies that for each p_χ we have $e_i(p_\chi) = v(E_i, \chi)$ for all $i \in 1, \dots, n$. But all the numbers $v(E_i, \chi)$ are positive by definition, which implies that each p_χ lies in σ^\vee and so $\mathcal{F}(A_\sigma) \subseteq \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$. Conversely, given any $m \in \sigma^\vee \cap \mathbb{Z}^n$,

$$e_i(m - p_{\rho(m)}) = e_i(m) - v(\rho(m), E_i) \geq 0$$

as $v(E_i)$, $\rho(m)$ is precisely the fractional part of $v_{E_i}(m) = e_i(m)$. Therefore $m - p_{\rho(m)} \in \sigma^\vee \cap M$ and so we have the inclusion in the other direction.

Geometrically, one could easily convince oneself in the truth of this statement by picturing the cone $\sigma^\vee = \{v \in \mathbb{R}^n \mid e_i(v) \geq 0\}$ in $\mathbb{Z}^n \otimes \mathbb{R}$ and observing that the set $\{p_\chi\}$ of the exponents of the reductor piece of \mathcal{F} on A_σ consists precisely of all the elements of \mathbb{Z}^n lying within the topmost area U of σ^\vee given by $1 > e_i(v) \geq 0$. $\sigma^\vee \cap \mathbb{Z}^n$ is then precisely $(U \cap \mathbb{Z}^n) + (\sigma^\vee \cap M)$. We can also see why reductor condition holds: as the cone \mathbb{R}_+^n lies within the cone σ^\vee , $p_\chi + m$ lies within $\sigma^\vee \cap \mathbb{Z}^n$ for any $x^m \in R$. \square

Example 5.12. The reductor set $\{D_\chi\}$ given in Example 5.8 specifies the canonical family on Y . Indeed, observe that all the numbers $q_{\chi,i}$ are between 0 and 1. The (3.2) in definition of a G -Weil divisor implies they must be $v(E_i, \chi)$.

Generally, to calculate the canonical family in a toric case, one needs to choose a monomial m_χ of weight χ for each $\chi \in G$. Then, for each $e_i \in \mathfrak{F}$, one calculates the rational number $e_i(m_\chi)$ and takes its fractional part, which is precisely $v(E_i, \chi)$. The G -Weil divisors $D_\chi = \sum_i v(E_i, \chi) E_i$ are then the reductor set for the canonical family.

For instance, the numbers for the canonical family in Example 5.8 were obtained as follows: take character $\chi_3 \in G^\vee$ and then take x^3 , a monomial of weight χ_3 . Calculating $e_5(3, 0, 0) = \frac{1}{8}(2 \times 3 + 4 \times 0 + 2 \times 0) = \frac{6}{8}$, we obtain the coefficient of E_5 in D_{χ_3} . Similarly $e_7(3, 0, 0) = \frac{15}{8}$ and its fractional part $\frac{7}{8}$ is the coefficient of E_7 in D_{χ_3} .

Observe also that given any other reductor set $\{D'_\chi\}$, its $q_{\chi',i}$ will differ from those of the canonical one by integer numbers.

Observe also that on the level of reductors $\{x^{p_\chi}\}$, the change introduced to the family by adding an integer n to $q_{\chi,i}$ amounts precisely to shifting p_χ by $n\check{e}_i$ in the reductor of those open pieces A_σ where $e_i \in \sigma$. But note that \check{e}_i is a different vector in M for each such σ .

Having established that deformations of the generic orbit across Y always exist, we now consider symmetries which the set of them must possess.

Proposition 5.13 (Character shift). *Let $\{D_\chi\}$ be a reductor set. Then for any λ -Weil divisor N , the set $\{D_\chi + N\}$ also satisfies the reductor condition.*

Moreover, up to equivalence of families, the deformation \mathcal{F}' it specifies depends only on λ and not on the choice of N , and the unique normalized reductor set $\{D'_\chi\}$ specifying \mathcal{F}' is given by

$$D'_{\chi\lambda} = D_\chi - D_{\lambda^{-1}}.$$

Proof. It is trivial that the new set of divisors satisfies the reductor condition:

$$(D_\chi + N) + (m) - (D_{\chi\rho(m)} + N) \geq 0$$

is immediately equivalent to the statement that $\{D_\chi\}$ satisfy the reductor condition.

