The Heisenberg category of a category

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Contents

Abstract

Starting with a $\Bbbk$-linear or DG category admitting a (homotopy) Serre functor, we construct a $\Bbbk$-linear or DG 2-category categorifying the Heisenberg algebra of the numerical K-group of the original category. We also define a 2-categorical analogue of the Fock space representation of the Heisenberg algebra. Our construction generalises and unifies various categorical Heisenberg algebra actions appearing in the literature. In particular, we give a full categorical enhancement of the action on derived categories of symmetric quotient stacks introduced by Krug, which itself categorifies a Heisenberg algebra action proposed by Grojnowski.

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CHAPTER 1

Introduction

The Heisenberg algebra of a lattice is a much investigated object originating in quantum theory. It appears in many areas of mathematics, including the representation theory of affine Lie algebras. For a smooth projective surface, Grojnowksi and Nakajima [**24, 37**] identified the total cohomology of its Hilbert schemes of points with the Fock space representation of the Heisenberg algebra associated to the cohomology of the surface. As proposed by Grojnowksi [**24**, footnote 3] and proved by Krug [**33**], this occurs more generally for the symmetric quotient stacks of any smooth projective variety on the level of K-theory, and, more fundamentally, on the level of derived categories of coherent sheaves.

On the other hand, Khovanov [**31**] introduced a categorification of the infinite Heisenberg algebra associated to the free boson or, equivalently, a rank 1 lattice. It used a graphical construction involving planar diagrams. A related graphically defined category was constructed by Cautis and Licata [**13**] for ADE type root lattices. Both of these Heisenberg categories admit categorical representations on categorifications of the corresponding Fock spaces. They were much studied since [**20, 41, 9, 10, 42**].

In this paper we unify and generalise many of these constructions. We start with a $\Bbbk$-linear and Hom-finite category $\mathcal{V}$ equipped with a Serre functor S. That is, S is a $\Bbbk$-linear autoequivalence equipped with natural isomorphisms

$$\eta_{a,b}\colon \operatorname{Hom}_{\mathcal{V}}(b, Sa)^* \cong \operatorname{Hom}_{\mathcal{V}}(a,b) \qquad \forall\, a,\, b \in \mathcal{V}. \tag{1.1}$$

A typical example is the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ of a smooth and proper variety X with $S = (-) \otimes \omega_X[\dim X]$. We further allow $\mathcal{V}$ to be graded or a DG category. In the latter case, S only needs to be a homotopy Serre functor. The following summarises our main results:

Theorem (Summary of the main results). *There exists a* Heisenberg 2-category $\mathbf{H}_{\mathcal{V}}$ *of* $\mathcal{V}$ *defined using a graphical calculus, together with a* Fock space representation *on the categories of* S_N*-equivariant objects in* $\mathcal{V}^{\otimes N}$.

We now make this statement more precise.

1.1. Heisenberg algebras of categories

The numerical Grothendieck group $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V})$ has a bilinear pairing χ given by the dimension of $\operatorname{Hom}_{\mathcal{V}}(a,b)$ or its Euler characteristic in the graded or DG case, cf. Section 4.9. If χ is symmetric, we can define a Heisenberg algebra $H_{\mathcal{V}}$ with generators

$$\{a_b(n)\}_{b \in \mathrm{K}_0^{\mathrm{num}}(\mathcal{V}),\ n \in \mathbb{Z}\setminus\{0\}}$$

and relations

$$[a_b(m),\, a_c(n)] = m\delta_{m,-n}\langle b,\, c\rangle_{\chi}.$$

However, in practice χ is rarely symmetric, cf. Example 4.43[1].

As observed in [**31, 13**], it can be more convenient to choose a different set of generators

$$\left\{p_b^{(n)}, q_b^{(n)}\right\}_{b\in \mathrm{K}_0^{\mathrm{num}}(\mathcal{V}),\ n\in\mathbb{Z}_{\geq 0}}$$

and a different set of relations

$$p_b^{(0)} = q_b^{(0)} = 1 \tag{1.2}$$

$$p_{a+b}^{(n)} = \textstyle\sum_{k=0}^{n} p_a^{(k)} p_b^{(n-k)} \quad \text{and} \quad q_{a+b}^{(n)} = \textstyle\sum_{k=0}^{n} q_a^{(k)} q_b^{(n-k)}, \tag{1.3}$$

$$p_a^{(n)} p_b^{(m)} = p_b^{(m)} p_a^{(n)} \quad \text{and} \quad q_a^{(n)} q_b^{(m)} = q_b^{(m)} q_a^{(n)}, \tag{1.4}$$

$$q_a^{(n)} p_b^{(m)} = \textstyle\sum_{k=0}^{\min(m,n)} s^k\left(\langle a,b\rangle_\chi\right)\, p_b^{(m-k)} q_a^{(n-k)}, \tag{1.5}$$

and $s^k(n) = \dim \mathrm{Sym}^k \Bbbk^n$. These relations are consistent even when χ is non-symmetric. Thus the above defines the Heisenberg algebra $H_{\mathcal{V}}$ of any $\mathcal{V}$. We prove in Corollary 2.6 that it is always isomorphic to one induced by a symmetric pairing.

1.2. Categorification

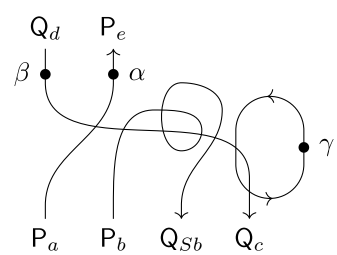
The goal is to define a monoidal category $\mathbf{H}_{\mathcal{V}}$ with objects generated by symbols P_a and Q_a for each $a \in \mathcal{V}$ and the morphisms set up so that we can define $\mathsf{P}_a^{(n)}$ and $\mathsf{Q}_a^{(n)}$ in terms P_a's and Q_a's and so that the relations above become isomorphisms of objects. For example, relation (1.5) should become an isomorphism

$$\mathsf{Q}_a^{(m)} \mathsf{P}_b^{(n)} \cong \bigoplus_{i=0}^{\min(m,n)} \mathrm{Sym}^i \operatorname{Hom}_{\mathcal{V}}(a,b) \otimes_{\Bbbk} \mathsf{P}_b^{(n-i)} \mathsf{Q}_a^{(m-i)}. \tag{1.6}$$

We construct $\mathbf{H}_{\mathcal{V}}$ as a 2-*category* with objects $\mathbb{Z}$, 1-morphisms generated by $\mathsf{P}_a\colon N \to N+1$ and $\mathsf{Q}_b\colon N \to N-1$, and appropriate 2-morphisms. A representation of $\mathbf{H}_{\mathcal{V}}$ is a 2-functor into the 2-category of categories, sending each integer to a "weight space category". This *idempotent modification* is done for convenience, and our construction can be easily repackaged into a monoidal category, cf. Section 8.3.

The crux of the categorification is to define "useable" 2-morphism spaces which imply only the necessary isomorphisms such as (1.6). We define these by planar string diagrams such as

(1.7)



read from bottom to top. These are built out of a handful of generators, subject to relations.

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[1] In particular, in [**33**, Corollary 1.5] the algebra $\mathsf{H}_{\mathrm{K}(X)}$ is a priori not well-defined for a general smooth and projective variety X. We are thankful to Pieter Belmans for this remark.

1.3. Main results

Our approach differs depending on whether our input datum $\mathcal{V}$ is a graded additive category with a genuine Serre functor or a DG category with only a homotopy Serre functor. We call these two setups the *additive* and *DG* settings, respectively. We construct the Heisenberg 2-category $\mathbf{H}_{\mathcal{V}}$ in the additive setting in Chapter 3 and in the DG setting in Chapter 5. We then prove Theorems A, B and C stated below for the DG setting and Theorem B in the additive setting. Theorems A and C are also expected to hold in the additive setting if the numerical Grothendieck group $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V})$ is a finitely generated abelian group. In such case, our DG proofs can be adapted and even simplified for the additive setting. Let us therefore state our main results in the language of the DG setting.

Let $\mathcal{V}$ be a smooth and proper DG category. We view it as a Morita enhanced triangulated category, cf. Chapter 4.4. It is the noncommutative analogue of a smooth and proper algebraic variety X: the enhanced derived category of X is an example of such $\mathcal{V}$. The graphical calculus described in Chapter 5 yields a DG bicategory $\mathbf{H}_{\mathcal{V}}$ together with maps of $\Bbbk$-algebras

$$\pi\colon H_{\mathcal{V}} \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk), \tag{1.8}$$

where $H_{\mathcal{V}}$ is the Heisenberg algebra of $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V})$. Here, a *bicategory* is a certain kind of weak 2-category. To be precise, we actually mean a bicategory enriched over the homotopy 2-category $\mathbf{Ho}(\mathbf{dgCat})$ of DG categories, see Chapter 4.1. We treat these subtle differences carefully in the main text of the paper, but here refer to these merely as DG bicategories.

As in the literature of Heisenberg categorification (numerical) Grothendieck groups appear more frequently, let us first state our results towards this direction. Our first main result shows that a 2-full subcategory of $\mathbf{H}_{\mathcal{V}}$ categorifies the Heisenberg algebra $H_{\mathcal{V}}$:

THEOREM A (Theorem 6.20). *The map $\pi\colon H_{\mathcal{V}} \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ is injective.*

Indeed, Theorem A implies that the 2-full subcategory of $\mathbf{H}_{\mathcal{V}}$ comprising the objects whose class in $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ lies in the image of π is a categorification of $H_{\mathcal{V}}$. Since this subcategory is 2-full and contains the objects P_a and Q_a for $a \in \mathcal{V}$, which generate $\mathbf{H}_{\mathcal{V}}$ under taking 1-compositions and perfect hulls, any 2-representation of this subcategory extends uniquely to one of $\mathbf{H}_{\mathcal{V}}$. Thus we work with $\mathbf{H}_{\mathcal{V}}$ instead.

Let $\mathbf{EnhCat}_{\mathrm{kc}}^{\mathrm{dg}}$ be the DG bicategory of enhanced triangulated categories, cf. Chapter 4.4. Here and throughout the paper the subscript kc means "Karoubi-complete". Let $\mathbf{F}_{\mathcal{V}}$ be its 2-full subcategory comprising the symmetric powers $\mathcal{S}^N\mathcal{V}$. If $\mathcal{V}$ is the derived category of a variety X, then $\mathcal{S}^N\mathcal{V}$ is the derived category of the symmetric quotient stack $[X^N/S_N]$.

Our second main result constructs a 2-action of $\mathbf{H}_{\mathcal{V}}$ on $\mathbf{F}_{\mathcal{V}}$ which implies that a 2-full subcategory of $\mathbf{F}_{\mathcal{V}}$ categorifies the classical Fock space representation $F_{\mathcal{V}}$ of $H_{\mathcal{V}}$:

THEOREM B (Theorem 7.30). *There is a 2-representation of $\mathbf{H}_{\mathcal{V}}$ on $\mathbf{F}_{\mathcal{V}}$. More precisely, there is a homotopy strong DG 2-functor $\Phi_{\mathcal{V}}\colon \mathbf{H}_{\mathcal{V}} \to \mathbf{F}_{\mathcal{V}}$.*

Indeed, this 2-action induces a representation of $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ and hence of $H_{\mathcal{V}}$ on $\mathrm{K}_0^{\mathrm{num}}(\mathbf{F}_{\mathcal{V}}, \Bbbk)$. We analyze it in Section 8.2 and show that it induces an embedding of $\phi\colon F_{\mathcal{V}} \hookrightarrow \mathrm{K}_0^{\mathrm{num}}(\mathbf{F}_{\mathcal{V}}, \Bbbk)$ as the subrepresentation generated by $1 \in$

$\mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^0\mathcal{V}, \Bbbk) \cong \Bbbk$. Thus the 2-full subcategory of $\mathbf{F}_\mathcal{V}$ comprising the objects whose class in $\mathrm{K}_0^{\mathrm{num}}(\mathbf{F}_\mathcal{V}, \Bbbk)$ lies in the image of ϕ gives a categorification of $F_\mathcal{V}$.

In many cases, for example if $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V})$ satisfies a Künneth-type formula for symmetric powers, the embedding ϕ above is an isomorphism. Then the whole of $\mathbf{F}_\mathcal{V}$ is a categorification of $F_\mathcal{V}$. In any case, we call $\mathbf{F}_\mathcal{V}$ the *categorical Fock space* of $\mathcal{V}$.

Our third main result gives another sufficient condition for $\mathbf{F}_\mathcal{V}$ to exactly categorify $F_\mathcal{V}$, while at the same time exhibiting an obstruction for π to be an isomorphism.

THEOREM C (Theorem 8.13). *If $\mathbf{H}_\mathcal{V}$ categorifies $H_\mathcal{V}$, that is, if π is an isomorphism, then $\mathbf{F}_\mathcal{V}$ categorifies $F_\mathcal{V}$. In particular, in such case for all $N \geq 0$*

$$\mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}) \cong \bigoplus_{1^{\lambda_1}2^{\lambda_2}\cdots\dashv N} \mathrm{Sym}^{\lambda_1}\mathrm{K}_0^{\mathrm{num}}(\mathcal{V}) \otimes \mathrm{Sym}^{\lambda_2}\mathrm{K}_0^{\mathrm{num}}(\mathcal{V}) \otimes \cdots$$

where the direct sum is taken over all integer partitions of N.

We conjecture that the converse of this statement holds as well.

CONJECTURE D. *If $\mathbf{F}_\mathcal{V}$ categorifies $F_\mathcal{V}$, then $\pi\colon H_\mathcal{V} \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_\mathcal{V}, \Bbbk)$ is an isomorphism.*

We provide examples in Section 8.2.2 where ϕ is an isomorphism. We also give an example in Section 8.2.3 where it fails to be an isomorphism. In the latter case π also can not be an isomorphism by Theorem C. In fact, the numerical Grothendieck group decategorifications of $\mathbf{H}_\mathcal{V}$ and $\mathbf{F}_\mathcal{V}$ are generally larger than the classical Heisenberg algebra $H_\mathcal{V}$ and its Fock space $F_\mathcal{V}$. However, our decategorifications always *contain* $H_\mathcal{V}$ and $F_\mathcal{V}$. It becomes an interesting new problem to compute the surplus and find ways to interpret it.

In the sequel paper [**27**], we show that our 2-category $\mathbf{H}_\mathcal{V}$ can also be decategorified using the Hochschild homology $\mathrm{HH}_\bullet$. Specifically, we settle some foundational issues to define the Heisenberg algebra $H^H_\mathcal{V}$ of the $\mathbb{Z}_2$-graded vector space $\mathrm{HH}_\bullet(\mathcal{V})$. We then prove the following:

THEOREM ([**27**]). *For any smooth and proper DG category $\mathcal{V}$:*

1. *There exists an injective map $\pi^H\colon H^H_\mathcal{V} \longrightarrow \mathrm{HH}_\bullet(\mathbf{H}_\mathcal{V})$.*
2. *The map π^H and the 2-representation $\Phi_\mathcal{V}$ induce an action of $H^H_\mathcal{V}$ on $\mathrm{HH}_\bullet(\mathbf{F}_\mathcal{V})$. There is an injective map $\phi^H\colon F^H_\mathcal{V} \hookrightarrow \mathrm{HH}_\bullet(\mathbf{F}_\mathcal{V})$ which embeds the Fock space $F^H_\mathcal{V}$ of $H^H_\mathcal{V}$ as the subrepresentation generated by $1 \in \mathrm{HH}_\bullet(\mathbf{F}_\mathcal{V})$.*
3. *The map ϕ^H is always an isomorphism and therefore $\mathbf{F}_\mathcal{V}$ always categorifies $F^H_\mathcal{V}$.*

This leads us to conjecture the following:

CONJECTURE. *The map π^H is always an isomorphism, so $\mathbf{H}_\mathcal{V}$ always categorifies $H^H_\mathcal{V}$.*

1.4. Relation to earlier results

Our results recover as special cases the earlier Heisenberg categorification and Fock space action results mentioned above. We bring forward these specialisations throughout the paper as sequences of examples; here we just preview them briefly.

For $\mathcal{V} = \Bbbk$, the field $\Bbbk$ considered as a single object DG category concentrated in degree 0, our category $\mathbf{H}_{\mathcal{V}}$ is a DG enhancement of Khovanov's original category [**31**]; see Examples 3.14, 5.9 and 6.22. When X is a smooth and projective variety and $\mathcal{V}$ its DG enhanced coherent derived category, a subcategory of $\mathbf{H}_{\mathcal{V}}$ categorifies the Heisenberg algebra modeled on the numerical K-theory of X. Its action on $\mathbf{F}_{\mathcal{V}}$ constructed in Theorem B coincides, after taking homotopy categories, with that of Krug [**33**]; see Examples 4.39, 7.1, 7.11 and 8.3. This answers the questions raised in [**33**, Section 3.5]. When X is Calabi-Yau, the direct sum of the Hochschild (co)homologies of X carries the structure of a Frobenius algebra. In this case our categories essentially coincide with those of [**41**], although we do not consider super-Frobenius algebras. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup and let $\mathcal{V}$ be the DG enhanced derived category of coherent sheaves supported on the exceptional divisor E of the minimal resolution X of the quotient singularity $\mathbb{C}^2/\Gamma$. Then our construction yields the Heisenberg category constructed by Cautis and Licata [**13**], see Examples 5.10, 6.16 and 7.16.

There are several advantages to our approach compared to the earlier ones. Our definition allows any DG category $\mathcal{V}$ as the input of the machinery. This fits well into the framework of noncommutative motives [**48**]. We do not need the form χ on the Grothendieck group to be symmetric. In particular, if $\mathcal{V}$ comes from a variety, the latter does not have to be a Calabi-Yau. In fact, our construction works with $\mathcal{V}$ being a DG enhancement of any smooth and proper scheme X, as opposed to the construction in [**13**] which is specific to the case where X is (a local model of) the minimal resolution of a Kleinian surface singularity. Finally, working with DG categories, we obtain a natural framework for working with complexes of operators, as is necessary when categorifying alternating sums which appear, for example, in the Frenkel–Kac construction [**17**, Chapter 7].

1.5. The additive construction

We now describe our construction of $\mathbf{H}_{\mathcal{V}}$ in more detail. We begin with the simpler additive construction.

In categorification, one often encounters the following diagram of categories and functors:

$$\mathcal{C} \underset{\mathsf{F}}{\overset{\mathsf{E}}{\rightleftarrows}} \mathcal{D} \tag{1.9}$$

Frequently, these functors are required to be biadjoint. For example, in Khovanov's Heisenberg category [**31**] the generating objects Q_+ and Q_- are biadjoint, while in the Cautis–Licata categorification [**13**] the 1-morphisms P_i and Q_i are biadjoint up to a shift.

The biadjointness assumption can be a powerful tool, but it can also be very restrictive. For example, in Krug's action of a Heisenberg algebra on derived categories of symmetric quotient stacks [**33**] the functors $Q_\beta^{(n)}$ are only *right* adjoint to $P_\beta^{(n)}$.

Inspired by [**5**], we use Serre functors to overcome this. In (1.9), if E is the left adjoint of F and $S_{\mathcal{C}}$ and $S_{\mathcal{D}}$ are Serre functors on $\mathcal{C}$ and $\mathcal{D}$, then $S_{\mathcal{D}}\mathsf{E}S_{\mathcal{C}}^{-1}$ is the right adjoint of F. We use this to relax Khovanov's biadjunction condition for our categorification.

Thus, let $\mathcal{V}$ be a Hom-finite graded $\Bbbk$-linear category endowed with a Serre functor S. To construct the *additive Heisenberg category* $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$ we first construct a simpler 2-category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$ whose objects are the integers $N \in \mathbb{Z}$ and whose 1-morphisms are freely generated by

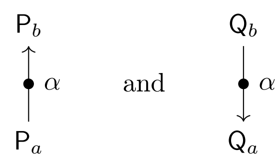
$$\mathsf{P}_a \colon N \to N+1 \quad \text{and } \mathsf{Q}_a \colon N \to N-1$$

for each $a \in \mathcal{V}$ and $N \in \mathbb{Z}$. The identity 1-morphism of each N is denoted by $\mathbb{1}$.

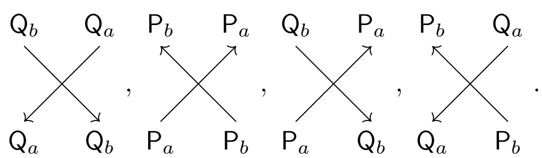
The 2-morphisms of $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$ we define below ensure that P_a is the left adjoint of Q_a. Motivated by the above, we also ensure that P_{Sa} is the right adjoint of Q_a. Thus, we have

$$\mathsf{P}_a \dashv \mathsf{Q}_a \dashv \mathsf{P}_{Sa}. \tag{1.10}$$

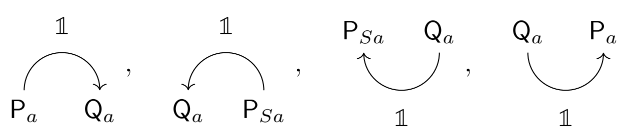
We define the 2-morphisms by planar string diagrams similar to those of Khovanov [**31**]; an example is (1.7) above. Similarly to the work of Cautis and Licata [**13**] our strings are decorated by morphisms of $\mathcal{V}$. For every $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a, b)$ we have vertical oriented strings



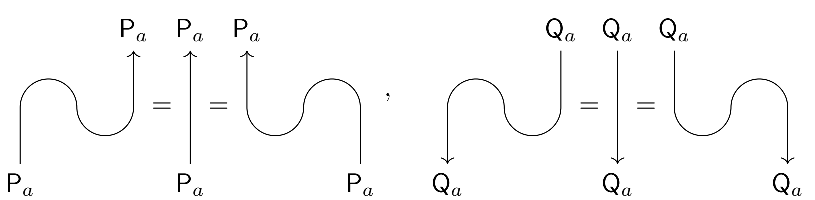
As a shorthand, the strings decorated by the identity morphism are drawn unadorned. Strings are also allowed to cross and bend. Thus, for any $a, b \in \mathcal{V}$ we have the crossings



The cups and caps that appear at the bends need to take into account the Serre functor. For any $a \in \mathcal{V}$ we have the following cups and caps

(1.11)


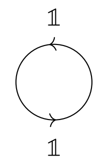
As in [**31**], the planar diagrams generated by the above are subject to a number of relations. The full list is in Chapter 3. For example, for any $a \in \mathcal{V}$ we have the straightening relations



ensuring the 2-categorical adjunctions (1.10) with units and counits given by the caps and cups (1.11).

The relations on the planar string diagrams take into account the Serre functor. The details are in Chapter 3, while here we give one representative example. In Khovanov's category one has the "biadjunction" or "bubble" relation specifying that the diagram composition

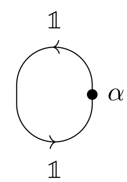
$$\mathbb{1} \xrightarrow{\text{unit}} \mathsf{QP} \xrightarrow{\text{counit}} \mathbb{1}, \qquad \text{pictorially}$$



is the identity. Here we set $\mathsf{Q} = Q_-$ and $\mathsf{P} = Q_+$ in the notation of [**31**], and the first map is the unit of (P, Q)-adjunction, while the second map is the counit of (Q, P)-adjunction.

In the absence of biadjunction, the above cannot possibly hold. Instead, we demand that for any $\alpha \in \operatorname{Hom}_{\mathcal{V}}(a, Sa)$ the composition

$$\mathbb{1} \xrightarrow{\text{unit}} \mathsf{Q}_a\mathsf{P}_a \xrightarrow{(\mathrm{id}_{\mathsf{Q}_a})\alpha} \mathsf{Q}_a\mathsf{P}_{Sa} \xrightarrow{\text{counit}} \mathbb{1}, \qquad \text{pictorially}$$



is the multiplication by the *Serre trace* $\operatorname{Tr}(\alpha) \in \Bbbk$, defined in Section 2.1.

Finally, as in some previous works on the categorification of Heisenberg algebras, having constructed the smaller 2-category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$ 1-generated only by $\mathsf{P}_a = \mathsf{P}_a^{(1)}$ and $\mathsf{Q}_a = \mathsf{Q}_a^{(1)}$ for $a \in \mathcal{V}$, we define $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$ to be its idempotent completion. The remaining elements $\mathsf{P}_a^{(n)}$ and $\mathsf{Q}_a^{(n)}$ are then the direct summands of 1-compositions P_a^n and Q_a^n defined by the symmetrising idempotents of the action of the permutation group S_n by braid diagrams. Thus, for constructing a 2-representation of $\mathbf{H}_{\mathcal{V}}$ one only needs to specify the actions of P_a and Q_a.

In Section 3.5 we give such an action on the categorical version of the Fock space, consisting of the categories of S_N-equivariant objects in $\mathcal{V}^{\otimes N}$.

1.6. The DG construction

From the viewpoint of algebraic geometry, we want to work with a DG category $\mathcal{V}$ which Morita enhances the derived category of an algebraic variety X. This means that the compact derived category $\mathrm{D_c}(\mathcal{V})$ of DG modules over $\mathcal{V}$ is equivalent to the bounded derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ of coherent sheaves on X. This is different from the older notion of a (non-Morita) DG enhancement, which required $\mathcal{V}$ to have special properties (being pre-triangulated) and the triangulated category it enhanced was $\mathrm{H}^0(\mathcal{V})$. The two notions are connected: if $\mathcal{V}$ Morita enhances $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, then the *perfect hull* $\mathcal{H}perf\,\mathcal{V}$ enhances it in the usual sense. On triangulated level, the perfect hull corresponds to taking the Karoubi-completed triangulated hull. Thus, with Morita enhancements we can work with smaller DG categories which explicitly enhance only a small part of the triangulated category from which the rest can be generated by taking cones, shifts, and idempotent completions.

A nice example is provided by the symmetric quotient stacks. A naive symmetric power of a triangulated category is not triangulated. In [**21**] Kapranov and

Gantner took a pretriangulated category $\mathcal{A}$ and defined its completed n-th symmetrical power $\widehat{\mathcal{S}}^n\mathcal{A}$ which ensured that $\mathrm{H}^0(\widehat{\mathcal{S}}^n\mathcal{A})$ is the correct symmetric power of $\mathrm{H}^0(\mathcal{A})$. In §4.8 we give for any DG category $\mathcal{A}$ a simpler construction $\mathcal{S}^n\mathcal{A}$ which ensures that $\mathrm{D_c}(\mathcal{S}^n\mathcal{A})$ is the correct symmetric power of $\mathrm{D_c}(\mathcal{A})$. It is a categorification of the skew group algebra construction and its perfect hull coincides with the Kapranov-Gantner's $\widehat{\mathcal{S}}^n\mathcal{A}$ on the DG level (see Lemma 4.41). It is, in a sense, the smallest natural DG category which does this job. In particular, when $\mathcal{V}$ Morita enhances $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, $\mathcal{S}^N\mathcal{V}$ Morita enhances the symmetric quotient stack $[X^N/S_N]$.

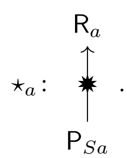
Let $\mathcal{V}$ be a smooth and proper DG category (see Chapter 4 for a review on DG categories). Then $\mathcal{H}perf\,\mathcal{V}$ always possesses a *homotopy Serre functor*, i.e. a quasi-autoequivalence S together with quasi-isomorphisms

$$\eta_{a,b}\colon\ \mathrm{Hom}_{\mathcal{V}}(a,b) \to \mathrm{Hom}_{\mathcal{V}}(b,Sa)^*,$$

natural in $a,b \in \mathcal{V}$ (see Section 4.7). In other words, S is only a Serre functor *up to homotopy.*

Thus the adjunction relation $\mathsf{Q}_a^{(n)} \dashv \mathsf{P}_{Sa}^{(n)}$ in the DG Heisenberg category $\mathbf{H}_{\mathcal{V}}$ needs to be homotopically weakened. One option would be to upgrade $\mathbf{H}_{\mathcal{V}}$ to an $(\infty,2)$-category and have the additional homotopical information come from the topology of string diagrams. However, at the moment the authors still find it difficult to construct $(\infty,2)$-categories by means of generators and relations. In this paper we take a different approach which stays entirely within the realm of DG categories.

Our main idea is to introduce three sets of generating objects P_a, Q_a and R_a, related by strict adjunctions $\mathsf{P}_a \dashv \mathsf{Q}_a$ and $\mathsf{Q}_a \dashv \mathsf{R}_a$. To relate the left and right adjoints of Q_a, we add for each $a \in \mathcal{V}$ the starred string 2-morphism



By the considerations above, all these $\star_a$ should be homotopy equivalences. To impose this in a consistent way, without having to specify the higher homotopies by hand, we take the Drinfeld quotient by the cone of $\star_a$. This makes $\star_a$ a homotopy equivalence, and thus makes each P_{Sa} a homotopy right adjoint of Q_a.

Thus, we first define a strict DG 2-category $\mathbf{H}'_{\mathcal{V}}$ with objects $N \in \mathbb{Z}$, 1-morphisms freely generated by P_a, Q_a and R_a, and 2-morphisms given by planar string diagrams similar to those in $\mathbf{H}^{\mathrm{add}\prime}_{\mathcal{V}}$ with the addition of the star-morphisms $\star_a\colon \mathsf{P}_{Sa} \to \mathsf{R}_a$. We then take the h-perfect hull $\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})$ to obtain a DG bicategory whose 1-morphism DG categories are pretriangulated and homotopy Karoubi complete. Finally, we define $\mathbf{H}_{\mathcal{V}}$ to be the Drinfeld quotient of $\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})$ by the two-sided ideal $I_{\mathcal{V}}$ generated by the cones of $\star_a$ and of another 2-relation we only want to hold up to homotopy. This is one of the subtler points of our construction: the original Drinfeld quotient construction **[14]** is very much incompatible with monoidal structures such as that of a 1-composition in a 2-category. However, this was already considered by Shoikhet **[44]** who refined Drinfeld's construction to obtain on it the structure of a *weak Leinster monoid*. We use this to define the notion of a *monoidal Drinfeld quotient* of a DG bicategory by a two-sided ideal of 1-morphisms. It has all the expected universal properties. The price is that

$\mathbf{H}_{\mathcal{V}}$ becomes a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory. In other words, its 1-composition is now given by quasi-functors: compositions of genuine DG functors with formal inverses of quasi-equivalences. However, the homotopy category of $\mathbf{H}_{\mathcal{V}}$ is a genuine 2-category whose 1-morphism categories are triangulated and Karoubi-complete. In particular, it recovers all the combinatorics of the additive setting.

In Chapter 7 we construct a categorical version of the Fock space for the DG setting. As noted in [**7**], the naive tensor product of categories does not behave well with respect to triangulated structures. In the DG enhanced setting this is solved by taking the h-perfect hull of the naive tensor product (often called the completed tensor product). This was one of our reasons to develop the machinery of Heisenberg categories on the level of DG categories.

We thus proceed in two steps again: first, we define a strict 2-functor $\Phi'_{\mathcal{V}}$ from $\mathbf{H}'_{\mathcal{V}}$ to the strict DG 2-category $\mathbf{dgModCat}$ of DG categories, DG functors between their module categories and natural transformations. The image of $\Phi'_{\mathcal{V}}$ is contained in the 1-full subcategory $\mathbf{F}'_{\mathcal{V}}$ whose objects are the symmetric powers $\mathcal{S}^N\mathcal{V}$. This concrete definition is at the heart of our categorical Fock space representation.

We next apply some abstract DG wizardry. We use the bimodule approximation 2-functor $\underline{\mathrm{Apx}}$ to approximate the 1-morphisms of $\mathbf{F}'_{\mathcal{V}}$ by DG bimodules. This yields a homotopy strong 2-functor from $\mathbf{H}'_{\mathcal{V}}$ into the bicategory $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ of enhanced triangulated categories. We next take perfect hulls and verify that on the homotopy level the resulting 2-functor $\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}}) \to \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ kills all 1-morphisms of $I_{\mathcal{V}}$ and thus descends to a homotopy strong 2-functor $\Phi_{\mathcal{V}} \colon \mathbf{H}_{\mathcal{V}} \to \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$. Its image is our categorical Fock space $\mathbf{F}_{\mathcal{V}}$.

1.7. Results on DG categories

To construct the DG Heisenberg algebra and its Fock space representation, we needed to develop several new results on DG categories. Most of these are 2-categorical analogues of common DG-categorical constructions. We hope that these results and techniques may have applications outside of our work. We thus summarise them here in the order in which we perceive them to be potentially useful to others. For the technical details, see the indicated sections.

In Section 4.6, we use Shoikhet's construction [**44**] to define a *monoidal Drinfeld quotient* $\mathbf{C}/\mathbf{I}$ of a DG bicategory $\mathbf{C}$ by a two-sided 1-morphism ideal $\mathbf{I}$. We want this to be a 2-category with the same objects as $\mathbf{C}$ whose 1-morphism categories are Drinfeld quotients of those of $\mathbf{C}$ by $\mathbf{I}$. The problem is to define the 1-composition, as the interchange law would force relations to exist between the contracting homotopies, which were freely introduced. Following Shoikhet [**44**], we define 1-composition by resolving tensor products of Drinfeld quotients of 1-morphism categories of $\mathbf{C}$ by a refined construction which admits a natural 1-composition functor. The resulting 1-composition is then a quasi-functor in the homotopy category $\mathbf{Ho}(\mathbf{dgCat})$ of DG categories. In Theorem 4.37 we prove that the resulting $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory $\mathbf{C}/\mathbf{I}$ has the expected universal property with respect to the 2-functors out of $\mathbf{C}$ which are null-homotopic on the 1-morphisms of $\mathbf{I}$.

In Chapter 4.4, we define the DG bicategory $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ of enhanced triangulated categories. It is where the main action of this paper takes place. Its homotopy category, the strict 1-triangulated 2-category $\mathbf{EnhCat}_{\mathrm{kc}}$ has been understood for a while [**51**],[**36**]. However, there are well-known technical difficulties in constructing a DG bicategory enhancing it. We propose two constructions which are both almost a DG bicategory. One uses the technology of bar-categories of modules [**2**]. The result is a homotopy unital DG bicategory. Its unitor morphisms are homotopy equivalences with canonical homotopy inverses which are genuine inverses on one side. This approach is more elegant and its structures are explicitly defined and thus easily computable. Alternatively, we use our new notion of the monoidal Drinfeld quotient to construct $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ as the quotient of the Morita 2-category of DG bimodules by acyclics. The result is a bicategory, but enriched over $\mathbf{Ho}(\mathbf{dgCat})$ and not $\mathbf{dgCat}$. This definition is simpler, not requiring familiarity with [**2**], but less explicit and less practical to compute with. Either construction works well for the purposes of this paper.

In Section 4.5, we define the *h-perfect hull* of a DG bicategory $\mathbf{C}$. It is a DG bicategory with the same objects as $\mathbf{C}$ whose 1-morphism categories are h-perfect hulls of those of $\mathbf{C}$.

In Section 4.3, we define the *bimodule approximation* 2-functor $\underline{\mathrm{Apx}}$ which approximates DG functors by DG bimodules. Some of these formalities are well-known to experts [**30**, Section 6.4], but it may be useful to have them written down.

In Section 4.7 we define the notion of a *homotopy Serre functor* and show that every smooth and proper DG category $\mathcal{V}$ admits one on $\mathcal{H}perf\,\mathcal{V}$. Again, this is well-known to experts, but the point is that the genuine Serre functor constructed on $\mathrm{H}^0(\mathcal{H}perf\,\mathcal{V})$ in [**43**] lifts together with all its natural morphisms to $\mathcal{H}perf\,\mathcal{V}$ itself.

1.8. Further questions and remarks

Next, we outline some further questions and related results that we believe to be interesting for future investigations.

Gal [**20**] showed that the structure of a Hopf category on a semisimple symmetric monoidal abelian category implies the existence of a categorical Heisenberg action in the sense of Khovanov. It would be interesting to see whether this construction can be generalised to obtain a category isomorphic to $\mathbf{H}_{\mathcal{V}}$ for any $\mathcal{V}$. Several examples of categorifications of algebraic structures seemingly related to ours carry actions of braid groups. It would also be interesting to see if there is a deeper relationship between our categorification, Hopf categories and braid group actions.

Extending the work of Grojnowski and Nakajima, Lehn [**34**] constructed Virasoro operators on the cohomology of Hilbert schemes of points of smooth projective surfaces. The present article is motivated in part by a desire to generalise this construction to the Heisenberg algebra action on derived categories of symmetric quotient stacks. Such operators should arise as convolutions of certain complexes of 2-morphisms on $\mathbf{H}_{\mathcal{V}}$. The desire to obtain a good framework for working with such complexes is one of the reasons we work with DG categories in this paper. We intend to return to this question in future work.

In a different direction, the BGG category $\mathcal{O}$ of prominence in representation theory has a Serre functor (see Example 2.2 and [**32**]). It would be enlightening to understand the associated Heisenberg category and its Fock space in detail.

Theorem C shows that it is interesting to consider when the morphism (1.8) is an isomorphism. Following [**10**], one way to understand surjectivity of this morphism seems to be via a suitable generalisation of degenerate affine Hecke algebras and their categorifications. This may also lead to the answers for the questions raised in [**13**, Section 10.3].

1.9. Structure of the paper

The structure of the paper is as follows. In Chapter 2 we give preliminaries relevant to both the additive and the DG settings. We recall the concept of Serre functors and introduce the idempotent modification of Heisenberg algebras which we categorify. In Chapter 3 we construct the additive Heisenberg 2-category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$ and investigate its properties.

In Chapter 4 we give preliminaries required for the DG setting. We encourage the reader uninterested in DG technicalities to skip this section and refer back to it when needed.

