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Numerical Computation of the Rosenblatt Distribution and Applications

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ABSTRACT

The Rosenblatt distribution plays a key role in the limit theorems for non-linear functionals of stationary Gaussian processes with long-range dependence. We derive new expressions for the characteristic function of the Rosenblatt distribution. Also we present a novel accurate approximation of all eigenvalues of the Riesz integral operator associated with the correlation function of the Gaussian process and propose an efficient algorithm for computation of the density of the Rosenblatt distribution. We perform Monte-Carlo simulation for small sample sizes to demonstrate the appearance of the Rosenblatt distribution for several functionals of stationary Gaussian processes with long-range dependence.

1 | Introduction

The phenomenon of long-range dependence (also called long memory) is one of exciting area of research in the probability theory for last few decades due to non-standard normalizations and non-Gaussian limiting distributions of nonlinear functionals (Pipiras and Taqqu 2017). The Rosenblatt distribution serves a significant role in the study of this phenomenon which occurs in economics, finance, hydrology, turbulence, cosmology and physics (Doukhan et al. 2002; Beran 2017).

The Rosenblatt distribution was introduced in (Rosenblatt 1961) and later it was investigated by many researchers, see (Taqqu 1975, 1979; Dobrushin and Major 1979) among many others. The Rosenblatt distribution appears as the limiting distribution of some popular functionals, see (Doukhan et al. 2002; Berman 1979; Rosenblatt 1979) and references therein.

The known analytical form of the characteristic function of the Rosenblatt distribution contains a series which converges in a neighbourhood of zero, see (Rosenblatt 1961; Albin 1998; Veillette and Taqqu 2013). It turns out that the Rosenblatt distribution is infinitely divisible (Veillette and Taqqu 2013) and self-decomposable, moreover, it belongs to the Thorin class (Maejima and Tudor 2013; Leonenko et al. 2017a, 2017b).

The density of the Rosenblatt distribution exists and bounded, however, the closed analytical form of the density is unknown, see (Veillette and Taqqu 2013; Maejima and Tudor 2013; Leonenko et al. 2017a, 2017b). The Edgeworth expansion was used to approximate the density of the Rosenblatt distribution (Veillette and Taqqu 2012). The numerical evaluation of the density of the Rosenblatt distribution was proposed in (Veillette and Taqqu 2013) and serves as a competing method to our own. The difference lies in the fact that we use the accurate analytical approximation of eigenvalues instead of their numerical calculation for a large matrix and compute the density directly from the Fourier transform of a simple characteristic function instead of the numerical evaluation of the convolution of two densities, where one density is computed via the Edgeworth expansion and another density is computed via the Fourier transform of some simple characteristic function. Overall, our numerical computation of the density of the Rosenblatt distribution is uninvolved and faster than the method from (Veillette and Taqqu 2013), while the results of two methods coincide.

The present paper is organized as follows. In Section 2 we review the known facts on the Rosenblatt distribution. In Section 3 we provide the novel analytic forms of the characteristic function of the Rosenblatt distribution on the entire line and a compelling viewpoint on the structure of the Rosenblatt distribution. In Section 4 we propose an accurate approximation of all eigenvalues of the Riesz integral operator, which allows us to perform fast numerical computation of the density of the Rosenblatt distribution.

In Section 5 we propose a time-efficient algorithm for simulation of the stationary Gaussian sequences with long-range dependence with the power correlation function and the Mittag-Lefler correlation function. Furthermore, we demonstrate small sample properties of four popular functionals for analysis of sequences of correlated random variables including their non-standard normalization and convergence to the Rosenblatt distribution. Specifically, we consider estimation of the mean in Section 5.3, the correlation function in Section 5.4, the sojourn functionals in Section 5.5 and path roughness in Section 5.6. All these problems are essentials of the statistical analysis of stationary Gaussian sequences with long-range dependence.

2 | Formal Statement

We consider the Rosenblatt distribution with shape parameter a, zero mean, unit variance and the characteristic function

$$\phi(z) = \exp\left(\frac{1}{2}\sum_{k=2}(2iz)^k \frac{c_{a,k}\sigma_a^k}{k}\right), a \in \left[0, 1/2\right], z \in S_0 \subset \mathbb{R}, \tag{1}$$

where S_0 is a small neighbourhood of zero,

$$\sigma_a = \sqrt{(1-2a)(1-a)/2}$$

and

$$c_{a,k} = \int_0^1 \cdots \int_0^1 |x_1 - x_2|^{-a}|x_2 - x_3|^{-a} \cdots |x_{k-1} - x_k|^{-a}|x_k - x_1|^{-a} dx_1 \cdots dx_k.$$

In the rest of this section we describe the Rosenblatt distribution following (Veillette and Taqqu 2013). The random variable V from the Rosenblatt distribution can be given as

$$V = \sum_{n=1}^{\infty} \lambda_{a,n}\left(\varepsilon_n^2 - 1\right), \tag{2}$$

where ε_n are i.i.d. random variables from the standard normal distribution and $\lambda_{a,1}, \lambda_{a,2}, \ldots$ are eigenvalues of the Riesz integral operator $\widetilde{K}_a : L^2(0,1) \to L^2(0,1)$ defined as

$$\left(\widetilde{K}_a f\right)(x) = \sigma_a \int_0^1 |x-u|^{-a} f(u) du.$$

The operator $\widetilde{K}_a$ is known to be positive and compact, and therefore has positive eigenvalues $\lambda_{a,n}$, see (Reade 1979). The representation (2) means that V is a specific instance of second-order Wiener chaos (Nourdin and Poly 2012). These eigenvalues satisfy the relation

$$\sum_{n=1}^{\infty} \lambda_{a,n}^k = \begin{cases} \infty, & k = 1, \\ c_{a,k}\sigma_a^k, & k = 2, 3, \ldots. \end{cases}$$

Moreover, we have

$$\sum_{n=1}^{\infty} \lambda_{a,n}^2 = 1/2,$$
$$\sum_{n=1}^{\infty} \lambda_{a,n}^3 = \frac{2\sigma_a^3}{(1-a)(2-3a)} \beta(1-a, 1-a),$$

where $\beta(u,v) = \int_0^1 x^{u-1}(1-x)^{v-1} dx$ is the beta function. The Rosenblatt distribution is infinitely divisible with the Lévy density

$$m(x) = \frac{1}{2x} \sum_{n=1}^{\infty} \exp\left(-\frac{x}{2\lambda_{a,n}}\right), x > 0,$$

that shows self-decomposability of the Rosenblatt distribution (Veillette and Taqqu 2013).

