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ORIGINAL PAPER

Probability of ruin within finite time and Cramér–Lundberg inequality for fractional risk processes

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Abstract

While the interarrival times of the classical Poisson process are exponentially distributed, complex systems often exhibit non-exponential patterns, motivating the use of the fractional Poisson process, in which interarrival times follow a Mittag–Leffler distribution. This paper investigates the associated risk process, describes its Cramér–Lundberg formula and establishes a relationship between the continuous premium rate and the fractional claim frequency. For a compound fractional risk process with exponential claims, we derive closed-form expressions for the finite-time ruin probability. Furthermore, for a general claim distribution, we provide ruin probability estimates that can serve as a basis for developing reinsurance strategies.

Keywords Fractional Poisson process · Finite-time ruin probability · Cramér–Lundberg model · Value-at-risk

Mathematics Subject Classification 91B30 · 91G70 · 62P05 · 62P20 · 60K05 · 60G22

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1 Introduction

Insurance, particularly non-life insurance companies and reinsurers rely on proper assumptions for claim sizes and claim frequencies. The classical assumption for claim frequencies is the Poisson process with exponentially distributed waiting times. Unlike the memoryless Poisson process, the fractional Poisson process is non-Markovian, see Mainardi et al. (2004), Laskin (2003), Beghin and Orsingher (2009), Meerschaert et al. (2011). This is achieved through interarrival times that follow a Mittag–Leffler distribution, which, unlike the exponential distribution, has a heavier tail. This allows the model to account for periods of high and low claim activity, reflecting a more realistic clustering of events. Such periods are indeed observed empirically in complex systems (Laskin 2003; Kumar et al. 2020), where non-exponential interarrival times have been confirmed in insurance and internet traffic data.

A key feature of the fractional Poisson process is its ability to model long-range dependence. This means that the number of claims in distant time intervals can be correlated. This is particularly relevant in lines of business like catastrophic insurance, where the effects of a major event can linger and influence claim patterns for an extended period. In many insurance contexts, claims do not arrive independently. External factors like weather patterns (hurricane seasons, freezing winters), economic cycles, or pandemic waves can cause claims to cluster in a way that influences the process for a long time. This is crucial for catastrophe insurance (hurricanes, earthquakes), crop insurance (droughts), and health insurance (epidemics), where events are clearly not memoryless.

Let us now formally introduce fractional generalizations of the classical risk model. Biard and Saussereau (2016) and Beghin and Macci (2013) studied the renewal risk process in which the surplus of an insurance company is modelled by a *fractional risk process*

$$R_\alpha(t) = u + c\,t - \sum_{i=1}^{N_\alpha(t)} Z_i, \quad t \geq 0, \tag{1.1}$$

where $u > 0$ is the initial capital, $c > 0$ is the constant premium rate, the sequence $Z_1, Z_2, \ldots$ is the successive claims modelled by independent nonnegative random variables with mean μ. The renewal process $N_\alpha(t)$ is independent of the sequence $Z_1, Z_2, \ldots$ and has the form

$$N_\alpha(t) = \max\{n\colon\ V_1 + \ldots + V_n \leq t\}, \quad t \geq 0,\ 0 < \alpha \leq 1, \tag{1.2}$$

where the interarrival times $V_1, \ldots, V_n$ between claims are independent identically distributed random variables with the Mittag–Leffler distribution, that is,

$$\mathbb{P}[V_i \leq t] = 1 - E_\alpha\left(-\lambda t^\alpha\right), \quad t \geq 0,\ 0 < \alpha \leq 1, \tag{1.3}$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C},\ \Re(\alpha) > 0, \tag{1.4}$$

is the Mittag–Leffler function and $\Gamma(\cdot)$ is the gamma function. Note that

$$\mathbb{E}V_i = \infty \tag{1.5}$$

for $0 < \alpha < 1$. As known, $N_\alpha(t)$ is also called the fractional Poisson process (FPP), see Beghin and Orsingher (2009), Meerschaert et al. (2011) for other approaches to the definition of the FPP. Biard and Saussereau (2014), Biard and Saussereau (2016) derived some expressions for the ruin probabilities of the fractional risk process (1.1) in the case of light-tailed and heavy-tailed claim sizes, see "Appendix" for details.

For $\alpha = 1$, the model (1.1) becomes the classical collective risk Cramér–Lundberg model denoted by $R(t)$, where $\mathbb{P}(V_i \le t) = 1 - e^{-\lambda t}$, $t \ge 0$, see, for example, Malinovskii (2021). To avoid ruin with certainty under the net profit condition, we assume that Lundberg's constant

$$\beta = \frac{\lambda\mu}{c} = \frac{\mathbb{E}Z_i}{c \cdot \mathbb{E}V_i} \tag{1.6}$$

satisfies the constraint $\beta < 1$.

Recently, Kumar et al. (2020) introduced and studied the *compound fractional risk process*

$$R_\alpha^\bullet(t) = u + c\,Y_\alpha(t) - \sum_{i=1}^{N_\alpha(t)} Z_i, \quad t \ge 0,\ 0 < \alpha < 1, \tag{1.7}$$

where $Y_\alpha(t)$, $t \ge 0$, $0 < \alpha < 1$, is the inverse stable subordinator defined below by (2.1). They investigate the infinite horizon probability of ruin of the risk process (1.7) and show that it is the same as for the classical risk process for exponentially distributed claims.

Furthermore, Kataria and Khandakar (2021) introduced and investigated a mixed fractional risk process, while Pirozzi (2022) studied some other fractional insurance models.

The present paper investigates both fractional risk processes (1.1) and (1.7). We study the probability of ruin within finite time for the compound fractional risk process (1.7) and present closed forms, which are only available for specific claim distributions. For the general situation, we discuss the Cramér–Lundberg inequality in the framework of fractional risk processes (1.1). For a general discussion on the probability of ruin within finite time, we refer to the papers and books of Asmussen and Albrecher (2010) and Malinovskii (2021), see also the references therein.

Outline of the paper. The paper presents preliminaries on the α-stable subordinator, its inverse and the fractional Poisson process in Section 2. Section 3 introduces the probability of ruin within finite time for the compound fractional risk process $R^{\bullet}_{\alpha}(t)$, while Section 4 discusses the time of ruin itself. As closed-form formulae are not always available, Section 5 highlights the Cramér–Lundberg formula for the fractional risk process $R_\alpha(t)$. Its perspective for insurance is considered in Section 6. The simulation of the compound fractional risk process $R^{\bullet}_{\alpha}(t)$ is discussed in Section 7. Proofs of our results are deferred to "Appendix", and the R code with demonstration of our findings is placed in online supplement.

2 Preliminaries

This section presents some known results, which will be required later.

An α-stable subordinator $\{D_\alpha(t)\}_{t\geq 0}$ is a one-dimensional Lévy process whose sample paths are non-decreasing and continuous from the right with left limits (càdlàg). It is characterized by the Laplace transform $\mathbb{E}\left[e^{-sD_\alpha(t)}\right] = e^{-t\,\phi(s)}, \quad s \geq 0$, where $\phi(s) = s^\alpha$, which can also be written in the integral form $\phi(s) = \int_0^\infty \left(1 - e^{-sx}\right)\nu(dx), \quad 0 < \alpha < 1$, with the Lévy measure $\nu(dx) = \frac{\alpha}{\Gamma(1-\alpha)\,x^{1+\alpha}}dx$, $x > 0$. The first-passage time of $\{D_\alpha(t)\}_{t\geq 0}$ is a non-decreasing process $Y_\alpha(t)$ known as the inverse subordinator. It is defined as

$$Y_\alpha(t) = \inf\{s > 0\colon D_\alpha(s) > t\}, \quad t \geq 0. \tag{2.1}$$

For further details on the subordinator and its inverse, we refer to Meerschaert and Sikorskii (2019). We only note that the density of $Y_\alpha(t)$ is given by

$$f_\alpha(x,t) = \frac{t}{\alpha}\frac{1}{x^{1+\frac{1}{\alpha}}}g_\alpha\left(\frac{t}{x^{1/\alpha}}\right), \quad x > 0,\ t > 0, \tag{2.2}$$

where $g_\alpha(x)$ is the density of $D_\alpha(1)$, see Meerschaert and Sikorskii (2019, Eq. 4.47, p. 111), and can be written explicitly as

$$g_\alpha(x) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\Gamma(\alpha k+1)}{k!}\frac{\sin(\pi\alpha k)}{x^{\alpha k+1}}, \quad x \geq 0, \tag{2.3}$$

see Feller (1991, p. 583). Figure 1 displays the density of $Y_\alpha(1)$. We note that the density $f_\alpha(x,1)$ cannot be well numerically computed at the right tail due slow convergence of the series in (2.3). Other representations of the density $f_\alpha(x,t)$ are given in "Appendix", while the representation (8.1) using the M-Wright function is more computationally suitable.