In Chapter 5 we construct the Heisenberg 2-category $\mathbf{H}_{\mathcal{V}}$ in the DG setting. In Chapter 6, we investigate the structure of $\mathbf{H}_{\mathcal{V}}$ and, in particular, deduce the categorical version of the Heisenberg commutation relations and prove Theorem A. In Chapter 7 we construct the categorical Fock space representation $\mathbf{F}_{\mathcal{V}}$ and the 2-functor $\mathbf{H}_{\mathcal{V}} \to \mathbf{F}_{\mathcal{V}}$, and prove Theorem B. We note that the proof of Theorem A depends on Theorem B. Finally, in Chapter 8 we investigate the properties of $\mathbf{F}_{\mathcal{V}}$ and prove Theorem C.

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1.11. Notation

Throughout the paper, $\Bbbk$ is an algebraically closed field of characteristic 0. All categories and functors are assumed to be $\Bbbk$-linear. By a variety we mean an integral, separated scheme of finite type over $\Bbbk$. All of our tensor products are over $\Bbbk$, unless indicated otherwise. The tensor product of two complexes over $\Bbbk$ is understood as the total complex of the double complex containing the tensor products of the terms.

We always denote 2-categories in bold (such as $\mathbf{H}_{\mathcal{V}}$ or **dgCat**) and 1-categories in calligraphic letters (such as $\mathcal{V}$). Objects in a 1-category are denoted by lowercase Latin letters, while morphisms are denoted in lowercase Greek letters.

CHAPTER 2

Preliminaries

2.1. Serre functors

Let $\mathcal{A}$ be a graded $\Bbbk$-linear category with finite-dimensional Hom-spaces. A "graded $\Bbbk$-linear" category means a category enriched in graded vector spaces.

A *Serre functor* on $\mathcal{A}$ is a degree zero autoequivalence S of $\mathcal{A}$ equipped with isomorphisms

$$\eta_{a,b}\colon \operatorname{Hom}_{\mathcal{A}}(a,b) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(b,Sa)^*,$$

natural in $a,b \in \mathcal{A}$ [**5**]. If a Serre functor exists, then it is unique up to an isomorphism [**8**, Proposition 1.5].

Example 2.1. If X is a smooth and proper variety over $\Bbbk$, then $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ admits a Serre functor $S = (-) \otimes_X \omega_X[\dim X]$, where ω_X is the canonical line bundle of X.

Example 2.2. Let G be a reductive algebraic group over $\Bbbk$, with Borel subgroup B. Then the category of Schubert-constructible sheaves on the flag variety G/B has a Serre functor given by the square of the intertwining operator associated to the longest element of the Weyl group [**3**]. We note that by Beilinson–Bernstein localisation and the Riemann–Hilbert equivalence this category is the same as the principal block of the Beilinson–Gelfand–Gelfand category $\mathcal{O}$ associated to the Lie algebra of G. The Serre functors for similar categories of importance to representation theory are further explored in [**19**].

Remark 2.3. Serre functors are particularly useful for producing adjoint functors. If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between $\Bbbk$-linear categories with Serre functors $S_{\mathcal{C}}$ and $S_{\mathcal{D}}$ respectively, then

$$F^L \cong S_{\mathcal{C}}^{-1} F^R S_{\mathcal{D}},$$

where F^R and F^L are the right and left adjoint of F. Indeed, for $x \in \mathcal{C}$ and $y \in \mathcal{D}$ one has

$$\operatorname{Hom}_{\mathcal{D}}(y,\, Fx) \cong \operatorname{Hom}_{\mathcal{D}}(Fx,\, S_{\mathcal{D}}y)^* \cong \operatorname{Hom}_{\mathcal{C}}(x,\, F^R S_{\mathcal{D}}y)^* \cong \operatorname{Hom}_{\mathcal{C}}(S_{\mathcal{C}}^{-1} F^R S_{\mathcal{D}}y,\, x).$$

Our usage of the Serre functor in the definition of the Heisenberg category is closely related to this observation.

The Serre functor S induces a *Serre trace* map

$$\operatorname{Tr}\colon \operatorname{Hom}_{\mathcal{A}}(a,\, Sa) \to \Bbbk, \qquad \alpha \mapsto \eta_{a,a}(\mathrm{id}_a)(\alpha). \tag{2.1}$$

Proposition 2.4. *Let $\mathcal{C}$ be a Hom-finite $\Bbbk$-linear category which admits a Serre functor S. For any $a, b \in \mathcal{C}$ and any $\alpha \in \operatorname{Hom}_{\mathcal{C}}(a,\, b)$, $\beta \in \operatorname{Hom}_{\mathcal{C}}(b,\, Sa)$ we have*

$$\operatorname{Tr}(\beta \circ \alpha) = (-1)^{\deg\alpha \deg\beta} \operatorname{Tr}(S\alpha \circ \beta).$$

Proof. We note that if $\mathcal{C}$ is a graded category, then composition of two morphisms α and β in $\mathcal{C}^{\mathrm{opp}}$ is twisted by $(-1)^{\deg\alpha\deg\beta}$. Thus acting on the first argument of the bifunctor $\mathrm{Hom}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{opp}}\times\mathcal{C}\to \text{gr-Vect}_{\Bbbk}$ involves a sign twist. Naturality of η therefore implies that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{A}}(b,b) & \xrightarrow[\sim]{\eta} & \mathrm{Hom}_{\mathcal{A}}(b,Sb)^* \\
{\scriptstyle (-1)^{\deg(-)\deg(\alpha)}(-)\circ\alpha}\downarrow & & \downarrow{\scriptstyle f(-)\mapsto(-1)^{\deg(f)\deg(\alpha)}f(S\alpha\circ(-))} \\
\mathrm{Hom}_{\mathcal{A}}(a,b) & \xrightarrow[\sim]{\eta} & \mathrm{Hom}_{\mathcal{A}}(b,Sa)^* \\
{\scriptstyle \alpha\circ(-)}\uparrow & & \uparrow{\scriptstyle f(-)\mapsto(-1)^{(\deg(f)+\deg(-))\deg(\alpha)}f((-)\circ\alpha)} \\
\mathrm{Hom}_{\mathcal{A}}(a,a) & \xrightarrow[\sim]{\eta} & \mathrm{Hom}_{\mathcal{A}}(a,Sa)^*.
\end{array}$$

commutes.

Chasing id_b through the upper square and id_a through the lower square yields

$$\mathrm{Tr}(S\alpha\circ -)=\eta(\alpha)(-)=(-1)^{\deg\alpha\deg(-)}\,\mathrm{Tr}(-\circ\alpha),$$

whence the desired assertion follows. □

2.2. Heisenberg algebras

Recall that a lattice is a free $\mathbb{Z}$-module M of finite rank equipped with a bilinear form

$$\chi\colon M\times M\to\mathbb{Z},\quad v,w\mapsto\langle v,w\rangle_\chi.$$

We do not require the form χ to be symmetric or antisymmetric; to the knowledge of the authors no treatment of Heisenberg algebras has been this general. If the bilinear form χ on M is degenerate, then the Heisenberg algebra defined as below has a non-trivial centre. Thus it is common to assume that χ is non-degenerate and we do so from now on.

Let (M,χ) be a lattice. As a preliminary definition of the Heisenberg algebra we let $\underline{H}_M := \underline{H}_{(M,\chi)}$ to be the unital $\Bbbk$-algebra with generators $p_a^{(n)}$, $q_a^{(n)}$ for $a\in M$ and integers $n\geq 0$ modulo the following relations for all $a,b\in M$ and $n,m\geq 0$:

$$p_a^{(0)}=1=q_a^{(0)}, \tag{2.2}$$

$$p_{a+b}^{(n)}=\sum_{k=0}^{n}p_a^{(k)}p_b^{(n-k)}\quad\text{and}\quad q_{a+b}^{(n)}=\sum_{k=0}^{n}q_a^{(k)}q_b^{(n-k)}, \tag{2.3}$$

$$p_a^{(n)}p_b^{(m)}=p_b^{(m)}p_a^{(n)}\quad\text{and}\quad q_a^{(n)}q_b^{(m)}=q_b^{(m)}q_a^{(n)}, \tag{2.4}$$

$$q_a^{(n)}p_b^{(m)}=\sum_{k=0}^{\min(m,n)}s^k\langle a,b\rangle_\chi\,p_b^{(m-k)}q_a^{(n-k)}. \tag{2.5}$$

Here for any pair of integers $k\geq 0$ and r we set

$$s^k r:=\binom{r+k-1}{k}=\frac{1}{k!}(r+k-1)(r+k-2)\cdots(r+1)r,$$

which for positive r coincides with the dimension of the k-th symmetric power of a vector space of dimension r, that is,

$$s^k r=\dim(S^k(\mathbb{C}^r)),$$

and for negative r analogously

$$s^k r = (-1)^k \dim(\Lambda^k(\mathbb{C}^{-r})).$$

We use the convention that $p_a^{(n)} = q_b^{(n)} = 0$ for $n < 0$.

Let $r = \operatorname{rank} M$ and fix an identification $M \cong \mathbb{Z}^r$. Let S and T be integral $r \times r$ matrices which are invertible over $\mathbb{Z}$. In particular, both S and T are unimodular. Moreover, the form

$$\langle a, b\rangle_{S\chi T} := \langle S^t a, Tb\rangle_\chi \tag{2.6}$$

gives again a new pairing on M. It is non-degenerate if and only if χ is non-degenerate. If X denotes the matrix of χ in the chosen basis of M, then the matrix of $S\chi T$ is SXT. To the knowledge of the authors, the following observation has not yet appeared in the literature.

Lemma 2.5. *Let S and T be as above. The algebras $\underline{H}_{(M,\chi)}$ and $\underline{H}_{(M,S\chi T)}$ are isomorphic.*

Proof. Define a map $\underline{H}_{(M,S\chi T)} \to \underline{H}_{(M,\chi)}$ on generators by

$$q_a^{(n)} \mapsto q_{S^t a}^{(n)} \qquad \text{and} \qquad p_a^{(n)} \mapsto p_{Ta}^{(n)}. \tag{2.7}$$

As S and T are invertible, it is a bijection on the sets of generators. It remains to show that it respects the relations. This is immediate for relations (2.2)–(2.4), while for relation (2.5) it follows from (2.6). □

Corollary 2.6. *The Heisenberg algebra on every lattice is isomorphic to one which is induced by a symmetric (in fact, a diagonal) form.*

Proof. The Smith normal form of χ (or more precisely of its matrix X) provides matrices S and T, such that $S\chi T$ (in fact, SXT) is diagonal. □

Remark 2.7. The result above says that every Heisenberg algebra arises as the Heisenberg algebra of a lattice with a symmetric pairing. In the geometrical context, our lattice is the numerical Grothendieck group of an algebraic variety and our pairing is the Euler pairing. Drawing loose parallels, it is tempting to interpret the result above as saying that Heisenberg algebra is an intrinsically Calabi–Yau construction. It is certainly the case in the original constructions by Khovanov [**31**] who works on a point, by Cautis and Licata [**13**] who work on a minimal resolution of an ADE singularity, and by Grojnowski and Nakajima [**24, 37**] who make use of the Poincaré duality on cohomology.

The authors hope to revisit this issue in a future work which would extend our categorification from Heisenberg agebras to the associated vertex algebras.

Remark 2.8. When χ is symmetric, the matrices S and T can be chosen to be equal. Hence, they represent a base change on the underlying lattice M. Moreover, in this case there is another common set of generators of the Heisenberg algebra. It is given by polynomials (possibly with constant term) on the symbols $a_b(n)$ for $n \in \mathbb{Z} \setminus \{0\}$, $b \in M$. The set of relations between these is given by

$$\big[a_b(m), a_c(n)\big] = \delta_{m,-n} m \langle b, c\rangle_\chi.$$

The proof that these (in the symmetric case) define the same algebra is given for example in [**33**, Lemma 1.2]. The advantage of using the presentation (2.2)–(2.5) is that it also makes sense when χ is not symmetric. Hence, it is more natural in our context.

2.2.1. Idempotent modification. In this paper, we do not work with the Heisenberg algebra $\underline{H}_M$ itself, but with its idempotent modification H_M. We define it as follows.

Recall that a unital $\Bbbk$-algebra R is the same as a $\Bbbk$-linear category $\mathcal{C}$ with a single object whose endomorphism space is R. Similarly, a unital algebra R with a choice of a decomposition $1_R = \sum_1^n 1_i$ of its unit into a finite sum of orthogonal idempotents can be viewed as a $\Bbbk$-linear category $\mathcal{C}$ whose objects are $\{1, \dots, n\}$ and whose Hom-spaces are given by $\mathrm{Hom}_{\mathcal{C}}(i,j) = 1_j R 1_i$. Conversely, we can recover R from $\mathcal{C}$ as a direct sum of its Hom-spaces.

We would like to decompose the unit of $\underline{H}_M$ into an infinite sum of idempotents $\sum_{i\in\mathbb{Z}} 1_i$. This is not possible directly, as infinite sums of elements are not well-defined. However, the categorical analogy above suggests the following construction.

Introduce a $\mathbb{Z}$-grading on $\underline{H}_M$ by setting $\deg p_a^{(m)} = m$ and $\deg q_a^{(n)} = -n$ for all $n, m \in \mathbb{Z}$ and $a \in M$. Let $\mathcal{C}_M$ be a category whose object set is $\mathbb{Z}$ and whose Hom-space $\mathrm{Hom}_{\mathcal{C}_M}(i,j)$ is the degree $j-i$ part of $\underline{H}_M$. The identity element 1_i in each $\mathrm{Hom}_{\mathcal{C}_M}(i,i)$ is the corresponding copy of the unit 1 of $\underline{H}_M$. The composition is given by multiplication in $\underline{H}_M$. For any element $x \in \underline{H}_M$ of degree $j-i$ we write $1_j x$, $1_j x 1_i$ or $x 1_i$ to differentiate the copy of x in $\mathrm{Hom}_{\mathcal{C}_M}(i,j)$ from its counterparts in any other $\mathrm{Hom}_{\mathcal{C}_M}(l, l+j-i)$.

Now let H_M be the direct sum of Hom-spaces of $\mathcal{C}_M$:

$$H_M \coloneqq \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}_M}(i,j).$$

This is a non-unital algebra as it does not contain the infinite sum $\sum_{i\in\mathbb{Z}} 1_i$. Instead, it has a collection of orthogonal idempotents $\{1_i\}_{i\in\mathbb{Z}}$ and each defining relation (2.2)–(2.5) of the unital algebra $\underline{H}_M$ gives rise, for each $i \in \mathbb{Z}$, to a relation in H_M. Namely, take the original relation and add the idempotent 1_k at the end of each expression. For example,

$$p_a^{(n)} p_b^{(m)} 1_i = p_b^{(m)} p_a^{(n)} 1_i, \quad a, b \in M,\ n, m \in \mathbb{N},\ i \in \mathbb{Z}.$$

Note that elements $p_a^{(m)}$ and $q_a^{(n)}$ themselves do not exist in H_M anymore, as they should correspond to infinite sums $\sum_{i\in\mathbb{Z}} p_a^{(m)} 1_i$ and $\sum_{i\in\mathbb{Z}} q_a^{(n)} 1_i$.

We have a canonical projection $H_M \to \underline{H}_M$ given by sending each idempotent 1_i to the unit $1_{\underline{H}_M}$. A representation of the category $\mathcal{C}_M$ into the category of vector spaces is the same as a graded module over H_M. Moreover, any graded module over $\underline{H}_M$ induces a representation of H_M via restriction of scalars.

2.2.2. The transposed generators. Fix $a \in M$ and let z be a formal variable. Let

$$\sum_{n\geq 0} p_a^{(n)} z^n \quad \text{and} \quad \sum_{n\geq 0} q_a^{(n)} z^n$$

be the generating series of the p, resp. q elements associated with a. Define a new set of elements $p_a^{(1^n)}$ and $q_a^{(1^n)}$, $n \in \mathbb{Z}_{>0}$ so that the generating series

$$\sum_{n\geq 0} (-1)^n p_a^{(1^n)} z^n \quad \text{and} \quad \sum_{n\geq 0} (-1)^n q_a^{(1^n)} z^n$$

are the inverses of those of $p^{(n)}$ and $q^{(n)}$ respectively:

$$\left(\sum_{n\geq 0} p_a^{(n)} z^n\right)\left(\sum_{n\geq 0} (-1)^n p_a^{(1^n)} z^n\right) = 1$$

and

$$\left(\sum_{n\geq 0} q_a^{(n)} z^n\right)\left(\sum_{n\geq 0} (-1)^n q_a^{(1^n)} z^n\right) = 1.$$

Compare [**13**, Section 2.2.2] and [**33**, Section 3.2]. One can show that the relations among these generators are exactly the same as those between the $p_a^{(n)}$ and $q_a^{(n)}$, just replace (n) by (1^n) everywhere. In particular, they also give a set of generators of $\underline{H}_M$. Additionally, for all $a, b \in M$ one has the following relations:

$$p_a^{(n)} p_b^{(1^n)} = p_b^{(1^n)} p_a^{(n)}, \quad q_a^{(n)} q_b^{(1^n)} = q_b^{(1^n)} q_a^{(n)}$$
$$q_a^{(1^n)} p_b^{(m)} = \textstyle\sum_{k=0}^{\min(m,n)} s^k\left(-\langle a, b\rangle_\chi\right) p_b^{(n-k)} q_a^{(1^{m-k})}.$$

2.2.3. The Fock space. Let $H_M^- \subset H_M$ denote the subalgebra generated by the set

$$\left\{q_a^{(n)} 1_k : a \in M,\, k \leq 0,\, n \geq 0\right\}.$$

Let triv_0 denote the trivial representation of H_M^-, where 1_0 acts as identity and 1_k acts by zero for $k < 0$. The Fock space representation of the Heisenberg algebra H_M is defined as the induced representation

$$F_M = \mathrm{Ind}_{H_M^-}^{H_M}(\mathrm{triv}_0) \cong H_M \otimes_{H_M^-} \Bbbk.$$

We note that in F_M one has $1_k \otimes 1 = 1_k \otimes (1_0 \cdot 1) = 1_k 1_0 \otimes 1 = 0$ for all $k \neq 0$. It follows that F_M is generated by elements $p_a^{(n)} 1_0$ for $a \in M$ and $n \geq 0$. The $\mathbb{Z}$-grading on $\underline{H}_M$ induces a grading on F_M where the degree k part is canonically isomorphic to

$$F_M^k \cong \bigoplus_{k_1+2k_2+\cdots=k} \bigotimes_i \mathrm{Sym}^{k_i}(M \otimes_{\mathbb{Z}} \Bbbk). \tag{2.8}$$

The idempotent $1_k \in H_M$ acts by projection onto F_M^k. Alternatively, the Fock space can be described as $F_M = H_M / I$ where I is the left ideal generated by the operators 1_k for $k \neq 0$ and $q_a^{(n)} 1_k$ for $k = 0$ and $n > 0$.

For χ non-degenerate, the Fock space is an irreducible and faithful representation of H_M with highest weight vector 1. If χ is of rank 1, irreducibility and faithfulness follows from the description of the Fock space representation as differential operators on an infinite polynomial algebra [**16**, Section 2]. As the form can be chosen to be diagonal, the higher rank case follows by taking a direct sum; the Fock space of the Heisenberg algebra of a direct sum of lattices is the tensor product of the Fock spaces of the Heisenberg algebras of the summands. Hence the representation can be described as differential operators on a polynomial algebra.

The next claim follows from the definition and irreducibility of the Fock space.

LEMMA 2.9. *Let $H_M \to \mathrm{End}(V)$ be a representation and let $v \in V$ be an element annihilated by $H_M^- \setminus \{1_0\}$ which is invariant under 1_0. Then the map $1 \mapsto v$ induces an embedding $F_M \to V$ of H_M-representations.*

CHAPTER 3

The Additive Heisenberg 2-category

In this section, we fix a Hom-finite graded $\Bbbk$-linear category $\mathcal{V}$ which is closed under shifts and has a Serre functor S. We then define a 2-category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$, the *(additive) Heisenberg category* of $\mathcal{V}$. We present the results in this section for graded categories for comparison with the homotopy category of the dg version in Chapter 5. Any $\Bbbk$-linear category can be seen as graded $\Bbbk$-linear by viewing the Hom-spaces as placed in degree 0. In such case all sign rules in this section can be ignored.

The category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$ is the Karoubi completion of a simpler 2-category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$ which we set up in the following first two subsections. This additive version of the Heisenberg category is less powerful than the DG version constructed in Chapter 5. We include it in the paper as it might be of wider interest and because the similarities and differences to the earlier constructions are more readily apparent in the purely $\Bbbk$-linear setting.

In our constructions, we want to work with objects of the form $a \otimes V$ where $a \in \mathcal{V}$ and $V \in \mathcal{G}rVect^{\mathrm{fin}}$, the category of finite-dimensional graded vector spaces. By this we mean a direct sum of $\dim V$ shifted copies of a indexed by a choice of basis of V. The maps between two such objects $a \otimes V$ and $b \otimes W$ then correspond to matrices with values in $\mathrm{Hom}_{\mathcal{V}}(a, b)$.

To do this without having to choose a basis, we replace $\mathcal{V}$ by the category $\mathcal{V} \otimes_k \mathcal{G}rVect^{\mathrm{fin}}$ which is (non-canonically) equivalent to $\mathcal{V}$. The equivalence is defined by choosing a homogeneous basis $\{e_1, \dots, e_n\}$ for every $V \in \mathcal{G}rVect^{\mathrm{fin}}$ and setting

$$(a, V) \mapsto \bigoplus_{e_i \in \{e_1, \dots, e_n\}} a[\deg(e_i)] \qquad a \in \mathcal{V}, V \in \mathcal{G}rVect^{\mathrm{fin}}$$

$$\alpha \otimes \beta \mapsto \sum \beta_{ij} \left(a[\deg(e_j)] \xrightarrow{\alpha} b[\deg(f_i)] \right) \qquad \alpha \in \mathrm{Hom}_{\mathcal{V}}(a, b), \beta \in \mathrm{Hom}(V, W)$$

where (β_{ij}) is the matrix of β with respect to the chosen bases.

The inverse equivalence is given by

$$a \mapsto (a, \Bbbk) \qquad a \in \mathcal{V}$$

$$\alpha \mapsto \alpha \otimes \mathrm{id} \qquad \alpha \in \mathrm{Hom}_{\mathcal{V}}(a, b).$$

3.1. The category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$: generators

We now define a (strict) 2-category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$. The objects of $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$ are the integers $N \in \mathbb{Z}$.

The 1-morphism categories are additive graded $\Bbbk$-linear categories whose 1-morphisms are freely generated under 1-composition by symbols

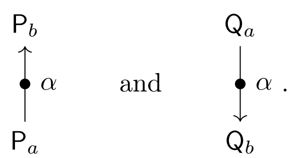
$$\mathsf{P}_a\colon N \to N+1 \quad \text{and} \quad \mathsf{Q}_a\colon N+1 \to N$$

for each $a \in \mathcal{V}$ and $N \in \mathbb{Z}$. Thus the objects of $\mathrm{Hom}_{\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}}(N,\, N')$ are direct sums of finite strings generated by the symbols P_a and Q_a with $a \in \mathcal{V}$, such that the difference of the number of P's and the number of Q's in each summand is $N' - N$. The identity 1-morphism of any $N \in \mathbb{Z}$ is denoted by $\mathbb{1}$.

Strictly speaking, one should distinguish between 1-morphisms with different sources in the notation, i.e. write $\mathsf{P}_a \mathbb{1}_N$ and $\mathbb{1}_N \mathsf{Q}_a$. However, we will have $\mathrm{Hom}_{\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}}(N,\, N') = \mathrm{Hom}_{\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}}(N+i,\, N'+i)$ for each integer i, and do not distinguish these in our notation.

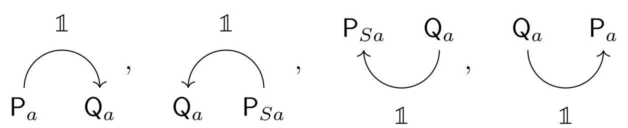
The 2-morphisms between a pair of 1-morphisms form a $\Bbbk$-vector space. These vector spaces are freely generated by a number of generators listed below, subject to the axioms of a strict 2-category as well as certain relations which we detail in the next subsection. We usually represent these 2-morphisms as planar diagrams. This requires certain sign rules, see Remark 3.1 below. The diagrams are read bottom to top, i.e. the source of a given 2-morphism lies on the lower boundary, while the target lies on the upper boundary.

The 2-morphism spaces are generated by three types of symbols. Firstly, for every $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a, b)$ there are arrows

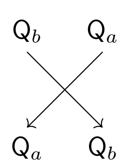


These 2-morphisms are homogeneous of degree $|\alpha|$. The remaining generators listed below are all of degree 0. By convention a strand without a dot is the same as one marked with the identity morphism. Any such unmarked strand is an identity 2-morphism in $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$. The identity 2-morphisms of the identity 1-morphism $\mathbb{1}$ are denoted by blank space.

Secondly, for any object $a \in \mathcal{V}$ there are cups and caps



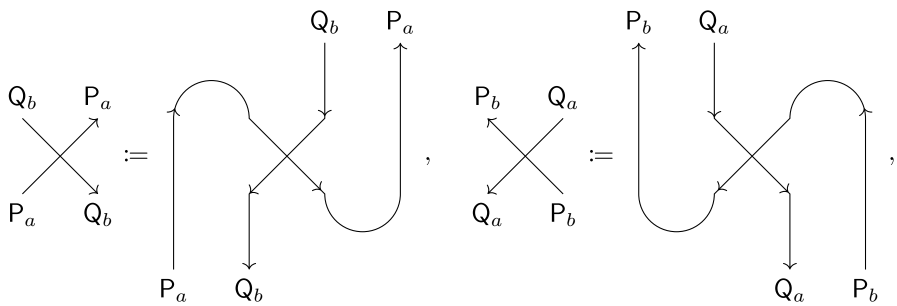
Thirdly, for any pair of objects $a,\, b \in \mathcal{V}$ there is a crossing of two downward[1] strands:



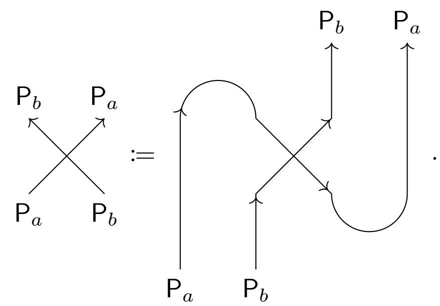
[1] We use the downward crossing rather than the upward crossing as a basic generator since in the DG version of the Heisenberg category described in Chapter 5 this will lead to a more symmetric presentation.

For convenience, we define three further types of strand crossings from this basic one by composition with cups and caps:

(3.1)



(3.2)

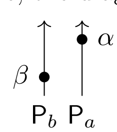


Remark 3.1. We draw compositions of basic 2-morphisms as planar diagrams, as in (3.1)-(3.2). In the ungraded case, the interchange law of 2-categories guarantees that such diagrams can be read without ambiguity.

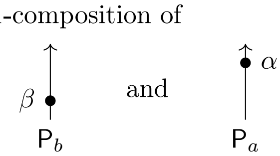
However, the interchange law for graded 2-categories includes a sign:

$$(\alpha \circ_1 \beta) \circ_2 (\gamma \circ_1 \delta) = (-1)^{|\beta||\gamma|}(\alpha \circ_2 \gamma) \circ_1 (\beta \circ_2 \delta), \tag{3.3}$$

where we write $\circ_1$ and $\circ_2$ for the 1- and 2-composition operations respectively. This can lead to ambiguities. For example, the diagram



could be read either as the 1-composition of



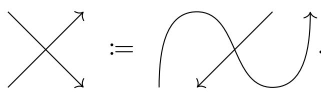
or the 2-composition of

These differ by a factor of $(-1)^{|\alpha||\beta|}$.

We impose the latter convention. Thus to read a diagram, one first slices it into lines containing no 2-composition of basic 2-morphisms and no dots at different heights. Every such line is a 1-composition of basic 2-morphisms, and the overall diagram is then the 2-composition of these 1-compositions.

With this convention, a diagram with two or more dots at the same height represents the same 2-morphism as the diagram with the rightmost of these dots moved a small distance downwards. Graphically, 1-composition corresponds to placing diagrams side-by-side and 2-composition corresponds to stacking diagrams on top of each other.

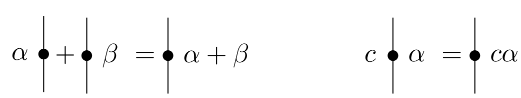
Remark 3.2. When the domain or target of a diagram is irrelevant or evident from the context, we may omit the labels. This is the case usually with the empty string occurring as the target of caps and the domain of cups. We also usually smooth out the strings in the diagram. For example, we may draw the left definition of (3.1) more succinctly as



3.2. The category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$: relations between 2-morphisms

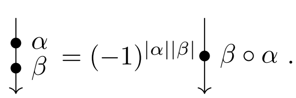
In Section 3.1 we gave a list of generating symbols. The 2-morphisms in $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$ are 1- and 2-compositions of these symbols, subject to the following list of relations. As a shorthand, a relation specified by an unoriented diagram holds for all permissible orientations of this diagram.

First, we impose the linearity relations



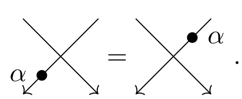
for any $\alpha, \beta \in \mathrm{Hom}(a, b)$ and any scalar $c \in \Bbbk$ for any compatible orientation of the strings.

Neighboring dots along a downward string can merge with a sign twist:

(3.4)


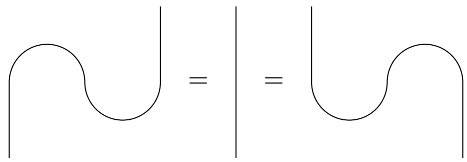
Dots may "slide" through caps and downwards crossings as follows:

(3.5)

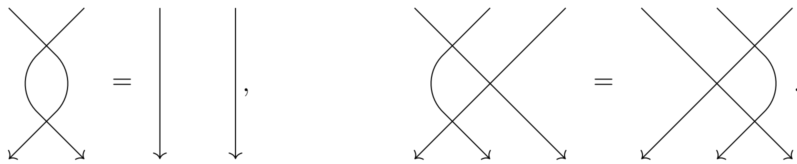

(3.6)


Note that when drawing diagrams, dots need to keep their relative heights when doing these operations in order to avoid accidentally introducing signs (cf. Lemma 3.3 below).

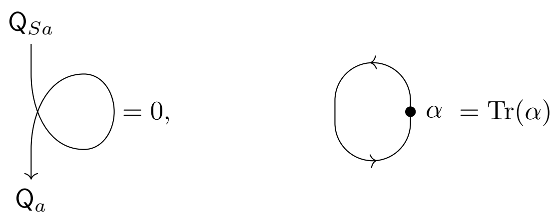
Next, there are two sets of local relations for unmarked strings: the *adjunction relations*

(3.7) 

and the *symmetric group* relations on downward strands

(3.8) 

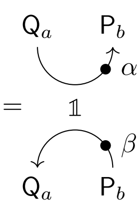
Further, for any $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a, Sa)$ and with Tr being the Serre trace (2.1) we have:

(3.9) 

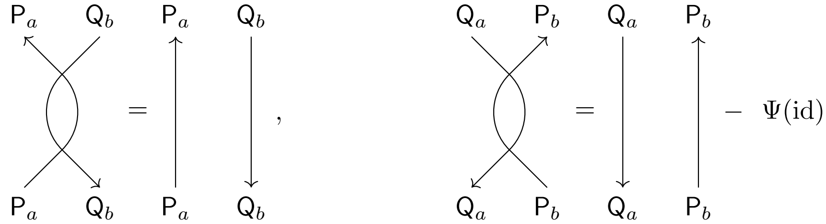
Finally, we have relations for crossings of opposite oriented strands. Consider the map

$$\Psi\colon \mathrm{Hom}_{\mathcal{V}}(a,b) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{V}}(a,b)^* \to \mathrm{Hom}(\mathsf{Q}_a\mathsf{P}_b,\, \mathsf{Q}_a\mathsf{P}_b)$$

sending $\alpha \otimes \beta \in \mathrm{Hom}(a,b) \otimes_{\Bbbk} \mathrm{Hom}(a,b)^* \cong \mathrm{Hom}(a,b) \otimes_{\Bbbk} \mathrm{Hom}(b, Sa)$ to

$\Psi(\alpha \otimes \beta) =$  .

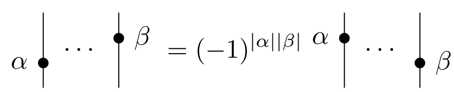
Consider $\mathrm{id} \in \mathrm{End}_{\Bbbk}\big(\mathrm{Hom}(a,b)\big) \cong \mathrm{Hom}(a,b) \otimes_{\Bbbk} \mathrm{Hom}(a,b)^*$. The final two relations are

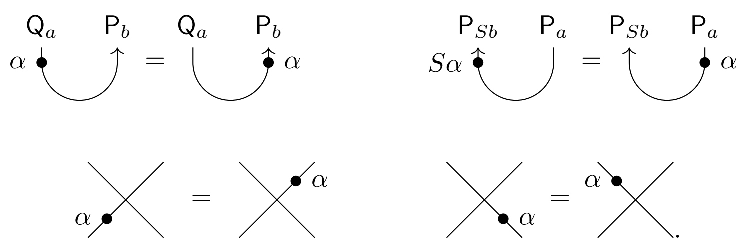
(3.10) 

3.3. Remarks on the 2-morphism relations in $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$

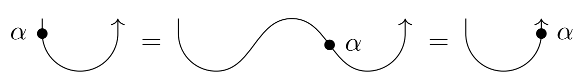
In order to reduce the number of relations necessary to verify when defining a representation of the Heisenberg category, we have chosen to keep the number of generators and relations on the definition of $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$ small. We now note some of their consequences. One such consequence is that essentially we can homotopy deform string diagrams. This is made precise in the following sequence of lemmas.

LEMMA 3.3. *Dots may freely "slide along" strands as well as through cups, caps and all types of crossings, picking up a sign when sliding past each other. That is, one has the following additional relations:*

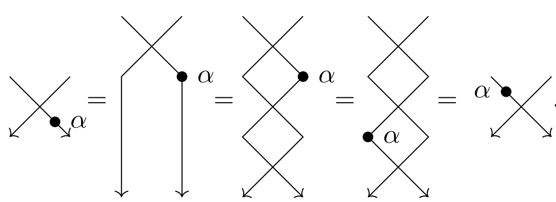




PROOF. The first relation is simply a graphical depiction of the interchange law in graded 2-categories. The relations in the second line follow from those in (3.5) by applying (3.7):

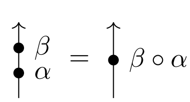


Relations (3.6) and (3.8) imply:

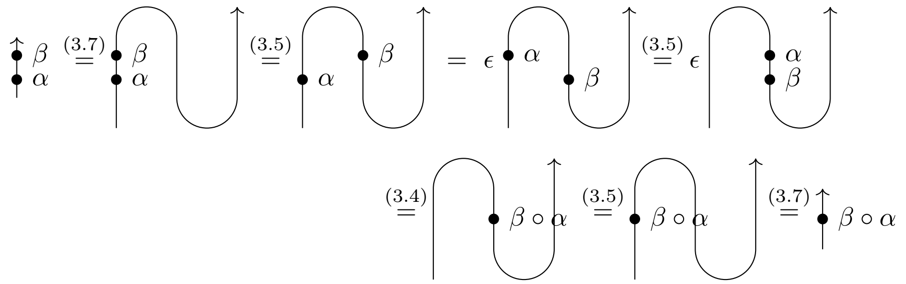


The remaining interactions of dots and crossings follow from the relations for downward crossings, cups and caps via the definition of the crossings. □

LEMMA 3.4. *Dots on upward strands merge without a sign twist:*



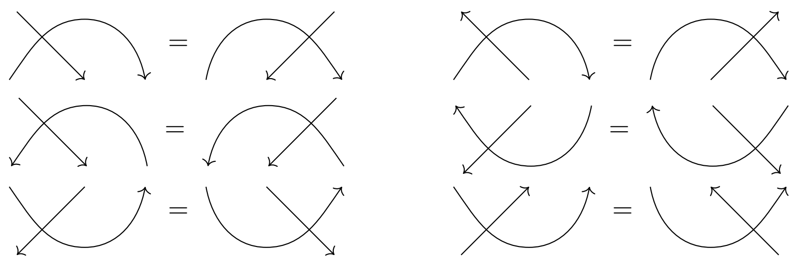
PROOF. With $\epsilon = (-1)^{|\alpha||\beta|}$ we have



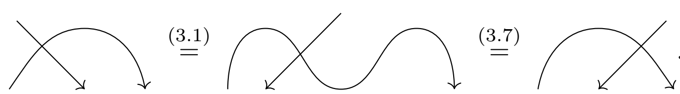
∎

REMARK 3.5. The adjunction relations (3.7) say that we have adjunctions of 1-morphisms $(\mathsf{P}_a, \mathsf{Q}_a)$ and $(\mathsf{Q}_a, \mathsf{P}_{Sa})$ for any $a \in \mathcal{V}$.