3 | Main Results

The Laplace transform of V defined in (2) is given by

$$\phi_{LT}(s) = \mathbb{E}\left(e^{-sV}\right) = \exp\left(-\sum_{n=1}^{\infty}\left(\frac{1}{2}\ln\left(1 + 2\lambda_{a,n}s\right) - \lambda_{a,n}s\right)\right), s > -\frac{1}{2\lambda_{a,1}},$$

see (Albin 1998). We note that the formal expansion of the sum in the above expression gives

$$-\frac{1}{2}\sum_{n=1}^{\infty} \ln\left(1 + 2\lambda_{a,n}s\right) + s\sum_{n=1}^{\infty} \lambda_{a,n},$$

where the latter summand equals infinity. Using the classical Taylor expansion of the logarithm

$$\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k, x \in (0, 2),$$

which does not converge for all positive x, we obtain

$$\phi_{LT}(s) = \exp\left(\frac{1}{2}\sum_{k=2}^{\infty} \frac{(-2s)^k}{k}\left(\sum_{n=1}^{\infty} \lambda_{a,n}^k\right)\right), \ |s| < \frac{1}{2\lambda_{a,1}}, \tag{3}$$

which is related to (1) as $\phi(z) = \phi_{LT}(-iz)$, $z \in \mathbb{R}$. We note that the expressions of the characteristic function (1) and the Laplace transform (3) cannot be used for numerical calculation because they are defined at a neighbourhood of zero.

In next theorem, we derive new expressions using three other expansions of the logarithm which converge on the full domain $(0, \infty)$ and one integral representation. In other words, we construct analytic continuations for the expressions (1) and (3).

Theorem 1. *The characteristic function $\phi(z)$ of the Rosenblatt distribution is given by $\phi(z) = \phi_{LT}(-iz)$, where the Laplace transform $\phi_{LT}(s)$ with $s > -\frac{1}{2\lambda_{a,1}}$ admits*

i. *the representation using the domain-scaled expansion*

$$\ln\left(\phi_{LT}(s)\right) = \sum_{n=1}^{\infty}\left(\frac{\left(\lambda_{a,n}s\right)^2}{1+\lambda_{a,n}s} - \sum_{k=2}^{\infty}\frac{1}{2k-1}\left(\frac{\lambda_{a,n}s}{1+\lambda_{a,n}s}\right)^{2k-1}\right),$$

ii. *the representation using the Ramanujan expansion*

$$\ln\left(\phi_{LT}(s)\right) = \sum_{n=1}^{\infty}\lambda_{a,n}s\ln\left(1+2\lambda_{a,n}s\right)\sum_{k=1}^{\infty}\frac{1}{2^k\left(1+\left(1+2\lambda_{a,n}s\right)^{2^{-k}}\right)}, \tag{4}$$

iii. *the representation using the Ramanujan-Bradley expansion*

$$\ln\left(\phi_{LT}(s)\right) = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}2^{k-1}\left(\left(1+2\lambda_{a,n}s\right)^{2^{-k}}-1\right)^2, \tag{5}$$

iv. *the representation using the integral form*

$$\ln\left(\phi_{LT}(s)\right) = \int_0^s \sum_{n=1}^{\infty}\frac{2\lambda_{a,n}^2 u}{2\lambda_{a,n}u+1}du.$$

The proof of Theorem 1 is deferred to the online supplement. We note that the evaluation of the characteristic function of the Rosenblatt distribution requires computation of the infinite sums with a special care because numerical computation of $\sum_{i=1}^{\infty}\lambda_{a,n}^2$ is problematic for $a \in (0.35, 0.5)$ due to very slow convergence of the series $\sum_{i=1}^{\infty} n^{2a-2}$. To overcome difficulties of numerical computation, we propose to find smallest integer M_ϵ such that

$$\sum_{n=M_\epsilon+1}^{\infty}\lambda_{a,n}^k < \epsilon$$

for $k = 3, 4, \ldots$, where ϵ is a small positive number, for example, $\epsilon = 0.0001$. Then we can write the random variable V defined in (2) in the form $V = W + Z$, where

$$W = \sum_{n=1}^{M_\epsilon}\lambda_{a,n}\left(\varepsilon_n^2 - 1\right), Z = \sum_{n=M_\epsilon+1}^{\infty}\lambda_{a,n}\left(\varepsilon_n^2 - 1\right).$$

Due to the choice of M_ϵ, the random variable Z has approximately the normal distribution with mean zero and variance

$$\sigma_\epsilon^2 = 1 - 2\sum_{n=1}^{M_\epsilon}\lambda_{a,n}^2 = 2\sum_{n=M_\epsilon+1}^{\infty}\lambda_{a,n}^2.$$

Indeed, the characteristic function of Z has the form

$$\begin{aligned}\phi_Z(z) &= \exp\left(-\sum_{n=M_\epsilon+1}^{\infty}\left(\frac{1}{2}\ln\left(1-2\lambda_{a,n}iz\right)+\lambda_{a,n}iz\right)\right)\\ &\overset{|z|<\frac{1}{2\lambda_{a,M_\epsilon}}}{=} \exp\left(\frac{1}{2}\sum_{k=2}^{\infty}\frac{(2iz)^k}{k}\left(\sum_{n=M_\epsilon+1}^{\infty}\lambda_{a,n}^k\right)\right)\\ &\approx \exp\left(-z^2\left(\sum_{n=M_\epsilon+1}^{\infty}\lambda_{a,n}^2\right)+\frac{4}{3}i^3z^3\epsilon\right)\\ &\approx \exp\left(-\frac{1}{2}z^2\sigma_\epsilon^2\right).\end{aligned}$$

Moreover, we have

$$\max_{|z|\leq\frac{1}{2\lambda_{a,M_\epsilon}}} \left|\,\phi_Z(z) - \exp\left(-\frac{1}{2}z^2\sigma_\epsilon^2\right)\right| \to 0 \text{ as } \epsilon \to 0.$$

This argument provides a clear constructive view on the Rosenblatt distribution. Specifically, the random variable from the Rosenblatt distribution can be simulated as

$$V_\epsilon = \sigma_\epsilon\varepsilon_0 + \sum_{n=1}^{M_\epsilon}\lambda_{a,n}\left(\varepsilon_n^2 - 1\right),$$

where ε_n are i.i.d. random variables from the standard normal distribution. The density of the Rosenblatt distribution can be computed by the inverse Fourier transform of the characteristic function

$$\phi_\epsilon(z) = \exp\left(-\frac{1}{2}z^2\sigma_\epsilon^2 - \sum_{n=1}^{M_\epsilon}\left(\frac{1}{2}\ln\left(1-2\lambda_{a,n}iz\right)+\lambda_{a,n}iz\right)\right).$$

Applying the Stein method for the random variable V_ϵ and taking the limit as $\epsilon \to 0$, we obtain the following characterizing identity, which follows from (Arras and Houdré 2019, Ch. 3).