We note that both processes $\{D_\alpha(t)\}_{t\geq 0}$ and $\{Y_\alpha(t)\}_{t\geq 0}$ are self-similar, that is, there are equalities

$$D_\alpha(at) \stackrel{d}{=} a^{1/\alpha}D_\alpha(t), \quad Y_\alpha(at) \stackrel{d}{=} a^\alpha Y_\alpha(t), \quad t \geq 0,\ a \geq 0,$$

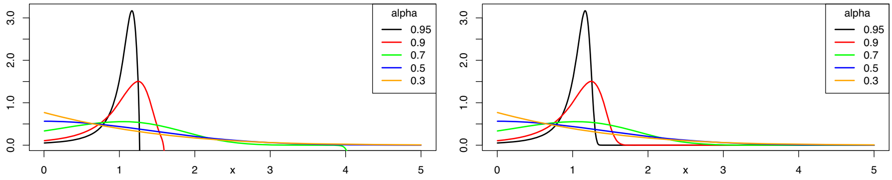

Fig. 1 The density of $Y_\alpha(t)$ given in (2.2) for $t = 1$ and varying α using the truncated series with 120 terms of (2.3) (left) and the M-Wright function (right)

where " $\stackrel{d}{=}$ " means equality in the sense of finite-dimensional distributions. We also note that $t \Big/ [Y_\alpha(t)]^{1/\alpha} \stackrel{d}{=} D_\alpha(1)$, the subordinator $\{D_\alpha(t)\}_{t\geq 0}$ has independent increments but the inverse subordinator $\{Y_\alpha(t)\}_{t\geq 0}$ does not have independent increments. The Laplace transform of $Y_\alpha(t)$ is given by

$$\mathbb{E}\left[e^{-s\,Y_\alpha(t)}\right] = \int_0^\infty e^{-sh} f_\alpha(h,t)dh = E_\alpha\left(-s\,t^\alpha\right), \quad s > 0, \tag{2.4}$$

where $E_\alpha(\cdot)$ is the Mittag–Leffler function defined by (1.4). Note that

$$\mathbb{E}\left[Y_\alpha(t)^\nu\right] = \frac{\Gamma(\nu+1)}{\Gamma(\alpha\nu+1)} t^{\alpha\nu}, \quad \nu > 0,$$

and

$$\operatorname{Cov}\left(Y_\alpha(s), Y_\alpha(t)\right) = \int_0^{\min\{s,t\}} \frac{\left((t-\tau)^\alpha + (s-\tau)^\alpha\right)\tau^{\alpha-1}}{\Gamma(\alpha)\Gamma(1+\alpha)} d\tau - \frac{(s\,t)^\alpha}{\Gamma^2(1+\alpha)} \tag{2.5}$$

see Leonenko et al. (2014). For $0 < s < t$, we have

$$\operatorname{Cov}\left(Y_\alpha(s), Y_\alpha(t)\right) = \frac{1}{\Gamma^2(\alpha+1)}\left(\alpha s^{2\alpha} B(\alpha,\alpha+1) + \alpha t^{2\alpha} B(\alpha,\alpha+1,s/t) - (st)^\alpha\right),$$

where $B(\alpha, \alpha+1)$ is the beta function and $B(\alpha, \alpha+1, s/t)$ is the incomplete beta function. For fixed s and large t, we obtain

$$\operatorname{Cov}\left(Y_\alpha(s), Y_\alpha(t)\right) \cong \frac{1}{\Gamma^2(\alpha+1)}\left(\alpha s^{2\alpha} B(\alpha,\alpha+1) - \frac{\alpha^2 s^{\alpha+1}}{(\alpha+1)t^{1-\alpha}}\right), \; 0 < s < t. \tag{2.6}$$

The stochastic process $\{Y_\alpha(t)\}_{t\geq 0}$ is not stationary, but we will refer to the property (2.6) as long-range dependence, see Maheshwari and Vellaisamy (2016) for details.

Finally, the FPP $N_\alpha(t)$ which is introduced in (1.2) can be defined as

$$N_\alpha(t) = N(Y_\alpha(t)), \quad t \geq 0, \tag{2.7}$$

as shown in Meerschaert et al. (2011), where $\{N(t)\}_{t>0}$ is the homogeneous Poisson process with intensity $\lambda > 0$, and $\{Y_\alpha(t)\}_{t\geq 0}$ is the inverse stable subordinator defined by (2.1), independent of $\{N(t)\}_{t\geq 0}$. We have

$$\begin{aligned}
\mathbb{E}N_\alpha(t) &= \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \\
\operatorname{Var} N_\alpha(t) &= \frac{\lambda^2 t^{2\alpha}}{\Gamma(1+\alpha)}\left(\frac{\alpha\Gamma(\alpha)}{\Gamma(2\alpha)} - 1\right) + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \quad t \geq 0,
\end{aligned}$$

and

$$\operatorname{Cov}\left(N_\alpha(t), N_\alpha(s)\right) = \frac{\lambda(\min\{t,s\})^\alpha}{\Gamma(1+\alpha)} + \lambda^2 \operatorname{Cov}\left(Y_\alpha(t), Y_\alpha(s)\right),$$

where $\operatorname{Cov}(Y_\alpha(t), Y_\alpha(s))$ is given by (2.5).

The state probabilities of the FPP are of the form:

$$\begin{aligned}
P\{N_\alpha(t) = k\} &= \int_0^\infty \frac{e^{-\lambda x}(\lambda x)^k}{k!} f_\alpha(t,x)dx \\
&= \frac{(\lambda t^\alpha)^k}{k!} \sum_{j=0}^\infty \frac{(k+j)!}{j!} \frac{(-\lambda t^\alpha)^j}{\Gamma(\alpha(j+k)+1)} \\
&= \frac{(\lambda t^\alpha)^k}{k!} E_\alpha^{(k)}(-\lambda t^\alpha),
\end{aligned} \tag{2.8}$$

where

$$E_\alpha^{(k)}(z) = \frac{d^k}{dz^k} E_\alpha(z), \quad z \in \mathbb{C}, \tag{2.9}$$

is the k-th derivative of $E_\alpha(z)$, which is given in (1.4). From Podlubny (1998), we have $E_{\alpha,\rho}^{(k)}(z) = k!\, E_{\alpha,k\alpha+\rho}^{k+1}(z),\ k \geq 0$, where

$$E_{\alpha,\rho}^\gamma(z) = \sum_{k=0}^\infty \frac{(\gamma)_k\, z^k}{\Gamma(\alpha k + \rho) k!}, \quad z \in \mathbb{C},\ \alpha, \rho, \gamma \in \mathbb{C},\ \Re(\alpha) > 0, \tag{2.10}$$

is the 3-parameter Mittag–Leffler function and $(\gamma)_k$ is the Pochhammer symbol defined by $(\gamma)_0 = 1$, $(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)} = \gamma(\gamma+1)\cdots(\gamma+k-1)$. Note that $E_\alpha(z) = E_{\alpha,1}^1(z)$.

3 The probability of ruin within finite time

By involving the fractional Poisson process, we address the compound fractional risk process $R_\alpha^\bullet(t)$ defined in (1.7),

$$R_\alpha^\bullet(t) := R(Y_\alpha(t)) = u + c\,Y_\alpha(t) - \sum_{i=1}^{N(Y_\alpha(t))} Z_i, \quad t \geq 0,\ 0 < \alpha < 1, \tag{3.1}$$

where the classical risk process $R(t)$ and the inverse stable subordinator $Y_\alpha(t)$, defined in (2.1), are independent. For $\alpha = 1$, we define $R_1(t) := R(t)$.

From Wald's formula, we have

$$\mathbb{E}R_\alpha^\bullet(t) = u + c\frac{t^\alpha}{\Gamma(1+\alpha)} - \mu\lambda\frac{t^\alpha}{\Gamma(1+\alpha)}$$

and

$$\operatorname{Cov}\left(R_\alpha^\bullet(t), R_\alpha^\bullet(s)\right) = \frac{\lambda\cdot\min(s,t)^\alpha}{\Gamma(1+\alpha)}\left[\mu^2+\sigma^2\right] + (\lambda\mu)^2\operatorname{Cov}\left(Y_\alpha(t), Y_\alpha(s)\right),$$

where $\sigma^2 = \operatorname{Var} Z_i$, and $\operatorname{Cov}(Y_\alpha(t), Y_\alpha(s))$ is given by (2.5), where $\mathbb{P}[Z_i \leq x] = 1 - e^{-x/\mu}$, $x > 0$ and $\mathbb{E}Z_i = \mu$.

The probability of ruin associated with the process $R_\alpha^\bullet(t)$ within finite time is denoted by

$$\psi^\alpha(t) = \mathbb{P}\left[\inf_{0\leq s\leq t} R_\alpha^\bullet(s) < 0\right], \tag{3.2}$$

while the probability of ultimate ruin is

$$\psi_\infty^\alpha = \lim_{t\to\infty}\psi^\alpha(t). \tag{3.3}$$

It follows from Kumar et al. (2020) that for any $0 < \alpha < 1$ and exponentially distributed claims Z_k with mean μ, we have

$$\psi_\infty^\alpha = \psi_\infty := \begin{cases} \beta\exp\left(-\frac{u}{\mu}(1-\beta)\right), & \text{if } \beta < 1, \\ 1, & \text{if } \beta \geq 1, \end{cases} \tag{3.4}$$

where $\beta = \frac{\lambda\mu}{c}$ is Lundberg's constant; note, however, that $\mathbb{E}\,V_i$ is infinite for the fractional models, while $\mathbb{E}\,V_i$ is finite for the classical risk process. It is also known [see, for example, Malinovskii (2021, Theorems C.14–16) and the references therein] that there are three expressions for the ruin probability $\psi(t)$ if the claims Z_k, $k \geq 1$, are exponentially distributed with mean $\mu > 0$.