LEMMA 3.6 (Pitchfork relations, part I). *The following relations hold in* $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$*:*



PROOF. These relations follow immediately from the definition of the crossings in (3.1) and (3.2) together with the adjunction relations (3.7). For example, for the first relation one has



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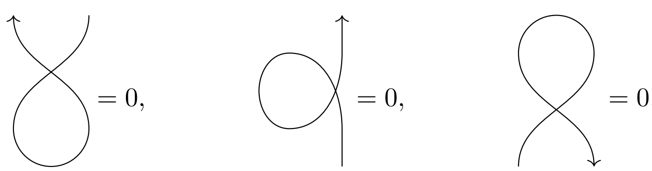
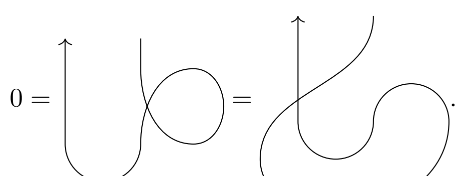
LEMMA 3.7 (Counter-clockwise loops). *The following relations hold in* $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$*:*

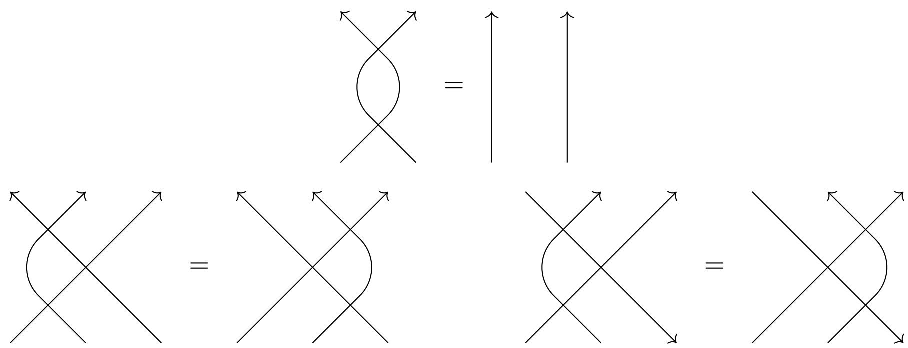


PROOF. Using the left relation in (3.9) and a pitchfork move across the bottom cup, we have

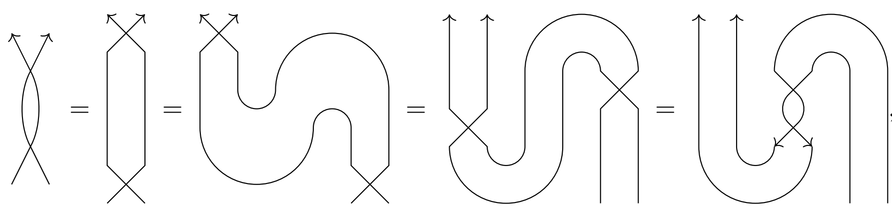


Straightening out via (3.7), one obtains the first relation. The other two relations are obtained in a similar manner. ∎

Lemma 3.8. *The following relations hold in* $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$:



Proof. These relations are obtained by adding appropriate cups and caps to (3.8) and using the pitchfork and adjunction relations. For example, for the first relation, one has

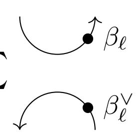


where the second equality is relation (3.7), the third equality is the interchange law in the 2-category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$ and the fourth is obtained by applying the pitchfork relations twice at the top and twice at the bottom. The first relation now follows by (3.8). $\square$

Remark 3.9. Relations (3.8) imply that we have an action of the symmetric group S_n on Q_a^n by twisted unmarked downward strands, i.e., we have a morphism $\Bbbk[S_n] \to \mathrm{End}(\mathsf{Q}_a^n)$. Similarly, Lemma 3.8 shows that there exists an action of the symmetric group on P_a^n.

Fixing a basis $\{\beta_\ell\}$ of $\mathrm{Hom}(a, b)$ one can write the term $\Psi(\mathrm{id})$ in (3.10) as

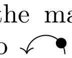
$$\Psi(\mathrm{id}) = \sum_{\ell}$$



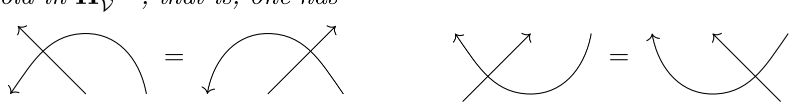
,

where $\{\beta_\ell^\vee\}$ is the dual basis of $\mathrm{Hom}(b, Sa) \cong \mathrm{Hom}(a, b)^\vee$. It can also be written as the composition of 2-morphisms

(3.11) $$\psi_1 \colon \mathsf{Q}_a\mathsf{P}_b \to \mathrm{Hom}(a, b) \otimes_{\Bbbk} \mathbb{1} \quad \text{and} \quad \psi_2 \colon \mathrm{Hom}(a, b) \otimes_{\Bbbk} \mathbb{1} \to \mathsf{Q}_a\mathsf{P}_b.$$

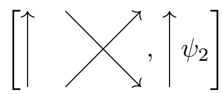
Here ψ_1 is obtained form the map $\mathrm{Hom}(a, b)^\vee \to \mathrm{Hom}(\mathsf{Q}_a\mathsf{P}_b, \mathbb{1})$ sending $\beta \in \mathrm{Hom}(b, Sa) \cong \mathrm{Hom}(a, b)^\vee$ to  β and ψ_2 is similarly obtained from the natural map $\mathrm{Hom}(a, b) \to \mathrm{Hom}(\mathbb{1}, \mathsf{Q}_a\mathsf{P}_b)$. We note that the right relation in (3.9) implies that the composition $\psi_1 \circ \psi_2$ is the identity on $\mathrm{Hom}(a, b) \otimes_{\Bbbk} \mathbb{1}$.

Lemma 3.10 (Pitchfork relations, part II). *The two remaining pitchfork relations hold in* $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$, *that is, one has*

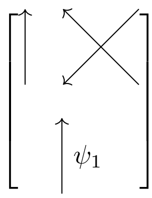


In particular, these relations show that we could have defined the upward crossing as a rotation of the left-wards crossing (instead of the right-wards one in (3.2)) and obtained the same 2-morphism. The proof is inspired by the proof of [**9**, Lemma 2.6].

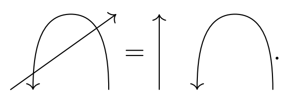
PROOF. These two pitchfork relations are slightly harder to see than the ones in Lemma 3.6. First, the relations in Section 3.2 imply that for any $a, b \in \mathcal{V}$ the morphism

(3.12)  $: \mathsf{P}_a\mathsf{P}_{Sb}\mathsf{Q}_b \oplus \big(\mathrm{Hom}(b, Sb) \otimes_{\Bbbk} \mathsf{P}_a\big) \to \mathsf{P}_a\mathsf{Q}_b\mathsf{P}_{Sb}$

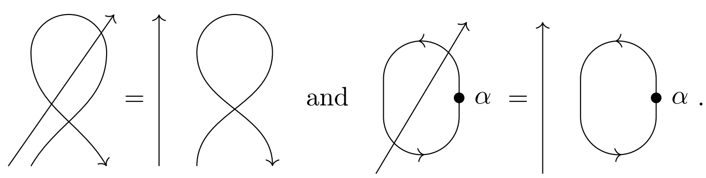
is an isomorphism with inverse

 $: \mathsf{P}_a\mathsf{Q}_b\mathsf{P}_{Sb} \to \mathsf{P}_a\mathsf{P}_{Sb}\mathsf{Q}_b \oplus \big(\mathrm{Hom}(b, Sb) \otimes_{\Bbbk} \mathsf{P}_a\big).$

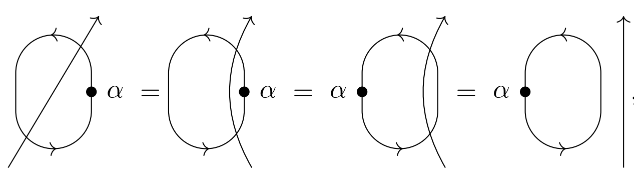
Next, we show that

(3.13) 

Precomposing with isomorphism (3.12), it remains to show that for any $\alpha \in \mathrm{Hom}(b, Sb)$:

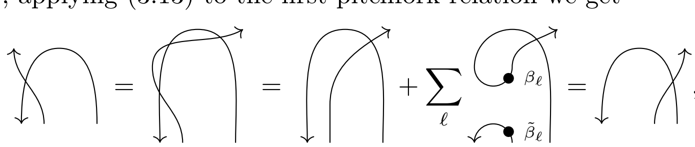


The right diagram of the left equality has a counter-clockwise curl, hence is vanishing. Applying the third equality (read from its right to left) of Lemma 3.8 to the left diagram of the left equality, we can move the upward diagonal arrow to below the counter-clockwise curl. Hence, this diagram also equals zero. Further we have



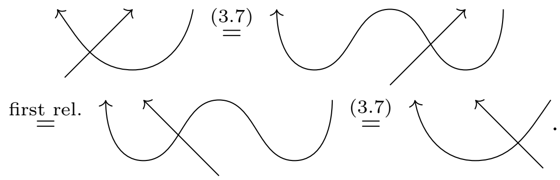
which is $\mathrm{Tr}(\alpha)$ times the identity 2-morphism and thus agrees with the rightmost 2-morphism.

Finally, applying (3.13) to the first pitchfork relation we get



where the last equality holds because of the presence of counter-clockwise curls.

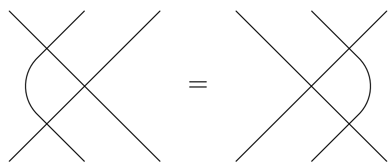
The second relation immediately follows from the first one:



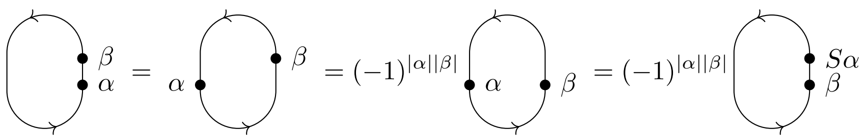
□

Using the pitchfork relations one shows that the remaining triple moves also hold.

LEMMA 3.11 (Triple moves). *The following relations holds in* $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$*:*



REMARK 3.12. For any object $a \in \mathcal{V}$, $\alpha \in \mathrm{Hom}(a, a)$ and $\beta \in \mathrm{Hom}(a, Sa)$ one has



This matches the identity of Proposition 2.4.

3.4. The category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$: Karoubi-completion

A category is *Karoubian* or *idempotent complete* if all its idempotents are split. Given a category $\mathcal{C}$, its *Karoubi envelope* or *idempotent completion* is the universal pair $(\mathrm{kar}(\mathcal{C}), \iota)$ where $\mathrm{kar}(\mathcal{C})$ a Karoubian category and ι is a functor $C \to \mathrm{kar}(\mathcal{C})$. The functor ι is necessarily fully faithful, see [**26**, Exercice 7.5].

DEFINITION 3.13. The *(additive) Heisenberg category* $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$ of $\mathcal{V}$ is the Karoubi envelope of $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$.

The objects of $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$ are those of $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$. Its 1-morphisms are pairs (R, e), where R is a 1-morphism of $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$ and $e \colon R \to R$ is a idempotent in $\mathrm{End}_{\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}}(R)$. Its 2-morphisms $(R_1, e_1) \to (R_2, e_2)$ are 2-morphisms $f \colon R_1 \to R_2$ from $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}\prime}$ which satisfy $f = e_2 \circ f \circ e_1$.

EXAMPLE 3.14. Let $\mathcal{V} = \mathrm{Vect}_{\Bbbk}^{\mathrm{f}}$ be the category of finite-dimensional vector spaces over $\Bbbk$. It is the additive hull of the field $\Bbbk$ considered as a single-object category. Then the Serre functor on $\mathcal{V}$ is the identity, and the category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$ reproduces Khovanov's categorification of the infinite Heisenberg algebra [**31**]. More precisely, collapsing our category $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$ to a monoidal 1-category by identifying the objects, the morphism $\mathsf{P}_{\Bbbk}$ corresponds to Q_+ in [**31**], while $\mathsf{Q}_{\Bbbk}$ corresponds to Q_-. Since $\mathsf{P}_{\Bbbk\oplus\Bbbk} \cong \mathsf{P}_{\Bbbk} \oplus \mathsf{P}_{\Bbbk}$, and similarly for Q, all data is encoded in the relations between these two morphisms.

By Remark 3.9, for each object $a \in \mathcal{V}$ there are canonical morphisms $\Bbbk[S_n] \to \mathrm{End}(\mathsf{P}_a^n)$ and $\Bbbk[S_n] \to \mathrm{End}(\mathsf{Q}_a^n)$. Let

$$e_{\mathrm{triv}} = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \in \Bbbk[S_n]$$

be the symmetriser idempotent of $\Bbbk[S_n]$. Abusing notation, we denote the image of the symmetriser under either of the above maps again by e_{triv}. The 2-morphisms e_{triv} are idempotent endomorphisms of P_a^n and Q_a^n respectively, and hence split in $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$. We write $\mathsf{P}_a^{(n)}$ and $\mathsf{Q}_a^{(n)}$ for the corresponding 1-morphisms $(\mathsf{P}_a^n, e_{\mathrm{triv}})$ and $(\mathsf{Q}_a^n, e_{\mathrm{triv}})$.

THEOREM 3.15. *For any $a, b \in \mathcal{V}$ and $n, m \in \mathbb{N}$ we have the following relations in $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$:*

$$\mathsf{P}_a^{(m)} \mathsf{P}_b^{(n)} \cong \mathsf{P}_b^{(n)} \mathsf{P}_a^{(m)}, \quad \mathsf{Q}_a^{(m)} \mathsf{Q}_b^{(n)} \cong \mathsf{Q}_b^{(n)} \mathsf{Q}_a^{(m)},$$

$$\mathsf{Q}_a^{(m)} \mathsf{P}_b^{(n)} \cong \bigoplus_{i=0}^{\min(m,n)} \mathrm{Sym}^i \mathrm{Hom}_{\mathcal{V}}(a,b) \otimes_{\Bbbk} \mathsf{P}_b^{(n-i)} \mathsf{Q}_a^{(m-i)}.$$

The symmetric powers of $\mathrm{Hom}_{\mathcal{V}}(a,b)$ in the last isomorphism of Theorem 3.15 categorify the coefficient $s^k\langle a, b\rangle$ in (2.5). In Remark 6.5 we explain that from any $\mathsf{P}_a^{(i)}$ to any $\mathsf{P}_b^{(i)}$ there are morphisms which correspond to i parallel strands labelled by elements of $\mathrm{Sym}^i \mathrm{Hom}_{\mathcal{V}}(a,b)$. The last isomorphism of Theorem 3.15 is then naturally expressed in terms of these morphisms. In particular, in the case $m = n = 1$, the 1-precomposition of this 2-isomorphism with $\mathrm{id}_{\mathsf{P}_a}$ on the left is the isomorphism used in the proof of Lemma 3.10.

The proof of Theorem 3.15 is entirely combinatorial and virtually the same as the one for the DG version, Theorem 6.3. We thus skip it. Similarly, the constructions and the results of Section 6.3 have obvious analogues in the additive setting.

3.5. The categorical Fock space in the additive case

In this section we construct a categorical Fock space $\mathbf{F}_{\mathcal{V}}^{\mathrm{add}}$ of the base category $\mathcal{V}$. It consists of the categorical symmetric powers of $\mathcal{V}$. We show that $\mathbf{H}_{\mathcal{V}}^{\mathrm{add}}$ has a representation on the categorical Fock space.

Once this is established, the same decategorification argument as in Section 8.2 shows that $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}^{\mathrm{add}})$ acts on $\mathrm{K}_0^{\mathrm{num}}(\mathbf{F}_{\mathcal{V}}^{\mathrm{add}})$. Theorem 3.15, we have a group homomorphism from the classical Heisenberg algebra $H_{\mathcal{V}}$ to $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}^{\mathrm{add}})$. Thus $H_{\mathcal{V}}$ acts on $\mathrm{K}_0^{\mathrm{num}}(\mathbf{F}_{\mathcal{V}}^{\mathrm{add}})$ and the same argument as in Section 8.2.2 shows that the subrepresentation of $\mathrm{K}_0^{\mathrm{num}}(\mathbf{F}_{\mathcal{V}}^{\mathrm{add}})$ generated by $1 \in \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^0\mathcal{V}) \cong \Bbbk$ is the Fock space representation $F_{\mathcal{V}}$ of $H_{\mathcal{V}}$.

If $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V})$ is a finitely generated abelian group and if we have for all $N \geq 0$

$$\mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}) \cong \bigoplus_{1^{\lambda_1} 2^{\lambda_2} \cdots \dashv N} \mathrm{Sym}^{\lambda_1} \mathrm{K}_0^{\mathrm{num}}(\mathcal{V}) \otimes \mathrm{Sym}^{\lambda_2} \mathrm{K}_0^{\mathrm{num}}(\mathcal{V}) \otimes \cdots$$

then a dimension count shows that $F_{\mathcal{V}}$ is the whole of $\mathrm{K}_0^{\mathrm{num}}(\mathbf{F}_{\mathcal{V}}^{\mathrm{add}})$. In other words, our categorical Fock space categorifies the classical Fock space.

The N-fold tensor power $\mathcal{V}^{\otimes N}$ is the additive hull (that is, the closure under finite direct sums) of the category of N-tuples $a_1 \otimes \cdots \otimes a_N$ of objects of $\mathcal{V}$ with morphism spaces

$$\mathrm{Hom}_{\mathcal{V}^{\otimes N}}(a_1 \otimes \cdots \otimes a_N,\, b_1 \otimes \cdots \otimes b_N) \coloneqq \mathrm{Hom}_{\mathcal{V}}(a_1, b_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{V}}(a_N, b_N).$$

The category $\mathcal{V}^{\otimes N}$ can be endowed with an action of S_N, given on objects by

$$\sigma(a_1 \otimes \cdots \otimes a_N) \coloneqq a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(N)}. \tag{3.14}$$

The category of S_N-equivariant objects in $\mathcal{V}^{\otimes N}$

$$\mathcal{S}^N\mathcal{V} := (\mathcal{V}^{\otimes N})^{S_N}$$

has as objects all tuples $\big(\underline{a},\, (\epsilon_\sigma)_{\sigma\in S_N}\big)$ with $\underline{a} \in \mathcal{V}^{\otimes N}$ and $\epsilon_\sigma\colon \underline{a} \xrightarrow{\sim} \sigma(\underline{a})$ isomorphisms compatible with the S_N-action. A morphism $\big(\underline{a},\, \epsilon_\sigma\big) \to \big(\underline{b},\, \tau_\sigma\big)$ is a morphism $\alpha\colon \underline{a} \to \underline{b}$ in $\mathcal{V}^{\otimes N}$ such that $\sigma(\alpha) \circ \epsilon_\sigma = \tau_\sigma \circ \alpha$ for all $\sigma \in S_N$. We refer to [**21**, Section 2] for details. For ease of notation, we set $\mathcal{S}^0\mathcal{V} = \mathrm{Vect}^{\mathrm{f}}_{\Bbbk}$ and $\mathcal{S}^N\mathcal{V} = 0$ for $N < 0$.

Remark 3.16. If $\mathcal{V}$ is a $\Bbbk$-linear category equipped with additional structure and/or conditions, e.g. an abelian category, then $\mathcal{V}^{\otimes N}$ will not automatically also have these. In such case, in the definition above one should replace the additive hull with an appropriate completion. For example, Deligne's tensor product of abelian categories takes the abelian hull of N-tuples. We are particularly interested in the case of DG enhanced triangulated categories, which we discuss in detail in Section 4.8 and Chapter 7.

Let $\mathbf{F}^{\mathrm{add}\prime}_{\mathcal{V}}$ be the strict 2-category with objects $\mathcal{S}^N\mathcal{V}$, 1-morphisms $\Bbbk$-linear functors and 2-morphisms natural transformations. We want to define a 2-functor $\Psi'_{\mathcal{V}}\colon \mathbf{H}^{\mathrm{add}\prime}_{\mathcal{V}} \to \mathbf{F}^{\mathrm{add}\prime}_{\mathcal{V}}$. For this, we need the functors of restriction and induction. Let $1 \times S_{N-1} \leq S_N$ be the subgroup comprising the elements fixing the first letter. The restriction functor is defined as

$$\begin{array}{rccc} \mathrm{Res}^{1\times S_{N-1}}_{S_N}\colon & \mathcal{S}^N\mathcal{V} & \to & \big(\mathcal{V}^{\otimes N}\big)^{1\times S_{N-1}} \\ & \big(\underline{a},\, (\epsilon_g)_{g\in S_N}\big) & \mapsto & \big(\underline{a},\, (\epsilon_g)_{g\in 1\times S_{N-1}}\big) \end{array}$$

on objects and by id on morphisms. Its left and right adjoint, the induction functor, is

$$\begin{array}{rccc} \mathrm{Ind}^{S_N}_{1\times S_{N-1}}\colon & \big(\mathcal{V}^{\otimes N}\big)^{1\times S_{N-1}} & \to & \mathcal{S}^N\mathcal{V} \\ & \big(\underline{a},\, (\epsilon_h)_{h\in 1\times S_{N-1}}\big) & \mapsto & \big(\bigoplus_{[f]\in S_N/(1\times S_{N-1})} f(\underline{a}),\, (\varepsilon_g)_{g\in S_N}\big) \end{array}$$

on objects. Here $S_N/(1 \times S_{N-1})$ is the set of left cosets, the summation happens over a fixed choice of their representatives f, and the isomorphism

$$\varepsilon_g\colon \bigoplus_{[f]\in S_N/(1\times S_{N-1})} f(\underline{a}) \to \bigoplus_{[f']\in S_N/(1\times S_{N-1})} gf'(\underline{a})$$

maps each summand $f(\underline{a})$ to the summand $gf'(\underline{a})$ with $[f] = [gf']$ via the isomorphism $f(\epsilon_h)$ where $h \in 1 \times S_{N-1}$ is such that $gf' = fh$. On morphisms, $\mathrm{Ind}^{S_N}_{1\times S_{N-1}}$ is given by

$$\alpha \to \sum_{[f]\in S_N/(1\times S_{N-1})} f(\alpha).$$

A more general treatment of these functors is given in Section 4.8 below.

On objects, we define $\Psi'_{\mathcal{V}}$ as

$$\Psi'_{\mathcal{V}}(N) = \mathcal{S}^N\mathcal{V}, \qquad \forall\, N \in \mathbb{Z}.$$

On 1-morphisms $\Psi'_{\mathcal{V}}$ sends $\mathsf{P}_a\colon (N-1) \to N$ to the composition

$$P_a\colon \mathcal{S}^{N-1}\mathcal{V} \xrightarrow{a\otimes -} \left(\mathcal{V}^{\otimes N}\right)^{1\times S_{N-1}} \xrightarrow{\operatorname{Ind}_{1\times S_{N-1}}^{S_N}} \mathcal{S}^N\mathcal{V},$$

and $\mathsf{Q}_a\colon N \to N-1$ to the composition

$$Q_a\colon \mathcal{S}^N\mathcal{V} \xrightarrow{\operatorname{Res}_{S_N}^{1\times S_{N-1}}} \left(\mathcal{V}^{\otimes N}\right)^{1\times S_{N-1}} \xrightarrow{\operatorname{Hom}_{\mathcal{V}}(a,-)\otimes \mathrm{id}} \mathcal{S}^{N-1}\mathcal{V}.$$

Tensor-Hom adjunction implies that P_a is left adjoint to Q_a and the definition of a Serre functor further implies that Q_a is left adjoint to P_{Sa}.

EXAMPLE 3.17. Let $(a_1 \otimes \cdots \otimes a_N,\, (\epsilon_\sigma)_{\sigma\in S_N})$ be an object in $\mathcal{S}^N\mathcal{V}$. There are $N+1$ cosets of the subgroup $S_N < S_{N+1}$ fixing the symbol 1. A set of representatives of these cosets is given by the cycles $\{\big(i\ldots 1\big)\}_{1\le i\le N+1}$. Denote each $\big(i\ldots 1\big)$ by ξ_i.

By definition of the P_b, we have

$$P_b(a_1 \otimes \cdots \otimes a_N) = \operatorname{Ind}_{1\times S_N}^{S_N+1}\left(b\otimes a_1 \otimes \cdots \otimes a_N\right) = \bigoplus_{i=1}^{N+1} \xi_i\left(b\otimes a_1 \otimes \cdots \otimes a_N\right).$$

By the definition (3.14) of the action of S_{N+1} on $\mathcal{V}^{\otimes N+1}$, ξ_i acts by placing the $\xi_i^{-1}(j)$th factor into jth place. Thus we have

$$\xi_i\left(b\otimes a_1 \otimes \cdots \otimes a_N\right) = a_1 \otimes \cdots \otimes a_{i-1} \otimes b \otimes a_i \otimes \cdots \otimes a_N$$

and therefore

$$P_b = \bigoplus_{i=1}^{N+1} a_1 \otimes \cdots \otimes a_{i-1} \otimes b \otimes a_i \otimes \cdots \otimes a_N. \tag{3.15}$$

We describe the S_{N+1}-equivariant structure on this direct sum. Let $\sigma \in S_{N+1}$. For each ξ_i, the element $\sigma\xi_{\sigma^{-1}(i)}$ lies in the same coset as they both send 1 to i. Thus

$$\xi_i^{-1}\sigma\xi_{\sigma^{-1}(i)} = \big(1\cdots i\big)\sigma\big(\sigma^{-1}(i)\cdots 1\big) \in 1\times S_N \subset S_{N+1}.$$

Let τ_i be the corresponding element of S_N. By definition, the isomorphism

$$\varepsilon_\sigma\colon \quad \bigoplus_{i=1}^{N+1} \xi_i\left(b\otimes a_1 \otimes \cdots \otimes a_N\right) \longrightarrow \bigoplus_{i=1}^{N+1} \sigma\xi_i\left(b\otimes a_1 \otimes \cdots \otimes a_N\right)$$

is a sum of components

$$\xi_i \circ (b\otimes -)(\epsilon_{\tau_i})\colon \xi_i\left(b\otimes a_1 \otimes \cdots \otimes a_N\right) \to \xi_i\left(b\otimes\left(\tau_i\left(a_1 \otimes \cdots \otimes a_N\right)\right)\right).$$

Hence, in terms of (3.15), ε_σ is the sum of the components

$$a_1 \otimes \cdots \otimes a_{i-1} \otimes b \otimes a_i \otimes \cdots \otimes a_N$$
$$\xrightarrow{\xi_i\circ(b\otimes -)(\epsilon_{\tau_i})} a_{\tau_i^{-1}(1)} \otimes \cdots \otimes a_{\tau_i^{-1}(i-1)} \otimes b \otimes a_{\tau_i^{-1}(i)} \otimes \cdots \otimes a_{\tau_i^{-1}(N)}.$$

It follows that

$$\begin{aligned} Q_aP_b(a_1 \otimes \cdots \otimes a_N) &= Q_a\left(\bigoplus_{i=1}^{N+1} \xi_i(b \otimes a_1 \otimes \cdots \otimes \cdots \otimes a_N)\right) \\ &= Q_a\left(\bigoplus_{i=1}^{N+1} a_1 \otimes \cdots \otimes a_{i-1} \otimes b \otimes a_i \otimes \cdots \otimes a_N\right) \\ &= \mathrm{Hom}(a,b) \otimes_{\Bbbk} a_1 \otimes \cdots \otimes a_N \oplus \\ &\quad \bigoplus_{i=1}^{N} \mathrm{Hom}(a,a_1) \otimes_{\Bbbk} a_2 \otimes \cdots \otimes a_i \otimes b \otimes a_{i+1} \otimes \cdots \otimes a_N. \end{aligned}$$

We describe the S_N-equivariant structure on this direct sum. Let $\sigma \in S_N$. Let $1 \times \sigma$ be the corresponding element of $1 \times S_N \subset S_{N+1}$ and note that $1 \times \sigma(i) = 1$ if $i = 1$ and $1 + \sigma(i-1)$ if $i > 1$. As before, we have $\xi_i^{-1}(1 \times \sigma)\xi_{(1\times\sigma^{-1})(i)} \in 1 \times S_N$, so let τ_i be the corresponding element of S_N.

Restricting the S_{N+1}-equivariant structure on $P_b(a_1 \otimes \cdots \otimes a_N)$ described above, we see that the isomorphism

$$\varepsilon'_\sigma \colon Q_aP_b(a_1 \otimes \cdots \otimes a_N) \longrightarrow \sigma\left(Q_aP_b(a_1 \otimes \cdots \otimes a_N)\right)$$

is the sum

$$\sum_{i=1}^{N+1} (\mathrm{Hom}(a,-) \otimes \mathrm{id}) \circ \xi_i \circ (b \otimes -)(\epsilon_{\tau_i}).$$

When $i = 1$ we have $\xi_1 = \xi_{(1\times\sigma^{-1})(1)} = \mathrm{id}$, so $\tau = \sigma$ and the corresponding summand of ε'_σ is

$$\mathrm{Hom}(a,b) \otimes_{\Bbbk} a_1 \otimes \cdots \otimes a_N \xrightarrow{\mathrm{id} \otimes \epsilon_\sigma} \mathrm{Hom}(a,b) \otimes_{\Bbbk} a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(N)}.$$

When $i > 1$, observe that $\tau_i(1) = 1$. This is because

$$\tau_i(1) = \xi_i^{-1}(1 \times \sigma)\xi_{(1\times\sigma^{-1})(i)}(2) - 1 = \xi_i^{-1}(1 \times \sigma)(1) - 1 = \xi_i^{-1}(1) - 1 = 2 - 1 = 1.$$

The corresponding summand ε'_σ is therefore

$$\begin{array}{c} \mathrm{Hom}(a,a_1) \otimes_{\Bbbk} a_2 \otimes \cdots \otimes a_{i-1} \otimes b \otimes a_i \otimes \cdots \otimes a_N \\ \Big\downarrow {\scriptstyle (\mathrm{Hom}(a,-)\otimes\mathrm{id})\circ\xi_i\circ(b\otimes -)\left(\epsilon_{\tau_i}\right)} \\ \mathrm{Hom}(a,a_1) \otimes_{\Bbbk} a_{\tau^{-1}(2)} \otimes \cdots \otimes a_{\tau^{-1}(i-1)} \otimes b \otimes a_{\tau^{-1}(i)} \otimes \cdots \otimes a_{\tau^{-1}(N)}. \end{array}$$

If $a = b$, then the adjunction unit

$$a_1 \otimes \cdots \otimes a_N \to Q_aP_a(a_1 \otimes \cdots \otimes a_N)$$

embeds $a_1 \otimes \cdots \otimes a_N$ into the first summand as $\{\mathrm{id}_a\} \otimes_{\Bbbk} a_1 \otimes \cdots \otimes a_N$.

Example 3.18. In the same way, we obtain

$$\begin{aligned} P_bQ_a(a_1 \otimes \cdots \otimes a_N) &= P_b\big(\mathrm{Hom}(a,a_1) \otimes_{\Bbbk} a_2 \otimes \cdots \otimes a_N\big) \\ &= \bigoplus_{i=1}^{N} \mathrm{Hom}(a,a_1) \otimes_{\Bbbk} a_2 \otimes \cdots \otimes a_i \otimes b \otimes a_{i+1} \otimes \cdots \otimes a_N. \end{aligned}$$

The equivariant structure is the same as in the preceding example, keeping in mind that

$$(\mathrm{Hom}(a,-) \otimes \mathrm{id}) \circ ((i+1) \cdots 1) \circ (b \otimes -) = (i...1) \circ (b \otimes -) \circ (\mathrm{Hom}(a,-) \otimes \mathrm{id}).$$

The adjunction counit

$$P_aQ_a(a_1 \otimes \cdots \otimes a_N) \to a_1 \otimes \cdots \otimes a_N$$

first applies the adjunction map $\mathrm{Hom}(a, a_1) \otimes a \to a_1$ on each summand yielding

$$\bigoplus_{i=1}^{N} a_2 \otimes \cdots \otimes a_i \otimes a_1 \otimes a_{i+1} \cdots \otimes a_N.$$

Then the equivariant structure of $a_1 \otimes \cdots \otimes a_N$ provides a morphism

$$\bigoplus_{i=1}^{N} a_2 \otimes \cdots \otimes a_i \otimes a_1 \otimes a_{i+1} \cdots \otimes a_l \xrightarrow{\sum \epsilon_{(12\cdots i)}} a_1 \otimes \cdots \otimes a_N.$$

Example 3.19. In the same way one sees that the unit of the adjunction $Q_a \dashv P_{Sa}$ is given by the canonical map $a_1 \to \mathrm{Hom}(a, a_1) \otimes_{\Bbbk} Sa$ coming from the $\mathrm{Hom}(a, -) \dashv - \otimes_{\Bbbk} Sa$ adjunction followed by the diagonal map into the product. The counit is projection onto the factor $\mathrm{Hom}(a, Sa) \otimes_{\Bbbk} a_1 \otimes \cdots \otimes a_N$ followed by the Serre trace applied to $\mathrm{Hom}(a, Sa)$.

It follows from the explicit computations above that there is an isomorphism

$$Q_aP_b \cong \big(\mathrm{Hom}_{\mathcal{V}}(a, b) \otimes_{\Bbbk} \mathbb{1}\big) \oplus P_bQ_a \tag{3.16}$$

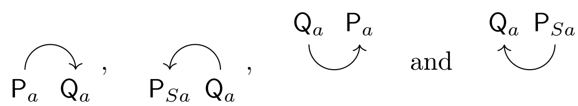
natural in $a, b \in \mathcal{V}$.

We can now define the action of $\Psi'_{\mathcal{V}}$ on 2-morphisms. Firstly, the dotted strings $\overset{\bullet}{\uparrow}\,\alpha$ and $\overset{\bullet}{\downarrow}\,\alpha$ for $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a, b)$ are sent to the natural transformations $P_a \Rightarrow P_b$ and $Q_b \Rightarrow Q_a$ induced by the natural transformations

$$a \otimes - \xRightarrow{\alpha \otimes \mathrm{id}} b \otimes - \qquad \text{and} \qquad \mathrm{Hom}_{\mathcal{V}}(b, -) \xRightarrow{\alpha \circ -} \mathrm{Hom}_{\mathcal{V}}(a, -)$$

respectively.

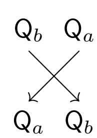
Next, the caps and cups



are sent to the adjunction maps

$$P_aQ_a \to \mathrm{id}, \quad Q_aP_{Sa} \to \mathrm{id}, \quad \mathrm{id} \to Q_aP_a, \quad \text{and} \quad \mathrm{id} \to P_{Sa}Q_a.$$

Finally, the downward crossing



is sent to the following functorial isomorphism. As functors $\mathcal{S}^N\mathcal{V} \to \mathcal{S}^{N-2}\mathcal{V}$ we have

$$\mathsf{Q}_a\mathsf{Q}_b \cong \big(\mathrm{Hom}_{\mathcal{V}}(a, -) \otimes \mathrm{Hom}_{\mathcal{V}}(b, -) \otimes \mathrm{id}_{\mathcal{S}^{N-2}\mathcal{V}}\big) \circ \mathrm{Res}_{S_N}^{1\times1\times S_{N-2}},$$

$$\mathsf{Q}_b\mathsf{Q}_a \cong \big(\mathrm{Hom}_{\mathcal{V}}(b, -) \otimes \mathrm{Hom}_{\mathcal{V}}(a, -) \otimes \mathrm{id}_{\mathcal{S}^{N-2}\mathcal{V}}\big) \circ \mathrm{Res}_{S_N}^{1\times1\times S_{N-2}}.$$

The latter can be further rewritten as

$$\mathsf{Q}_b\mathsf{Q}_a \cong \big(\mathrm{Hom}_{\mathcal{V}}(a, -) \otimes \mathrm{Hom}_{\mathcal{V}}(b, -) \otimes \mathrm{id}_{\mathcal{S}^{N-2}\mathcal{V}}\big) \circ (12) \circ \mathrm{Res}_{S_N}^{1\times1\times S_{N-2}}.$$

With these identifications in mind, we send the downward crossing to the functorial isomorphism $\mathsf{Q}_a\mathsf{Q}_b \xrightarrow{\sim} \mathsf{Q}_b\mathsf{Q}_a$ induced by the natural isomorphism

$$\mathrm{Res}^{1\times 1\times S_{N-2}}_{S_N} \xrightarrow{\sim} (12)\circ \mathrm{Res}^{1\times 1\times S_{N-2}}_{S_N}$$

given on any object $(\underline{a}, \epsilon_\sigma)$ by $\epsilon_{(12)}$.

Explicit computations (making particular use of the decomposition (3.16)) show that this definition of $\Psi'_{\mathcal{V}}$ is compatible with all 2-relations on $\mathbf{H}^{\mathrm{add}\prime}_{\mathcal{V}}$. Thus we have the following result:

PROPOSITION 3.20. *The above definition gives a* 2*-functor*

$$\Psi'_{\mathcal{V}} \colon \mathbf{H}^{\mathrm{add}\prime}_{\mathcal{V}} \to \mathbf{F}^{\mathrm{add}\prime}_{\mathcal{V}}.$$

Let $\mathbf{F}^{\mathrm{add}}_{\mathcal{V}}$ be the 2-category with objects the Karoubi completions $\mathrm{Kar}(\mathcal{S}^N\mathcal{V})$, 1-morphisms $\Bbbk$-linear functors, and 2-morphisms natural transformations. We call $\mathbf{F}^{\mathrm{add}}_{\mathcal{V}}$ the *Fock category* or, equivalently, the *categorical Fock space* of $\mathbf{H}^{\mathrm{add}}_{\mathcal{V}}$. By the universal property of the Karoubi envelope, we have:

COROLLARY 3.21. *The functor* $\Psi'_{\mathcal{V}}$ *induces a* 2*-functor*

$$\Psi_{\mathcal{V}} \colon \mathbf{H}^{\mathrm{add}}_{\mathcal{V}} \to \mathbf{F}^{\mathrm{add}}_{\mathcal{V}}.$$

REMARK 3.22. The functors P_a and Q_a in the above definition have both a right and left adjoint. Hence, if $\mathcal{V}$ is abelian they are exact. Thus they extend to the Deligne tensor product, i.e. there exists an action of $\mathbf{H}^{\mathrm{add}}_{\mathcal{V}}$ on the 2-category with objects $\widehat{\mathcal{S}}^N\mathcal{V} = (\mathcal{V}^{\widehat{\otimes} N})^{S_N}$, where $\widehat{\otimes}$ is the Deligne tensor product of abelian categories [**15**, Proposition 1.46.2].