Theorem 2. *Let X be a random variable with zero mean. The Stein equation*

$$\mathbb{E}Xf(X) = \mathbb{E}\int_0^{\infty}(f(X+x)-f(X))\frac{1}{2}\sum_{n=1}^{\infty}\exp\left(-\frac{x}{2\lambda_{a,n}}\right)dx \tag{6}$$

holds for all bounded Lipschitz function $f(\cdot)$ if and only if X has the Rosenblatt distribution with parameter a.

Let us use the Stein equation (6) with $f(x) = x$, we refer to (Arras and Houdré 2019, Ch. 3) for a discussion on a class of suitable functions. Then the right hand side of the Stein equation is

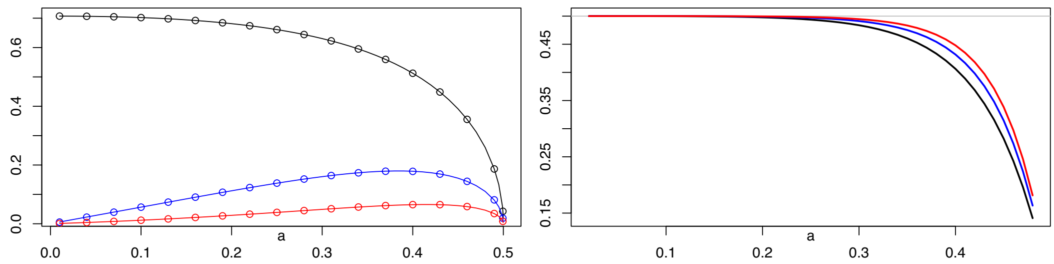

FIGURE 1 | Left: Eigenvalues $\lambda_{a,1}$ (black), $\lambda_{a,2}$ (blue) and $\lambda_{a,8}$ (red) which are computed numerically (circles) and via approximation (7) (solid line) for $a \in (0,0.5)$. Right: $S_{a,M} = \sum_{n=1}^{M} \lambda_{a,n}^2$ as a function of a for $M = 100$ (black), $M = 500$ (blue) and $M = 2000$ (red).

$$\mathbb{E}\int_0^\infty (X + x - X)\frac{1}{2}\sum_{n=1}^\infty \exp\left(-\frac{x}{2\lambda_{a,n}}\right)dx = 2\sum_{n=1}^\infty \lambda_{a,n}^2 = 1$$

and the left hand side of the Stein equation is the second moment, which equals one from the definition of the Rosenblatt distribution. Thus, the Stein equation for $f(x) = x$ and $X = V$ becomes $\mathbb{E}V^2 = 1$. Taking $f(x) = x^2$ and $X = V$ in the Stein equation (6) we obtain

$$\mathbb{E}V^3 = \mathbb{E}\int_0^\infty \left(2Vx + x^2\right)\frac{1}{2}\sum_{n=1}^\infty \exp\left(-\frac{x}{2\lambda_{a,n}}\right)dx = 8\sum_{n=1}^\infty \lambda_{a,n}^3.$$

For $f(x) = x^3$ and $X = V$ in the Stein equation (6) we obtain

$$\mathbb{E}V^4 = \mathbb{E}\int_0^\infty \left(3V^2x - 3Vx + x^3\right)\frac{1}{2}\sum_{n=1}^\infty \exp\left(-\frac{x}{2\lambda_{a,n}}\right)dx = 48\sum_{n=1}^\infty \lambda_{a,n}^4 + 3.$$

The values of the third and fourth moments demonstrate that the Rosenblatt distribution is non-Gaussian.

4 | Computational Aspects

Although moments of the Rosenblatt distribution depend on the multidimensional integrals $c_{a,k}$,

$$\mathbb{E}V^k = \frac{1}{i^k}\frac{d^k}{dz^k}\phi(z)\Big|_{z=0} = \begin{cases} 0, & k=1,\\ 1, & k=2,\\ 8\sigma_a^3 c_{a,3}, & k=3\\ 48\sigma_a^4 c_{a,4} + 12\sigma_a^4 c_{a,2}^2, & k=4\\ \vdots & k=5,6,\ \dots, \end{cases}$$

simulation and computation of the density of the Rosenblatt distribution is not possible without the eigenvalues $\lambda_{a,n}$. It was shown in (Dostani'c 1998) that these eigenvalues have the asymptotic behaviour $\lambda_{a,n} = C_a n^{a-1}\left(1 + o\left(n^{-\delta}\right)\right)$ as $n \to \infty$ for any $\delta \in (0,1)$ and admit the accurate approximation

$$\lambda_{a,n} \cong C_a n^{a-1}, n > 20,$$

TABLE 1 | Values of M_ϵ for various ϵ and a.

	$a=0.1$	$a=0.2$	$a=0.3$	$a=0.35$	$a=0.4$	$a=0.44$	$a=0.48$
$M_\epsilon\big\|_{\epsilon=10^{-3}}$	2	3	7	13	24	34	13
$M_\epsilon\big\|_{\epsilon=10^{-4}}$	3	9	48	133	409	909	630

where

$$C_a = \frac{2\sigma_a}{\pi^{1-a}}\Gamma(1-a)\sin(\pi a/2),$$

see (Veillette and Taqqu 2013) for details. From extensive numerical calculation of eigenvalues for various $a \in (0,0.5)$, we obtain that the eigenvalues $\lambda_{a,n}$ admit the accurate approximation

$$\lambda_{a,n} \cong \begin{cases} (1+0.1409a)\sqrt{\pi^a\Gamma(1-a)}\sqrt{\frac{1}{2}-a}, & n=1,\\ C_a n^{a-1} + \frac{5}{4}a^{1.05}\sqrt{\Gamma\left(a+\frac{1}{2}\right)-1}\,n^{a-2.2}, & n=2,3,\ \dots. \end{cases} \tag{7}$$

In Figure 1 we demonstrate that eigenvalues $\lambda_{a,n}$ for various a and n computed via approximation (7) are close to values computed numerically using the algorithm described in (Veillette and Taqqu 2013). Figure 1 also shows the behaviour of $S_{a,M} = \sum_{n=1}^{M} \lambda_{a,n}^2$ which show the contribution of W. We note that $\sigma_\epsilon^2 = 1 - 2S_{a,M}$ describes the contribution of the normal component Z. We note that for $a \approx 0.45$ we have the situation, where both components W and Z have approximately equal contribution for making the shape of the Rosenblatt distribution, that is, $S_{a,M} \approx \sigma_\epsilon^2$ for $a \approx 0.45$.