One expression is given by

$$\psi(t) = \psi_\infty - \frac{1}{\pi}\int_0^\pi F_t(x)dx,$$

where ψ_∞ is given by (3.4), and $F_t(x) = U_t(x)V(x)$, where

$$U_t(x) = \exp\left(-t \cdot \frac{c}{\mu}\left(1+\beta-2\sqrt{\beta}\cos(x)\right)\right) \tag{3.5}$$

and

$$V(x) = \frac{\beta}{1+\beta-2\sqrt{\beta}\cos x}\exp\left(\frac{u}{\mu}\left(\sqrt{\beta}\cos x - 1\right)\right) \tag{3.6}$$

$$\times\left(\cos\left(\frac{u}{\mu}\sqrt{\beta}\sin x\right) - \cos\left(\frac{u}{\mu}\sqrt{\beta}\sin x + 2x\right)\right). \tag{3.7}$$

Other two expressions for $\psi(t)$ can be written as

$$\psi(t) = \beta e^{-\frac{u}{\mu}}\int_0^{tc/\mu} e^{-(1+\beta)x}G(x)dx,$$

where

$$G(x) = I_0\left(2\sqrt{\beta x(x+u/\mu)}\right) - \frac{x}{x+u/\mu}I_2\left(2\sqrt{\beta x(x+u/\mu)}\right),$$

which follows from Malinovskii (2021, Th. C.14), or alternately

$$G(x) = \sum_{n=0}^{\infty}\left(\sqrt{\beta}\frac{u}{\mu}\right)^n \frac{n+1}{x\sqrt{\beta}\,n!}I_{n+1}\left(2x\sqrt{\beta}\right), \tag{3.8}$$

which follows from Malinovskii (2021, Th. C.15), where

$$I_n(z) = \sum_{k=0}^{\infty}\frac{1}{k!(n+k)!}\left(\frac{z}{2}\right)^{n+2k}, \qquad z \in \mathbb{C},$$

$k = 0, 1, 2, \ldots$, is the modified Bessel function of the first kind of order n.

3.1 Explicit expression of the probability of ruin

This section presents the density of ruin probabilities of the compound fractional risk model $R_\alpha^\bullet(t)$ and provides explicit expressions for exponentially distributed claims.

Theorem 1 *Let $\psi(t)$ be the ruin probability for the Cramèr–Lundberg model. Then, the ruin probability for the compound fractional risk model* (3.1) *is given by*

$$\psi^{\alpha}(t) = \int_0^{\infty} \psi(h) f_{\alpha}(h, t) dh, \tag{3.9}$$

where $f_{\alpha}(\cdot, t)$ is the density of $Y_{\alpha}(t)$ provided by (2.2).

According to Theorem 1, the numerical computation of the ruin probability $\psi^{\alpha}(t)$ requires double integration and all three expressions of the ruin probability $\psi(t)$ given above can be used. Fortunately, for one expression of $\psi(t)$, the computation of the ruin probability $\psi^{\alpha}(t)$ can be simplified as follows.

Theorem 2 *In the compound fractional risk model* (3.1) *with exponentially distributed claims Z_i with mean μ, the probability of ruin within finite time is given by*

$$\psi^{\alpha}(t) = \psi_{\infty} - \frac{1}{\pi} \int_0^{\pi} F_t^{\alpha}(x) dx, \quad 0 < \alpha < 1,$$

where $F_t^{\alpha}(x) = V(x) \cdot U_t^{\alpha}(x)$ and ψ_{∞} is defined by (3.4)*, while $V(x)$ is defined by* (3.7)*, and $U_t^{\alpha}(x) = E_{\alpha}\big(-t^{\alpha} A(x)\big)$, where $E_{\alpha}(\cdot)$ is the Mittag–Leffler function given by* (1.4) *and*

$$A(x) = \frac{c}{\mu}\left(1 + \beta - 2\sqrt{\beta}\cos(x)\right). \tag{3.10}$$

Remark 3 For $\alpha = 1$, we have $E_1\big(-tA(x)\big) = e^{-tA(x)}$, and the formula from Theorem 2 is consistent with (3.5). Using Simon (2014, Th. 4), we obtain the two-sided inequality

$$\frac{1}{1 + \Gamma(1-\alpha)t^{\alpha} A(x)} \le E_{\alpha}\big(-t^{\alpha} A(x)\big) \le \frac{1}{1 + \frac{1}{\Gamma(1+\alpha)} t^{\alpha} A(x)}$$

for $\alpha \in (0, 1)$ uniformly in $t \ge 0$.

Figure 2 depicts the cumulative distribution function (cdf) of the Mittag–Leffler distribution and the ruin probability $\psi^{\alpha}(t)$ for various values of parameters, and the R code is deferred to online supplement.

4 Ruin time for the compound fractional risk process

Another characteristic associated with the probability of ruin in (3.2) and (3.3) is the *time* of ruin

$$\tau_{\alpha} = \inf\left\{t > 0\colon R_{\alpha}^{\bullet}(t) < 0\right\}, \quad 0 < \alpha < 1, \tag{4.1}$$

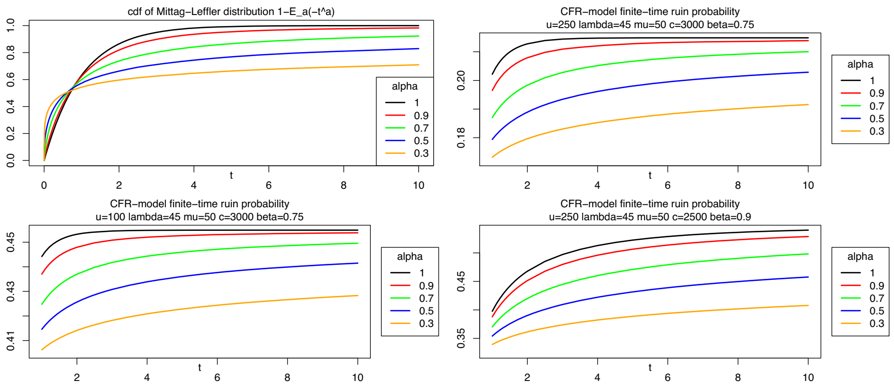

Fig. 2 The cdf of the Mittag–Leffler distribution and the ruin probability $\psi^\alpha(t)$ for various values of parameters

which reduces to

$$\tau = \inf\{t > 0\colon\ R(t) < 0\} \tag{4.2}$$

for $\alpha = 1$, where $R(t)$ is the classical risk model and $R^\bullet_\alpha(t)$ is defined by (1.7). We define $\tau_\alpha = \infty$ and $\tau = \infty$ if $R(t) > 0$ and $R^\bullet_\alpha(t) > 0$ for all $t > 0$. Next proposition establishes a relation between the ruin times τ_α and τ.

Proposition 4 *For the given random paths of $Y_\alpha(t)$ and the classical risk model $R(t)$, we have*

$$Y_\alpha(\tau_\alpha) \stackrel{d}{=} \tau, \tag{4.3}$$

and $\tau_\alpha \le D_\alpha(\tau)$, where $Y_\alpha(t)$ is the inverse subordinator of $D_\alpha(t)$, cf. (2.1).

Let $p_\tau(t)$ be the density of τ and $p_{\tau_\alpha}(t)$ be the density of τ_α. As stated in Dickson (2016), we have $p_\tau(t) = \frac{d}{dt}\psi(t)$ which implies that the density of the time of ruin can be computed as the derivative of the probability of ruin within finite time. Similarly, for $p_{\tau_\alpha}(t) = \frac{d}{dt}\psi^\alpha(t)$ we have

$$p_{\tau_\alpha}(t) = -\frac{1}{\pi}\int_0^\pi V(x)\frac{\partial}{\partial t}U_t^\alpha(x)dx = \frac{\alpha t^{\alpha-1}}{\pi}\int_0^\pi V(x)A(x)E^2_{\alpha,\alpha+1}(-t^\alpha A(x))dx,$$

where $\psi^\alpha(t)$ is given in Theorem 2, $V(x)$ is defined in (3.7), $A(x)$ is defined in (3.10), and $E^2_{\alpha,\alpha+1}(\cdot)$ is defined in (2.10).