CHAPTER 4

Preliminaries on DG Categories

In this section, we review the existing formalism of DG categories and enriched bicategories and introduce several new results we need for our construction of a DG Heisenberg 2-category. Below we summarise the key items of notation we employ.

Given a DG category $\mathcal{A}$, we denote by $\mathcal{M}od\text{-}\mathcal{A}$ its DG category of right $\mathcal{A}$-modules. We denote by $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}erf(\mathcal{A})$ the full subcategories of $\mathcal{M}od\text{-}\mathcal{A}$ comprising h-projective modules and perfect modules, respectively. We write $\mathcal{H}perf(\mathcal{A})$ for their intersection. We denote by $\mathrm{D}(\mathcal{A})$ the derived category of right $\mathcal{A}$-modules, and by $\mathrm{D}_{\mathrm{c}}(\mathcal{A})$ its full subcategory of compact objects. Note that $\mathrm{D}(\mathcal{A}) \cong \mathrm{H}^0(\mathcal{P}(\mathcal{A}))$ and $\mathrm{D}_{\mathrm{c}}(\mathcal{A}) \cong \mathrm{H}^0(\mathcal{H}perf(\mathcal{A}))$.

Given a scheme X, we write $\mathrm{D}_{\mathrm{qc}}(X)$ for the derived category of complexes of sheaves on X with quasi-coherent cohomology and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ for its full subcategory of complexes with bounded, coherent cohomology. Let $\mathcal{I}(X)$ be the standard DG enhancement of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

In this paper we arrange DG categories into a ménagerie of 1-categories, strict 2-categories and DG bicategories. Figure 1 gives an overview of these and their relation to each other:

- $\mathbf{dgCat}^1$ is the 1-category of DG categories and DG functors between them, see Section 4.1, Definition 4.1,
- $\mathbf{Ho}(\mathbf{dgCat}^1)$ is the localisation of $\mathbf{dgCat}^1$ by quasi-equivalences, see Section 4.4 and [**51**],
- $\mathbf{EnhCat}^1$ is the full subcategory of $\mathbf{Ho}(\mathbf{dgCat}^1)$ comprising pretriangulated DG categories. We view it as the 1-category of enhanced triangulated categories, see Section 4.4.
- $\mathbf{Mor}(\mathbf{dgCat}^1)$ is the localisation of $\mathbf{dgCat}^1$ by Morita equivalences. We view it as the 1-category of Morita enhanced triangulated categories, see Section 4.4 and [**46**],
- **dgCat** is the strict 2-category of the isomorphism classes of DG categories, DG functors, and closed degree zero DG natural transformations, see Section 4.1, Definition 4.1,
- **Ho**(**dgCat**) is a strict 2-categorical version of $\mathbf{Ho}(\mathbf{dgCat}^1)$ constructed using the main results of [**51**], see Section 4.1, Definition 4.1 and [**51**],
- **EnhCat** is the strict 2-category of enhanced triangulated categories, see [**36**, Sec. 1]. It is the 1-full subcategory of **Ho**(**dgCat**) comprising pretriangulated DG categories.
- **Mor**(**dgCat**) is the strict 2-category of Morita enhanced triangulated categories, see Section 4.4, Definition 4.14. It is also known as $\mathbf{EnhCat}_{\mathrm{kc}}$, because it can be realised as the 1-full subcategory of **EnhCat** comprising homotopy Karoubi complete DG categories. Here and throughout the paper the subscript kc means ‘Karoubi complete’.

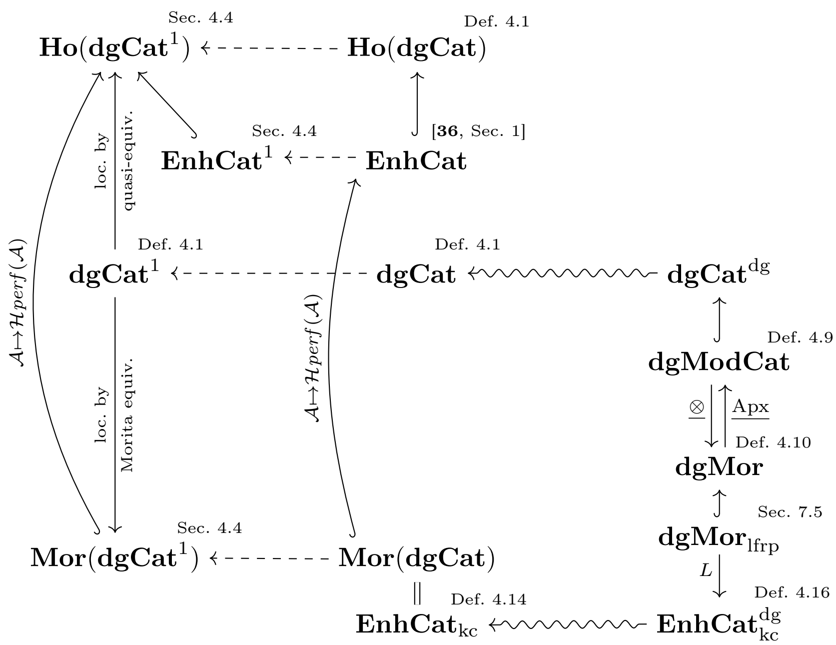


FIGURE 1. Summary of various categories of DG categories. Dashed arrows represent 1-categorical truncation and the squiggly arrow represents taking homotopy categories of the 1-morphism categories.

- $\mathbf{dgCat}^{\mathrm{dg}}$ is the strict DG 2-category of DG categories, DG functors, and (all) DG natural transformations,
- **dgModCat** is the strict DG 2-category of DG module categories. It is the 1-full subcategory of $\mathbf{dgCat}^{\mathrm{dg}}$ consisting of all DG categories of form $\mathcal{M}od\text{-}\mathcal{A}$ for some small DG category $\mathcal{A}$, see Section 4.3, Definition 4.9,
- **dgMor** is the DG bicategory whose objects are small DG categories and whose 1-morphism categories are DG categories of DG bimodules, see Section 4.3, Definition 4.10,
- $\mathbf{dgMor}_{\mathrm{lfrp}}$ is the 2-full subcategory of **dgMor** comprising the same objects and the 1-morphisms given by left-h-flat and right-perfect bimodules, see Sec. 7.5,
- $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ is the lax-unital DG bicategory of Morita enhanced triangulated categories. It is a DG enhancement of $\mathbf{EnhCat}_{\mathrm{kc}}$ and is a new object introduced in this paper, see Definition 4.16. Alternatively, it can be constructed as a strictly unital **Ho**(**dgCat**)-enriched bicategory, see Section 4.4, Definition 4.17.

4.1. Enriched bicategories

The DG version of the Heisenberg category, which we define in Chapter 5, is a certain weak 2-category and its representations are given by weak 2-functors. The notion of a weak 2-category we use is a *bicategory*. We refer to [**4**] for the original definition and a comprehensive technical treatment of bicategories.

We need to work with *enriched bicategories*. The natural structure to enrich bicategories over is a monoidal bicategory or, more generally, a tricategory. The formal definitions can be found in [**23**], and they are rather involved. However, a reader comfortable with the properties of cartesian products of categories and tensor products of DG categories need not consider the formal definition of a tricategory for the purposes of reading this paper. We only work with enrichments over one of the following three strictly monoidal strict 2-categories:

DEFINITION 4.1.

(1) **Cat**: The 2-category of isomorphism classes of small categories, of functors, and of natural transformations. The monoidal structure is the cartesian product $\times$.
(2) **dgCat**: The 2-category of isomorphism classes of small $\Bbbk$-linear DG categories, of DG functors, and of (closed degree zero) DG natural transformations. The monoidal structure is the tensor product $\otimes_{\Bbbk}$ over $\Bbbk$. We further write $\mathbf{dgCat}^1$ for the underlying 1-category of **dgCat**, where we only consider DG categories and DG functors between them.
(3) **Ho**(**dgCat**): The 2-categorical version considered in [**51**] of the localisation of $\mathbf{dgCat}^1$ by quasi-equivalences. Its objects are the isomorphism classes of small $\Bbbk$-linear DG categories, its 1-morphisms are the isomorphism classes of right quasi-representable bimodules in $\mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$, and its 2-morphisms are the morphisms between these in $\mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$. The monoidal structure is given by the tensor product $\otimes_{\Bbbk}$.

For the general definition of an enriched bicategory we refer the reader to [**22**, Section 3]. Considering only enrichments over strictly monoidal strict 2-categories allows us to give a simpler definition which is nearly identical to the original definition of a bicategory in [**4**].

DEFINITION 4.2. Let $(\mathbf{M}, \otimes, 1_{\mathbf{M}})$ be a strictly monoidal strict 2-category. A *bicategory* $\mathbf{C}$ *enriched over* $\mathbf{M}$ comprises the following data:

(1) a collection of *objects* $\operatorname{Ob}\mathbf{C}$;
(2) $\forall\, a, b \in \operatorname{Ob}\mathbf{C}$ a 1-*morphism object* $\mathbf{C}(a, b)$, which is an object in $\mathbf{M}$;
(3) $\forall\, a \in \operatorname{Ob}\mathbf{C}$ an *identity element* $1_a : 1_{\mathbf{M}} \to \mathbf{C}(a, a)$, which is a 1-morphism in $\mathbf{M}$;
(4) $\forall\, a, b, c \in \operatorname{Ob}\mathbf{C}$ the 1-*morphism composition*, which is a 1-morphism in $\mathbf{M}$:

$$\mu\colon \mathbf{C}(b, c) \otimes \mathbf{C}(a, b) \to \mathbf{C}(a, c);$$

(5) $\forall\, a, b, c, d \in \operatorname{Ob}\mathbf{C}$ the *associator* α which is a 2-isomorphism in $\mathbf{M}$:

$$\begin{array}{ccc} \mathbf{C}(c,d) \otimes \mathbf{C}(b,c) \otimes \mathbf{C}(a,b) & & \mathbf{C}(c,d) \otimes \mathbf{C}(b,c) \otimes \mathbf{C}(a,b) \\ \downarrow \mu\otimes\mathrm{id} & & \downarrow \mathrm{id}\otimes\mu \\ \mathbf{C}(b,d) \otimes \mathbf{C}(a,b) & \xrightarrow[\sim]{\alpha} & \mathbf{C}(c,d) \otimes \mathbf{C}(a,c) \\ \downarrow \mu & & \downarrow \mu \\ \mathbf{C}(a,d) & & \mathbf{C}(a,d); \end{array}$$

(6) $\forall\ a,b \in \operatorname{Ob}\mathbf{C}$ the *unitors* ρ and λ which are 2-isomorphisms of 1-morphisms in $\mathbf{M}$:

$$\begin{array}{ccc} \mathbf{C}(a,b) & & \mathbf{C}(a,b) \\ \| & & \\ \mathbf{C}(a,b)\otimes 1_{\mathbf{M}} & & \\ \downarrow \mathrm{id}\otimes 1_a & \xrightarrow[\sim]{\rho} & \Big\downarrow \mathrm{id} \\ \mathbf{C}(a,b)\otimes\mathbf{C}(a,a) & & \\ \downarrow \mu & & \\ \mathbf{C}(a,b) & & \mathbf{C}(a,b) \end{array}$$

and

$$\begin{array}{ccc} \mathbf{C}(a,b) & & \mathbf{C}(a,b) \\ \| & & \\ 1_{\mathbf{M}}\otimes\mathbf{C}(a,b) & & \\ \downarrow 1_a\otimes\mathrm{id} & \xrightarrow[\sim]{\lambda} & \Big\downarrow \mathrm{id} \\ \mathbf{C}(a,a)\otimes\mathbf{C}(a,b) & & \\ \downarrow \mu & & \\ \mathbf{C}(a,b) & & \mathbf{C}(a,b) \end{array}$$

which must satisfy the following conditions:

(7) $\forall\ a,b,c,d,e \in \operatorname{Ob}\mathbf{C}$ the following diagram of 2-morphisms between 1-morphisms $\mathbf{C}(d,e)\otimes\mathbf{C}(c,d)\otimes\mathbf{C}(b,c)\otimes\mathbf{C}(a,b)\to\mathbf{C}(a,e)$ must commute in $\mathbf{M}$:

$$\begin{array}{ccccc} \mu\circ(\mu\otimes\mathrm{id})\circ(\mu\otimes\mathrm{id}\otimes\mathrm{id}) & \xrightarrow{\mu\circ(\alpha\otimes\mathrm{id})} & \mu\circ(\mu\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mu\otimes\mathrm{id}) & \xrightarrow{\alpha\circ\mathrm{id}} & \mu\circ(\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes\mu\otimes\mathrm{id}) \\ \Big\downarrow \alpha\circ\mathrm{id} & & & & \Big\downarrow \mu\circ(\mathrm{id}\otimes\alpha) \\ \mu\circ(\mathrm{id}\otimes\mu)\circ(\mu\otimes\mathrm{id}\otimes\mathrm{id}) & & \xrightarrow{\alpha\circ\mathrm{id}} & & \mu\circ(\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mu); \end{array}$$

(8) $\forall\ a,b,c \in \operatorname{Ob}\mathbf{C}$ the following diagram of 2-morphisms between 1-morphisms $\mathbf{C}(b,c)\otimes\mathbf{C}(a,b)\to\mathbf{C}(a,c)$ must commute in $\mathbf{M}$:

$$\begin{array}{ccc} \mu\circ(\mu\otimes\mathrm{id})\circ(\mathrm{id}\otimes 1_b\otimes\mathrm{id}) & \xrightarrow{\alpha\circ(\mathrm{id}\otimes 1_b\,\mathrm{id})} & \mu\circ(\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes 1_b\otimes\mathrm{id}) \\ & {\scriptstyle \mu\circ(\rho\otimes\mathrm{id})}\searrow \quad \swarrow{\scriptstyle \mu\circ(\mathrm{id}\otimes\lambda)} & \\ & \mu & \end{array}.$$

Remark 4.3. The objects of the three 2-categories we define in Definition 4.1 are isomorphism classes of categories. This is to make the strictly associative monoidal structures provided by $\times$ and $\otimes_{\Bbbk}$ also strictly unital. To work with individual categories one only needs to adjust the definition above to allow the monoidal structure on $\mathbf{M}$ to be lax-unital.

Examples 4.4.

(1) A bicategory enriched over **Cat** is an ordinary bicategory in the sense of [**4**]. We refer to these simply as *bicategories*. Special cases are:
  (a) A bicategory with a single object is a *monoidal category*.
  (b) A bicategory whose associator and unitor isomorphisms are identity maps is a *strict* 2-*category*.

(2) A bicategory enriched over **dgCat** is a *DG bicategory*.

REMARK 4.5. Consider a DG bicategory $\mathbf{C}$. Then the data of the 1-composition functor $\mu = \circ_1 \colon \mathbf{C}(b,c) \otimes \mathbf{C}(a,b) \to \mathbf{C}(a,c)$ gives rise to the graded interchange law

$$(\alpha \circ_1 \beta) \circ_2 (\gamma \circ_1 \delta) = (-1)^{|\beta||\gamma|} (\alpha \circ_2 \gamma) \circ_1 (\beta \circ_2 \delta),$$

where we write $\circ_2$ for the 2-composition, i.e., the composition in the 1-morphism categories.

DEFINITION 4.6. Let $(\mathbf{M}, \otimes, 1_{\mathbf{M}})$ be a strictly monoidal strict 2-category. Let $\mathbf{C}$ and $\mathbf{D}$ be two bicategories enriched over $\mathbf{M}$. An *enriched* 2*-functor* $F \colon \mathbf{C} \to \mathbf{D}$ comprises

(1) a map $F \colon \operatorname{Ob} \mathbf{C} \to \operatorname{Ob} \mathbf{D}$;
(2) $\forall\, a, b \in \operatorname{Ob} \mathbf{C}$ a 1-morphism $F_{a,b}$ in $\mathbf{M}$,
$$F_{a,b} \colon \mathbf{C}(a,b) \to \mathbf{D}(Fa, Fb);$$
(3) $\forall\, a \in \operatorname{Ob} \mathbf{C}$ a *unit coherence* 2-morphism ι in $\mathbf{M}$ between the following 1-morphisms $1_{\mathbf{M}} \to \mathbf{D}(Fa, Fa)$:
$$\iota \colon 1_{Fa} \to F_{a,a} \circ 1_a;$$
(4) $\forall\, a, b, c \in \operatorname{Ob} \mathbf{C}$ a *composition coherence* 2-morphism ϕ in $\mathbf{M}$ between the following 1-morphisms $\mathbf{C}(b,c) \otimes \mathbf{C}(a,b) \to \mathbf{D}(Fa, Fc)$:
$$\phi \colon \mu_{\mathbf{D}} \circ (F_{b,c} \otimes F_{a,b}) \to F_{a,c} \circ \mu_{\mathbf{C}};$$

which must satisfy the following conditions:

(5) *associativity coherence:* $\forall\; a, b, c, d \in \operatorname{Ob} \mathbf{C}$ the following diagram of 2-morphisms between 1-morphisms $\mathbf{C}(c,d) \otimes \mathbf{C}(b,c) \otimes \mathbf{C}(a,b) \to \mathbf{D}(Fa, Fd)$ must commute in $\mathbf{M}$:

$$\begin{array}{ccc}
\mu_{\mathbf{D}} \circ (\mu_{\mathbf{D}} \otimes \mathrm{id}) \circ (F_{c,d} \otimes F_{b,c} \otimes F_{a,b}) & \xrightarrow{\alpha_{\mathbf{D}} \circ (F_{c,d} \otimes F_{b,c} \otimes F_{a,b})} & \mu_{\mathbf{D}} \circ (\mathrm{id} \otimes \mu_{\mathbf{D}}) \circ (F_{c,d} \otimes F_{b,c} \otimes F_{a,b}) \\
\downarrow{\scriptstyle \mu_{\mathbf{D}} \circ (\phi \otimes F_{a,b})} & & \downarrow{\scriptstyle \mu_{\mathbf{D}} \circ (F_{c,d} \otimes \phi)} \\
\mu_{\mathbf{D}} \circ (F_{b,d} \otimes F_{a,b}) \circ (\mu_{\mathbf{C}} \otimes \mathrm{id}) & & \mu_{\mathbf{D}} \circ (F_{c,d} \otimes F_{a,c}) \circ (\mathrm{id} \otimes \mu_{\mathbf{C}}) \\
\downarrow{\scriptstyle \phi \circ (\mu_{\mathbf{C}} \otimes \mathrm{id})} & & \downarrow{\scriptstyle \phi \circ (\mathrm{id} \otimes \mu_{\mathbf{C}})} \\
F_{a,d} \circ \mu_{\mathbf{C}} \circ (\mu_{\mathbf{C}} \otimes \mathrm{id}) & \xrightarrow{F_{a,d} \circ \alpha_{\mathbf{C}}} & F_{a,d} \circ \mu_{\mathbf{C}} \circ (\mathrm{id} \otimes \mu_{\mathbf{C}});
\end{array}$$

(6) *unitality coherence:* $\forall\, a, b \in \operatorname{Ob} \mathbf{C}$ the following diagrams of 2-morphisms between 1-morphisms $\mathbf{C}(a,b) \to \mathbf{D}(Fa, Fb)$ must commute in $\mathbf{M}$:

$$\begin{array}{ccc}
\mu_{\mathbf{D}} \circ (\mathrm{id} \otimes 1_{Fa}) \circ F_{a,b} & \xrightarrow{\mu_{\mathbf{D}} \circ (\mathrm{id} \otimes \iota) \circ F_{a,b}} & \mu_{\mathbf{D}} \circ (F_{a,b} \otimes F_{a,a}) \circ (\mathrm{id} \otimes 1_a) \\
\downarrow{\scriptstyle \rho_{\mathbf{D}} \circ F_{a,b}} & & \downarrow{\scriptstyle \phi \circ (\mathrm{id} \otimes 1_a)} \\
F_{a,b} & \xleftarrow{F_{a,b} \circ \rho_{\mathbf{C}}} & F_{a,b} \circ \mu_{\mathbf{C}} \circ (\mathrm{id} \otimes 1_a)
\end{array}$$

$$\begin{array}{ccc}
\mu_{\mathbf{D}} \circ (1_{Fb} \otimes \mathrm{id}) \circ F_{a,b} & \xrightarrow{\mu_{\mathbf{D}} \circ (\iota \otimes \mathrm{id}) \circ F_{a,b}} & \mu_{\mathbf{D}} \circ (F_{b,b} \otimes F_{a,b}) \circ (1_b \otimes \mathrm{id}) \\
\downarrow{\scriptstyle \lambda_{\mathbf{D}} \circ F_{a,b}} & & \downarrow{\scriptstyle \phi \circ (\mathrm{id} \otimes 1_a)} \\
F_{a,b} & \xleftarrow{F_{a,b} \circ \lambda_{\mathbf{C}}} & F_{a,b} \circ \mu_{\mathbf{C}} \circ (1_b \otimes \mathrm{id})
\end{array}$$

DEFINITION 4.7. A 2-functor is said to be:

- *strict* if its unit and composition coherence maps are the identity maps;
- *strong* if its unit and composition coherence maps are isomorphisms;

- *homotopy strong* if its unit and composition coherence maps are homotopy equivalences.
- *lax* if its unit and composition maps are not necessarily isomorphisms;

4.2. DG-categories

For an introduction to DG categories, DG modules, and the related technical notions, we refer the reader to [**1**, Section 2]. For an in-depth treatment in the language of model categories see [**52**]. Below we review the main notions we use.

4.2.1. DG categories and DG modules. A *DG (differential graded)* category $\mathcal{A}$ is a category enriched over the monoidal category $\mathcal{M}od\text{-}\Bbbk$ of complexes of $\Bbbk$-modules. A (right) module E over $\mathcal{A}$ is a functor $E\colon \mathcal{A}^{\text{opp}} \to \mathcal{M}od\text{-}\Bbbk$. For any $a \in \mathcal{A}$ we write E_a for the complex $E(a) \in \mathcal{M}od\text{-}\Bbbk$, the *fibre of E over a*. We write $\mathcal{M}od\text{-}\mathcal{A}$ for the DG category of (right) $\mathcal{A}$-modules. Similarly, a left $\mathcal{A}$-module F is a functor $F\colon \mathcal{A} \to \mathcal{M}od\text{-}\Bbbk$. We write ${}_aF$ for the fibre $F(a) \in \mathcal{M}od\text{-}\Bbbk$ of F over $a \in \mathcal{A}$ and $\mathcal{A}\text{-}\mathcal{M}od$ for the DG category of left $\mathcal{A}$-modules. For any $a \in \mathcal{A}$ define the right and left *representable* modules corresponding to a to be $h^r(a) = \operatorname{Hom}_{\mathcal{A}}(-, a) \in \mathcal{M}od\text{-}\mathcal{A}$ and $h^l(a) = \operatorname{Hom}_{\mathcal{A}}(a, -) \in \mathcal{A}\text{-}\mathcal{M}od$. We further have Yoneda embeddings $\mathcal{A} \hookrightarrow \mathcal{M}od\text{-}\mathcal{A}$ and $\mathcal{A}^{\text{opp}} \hookrightarrow \mathcal{A}\text{-}\mathcal{M}od$ whose images are the representable modules.

Given another DG category $\mathcal{B}$, an $\mathcal{A}$-$\mathcal{B}$-bimodule M is an $\mathcal{A}^{\text{opp}} \otimes_{\Bbbk} \mathcal{B}$-module, that is, a functor $M\colon \mathcal{A} \otimes_{\Bbbk} \mathcal{B}^{\text{opp}} \to \mathcal{M}od\text{-}\Bbbk$. For any $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we write ${}_aM \in \mathcal{M}od\text{-}\mathcal{B}$ for the fibre $M(a, -)$ of M over a, $M_b \in \mathcal{A}\text{-}\mathcal{M}od$ for the fibre $M(-, b)$ of M over b, and ${}_aM_b \in \mathcal{M}od\text{-}\Bbbk$ for the fibre of M over (a, b). We write $\mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B}$ for the DG category of $\mathcal{A}$-$\mathcal{B}$-bimodules. The categories $\mathcal{M}od\text{-}\mathcal{A}$ and $\mathcal{A}\text{-}\mathcal{M}od$ of right and left $\mathcal{A}$-modules can therefore be considered as the categories of $\Bbbk$-$\mathcal{A}$- and $\mathcal{A}$-$\Bbbk$-bimodules. For any DG category $\mathcal{A}$, we write $\mathcal{A}$ for the diagonal $\mathcal{A}$-$\mathcal{A}$-bimodule defined by ${}_a\mathcal{A}_b = \operatorname{Hom}_{\mathcal{A}}(b, a)$ for all $a, b \in \mathcal{A}$ and morphisms of $\mathcal{A}$ acting on the right and on the left by pre- and post-composition, respectively:

(4.1)
$$\mathcal{A}(\alpha \otimes \beta) = (-1)^{\deg(\beta)\deg(-)} \alpha \circ (-) \circ \beta, \qquad \forall\, \alpha \in \operatorname{Hom}_{\mathcal{A}}(a, a'), \beta \in \operatorname{Hom}_{\mathcal{A}}(b', b).$$

DG bimodules over DG categories admit a closed symmetric monoidal structure. Given three DG categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, we define functors

$$(-) \otimes_{\mathcal{B}} (-)\colon \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B} \otimes \mathcal{B}\text{-}\mathcal{M}od\text{-}\mathcal{C} \to \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{C},$$
$$\operatorname{Hom}_{\mathcal{B}}(-,-)\colon \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B} \otimes \mathcal{C}\text{-}\mathcal{M}od\text{-}\mathcal{B} \to \mathcal{C}\text{-}\mathcal{M}od\text{-}\mathcal{A},$$
$$\operatorname{Hom}_{\mathcal{B}}(-,-)\colon \mathcal{B}\text{-}\mathcal{M}od\text{-}\mathcal{A} \otimes \mathcal{B}\text{-}\mathcal{M}od\text{-}\mathcal{C} \to \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{C},$$

by

$$M \otimes_{\mathcal{B}} N = \operatorname{Coker}(M \otimes_{\Bbbk} \mathcal{B} \otimes_{\Bbbk} N \xrightarrow{\text{act} \otimes \text{id} - \text{id} \otimes \text{act}} M \otimes_{\Bbbk} N),$$

$${}_c\left(\operatorname{Hom}_{\mathcal{B}}(M, N)\right)_a = \operatorname{Hom}_{\mathcal{B}}({}_aM, {}_cN)$$

for M, N with right $\mathcal{B}$-action, and

$${}_a\left(\operatorname{Hom}_{\mathcal{B}}(M, N)\right)_c = \operatorname{Hom}_{\mathcal{B}}(M_c, N_a)$$

for M, N with left $\mathcal{B}$-action, cf. [**1**, Section 2.1.5].

4.2.2. The derived category of a DG category. Let $\mathcal{A}$ be a DG category. Its *homotopy category* $\mathrm{H}^0(\mathcal{A})$ is the $\Bbbk$-linear category whose objects are the same as those of $\mathcal{A}$ and whose morphism spaces are $\mathrm{H}^0(-)$ of the morphism complexes of $\mathcal{A}$.

The category $\mathrm{H}^0(\mathcal{M}od\text{-}\mathcal{A})$ has a natural structure of a triangulated category defined fibrewise in $\mathcal{M}od\text{-}\Bbbk$, that is: the homotopy category $\mathrm{H}^0(\mathcal{M}od\text{-}\Bbbk)$ of complexes of $\Bbbk$-modules has a natural triangulated structure, and we apply it in each fibre over each $a \in \mathcal{A}$ to define the triangulated structure on $\mathrm{H}^0(\mathcal{M}od\text{-}\mathcal{A})$. A DG category $\mathcal{A}$ is *pretriangulated* if $\mathrm{H}^0(\mathcal{A})$ is a triangulated subcategory of $\mathrm{H}^0(\mathcal{M}od\text{-}\mathcal{A})$ under the Yoneda embedding.

An $\mathcal{A}$-module E is *acyclic* if it is acyclic fibrewise in $\mathcal{M}od\text{-}\Bbbk$. We denote by $\mathcal{A}c\,\mathcal{A}$ the full subcategory of $\mathcal{M}od\text{-}\mathcal{A}$ consisting of acyclic modules. A morphism of $\mathcal{A}$-modules is a *quasi-isomorphism* if it is one levelwise in $\mathcal{M}od\text{-}\Bbbk$. The derived category $\mathrm{D}(\mathcal{A})$ is the localisation of $\mathrm{H}^0(\mathcal{M}od\text{-}\mathcal{A})$ by quasi-isomorphisms, constructed as the Verdier quotient $\mathrm{H}^0(\mathcal{M}od\text{-}\mathcal{A})/\,\mathcal{A}c\,\mathcal{A}$.

The derived category can also be constructed on the DG level. An $\mathcal{A}$-module P is *h-projective* (resp. *h-flat*) if $\mathrm{Hom}_{\mathcal{A}}(P, C)$ (resp. $P \otimes_{\mathcal{A}} C$) is an acyclic complex of $\Bbbk$-modules for any acyclic $C \in \mathcal{M}od\text{-}\mathcal{A}$ (resp. $C \in \mathcal{A}\text{-}\mathcal{M}od$). We denote by $\mathcal{P}(\mathcal{A})$ the full subcategory of $\mathcal{M}od\text{-}\mathcal{A}$ consisting of h-projective modules. It follows from the definition that in $\mathcal{P}(\mathcal{A})$ every quasi-isomorphism is a homotopy equivalence, and therefore we have $\mathrm{D}(\mathcal{A}) \cong \mathrm{H}^0(\mathcal{P}(\mathcal{A}))$. Alternatively, one uses Drinfeld quotients [**14**]: Given a DG category $\mathcal{A}$ with a full subcategory $\mathcal{C} \subset \mathcal{A}$, we can form the Drinfeld quotient $\mathcal{A}/\mathcal{C}$. When $\mathcal{A}$ and $\mathcal{C}$ are pretriangulated, this recovers the Verdier quotient as $\mathrm{H}^0(\mathcal{A}/\mathcal{C}) \cong \mathrm{H}^0(\mathcal{A})/\mathcal{C}$. Thus $\mathrm{D}(\mathcal{A}) = \mathrm{H}^0(\mathcal{M}od\text{-}\mathcal{A}/\,\mathcal{A}c\,\mathcal{A})$.

An object a of a triangulated category $\mathcal{T}$ is *compact* if $\mathrm{Hom}_{\mathcal{T}}(a, -)$ commutes with infinite direct sums. We write $\mathrm{D_c}(\mathcal{A})$ for the full subcategory of $\mathrm{D}(\mathcal{A})$ comprising compact objects. An $\mathcal{A}$-module E is *perfect* if $E \in \mathrm{D_c}(\mathcal{A})$. We write $\mathcal{P}erf(\mathcal{A})$ and $\mathcal{H}perf(\mathcal{A})$ for the full subcategories of $\mathcal{M}od\text{-}\mathcal{A}$ comprising perfect modules and h-projective, perfect modules.

Let $\mathcal{A}$ be a DG category. We denote by $\mathcal{P}re\text{-}\mathcal{T}r\,\mathcal{A}$ the DG category of one-sided twisted complexes over $\mathcal{A}$, see [**1**, Section 3.1]. It is a DG version of the notion of triangulated hull. There is a natural fully faithful inclusion $\mathcal{P}re\text{-}\mathcal{T}r\,\mathcal{A} \hookrightarrow \mathcal{M}od\text{-}\mathcal{A}$ and $\mathrm{H}^0(\mathcal{P}re\text{-}\mathcal{T}r\,\mathcal{A})$ is the triangulated hull of $\mathrm{H}^0(\mathcal{A})$ in $\mathrm{H}^0(\mathcal{M}od\text{-}\mathcal{A})$. Moreover, we have $\mathcal{P}re\text{-}\mathcal{T}r\,\mathcal{A} \subset \mathcal{H}perf(\mathcal{A})$ and $\mathrm{D_c}(\mathcal{A}) \cong \mathrm{H}^0(\mathcal{H}perf(\mathcal{A}))$ is the Karoubi completion of $\mathrm{H}^0(\mathcal{P}re\text{-}\mathcal{T}r\,\mathcal{A})$ in $\mathrm{H}^0(\mathcal{M}od\text{-}\mathcal{A})$. We say that $\mathcal{A}$ is *strongly pretriangulated* if $\mathcal{A} \hookrightarrow \mathcal{P}re\text{-}\mathcal{T}r\,\mathcal{A}$ is an equivalence.

Let $\mathcal{A}$ and $\mathcal{B}$ be DG categories and let M be an $\mathcal{A}$-$\mathcal{B}$-bimodule. We say that M is *$\mathcal{A}$-perfect* (resp. *$\mathcal{B}$-perfect*) if it is perfect levelwise in $\mathcal{A}$ (resp. $\mathcal{B}$). That is, ${}_aM$ (resp. M_b) is a perfect module for all $a \in \mathcal{A}$ (resp. $b \in \mathcal{B}$). Similarly, for other properties of modules such as h-projective, h-flat, or representable.

A DG category $\mathcal{A}$ is *smooth* if the diagonal bimodule $\mathcal{A}$ is perfect as an $\mathcal{A}$-$\mathcal{A}$-bimodule. It is *proper* if $\mathcal{A}$ is Morita equivalent (see Section 4.4) to a DG algebra which is perfect over $\Bbbk$. Equivalently, $\mathcal{A}$ is proper if and only if the total cohomology of each Hom-complex is finitely-generated and $\mathrm{D}(\mathcal{A})$ is compactly generated. See [**53**, Section 2.2] for further details on these two notions.

4.2.3. Restriction and extension of scalars. Let $\mathcal{A}$ and $\mathcal{B}$ be two DG categories and let M be an $\mathcal{A}$-$\mathcal{B}$-bimodule. Moreover, let $\mathcal{A}'$ and $\mathcal{B}'$ be another two DG categories and let $f\colon \mathcal{A}' \to \mathcal{A}$ and $g\colon \mathcal{B}' \to \mathcal{B}$ be DG functors. Define the *restriction of scalars of M along f and g* to be the $\mathcal{A}'$-$\mathcal{B}'$-bimodule ${}_fM_g$ defined as $M \circ (f \otimes_{\Bbbk} g)$. In particular, for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have ${}_a({}_fM_g)_b = {}_{f(a)}M_{g(b)}$. We write ${}_fM$ and M_g for ${}_fM_{\mathrm{id}}$ and ${}_{\mathrm{id}}M_g$, respectively.

Let $\mathcal{A}$ and $\mathcal{B}$ be two DG categories and let $f : \mathcal{A} \to \mathcal{B}$ be a DG functor. We have:

(1) The *restriction of scalars* functor

$$f_* \colon \mathcal{M}od\text{-}\mathcal{B} \to \mathcal{M}od\text{-}\mathcal{A},$$

is defined to be $(-) \otimes_{\mathcal{B}} \mathcal{B}_f$. It sends each $E \in \mathcal{M}od\text{-}\mathcal{B}$ to its restriction $E_f \in \mathcal{M}od\text{-}\mathcal{A}$, and therefore sends acyclic modules to acyclic modules.

(2) The *extension of scalars* functor

$$f^* \colon \mathcal{M}od\text{-}\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B},$$

is defined to be $(-) \otimes_{\mathcal{A}} {}_f\mathcal{B}$. For each $a \in \mathcal{A}$ it sends the representable module $h^r(a) \in \mathcal{M}od\text{-}\mathcal{A}$ to the representable module $h^r(f(a)) \in \mathcal{M}od\text{-}\mathcal{B}$. It follows that f^* restricts to a functor $\mathcal{H}perf(\mathcal{A}) \to \mathcal{H}perf(\mathcal{B})$.

(3) The *twisted extension of scalars* functor

$$f^! \colon \mathcal{M}od\text{-}\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B},$$

is defined to be $\mathrm{Hom}_{\mathcal{A}}(\mathcal{B}_f, -)$.

By Tensor-Hom adjunction, (f^*, f_*) and $(f_*, f^!)$ are adjoint pairs. As f_* preserves acyclic modules, f^* preserves h-projective modules and $f^!$ preserves h-injective modules.

4.3. Bimodule approximation

In this section, we define and describe the basic properties of a lax 2-functor $\underline{\mathrm{Apx}}$ which approximates DG functors between DG module categories by (the tensor functors defined by) DG bimodules. On per-functor basis, this was already examined by Keller in [**30**, Section 6.4]. We will apply the bimodule approximation to the first step in our construction of a categorical Fock space for our Heisenberg DG bicategory $\mathbf{H}_{\mathcal{V}}$ (see Section 7.5). At this first step, a representation of a simpler strict DG 2-category $\mathbf{H}'_{\mathcal{V}}$ is constructed with (non-derived) DG functors. The bimodule approximation turns these into DG bimodules which are then considered as enhanced exact functors, see Section 4.4.

We first look at bimodule approximation on the 1-categorical level.