Let us find M_ϵ from the condition $\sum_{n=M_\epsilon+1}^{\infty} \lambda_{a,n}^3 \cong \epsilon$. We can see from Table 1 that the case of $a \cong 0.44$ requires the largest value of M_ϵ. This means that the accurate computation of the Rosenblatt distribution requires a big number of eigenvalues for $a \approx 0.44$.

In Figure 2 we depict the characteristic function $\phi_\epsilon(z)$ for various a. We can see that the shape of $\phi_\epsilon(z)$ tends to the characteristic function of the normal distribution as $a \to 0.5$.

In Figure 3 we depict the density, the logarithm of the density and the cumulative distribution function (cdf) of the

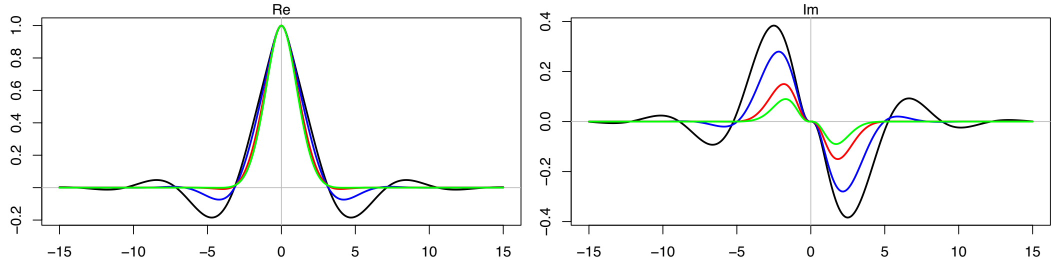

FIGURE 2 | The real part (left) and the imaginary part (right) of the characteristic function $\phi_\epsilon(z)$ with $M_\epsilon = 100$ for $a = 0.2$ (black), $a = 0.3$ (blue), $a = 0.4$ (red) and $a = 0.44$ (green).

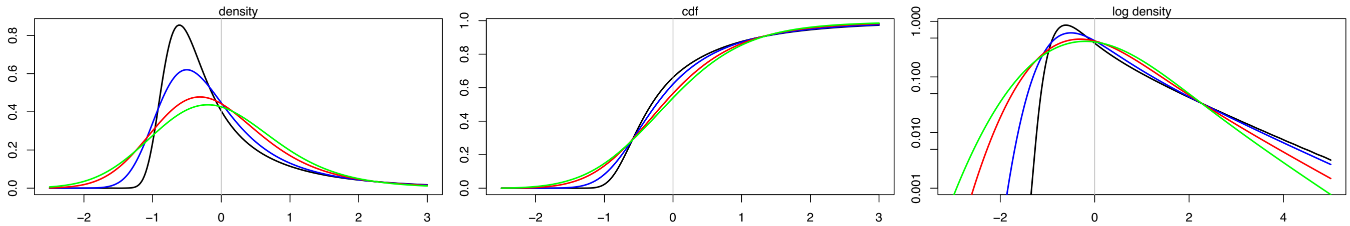

FIGURE 3 | The density (left), the logarithm of the density (right) and the cdf (middle) of the Rosenblatt distribution which is computed via the inverse Fourier transform of the characteristic function $\phi_\epsilon(z)$ with $M_\epsilon = 100$ for $a = 0.2$ (black), $a = 0.3$ (blue), $a = 0.4$ (red) and $a = 0.44$ (green).

Rosenblatt distribution for various a. We can see that the density of Rosenblatt distribution is close to the density of shifted chi-square distribution for $a \in (0,0.2)$ and close to the density of the normal distribution for $a \in (0.4,0.5)$.

5 | Numerical Study With Applications of the Rosenblatt Distribution

The Rosenblatt distribution with parameter a appears as a limiting distribution of several functionals for stationary Gaussian sequences with the correlation function of the form

$$r(t) = \frac{l(|t|)}{|t|^a},$$

where $l(\cdot)$ is a slowly varying function, that implies long-range dependence if $a \in (0,1)$. Typical examples of such correlation function are $r(t) = \left(1+t^2\right)^{-a/2}$, $r(t) = (1+|t|)^{-a}$ and $r(t) = E_a\left(-|t|^a\right)$, where $E_a(v) = \sum_{k=0}^{\infty} \frac{v^k}{\Gamma(1+ak)}, v \in \mathbb{C}$, is the Mittag-Leffler function. For running our numerical study, we firstly propose an efficient algorithm for simulation of long sequences with LRD, see (Bardet et al. 2003) for review of simulation algorithms, which were found to be very time consuming in our settings.

5.1 | Simulation of Long Gaussian Sequences

The traditional way of simulation of a stationary Gaussian sequence $X_1, \ldots, X_n$ with zero mean, unit variance and the specified correlation function is via simulation of the vector from the multivariate normal distribution with the use of the Cholesky decomposition or the eigenvalue decomposition of the covariance matrix. However, this way is computationally infeasible if the length of Gaussian sequence is larger than 10,000 due to the problem with the decomposition of the covariance matrix of large size.

The second way of simulation is to approximate the Gaussian sequence by the autoregressive process. This approach works well when the specified correlation function is close to zero for large lags and is not suitable for simulation of sequences with LRD.

The third way of simulation is based on the approximation

$$X_j \approx \sum_{k=1}^{M} \sqrt{b_k} X_j^{(k)}, j = 1, \ldots, n,$$

where $X_1^{(k)}, \ldots, X_n^{(k)}$ is a Gaussian autoregressive sequence of order 1 with zero mean, unit variance and the correlation function $e^{-\lambda_k |t|}$ and the specified correlation function $r(t)$ allows the approximation in the form

$$r(t) \approx \sum_{k=1}^{M} b_k e^{-\lambda_k |t|}. \tag{8}$$

We note that the coefficients b_k could be computed via the minimization of an error of the approximation (8) with some λ_k, for example, $\lambda_k = e^{-(k-1)}$. To avoid this high-dimensional optimization, we propose the following methodology for finding b_k and λ_k in the approximation (8).

Suppose that the correlation function $r(t)$ has the representation

$$r(t) = \int_0^{\infty} e^{-tx} p(x) dx,$$

where $p(x)$ is a density. Then we can construct the approximation (8) with $b_k = 1/M$ and λ_k be random values from the distribution with the density $p(x)$. Alternatively, λ_k can be chosen as the $(k-0.5)/M$-quantile of the distribution with the density $p(x)$.