Alternatively, using Borovkov and Dickson (2008, Th. 1), we have

$$
\begin{aligned}
p_\tau(t) = & \frac{u\lambda e^{-\frac{u+ct}{\mu}-\lambda t}}{u+ct}\sum_{m=0}^{\infty}\frac{(u+ct)^m}{m!\mu^m}\frac{(\lambda t)^m}{\Gamma(m+1)} \\
& + \lambda t\frac{ce^{-\frac{u+ct}{\mu}-\lambda t}}{u+ct}\sum_{m=0}^{\infty}\frac{(u+ct)^m(\lambda t)^m}{m!\mu^m\Gamma(m+2)}
\end{aligned}
$$

and $p_{\tau_\alpha}(t) = \frac{d}{dt}\mathbb{P}[\tau_\alpha \le t]$, where

$$
\begin{aligned}
\mathbb{P}[\tau_\alpha \le t] &= \mathbb{P}\left[\inf\left\{v \ge 0\colon R_\alpha^\bullet(v) \le 0\right\}\right] \\
&= \int_0^\infty \mathbb{P}\left[\inf_{0\le s\le h} R(s) \le 0\right] f_\alpha(h,t)dh \\
&= \int_0^\infty \left[\int_0^h p_\tau(z)dz\right] f_\alpha(h,t)dh \\
&= \int_0^\infty \int_0^h \left[\frac{u\lambda e^{-\frac{u+cz}{\mu}-\lambda z}}{u+cz}\cdot\sum_{m=0}^{\infty}\frac{(u+cz)^m}{m!\mu^m}\cdot\frac{(\lambda z)^m}{\Gamma(m+1)}\right. \\
&\quad \left. + (\lambda z)\frac{ce^{-\frac{u+cz}{\mu}-\lambda z}}{u+cz}\sum_{m=0}^{\infty}\frac{(u+cz)^m}{m!\mu^m}\frac{(\lambda z)^m}{\Gamma(m+z)}\right] dz f_\alpha(h,t)dh\Bigg].
\end{aligned}
$$

5 The Cramér–Lundberg inequality for the fractional risk process

In this section, we discuss the general situation where explicit expressions for the probability of ruin are not available. In particular, we present bounds and premiums for the fractional risk process $R_\alpha(t)$.

5.1 Cramér–Lindberg coefficient of the fractional risk process

We will now consider the Cramér–Lundberg inequality for general claim distributions, as it provides an upper bound for the ruin probability in cases where explicit expressions are not available.

As in (1.2), consider the interarrival times $V_1, V_2, \ldots$ in the risk process with claim Z_i after waiting time V_i, $i = 1, 2, \ldots$. At the times of claim occurrences, the fractional risk process (1.1) is

$$R_\alpha(V_1 + \cdots + V_n) = u + c\,V_1 + \cdots + c\,V_n - \sum_{i=1}^{n} Z_i = u + \sum_{i=1}^{n}(c\,V_i - Z_i). \quad (5.1)$$

The classical *net profit condition* in non-life insurance mathematics [cf. Mikosch (2003, Definition 4.1.4)] reads $\mathbb{E}(c\,V_i - Z_i) \ge 0$, but this condition is void as $\mathbb{E}\,V_i$

is infinite by (1.5) (cf. also (1.6)). Alike, Lundberg's constant $\beta = \frac{\mathbb{E}\, Z_i}{c \cdot \mathbb{E}\, V_i}$ is not well defined, cf. (1.6). However, the Cramér–Lundberg inequality also holds true in the framework of the fractional risk process.

In what follows, we include the initial capital u of (5.1) in (3.2) in the notation, and we write

$$\psi_u^\alpha := \mathbb{P}\left[\inf_{s \geq 0} R_\alpha(s) \leq 0\right]. \tag{5.2}$$

Theorem 5 *(Cramér–Lundberg inequality for the fractional risk process $R_\alpha(t)$) For the probability of ruin (cf. (3.2) and (3.3)), it holds that*

$$\psi_u^\alpha \leq e^{-r_\alpha u}, \tag{5.3}$$

where $r_\alpha > 0$ is the adjustment or Cramér–Lundberg coefficient, which is defined to be the unique and positive solution of

$$\mathbb{E}\, e^{r_\alpha (Z_i - c\, V_i)} = 1. \tag{5.4}$$

We shall apply this bound to provide a basis for premiums for both, insurance and reinsurance.

5.2 Premium adjustment for higher claim frequencies

For independent random variables Z_i and V_i, the moment generating function in (5.4) decomposes to

$$\mathbb{E}\, e^{r_\alpha (Z - c\, V)} = \mathbb{E}\, e^{r_\alpha Z} \cdot \mathbb{E}\, e^{-r_\alpha c\, V},$$

where the first expression involves the amount of the claim only, and the second expression involves the interarrival times of the fractional risk process. We have the following explicit form for this expression.

Proposition 6 *Suppose the interarrival times V_i and the claims Z_i are independent. Then, the Cramér–Lundberg coefficient r_α of the fractional risk process $R_\alpha(t)$ with intensity λ satisfies the nonlinear equation*

$$\mathbb{E}\, e^{r_\alpha Z} = 1 + \frac{(c\, r_\alpha)^\alpha}{\lambda}. \tag{5.5}$$

Remark 7 (Premium adjustment for increased claim frequencies) Suppose the frequency of claims increases from λ to λ'. Then, the Cramér–Lundberg coefficient r_α does not change, provided that the premium increases from c to $c' := c \cdot \left(\lambda'/\lambda\right)^{1/\alpha}$. Indeed, from (5.5) we obtain that $(c'\, r_\alpha)^\alpha / \lambda' = (c\, r_\alpha)^\alpha / \lambda$ and thus the processes with parameters (c, λ) and (c', λ') are equivalent with respect to (5.5).

The following section combines the results to assess the loss in the case of ruin.

6 Quantification of the loss at ruin

The preceding sections and the Cramér–Lundberg formula (5.3) study the probability of ruin. This assessment does not give access to the expected loss in the case of ruin. Let us extend the discussion by describing the expected loss in the case of ruin at time t, as well as the expected maximum loss over time. Our results will constitute the basis to thoroughly assess the price for reinsurance for the fractional risk process.

6.1 Loss

To study the random losses, we associate the random variable

$$R_\alpha := \inf_{t>0} R_\alpha(t)$$

with the fractional risk process $R_\alpha(t)$. The worst loss constitutes the left tail of R_α, and the expectation of R_α can be written as

$$\mathbb{E}\, R_\alpha = u - \int_{-\infty}^{u} \mathbb{P}(R_\alpha \le r)\, dr,$$

which is obtained by integration by parts. Also, we note that $R_\alpha \le R_\alpha(0) = u$. Employing Markov's inequality, the expected loss is an upper bound for this expression, and the probability $\mathbb{P}(R_\alpha \le r)$ is bounded as

$$\begin{aligned}
\mathbb{P}(R_\alpha \le r) &\le \inf_{s>0} e^{sr} \cdot \mathbb{E}\, e^{-sR_\alpha} \\
&= \inf_{s>0} e^{sr+\log \mathbb{E}\, e^{-sR_\alpha}} \\
&= e^{\inf_{s>0}\{sr - K_{R_\alpha}(s)\}} \\
&= e^{-K^*_{R_\alpha}(r)},
\end{aligned} \tag{6.1}$$

where $K_{R_\alpha}(s) := -\log \mathbb{E}\, e^{-sR_\alpha}$ is the cumulant-generating function of the random variable R_α, and $K^*_{R_\alpha}(r) := \inf_{s>0}\{rs - K_{R_\alpha}(s)\}$ its convex conjugate function (Legendre–Fenchel transformation). Combining the components above and the expression (6.1)— well-known from large deviations theory —we obtain the lower bound

$$\mathbb{E}\, R_\alpha \ge u - \int_{-\infty}^{u} e^{-K^*_{R_\alpha}(r)} dr.$$

The following proposition addresses the Laplace transform $\mathbb{E}\, e^{-sR_\alpha}$ employed in (6.1) and the preceding upper bound.

Proposition 8 *The Laplace transform of the fractional risk process $R_\alpha(t)$ is*

$$\mathbb{E}\, e^{-sR_\alpha(t)} = e^{-su} \cdot E_\alpha\left(\lambda t^\alpha \left(\frac{\lambda\, \varphi_Z(s)}{\lambda + (sc)^\alpha} - 1\right)\right), \tag{6.2}$$

where $\varphi_Z(s) = \mathbb{E}\, e^{sZ_i}$ is the moment generating function of the claims Z_i and $s > 0$, and the interarrival times V_i are independent from the claims Z_i.

The Laplace transform in Proposition 8 is essential in elaborating the Legendre–Fenchel transformation, which is critical, by (6.1), in providing estimates for the random loss R_α. The quantity involves the individual claims Z_i via its moment generating function $\varphi_Z(\cdot)$.

6.2 Reinsurance against ruin

The fractional risk process $R_\alpha(t)$ is of fundamental importance in insurance, as it models the accumulated premiums $c \cdot t$ and losses $Z_1, \dots, Z_{N_\alpha(t)}$ up to time t. The right tail of the distribution of $R_\alpha(t)$ corresponds to the accumulated profit of the insurer at time t, whereas its left tail represents potential losses. The expectation $\mathbb{E}\big(-R_\alpha(t)\big)$ defines the *fair premium* for insuring against the loss $R_\alpha(t)$ [cf. Deprez and Gerber (1985) for an early discussion, or Young (2006)]. To incorporate a customary safety margin γ, the premium can be determined via

$$\mathsf{AV@R}_\gamma\big(-R_\alpha(t)\big), \tag{6.3}$$

where

$$\mathsf{AV@R}_\gamma(Y) = \inf_{t\in\mathbb{R}} \left(t + \frac{1}{1-\gamma}\,\mathbb{E}(Y-t)^+\right)$$

is called the *average value-at-risk* of the random variable Y at risk level $\gamma \in [0, 1)$.