DEFINITION 4.8. Let $\mathcal{A}$ and $\mathcal{B}$ be DG categories. The *bimodule approximation* functor is

$$\underline{\mathrm{Apx}} \colon \mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{A},\, \mathcal{M}od\text{-}\mathcal{B}) \to \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B}, \qquad F \mapsto F(\mathcal{A}),$$

where $F(\mathcal{A}) \in \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B}$ is the evaluation of F at the diagonal bimodule $\mathcal{A}$. In other words, ${}_aF(\mathcal{A})_b = F({}_a\mathcal{A})_b$ for all $a \in \mathcal{A}, b \in \mathcal{B}$.

The bimodule approximation functor $\underline{\mathrm{Apx}}$ is right adjoint to the *'tensor functor'* functor:

$$\underline{\otimes} \colon \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B} \to \mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{A},\, \mathcal{M}od\text{-}\mathcal{B}), \qquad M \mapsto (-) \otimes_{\mathcal{A}} M.$$

The adjunction unit $\eta\colon \mathrm{id} \to \underline{\mathrm{Apx}} \circ \underline{\otimes}$ is given by the natural isomorphism

$$M \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{A}} M \qquad \text{for } M \in \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B},$$

and thus $\underline{\otimes}$ is a fully faithful embedding.

The adjunction counit $\epsilon\colon \underline{\otimes} \circ \underline{\mathrm{Apx}} \to \mathrm{id}$ is given by the natural transformation

$$(-) \otimes_{\mathcal{A}} F(\mathcal{A}) \to F \qquad \text{for } F \in \mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{A},\, \mathcal{M}od\text{-}\mathcal{B}), \tag{4.2}$$

defined by the map

$$E \otimes_{\mathcal{A}} F(\mathcal{A}) \to F(E), \qquad \text{for } E \in \mathcal{M}od\text{-}\mathcal{A} \tag{4.3}$$

which is adjoint to the composition

$$E \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, E) \xrightarrow{F} \mathrm{Hom}_{\mathcal{B}}(F(\mathcal{A}),\, F(E)).$$

The map (4.3) is an isomorphism for representable E, and thus a homotopy equivalence for $E \in \mathcal{H}perf(\mathcal{A})$. This implies, as noted in [**30**, Section 6.4], that (4.2) yields an isomorphism of derived functors $\mathrm{D_c}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ and hence, for F continuous, of functors $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$.

We now consider two DG bicategories whose 1-morphisms are DG functors and DG bimodules, respectively. The objects of these bicategories are the same: morally, they are the categories of DG modules over small DG categories. For brevity, however, we define these objects to be the small DG categories themselves:

Definition 4.9. Define **dgModCat** to be the strict DG 2-category whose objects are small DG categories and whose 1-morphism categories **dgModCat**$(\mathcal{A}, \mathcal{B})$ are the DG categories $\mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{A},\, \mathcal{M}od\text{-}\mathcal{B})$ of DG functors between their DG module categories.

Definition 4.10. Define **dgMor** to be the following DG bicategory:

(1) Its *objects* are small DG categories.
(2) $\forall\, \mathcal{A}, \mathcal{B} \in \mathrm{Ob}$, the DG category of 1-*morphisms* from $\mathcal{A}$ to $\mathcal{B}$ is $\mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B}$.
(3) $\forall\, \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Ob}$ the 1-*composition functor* is the tensor product of bimodules:

$$\begin{aligned} \mathcal{B}\text{-}\mathcal{M}od\text{-}\mathcal{C} \otimes \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B} &\to \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{C} \\ (N, M) &\mapsto M \otimes_{\mathcal{B}} N. \end{aligned}$$

(4) $\forall\, \mathcal{A} \in \mathrm{Ob}$ the *identity* 1-*morphism* of $\mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{A}$ is the diagonal bimodule $\mathcal{A}$.
(5) The *associator* isomorphisms are the canonical isomorphisms

$$(M \otimes_{\mathcal{B}} N) \otimes_{\mathcal{C}} L \xrightarrow{\sim} M \otimes_{\mathcal{B}} (N \otimes_{\mathcal{C}} L).$$

(6) $\forall\, M \in \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B}$ the *left and right unitor* isomorphisms are the natural maps

$$\mathcal{A} \otimes_{\mathcal{A}} M \xrightarrow{\sim} M \qquad \text{and} \qquad M \otimes_{\mathcal{B}} \mathcal{B} \xrightarrow{\sim} M.$$

The 1-categorical functors $\underline{\otimes}$ package up into an obvious strong 2-functor.

Definition 4.11. Define the strong 2-functor

$$\underline{\otimes}\colon \mathbf{dgMor} \to \mathbf{dgModCat},$$

(1) On *objects*, $\underline{\otimes}$ is the identity map,
(2) On 1-*morphism categories*, $\underline{\otimes}$ is the 1-categorical functor $\underline{\otimes}$ defined above.

(3) For any small DG category $\mathcal{A}$, the *unit coherence* 2-morphism

$$1_{\underline{\otimes}\mathcal{A}} \to \underline{\otimes}(1_{\mathcal{A}}),$$

is the natural isomorphism:

$$\mathrm{id}_{\mathcal{M}od\text{-}\mathcal{A}} \xrightarrow{\sim} (-) \otimes_{\mathcal{A}} \mathcal{A}.$$

(4) For any small DG categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and any $M \in \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B}$ and $N \in \mathcal{B}\text{-}\mathcal{M}od\text{-}\mathcal{C}$, the *composition coherence* 2-morphism

$$\underline{\otimes}(N) \circ_1 \underline{\otimes}(M) \to \underline{\otimes}(N \circ_1 M),$$

is the natural transformation defined by canonical isomorphisms

$$(- \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} N \xrightarrow{\sim} (-) \otimes_{\mathcal{A}} (M \otimes_{\mathcal{B}} N).$$

Since the 2-functor $\underline{\otimes}$ is strong, it induces a natural structure of a (lax) 2-functor on the right adjoints of its 1-categorical components.

Definition 4.12. Define the lax 2-functor

$$\underline{\mathrm{Apx}} \colon \mathbf{dgModCat} \to \mathbf{dgMor},$$

as follows:

(1) On *objects*, $\underline{\mathrm{Apx}}$ is the identity map,
(2) On 1-*morphism categories*, $\underline{\mathrm{Apx}}$ is the 1-categorical functor $\underline{\mathrm{Apx}}$ defined above.
(3) For any small DG category $\mathcal{A}$, the *unit coherence* 2-morphism

$$1_{\underline{\mathrm{Apx}}\mathcal{A}} \to \underline{\mathrm{Apx}}(1_{\mathcal{A}}),$$

is the identity morphism of the diagonal bimodule $\mathcal{A}$.

(4) For any small DG categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and any

$$F \in \mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{A}, \mathcal{M}od\text{-}\mathcal{B})$$

and

$$G \in \mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{B}, \mathcal{M}od\text{-}\mathcal{C}),$$

the *composition coherence* 2-morphism

$$\underline{\mathrm{Apx}}(G) \circ_1 \underline{\mathrm{Apx}}(F) \to \underline{\mathrm{Apx}}(G \circ_1 F),$$

is given by the adjunction counit of $(\underline{\otimes}, \underline{\mathrm{Apx}})$ for G:

$$F(\mathcal{A}) \otimes_B G(\mathcal{B}) \xrightarrow{\epsilon_G} GF(\mathcal{A}). \tag{4.4}$$

In general, the 2-functor $\underline{\mathrm{Apx}}$ is not even homotopy strong. However, its unit coherence morphism is the identity map, while below we show that under certain assumptions on the DG functors F and G their composition coherence morphism (4.4) is a fibrewise homotopy equivalence or quasi-isomorphism. This is important for us, because then the composition of $\underline{\mathrm{Apx}}$ with one of the homotopy quotients of **dgMor** by acyclics discussed in Section 4.4 below becomes homotopy strong when restricted to such DG functors.

Proposition 4.13. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be small DG categories, and let*

$$F \in \mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{A}, \mathcal{M}od\text{-}\mathcal{B}) \quad \text{and} \quad G \in \mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{B}, \mathcal{M}od\text{-}\mathcal{C}).$$

Then

(1) If $G \cong (-) \otimes_{\mathcal{B}} M$ for some $M \in \mathcal{B}\text{-}\mathcal{M}od\text{-}\mathcal{C}$, then (4.4) *is an isomorphism.*

(2) If $G \cong \mathrm{Hom}_{\mathcal{B}}(N,-)$ for some $N \in \mathcal{C}$-$\mathcal{M}od$-$\mathcal{B}$ which is B-perfect and $\mathcal{B}$-h-projective, then (4.4) *is fibrewise a homotopy equivalence in $\mathcal{M}od$-$\mathcal{A}$.*

(3) If F restricts to a functor $\mathcal{H}perf(\mathcal{A}) \to \mathcal{H}perf(\mathcal{B})$, then (4.4) *is fibrewise a homotopy equivalence in $\mathcal{M}od$-$\mathcal{C}$.*

Proof. Assertion (1) is clear.
(2): If $G \cong \mathrm{Hom}_B(N,-)$, then the morphism (4.4) is the evaluation map

$$F(\mathcal{A}) \otimes_B \mathrm{Hom}_B(N,\, B) \xrightarrow{\text{eval}} \mathrm{Hom}_B(N,\, F(\mathcal{A})).$$

Since the fibres of $\mathcal{N}$ over $\mathcal{C}$ are perfect and h-projective $\mathcal{B}$-modules, the fibres of this map over $\mathcal{C}$ are homotopy equivalences in $\mathcal{M}od$-$\mathcal{A}$.

(3): Morphism (4.4) is the $(\underline{\otimes}, \underline{\mathrm{Apx}})$-adjunction counit for G applied to $F(\mathcal{A})$. Thus the fibres of (4.4) in $\mathcal{M}od$-$\mathcal{C}$ are given by applying the natural transformation

$$\mathrm{id} \otimes_{\mathcal{B}} G(\mathcal{B}) \xrightarrow{(4.3)} G,$$

to the fibres of $F(\mathcal{A})$ in $\mathcal{M}od$-$\mathcal{B}$. By assumption these fibres lie in $\mathcal{H}perf(\mathcal{B})$. Hence (4.3) is a homotopy equivalence. □

4.4. DG enhanced triangulated categories

DG enhancements were introduced by Bondal and Kapranov in [**6**]. A *DG enhancement* of a triangulated category $\mathcal{T}$ is a pretriangulated DG category $\mathcal{A}$ together with an exact equivalence $\mathrm{H}^0(\mathcal{A}) \cong \mathcal{T}$. These are considered up to quasi-equivalences and are naturally objects in $\mathbf{Ho}(\mathbf{dgCat}^1)$, the localisation of $\mathbf{dgCat}^1$ by quasi-equivalences [**51**]. We write $\mathbf{EnhCat}^1$ for the full subcategory of $\mathbf{Ho}(\mathbf{dgCat}^1)$ comprising pretriangulated DG categories and consider this to be the 1-category of enhanced triangulated categories.

A *Morita DG enhancement* of a triangulated category $\mathcal{T}$ is a small DG category $\mathcal{A}$ whose compact derived category $\mathrm{D_c}(\mathcal{A})$ is equivalent to $\mathcal{T}$. These are considered up to *Morita equivalences:* functors $\phi\colon \mathcal{A} \to \mathcal{B}$ such that $\phi^* : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ restricts to an equivalence $\mathrm{D_c}(\mathcal{A}) \to \mathrm{D_c}(\mathcal{B})$. They are thus naturally the objects of $\mathbf{Mor}(\mathbf{dgCat}^1)$, the localisation of $\mathbf{dgCat}^1$ by Morita equivalences [**46**].

Let $\mathcal{A}$ be a DG category. The Yoneda embedding $\mathcal{A} \hookrightarrow \mathcal{H}perf(\mathcal{A})$ is a Morita equivalence. Moreover, it identifies $\mathbf{Mor}(\mathbf{dgCat}^1)$ with the full subcategory of $\mathbf{Ho}(\mathbf{dgCat}^1)$ consisting of pretriangulated categories whose homotopy categories are Karoubi-complete. Thus working in the Morita setting means working with small Karoubi-complete triangulated categories, such as bounded derived categories of abelian categories. If $\mathcal{A}$ is an enhancement of a Karoubi-complete triangulated category $\mathcal{T}$, then it is also its Morita enhancement. Conversely, if $\mathcal{A}$ is a Morita enhancement of a triangulated category $\mathcal{T}$, then $\mathcal{T}$ is Karoubi-complete and $\mathcal{H}perf(\mathcal{A})$ is an ordinary enhancement of $\mathcal{T}$.

The advantage of Morita enhancements is that morphisms in $\mathbf{Mor}(\mathbf{dgCat}^1)$ admit a nice description. The morphisms from $\mathcal{A}$ to $\mathcal{B}$ in $\mathbf{Mor}(\mathbf{dgCat}^1)$ are in bijection with the isomorphism classes in $\mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$ of $\mathcal{B}$-perfect $\mathcal{A}$-$\mathcal{B}$ bimodules [**51**, Theorems 4.2, 7.2]. We define an *enhanced exact functor* $\mathcal{A} \to \mathcal{B}$ to be a B-perfect bimodule $M \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$. The underlying exact functor between the underlying triangulated categories is $(-) \otimes^{\mathbf{L}} M\colon \mathrm{D_c}(\mathcal{A}) \to \mathrm{D_c}(\mathcal{B})$. An *enhanced natural transformation* is a morphism in $\mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$ between $\mathcal{B}$-perfect bimodules.

The 1-category $\mathbf{Mor}(\mathbf{dgCat}^1)$ is thus refined to the following strict 2-category of Morita enhanced triangulated categories.

DEFINITION 4.14. Define the strict 2-category $\mathbf{EnhCat}_{\mathrm{kc}}$, with kc referring to Karoubi-complete, also denoted by $\mathbf{Mor}(\mathbf{dgCat})$, to consist of the following data:

(1) Its set of *objects* is the set of all small DG categories.
(2) For any two $\mathcal{A}, \mathcal{B} \in \operatorname{Ob} \mathbf{EnhCat}_{\mathrm{kc}}$, the category $\mathbf{EnhCat}_{\mathrm{kc}}(\mathcal{A}, \mathcal{B})$ of 1-*morphisms* from $\mathcal{A}$ to $\mathcal{B}$ is the skeleton of $D_{\mathcal{B}\text{-}\mathcal{P}erf}(\mathcal{A}\text{-}\mathcal{B})$.
(3) For any triple $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \operatorname{Ob} \mathbf{EnhCat}_{\mathrm{kc}}$ the 1-*composition* functor is given by the derived tensor product of bimodules:
$$\mathbf{EnhCat}_{\mathrm{kc}}(\mathcal{B}, \mathcal{C}) \times \mathbf{EnhCat}_{\mathrm{kc}}(\mathcal{A}, \mathcal{B}) \to \mathbf{EnhCat}_{\mathrm{kc}}(\mathcal{A}, \mathcal{C})$$
$$(M, N) \mapsto M \overset{\mathbf{L}}{\otimes}_{\mathcal{B}} N.$$
(4) For any $\mathcal{A} \in \operatorname{Ob} \mathbf{EnhCat}_{\mathrm{kc}}$ the *identity* 1-*morphism* of $\mathbf{EnhCat}_{\mathrm{kc}}(\mathcal{A}, \mathcal{A})$ is the diagonal bimodule $\mathcal{A}$.

We have the 2-functor $\mathbf{EnhCat}_{\mathrm{kc}} \to \mathbf{Cat}$ which sends each Morita enhancement $\mathcal{A}$ to its underlying triangulated category $\mathrm{D_c}(\mathcal{A})$, each enhanced functor $M \in D_{\mathcal{B}\text{-}\mathcal{P}erf}(\mathcal{A}\text{-}\mathcal{B})$ to its underlying exact functor $(-) \otimes^{\mathbf{L}} M$, and each morphism in $D_{\mathcal{B}\text{-}\mathcal{P}erf}(\mathcal{A}\text{-}\mathcal{B})$ to the induced natural transformation of these underlying exact functors.

The 2-category $\mathbf{EnhCat}_{\mathrm{kc}}$ can be identified, via the assignment $\mathcal{A} \mapsto \mathcal{H}perf(\mathcal{A})$ with the 2-full subcategory of Karoubi-complete categories in the strict 2-category $\mathbf{EnhCat}$ of enhanced triangulated categories defined in [**36**, Section 1]. The strict 2-category $\mathbf{EnhCat}$ is a 2-categorical refinement of the 1-category $\mathbf{EnhCat}^1$. It coincides with the homotopy category of the $(\infty, 2)$-category of DG categories in [**18**].

We next introduce a DG enhancement $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ of $\mathbf{EnhCat}_{\mathrm{kc}}$:

DEFINITION 4.15. A DG enhancement of a strict 2-category $\mathbf{A}$ is a DG bicategory $\mathbf{C}$ such that $\mathbf{A}$ is 2-equivalent to the strictification $\tilde{H}^0(\mathbf{C})$ of the bicategory $\mathrm{H}^0(\mathbf{C})$ obtained by taking skeletons of its 1-morphism categories.

We define the DG bicategory $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ in terms of the bar categories of modules and bimodules introduced in [**2**]. Given small DG categories $\mathcal{A}$ and $\mathcal{B}$, the bar-category $\mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{B}$ of $\mathcal{A}$-$\mathcal{B}$-bimodules is isomorphic to the DG category of DG $\mathcal{A}$-$\mathcal{B}$-bimodules with A_∞-morphisms between them [**2**, Prop. 3.5]. However, the bar-category has a simpler definition which avoids the complexities of the fully general A_∞-machinery.

We have $\mathrm{H}^0(\mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{B}) \cong \mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$, since all quasi-isomorphisms in $\mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{B}$ are homotopy equivalences. The bar-category $\mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{B}$ can be viewed as a more natural way to factor out the acyclic modules than taking the Drinfeld quotient: one does not introduce formal contracting homotopies which do not interact with the old morphisms, and thus retains the natural monoidal structure in the form of the bar-tensor product $\overline{\otimes}$ of bimodules. It corresponds to the A_∞-tensor product of A_∞-bimodules under the identification of the bar-category with the category of DG bimodules with A_∞-morphisms, see [**2**, Section 3.2].

DEFINITION 4.16. Define the homotopy unital DG bicategory $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ as follows:

(1) Its set of *objects* is the set of small DG categories.
(2) For any two $\mathcal{A}, \mathcal{B} \in \operatorname{Ob} \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$, the category of 1-*morphisms* from $\mathcal{A}$ to $\mathcal{B}$ is the full subcategory of $\mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{B}$ comprising $\mathcal{B}$-perfect bimodules.
(3) For any triple $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \operatorname{Ob} \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ the 1-composition functor is given by the bar tensor product of bimodules:

$$\mathcal{B}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{C} \otimes \mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{B} \to \mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{C}$$
$$(N, M) \mapsto M \mathbin{\overline{\otimes}_{\mathcal{B}}} N.$$

(4) For any $\mathcal{A} \in \operatorname{Ob} \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ the *identity* 1-*morphism* of $\mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{A}$ is the diagonal bimodule $\mathcal{A}$.
(5) The *associator* isomorphisms are the natural isomorphisms

$$(M \mathbin{\overline{\otimes}_{\mathcal{B}}} N) \mathbin{\overline{\otimes}_{\mathcal{C}}} L \xrightarrow{\sim} M \mathbin{\overline{\otimes}_{\mathcal{B}}} (N \mathbin{\overline{\otimes}_{\mathcal{C}}} L).$$

(6) The *left and right unitor* morphisms are given for any 1-morphism $M \in \mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{B}$ by the natural homotopy equivalences defined in [**2**, Section 3.3]:

$$\mathcal{A} \mathbin{\overline{\otimes}_{\mathcal{A}}} M \xrightarrow{\alpha_{\mathcal{A}}} M \qquad \text{and} \qquad M \mathbin{\overline{\otimes}_{\mathcal{B}}} \mathcal{B} \xrightarrow{\alpha_{\mathcal{B}}} M.$$

Note that the DG bicategory $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ is *homotopy unital*: its unitor morphisms are not isomorphisms, but only homotopy equivalences. On the homotopy level, such bicategories become genuine bicategories. Indeed, the strictified homotopy bicategory $\tilde{H}^0(\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}})$ is 2-isomorphic to $\mathbf{EnhCat}_{\mathrm{kc}}$. This is because $\mathrm{H}^0(\mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{B}) \cong \mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$ and $\mathrm{H}^0(\overline{\otimes}) \cong \otimes^{\mathbf{L}}$, see [**2**, Section 3.2].

The homotopy unitality of $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ does not interfere with our constructions. Its unitor morphisms have homotopy inverses which are genuine right inverses, see [**2**, Section 3.3].

We offer the following alternative construction of $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ where we use the monoidal Drinfeld quotient instead of bar-categories to kill the acyclic bimodules in **dgMor**. The original Drinfeld quotient [**14**] is not compatible with monoidal structures such as 1-composition in a bicategory. A construction by Shoikhet [**44**] fixes this, and in Section 4.6 we define the monoidal Drinfeld quotient of a DG bicategory. The price is the 1-composition no longer being a DG functor but a quasifunctor, that is, a 1-morphism in **Ho**(**dgCat**). Thus, in this alternative construction $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ is only a **Ho**(**dgCat**)-enriched bicategory.

Definition 4.17 (Alternative construction of $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$). Let $\mathbf{dgMor}_{\mathrm{lfrp}}$ denote the 2-full subcategory of **dgMor** comprising all objects and the 1-morphisms given by left-h-flat and right-perfect bimodules. The **Ho**(**dgCat**)-enriched bicategory $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ is the Drinfeld quotient of $\mathbf{dgMor}_{\mathrm{lfrp}}$ by its two-sided ideal of 1-morphisms given by acyclic bimodules.

For this paper, it does not matter which of the two constructions one uses. We use $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ as the target for a 2-represention of our Heisenberg **Ho**(**dgCat**)-enriched bicategory $\mathbf{H}_{\mathcal{V}}$. First, we construct a 2-functor from a simpler strict DG 2-category $\mathbf{H}'_{\mathcal{V}}$ to $\mathbf{dgMor}_{\mathrm{lfrp}}$, which is naturally a subcategory of both above versions of $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$. The two constructions should be viewed merely as two different ways to kill the acyclics in $\mathbf{dgMor}_{\mathrm{lfrp}}$. Thus we obtain the (same) 2-functor $\mathbf{H}'_{\mathcal{V}} \to \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ whichever version of the latter we use. This 2-functor is turned into the desired 2-representation of $\mathbf{H}_{\mathcal{V}}$ via a formal construction for which it is only important that acyclics are null-homotopic in $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$.

4.5. The perfect hull of a DG bicategory

In this section, describe the formalism of taking the perfect hull of a **dgCat**- or **Ho**(**dgCat**)-enriched bicategory. On the homotopy level, this corresponds to taking a Karoubi-completed triangulated hull of each 1-morphism category.

Let $\mathcal{A}$ and $\mathcal{B}$ be DG categories. We have a natural functor

$$\mathcal{M}od\text{-}\mathcal{A} \otimes \mathcal{M}od\text{-}\mathcal{B} \to \mathcal{M}od\text{-}(\mathcal{A} \otimes \mathcal{B}) \tag{4.5}$$

which is defined as the composition

$$\begin{array}{c} \mathcal{DGF}un(\mathcal{A}^{\mathrm{opp}},\, \mathcal{M}od\text{-}\Bbbk) \otimes \mathcal{DGF}un(\mathcal{B}^{\mathrm{opp}},\, \mathcal{M}od\text{-}\Bbbk) \\ \downarrow \\ \mathcal{DGF}un(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{B}^{\mathrm{opp}},\, \mathcal{M}od\text{-}\Bbbk \otimes \mathcal{M}od\text{-}\Bbbk) \\ \downarrow \\ \mathcal{DGF}un(\mathcal{A}^{\mathrm{opp}} \otimes \mathcal{B}^{\mathrm{opp}},\, \mathcal{M}od\text{-}\Bbbk) \end{array}$$

whose first map is due to functoriality of the tensor product of DG categories, and whose second map is due to the natural monoidal structure on $\mathcal{M}od\text{-}\Bbbk$ given by the tensor product over $\Bbbk$. Explicitly, given $E \in \mathcal{M}od\text{-}\mathcal{A}$ and $F \in \mathcal{M}od\text{-}\mathcal{B}$ the functor (4.5) maps $E \otimes F$ to an $\mathcal{A} \otimes \mathcal{B}$-module whose fibre over $(a, b) \in \mathcal{A} \otimes \mathcal{B}$ is the tensor product $E_a \otimes F_b$.

Let $\mathcal{C}$ be a DG category and let $\mu\colon \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ be a DG functor. It extends naturally to

$$\mu\colon \mathcal{M}od\text{-}\mathcal{A} \otimes \mathcal{M}od\text{-}\mathcal{B} \to \mathcal{M}od\text{-}\mathcal{C}$$

which is defined as the composition

$$\mathcal{M}od\text{-}\mathcal{A} \otimes \mathcal{M}od\text{-}\mathcal{B} \xrightarrow{(4.5)} \mathcal{M}od\text{-}(\mathcal{A} \otimes \mathcal{B}) \xrightarrow{\mu^*} \mathcal{M}od\text{-}\mathcal{C}.$$

Explicitly, given $E \in \mathcal{M}od\text{-}\mathcal{A}$ and $F \in \mathcal{M}od\text{-}\mathcal{B}$ we have for all $c \in \mathcal{C}$

$$\mu(E \otimes F)_c = \bigoplus_{a \in \mathcal{A},\, b \in \mathcal{B}} (E_a \otimes F_b) \otimes \mathrm{Hom}_{\mathcal{C}}\big(c,\, \mu(a \otimes b)\big)/\text{relations},$$

where the relations identify the actions of $\mathcal{A} \otimes \mathcal{B}$ on $E_a \otimes \mathcal{F}_b$ and on $\mu(a \otimes b)$.

The above generalises to the following.

Definition 4.18. Let $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{C}$ be DG categories.

(1) Define the functor

$$\varpi\colon \mathcal{M}od\text{-}\mathcal{A}_1 \otimes \cdots \otimes \mathcal{M}od\text{-}\mathcal{A}_n \to \mathcal{M}od\text{-}(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)$$

to be the composition

$$\bigotimes_{i=1}^{n} \mathcal{DGF}un\big(\mathcal{A}_i^{\mathrm{opp}},\, \mathcal{M}od\text{-}\Bbbk\big) \to \mathcal{DGF}un\Big(\bigotimes_{i=1}^{n} \mathcal{A}_i^{\mathrm{opp}},\, \bigotimes_{i=1}^{n} \mathcal{M}od\text{-}\Bbbk\Big)$$
$$\to \mathcal{DGF}un\Big(\bigotimes_{i=1}^{n} \mathcal{A}_i^{\mathrm{opp}},\, \mathcal{M}od\text{-}\Bbbk\Big),$$

whose first map is due to the functoriality of tensor product of DG categories and whose second map is due to the natural monoidal structure on $\mathcal{M}od\text{-}\Bbbk$.

(2) Define the functor

$$\Upsilon\colon \mathcal{DGF}un\big(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n,\, \mathcal{C}\big) \to \mathcal{DGF}un\big(\mathcal{M}od\text{-}\mathcal{A}_1 \otimes \cdots \otimes \mathcal{M}od\text{-}\mathcal{A}_n,\, \mathcal{M}od\text{-}\mathcal{C}\big)$$

to be the composition of the extension of scalars functor

$$(-)^*\colon \mathcal{DGF}un\big(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n,\, \mathcal{C}\big) \to \mathcal{DGF}un\big(\mathcal{M}od\text{-}(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n),\, \mathcal{M}od\text{-}\mathcal{C}\big)$$

with the functor of precomposition with ϖ.

Lemma 4.19. *For any DG categories $\mathcal{A}_1, \dots, \mathcal{A}_n$, $\mathcal{C}$ we have:*

(1) The functor Υ commutes with Yoneda embeddings, i.e. the following diagram commutes for any $\mu \in \mathcal{DGF}un(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n,\, \mathcal{C})$:

$$\begin{array}{ccc} \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n & \xrightarrow{\quad\mu\quad} & \mathcal{C} \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}od\text{-}\mathcal{A}_1 \otimes \cdots \otimes \mathcal{M}od\text{-}\mathcal{A}_n & \xrightarrow{\Upsilon(\mu)} & \mathcal{M}od\text{-}\mathcal{C}. \end{array}$$

(2) When $n = 1$, for any $\mu_1\colon \mathcal{A}_1 \to C$ we have $\Upsilon(\mu_1) = \mu_1^*$.

(3) $\Upsilon(\mathrm{id}) = \varpi$.

(4) Let $\mathcal{C}_1, \dots, \mathcal{C}_n$ be DG categories and $\mu_1, \dots, \mu_n$ be DG functors

$$\mu_i\colon \mathcal{A}_i \to \mathcal{C}_i.$$

Then

$$(\mu_1 \otimes \cdots \otimes \mu_n)^* \circ \varpi \;\cong\; \varpi \circ (\mu_1^* \otimes \cdots \otimes \mu_n^*).$$

(5) Let $\mu \in \mathcal{DGF}un(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n,\, \mathcal{C})$. Let $m_1, \dots, m_n \in \mathbb{Z}$, let

$$\{\mathcal{B}_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq m_i}$$

be DG categories, and $\{\lambda_i\}$ be DG functors

$$\lambda_i\colon \mathcal{B}_{i1} \otimes \cdots \otimes \mathcal{B}_{im_i} \to \mathcal{A}_i.$$

Then

$$\Upsilon\big(\mu \circ (\lambda_1 \otimes \cdots \otimes \lambda_n)\big) \cong \Upsilon(\mu) \circ \big(\Upsilon(\lambda_1) \otimes \cdots \otimes \Upsilon(\lambda_n)\big).$$

Proof. This is a straightforward verification.

For example, to establish (1), let μ be a functor $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to \mathcal{C}$. Then for any $a_1 \in \mathcal{A}_1, \dots, a_n \in \mathcal{A}_n$ we have

$$\Upsilon(\mu)(h^r(a_1) \otimes \cdots \otimes h^r(a_n)) = \mu^*(h^r(a_1 \otimes \cdots \otimes a_n)) = h^r\left(\mu(a_1 \otimes \cdots \otimes a_n)\right). \quad \square$$

Lemma 4.20. *For any DG categories $\mathcal{A}_1, \dots, \mathcal{A}_n$, $\mathcal{C}$ the functor*

$$\Upsilon\colon \mathcal{DGF}un(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n,\, \mathcal{C}) \to \mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{A}_1 \otimes \cdots \otimes \mathcal{M}od\text{-}\mathcal{A}_n,\, \mathcal{M}od\text{-}\mathcal{C}).$$

restricts to a functor

$$\Upsilon\colon \mathcal{DGF}un(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n,\, \mathcal{C}) \to \mathcal{DGF}un(\mathcal{H}perf\, \mathcal{A}_1 \otimes \cdots \otimes \mathcal{H}perf\, \mathcal{A}_n,\, \mathcal{H}perf\, \mathcal{C}).$$

Proof. For any $\mu\colon \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to C$ the functor $\Upsilon(\mu)\colon \mathcal{M}od\text{-}\mathcal{A}_1 \otimes \cdots \otimes \mathcal{M}od\text{-}\mathcal{A}_n \to \mathcal{M}od\text{-}\mathcal{C}$ takes tensor products of representables to representables, and therefore tensor products of h-projective, perfect modules to h-projective perfect modules. $\square$

We have the following key result.

PROPOSITION 4.21. *Let* $\mathbf{C}$ *be a DG (resp.* $\mathbf{Ho}(\mathbf{dgCat})$*-enriched) bicategory. The following set of data defines a DG (resp.* $\mathbf{Ho}(\mathbf{dgCat})$*-enriched) bicategory* $\tilde{\mathbf{C}}$*:*

- $\operatorname{Ob}\tilde{\mathbf{C}} \coloneqq \operatorname{Ob}\mathbf{C}$.
- *For each* $a, b \in \operatorname{Ob}\tilde{\mathbf{C}}$*,*
$$\tilde{\mathbf{C}}(a,b) \coloneqq \mathcal{H}perf\,\mathbf{C}(a,b).$$
- *For each* $a \in \tilde{\mathbf{C}}$*,*
$$\tilde{1}_a \coloneqq h^r(1_a).$$
That is, it is the representable module defined by the identity 1*-morphism of* a *in* $\mathbf{C}$.
- *For each* $a, b, c \in \operatorname{Ob}\tilde{\mathbf{C}}$ *the* 1*-composition functor*
$$\tilde{\mu}\colon \mathcal{H}perf\,\mathbf{C}(b,c) \otimes \mathcal{H}perf\,\mathbf{C}(a,b) \to \mathcal{H}perf\,\mathbf{C}(a,c)$$
is the extension $\Upsilon(\mu)$ *given in Lemma 4.20 of the* 1*-composition functor of* $\mathbf{C}$
$$\mu\colon C(b,c) \otimes \mathbf{C}(a,b) \to \mathbf{C}(a,c).$$
- *For each* $a, b, c, d \in \operatorname{Ob}\tilde{\mathbf{C}}$ *the natural associator isomorphism*
$$\tilde{\alpha}\colon \tilde{\mu}(\tilde{\mu} \otimes \mathrm{id}) \cong \tilde{\mu}(\mathrm{id} \otimes \tilde{\mu})$$
of functors
$$\mathcal{H}perf\,\mathbf{C}(c,d) \otimes \mathcal{H}perf\,\mathbf{C}(b,c) \otimes \mathcal{H}perf\,\mathbf{C}(a,b) \to \mathcal{H}perf\,\mathbf{C}(a,d)$$
is the conjugate of the extension $\Upsilon(\alpha)$ *of the associator isomorphism* α *of* $\mathbf{C}$ *by the isomorphism of Lemma 4.19 (5):*
$$\begin{array}{ccc} \Upsilon(\mu(\mu\otimes\mathrm{id})) & \xrightarrow{\Upsilon(\alpha)} & \Upsilon(\mu(\mathrm{id}\otimes\mu)) \\ \cong\uparrow & & \downarrow\cong \\ \Upsilon(\mu)\,(\Upsilon(\mu)\otimes\mathrm{id}) & \overset{\tilde{\alpha}}{\dashrightarrow} & \Upsilon(\mu)\,(\mathrm{id}\otimes\Upsilon(\mu))\,. \end{array}$$
- *Similarly, for each* $a, b \in \operatorname{Ob}\tilde{\mathbf{C}}$ *the unitor isomorphism* $\tilde{\lambda}$ *(resp.* $\tilde{\rho}$*) is the conjugate of the extension* $\Upsilon(\lambda)$ *(reps.,* $\Upsilon(\rho)$*) of the corresponding unitor isomorphism* λ *(resp.,* ρ*) of* $\mathbf{C}$ *by the isomorphism of Lemma 4.19 (5).*

PROOF. This is a straightforward verification. For example, to show that the diagram of Definition 4.2 (7) commutes for $\tilde{\mathbf{C}}$ we write, according to the definition, each instance of $\tilde{\alpha}$ in this diagram as a conjugate of $\Upsilon(\alpha)$ by the isomorphisms from Lemma 4.19 (5). The resulting diagram can then be simplified to the image under Υ of the the same associativity coherence diagram for $\mathbf{C}$. The claim then follows since the image of a commutative diagram under a functor is itself a commutative diagram. □

DEFINITION 4.22. Let $\mathbf{C}$ be a DG or $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory. The *perfect hull* of $\mathbf{C}$, denoted $\mathbf{Hperf}(\mathbf{C})$, is the bicategory defined in Proposition 4.21.

REMARK 4.23. Even when $\mathbf{C}$ is a strict 2-category, its perfect hull $\mathbf{Hperf}(\mathbf{C})$ is in general only a bicategory. Indeed, since $\Upsilon(\mathrm{id})$ is only isomorphic to the identity (being given by an extension of scalars), the unitor and associator isomorphisms of $\mathbf{Hperf}(\mathbf{C})$ will not be equal to the identity.

PROPOSITION 4.24. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be DG or* $\mathbf{Ho}(\mathbf{dgCat})$*-enriched bicategories and* $F\colon \mathbf{C} \to \mathbf{D}$ *a* 2*-functor. Then the following set of data defines a* 2*-functor*

$$\mathbf{Hperf}(F) : \mathbf{Hperf}(\mathbf{C}) \to \mathbf{Hperf}(\mathbf{D}).$$

- *The map*
$$F\colon \operatorname{Ob} \mathbf{Hperf}(\mathbf{C}) \to \operatorname{Ob} \mathbf{Hperf}(\mathbf{D})$$
which equals the map $F\colon \operatorname{Ob} \mathbf{C} \to \operatorname{Ob} \mathbf{D}$ *as taking the perfect hull of a bicategory does not change the objects.*
- *For every* $a, b \in \operatorname{Ob} \mathbf{Hperf}(\mathbf{C})$ *the functor*
$$\mathbf{Hperf}(F)_{a,b}\colon \mathbf{Hperf}\,\mathbf{C}(a,b) \to \mathbf{Hperf}\,\mathbf{D}(Fa, Fb)$$
is defined to be the extension of scalars functor $F^*_{a,b}$*.*
- *For every* $a \in \operatorname{Ob} \mathbf{Hperf}(\mathbf{C})$ *the* 2*-morphism*
$$\iota\colon 1_{Fa} \to \mathcal{H}perf\; F(1_a)$$
is the image under the Yoneda embedding of the corresponding 2*-morphism* ι_F *for* F*.*
- *For each* $a, b, c, \in \operatorname{Ob} \mathbf{Hperf}(\mathbf{C})$ *a natural transformation*
$$\phi\colon \mu_{\mathbf{Hperf}(\mathbf{D})} \circ (\mathbf{Hperf}(F)_{b,c} \otimes \mathbf{Hperf}(F)_{a,b}) \to \mathbf{Hperf}(F)_{a,c} \circ \mu_{\mathbf{Hperf}(\mathbf{C})},$$
which is the conjugate by the isomorphisms from Lemma 4.19 (5) of the extension $\Upsilon(\phi_F)$ *of the corresponding natural transformation for* F*.*

PROOF. This is a straightforward verification. □

REMARK 4.25. By replacing the perfect hull $\mathcal{H}perf\,\mathbf{C}(a,b)$ with the pretriangulated hull $\mathcal{P}re\text{-}\mathcal{T}r\,\mathbf{C}(a,b)$ in Proposition 4.21 and Definition 4.22, one obtains the pretriangulated hull $\mathbf{Pre\text{-}Tr}(\mathbf{C})$ of a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory $\mathbf{C}$.