Let us take a decreasing positive sequence $q_0, q_1, q_2, \ldots$ such that $1 = q_0 > q_1 > q_2 > \ldots > 0$ and $\lim_{k\to\infty} q_k = 0$. Define τ_k as the q_k-quantile of the distribution with density $p(x)$, that is,

$$q_k = \int_0^{\tau_k} p(x)dx, k = 0,1,2, \ldots .$$

We note that $\tau_1, \tau_2, \ldots$ is a decreasing sequence, $\tau_1 > \tau_2 > \ldots$ and $\lim_{k\to\infty} \tau_k = 0$. Then by splitting the integration domain $(0, \infty)$ with breakpoints $\tau_1, \tau_2, \ldots$ we obtain that

$$r(t) = \sum_{k=1}^{\infty} \int_{\tau_k}^{\tau_{k-1}} e^{-tx} p(x)dx$$

and, consequently, the correlation function $r(t)$ has the approximation (8) with

$$b_k = q_{k-1} - q_k \text{ and some } \lambda_k \in (\tau_{k-1}, \tau_k),$$

that follows from

$$\int_{\tau_k}^{\tau_{k-1}} e^{-tx} p(x)dx \approx e^{-t\lambda_k} \int_{\tau_k}^{\tau_{k-1}} p(x)dx = (q_{k-1} - q_k) e^{-t\lambda_k}.$$

We recommend to take $\lambda_1 = \tau_1$ and

$$\lambda_k = \sqrt{\tau_{k-1}\tau_k}, k = 2, 3, \ldots .$$

The approximation (8) is accurate if M is large and $\max\{b_1, \ldots, b_M\}$ is small.

5.2 | Simulation of Long Gaussian Sequences With LRD

Suppose that

$$p(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} l(1/x), \quad \alpha > 0, x > 0, x \to 0, \tag{9}$$

where $l(\cdot)$ is a slowly varying function, which describes the behaviour of the density $p(x)$ at zero. It follows from the Tauberian-Abelian theorem that the property (9) is equivalent to

$$r(t) = \frac{l(|t|)}{|t|^{\alpha}}, \quad t \to \infty, \tag{10}$$

which describes the behaviour of the correlation function $r(t)$ at infinity, see e.g. (Feller and Morse 1958). We note that LRD occurs if $\alpha \in (0, 1)$.

For simulation of sequences with $r(t)$ in the form (10), it was proposed in (Leonenko and Taufer 2005) to take

$$b_k = c/k^{(1+a)}, \lambda_k = 1/k,$$

where c is a constant such that $\sum_{k=1}^{M} b_k = 1$. This choice of λ_k is not convenient in numerical studies due to slow convergence to zero.

Let us develop an approximation for the correlation function $r(t) = 1/(1+|t|)^a$. From (Barndorff-Nielsen and Leonenko 2005) we have that

$$\frac{1}{(1+|t|)^a} = \int_0^{\infty} e^{-tx} p(x)dx, a \in (0, 1),$$

where

$$p(x) = \frac{1}{\Gamma(a)} x^{a-1} e^{-x}.$$

We recommend to choose the sequence q_k to be fast-decreasing. Specifically, we propose to take

$$q_k = \begin{cases} 0.98, & k = 1, \\ 0.9\gamma^{k-2}, & k = 2, 3, \ldots, \end{cases} \quad \gamma = e^{-(2-a)a}, M = \left\lceil \frac{2}{a} \right\rceil + 8, \tag{11}$$

The above choice of q_k provides the reasonable accuracy of the approximation (8) even with small M. The simulation algorithm is deferred to the online supplement.

In Figure 4 we demonstrate the good accuracy of the approximation (8) with parameters (11) of the correlation function $r(t) = 1/(1+|t|)^a$ for various $a \in (0,0.5)$.

In Figure 5 we depict several realizations of a stationary Gaussian sequence with zero mean, unit variance and the correlation function $r(t) = 1/(1+|t|)^a$ to illustrate the phenomenon of long-range dependence. We can see that the deviation of a local trend of realizations from zero increases as a decreases. We note that a sequence of length 2,000,000 is needed for rather good accuracy of estimation of the parameter a if $a \approx 0.25$.

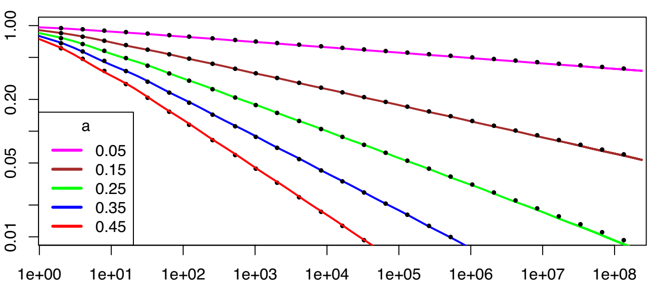

FIGURE 4 | The approximation (8) with parameters (11) (solid line) of the correlation function $r(t) = 1/(1+|t|)^a$ (dotted line) for various a in log–log scale.

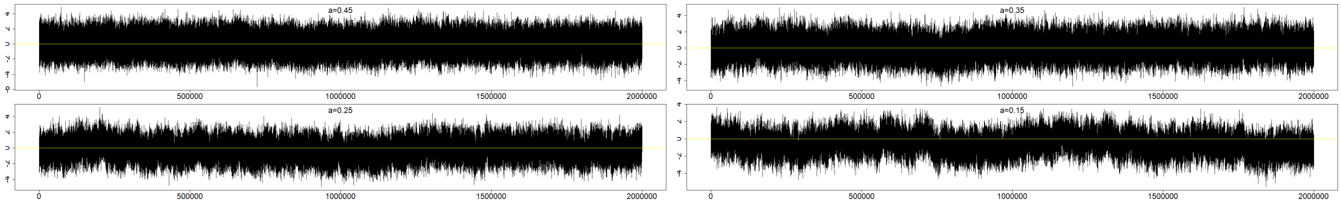

FIGURE 5 | The realizations of a stationary Gaussian sequence with zero mean, unit variance and the correlation function $r(t) = 1/(1+|t|)^a$ for $a = 0.15, 0.25, 0.35$ and 0.45.