The average value-at-risk is a risk functional, and it satisfies the axioms in the following definition.

Definition 9 (Cf. Artzner et al. 1997, 1999) Let L be a set of random variables. The mapping $\mathcal{R}\colon L \to \mathbb{R}$ is a *risk functional*, provided that the following axioms hold true:

- (i) For $X \le Y$ almost everywhere, the risk measure $\mathcal{R}$ satisfies $\mathcal{R}(X) \le \mathcal{R}(Y)$ (monotonicity);
- (ii) for $\lambda > 0$, it holds that $\mathcal{R}(\lambda\, Y) = \lambda\, \mathcal{R}(Y)$ (positive homogeneity);
- (iii) for random variables X and Y, it holds that $\mathcal{R}(X + Y) \le \mathcal{R}(X) + \mathcal{R}(Y)$ (convexity);
- (iv) for $c \in \mathbb{R}$, $\mathcal{R}(Y + c) = c + \mathcal{R}(Y)$ (translation equivariance).

The domain L of the risk function $\mathcal{R}$ in Definition 9 can be specified, in general, as a specific Banach space, which is associated with the risk function $\mathcal{R}$, cf. Pichler (2013). For our study, it is sufficient to consider the domain $L = L^\infty(P)$, the space of uniformly bounded random variables.

The average value-at-risk allows expressing the expected loss in the case of ruin. Indeed, from the identity

$$\mathbb{E}(Y |\, Y \ge \mathsf{V@R}_\gamma(Y)) = \mathsf{AV@R}_\gamma(Y) \tag{6.4}$$

where $\mathsf{V@R}_\gamma(Y) := F_Y^{-1}(\gamma) = \inf\{y\colon \mathbb{P}(Y \le y) \ge \gamma\}$ is the generalized inverse function, we obtain that

$$\mathsf{AV@R}_{1-\psi_u^\alpha}(-R_\alpha(t)) = \frac{\mathbb{E}\max(0, -R_\alpha(t))}{\psi_u^\alpha}, \tag{6.5}$$

where $\psi_u^\alpha = \mathbb{P}(R_\alpha(t) \le 0)$ is the probability of ruin introduced in (5.2), justifying the premium (6.3). We note that the average value-at-risk is strongly related to the probability of ruin of the fractional risk process.

Similarly, the loss up to fixed time $t > 0$ can be expressed via the average value-at-risk as well by involving (6.4). Specifically, we obtain that

$$\mathsf{AV@R}_{1-\gamma}(-R_\alpha(t)) = -\frac{\mathbb{E}\min(0, R_\alpha(t))}{\gamma},$$

where $\gamma = \mathbb{P}(R(t) \le 0)$, the crucial probability at time t.

6.3 Entropic value-at-risk

The total loss $R_\alpha(t)$, necessary in (6.5) to provide a premium, is usually not accessible for computations and does not coincide with the average value-at-risk in (6.3). However, we have explicit access to the Laplace transform of the fractional risk process $R_\alpha(t)$ via (6.2) in Proposition 8. The following entropic value-at-risk takes advantage of this knowledge components.

Definition 10 The *entropic value-at-risk* of a random variable Y at risk level γ is

$$\mathsf{EV@R}_\gamma(Y) := \inf_{t>0} \frac{1}{t} \log \frac{1}{1-\gamma}\, \mathbb{E}\, e^{tY}, \quad \gamma \in [0, 1),$$

cf. Ahmadi-Javid (2012) and Ahmadi-Javid and Pichler (2017).

For the fractional risk process $R_\alpha(t)$, the entropic value-at-risk is explicitly given by

$$\begin{aligned}\mathsf{EV@R}_\gamma\big(R_\alpha(t)\big) &= \inf_{s>0} \frac{1}{s} \log \frac{1}{1-\gamma}\, \mathbb{E}\, e^{-s\cdot R_\alpha(t)}\\ &= \inf_{s>0} \frac{1}{s} \log \frac{1}{1-\gamma} e^{-su} E_\alpha\left(\lambda t^\alpha \left(\frac{\lambda \varphi_Z(s)}{\lambda + (sc)^\alpha} - 1\right)\right),\end{aligned}$$

where we have used the explicit form (6.2).

The entropic value-at-risk is a convex premium, as the average value-at-risk. Their relation is established in the following proposition.

Proposition 11 *For every random variable Y and $\gamma \in [0, 1)$, we have that*

$$\mathsf{AV@R}_\gamma(Y) \le \mathsf{EV@R}_\gamma(Y).$$

The general relationship between the average value-at-risk and the entropic value-at-risk is given by its Kusuoka representation (cf. Kusuoka 2001), which is

$$\mathsf{EV@R}_\gamma(Y) = \sup_\mu \int_0^1 \mathsf{AV@R}_x(Y)\mu(dx),$$

where the supremum is along all measures μ on $[0, 1]$, for which the associated density $\sigma_\mu(p) := \int_0^p \frac{1}{1-x}\mu(dx)$ satisfies $\int_0^1 \sigma_\mu(u)\log\sigma_\mu(u)du \le \log\frac{1}{1-\gamma}$; the latter relationship is related to the Donsker and Varadhan's variational formula, cf. Pichler and Schlotter (2020), and gives the upper bound for the entropy of the distribution with density σ_μ.

7 Simulation

We propose the simulation algorithm of the compound fractional risk process $R_\alpha^\bullet(t)$ in the form of the direct superposition of trajectories of the classical risk process $R(t)$ and the inverse subordinator $Y_\alpha(t)$, see Meerschaert and Sikorskii (2019, Sect. 5.2) and "Appendix".

The trajectory of the classical risk process $R(t)$ starts with an initial capital u and then continuously increases the surplus with a constant premium rate c. At random, exponentially distributed intervals with mean $1/\lambda$, the surplus is instantaneously reduced by the size of a claim, which is also a random variable drawn from the exponential distribution with mean μ.

The trajectory of the inverse subordinator $Y_\alpha(t)$ can be obtained as follows. Let Δ be a level of discretization, for example, we take $\Delta = 0.01$. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables with the distribution $D_\alpha(1)$. As it is shown in Gupta et al. (2021), the random variable with the distribution $D_\alpha(1)$ can be expressed as

$$\frac{\sin(\alpha\pi U)\big(\sin((1-\alpha)\pi U)\big)^{\frac{1}{\alpha}-1}}{\big(\sin(\pi U)\big)^{\frac{1}{\alpha}}\big|\ln(V)\big|^{\frac{1}{\alpha}-1}},$$

where U and V are independent random variables with the uniform distribution on the interval $[0, 1]$. Let $y_i = i\Delta, i = 0, 1, 2, \ldots$, and $t_0 = 0$. Define $t_i = \Delta^{\frac{1}{\alpha}}\sum_{j=1}^{i}\xi_j, i = 1, 2, \ldots$. Finally, the step plot for points (t_0, y_0), (t_1, y_1), (t_2, y_2), $\ldots$ is a discretized trajectory of $Y_\alpha(t)$. We note that the step plot for points (y_0, t_0), (y_1, t_1), (y_2, t_2), $\ldots$ is a discretized trajectory of $D_\alpha(t)$. We can notice that the discretized trajectory of $Y_\alpha(t)$ is similar to a trajectory of a Poisson process but jumps have height Δ and time between jumps is distributed as $\Delta^{\frac{1}{\alpha}}D_\alpha(1)$.

In Figure 3, we show 50 realizations of the inverse subordinator $Y_\alpha(t)$ for several values of α, and the R code is deferred to online supplement. Unlike the α-stable subordinator itself, which is a pure jump process, its inverse is a continuous function of time. This is because it takes time for the subordinator to reach a certain level,

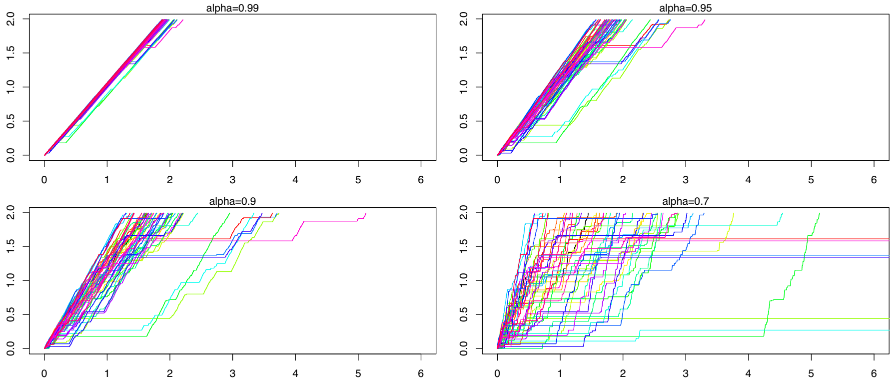

Fig. 3 Fifty realizations of the inverse subordinator $Y_\alpha(t)$ for selected values of α