4.6. Monoidal Drinfeld quotient

In this section we give a generalisation of the notion of the *Drinfeld quotient* of a DG category [**14**]. The original notion is not compatible with monoidal structures, which led Shoikhet to introduce in [**44**] the notion of a *refined Drinfeld quotient* and use it to construct the structure of a *weak Leinster monoid* on the Drinfeld quotient of a monoidal DG category by a two-sided ideal of objects.

Here, we use Shoikhet's construction to define the Drinfeld quotient of a DG bicategory by a two-sided ideal of 1-morphisms. The result is a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory. That is, the 1-composition is no longer given by DG functors, but by quasi-functors: compositions of genuine DG functors with formal inverses of quasi-equivalences.

We actually get a richer structure: 1-morphism spaces in the quotient bicategory are not abstract objects of $\mathbf{Ho}(\mathbf{dgCat})$, but concrete DG categories. These admit a multi-object analogue of a weak Leinster monoid structure. Localising by quasi-equivalences simplifies it to an ordinary, associative 1-composition, whence we obtain a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory.

Finally, our quotient construction works just as well with a bicategory that is already $\mathbf{Ho}(\mathbf{dgCat})$-enriched and produces again a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory.

Recall the original construction by Drinfeld:

DEFINITION 4.26 ([**14**, Section 3.1]). Let $\mathcal{C}$ be a DG category and $\mathcal{A} \subseteq \mathcal{C}$ a full DG subcategory. The *Drinfeld quotient* $\mathcal{C}/\mathcal{A}$ is the DG category freely generated over $\mathcal{C}$ by adding for each $a \in \mathcal{A}$ a degree -1 contracting homotopy $\xi_a : a \to a$ with $d\xi_a = \mathrm{id}_a$.

Explicitly:

(1) The objects of $\mathcal{C}/\mathcal{A}$ are those of $\mathcal{C}$.
(2) $\forall\, c, d \in \mathcal{C}$ the morphism complex $\mathrm{Hom}_{\mathcal{C}/\mathcal{A}}(c, d)$ comprises all composable words

$$f_n \xi_{a_n} f_{n-1} \cdots f_1 \xi_{a_1} f_0$$

with $a_1, \cdots, a_n \in \mathcal{A}$ and $f_0 \in \mathrm{Hom}_{\mathcal{C}}(c, a_1)$, $f_n \in \mathrm{Hom}_{\mathcal{C}}(a_n, d)$. Composable here means that $f_i \in \mathrm{Hom}_{\mathcal{C}}(a_i, a_{i+1})$ for $0 < i < n$. The degree of such a word is $(\sum \deg f_i) - n$. The differential is given by the Leibniz rule and, when differentiating one of the ξ_i, the subsequent composition of f_{i-1} and f_i in $\mathcal{C}$.
(3) The composition in $\mathcal{C}/\mathcal{A}$ is given by the concatenation of words and the subsequent composition in $\mathcal{C}$ of the two letters at which the concatenation happens.
(4) The identity morphisms in $\mathcal{C}/\mathcal{A}$ are the identity morphisms of $\mathcal{C}$.

We have the natural embedding $\mathcal{C} \to \mathcal{C}/\mathcal{A}$ which is the identity on objects. On morphisms, it considers morphisms of $\mathcal{C}$ as length 1 composable words; that is, $n = 0$ in the notation of Definition 4.26 (2). We thus have an embedding $\mathrm{D_c}(\mathcal{C}) \to \mathrm{D_c}(\mathcal{C}/\mathcal{A})$. It sends the objects of $\mathrm{D_c}(\mathcal{A}) \subset \mathrm{D_c}(\mathcal{C})$ to zero, and therefore by the universal property of the Verdier quotient it filters through a unique functor $\mathrm{D_c}(\mathcal{C})/\mathrm{D_c}(\mathcal{A}) \to \mathrm{D_c}(\mathcal{C}/\mathcal{A})$.

The main properties of the Drinfeld quotient are summarised as follows:

THEOREM 4.27 ([**14**, Theorem 1.6.2], [**47**, Theorem 4.0.3]). *Let $\mathcal{C}$ be a DG category and let $\mathcal{A} \subseteq \mathcal{C}$ be a full subcategory. Then:*

(1) The natural functor $\mathrm{D_c}(\mathcal{C})/\mathrm{D_c}(\mathcal{A}) \to \mathrm{D_c}(\mathcal{C}/\mathcal{A})$ *is an exact equivalence.*
(2) Let $\mathcal{B}$ be a DG category. The natural functor $\mathcal{C} \to \mathcal{C}/\mathcal{A}$ gives a fully faithful functor

$$\mathrm{Hom}_{\mathbf{Ho(dgCat)}}(\mathcal{C}/\mathcal{A}, \mathcal{B}) \to \mathrm{Hom}_{\mathbf{Ho(dgCat)}}(\mathcal{C}, \mathcal{B}),$$

whose image comprises the quasi-functors whose underlying functors

$$\mathrm{H}^0(\mathcal{C}) \to \mathrm{H}^0(\mathcal{B})$$

send the objects of $\mathcal{A}$ to zero.

Let $\mathbf{C}$ be a DG bicategory. For any collections $\mathbf{A}, \mathbf{B}$ of 1-morphisms of $\mathbf{C}$, write $\mathbf{A} \circ_1 \mathbf{B}$ for the collection of 1-morphisms of $\mathbf{C}$ 2-isomorphic to $a \circ_1 b$ with $a \in \mathbf{A}$, $b \in \mathbf{B}$. The *two-sided ideal* $\mathbf{I_A}$ *generated by* $\mathbf{A}$ is the 2-full subcategory of $\mathbf{C}$ supported on objects and 1-morphisms of $\mathbf{C} \circ_1 \mathbf{A} \circ_1 \mathbf{C}$. Here, by abuse of notation, $\mathbf{C}$ denotes the collection of all its 1-morphisms.

Let $\mathbf{C}$ be a DG bicategory and $\mathbf{A}$ a collection of 1-morphisms of $\mathbf{C}$. For any $a, b \in \mathbf{C}$ write $\mathbf{C}(a, b)/\mathbf{A}$ for the Drinfeld quotient $\mathbf{C}(a, b)/\mathbf{A}(a, b)$. These do not apriori form a bicategory. First of all, any 1-composition involving a contractible 1-morphism would have to be contractible. Were a bicategory structure to exist, for any $f \in \mathbf{A}(a, b)$ and any $g \in \mathbf{C}(b, c)$ the 1-composition $\mathrm{id}_g \circ_1 \xi_f$ would have to be a contracting homotopy for $g \circ_1 f$. Unless $g \circ_1 f$ lies in $\mathbf{A}(a,c)$, there is no reason for it to be contractible in $\mathbf{C}(a,c)/\mathbf{A}$.

This could be rectified by replacing $\mathbf{A}$ with two-sided ideal $\mathbf{I_A}$. We could then attempt to define the 1-composition $\mathrm{id}_g \circ_1 \xi_f$ to be contracting homotopy $\xi_{g\circ_1 f}$. However, the interchange law (3.3) for 1-composition demands that for any 2-morphism $\alpha\colon g \to h$ in $\mathbf{C}(b,c)$ we have:

$$(\mathrm{id}_g \circ_1 \xi_f) \circ_2 (\alpha \circ_1 \mathrm{id}_f) = \alpha \circ_1 \xi_f = (-1)^{\deg(\alpha)}(\alpha \circ_1 \mathrm{id}_f) \circ_1 (\mathrm{id}_g \circ_1 \xi_f).$$

If we define $\mathrm{id}_g \circ_1 \xi_f = \xi_{g\circ_1 f}$, this would then ask for $\xi_{g\circ_1 f}$ to supercommute with $(\alpha \circ_1 \mathrm{id}_f)$. But, by definition, there are no relations between $\xi_{g\circ_1 f}$ and any 2-morphisms in $\mathbf{C}(a,c)$!

This is why the original Drinfeld quotient works poorly with monoidal structures: it is freely generated by the contracting homotopies ξ_f over the original category. Thus ξ_f cannot satisfy the relations in the definition of 1-composition. The 1-composition functor

$$\circ_1\colon \mathbf{C}(b,c)/\mathbf{I_A} \otimes \mathbf{C}(a,b)/\mathbf{I_A} \to \mathbf{C}(a,c)/\mathbf{I_A}$$

could not exist because its target is a free category generated by ξ_f, while its source is not.

In [**44**], Shoikhet solves this by constructing a free resolution of

$$\mathbf{C}(b,c)/\mathbf{I_A} \otimes \mathbf{C}(a,b)/\mathbf{I_A},$$

which admits a natural 1-composition functor into $\mathbf{C}(a,b)/\mathbf{I_A}$. He defines:

Definition 4.28 ([**44**, Section 4.3]). Let $\mathcal{C}$ be a DG category and let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be full subcategories. The *refined Drinfeld quotient* $\mathcal{C}/(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is the DG category whose underlying graded category is freely generated over that of $\mathcal{C}$ by introducing for any

$$i_1 < i_2 < \cdots < i_k \text{ and } a \in \mathcal{A}_{i_1} \cap \cdots \cap \mathcal{A}_{i_k}$$

a new degree k element

$$\xi_a^{i_1 \dots i_k}.$$

The differential on these new elements is defined by setting

$$d\xi_a^{i_1} = \mathrm{id}_a$$

and for $k > 1$

$$d\xi_a^{i_1 \dots i_k} = \sum_{j=1}^{k} (-1)^{j-1} \xi_a^{i_1 \dots \hat{i}_j \dots i_k}.$$

Remark 4.29. When $n = 1$, the refined Drinfeld quotient $\mathcal{C}/\mathcal{A}_1$ coincides with the ordinary Drinfeld quotient. In this case we therefore omit the superscript in the notation above and write ξ_a for ξ_a^1.

The reason for considering the above as a refinement of the original Drinfeld quotient is the following theorem by Shoikhet:

Theorem 4.30 ([**44**, Lemma 4.3]). *Let $\mathcal{C}$ be a DG category and let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be full subcategories. The functor*

$$\Psi\colon \mathcal{C}/(\mathcal{A}_1, \dots, \mathcal{A}_n) \to \mathcal{C}/\bigcup_{i=1}^{n} \mathcal{A}_i$$

defined as the identity on objects and morphisms of $\mathcal{C}$ *and as*

$$\Psi(\xi_a^{i_1}) = \xi_a,$$
$$\Psi(\xi_a^{i_1\dots i_k}) = 0 \qquad \textit{for } k > 1,$$

is a quasi-equivalence of DG categories.

Observe that $\mathcal{C}/(\mathcal{A}_1, \dots, \mathcal{A}_n)$ and $\mathcal{C}/\bigcup_{i=1}^n \mathcal{A}_i$ are therefore isomorphic in $\mathbf{Ho}(\mathbf{dgCat})$. It follows from Theorem 4.27 that the former enjoys the same unique lifting property as the latter with respect to quasifunctors out of $\mathcal{C}$ which kill $\bigcup_{i=1}^n \mathcal{A}_i$ on the homotopy level.

At the same time, the next example shows that the refined Drinfeld quotient serves as a free resolution of the tensor product of ordinary Drinfeld quotients.

Example 4.31. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be DG categories and $\mathcal{A}_i \subset \mathcal{C}_i$ be full subcategories. Let

$$\beta_{\mathrm{Dr}} \colon \mathcal{C}_1 \otimes \mathcal{C}_2/(\mathcal{A}_1 \otimes \mathcal{C}_2, \mathcal{C}_1 \otimes \mathcal{A}_2) \to (\mathcal{C}_1/\mathcal{A}_1) \otimes (\mathcal{C}_2/\mathcal{A}_2)$$

be the functor defined as the identity on objects and the morphisms of $\mathcal{C}_1 \otimes \mathcal{C}_2$ and as follows on the contracting homotopies:

$$\beta_{\mathrm{Dr}}(\xi^1_{a_1 \otimes c_2}) = \xi_{a_1} \otimes \mathrm{id}_{c_2},$$
$$\beta_{\mathrm{Dr}}(\xi^2_{c_1 \otimes a_2}) = \mathrm{id}_{c_1} \otimes \xi_{a_2},$$
$$\beta_{\mathrm{Dr}}(\xi^{12}_{a_1 \otimes a_2}) = \xi_{a_1} \otimes \xi_{a_2}.$$

It can be readily verified that β_{Dr} is a quasi-equivalence of DG categories.

The above example can be formalised as follows:

Definition 4.32 ([**44**, Section 4.4]). Let $\mathcal{P}dg\mathcal{C}at$ be the following category:

- Its objects are pairs $(\mathcal{C};\, \mathcal{A}_1, \dots, \mathcal{A}_n)$ where $\mathcal{C}$ is a DG category and $\mathcal{A}_1, \dots, \mathcal{A}_n$ is an ordered n-tuple of full subcategories of $\mathcal{C}$.
- A morphism

$$(\mathcal{C};\, \mathcal{A}_1, \dots, \mathcal{A}_n) \to (\mathcal{D};\, \mathcal{B}_1, \dots, \mathcal{B}_m)$$

  is a pair (F, f) where $f \colon \{1, \dots, n\} \to \{1, \dots, m\}$ is a map of sets and $F \colon \mathcal{C} \to \mathcal{D}$ is a DG functor such that $F(\mathcal{A}_i) \subset \mathcal{B}_{f(j)}$.

We define a monoidal structure on $\mathcal{P}dg\mathcal{C}at$ by setting

$$(\mathcal{C};\, \mathcal{A}_1, \dots, \mathcal{A}_n) \otimes (\mathcal{D};\, \mathcal{B}_1, \dots, \mathcal{B}_m)$$

to be

$$(\mathcal{C} \otimes \mathcal{D};\, \mathcal{A}_1 \otimes \mathcal{D}, \dots, \mathcal{A}_n \otimes \mathcal{D}, \mathcal{C} \otimes \mathcal{B}_1, \dots, \mathcal{C} \otimes \mathcal{B}_m)$$

and the unit to be $(\Bbbk; \emptyset)$.

Theorem 4.33 ([**44**, Section 4.4]). *The refined Drinfeld quotient defines a functor:*

$$Dr \colon \mathcal{P}dg\mathcal{C}at \to \mathbf{dgCat}^1,$$

which has a natural homotopy monoidal structure given by quasi-equivalences

$$\beta \colon Dr\big((\mathcal{C};\, \mathcal{A}_1, \dots, \mathcal{A}_n) \otimes (\mathcal{D};\, \mathcal{B}_1, \dots, \mathcal{B}_m)\big)$$
$$\to Dr\,(\mathcal{C};\, \mathcal{A}_1, \dots, \mathcal{A}_n) \otimes Dr\,(\mathcal{D};\, \mathcal{B}_1, \dots, \mathcal{B}_m)\,.$$

The case considered in Example 4.31 follows by observing that in $\mathcal{P}dg\mathcal{C}at$ we have

$$(\mathcal{C};\,\mathcal{A})\otimes(\mathcal{D};\,\mathcal{B}) = (\mathcal{C}\otimes\mathcal{D};\,\mathcal{A}\otimes\mathcal{D},\mathcal{C}\otimes\mathcal{B}).$$

We now return to the problem of constructing the Drinfeld quotient of a DG bicategory. Let $\mathbf{C}$ be a DG bicategory and $\mathbf{A}$ a collection of its 1-morphisms. The 1-composition functor

$$\circ_1\colon \mathbf{C}(b,c)/\mathbf{I_A}\otimes\mathbf{C}(a,b)/\mathbf{I_A}\to\mathbf{C}(a,c)/\mathbf{I_A},$$

which does not exist in $\mathbf{dgCat}$, can now be defined in $\mathbf{Ho}(\mathbf{dgCat})$ as follows. The homotopy monoidal structure of the refined Drinfeld quotient functor gives us a quasi-equivalence

$$\beta_{\mathrm{Dr}}\colon \mathbf{C}(b,c)\otimes\mathbf{C}(a,b)/\left(\mathbf{I_A}\otimes\mathbf{C},\mathbf{C}\otimes\mathbf{I_A}\right)\longrightarrow\mathbf{C}(b,c)/\mathbf{I_A}\otimes\mathbf{C}(a,b)/\mathbf{I_A}.$$

On the other hand, since $\mathbf{I_A}$ is a two-sided ideal, the original 1-composition functor

$$\circ_1^{\mathrm{old}}\colon \mathbf{C}(b,c)\otimes\mathbf{C}(a,b)\to\mathbf{C}(a,c),$$

takes $\mathbf{I_A}(b,c)\otimes\mathbf{C}(a,b)$ and $\mathbf{C}(b,c)\otimes\mathbf{I_A}(a,b)$ to $\mathbf{I_A}(a,c)$, and thus uniquely extends in $\mathbf{Ho}(\mathbf{dgCat})$ to a quasi-functor

$$\circ_1^{\mathrm{old}}\colon \mathbf{C}(b,c)\otimes\mathbf{C}(a,b)/\left(\mathbf{I_A}\otimes\mathbf{C},\mathbf{C}\otimes\mathbf{I_A}\right)\to\mathbf{C}(a,c)/\mathbf{I_A}.$$

We can therefore define $\circ_1$ in $\mathbf{Ho}(\mathbf{dgCat})$ as the composition

$$\mathbf{C}(b,c)/\mathbf{I_A}\otimes\mathbf{C}(a,b)/\mathbf{I_A}\xrightarrow{\beta_{\mathrm{Dr}}^{-1}}\mathbf{C}(b,c)\otimes\mathbf{C}(a,b)/\left(\mathbf{I_A}\otimes\mathbf{C},\mathbf{C}\otimes\mathbf{I_A}\right)\xrightarrow{\circ_1^{\mathrm{old}}}\mathbf{C}(a,c)/\mathbf{I}.$$

THEOREM 4.34. *Let $\mathbf{C}$ be a DG bicategory, or more generally a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory. Let $\mathbf{A}$ be a collection of 1-morphisms in $\mathbf{C}$, and let $\mathbf{I_A}$ be the two-sided ideal generated by $\mathbf{A}$. Then the following data defines a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory:*

- *The same set of objects as $\mathbf{C}$.*
- *For any $a,b\in\mathbf{C}$, the DG category of 1-morphisms from a to b is $\mathbf{C}(a,b)/\mathbf{I_A}$.*
- *For any $a,b,c\in\mathbf{C}$, the 1-composition functor*

$$\circ_1\colon \mathbf{C}(b,c)/\mathbf{I_A}\otimes\mathbf{C}(a,b)/\mathbf{I_A}\xrightarrow{\beta_{\mathrm{Dr}}^{-1}}\mathbf{C}(b,c)\otimes\mathbf{C}(a,b)/\left(\mathbf{I_A}\otimes\mathbf{C},\mathbf{C}\otimes\mathbf{I_A}\right)\xrightarrow{\circ_1^{\mathrm{old}}}\mathbf{C}(a,c)/\mathbf{I_A},$$

- *The associator and unitor 2-isomorphisms in $\mathbf{Ho}(\mathbf{dgCat})$ which are similarly obtained from the associator and unitor 2-isomorphisms of $\mathbf{C}$ via the precomposition with β_{Dr}^{-1}.*

  For example, for any $a,b,c,d\in\mathbf{C}$, the quasi-functors $\circ_1(\circ_1\otimes\mathrm{id})$ and $\circ_1(\mathrm{id}\otimes\circ_1)$:

$$\mathbf{C}(c,d)/\mathbf{I_A}\otimes\mathbf{C}(b,c)/\mathbf{I_A}\otimes\mathbf{C}(a,b)/\mathbf{I_A}\to\mathbf{C}(a,d)/\mathbf{I_A}$$

  are the composition of the quasi-functor β_{Dr}^{-1} :

$$\mathbf{C}(c,d)/\mathbf{I_A}\otimes\mathbf{C}(b,c)/\mathbf{I_A}\otimes\mathbf{C}(a,b)/\mathbf{I_A}\to \mathbf{C}(c,d)\otimes\mathbf{C}(b,c)\otimes\mathbf{C}(a,b)/\left(\mathbf{I_A}\otimes\mathbf{C}\otimes\mathbf{C},\mathbf{C}\otimes\mathbf{I_A}\otimes\mathbf{C},\mathbf{C}\otimes\mathbf{C}\otimes\mathbf{I_A}\right)$$

  with the quasi-functors $\circ_1^{\mathrm{old}}(\circ_1^{\mathrm{old}}\otimes\mathrm{id})$ and $\circ_1^{\mathrm{old}}(\mathrm{id}\otimes\circ_1^{\mathrm{old}})$:

$$\mathbf{C}(c,d)\otimes\mathbf{C}(b,c)\otimes\mathbf{C}(a,b)/\left(\mathbf{I_A}\otimes\mathbf{C}\otimes\mathbf{C},\mathbf{C}\otimes\mathbf{I_A}\otimes\mathbf{C},\mathbf{C}\otimes\mathbf{C}\otimes\mathbf{I_A}\right)\to\mathbf{C}(a,d)/\mathbf{I_A}.$$

We thus define the new associator by precomposing the old associator with β_{Dr}^{-1}.

Proof. Shoikhet began his proof of [**44**, Theorem 5.4] by constructing a Leinster monoid $F_{\mathcal{A}}$ in $\mathcal{P}dg\mathcal{C}at$ out of a certain monoidal DG category $\mathcal{A}_0$ and the two-sided ideal $\mathcal{I}_0$ of acyclic objects in it. His construction works exactly the same for an arbitrary monoidal DG category $\mathcal{A}$ and an arbitrary two-sided ideal $\mathcal{I}$ in $\mathcal{A}$.

In general, a Leinster monoid $L_\bullet$ is a simplicial structure, generalising the notion of an algebra in a monoidal category, cf. [**44**, Defn. 2.1]. It has colax maps $\beta_{m,n}\colon L_{m+n} \to L_m \otimes L_n$ which are weak equivalences, and thus each L_n is weakly equivalent to $(L_1)^{\otimes n}$. The non-extremal face maps $L_n \to L_{n-1}$ should be thought of as analogues of applying the algebra operation to subsequent pairs of L_1's in $(L_1)^{\otimes n}$, and the degeneracy maps $L_n \to L_{n+1}$ as applying the algebra unit in between two subsequent L_1's. It follows that if the colax maps are not just weak equivalences, but isomorphisms, we have a unital algebra structure on L_1 whose algebra operation is

$$L_1 \otimes L_1 \xrightarrow{\beta_{1,1}^{-1}} L_2 \xrightarrow{\text{the unique non-extremal face}} L_1$$

and whose unit is the degeneracy map $\mathbb{1} \cong L_0 \to L_1$.

The colax maps of the Leinster monoid $F_{\mathcal{A}}$ in $\mathcal{P}dg\mathcal{C}at$ constructed by Shoikhet are the identity maps and $(F_{\mathcal{A}})_1 = (\mathcal{A};\mathcal{I})$. Applying the refined Drinfeld quotient functor, we obtain Leinster monoid $\mathrm{Dr}(F_{\mathcal{A}})$ in $\mathbf{dgCat}^1$ whose colax maps are the quasi-equivalences β_{Dr} and $(\mathrm{Dr}(F_{\mathcal{A}}))_1 = \mathcal{A}/\mathcal{I}$. We then view it as a Leinster monoid in $\mathbf{Ho}(\mathbf{dgCat}^1)$. There its colax maps become invertible, and we obtain the induced structure of unital algebra on $\mathcal{A}/\mathcal{I}$ in $\mathbf{Ho}(\mathbf{dgCat}^1)$. This structure is the one claimed in the assertion of this Theorem. Thus we have proved the Theorem for an arbitrary monoidal DG category, i.e. a DG bicategory with a single object. The proof for a general DG bicategory works identically, but with a more cumbersome notation. □

Definition 4.35. Let $\mathbf{C}$ be a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory. Let $\mathbf{A}$ be a collection of 1-morphisms in $\mathbf{C}$ and $\mathbf{I_A}$ be the two-sided ideal generated by $\mathbf{A}$. The *monoidal Drinfeld quotient* $\mathbf{C}/\mathbf{I_A}$ is the $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory constructed in Theorem 4.34.

We have a natural functor $\mathbf{C} \to \mathbf{C}/\mathbf{I_A}$ which is a strict 2-functorial embedding:

Definition 4.36. Let $\mathbf{C}$ be a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory. Let $\mathbf{A}$ be a collection of 1-morphisms in $\mathbf{C}$, and let $\mathbf{I_A}$ be the two-sided ideal generated by $\mathbf{A}$. Define a strict 2-functor

$$\iota\colon \mathbf{C} \hookrightarrow \mathbf{C}/\mathbf{I_A},$$

to be the identity on the objects. On 1-morphisms, for any $a, b \in \mathbf{C}$ define

$$\iota\colon \mathbf{C}(a,b) \to \mathbf{C}/\mathbf{I_A}(a,b),$$

to be the natural inclusion of the category into its Drinfeld quotient

$$\mathbf{C}(a,b) \hookrightarrow \mathbf{C}(a,b)/\mathbf{I_A}(a,b).$$

We can now formulate an analogue of Theorem 4.27 summarising the main properties of our monoidal Drinfeld quotient:

THEOREM 4.37. *Let* $\mathbf{C}$ *be a* $\mathbf{Ho}(\mathbf{dgCat})$*-enriched bicategory. Let* $\mathbf{A}$ *be a collection of* 1*-morphisms in* $\mathbf{C}$*, and let* $\mathbf{I_A}$ *be the two-sided ideal generated by* $\mathbf{A}$*. Then:*

(1) *For any* $a, b \in \mathbf{C}$*, the following natural functor is an exact equivalence*

$$D_c\big(\mathbf{C}/\mathbf{I_A}(a,b)\big) \to D_c\big(\mathbf{C}(a,b)\big)\Big/ D_c\big(\mathbf{I_A}(a,b)\big).$$

(2) *Let* $\mathbf{D}$ *be another* $\mathbf{Ho}(\mathbf{dgCat})$*-enriched bicategory and let* $F\colon \mathbf{C} \to \mathbf{D}$ *be a* 2*-functor. If* $F(\mathbf{I_A})$ *is null-homotopic in the* 1*-morphism categories of* $\mathbf{D}$*, then there exists a unique lift of* F *to a* 2*-functor* $F'\colon \mathbf{C}/\mathbf{I_A} \to \mathbf{D}$*:*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ & \searrow^{\iota} & \uparrow{\scriptstyle \exists!\ F'} \\ & & \mathbf{C}/\mathbf{I_A}. \end{array}$$

PROOF. (1): This is immediate from the corresponding result for ordinary Drinfeld quotients.

(2): This is due to the 2-categorical unique lifting property of ordinary Drinfeld quotients (Theorem 4.27), as follows:

The data of a 2-functor consists of a map of object sets, a collection of functors between 1-morphisms categories and composition/unit coherence morphisms. Since the embedding $\iota\colon \mathbf{C} \to \mathbf{C}/\mathbf{I_A}$ is the identity on object sets, the condition $F = F' \circ \iota$ completely determines the action of F' on objects. Next, let $a, b \in \mathbf{C}$ be any pair of objects. Since

$$\iota_{a,b}\colon \mathbf{C}(a,b) \hookrightarrow \mathbf{C}(a,b)/\mathbf{I_A}(a,b)$$

is the canonical embedding of a category into its Drinfeld quotient, and since, by assumption, $\mathrm{H}^0(F_{a,b})$ kills $\mathbf{I_A}(a,b)$, the quasifunctor

$$F_{a,b}\colon \mathbf{C}(a,b) \to \mathbf{D}(Fa, Fb),$$

lifts to a unique quasifunctor

$$F'_{a,b}\colon \mathbf{C}(a,b)/\mathbf{I_A}(a,b) \to \mathbf{D}(Fa, Fb),$$

such that $F'_{a,b} \circ \iota_{a,b} = F_{a,b}$.

It remains to show that composition and unit coherence morphisms exist and are unique. This is due to the lifting property in Theorem 4.27 being 2-categorical in $\mathbf{Ho}(\mathbf{dgCat})$. We treat the composition coherence morphism below, the proof for unit coherence is similar.

Let $a, b, c \in \mathbf{C}$ be objects. Consider the diagram

$$\begin{array}{ccc} \mathbf{C}(b,c) \otimes \mathbf{C}(a,b) & \xrightarrow{\mu_{\mathbf{C}}} & \mathbf{C}(a,c) \\ \downarrow{\scriptstyle \iota_{b,c}\otimes\iota_{a,b}} & & \downarrow{\scriptstyle \iota_{a,c}} \\ \mathbf{C}(b,c)/\mathbf{I_A}(b,c) \otimes \mathbf{C}(a,b)/\mathbf{I_A}(a,b) & \xrightarrow{\mu_{\mathbf{C}/\mathbf{I_A}}} & \mathbf{C}(a,c)/\mathbf{I_A}(a,c) \\ \downarrow{\scriptstyle F'_{b,c}\otimes F'_{a,b}} & & \downarrow{\scriptstyle F'_{a,c}} \\ \mathbf{D}(Fb,Fc) \otimes \mathbf{D}(Fa,Fb) & \xrightarrow{\mu_{\mathbf{D}}} & \mathbf{D}(Fa,Fc). \end{array}$$

It follows from our definition of $\mu_{\mathbf{C}/\mathbf{I_A}}$ that the top square commutes on the nose. Indeed, this can be takes as an alternative definition of $\mu_{\mathbf{C}/\mathbf{I_A}}$ since $\mathbf{C}(b,c)/\mathbf{I_A}(b,c)\otimes$

$\mathbf{C}(a,b)/\mathbf{I}_{\mathbf{A}}(a,b)$ is quasi-equivalent to $\mathbf{C}(b,c)\otimes\mathbf{C}(a,b)/(\mathbf{I}_{\mathbf{A}}(b,c)\otimes\mathbf{C}(b,c),\mathbf{C}(a,b)\otimes\mathbf{I}_{\mathbf{A}}(a,b))$ and thus enjoys its unique lifting property with respect to the quasi-functors out of $\mathbf{C}(b,c)\otimes\mathbf{C}(a,b)$.

On the other hand, by our definition of $F'_{a,b}$ and $F'_{b,c}$ it follows that the outer perimeter of the above diagram composes to

$$\begin{array}{ccc} \mathbf{C}(b,c)\otimes\mathbf{C}(a,b) & \xrightarrow{\mu_{\mathbf{C}}} & \mathbf{C}(a,c) \\ \big\downarrow{\scriptstyle F_{b,c}\otimes\mathcal{F}_{a,b}} & & \big\downarrow{\scriptstyle F_{a,c}} \\ \mathbf{D}(Fb,Fc)\otimes\mathbf{D}(Fa,Fb) & \xrightarrow{\mu_{\mathbf{D}}} & \mathbf{D}(Fa,Fc). \end{array}$$

The composition coherence morphism ϕ_F is a choice of a 2-morphism in $\mathbf{Ho}(\mathbf{dgCat})$ which makes this diagram commute. Since the lifting property of Drinfeld quotients is 2-categorical, there exists a unique 2-morphism ϕ'_F which makes the bottom square in the first diagram commute, and composes with $\iota_{b,c}\otimes\iota_{a,b}$ to give ϕ_F. □

4.7. Homotopy Serre functors

Let $\mathcal{A}$ be a pretriangulated DG category. A *homotopy Serre functor* on $\mathcal{A}$ is a quasi-autoequivalence S of $\mathcal{A}$ equipped with a closed degree zero $\mathcal{A}$-$\mathcal{A}$-bimodule quasi-isomorphism

$$\eta\colon \mathcal{A}\to ({}_S\mathcal{A})^*\,,$$

such that S and η induce a Serre functor on $\mathrm{H}^0(\mathcal{A})$ in the sense of Section 2.1. Here $(-)^*$ denotes the dualisation over $\Bbbk$ and ${}_S$ denotes the twist of the left $\mathcal{A}$-action by S. Explicitly, the data of η can be thought of as a collection of quasi-isomorphisms natural in $a,b\in\mathcal{A}$:

$$\eta_{a,b}\colon\ \mathrm{Hom}_{\mathcal{A}}(a,b)\cong\mathrm{Hom}_{\mathcal{A}}(b,Sa)^*,$$

Since the $\Bbbk$-dualisation $(-)^*$ commutes with taking cohomologies, the dual of a quasi-isomorphism is a quasi-isomorphism. It also follows that the natural map

$${}_S\mathcal{A}\to({}_S\mathcal{A})^{**}\,.$$

is a quasi-isomorphism if $\mathcal{A}$ is proper. The composition

$${}_S\mathcal{A}\to{}_S\mathcal{A}^{**}\xrightarrow{\eta^*}\mathcal{A}^*,$$

is then also a quasi-isomorphism. By abuse of notation, we also denote it by η^*.

Lemma 4.38. *Let $\mathcal{A}$ be a smooth and proper DG category. Then $\mathcal{H}perf(\mathcal{A})$ admits a homotopy Serre functor given by $S=(-)\otimes_{\mathcal{A}}\mathcal{A}^*$.*

Proof. It was shown in [**43**] that S descends to a Serre functor on $\mathrm{H}^0(\mathcal{H}perf\,\mathcal{A})\cong\mathrm{D_c}(\mathcal{A})$. It remains to demonstrate that there is a quasi-isomorphism $\eta\colon\mathcal{H}perf(\mathcal{A})\to({}_S\mathcal{H}perf(\mathcal{A}))^*$. Since Serre functors are unique, such η would then necessarily be a DG lift of the bifunctorial isomorphisms η of the Serre functor on $\mathrm{D_c}(\mathcal{A})$.

We prove a more general statement. Let $P\in\mathcal{H}perf(\mathcal{A})$ and Q be any DG $\mathcal{A}$-module. Consider the natural morphism functorial in P

$$P\otimes_{\mathcal{A}}\mathrm{Hom}_{\Bbbk}(\mathcal{A},\Bbbk)\longrightarrow\mathrm{Hom}_{\Bbbk}\big(\mathrm{Hom}_{\mathcal{A}}(P,\mathcal{A}),\,\Bbbk\big).$$

It is an isomorphism on representable P and hence a homotopy equivalence on $P\in\mathcal{H}perf(\mathcal{A})$. We thus obtain a bifunctorial homotopy equivalence

$$\beta\colon\ \mathrm{Hom}_{\mathcal{A}}\big(Q,\,P\otimes_{\mathcal{A}}\mathrm{Hom}_{\Bbbk}(\mathcal{A},\Bbbk)\big)\longrightarrow\mathrm{Hom}_{\mathcal{A}}\big(Q,\,\mathrm{Hom}_{\Bbbk}(\mathrm{Hom}_{\mathcal{A}}(P,\mathcal{A}),\,\Bbbk)\big).$$

By Tensor-Hom adjunction, the RHS is canonically isomorphic to

$$\operatorname{Hom}_{\Bbbk}\big(Q \otimes_{\mathcal{A}} \operatorname{Hom}_{\mathcal{A}}(P, \mathcal{A}),\, \Bbbk\big),$$

and since $P \in \mathcal{H}perf(\mathcal{A})$, the natural morphism $Q \otimes_{\mathcal{A}} \operatorname{Hom}_{\mathcal{A}}(P, \mathcal{A}) \to \operatorname{Hom}_{\mathcal{A}}(P, Q)$ is a homotopy equivalence. Thus β can be rewritten as a homotopy equivalence

$$\eta\colon \operatorname{Hom}_{\mathcal{A}}\big(Q,\, P \otimes_{\mathcal{A}} \operatorname{Hom}_{\Bbbk}(\mathcal{A}, \Bbbk)\big) \longrightarrow \operatorname{Hom}_{\Bbbk}\big(\operatorname{Hom}_{\mathcal{A}}(P, Q),\, \Bbbk\big),$$

or, in other words, as

$$\eta\colon \operatorname{Hom}_{\mathcal{A}}(Q,\, SP) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(P,\, Q)^*. \qquad \square$$

EXAMPLE 4.39. For X a smooth and proper scheme over $\Bbbk$, the enhanced derived category $\mathcal{I}(X)$ is smooth and proper, and hence admits a homotopy Serre functor lifting the Serre functor on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ from Example 2.1.

As before, a homotopy Serre functor S induces a *Serre trace map*

$$\operatorname{Tr}\colon \operatorname{Hom}_{\mathcal{A}}(a, Sa) \to \Bbbk, \qquad \alpha \mapsto \eta_{a,a}(\mathrm{id}_a)(\alpha) = \eta_{a,Sa}(\alpha)(\mathrm{id}_a).$$

As in Proposition 2.4, we have

$$\operatorname{Tr}(\beta \circ \alpha) = (-1)^{\deg(\alpha)\deg(\beta)} \operatorname{Tr}(S\alpha \circ \beta).$$

for any $a, b \in \mathcal{A}$ and any $\alpha \in \operatorname{Hom}_{\mathcal{A}}(a, b)$, $\beta \in \operatorname{Hom}_{\mathcal{A}}(b, Sa)$.

4.8. G-equivariant DG categories for strong group actions

Let $\mathcal{A}$ be a small DG category with a *strong* action of a finite group G. That is, with an embedding of G into the group of DG automorphisms of $\mathcal{A}$.