For the second claim, observe that the divisor in the trivial character class is now $(D_{\lambda^{-1}} + N)$. Normalising by it we obtain in character class $\chi + \lambda$:

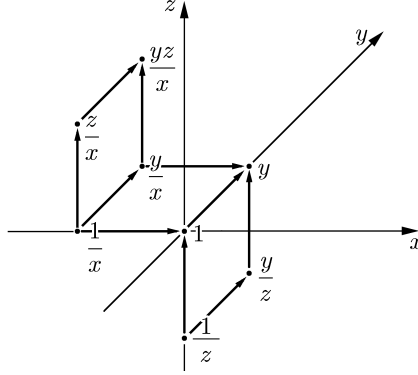
$$D_\chi + N - D_{\lambda^{-1}} - N,$$

which establishes the claim. □

Definition 5.14. Given a normalized reductor set $\{D_\chi\}$, we shall call normalized reductor set $\{D_\chi - D_{\lambda^{-1}}\}$ the λ -shift of $\{D_\chi\}$.

Example 5.15. On the level of reductors $\{x^{p_\chi}\}$, λ -shift leaves the geometrical configuration of p_χ 's in the lattice \mathbb{Z}^n the same, but permutes them and shifts the origin to the new location of p_{χ_0} .

For example, consider the case of the reductor piece calculated in Example 5.8. After a χ_4 -shift it becomes



Proposition 5.16 (Reflection). *Let $\{D_\chi\}$ be a reductor set. Then the set $\{-D_\chi\}$ also satisfies the reductor condition.*

Proof. We need to show that

$$-D_{\chi^{-1}} + (m) - (-D_{\chi^{-1}\rho(m)^{-1}}) \geq 0.$$

Rearranging we get

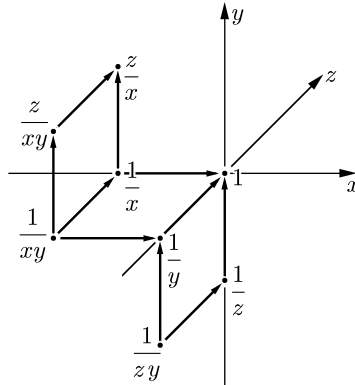
$$D_{\chi^{-1}\rho(m)^{-1}} + (m) - D_{\chi^{-1}\rho(m)^{-1}\rho(m)} \geq 0,$$

which is one of the reductor equations the original set $\{D_\chi\}$ must satisfy. \square

Definition 5.17. Given a reductor set $\{D_\chi\}$, we shall call the reductor set $\{-D_\chi\}$ the *reflection* of $\{D_\chi\}$.

Example 5.18. On the level of reductors $\{x^{p_\chi}\}$, the reflection is precisely the reflection of p_χ about the origin in the lattice \mathbb{Z}^n .

For example, consider the case of the reductor piece calculated in Example 5.8. After a reflection it becomes



5.5. Maximal shifts

We now examine the individual line bundles $\mathcal{L}(-D_\chi)$ in a deformation of the generic orbit and show that the reductor condition imposes a restriction on how far apart from each other they can be.

Lemma 5.19. *Let $\{D_\chi\}$ be a reductor set. Write each D_χ as $\sum q_{\chi,P}P$, where P ranges over all the prime Weil divisors on Y . Then we necessarily have for any $\chi_1, \chi_2 \in G^\vee$ and for any prime Weil divisor P ,*

$$\min_{f \in R_{\chi_1/\chi_2}} v_P(f) \geq q_{\chi_1,P} - q_{\chi_2,P} \geq - \min_{f \in R_{\chi_2/\chi_1}} v_P(f),$$

where R_χ is the set of all the χ -homogeneous functions in R .

Proof. Both inequalities follow directly from the reductor condition (5.2): the right inequality by setting $\chi = \chi_1 \in G^\vee$, $\rho(f) = \frac{\chi_2}{\chi_1}$ and letting f vary within $R_{\rho(f)}$; the left inequality by setting $\chi = \chi_2$ and $\rho(f) = \frac{\chi_1}{\chi_2}$. \square

This suggests the following definition.