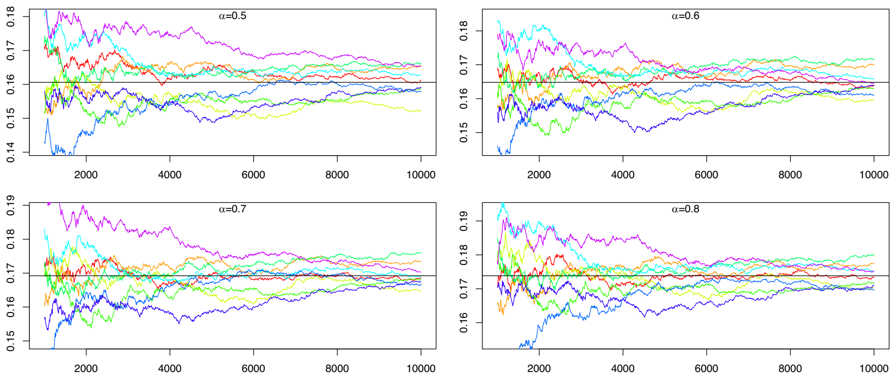

Fig. 4 Convergence of the empirical finite-time ruin probability $\hat{p}_k$, $k = 1000, \ldots, 10000$, for $u = 1.2$, $c = 3.5$, $\lambda = 1.4$, $\mu = 1.1$, $T = 1.8$, $\Delta = 0.01$ and $\alpha = 0.5,\ 0.6,\ 0.7,\ 0.8$. The theoretical finite-time ruin probability is given by the horizontal line

and the inverse process reflects this. A key feature of the realizations is the presence of flat periods or "resting times". These flat periods correspond to the large jumps in the original subordinator. When the subordinator makes a large jump, the inverse subordinator's value remains constant for a period, as it waits for the subordinator to climb to the next level. The length of flat periods depends on α. We can see that the process $Y_\alpha(t)$ is almost equal to t if α is very close to 1. If α decreases, the process $Y_\alpha(t)$ stays for longer times at some values.

Let us now demonstrate that the empirical finite-time ruin probability converges to the theoretical finite-time ruin probability as the number of realizations of the compound fractional risk process $R^\bullet_\alpha(t)$ on the interval $[0, T]$ tends to infinity. Let $b_k = 1$ if ruin was observed for the k-th realization and $b_k = 0$ otherwise, $k = 1, 2, \ldots$. Define $\hat{p}_k = \frac{1}{k}\sum_{j=1}^{k} b_j$. By doing the Monte Carlo simulation in Figure 4, we show sequences $\hat{p}_k$, $k = 1000, \ldots, 10000$, for $\alpha = 0.5,\ 0.6,\ 0.7,\ 0.8$, while other parameters are fixed as $u = 1.2$, $c = 3.5$, $\lambda = 1.4$, $\mu = 1.1$, $T = 1.8$, $\Delta = 0.01$. We can see that the empirical probability $\hat{p}_k$ converges to the theoretical one shown by the horizontal line in the similar manner for various values of α as $k \to \infty$; note that the theoretical probability depends on α. According to the theory of the Monte Carlo simulation, see, for example, Korn et al. (2010), we have also to look on the 95%-confidence interval $(\hat{p}_k - 1.96\sqrt{\hat{p}_k(1-\hat{p}_k)}/\sqrt{k}, \hat{p}_k + 1.96\sqrt{\hat{p}_k(1-\hat{p}_k)}/\sqrt{k})$, where 1.96 is the 97.5%-quantile of the standard normal distribution. In Figure 4, we observe that the variance of curves at each k is close to $p(1-p)/k$, where $p = \psi^\alpha(T)$ is the theoretical ruin probability, and the 95%-confidence intervals include p for various k, confirming the agreement of the theoretical finite-time ruin probability with the Monte Carlo simulation of it.

8 Appendix

8.1 Connections with results of Biard and Saussereau (2016)

Let us compare our findings with results of Biard and Saussereau (2016) who considered the model (1.1) in the framework of the Sparre–Andersen model with exponential claims and the renewal counting process (1.2). For the infinite-time ruin probability $\psi_\infty = \mathbb{P}\{R_\alpha(s) < 0 \text{ for some } s > 0\}$, they derived that the probability ψ_∞ can be expressed as $\psi_\infty = (1-\gamma\mu)e^{-\gamma u}$, $u > 0$, where γ is the unique solution of the equation $\gamma^\alpha - \gamma^{\alpha-1}/\mu + \lambda/c^\alpha = 0$. For the finite-time ruin probability $\psi_\alpha(t) = \mathbb{P}\{\inf_{0\le s\le t} R_\alpha(s) < 0\}$, they derived that its Laplace transform is given by

$$\xi \int_0^\infty e^{-\xi t}\psi^\alpha(t) = (1-y(\xi))e^{-\frac{u}{\mu}(1-y(\xi))},\ \xi > 0,$$

where $y(\xi)$ is the unique solution of the equation

$$y(\xi) = \lambda\left(\lambda + \xi + \frac{c}{\mu}\Big(1-y(\xi)\Big)\right)^{-\alpha},\ \xi > 0.$$

They also derived that the density of the ruin time $\tau_\alpha^* = \inf\{t > 0 :\ R_\alpha(t) < 0\}$ in the model (1.1) with exponential claims has the series representation

$$p_{\tau_\alpha^*}(t) = e^{-(u+ct)/\mu}\sum_{m=0}^{\infty}\frac{(u+ct)^{m-1}}{m!\mu^m}\left(u + \frac{ct}{m+1}\right) f_\alpha^{*(m+1)}(t),$$

where $f_\alpha^{*m}(t)$ is the m-fold convolution of the probability density function of the random variables defined in (1.3). They also derived some inequalities for the finite-time ruin probability for the risk process (1.1) with light-tailed claim sizes. We note that our results on the finite-time ruin probability are obtained for the risk process $R_\alpha^\bullet(t)$ defined in (1.7) and have different nature because we have used a different approach.

8.2 Further properties of the density of the inverse stable subordinators

We give some alternative expressions for the density $f_\alpha(x,t)$ of the inverse stable subordinators; see, for example, Leonenko and Pirozzi (2022) and their references.

Instead of (2.2), the density of $X_\alpha(t)$ can be also given in the form

$$f_\alpha(x,t) = \frac{1}{t^\alpha} M_\alpha\left(\frac{x}{t^\alpha}\right), \quad x > 0,\ t > 0, \tag{8.1}$$

where the M-Wright function $M_\alpha(z)$ is defined as

$$M_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + (1-\alpha))}, \quad z \in \mathbb{C},\ 0 < \alpha < 1,$$

see Mainardi et al. (2010) for the properties of $M_\alpha(z)$. The further expression of the density of $X_\alpha(t)$ is given by

$$f_\alpha(x,t) = \frac{1}{\pi}\int_0^\infty u^{\alpha-1} e^{-tu+xu^\alpha \cos(\pi\alpha)} \sin\left(\pi\alpha - xu^\alpha \sin(\pi\alpha)\right) du.$$

Also, in (2.2) the density $g_\alpha(x)$ has the Mikusinski's representation

$$g_\alpha(x) = \frac{\alpha}{1-\alpha}\frac{1}{\pi x}\int_0^\infty u(\phi) e^{-u(\phi)} d\phi,$$

where

$$u(\phi) = \frac{\sin((1-\alpha)\phi)}{\sin\phi}\left(\frac{\sin(\alpha\phi)}{\sin\phi}\right)^{\frac{\alpha}{1-\alpha}}$$

with the following asymptotics,

$$g_\alpha(x) \sim C_2 \frac{e^{-C_1 \cdot x^{\frac{\alpha}{1-\alpha}}}}{x^{\frac{2-\alpha}{2-2\alpha}}}, \quad x \to 0,$$
$$g_\alpha(x) \sim \frac{C_3}{x^{1+\alpha}}, \quad x \to \infty;$$

where

$$C_1 = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}, \quad C_2 = \frac{\alpha^{\frac{1}{2-2\alpha}}}{\sqrt{2\pi(1-\alpha)}}, \quad C_3 = \frac{\sin(\pi\alpha)}{\pi}\Gamma(1+\alpha).$$

Thus, we have

$$
\begin{aligned}
&f_\alpha(x,t) = C_2 \frac{t^\alpha\left(1+\frac{2\alpha-1}{2-2\alpha}\right)}{\alpha} x^{\frac{2\alpha-1}{2-2\alpha}} \exp\left\{-\frac{C_1}{t^{\frac{\alpha}{1-\alpha}}} x^{\frac{1}{1-\alpha}}\right\}, \quad \text{as } x \to \infty, \text{ and} \\
&f_\alpha(x,t) \to \frac{1}{t^\alpha}\frac{\sin(\pi\alpha)}{\pi\alpha}\Gamma(1+\alpha), \quad \text{as } x \to 0.
\end{aligned}
$$

8.3 Proof of Theorem 1

Note that

$$
R^\bullet_\alpha(t) = R\left(Y_\alpha(t)\right), \quad t \geq 0,\ 0<\alpha<1, \tag{8.2}
$$

where $R(t)$ is the classical risk process, $Y_\alpha(t)$ is given by (2.1), $R(t)$ and $Y_\alpha(t)$ are independent. Then, we have

$$
\begin{aligned}
\psi^\alpha(t) &= \mathbb{P}\left[\inf_{0\leq s\leq t} R^\bullet_\alpha(s) < 0\right] \\
&= \int_0^\infty \mathbb{P}\left[\inf_{0\leq s\leq t} R\left(Y_\alpha(s)\right) < 0 \mid Y_\alpha(t) = h\right] f_\alpha(h,t)dh \\
&= \int_0^\infty \mathbb{P}\left[\inf_{0\leq b\leq h} R(b) < 0\right] f_\alpha(h,t)dh = \int_0^\infty \psi(h) f_\alpha(h,t)dh,
\end{aligned}
$$

that completes the proof.