As a second example, we develop an approximation for the Mittag-Leffler correlation function $r(t) = E_a(-|t|^a)$. From (Barndorff-Nielsen and Leonenko 2005) we have that

$$E_a(-|t|^a) = \int_0^\infty e^{-tx} p_{ml}(x)dx, a \in (0, 1),$$

where

$$p_{ml}(x) = \frac{\sin(a\pi)}{\pi} \frac{x^{a-1}}{1 + 2\cos(a\pi)x^a + x^{2a}}, x > 0,$$

is the density of the Lamperti distribution, which has the quantile function

$$Q_{ml}(u) = \left(\frac{\sin(ua\pi)}{\sin((1-u)a\pi)} \right)^{1/a}, u \in (0, 1),$$

see (James 2010). We recommend to choose the sequence q_k in the form

$$q_k = \begin{cases} 0.98, & k=1, \\ 0.9, & k=2, \\ 0.7, & k=3, \\ 0.5\gamma^{k-4}, & k=4, 5, \ldots, \end{cases} \qquad \gamma = e^{-(2-a)a}, M = \left\lceil \frac{2}{a} \right\rceil + 8, \tag{12}$$

The above choice of q_k for the Mittag-Lefler correlation function provides the reasonable accuracy of the approximation (8) even with small M. In Figure 6 we demonstrate the good accuracy of the approximation (8) with parameters (12) of the correlation function $r(t) = E_a(-|t|^a)$ for various $a \in (0,0.5)$.

5.3 | Estimation of the Mean for a Sequence of Special Structure

Consider a stationary sequence $Y_1, Y_2, \ldots$, which is given by

$$Y_k = \theta + \sigma H_2(E_k),$$

where $H_2(x) = x^2 - 1$ and $E_1, E_2, \ldots$ is a stationary Gaussian sequence with zero mean, unit variance and correlation function $r(k)$. The sequence $\{E_k\}$ can be viewed as discretization of a stationary Gaussian process and the sequence $\{H_2(E_k)\}$ is a stationary sequence with zero mean and variance 2. Consider the estimator of θ in the form

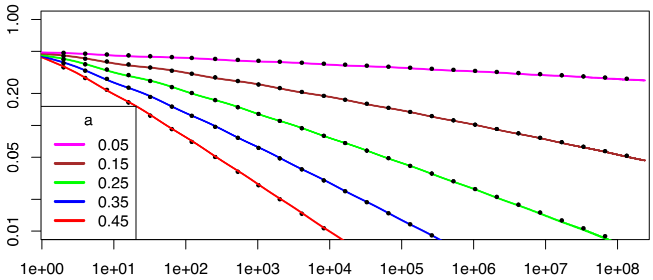

FIGURE 6 | The approximation (8) with parameters (12) (solid line) of the Mittag-Leffler correlation function $r(t) = E_a(-|t|^a)$ (dotted line) for various a in log–log scale.

$$\hat{\theta}_n = \frac{1}{n} \sum_{k=1}^n Y_k.$$

The estimator $\hat{\theta}_n$ has asymptotically the scaled Rosenblatt distribution if the sequence $\{E_k\}$ has the correlation function $r(t) = \frac{l(|t|)}{|t|^a}$ with $a \in (0,0.5)$ as $t \to \infty$ and has the normal distribution otherwise, see (Ivanov and Leonenko 2002; Leonenko and Taufer 2006).

Let us make a numerical study to obtain the distribution of the estimator for small sample sizes. Specifically, we consider the random variable

$$Z_n = \frac{\sigma_a}{n^{1-a}} \sum_{k=1}^n H_2(E_k), a \in (0,0.5),$$

which has asymptotically the Rosenblatt distribution with parameter $a \in (0,0.5)$.

In Figure 7 we can see that the empirical density of Z_n for $a = 0.25$ and various n is very close to the density of the Rosenblatt distribution and this holds for all $a \in (0,0.25)$. However, such proximity is not observed for larger values of a. Nevertheless, the empirical density of Z_n for $a = 0.45$ becomes more close to the density of the Rosenblatt distribution as n increases. We would like to note that the empirical density of Z_n for $a = 0.45$ and small n is close to the Rosenblatt distribution with some small value of a.

5.4 | Estimation of the Correlation Function for a Stationary Gaussian Sequence With LRD

Let $E_1, E_2, \ldots$ be a stationary Gaussian sequence with zero mean, unit variance and correlation function

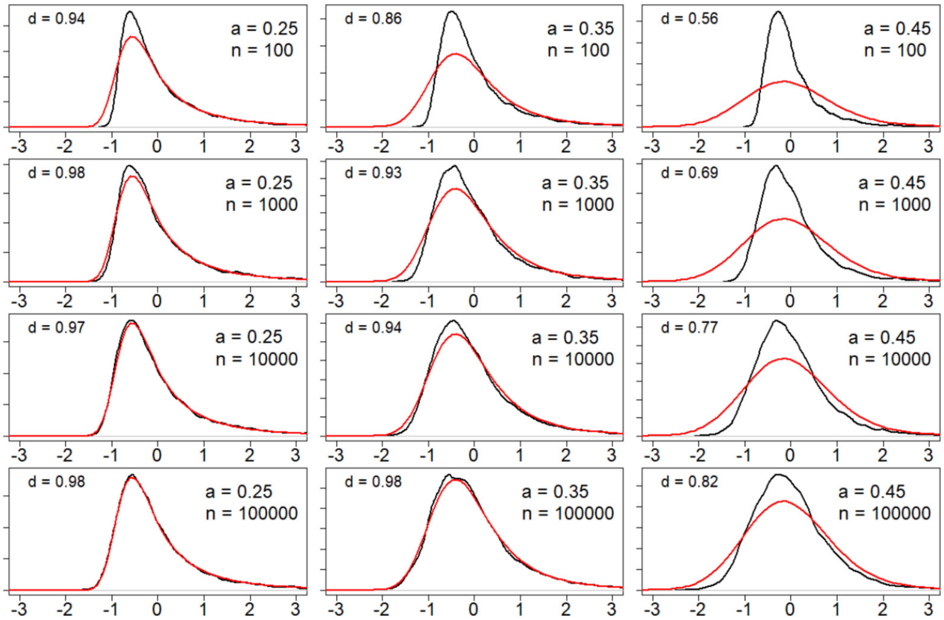

FIGURE 7 | The empirical density of Z_n (black) and the density of the Rosenblatt distribution (red) for various n and $a = 0.25$, 0.35 and 0.45; d is the empirical standard deviation of Z_n.