Definition 5.20. For each character $\chi \in G^\vee$, the *maximal shift χ -divisor* M_χ is defined to be

$$(5.6) \quad M_\chi = \sum_P \left(\min_{f \in R_\chi} v_P(f) \right) P,$$

where P ranges over all prime Weil divisors on Y .

Observe that the fact that the sum in (5.6) is finite follows directly from Corollary 3.14.

Lemma 5.21. *The G -Weil divisor set $\{M_\chi\}$ is a normalised reductor set.*

Proof. To show that the set $\{M_\chi\}$ satisfies the reductor condition, we need to show that for every $f \in R_G$ and any prime divisor P on Y ,

$$v_P(m_\chi) + v_P(f) - v_P(m_{\chi\rho(f)}) \geq 0,$$

where m_χ and $m_{\chi\rho(f)}$ are chosen to achieve the minimality in (5.6).

Observe that $m_\chi f$ is also a G -homogeneous element of R , therefore by the minimality of $v_P(m_{\chi\rho(f)})$ we have

$$v_P(m_\chi f) \geq v_P(m_{\chi\rho(f)}),$$

as required.

To establish that $M_{\chi_0} = 0$, we observe that $v_P(1) = 0$ for any prime Weil divisor P on Y and $v_P(f) \geq 0$ for any G -homogeneous $f \in R$. \square

Observe that with Lemma 5.21 we have established another deformation of the generic orbit of G which always exists across any resolution Y . While in some cases it coincides with the canonical family, the reader will see in Example 5.23 the case when the canonical family and the maximal shift family differ.

Putting together Lemmas 5.19 and 5.21 gives a result which shows that the reductor set $\{M_\chi\}$ and its reflection $\{-M_\chi\}$ provide bounds on the set of all normalized reductor sets on Y .

Proposition 5.22 (Maximal shifts). *Let $\{D_\chi\}$ be a normalized reductor set. Then for any $\chi \in G^\vee$,*

$$(5.7) \quad M_\chi \geq D_\chi \geq -M_{\chi^{-1}}.$$

Moreover both the bounds are achieved.

Proof. To establish that (5.7) holds, set $\chi_2 = \chi_0$ in Lemma 5.19. Lemma 5.21 shows that bounds are achieved. \square

Example 5.23. Let us calculate the maximal shift divisor set $\{M_\chi\}$ for the setup introduced in Example 4.1.

By definition, $M_\chi = \sum m_{\chi,P} P$ where $m_{\chi,P} = \min_{f \in R_\chi} v_P(f)$. By Corollary 3.14, the numbers $m_{\chi,P}$ are only non-zero for divisors corresponding to elements of \mathfrak{E} . Therefore for each $e_i \in \mathfrak{E}$, we need to find $m_{\chi,E_i} = \min e_i(p)$ where p ranges over elements of \mathbb{Z}_+^n such that $\rho(p) = \chi$.

It is only necessary to consider a finite number of choices for p to establish each $m_{\chi,P}$. Observe that it suffices to take the ones with $0 \leq p_i \leq |G|$ as $p' = p - (0, \dots, 0, |G|, 0, \dots, 0)$ is again an element of \mathbb{Z}^n with $\rho(p') = \rho(p)$ and $e_i(p') \leq e_i(p)$ for all $e_i \in \mathfrak{E}$.

For example, taking $e_5 = \frac{1}{8}(2, 4, 2)$ and considering all such p , we see that

$$\begin{aligned} m_{\chi_0, E_5} &= v_{E_5}(1) = e_5(0, 0, 0) = 0, & m_{\chi_1, E_5} &= v_{E_5}(x) = e_5(1, 0, 0) = \frac{2}{8}, \\ m_{\chi_2, E_5} &= v_{E_5}(x^2) = e_5(2, 0, 0) = \frac{4}{8}, & m_{\chi_3, E_5} &= v_{E_5}(x^3) = e_5(3, 0, 0) = \frac{6}{8}, \\ m_{\chi_4, E_5} &= v_{E_5}(x^4) = e_5(4, 0, 0) = 1, & m_{\chi_5, E_5} &= v_{E_5}(z) = e_5(0, 0, 1) = \frac{2}{8}, \\ m_{\chi_6, E_5} &= v_{E_5}(zx) = e_5(1, 0, 1) = \frac{4}{8}, & m_{\chi_7, E_5} &= v_{E_5}(zx^2) = e_5(2, 0, 1) = \frac{6}{8}. \end{aligned}$$

Observe that in case of χ_4 we have $m_{P, \chi} \neq v_{P, \chi}$. So the maximal shift family for this Y differs from the canonical family.