8.4 Proof of Theorem 2

Using Theorem 1, Fubini's theorem and (3.5), (3.7) and (3.9), we obtain

$$
\begin{aligned}
\psi^\alpha(t) &= \int_0^\infty \left[\psi_\infty - \frac{1}{\pi}\int_0^\pi F_h(x)dx\right] f_\alpha(h,t)dh \\
&= \psi_\infty - \int_0^\infty \left[\frac{1}{\pi}\int_0^\pi V(x)U_h(x)dx\right] f_\alpha(h,t)dh \\
&= \psi_\infty - \frac{1}{\pi}\int_0^\pi V(x)dx \int_0^\infty U_h(x) f_\alpha(h,t)dh \\
&= \psi_\infty - \frac{1}{\pi}\int_0^\pi V(x)dx \int_0^\infty e^{-h\cdot\frac{c}{\mu}(1+\beta-2\sqrt{\beta}\cos x)} f_\alpha(h,t)dh.
\end{aligned}
$$

From (2.4), we obtain that $\int_0^\infty e^{-A(x)h} f_\alpha(h,t)dh = E_\alpha\big(-A(x)t^\alpha\big)$, where $E_\alpha(\cdot)$ is the Mittag–Leffler function. Hence, from (3.5) we have

$$\begin{aligned}\psi^\alpha(t) &= \psi_\infty - \frac{1}{\pi}\int_0^\pi V(x)\int_0^\infty U_h^\alpha(x) f_\alpha(h,t)dhdx\\ &= \psi_\infty - \frac{1}{\pi}\int_0^\pi V(x)E_\alpha\big(-t^\alpha A(x)\big)dx\end{aligned}$$

that completes the proof.

8.5 Proof of Proposition 4

Note first that, by the definition of the inverse subordinator $Y_\alpha(t)$ in (2.1) and as $D_\alpha(t)$ is non-decreasing, that $Y_\alpha(D_\alpha(t)) = t$ and, as D_α is càdlàg, that $D_\alpha(Y_\alpha(t)) \geq t$ (cf. also Fig. 3 on page 16). Next, with definition (4.1) and (8.2), it holds that

$$\begin{aligned}\tau_\alpha &= \inf\{t > 0\colon R_\alpha^\bullet(t) < 0\}\\ &= \inf\{t > 0\colon R(Y_\alpha(t)) < 0\}\\ &\leq \inf\{D_\alpha(s)\colon R(Y_\alpha(D_\alpha(s))) < 0\}\\ &= \inf\{D_\alpha(s)\colon R(s) < 0\}\\ &\leq D_\alpha\big(\inf\{t\colon R(t) < 0\}\big)\\ &= D_\alpha(\tau),\end{aligned} \tag{8.3}$$

where we have used that D_α is càdlàg in (8.3), but (4.3) follows from the definition (4.1).

8.6 Proof of Theorem 5

The proof in Mikosch (2009, Theorem 4.2.3) applies with minor adjustments only. Indeed, with $Y_i := Z_i - c\,V_i$, consider the combined loss $S_n := Y_1 + \cdots + Y_n$, which occurs at time $V_1 + \cdots + V_n$. From the definition of the fractional risk process $R_\alpha(t)$ and (5.1), we note that

$$\psi_u^\alpha = \mathbb{P}\left(\max_{k\geq 1} S_k \geq u\right).$$

To prove the assertion of the theorem, we will show by induction that the inequality

$$\psi_{u;n}^\alpha := \mathbb{P}\left(\max_{k\leq n} S_k \geq u\right) \leq e^{-r_\alpha u} \tag{8.4}$$

holds for all $n = 1, 2, \ldots$. With Markov's inequality (Chernoff's bound), we obtain that

$$\mathbb{P}(S_1 \geq u) = \mathbb{P}\left(e^{r_\alpha Y_1} \geq e^{r_\alpha u}\right) \leq e^{-r_\alpha u}\,\mathbb{E}\,e^{r_\alpha Y_1} = e^{-r_\alpha u}\,\mathbb{E}\,e^{r_\alpha (Z_1 - cV_1)} = e^{-r_\alpha u},$$

where we have used the characterizing equation (5.4). Thus, we note that the inequality (8.4) holds for $n = 1$. To establish the induction step from n to $n + 1$, we firstly observe that

$$\begin{aligned}
\psi^{\alpha}_{u;n+1} &= \mathbb{P}\left(\max_{k\le n+1} S_k \ge u\right) \\
&\le \mathbb{P}(Y_1 \ge u) + \mathbb{P}\left(Y_1 < u \text{ and } \max_{k=2,\dots,n+1} Y_1 + (S_k - Y_1) \ge u\right) \\
&= \int_u^{\infty} \mathrm{d}F_{Y_1}(y) + \int_{-\infty}^{u} \mathbb{P}\left(y + \max_{k\le n} S_k \ge u\right) \mathrm{d}F_{Y_1}(y).
\end{aligned}$$

These expressions can be bounded from above by

$$\int_u^{\infty} \mathrm{d}F_{Y_1}(y) \le \int_u^{\infty} e^{-r_\alpha(u-y)} \mathrm{d}F_{Y_1}(y), \tag{8.5}$$

and

$$\begin{aligned}
\int_{-\infty}^{u} \mathbb{P}\left(y + \max_{k\le n} S_k \ge u\right) \mathrm{d}F_{Y_1}(y) &= \int_{-\infty}^{u} \psi^{\alpha}_{u-y;n}\, \mathrm{d}F_{Y_1}(y) && (8.6)\\
&\le \int_{-\infty}^{u} e^{-r_\alpha(u-y)}\, \mathrm{d}F_{Y_1}(y), && (8.7)
\end{aligned}$$

where we have used (8.4) in (8.6). Combination of (8.5) and (8.7) gives

$$\psi^{\alpha}_{u;n+1} \le \mathbb{E}\, e^{-r_\alpha(u-Y_1)} = e^{-r_\alpha u}\, \mathbb{E}\, e^{r_\alpha(Z_1 - c V_1)} \le e^{-r_\alpha u},$$

where we have used (5.4) again. This completes the induction step. Thus, the assertion of the theorem is proved.

8.7 Proof of Proposition 6

Using Pillai (1990, p. 272), we obtain that the Laplace transform of the random variable V with cumulative distribution function (1.3) is

$$\mathbb{E}\, e^{-r V} = \frac{\lambda}{\lambda + r^{\alpha}}, \quad r > 0. \tag{8.8}$$

With (8.8), the Cramér–Lindberg coefficient (5.4) satisfies

$$1 = \mathbb{E}\, e^{r_\alpha(Z - cV)} = \mathbb{E}\, e^{r_\alpha Z} \cdot \mathbb{E}\, e^{-r_\alpha c V} = \mathbb{E}\, e^{r_\alpha Z} \cdot \frac{\lambda}{\lambda + (c r_\alpha)^{\alpha}}$$

and thus the assertion (5.5) holds.

8.8 Proof of Proposition 8

For the fractional risk process $R_\alpha(t)$ defined in (1.1), the moment generating function is

$$\begin{aligned}\mathbb{E}\, e^{-sR_\alpha(t)} &= \mathbb{E}\, e^{-s(u+ct-\sum_{i=1}^{N_\alpha(t)} Z_i)}\\ &= e^{-su}\,\mathbb{E}\,\mathbb{E}\left(e^{-s\cdot\sum_{i=1}^{N_\alpha(t)}(cV_i-Z_i)}\,\Big|\, N_\alpha(t)\right),\end{aligned}$$

where we have employed the tower property of the expectation. With the explicit expression (2.8), we obtain that

$$\begin{aligned}\mathbb{E}\, e^{-sR_\alpha(t)} &= e^{-su}\sum_{n=0}^{\infty}\mathbb{E}\, e^{-\sum_{i=0}^{n}(scV_i-sZ_i)}\cdot P(N_\alpha(t)=n)\\ &= e^{-su}\sum_{n=0}^{\infty}\mathbb{E}\, e^{-\sum_{i=0}^{n}scV_i}\cdot\mathbb{E}\, e^{\sum_{i=0}^{n}sZ_i}\cdot P(N_\alpha(t)=n) \qquad (8.9)\\ &= e^{-su}\sum_{n=0}^{\infty}\left(\mathbb{E}\, e^{-scV}\right)^n\cdot\left(\mathbb{E}\, e^{sZ}\right)^n\cdot\frac{(\lambda t^\alpha)^n}{n!}\cdot E_\alpha^{(n)}(-\lambda t^\alpha), \qquad (8.10)\end{aligned}$$

where we have used that the claims $Z_1, Z_2, \ldots$ and the interarrival times $V_1, V_2, \ldots$ are independent in (8.9) and (8.10); $E^{(n)}$ in (8.10) is the nth derivative of the Mittag–Leffler function, cf. (2.9). Furthermore, we note that

$$E_\alpha\left(-\lambda t^\alpha+\omega t^\alpha\,\mathbb{E}\, e^{-scV}\,\mathbb{E}\, e^{sZ}\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\mathbb{E}\, e^{-scV}\,\mathbb{E}\, e^{sZ}\cdot\omega t^\alpha\right)^n E_\alpha^{(n)}(-\lambda t^\alpha), \tag{8.11}$$

where the Taylor expansion is applied to the function $E_\alpha(q_1+\omega q_2)$ at $\omega=0$, where $q_1=-\lambda t^\alpha$ and $q_2=t^\alpha\,\mathbb{E}\, e^{-scV}\,\mathbb{E}\, e^{sZ}$, see Podlubny (1998, pp. 16, 100). Recall now from (8.8) that $\mathbb{E}\, e^{-rV}=\frac{\lambda}{\lambda+r^\alpha}$. Collection of terms in (8.10) using (8.11) finally reveals (6.2) that completes the proof.