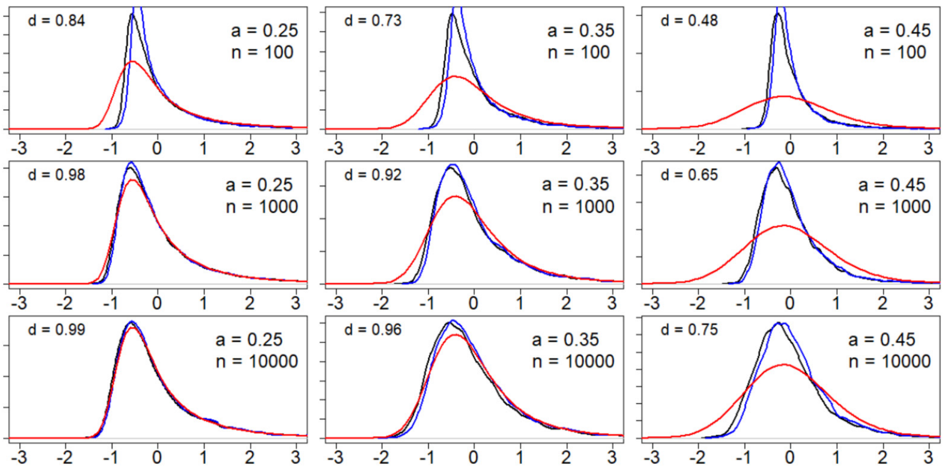

FIGURE 8 | The empirical density of $R_{10,n}$ (black) and $R_{20,n}$ (blue) and the density of the Rosenblatt distribution (red) for various n and $a = 0.25$, 0.35 and 0.45; d is the empirical standard deviation of $R_{10,n}$.

$r(k) = \mathrm{Cov}\left(E_s, E_{s+k}\right) = \mathbb{E}\left(E_s E_{s+k}\right)$. Suppose that the correlation function $r(k)$ has the shape

$$r(k) = \frac{l(k)}{k^a}.$$

As well known, the classical estimator of $r(k)$ is given by

$$\hat{r}(k) = \frac{1}{n}\sum_{j=1}^{n-k} E_j E_{j+k},$$

which has asymptotically the scaled Rosenblatt distribution if $a \in (0{,}0.5)$ and the normal distribution otherwise, see (Rosenblatt 1961, 1979). The fact that the asymptotic distribution of $\hat{r}(k)$ does not depend on k for $a \in (0{,}0.5)$ is very remarkable. Specifically, the limiting distribution of

$$R_{k,n} = \frac{\sigma_a}{n^{1-a}}\sum_{j=1}^{n-k}\left(E_j E_{j+k} - r(k)\right), a \in (0{,}0.5), k = 0, 1, \ \ldots \ .$$

is given by the Rosenblatt distribution.

Let us make a numerical study to obtain the distribution of $R_{k,n}$ for small sample sizes. We note that $R_{0,n} = Z_n$ and, therefore, the empirical distribution of $R_{0,n}$ is shown in Figure 7.

In Figure 8 we can see that the empirical distribution of $R_{k,n}$ for small n almost does not depends on k, that confirms the theoretical statement. In particular, the empirical distribution of $R_{k,n}$ is similar to the empirical distribution of Z_n for various a and n.

5.5 | Estimation of Sojourn Functionals

Let $E(t)$ be a stationary Gaussian process with zero mean, unit variance and correlation function $r(t) = \frac{l(|t|)}{|t|^a}$. Consider the sojourn functional

$$M_t(u) = \int_0^t \mathbf{1}_{\{|E(s)|>u\}}(s)ds,$$

where $u > 0$ and

$$\mathbf{1}_{\{|E(s)|>u\}}(s) = \begin{cases} 1, & |E(s)| > u, \\ 0, & |E(s)| \le u, \end{cases}$$

is the indicator function. We interpret $M_t(u)$ as the time spent by the process $|E(s)|$ above the level u for $s \in [0, t]$. Following (Berman 1979), the expansion of $M_t(u)$ is the form

$$M_t(u) = 2(1-\Phi(u)) + u\phi(u)\int_0^t \left(E^2(s)-1\right)ds + 2\phi(u)\sum_{j=2}^{\infty}\frac{H_{2j-1}}{2j!}\int_0^t H_{2j}(E(s))ds,$$

where $H_j(\cdot)$ is the j-th Hermit polynomial, and

$$\mathrm{Var}\left(M_t(u)\right) = 4\int_0^t (t-s)\int_0^{r(s)} (\phi(u,u;q) - \phi(u,u;-q))dqds,$$

where

$$\phi(u,v;q) = \frac{1}{2\pi\sqrt{1-q^2}}\exp\left(-\frac{x^2-2quv+v^2}{2\left(1-q^2\right)}\right)$$

is the standard bivariate normal density with correlation q. From (Berman 1979) we also have that the functional

$$\frac{M_t(u) - 2t(1-\Phi(u))}{2u\phi(u)\sqrt{\int_0^t (t-s)r^2(s)ds}}$$

has asymptotically the Rosenblatt distribution.

Let us make a numerical study to obtain the distribution of

$$S_{u,n} = \frac{\sigma_a}{n^{1-a}} \cdot \frac{1}{u\phi(u)}\left(\sum_{j=1}^{n}\mathbf{1}_{\{|E_j|>u\}}(j) - 2n(1-\Phi(u))\right)$$

for small sample sizes, where $E_1, \ldots, E_n$ is a stationary Gaussian sequence with zero mean and correlation function $r(t) = 1/(1+|t|)^a$.

In Figure 9 we can see that the empirical distribution of $S_{u,n}$ for small n depends slightly on u. In particular, the empirical distribution of $S_{u,n}$ tends to the Rosenblatt distribution as n increases.

5.6 | Roughness of the fBm Path

Let $X(t)$ be a stochastic process, $t \in [0,1]$. Define the quadratic variation of $X(t)$ by

$$V_n = \sum_{j=1}^{n}\left(X\left(\frac{j}{n}\right) - X\left(\frac{j-1}{n}\right)\right)^2,$$

which can be interpreted as the path roughness and often used in finance and geophysics.