DEFINITION 4.40. The *semi-direct product* $\mathcal{A} \rtimes G$ is the following DG category:

- $\operatorname{Ob} \mathcal{A} \rtimes G = \operatorname{Ob} \mathcal{A}$,
- For any $a, b \in \operatorname{Ob}(\mathcal{A} \rtimes G)$ their morphism complex is

  $$\operatorname{Hom}^i_{\mathcal{A}\rtimes G}(a, b) := \big\{(\alpha, g) \bigm| \alpha \in \operatorname{Hom}^i_{\mathcal{A}}(g.a, b), g \in G\big\}$$

  with $\deg_{\mathcal{A}\rtimes G}(\alpha, g) = \deg_{\mathcal{A}} \alpha$ and $d_{\mathcal{A}\rtimes G}(\alpha, g) = (d_{\mathcal{A}}\alpha, g)$,
- The composition in $\mathcal{A} \rtimes G$ is given by

  $$(\alpha_1, g_1) \circ (\alpha_2, g_2) = (\alpha_1 \circ g_1.\alpha_2,\, g_1 g_2).$$

- For any $a \in \operatorname{Ob}(\mathcal{A} \rtimes G)$ the identity morphism of a is $(\mathrm{id}_a, 1_G)$.

One can think of this as taking $\mathcal{A}$ and formally adding for every object $a \in \mathcal{A}$ a closed degree 0 isomorphism $a \to g.a$ for every $g \in G$. We then impose relations: these isomorphisms compose via the multiplication in G, and their composition with the native morphisms of $\mathcal{A}$ is subject to the relations

$$g \circ \alpha = g.\alpha \circ g \qquad \forall\, g \in G, \alpha \in \mathcal{A}. \tag{4.6}$$

Therefore an action of $\mathcal{A} \rtimes G$ is equivalent to the action of $\mathcal{A}$ and an action of G subject to (4.6). Here by action of G we mean the action of the above tautological isomorphisms corresponding to the elements of G.

We have a natural embedding

$$\eta\colon \mathcal{A} \hookrightarrow \mathcal{A} \rtimes G$$

given by the identity on objects and $\alpha \mapsto (\alpha, \mathrm{id}_G)$ on morphisms. On the other hand, the projections $(\alpha, g) = \alpha \circ g \mapsto \alpha$ and $(\alpha, g) = g \circ g^{-1}.\alpha \mapsto g^{-1}\alpha$ give rise to the decompositions

$$\mathrm{Hom}_{\mathcal{A}\rtimes G}(a,\, b) \cong \bigoplus_{g\in G} \mathrm{Hom}_{\mathcal{A}}(g.a,\, b) \cong \bigoplus_{g\in G} \mathrm{Hom}_{\mathcal{A}}(a,\, g^{-1}.b). \tag{4.7}$$

We can think of these as decompositions of the diagonal bimodule:

$$\mathcal{A} \rtimes G \cong \bigoplus_{g\in G} \mathcal{A}_g \cong \bigoplus_{g\in G} {}_g\mathcal{A}, \tag{4.8}$$

where g denotes the autoequivalence $g\colon \mathcal{A} \to \mathcal{A}$. Both decompositions respect the $\mathcal{A}$-$\mathcal{A}$-action and so the direct summands are themselves $\mathcal{A}$-$\mathcal{A}$-bimodules. The induced right and left actions of any $h \in G$ on the first decomposition are given by

$$\mathcal{A}_g \xrightarrow{\mathrm{id}} \mathcal{A}_{gh},$$

$$\mathcal{A}_g \xrightarrow{h.(-)} \mathcal{A}_{hg},$$

and similarly for the second decomposition.

The action of G on $\mathcal{A}$ induces the action of G on $\mathcal{M}od$-$\mathcal{A}$ where each $g \in G$ acts via the extension of scalars functor g^* with respect to the action functor $g\colon \mathcal{A} \to \mathcal{A}$. A *$G$-equivariant $\mathcal{A}$-module* is a pair (E, ϵ) where $E \in \mathcal{M}od$-$\mathcal{A}$ and $\epsilon = (\epsilon_g)_{g\in G}$ is a collection of isomorphisms

$$\epsilon_g\colon E \xrightarrow{\sim} g^*E \qquad g \in G$$

such that

$$\epsilon_{hg} = E \xrightarrow{\epsilon_g} g^*E \xrightarrow{g^*\epsilon_h} g^*h^*E = (hg)^*E \qquad g, h \in G.$$

The DG category $\mathcal{M}od^G$-$\mathcal{A}$ has as its objects G-equivariant $\mathcal{A}$-modules and as its morphisms the morphisms between the underlying $\mathcal{A}$-modules which commute with the isomorphisms ψ_g. See [**21**, Section 2.1] for further details. The following generalises the classical correspondence between representations of a group and modules over the associated skew group algebra [**35**, Chapter 5, Remark 5.56]:

LEMMA 4.41. *There are mutually inverse isomorphisms of categories*

$$\mathcal{M}od\text{-}(\mathcal{A} \rtimes G) \leftrightarrows \mathcal{M}od^G\text{-}\mathcal{A}.$$

PROOF. Given a G-equivariant $\mathcal{A}$-module (E, ϵ) we can extend the action of $\mathcal{A}$ on E to the action of $\mathcal{A} \rtimes G$ by having g act by $\epsilon_g\colon E_a \to E_{g^{-1}.a} = (g^*E)_a$. Conversely, given a $\mathcal{A} \rtimes G$-module E we can define a G-equivariant structure on the $\mathcal{A}$-module η_*E by defining ϵ_g to be the action of g. These operations are functorial and mutually inverse. □

Generalizing the setting from Section 3.5, for any subgroup $H \subset G$ there is a functor

$$\iota\colon \mathcal{A} \rtimes H \to \mathcal{A} \rtimes G$$

given by the identity on objects, and by the identity times the inclusion on morphisms. This functor induces restriction and induction functors

$$\mathrm{Res}^H_G := \iota_*\colon \mathcal{M}od\text{-}(\mathcal{A} \rtimes G) \to \mathcal{M}od\text{-}(\mathcal{A} \rtimes H),$$

$$\mathrm{Ind}^G_H := \iota^*\colon \mathcal{M}od\text{-}(\mathcal{A} \rtimes H) \to \mathcal{M}od\text{-}(\mathcal{A} \rtimes G).$$

From the viewpoint of equivariant modules, the restriction functor can be written as

$$\begin{array}{rccc} \mathrm{Res}^H_G\colon & \mathcal{M}od^G\text{-}\mathcal{A} & \to & \mathcal{M}od^H\text{-}\mathcal{A} \\ & \big(E,\,(\epsilon_g)_{g\in G}\big) & \mapsto & \big(E,\,(\epsilon_g)_{g\in H}\big) \end{array}$$

on objects and as the identity on morphisms. Similarly, the induction functor is

$$\begin{array}{rccc} \mathrm{Ind}^G_H\colon & \mathcal{M}od^H\text{-}\mathcal{A} & \to & \mathcal{M}od^G\text{-}\mathcal{A} \end{array}$$

$$\big(E,\,(\epsilon_h)_{h\in H}\big) \mapsto \big(\bigoplus\nolimits_{[f]\in G/H} f^*E,\,(\epsilon_g)_{g\in G}\big)$$

on objects where for every $g \in G$

$$\epsilon_g\colon \bigoplus\nolimits_{[f]\in G/H} f^*E \to \bigoplus\nolimits_{[f']\in G/H} g^*f'^*E$$

maps the f-permuted component in the domain to the gf'-permuted component in the target via $f^*\epsilon_h$ when $[f] = [gf'] \in G/H$ and $h \in H$ is such that $gf' = fh$. A similar formula applies to morphisms.

For us actions by symmetric groups, and in particular symmetric powers of categories, will be of interest. The n-fold tensor power $\mathcal{A}^{\otimes n}$ of a DG category $\mathcal{A}$ has as objects n-tuples $a_1 \otimes \cdots \otimes a_n$ of objects of $\mathcal{A}$ and has morphism complexes

$$\mathrm{Hom}_{\mathcal{A}^{\otimes n}}(a_1 \otimes \cdots \otimes a_n,\, b_1 \otimes \cdots \otimes b_n) := \mathrm{Hom}_{\mathcal{A}}(a_1, b_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(a_n, b_n).$$

We therefore have a natural strong action of the symmetric group S_n on $\mathcal{A}^{\otimes n}$ by permuting the factors of objects and the factors of morphisms.

DEFINITION 4.42. Let $\mathcal{A}$ be an enhanced triangulated category. We define $\mathcal{S}^n\mathcal{A}$, the *n-th symmetric power* of $\mathcal{A}$, to be the semidirect product $\mathcal{A}^{\otimes n} \rtimes S_n$.

The corresponding triangulated category is $\mathrm{D_c}(\mathcal{S}^n\mathcal{A}) \cong \mathrm{H}^0\big(\mathcal{H}perf(\mathcal{S}^n\mathcal{A})\big)$. We have

$$\mathcal{M}od(\mathcal{S}^n\mathcal{A}) \cong \mathcal{M}od^{S_n}(\mathcal{A}^{\otimes n})$$

by Lemma 4.41. It follows from the decomposition (4.7) that this further restricts to

$$\mathcal{H}perf(\mathcal{S}^n\mathcal{A}) \cong \mathcal{H}perf^{S_n}(\mathcal{A}^{\otimes n}), \tag{4.9}$$

where $\mathcal{H}perf^{S_n}(\mathcal{A}^{\otimes n})$ is the full subcategory of $\mathcal{M}od^{S_n}(\mathcal{A}^{\otimes n})$ consisting of the equivariant modules which are perfect in $\mathcal{M}od(\mathcal{A}^{\otimes n})$ after forgetting the equivariant structure. Since $\mathcal{H}perf^{S_n}(\mathcal{A}^{\otimes n})$ was the definition of the completed n-th symmetrical power $\widehat{\mathcal{S}}^n\mathcal{A}$ of $\mathcal{A}$ in [**21**, Section 2.2.7], that category is equivalent to the $\mathcal{H}perf$ hull of our $\mathcal{S}^n\mathcal{A}$. This discrepancy is due to us working in the Morita enhancement setting, where to pass to the underlying triangulated category one first takes the $\mathcal{H}perf$ hull, and then its homotopy category.

4.9. The numerical Grothendieck group and the Heisenberg algebra of a DG category

Consider a smooth and proper DG category $\mathcal{V}$. The Grothendieck group of $\mathcal{V}$,

$$\mathrm{K}_0(\mathcal{V}) = \mathrm{K}_0(\mathrm{D_c}(\mathcal{V})),$$

comes equipped with the *Euler* (or *Mukai*) *pairing*

$$\big\langle [a], [b] \big\rangle_\chi := \chi\big(\mathrm{Hom}_{\mathcal{H}perf\,\mathcal{V}}(a,b)\big) = \sum_{n\in\mathbb{Z}} (-1)^n \dim \mathrm{Hom}^n_{\mathrm{D_c}(\mathcal{V})}(a,b).$$

Example 4.43. The Euler pairing is in general neither symmetric nor anti-symmetric. A simple example is given by the Grothendieck group of $\mathrm{K}_0(\mathbb{P}^1) = \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^1))$. It has a semiorthogonal basis given by the classes $\{[\mathcal{O}], [\mathcal{O}(1)]\}$ for which the matrix of χ is

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

This matrix is clearly not diagonalisable over the integers.

Proposition 4.44. *Let $\mathcal{V}$ be a smooth and proper DG category.*

(1) For every pair of objects a, b of $\mathrm{D_c}(\mathcal{V})$,

$$\big\langle [a], [b] \big\rangle_\chi = \big\langle [b], [Sa] \big\rangle_\chi = \big\langle [S^{-1}b], [a] \big\rangle_\chi,$$

where S is the Serre functor on $\mathrm{D_c}(\mathcal{V})$.

(2) The left and right kernels of χ agree.

Proof. This is Lemma 4.25 and Proposition 4.24 of [**48**]. □

The *numerical Grothendieck group* $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V})$ of a smooth and proper DG category $\mathcal{V}$ is $\mathrm{K}_0(\mathcal{V})/\ker(\chi)$. We further set $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V}, \Bbbk) := \mathrm{K}_0^{\mathrm{num}}(\mathcal{V}) \otimes_{\mathbb{Z}} \Bbbk$.

Proposition 4.45 ([**50**, Theorem 1.2], [**49**, Theorem 1.2]). *The numerical Grothendieck group $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V})$ of a smooth and proper DG category $\mathcal{V}$ is a finitely generated free abelian group.*

As χ is non-degenerate and integral on $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V})$, we call the pair $(\mathrm{K}_0^{\mathrm{num}}(\mathcal{V}), \chi)$ the *Mukai lattice* of $\mathcal{V}$.

Example 4.46. For $\mathcal{V} = \mathcal{I}(X)$, where X is smooth and projective, the Euler form can be computed by Hirzebruch–Riemann–Roch theorem (see, for example, [**11**, Section 6.3]):

$$\chi\big(\mathrm{Hom}(a, b)\big) = \chi(a^\vee \otimes b) = \int_X \mathrm{ch}(a^\vee \otimes b) \cdot \mathrm{td}(T_X).$$

This implies that the kernel of χ equals the kernel of the Chern character map to Chow groups tensored with $\mathbb{Q}$.

Definition 4.47. Let $\mathcal{V}$ be a smooth and proper DG category. We write $H_{\mathcal{V}}$ for the idempotent modified Heisenberg algebra $H_{(\mathrm{K}_0^{\mathrm{num}}(\mathcal{V}), \chi)}$. The corresponding Fock space representation is denoted by $F_{\mathcal{V}}$.

Example 4.48. For $\mathcal{V} = \mathcal{I}(\mathbb{P}^1)$ as in Example 4.43, χ is nondegenerate and its Smith normal form is the unit 2×2 matrix Id_2. Therefore,

$$H_{\mathcal{I}(\mathbb{P}^1)} \cong H_{\mathbb{Z}^2, \mathrm{Id}_2} = H_{\mathcal{I}(\mathrm{pt} \sqcup \mathrm{pt})}$$

by Corollary 2.6.

Lemma 4.49. *Let $\mathcal{A}$, $\mathcal{B}$ be smooth and proper DG categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a DG functor. Then $F^*\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ and $F_*\colon \mathrm{D}(\mathcal{B}) \to \mathrm{D}(\mathcal{A})$ preserve compactness and induce*

$$F^*\colon \mathrm{K}_0^{\mathrm{num}}(\mathcal{A}) \to \mathrm{K}_0^{\mathrm{num}}(\mathcal{B}),$$

$$F_*\colon \mathrm{K}_0^{\mathrm{num}}(\mathcal{B}) \to \mathrm{K}_0^{\mathrm{num}}(\mathcal{A}).$$

PROOF. As explained in 4.2.3, for any $\mathcal{A}$ and $\mathcal{B}$, not necessarily smooth or proper, the extension of scalars functor $F^*\colon \mathcal{M}od\text{-}\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B}$ always restricts to a functor $\mathcal{H}perf(\mathcal{A}) \to \mathcal{H}perf(\mathcal{B})$. Thus its derived functor preserves compactness.

We now show that $F_*\colon \mathcal{M}od\text{-}\mathcal{B} \to \mathcal{M}od\text{-}\mathcal{A}$ restricts to $\mathcal{P}erf(\mathcal{A}) \to \mathcal{P}erf(\mathcal{B})$, whence its derived functor preserves compactness. As F_* is tensoring with the $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{B}_F$, it suffices to show $\mathcal{B}_F$ to be $\mathcal{A}$-perfect [**1**, Prop. 2.14]. Let $b \in \mathcal{B}$. For an $\mathcal{A}$-module ${}_b\mathcal{B}_F$ to be perfect it suffices, since $\mathcal{A}$ is smooth, for it to be $\Bbbk$-perfect [**1**, Cor. 2.15]. In other words, for any $a \in \mathcal{A}$ the total cohomology of the $\Bbbk$-module ${}_b\mathcal{B}_{Fa}$ has to be finite. This holds since $\mathcal{B}$ is proper.

The remaining assertions now follow by adjunction of F^* and F_*. Indeed, for any $a \in \mathrm{D_c}(\mathcal{A})$ and for any $b \in \mathrm{D_c}(\mathcal{B})$ we have

$$\begin{aligned}\chi(F^*(a),\, b) &= \sum (-1)^i \dim \operatorname{Hom}^i_{\mathrm{D_c}(\mathcal{B})}(F^*(a),\, b)\\ &= \sum (-1)^i \dim \operatorname{Hom}^i_{\mathrm{D_c}(\mathcal{A})}(a,\, F_*(b)) = \chi(a,\, F_*(b)).\end{aligned}$$

Thus F^* and F_* take $\ker\chi$ to $\ker\chi$ and so induce maps of numerical Grothendieck groups. □

CHAPTER 5

The DG Heisenberg 2-category

Let $\mathcal{V}$ be any smooth and proper DG category. We fix this choice throughout the rest of the paper. Now, recall from Section 4.4 that we work with DG categories up to Morita equivalence, viewing them as enhanced triangulated categories. Replace therefore $\mathcal{V}$ by its perfect hull $\mathcal{H}perf\,\mathcal{V}$. This doesn't change the Morita equivalence class of $\mathcal{V}$. However, it ensures that $\mathcal{V}$ is homotopy direct summand complete and admits a homotopy Serre functor. Note that, as explained in Section 4.7, any homotopy Serre functor S induces a Serre trace map $\mathrm{Tr}\colon \mathrm{Hom}_{\mathcal{V}}(a, Sa) \to \Bbbk$ for any $a \in \mathcal{V}$.

In this section we define a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory $\mathbf{H}_{\mathcal{V}}$, the *Heisenberg category* of $\mathcal{V}$. This category is a monoidal Drinfeld quotient of the perfect hull of a simpler strict DG 2-category $\mathbf{H}'_{\mathcal{V}}$ which we set up in the following paragraphs. We take the Drinfeld quotient to impose certain relations in $\mathbf{H}_{\mathcal{V}}$ which we only expect to hold on the level of homotopy categories, unlike the relations we impose on $\mathbf{H}'_{\mathcal{V}}$ which must hold on the DG level.

5.1. The category $\mathbf{H}'_{\mathcal{V}}$: generators

The objects of $\mathbf{H}'_{\mathcal{V}}$ are the integers $N \in \mathbb{Z}$.

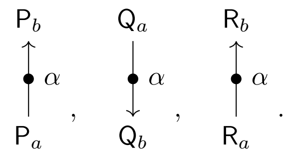
As in the additive setting of Chapter 3, we have 1-morphisms labeled Q_a for $a \in \mathcal{V}$. However, as we have only a homotopy Serre functor, we need to more carefully distinguish between the left and right duals of Q_a. The 1-morphisms are therefore freely generated by

- $\mathsf{P}_a\colon N \to N+1$,
- $\mathsf{Q}_a\colon N+1 \to N$,
- $\mathsf{R}_a\colon N \to N+1$,

for each $a \in \mathcal{V}$ and $N \in \mathbb{Z}$. Thus the objects of $\mathrm{Hom}_{\mathbf{H}'_{\mathcal{V}}}(N, N')$ are finite words in the symbols P_a, Q_a, and R_a with $a \in \mathcal{V}$, such that the difference of the number of Ps and Rs and the number of Qs is $N' - N$. The identity 1-morphism of any $N \in \mathbb{Z}$ is denoted as $\mathbb{1}$.

The 2-morphisms between two 1-morphisms form a complex of vector spaces. These vector spaces are freely generated by the generators listed below, subject to the axioms of a (strict) DG 2-category as well as the relations we detail in the next section. As before, we represent these 2-morphisms as planar diagrams, using the same sign rules as in Remark 3.1. We recall that diagrams are read bottom to top, i.e., the source of a given 2-morphism lies on the lower boundary, while the target lies on the upper boundary.

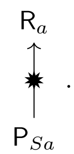
We now list the generating 2-morphisms. For every $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a,b)$ there are arrows



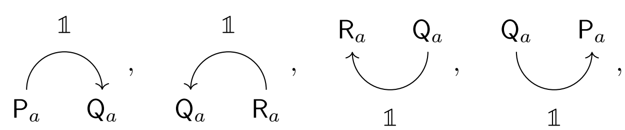
These 2-morphisms are homogeneous of degree $|\alpha|$. The remaining generators listed below are all of degree 0. By convention a strand without a dot is the same as one marked with the identity morphism. Any such unmarked string is an identity 2-morphism in $\mathbf{H}'_{\mathcal{V}}$. The identity 2-morphisms of the 1-morphisms $\mathbb{1}$ are usually pictured by a blank space.

For every $a \in \mathcal{V}$, there is a special arrow marked with a star:

(5.1)

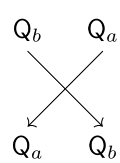


Furthermore, for any objects $a, b \in \mathcal{V}$ there are cups and caps



as well as crossings of two downward strands:

(5.2)

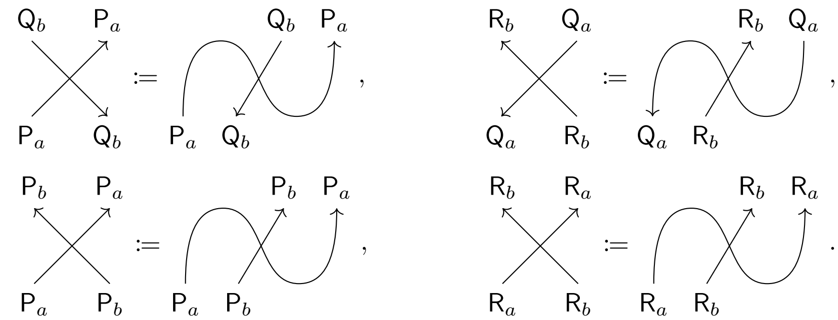


We recall again the sign convention for reading planar diagrams from Remark 3.1. As before, we often "prettify" diagrams by smoothing them out.

We give each 2-morphism space a DG structure. With the grading defined above, it remains to define the differential. If f is a single strand with one dot labelled α, then $d(f)$ is the same diagram with the label replaced by $d(\alpha)$. In particular, the differential of a strand labelled with the identity is $d(\mathrm{id}) = 0$. The differentials of the remaining generating 2-morphisms — the caps, the cups, the crossings, and the star — are zero. The differential of a general 2-morphism is then determined by the following graded Leibniz rules for 1- and 2-compositions. These follow from the definition of a DG bicategory:

- $d(h \circ_1 g) = d(h) \circ_1 g + (-1)^{|h|} h \circ_1 d(g)$,
- $d(h \circ_2 g) = d(h) \circ_2 g + (-1)^{|h|} h \circ_2 d(g)$.

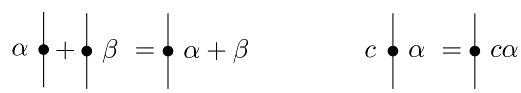
For convenience, we define four further types of strand crossings from the basic one in (5.2) by composition with cups and caps:

(5.3)


5.2. The category $\mathbf{H}'_{\mathcal{V}}$: relations between 2-morphisms

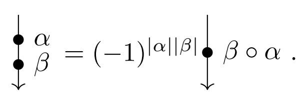
In the preceding subsection we gave the list of the generating symbols for 2-morphisms. We obtain all 2-morphisms in $\mathbf{H}'_{\mathcal{V}}$ 1- and 2-compositions of these symbols, subject to the axioms of a strict DG 2-category and a list of relations we impose in this section.

First, we impose the linearity relations:

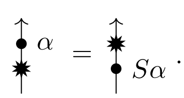
(5.4)


for any scalar $c \in \Bbbk$ and any compatible orientation of the strings.

Neighboring dots along a downward string can merge with a sign twist:

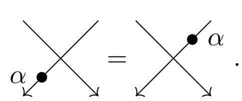
(5.5)


A dot can swap with a star according to the following rule:

(5.6)


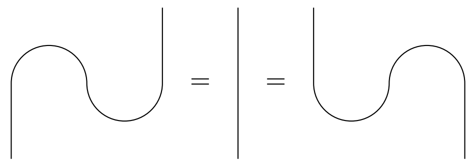
Dots may "slide" through the generating cups and crossing as follows:

(5.7)

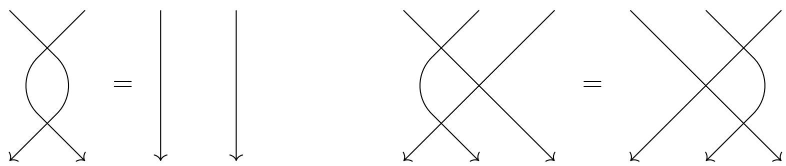

(5.8)


Note that when drawing diagrams, dots need to keep their relative heights when doing these operations in order to avoid accidentally introducing signs (cf. Remark 5.2 below).

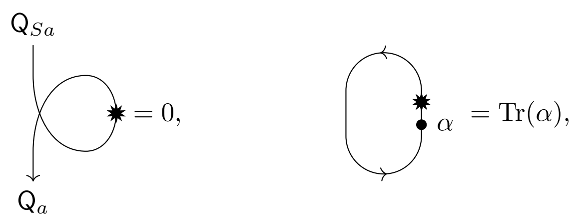
There are two sets of local relations for unmarked strings: the *adjunction relations*

(5.9)


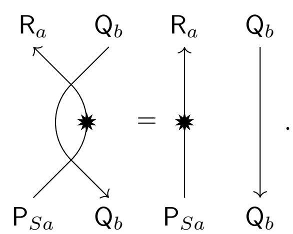
and the *symmetric group* relations on downward strands

(5.10)


Finally there are three relations involving a star-marked string

(5.11)


where $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a, Sa)$, and

(5.12)


The relations (5.11) are the analogues of the relations (3.9). As leftward caps involve an R but rightward cups involve a P, a star needs to be added between the two. Similarly, to get a consistent diagram a star must appear in both sides of (5.12), the analogue of the left relation from (3.10).

We do not have an equivalent of the right relation in (3.10) because to define the map Ψ we need the natural isomorphism $\mathrm{Hom}(b, Sa) \cong \mathrm{Hom}(a, b)^*$ afforded to us by the genuine Serre functor. In the present DG setup we only have a homotopy Serre functor which only gives us a natural homotopy equivalence $\mathrm{Hom}(b, Sa) \to \mathrm{Hom}(a, b)^*$, but not its natural inverse. We can't therefore define the map Ψ. More spefically, of the two composants ψ_1 and ψ_2 of the term $\Psi(\mathrm{id})$ described after Remark 3.9 in Section 3.3 we have ψ_2, but not ψ_1. However, the two relations in (3.9) and the left relation in (3.10) together are equivalent to the map

$$\mathsf{Q}_a\mathsf{P}_b \xrightarrow{\left[\times\,,\ \psi_1\right]} \mathsf{P}_b\mathsf{Q}_a \oplus \big(\mathrm{Hom}(a,b) \otimes_{\Bbbk} \mathbb{1}\big)$$

being the left inverse of the map

$$\mathsf{P}_b\mathsf{Q}_a \oplus \big(\mathrm{Hom}(a,b)\otimes_{\Bbbk}\mathbb{1}\big) \xrightarrow{\left[\,\times\,,\ \psi_2\right]} \mathsf{Q}_a\mathsf{P}_b,$$

while the right relation in (3.10) is equivalent to it being the right inverse.

Thus, having imposed the equivalents of the two relations in (3.9) and the left relation in (3.10), to have the equivalent of the right relation in (3.10) we only need the map $\left[\,\times\,,\ \psi_2\right]$ be a homotopy equivalence. We impose it in Section 5.4 by taking the Drinfeld quotient by its cone.

5.3. Remarks on the 2-morphism relations in $\mathbf{H}'_{\mathcal{V}}$

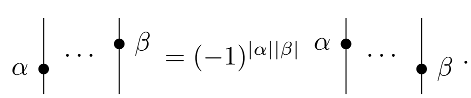
Let us remark on some of the above relations for 2-morphisms and their consequences.

Remark 5.1. The reader familiar with the categorifications of Khovanov and Cautis–Licata [**31, 13**] or the classical Heisenberg algebra might find the appearance of the third type of 1-morphisms, i.e. R_a, confusing. In the Fock space representation constructed in Chapter 7, the 1-morphism Q_a is sent to a pushforward functor $\phi_{a,*}$, while P_a and R_a are sent to the left adjoint ϕ_a^* and right adjoint $\phi_a^!$ respectively. In $\mathbf{H}'_{\mathcal{V}}$ this is expressed by the relations (5.9) which state that there are adjunctions of 1-morphisms $(\mathsf{P}_a,\ \mathsf{Q}_a)$ and $(\mathsf{Q}_a,\ \mathsf{R}_a)$ for any $a \in \mathcal{V}$.

Up to homotopy, the Serre functor lets us switch between left and right adjoints: ϕ_{Sa}^* and $\phi_a^!$ are identified in the homotopy category (note that in Khovanov's case the Serre functor is trivial, while in the Cautis–Licata setting it is a shift by 2, see Examples 5.9 and 5.10). However, on the DG level, there is only a canonical natural transformation $\phi_{Sa}^* \to \phi_a^!$. This natural transformation is represented by the starred arrow (5.1). In Section 5.4, we take the Drinfeld quotient by this arrow, forcing it to be an isomorphism on the homotopy level.

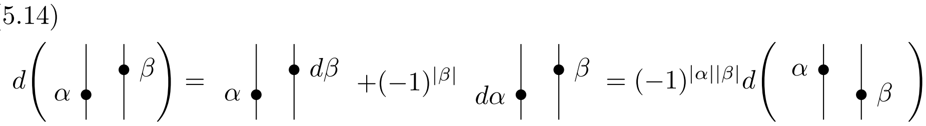
Remark 5.2. Since composing with the identity on either side doesn't change 2-morphisms, dots may freely "slide along" straight strands as long as the relative height of all dots is kept the same. The interchange law introduces a sign when two dots slide past each other:

(5.13)



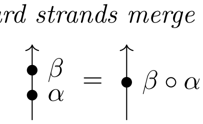
The axioms governing the differential in a DG 2-category are compatible with this super-commutativity:

(5.14)



In particular, it does not matter which dot one "moves" to the bottom of the diagram.

Lemma 5.3. *Dots on upward strands merge without a sign change:*

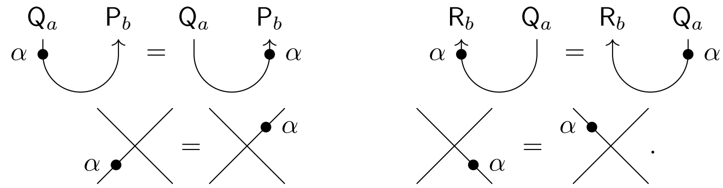


Proof. The same proof as in Lemma 3.4 applies. ☐

The sign rules also imply that merging of dots is compatible with the graded Leibniz rules for $\mathcal{V}$ and $\mathrm{Hom}_{\mathbf{H}'_{\mathcal{V}}}(N, N')$. See (5.14) and note that in $\mathcal{V}$ we have:

$$d(\beta \circ \alpha) = d(\beta) \circ \alpha + (-1)^{|\beta|} \beta \circ d(\alpha).$$

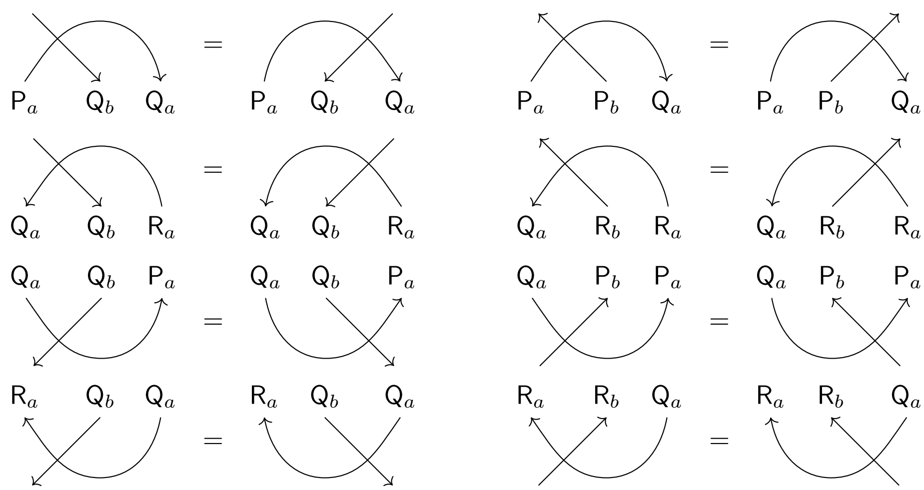
Lemma 5.4. *Dots may freely slide through cups, caps and all types of crossings:*



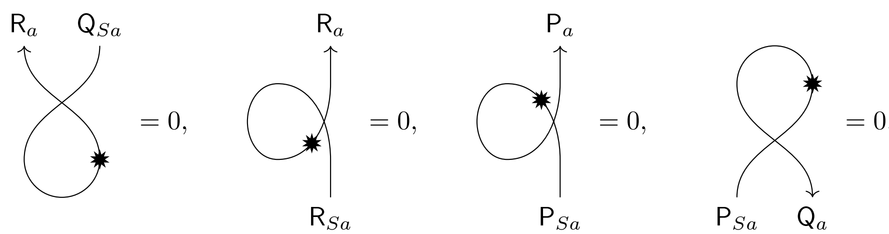
Proof. The same proof as in Lemma 3.3 applies. ☐

Lemma 5.5. *The following relations hold in $\mathbf{H}'_{\mathcal{V}}$ for all objects $a, b, c \in \mathcal{V}$.*

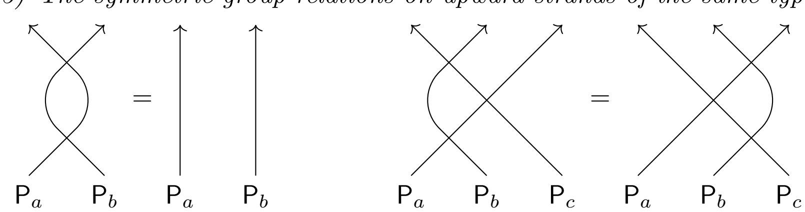
(1) All allowed pitchfork relations:

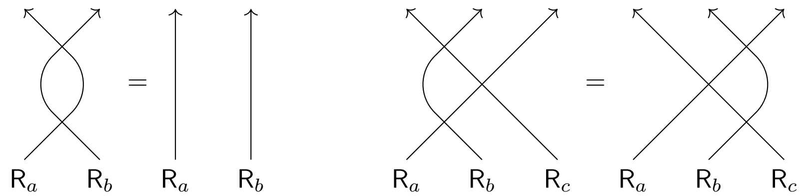


(2) All counterclockwise curls vanish:

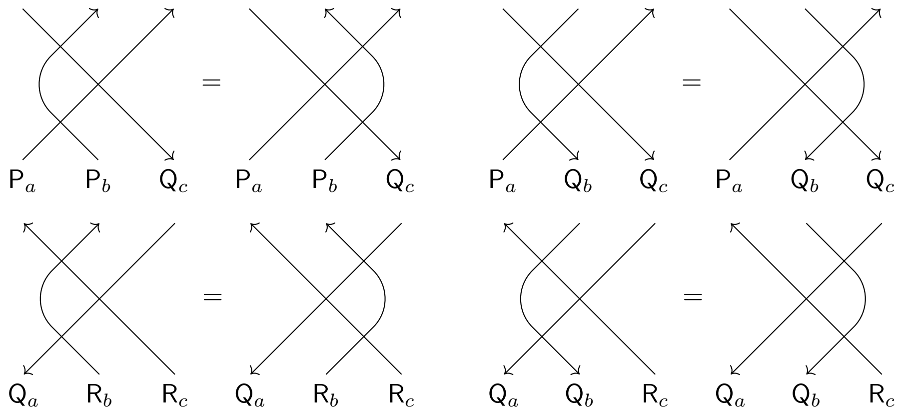


(3) The symmetric group relations on upward strands of the same type:





(4) The remaining allowed triple moves:



PROOF. These are proved similarly to Lemmas 3.6, 3.7, 3.8 and 3.11. One notes that the more complicated proof of Lemma 3.10 is not needed, as we do not require the left and right mates of the downward crossing to coincide. □

5.4. The category $\mathbf{H}_{\mathcal{V}}$: the perfect hull and homotopy relations

We construct the Heisenberg category $\mathbf{H}_{\mathcal{V}}$ out of category $\mathbf{H}'_{\mathcal{V}}$ in two steps. First, we apply Definition 4.22 to form the perfect hull $\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})$. This is no longer a strict 2-category, but a bicategory. It has the objects of $\mathbf{H}'_{\mathcal{V}}$, but the 1-morphism categories are replaced by their perfect hulls. In particular, they are strongly pre-triangulated and homotopy Karoubi-complete.

The pre-triangulated structure we obtain on 1-morphism categories of $\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})$ allows us to formulate the final relations we need to impose. Roughly, these postulate that certain 2-morphisms are isomorphisms in the homotopy category.

Let $a, b \in \mathcal{V}$. Since $\mathcal{V}$ is proper, $\operatorname{Hom}_{\mathcal{V}}(a,b)$ has finite dimensional cohomology and thus is a perfect DG $\Bbbk$-module. Hence for any 1-morphism $E \in \mathbf{H}'_{\mathcal{V}}$ the tensor product $\operatorname{Hom}_{\mathcal{V}}(a,b) \otimes_{\Bbbk} E$ lies in $\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})$. Indeed, since any complex of vector spaces is homotopy equivalent to the direct sum of its cohomologies $\operatorname{Hom}_{\mathcal{V}}(a,b) \otimes_{\Bbbk} E$ is homotopy equivalent to $\bigoplus_i H^i(\operatorname{Hom}_{\mathcal{V}}(a,b)) \otimes_{\Bbbk} E$ which is a direct sum of a finite number of copies of E.