If we repeat this calculation for all elements of \mathfrak{E} to obtain all numbers $m_{e_i, \chi}$, we will obtain

$$\begin{aligned} M_{\chi_0} &= 0, & M_{\chi_1} &= \frac{1}{8}E_4 + \frac{2}{8}E_5 + \frac{4}{8}E_6 + \frac{5}{8}E_7, \\ M_{\chi_2} &= \frac{2}{8}E_4 + \frac{4}{8}E_5 + \frac{2}{8}E_7, & M_{\chi_3} &= \frac{3}{8}E_4 + \frac{6}{8}E_5 + \frac{4}{8}E_6 + \frac{7}{8}E_7, \\ M_{\chi_4} &= \frac{4}{8}E_4 + E_5 + \frac{4}{8}E_7, & M_{\chi_5} &= \frac{5}{8}E_4 + \frac{2}{8}E_5 + \frac{4}{8}E_6 + \frac{1}{8}E_7, \\ M_{\chi_6} &= \frac{6}{8}E_4 + \frac{4}{8}E_5 + \frac{6}{8}E_7, & M_{\chi_7} &= \frac{7}{8}E_4 + \frac{6}{8}E_5 + \frac{4}{8}E_6 + \frac{3}{8}E_7. \end{aligned}$$

Compare it to the reductor set of the canonical family given in Example 5.8.

If we now want to calculate all the normalised reductor sets (and hence all the normalised deformations of the generic orbit), we simply need to check each of the finite number of prereductor sets between $\{M_\chi\}$ and its reflection $\{-M_\chi\}$ and pick out the ones which satisfy the reductor condition (5.2).

Recall now Remark 5.7 about checking reductor condition independently at each prime divisor in Y . Here, it means that for any reductor set $\{\sum_i q_{\chi,i} E_i\}_{\chi \in G^\vee}$, the numbers $\{q_{\chi,i}\}_{\chi \in G^\vee}$ satisfy or fail the reductor condition inequalities independently for each $e_i \in \mathfrak{E}$. This can be seen from the fact that each of the inequalities (5.4) features numbers $q_{\chi,i}$ all for the same i .

In particular it means that to list all the possible normalized reductor sets on Y , it is sufficient to list for each E_i all the sets $\{q_{\chi,i}\}_{\chi \in G^\vee}$ satisfying the inequalities (5.4). Then all the normalized reductor sets on Y are given by all the possible choices of one of these sets $\{q_{\chi,i}\}_{\chi \in G^\vee}$ for each E_i .

For our particular Y , we give such list below:

E_4 :

$$\frac{1}{8} \begin{pmatrix} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & -1 \\ 0 & 1 & 2 & 3 & 4 & 5 & -2 & -1 \\ 0 & 1 & 2 & 3 & 4 & -3 & -2 & -1 \\ 0 & 1 & 2 & 3 & -4 & -3 & -2 & -1 \\ 0 & 1 & 2 & -5 & -4 & -3 & -2 & -1 \\ 0 & 1 & -6 & -5 & -4 & -3 & -2 & -1 \\ 0 & -7 & -6 & -5 & -4 & -3 & -2 & -1 \end{pmatrix},$$

E_5 :

$$\frac{1}{8} \begin{pmatrix} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \\ 0 & 2 & 4 & 6 & 8 & 2 & 4 & 6 \\ 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\ 0 & 2 & 4 & -2 & 0 & 2 & 4 & 6 \\ 0 & 2 & 4 & 6 & 0 & 2 & 4 & -2 \\ 0 & 2 & 4 & -2 & 0 & 2 & 4 & -2 \\ 0 & 2 & -4 & -2 & 0 & 2 & 4 & -2 \\ 0 & 2 & 4 & -2 & 0 & 2 & -4 & -2 \\ 0 & 2 & -4 & -2 & 0 & 2 & -4 & -2 \\ 0 & -6 & -4 & -2 & 0 & 2 & -4 & -2 \\ 0 & 2 & -4 & -2 & 0 & -6 & -4 & -2 \\ 0 & -6 & -4 & -2 & 0 & -6 & -4 & -2 \\ 0 & -6 & -4 & -2 & -8 & -6 & -4 & -2 \end{pmatrix},$$