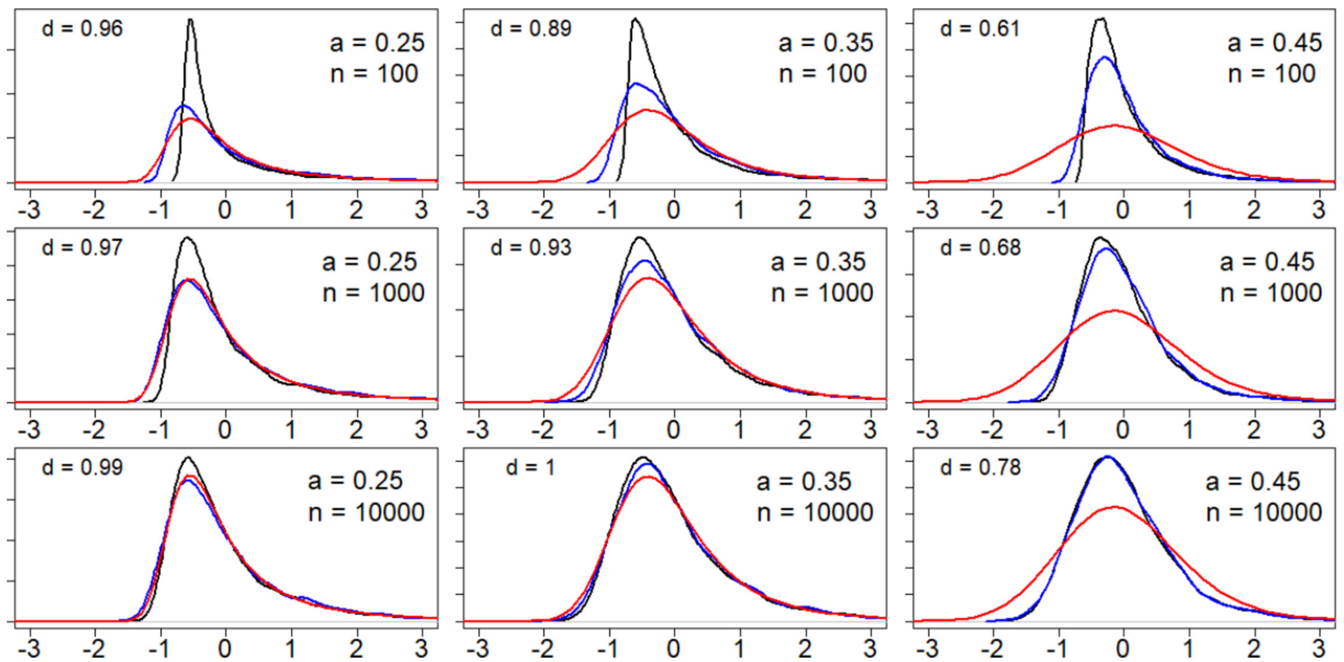

FIGURE 9 | The empirical density of $S_{2,n}$ (black) and $S_{1.5,n}$ (blue) and the density of the Rosenblatt distribution (red) for various n and $a = 0.25, 0.35$ and 0.45; d is the empirical standard deviation of $S_{2,n}$.

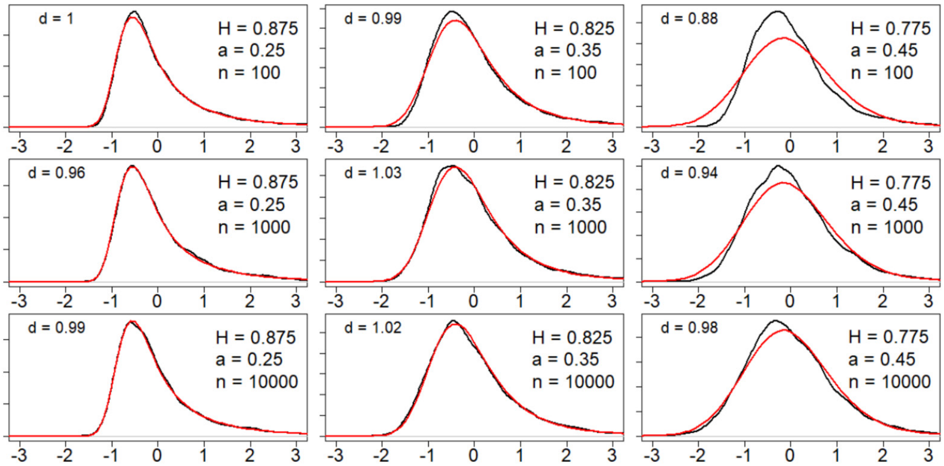

FIGURE 10 | The empirical density of G_n (black) and the density of the Rosenblatt distribution (red) for various n and $a = 0.25, 0.35$ and 0.45; d is the empirical standard deviation of G_n.

Suppose that $X(t)$ is the fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ and covariance function $r(t,s) = \left(s^{2H} + t^{2H} - |t-s|^{2H}\right)/2$. Then the distribution of

$$F_n = \frac{V_n - \mathbb{E}(V_n)}{\sqrt{\mathrm{Var}(V_n)}}.$$

is asymptotically normal when $H \in (0, 3/4]$ and converges to the Rosenblatt distribution with parameter $a = 2 - 2H$ when $H \in (3/4, 1)$ as n increases, see (Dobrushin and Major 1979; Nourdin 2012; Nourdin and Poly 2012). The quadratic variation V_n is useful in practice because the classical estimator of H is based on the fact that $n^{2H-1}V_n \to 1$ in probability as $n \to \infty$.

Let us make a numerical study to obtain the distribution of

$$G_n = \frac{\sigma_a}{n^{1-a}} \cdot \frac{n^{2-a}}{1.04 - 1.5a} \sum_{j=1}^{n} \left[\left(X\left(\frac{j}{n}\right) - X\left(\frac{j-1}{n}\right) \right)^2 - n^{a-2} \right]$$

for small sample sizes, where $X(t)$ is the fractional Brownian motion with $H = 1 - a/2$, which can be simulated using the FFT-based algorithm from (Wood and Chan 1994) implemented in the R package *SuperGauss*.

In Figure 10 we can see that the empirical distribution of G_n for small n is rather close to the Rosenblatt distribution for $a \le 0.35$. We also see that the empirical distribution of G_n tends to the Rosenblatt distribution as n increases for $a = 0.45$.

6 | Conclusion

We studied the Rosenblatt distribution which appears as a limiting distribution of several popular functionals of stationary Gaussian sequences with LRD and, therefore, it is required to construct the confidence intervals. The Rosenblatt distribution is difficult because it depends on eigenvalues of the Riesz integral operator. We derived new expressions for the characteristic function of the Rosenblatt distribution, which can be evaluated for any argument. We obtained the accurate approximation of all eigenvalues that enables easy evaluation of the Rosenblatt distribution. Also, we proposed an efficient algorithm for the simulation of stationary Gaussian sequences with the power correlation function $r(t) = 1/(1+|t|)^a$ and the Mittag-Leffler correlation function $r(t) = E_a\left(-|t|^a\right)$. Finally, we presented Monte-Carlo simulation on how the Rosenblatt distribution appears as a limiting distribution in estimation of several statistics: the mean, the correlation function, sojourn functionals and path roughness. We note that the Rosenblatt distribution also appears as a limiting distribution of some statistics for oscillating Gaussian processes and in ordinal pattern analysis of stationary Gaussian sequences with LRD. In addition, the Rosenblatt distribution is the distribution of the Rosenblatt process at unit time, see (Pipiras and Taqqu 2017; Tudor 2023; Ayache et al. 2025).

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Online supplement contains the proof of Theorem 1 and the R code with the computation of the density of the Rosenblatt distribution and the simulation of stationary Gaussian sequences with correlation functions $r(t) = (1+|t|)^{-a}$ and $r(t) = E_a\left(-|t|^a\right)$.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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Supporting Information

Additional supporting information can be found online in the Supporting Information section. **Data S1:** Supporting Information