8.9 Proof of Proposition 11

Let $a := \mathsf{V@R}_\gamma(Y)$, then $\gamma \ge \mathbb{P}(Y \le a)$. With Markov's inequality, we obtain that

$$1-\gamma \le \mathbb{P}(Y > a) \le e^{-at}\,\mathbb{E}\, e^{tY}.$$

Therefore, we have that

$$\mathsf{V@R}_\gamma(Y) = a \le \inf_{t>0}\frac{1}{t}\log\frac{1}{1-\gamma}\,\mathbb{E}\, e^{tY} = \mathsf{EV@R}_\gamma(Y).$$

Now, we recall (see, for example, Föllmer and Schied 2004) that the average value-at-risk is the smallest convex risk functional exceeding the value-at-risk. Thus, we obtain that the assertion

$$\mathsf{V@R}_\gamma(Y) \le \mathsf{AV@R}_\gamma(Y) \le \mathsf{EV@R}_\gamma(Y),$$

holds that completes the proof.

8.10 Simulation of the compound fractional process

Using the simulation algorithms of the classical risk process $R(t)$ and the inverse subordinator $Y_\alpha(t)$, the algorithm for testing the condition $R^\bullet_\alpha(t) < 0$ for $t \in [0, T]$ in the R package can be written as follows.

```
#Input: u, c, lambda, mu, alpha, T, Delta
#Output: IsRuined=TRUE if surplus is negative
 before time T
IsRuined = FALSE
CurrentTotalPay = 0
NextClaimYt = 0
CurrentYt = 0
Current.t = 0
while (Current.t <= T && !IsRuined) {
  gapPP = rexp(1, lam)
  NextClaimYt = NextClaimYt + gapPP
  Delta0=Delta
  while(CurrentYt<NextClaimYt)  {
    if(Delta > NextClaimYt - CurrentYt)
       Delta0 = NextClaimYt - CurrentYt
    U = runif(1)
    V = runif(1)
    numer = sin(alpha*pi*U) *
 (sin((1-alpha)*pi*U))^(1/alpha-1)
    denom = (sin(pi*U))^(1/alpha)
 * (abs(log(V)))^(1/alpha-1)
    D1 = numer / denom
    Current.t = Current.t + D1*Delta0^(1/alpha)
    CurrentYt = CurrentYt + Delta0
  }
  CurrentTotalPay = CurrentTotalPay + rexp(1, 1 / mu)
  CurrentSurplus = u + c * CurrentYt - CurrentTotalPay
  if (Current.t <= T && CurrentSurplus < 0) {
    IsRuined = TRUE
  }
}
return(IsRuined)
```

The detailed explanation of the above R code is given in online supplement.

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References

Ahmadi-Javid A (2012) Entropic value-at-risk: a new coherent risk measure. J Optim Theory Appl 155(3):1105–1123. https://doi.org/10.1007/s10957-011-9968-2

Ahmadi-Javid A, Pichler A (2017) An analytical study of norms and Banach spaces induced by the entropic value-at-risk. Math Financ Econ 11(4):527–550. https://doi.org/10.1007/s11579-017-0197-9

Artzner P, Delbaen F, Heath D (1997) Thinking coherently. Risk 10:68–71

Artzner P, Delbaen F, Eber JM, Heath D (1999) Coherent measures of risk. Math Financ 9:203–228. https://doi.org/10.1111/1467-9965.00068

Asmussen S, Albrecher H (2010) Ruin probabilities, vol 14. World Scientific, Singapore

Beghin L, Macci C (2013) Large deviations for fractional Poisson processes. Stat Probab Lett 83(4):1193–1202. https://doi.org/10.1016/j.spl.2013.01.017

Beghin L, Orsingher E (2009) Fractional Poisson processes and related planar random motions. Electron J Probab 14:1790

Biard R, Saussereau B (2014) Fractional Poisson process: long-range dependence and applications in ruin theory. J Appl Probab 51(3):727–740

Biard R, Saussereau B (2016) Fractional Poisson process: long-range dependence and applications in ruin theory—correction. J Appl Probab 53(4):1271–1272. https://doi.org/10.1017/jpr.2016.80

Borovkov KA, Dickson DC (2008) On the ruin time distribution for a Sparre Andersen process with exponential claim sizes. Insur Math Econ 42(3):1104–1108. https://doi.org/10.1016/j.insmatheco.2008.02.002

Deprez O, Gerber HU (1985) On convex principles of premium calculation. Insur Math Econ 4(3):179–189. https://doi.org/10.1016/0167-6687(85)90014-9

Dickson DC (2016) Insurance risk and ruin. Cambridge University Press, Cambridge

Feller W (1991) An introduction to probability theory and its applications, vol 2. Wiley, New York

Föllmer H, Schied A (2004) Stochastic finance: an introduction in discrete time. de Gruyter Studies in Mathematics 27. De Gruyter, Berlin

Gupta N, Kumar A, Leonenko N (2021) Stochastic models with mixtures of tempered stable subordinators. Math Commun 26(1):77–99

Kataria K, Khandakar M (2021) Mixed fractional risk process. J Math Anal Appl 504(1):125379. https://doi.org/10.1016/j.jmaa.2021.125379

Korn R, Korn E, Kroisandt G (2010) Monte Carlo methods and models in finance and insurance. CRC press

Kumar A, Leonenko N, Pichler A (2020) Fractional risk process in insurance. Math Financ Econ 14:43–65. https://doi.org/10.1007/s11579-019-00244-y

Kusuoka S (2001) On law invariant coherent risk measures. In: Advances in mathematical economics, volume 3, Chapter 4. Springer, Berlin, pp 83–95. https://doi.org/10.1007/978-4-431-67891-5

Laskin N (2003) Fractional Poisson process. Commun Nonlinear Sci Numer Simul 8:201–213. https://doi.org/10.1016/S1007-5704(03)00037-6

Leonenko NN, Meerschaert MM, Schilling RL, Sikorskii A (2014) Correlation structure of time-changed Lévy processes. Commun Appl Ind Math 6(1):e-483

Leonenko N, Pirozzi E (2022) First passage times for some classes of fractional time-changed diffusions. Stoch Anal Appl 40(4):735–763. https://doi.org/10.1080/07362994.2021.1953386

Maheshwari A, Vellaisamy P (2016) On the long-range dependence of fractional Poisson and negative binomial processes. J Appl Probab 53(4):989–1000

Mainardi F, Gorenflo R, Scalas E et al (2004) A fractional generalization of the Poisson processes. Vietnam J Math 32:53–64

Mainardi F, Mura A, Pagnini G (2010) The M-Wright function in time-fractional diffusion processes: a tutorial survey. Int J Differ Equ 2010(1):104505

Malinovskii VK (2021) Insurance planning models: price competition and regulation of financial stability. World Scientific, Singapore

Meerschaert MM, Sikorskii A (2019) Stochastic models for fractional calculus, vol 43. Walter de Gruyter GmbH & Co KG

Meerschaert M, Nane E, Vellaisamy P (2011) The fractional Poisson process and the inverse stable subordinator. Electron J Probab 16:1600–1620

Mikosch T (2003) Non-life insurance mathematics—a primer

Mikosch T (2009) Non-life insurance mathematics, 2nd edn. Springer, Berlin

Pichler A (2013) The natural Banach space for version independent risk measures. Insur Math Econ 53(2):405–415. https://doi.org/10.1016/j.insmatheco.2013.07.005

Pichler A, Schlotter R (2020) Entropy based risk measures. Eur J Oper Res 285(1):223–236. https://doi.org/10.1016/j.ejor.2019.01.016

Pillai RN (1990) On Mittag–Leffler functions and related distributions. Ann Inst Stat Math 42:157–161

Pirozzi E (2022) On a fractional stochastic risk model with a random initial surplus and a multi-layer strategy. Mathematics 10(4):570. https://doi.org/10.3390/math10040570

Podlubny I (1998) Fractional differential equations: an introduction to fractional derivatives. Elsevier, New York

Simon T (2014) Comparing Fréchet and positive stable laws. Electron J Probab 19:1–25. https://doi.org/10.1214/EJP.v19-3058

Young VR (2006) Premium principles, Encyclopedia of actuarial science, Encyclopedia of actuarial science, Chapter 3. Pennsylvania State University, Wiley. https://doi.org/10.1002/9780470012505.tap027