E_6 :

$$\frac{1}{8} \begin{pmatrix} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & -4 & 0 & -4 & 0 & -4 & 0 & -4 \end{pmatrix},$$

E_7 :

$$\frac{1}{8} \begin{pmatrix} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \\ 0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\ 0 & 5 & 2 & -1 & 4 & 1 & 6 & 3 \\ 0 & 5 & 2 & -1 & 4 & 1 & -2 & 3 \\ 0 & -3 & 2 & -1 & 4 & 1 & -2 & 3 \\ 0 & -3 & 2 & -1 & -4 & 1 & -2 & 3 \\ 0 & -3 & 2 & -1 & -4 & 1 & -2 & -5 \\ 0 & -3 & -6 & -1 & -4 & -7 & -2 & -5 \end{pmatrix}.$$

For one particular resolution Y , the family provided by the maximal shift divisors is already quite well-known.

Proposition 5.24. *Let $Y = G\text{-Hilb } \mathbb{C}^n$, the moduli space of G -clusters in \mathbb{C}^n . If Y is smooth, then $\bigoplus \mathcal{L}(-M_\chi)$ is the universal family \mathcal{F} of G -clusters parametrised by Y , up to the usual equivalence of families.*

Proof. Firstly \mathcal{F} is a deformation of the generic orbit, as over any set $U \subset X$ such that G acts freely on $q^{-1}(U)$ we have $\pi_*\mathcal{F}|_U \simeq q_*\mathcal{O}_{\mathbb{C}^n}|_U$. Hence write \mathcal{F} as $\bigoplus \mathcal{L}(-D_\chi)$ for some reductor set $\{D_\chi\}$. Take an open cover $\{U_i\}$ of Y and consider the generators $\{f_{\chi,i}\}$ of D_χ on each U_i . Working up to equivalence, we can consider $\{D_\chi\}$ to be normalised and so $f_{\chi_0,i} = 1$ for all U_i .

Now any G -cluster Z is given by some invariant ideal $I \subset R$ and so the corresponding G -constellation $H^0(\mathcal{O}_Z)$ is given by R/I . In particular note that R/I is generated by R -action on the generator of χ_0 -eigenspace. Therefore any $f_{\chi,i}$ is generated from $f_{\chi_0,i} = 1$ by R -action, which means that all $f_{\chi,i}$ lie in R .

But this means that for any prime Weil divisor P on Y we have

$$v_P(f_{\chi,i}) \geq \min_{f \in R_\chi} v_P(f),$$

and therefore $D_\chi \geq M_\chi$. Now Proposition 5.22 forces the equality. \square

5.6. Summary

Finally, we combine the results achieved thus far into a classification theorem.

Theorem 5.25 (Classification). *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X be the quotient of \mathbb{C}^n by the action of G and Y be a resolution of X . Then all deformations of the generic orbit across Y , up to isomorphism, are of form $\bigoplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$, where each D_χ is a χ -Weil divisor and the set $\{D_\chi\}$ satisfies the inequalities*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0$$

for all $\chi \in G^\vee$ and all G -homogeneous $f \in R$. Here $\rho(f)$ is the homogeneous weight of f . Conversely for any such set $\{D_\chi\}$, $\bigoplus \mathcal{L}(-D_\chi)$ is a deformation of the generic orbit.

Moreover, each equivalence class of families has precisely one family with $D_{\chi_0} = 0$. The divisor set $\{D_\chi\}$ corresponding to such a family satisfies inequalities

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}},$$

where $\{M_\chi\}$ is a fixed divisor set depending only on G and Y . In particular, the number of equivalence classes of families is finite.

Proof. Proposition 5.6 establishes the correspondence of isomorphism classes of deformations of the generic orbit and reductor sets. Corollary 5.4 lifts the correspondence to the level of equivalence classes and normalised reductor sets. Proposition 5.22 gives the bounds on the set of all normalised reductor sets, and as due to Corollary 3.14 each M_χ is a finite sum, this set is finite. \square

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