Similar to (3.11), we have the natural 2-morphism

$$\psi_2 \colon \operatorname{Hom}_{\mathcal{V}}(a,b) \otimes_{\Bbbk} \mathbb{1} \to \mathsf{Q}_a \mathsf{P}_b$$

in $\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})$ obtained by adjunction from the map of complexes of vector spaces

$$\psi_2^{\mathrm{adj}} \colon \operatorname{Hom}_{\mathcal{V}}(a,b) \to \operatorname{Hom}_{\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})}(\mathbb{1},\ \mathsf{Q}_a \mathsf{P}_b)$$

$\beta \quad \mapsto \quad$  β .

We no longer have its counterpart ψ_1 as we do not have a map

$$\mathrm{Hom}_{\mathcal{V}}(a,b)^{\vee} \to \mathrm{Hom}_{\mathcal{V}}(b, Sa).$$

The map ψ_2^{adj} is closed of degree 0 since for any $\beta \in \mathrm{Hom}_{\mathcal{V}}(a,b)$ we have

$$\begin{aligned} d\psi_2^{\mathrm{adj}}(\beta) &= d_{\mathrm{Hom}_{\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})}(\mathbb{1},\, \mathsf{Q}_a\mathsf{P}_b)}\left(\psi_2^{\mathrm{adj}}(\beta)\right) - \psi_2^{\mathrm{adj}}\left(d_{\mathrm{Hom}_{\mathcal{V}}(a,b)}\beta\right) \\ &= d\left(\smile\!\!\!\bullet\!\!\nearrow \beta\right) - \smile\!\!\!\bullet\!\!\nearrow d\beta = 0. \end{aligned}$$

Therefore the map ψ_2 is also closed of degree 0.

Together with a crossing, ψ_2 induces a natural degree zero closed 2-morphism

$$\mathsf{P}_b\mathsf{Q}_a \oplus \left(\mathrm{Hom}(a,b) \otimes_{\Bbbk} \mathbb{1}\right) \xrightarrow{\left[\times,\ \psi_2\right]} \mathsf{Q}_a\mathsf{P}_b, \tag{5.15}$$

and on the homotopy level, where ψ_1 does exist, we would like (5.15) to be an isomophism.

Secondly, in the homotopy category of $\mathcal{V}$, the functor S becomes an actual Serre functor. In terms of the graphical calculus, this means that on the homotopy level we would like

$$\mathsf{P}_{Sa} \xrightarrow{\ast} \mathsf{R}_a \tag{5.16}$$

to be isomorphisms for all $a \in \mathcal{V}$.

We therefore take the monoidal Drinfeld quotient (see Definition 4.35) of $\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})$ by the cones of (5.15) and (5.16). This produces a $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory where (5.15) and (5.16) are homotopy equivalences:

DEFINITION 5.6. The *Heisenberg category* $\mathbf{H}_{\mathcal{V}}$ of $\mathcal{V}$ is the Drinfeld quotient of the h-perfect hull of $\mathbf{H}'_{\mathcal{V}}$ by the two-sided ideal generated by the 1-morphisms

$$\mathrm{Cone}\left(\mathsf{P}_{Sa} \xrightarrow{\ast} \mathsf{R}_a\right)$$

$$\mathrm{Cone}\left(\mathsf{P}_b\mathsf{Q}_a \oplus \left(\mathrm{Hom}(a,b) \otimes_{\Bbbk} \mathbb{1}\right) \xrightarrow{\left[\times,\ \psi_2\right]} \mathsf{Q}_a\mathsf{P}_b\right)$$

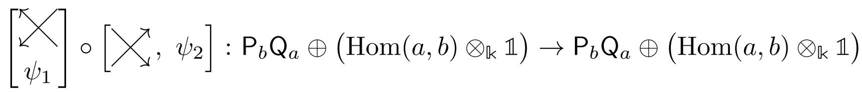
for all $a, b \in \mathcal{V}$.

The graded homotopy category $H^*(\mathcal{A})$ of a DG category $\mathcal{A}$ is defined to have the same objects as $\mathcal{A}$ and morphism spaces $\mathrm{Hom}_{H^*(\mathcal{A})}(a,b) = \bigoplus_{i \in \mathbb{Z}} H^i(\mathrm{Hom}_{\mathcal{A}}(a,b))$. The graded homotopy category $H^*(\mathbf{H}_{\mathcal{V}})$ of $\mathbf{H}_{\mathcal{V}}$ is similarly defined by replacing the 1-morphism categories with their graded homotopy categories. In particular, each $\mathrm{Hom}_{H^*(\mathbf{H}_{\mathcal{V}})}(N, N')$ is a Karoubian category. In $H^*(\mathbf{H}_{\mathcal{V}})$ one no longer has to distinguish between the 1-morphisms P and R and thus one recovers the formalism of Chapter 3, including the labels on cups and caps.

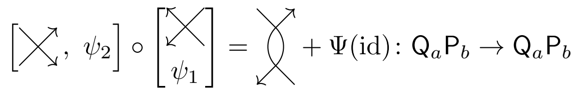
LEMMA 5.7. *Relations* (3.10) *hold in* $H^*(\mathbf{H}_{\mathcal{V}})$.

PROOF. The left-hand relation is just (5.12) after identifying R_a with P_{Sa} and relabeling.

Relations 5.11 and (5.12) together with the curl relations of Lemma 5.5 show that



is the identity. Since in $H^*(\mathbf{H}_{\mathcal{V}})$ the 2-morphism (5.15) is an isomorphism, the other composition



is also the identity, as required. □

COROLLARY 5.8. *There exists a canonical* 2*-functor*

$$\mathbf{H}^{\text{add}}_{H^*(\mathcal{V})} \to H^*(\mathbf{H}_{\mathcal{V}}).$$

PROOF. As all relations in $\mathbf{H}^{\text{add}\prime}_{H^*(\mathcal{V})}$ are satisfied in $H^*(\mathbf{H}_{\mathcal{V}})$, there exists a canonical functor $\mathbf{H}^{\text{add}\prime}_{H^*(\mathcal{V})} \to H^*(\mathbf{H}_{\mathcal{V}})$. Taking Karoubi completion gives the desired functor. □

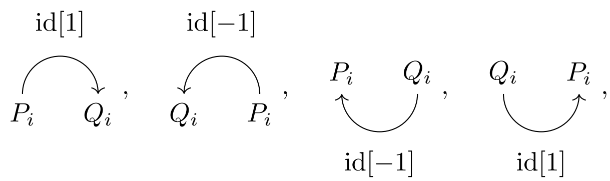
EXAMPLE 5.9. Let $\mathcal{V} = \Bbbk$, the field $\Bbbk$ considered as a single-object DG category concentrated in degree 0. The Serre functor S on $\mathcal{V}$ is the identity. We have $\mathcal{V} = H^*(\mathcal{V})$, the additive construction $\mathbf{H}^{\text{add}}_{\mathcal{V}}$ is Khovanov's categorification of the infinite Heisenberg algebra [**31**], and the 2-functor from Corollary 5.8 is a fully faithful embedding of graded 2-categories. In the DG construction we take the perfect hulls of the categories of 1-morphisms, so the 1-morphisms in $H^*(\mathbf{H}_{\mathcal{V}})$ are not only words in P and Q and their idempotents, but also finite complexes thereof. The category $\mathbf{H}_{\mathcal{V}}$ is hence a DG enhanced triangulated hull of Khovanov's categorification. The isomorphism (5.15) in $\mathrm{H}^0(\mathbf{H}_{\mathcal{V}})$ recovers the defining relation with central charge $k = -1$ from [**10**, (1.5)], which was shown to be an alternative of Khovanov's presentation. We expect that our construction has analogues for central charges $k \neq -1$.

EXAMPLE 5.10. Consider a finite subgroup Γ of $\mathrm{SL}(2, \mathbb{C})$ with corresponding simple surface singularity $Y = \mathbb{A}^2/\Gamma$ and minimal resolution X. Let $\mathcal{I}(X)$ be the DG enhanced bounded derived category of coherent sheaves on X. For $\mathcal{V}$ the full subcategory of $\mathcal{I}(X)$ consisting of sheaves supported on the exceptional divisor E, the category $\mathbf{H}_{\mathcal{V}}$ is a DG enhancement of the category $\mathcal{H}^\Gamma$ introduced by Cautis–Licata [**13**, Section 6].

Indeed, the exceptional divisor E decomposes into (-2)-curves E_i labeled by the non-trivial irreducible representations of Γ. Let I_Γ be the vertices of the McKay quiver of Γ. Denote by 0 the vertex corresponding to the trivial representation. For $i \in I_\Gamma$ define

$$\mathcal{E}_i = \begin{cases} \mathcal{O}_E[-1] & \text{if } i = 0 \\ \mathcal{O}_{E_i}(-1) & \text{otherwise.} \end{cases}$$

The generators P_i and Q_i of $\mathcal{H}^\Gamma$ for $i \in I_\Gamma$ correspond in $H^*(\mathbf{H}_{\mathcal{V}})$ to 1-morphisms $\mathsf{P}_{\mathcal{E}_i}$ and $\mathsf{Q}_{\mathcal{E}_i}[1]$, respectively. As X is Calabi–Yau, its Serre functor is [2]. Thus the shifts chosen above reproduce the grading on turns defined in [**13**, Section 6.1]:



One has

$$\mathrm{Hom}^*(\mathcal{E}_i, \mathcal{E}_j) = \begin{cases} \mathbb{C} \oplus \mathbb{C}[-2], & i = j \\ \mathbb{C}[-1], & |i - j| = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{5.17}$$

Thus a dot on a 2-morphism in $\mathcal{H}^\Gamma$ corresponds to a basis vector of either $\mathbb{C}[-2]$ or $\mathbb{C}[-1]$. Picking such a basis, one obtains a 2-functor

$$\mathcal{H}^\Gamma \to H^*(\mathbf{H}_\mathcal{V})$$

factoring through the 2-functor $\mathbf{H}^{\mathrm{add}}_{H^*(\mathcal{V})} \to H^*(\mathbf{H}_\mathcal{V})$ of Corollary 5.8.

Equivalently by [**29**, Theorem 2.3], instead of the sheaves $\mathcal{E}_i$, one could use the irreducible representations V_i of Γ considered as skyscraper sheaves at the origin on the quotient stack $[\mathbb{A}^2/\Gamma]$. In this setting one works in the ambient category $\mathcal{I}([\mathbb{A}^2/\Gamma])$, see also Example 7.16.

CHAPTER 6

Structure of the Heisenberg Category

In this section we deduce a number of properties of the Heisenberg category and we investigate its relationship with the classical Heisenberg algebra.

6.1. The Heisenberg commutation relations: DG level

As observed in Remark 3.9, the symmetric group relations (5.10) give us a canonical morphism $\Bbbk[S_n] \to \mathrm{End}(\mathsf{Q}_a^n)$. Similarly, by Lemma 5.5 there are morphisms to $\mathrm{End}(\mathsf{P}_a^n)$ and $\mathrm{End}(\mathsf{R}_a^n)$. Endomorphisms in the image of these maps are made up of unlabelled strands, thus they are closed and of degree 0.

Remark 6.1. The homomorphisms $\Bbbk[S_n] \to \mathrm{End}(\mathsf{P}_a^n)$ vary in a family over $\mathcal{V}^{\otimes n}$. That is, there exist natural functors

$$\Xi'^{\mathsf{P}}_{N,N+n} \colon \mathcal{S}^n\mathcal{V} \to \mathrm{Hom}_{\mathbf{H}_\mathcal{V}}(N,\, N+n),$$

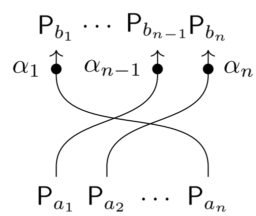
where $\mathcal{S}^n\mathcal{V} := \mathcal{V}^{\otimes n} \rtimes S_n$ is the semi-direct product of Definition 4.40. On objects $\Xi'^{\mathsf{P}}_{N,N+n}$ is given by

$$\Xi'^{\mathsf{P}}_{N,N+n}(a_1 \otimes \cdots \otimes a_n) = \mathsf{P}_{a_1} \cdots \mathsf{P}_{a_n}$$

and on morphisms by sending

$$(\alpha_1 \otimes \cdots \otimes \alpha_n,\, \sigma) \quad \text{for} \quad \alpha_i \in \mathrm{Hom}_\mathcal{V}(a_{\sigma^{-1}(i)},\, b_i),\, \sigma \in S_n$$

to the braid corresponding to σ followed by parallel vertical strands dotted with $\alpha_1, \ldots, \alpha_n$:



Similarly, we have canonical functors $\Xi'^{\mathsf{R}}_{N,N+n}$ sending $a_1 \otimes \cdots \otimes a_n$ to $\mathsf{R}_{a_1} \cdots \mathsf{R}_{a_n}$ and contravariant functors $\Xi'^{\mathsf{Q}}_{N,N+n}$ sending $a_1 \otimes \cdots \otimes a_n$ to $\mathsf{Q}_{a_1} \cdots \mathsf{Q}_{a_n}$.

We can further let N and n vary by defining a 2-category $\mathbf{Sym}_\mathcal{V}$ with objects $N \in \mathbb{Z}$, 1-morphism categories $\mathrm{Hom}_{\mathbf{Sym}_\mathcal{V}}(N,\, N+n) = \mathcal{S}^n\mathcal{V}$ and 1-composition given by the functors $\mathcal{S}^{n_1}\mathcal{V} \otimes \mathcal{S}^{n_2}\mathcal{V} \to \mathcal{S}^{n_1+n_2}\mathcal{V}$ induced by $S_{n_1} \times S_{n_2} \hookrightarrow S_{n_1+n_2}$. We then have a natural functor $\Xi^{\mathsf{P}} \colon \mathbf{Hperf}(\mathbf{Sym}_\mathcal{V}) \to \mathbf{H}_\mathcal{V}$ and similarly for Q and R.

Let

$$e := e_{\mathrm{triv}} := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \in \Bbbk[S_n]$$

be, as in Section 3.4, the symmetriser idempotent in $\Bbbk[S_n]$. Where we work with no other idempotents of $\Bbbk[S_n]$ and no confusion is possible, we use the shorter notation e for e_{triv}. Denote its image under any of the above maps again by e. The maps e are (strict) idempotent endomorphisms of P^n_a, Q^n_a and R^n_a respectively, and hence split in $H^*(\mathbf{H}_\mathcal{V})$. A standard construction gives natural representatives of the corresponding homotopy direct summands.

Definition 6.2. Let $\mathsf{P}^{(n)}_a$, $\mathsf{Q}^{(n)}_a$ and $\mathsf{R}^{(n)}_a$ be the convolutions of the twisted complexes

$$\mathsf{P}^{(n)}_a := \left\{ \dots \xrightarrow{e} \mathsf{P}^n_a \xrightarrow{1-e} \mathsf{P}^n_a \xrightarrow{e} \mathsf{P}^n_a \xrightarrow{1-e} \underset{\deg. 0}{\mathsf{P}^n_a} \right\},$$

$$\mathsf{Q}^{(n)}_a := \left\{ \dots \xrightarrow{e} \mathsf{Q}^n_a \xrightarrow{1-e} \mathsf{Q}^n_a \xrightarrow{e} \mathsf{Q}^n_a \xrightarrow{1-e} \underset{\deg. 0}{\mathsf{Q}^n_a} \right\},$$

$$\mathsf{R}^{(n)}_a := \left\{ \dots \xrightarrow{e} \mathsf{R}^n_a \xrightarrow{1-e} \mathsf{R}^n_a \xrightarrow{e} \mathsf{R}^n_a \xrightarrow{1-e} \underset{\deg. 0}{\mathsf{R}^n_a} \right\}.$$

These are h-projective and perfect modules over the 1-morphism categories of $\mathbf{H}'_\mathcal{V}$. They are h-projective since bounded above complexes of representable modules are semifree. They are perfect since in the homotopy categories they are the direct summands of P^n_a, Q^n_a, and R^n_a defined by the idempotents e. Thus, being h-projective and perfect, these modules define 1-morphisms of $\mathbf{H}_\mathcal{V}$ which we also denote by $\mathsf{P}^{(n)}_a$, $\mathsf{Q}^{(n)}_a$ and $\mathsf{R}^{(n)}_a$.

We can now state the main result of this section:

Theorem 6.3.

(1) For any $a, b \in \mathcal{V}$ and $n, m \in \mathbb{N}$ the following holds in $\mathbf{H}_\mathcal{V}$:

$$\mathsf{P}^{(m)}_a \mathsf{P}^{(n)}_b \cong \mathsf{P}^{(n)}_b \mathsf{P}^{(m)}_a, \quad \mathsf{Q}^{(m)}_a \mathsf{Q}^{(n)}_b \cong \mathsf{Q}^{(n)}_b \mathsf{Q}^{(m)}_a.$$

(2) For any $a, b \in \mathcal{V}$ and $n, m \in \mathbb{N}$ there exists a homotopy equivalence in $\mathbf{H}_\mathcal{V}$

$$\bigoplus_{i=0}^{\min(m,n)} \operatorname{Sym}^i \operatorname{Hom}_\mathcal{V}(a,b) \otimes_\Bbbk \mathsf{P}^{(n-i)}_b \mathsf{Q}^{(m-i)}_a \to \mathsf{Q}^{(m)}_a \mathsf{P}^{(n)}_b, \tag{6.1}$$

and thus the following holds in $H^*(\mathbf{H}_\mathcal{V})$:

$$\mathsf{Q}^{(m)}_a \mathsf{P}^{(n)}_b \cong \bigoplus_{i=0}^{\min(m,n)} \operatorname{Sym}^i \operatorname{Hom}_{H^*(\mathcal{V})}(a,b) \otimes_\Bbbk \mathsf{P}^{(n-i)}_b \mathsf{Q}^{(m-i)}_a.$$

Remark 6.4. Dually, one can formulate a version of Theorem 6.3, using the 1-morphisms R instead of P. That is, one has isomorphisms

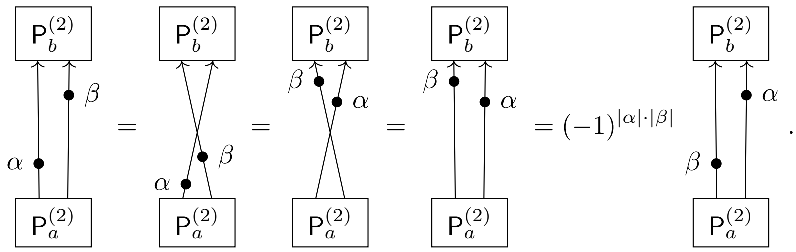
$$\mathsf{R}^{(m)}_a \mathsf{R}^{(n)}_b \cong \mathsf{R}^{(n)}_b \mathsf{R}^{(m)}_a$$

and a homotopy equivalence 2-morphism

$$\mathsf{Q}^{(m)}_a \mathsf{R}^{(n)}_b \to \bigoplus_{i=0}^{\min(m,n)} \operatorname{Sym}^i \operatorname{Hom}_\mathcal{V}(b,a)^* \otimes_\Bbbk \mathsf{R}^{(n-i)}_b \mathsf{Q}^{(m-i)}_a. \tag{6.2}$$

In the graded homotopy category, identifying $\operatorname{Hom}_{H^*(\mathcal{V})}(b,a)^*$ with $\operatorname{Hom}_{H^*(\mathcal{V})}(a, Sb)$ and R_b with P_{Sa} identifies (6.2) with (6.1).

REMARK 6.5. The appearance of the symmetric powers of $\mathrm{Hom}_{\mathcal{V}}(a,b)$ is related to the following observation. Since $\sigma e = e = e\sigma$ for any $\sigma \in S_n$, one sees that any crossings of parallel strands can be absorbed into the symmetrisers. In particular, for $\alpha, \beta \in \mathrm{Hom}(a,b)$ one has



Thus i parallel strands are naturally labeled by elements of $\mathrm{Sym}^i \mathrm{Hom}_{\mathcal{V}}(a,b)$ (using the Koszul sign convention as always).

In the remainder of this subsection we set up the maps occurring in Theorem 6.3 and prove the relations in Theorem 6.3(1) which hold on the DG level. In the next subsection we prove the relation in Theorem 6.3(2) which holds on the homotopy level.

We begin with several remarks detailing some DG 2-morphisms between $\mathsf{P}^{(n)}$s, $\mathsf{Q}^{(n)}$s and $\mathsf{R}^{(n)}$s which can be induced from those between P^ns, Q^ns and R^ns:

REMARK 6.6. We have the canonical 2-morphisms defined by e on degree 0 terms:

$$\mathsf{P}_a^n \xrightarrow{e} \mathsf{P}_a^{(n)} \xrightarrow{e} \mathsf{P}_a^n,$$

Any 2-morphism in or out of P_a^n induces via pre- or postcomposition a 2-morphism in or out of $\mathsf{P}_a^{(n)}$.

In the homotopy category, where as in any triangulated category all idempotents are split, $\mathsf{P}_a^{(n)}$ is a direct summand of P_a^n. The canonical 2-morphisms above become the morphisms of inclusion of and projection onto this direct summand. Thus, pre- or postcompositions with them are DG equivalents of taking the component corresponding to this direct summand.

The same holds for Qs, Rs, and any 1-composition of these.

REMARK 6.7. Let $\alpha\colon \mathsf{P}_a^n \to \mathsf{P}_a^n$ in $\mathbf{H}'_{\mathcal{V}}$. Recall that when illustrating maps of twisted complexes we only draw their non-zero components. In the homotopy category P_a^n splits as $\mathsf{P}_a^{(n)} \oplus \overline{\mathsf{P}_a^{(n)}}$, where $\overline{\mathsf{P}_a^{(n)}}$ is the complement summand. By Remark 6.6, the 2-morphism

$$e\alpha e := \quad \begin{array}{ccccccccc} \dots & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n \\ & & & & & & & & \big\downarrow{\scriptstyle e\alpha e} \\ \dots & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n. \end{array}$$

gives in the homotopy category the $\mathsf{P}_a^{(n)} \to \mathsf{P}_a^{(n)}$ component of α. Note, that so does

$$\alpha e := \quad \begin{array}{ccccccccc} \dots & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n \\ & & & & & & & & \big\downarrow{\scriptstyle \alpha e} \\ \dots & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n. \end{array}$$

However on the DG level αe contains extra information. Indeed, both morphisms are defined by a map of twisted complexes with a single component $\mathsf{P}_a^n \to \mathsf{P}_a^n$. These two maps $\mathsf{P}_a^n \to \mathsf{P}_a^n$, $\alpha e \alpha$ and αe, are different even in the homotopy category: $e\alpha e$ has a single component $\mathsf{P}_a^{(n)} \to \mathsf{P}_a^{(n)}$, while αe also has a $\mathsf{P}_a^{(n)} \to \overline{\mathsf{P}_a^{(n)}}$ component.

If α supercommutes with e we have another 2-morphism $\mathsf{P}_a^{(n)} \to \mathsf{P}_a^{(n)}$ given by

$$\tilde{\alpha} := \begin{array}{ccccccccc} \dots & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ \dots & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n & \xrightarrow{e} & \mathsf{P}_a^n & \xrightarrow{1-e} & \mathsf{P}_a^n. \end{array}$$

It is homotopic to αe and thus to $e\alpha e$: consider the degree -1 twisted complex map comprising degree $i \to (i-1)$ components given by α. As operations on α, both preserve the degree and commute with the differential. In particular, if α is closed of degree 0, so are αe and $\tilde{\alpha}$.

When α is an image of some $\sigma \in S_n$ under Ξ^{P}, it commutes with e because in $\Bbbk[S_n]$

$$\sigma e = e = e\sigma.$$

Hence $\tilde{\sigma}$ is well defined and homotopic to $e\sigma e = e$. Thus, all $\tilde{\sigma}$ are homotopic to $\mathrm{id} = \tilde{\mathrm{id}}$.

Similar considerations apply to a 1-composition of several powers of Ps. Let $\sigma \in S_m$ and let α be a 2-morphism

$$\alpha \colon \mathsf{P}_{a_1}^{n_1}\mathsf{P}_{a_2}^{n_2}\dots\mathsf{P}_{a_m}^{n_m} \to \mathsf{P}_{a_{\sigma(1)}}^{n_{\sigma(1)}}\mathsf{P}_{a_{\sigma(2)}}^{n_{\sigma(2)}}\dots\mathsf{P}_{a_{\sigma(m)}}^{n_{\sigma(m)}}.$$

If α supercommutes with the symmetriser e of each $\mathsf{P}_{a_i}^{n_i}$, then we have a map

$$\tilde{\alpha} \colon \mathsf{P}_{a_1}^{(n_1)}\mathsf{P}_{a_2}^{(n_2)}\dots\mathsf{P}_{a_m}^{(n_m)} \to \mathsf{P}_{a_{\sigma(1)}}^{(n_{\sigma(1)})}\mathsf{P}_{a_{\sigma(2)}}^{(n_{\sigma(2)})}\dots\mathsf{P}_{a_{\sigma(m)}}^{(n_{\sigma(m)})}$$

defined by the twisted complex map comprising degree $i \to i$ components $\sum \alpha$. Note that $\mathsf{P}_{a_1}^{(n_1)}\mathsf{P}_{a_2}^{(n_2)}\dots\mathsf{P}_{a_m}^{(n_m)}$ is the product of the twisted complexes defining each individual $\mathsf{P}_{a_j}^{(n_j)}$ and thus a twisted complex whose degree i element is the direct sum $\bigoplus_{i_1+\dots+i_n=i}\mathsf{P}_{a_1}^{n_1}\mathsf{P}_{a_2}^{n_2}\dots\mathsf{P}_{a_m}^{n_m}$ where the multi-index $(i_1,\dots,i_n)$ gives the degrees in each twisted complex of the product where each $\mathsf{P}_{a_j}^{(n_j)}$ comes from. By $\sum\alpha$ above we mean the map sending each $(i_1,\dots,i_n)$-indexed summand of the source to the $(i_1,\dots,i_n)$-indexed summand of the target via α. In the simple case when $\sigma = \mathrm{id}$ and $\alpha = \alpha_1\alpha_2\dots\alpha_n$ with $\alpha_i \colon \mathsf{P}_{a_i}^{n_i} \to \mathsf{P}_{a_i}^{n_i}$ we get $\tilde{\alpha} = \tilde{\alpha}_1\dots\tilde{\alpha}_n$.

Now, let $n = \sum_{i=0}^m n_i$, let $\phi : S_m \hookrightarrow S_n$ be the embedding of S_m as the permutation group of n_i-tuples of elements, and let $S_{n_1}\times\dots\times S_{n_m} \leq S_n$ be the subgroup of permutations which respect the partition $(n_1,\dots,n_m)$. If $\rho \in S_n$ is such that $\Xi^{\mathsf{P}}(\rho)$ is a morphism

$$\mathsf{P}_{a_1}^{n_1}\mathsf{P}_{a_2}^{n_2}\dots\mathsf{P}_{a_m}^{n_m} \to \mathsf{P}_{a_{\sigma(1)}}^{n_{\sigma(1)}}\mathsf{P}_{a_{\sigma(2)}}^{n_{\sigma(2)}}\dots\mathsf{P}_{a_{\sigma(m)}}^{n_{\sigma(m)}},$$

then $\rho = \phi(\sigma)\tau$ for some

$$\tau = (\tau_1,\dots,\tau_m) \in S_{n_1}\times\dots\times S_{n_m}.$$

Indeed, by its definition the 2-morphism $\Xi^{\mathsf{P}}(\rho)$ has only unmarked strings which can only go from P_{a_i} to P_{a_i} and not some other P_{a_j}. Thus ρ must send each n_i-tuple in the partition $(n_1,\dots,n_m)$ of n, in some order, to the corresponding $n_i = n_{\sigma(\sigma^{-1}(i))}$-tuple in the permuted partition $(n_{\sigma(1)},\dots,n_{\sigma(m)})$. Thus doing ρ is the same as individually permuting the elements of each n_i-tuple by some $\tau \in S_{n_1} \times \cdots \times S_{n_m}$ and then doing $\phi(\sigma)$ to permute the n_i-tuples.

Now, τ commutes with the symmetrisers $e_{n_i} \in \Bbbk[S_{n_i}] \subseteq \Bbbk[S_n]$ since

$$\tau e_{n_i} = (\tau_1, \dots, \tau_{i-1}, e_{n_i}, \tau_{i+1}, \dots, \tau_m) = e_{n_i}\tau.$$

The corresponding map

$$\tilde{\tau}\colon \mathsf{P}_{a_1}^{(n_1)}\mathsf{P}_{a_2}^{(n_2)}\dots\mathsf{P}_{a_m}^{(n_m)} \to \mathsf{P}_{a_1}^{(n_1)}\mathsf{P}_{a_2}^{(n_2)}\dots\mathsf{P}_{a_m}^{(n_m)},$$

is the 1-composition of the maps $\tilde{\tau}_i\colon \mathsf{P}_{a_i}^{n_i} \to \mathsf{P}_{a_i}^{n_i}$ described above, each of which is homotopic to id. Thus $\tilde{\tau}$ itself is homotopic to id.

On the other hand, $\phi(\sigma)$ commutes with the symmetrisers e_{n_i} since their action is contained within each n_i-tuple. The corresponding map

$$\widetilde{\phi(\sigma)}\colon \mathsf{P}_{a_1}^{(n_1)}\mathsf{P}_{a_2}^{(n_2)}\dots\mathsf{P}_{a_m}^{(n_m)} \to \mathsf{P}_{a_{\sigma(1)}}^{(n_{\sigma(1)})}\mathsf{P}_{a_{\sigma(2)}}^{(n_{\sigma(2)})}\dots\mathsf{P}_{a_{\sigma(m)}}^{(n_{\sigma(m)})}, \tag{6.3}$$

is then a 2-isomorphism, whose inverse is $\widetilde{\phi(\sigma^{-1})}$. In particular, in the simplest possible case $m = 2$ and $\sigma = (12)$, we get a 2-isomorphism

$$\widetilde{\phi(12)}\colon \mathsf{P}_{a_1}^{(n_1)}\mathsf{P}_{a_2}^{(n_2)} \xrightarrow{\sim} \mathsf{P}_{a_2}^{(n_2)}\mathsf{P}_{a_1}^{(n_1)}. \tag{6.4}$$

Similar considerations apply to 1-compositions of powers of Qs and Rs.

Remark 6.8. Let $\alpha\colon \mathsf{P}_a^n \to \mathsf{P}_b^n$. Arguing as in Remark 6.7 we see that if α commutes with the symmetrisers of P_a^n and P_b^n, it defines a 2-morphism

$$\tilde{\alpha}\colon \mathsf{P}_a^{(n)} \to \mathsf{P}_b^{(n)}.$$

Suppose such α lies in the image of the functor Ξ^{P}. Then, as per Remark 6.1, we have

$$\alpha = \Xi^{\mathsf{P}}(\beta)\,\Xi^{\mathsf{P}}(\sigma), \qquad \text{for } \beta \in \operatorname{Hom}_{\mathcal{V}^{\otimes n}}(a^n, b^n),\ \sigma \in S_n.$$

We saw in Remark 6.7 that $\Xi^{\mathsf{P}}(\sigma)\colon \mathsf{P}_a^n \to \mathsf{P}_a^n$ commutes with e and the corresponding map $\tilde{\sigma}\colon \mathsf{P}_a^{(n)} \to \mathsf{P}_a^{(n)}$ is homotopic to the identity.

For any $\tau \in S_n$ we have in $\mathcal{S}^n\mathcal{V}$

$$\tau \circ \beta = \tau(\beta) \circ \tau,$$

and hence β commutes with e if and only if $e(\beta) = \beta$. In other words, if and only if β lies in the image of the canonical embedding

$$\psi\colon \operatorname{Sym}^n \operatorname{Hom}_{\mathcal{V}}(a,b) \to \operatorname{Hom}_{\mathcal{V}^{\otimes n}}(a^n, b^n).$$

In particular, for any $\gamma \in \operatorname{Sym}^n \operatorname{Hom}_{\mathcal{V}}(a,b)$, $\psi(\gamma)$ commutes with e and hence its image under Ξ^{P} gives a well-defined map

$$\widetilde{\psi(\gamma)}\colon \mathsf{P}_a^{(n)} \to \mathsf{P}_b^{(n)}, \qquad \gamma \in \operatorname{Sym}^n \operatorname{Hom}_{\mathcal{V}}(a,b).$$

By the above, up to homotopy, all the maps $\mathsf{P}_a^{(n)} \to \mathsf{P}_b^{(n)}$ induced from those in the image of Ξ^{P} are of this form.

Throughout this section, we draw diagrams to define morphisms between 1-compositions of $\mathsf{P}^{(n)}$s and $\mathsf{Q}^{(n)}$s.

Any diagram defining a morphism α between the corresponding 1-compositions of P^ns and Q^ns defines a morphism $e\alpha e$ between those of $\mathsf{P}^{(n)}$s and $\mathsf{Q}^{(n)}$s as detailed in Remark 6.6.

If α commutes with the symmetriser differentials of source and target 1-compositions of P^ns and Q^ns, it furthermore defines a morphism $\tilde{\alpha}$ between 1-compositions of $\mathsf{P}^{(n)}$s and $\mathsf{Q}^{(n)}$s as detailed in Remark 6.7.

The two morphisms $e\alpha e$ and $\tilde{\alpha}$ thus produced are homotopic. In this subsection, working on DG level, we only want to work with the $(\widetilde{-})$ construction of Remark 6.7, as it can produce termwise DG isomorphisms of twisted complexes. Thus we only consider the diagrams which commute with the symmetriser differentials. In Section 6.2, working in the homotopy category, we employ arbitrary diagrams and use the symmetrising $e(-)e$ construction of Remark 6.6. It produces twisted complex maps concentrated in degree 0, which can only be homotopy equivalences. We stress again, that in the homotopy category there is no difference between the two constructions.

It is crucial for our proofs that the construction of a morphism between 1-compositions of $\mathsf{P}^{(n)}$s and $\mathsf{Q}^{(n)}$s from a diagram defining the morphism between the corresponding 1-compositions of $\mathsf{P}^{(n)}$s and $\mathsf{Q}^{(n)}$s is compatible with 2-composition, that is — with vertical concatenation of diagrams. For the $(\widetilde{-})$ construction this is automatic. For the $e(-)e$ construction this means that any two diagrams α and β we compose must satisfy

$$e\alpha e\beta e = e\alpha\beta e \tag{6.5}$$

In this subsection, we use diagrams which commute with the symmetrisers and use the $(\widetilde{-})$ construction, so this is not an issue. In §6.2 we use arbitrary diagrams and use the $e(-)e$ construction, so we check the condition (6.5) by hand. In Section 6.2.1, this is a simple idempotent absorption argument: the symmetriser idempotent of a subgroup can be absorbed into the symmetriser idempotent of the group. In Section 6.2.2 a more elaborate argument is necessary and we show that (6.5) only holds up to a desired numerical coefficient.

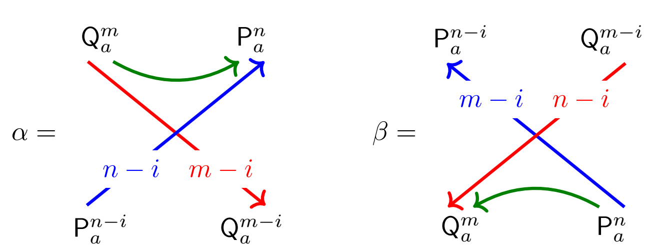
We use the following conventions to simplify the diagrams in the context of this section:

(1) A box containing a^n at the top or the bottom of the diagram denotes both 1-morphisms $\mathsf{P}_a^{(n)}$ and $\mathsf{Q}_a^{(n)}$:

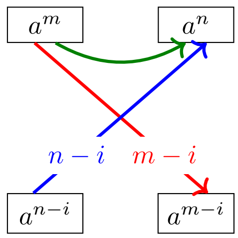
$$\boxed{a^n}$$

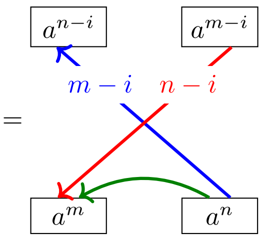
We never use type R 1-morphisms, so the orientation of the attached strands makes clear what is meant.

When such box occurs inside the diagram, it is the symmetriser idempotent e_{S_n}. Note that in the context of $e(\widetilde{-})e$ construction, the boxes at the top and the bottom can also be viewed as occurences of symmetriser idempotents. We mainly use this notation to differentiate between the LHS and the RHS of the condition (6.5). For example, if we start with diagrams

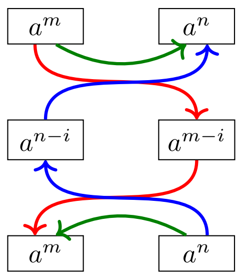


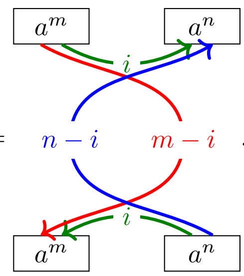
then the induced morphisms between $\mathsf{Q}_a^{(m)}\mathsf{P}_a^{(n)}$ and $\mathsf{P}_a^{(n-i)}\mathsf{Q}_a^{(m-i)}$ are

$e\alpha e =$ 

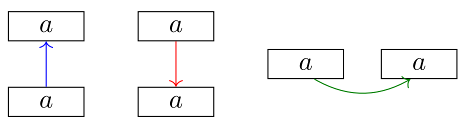
$e\beta e =$ 

and the LHS and the RHS of (6.5) are

$e\alpha e\beta e =$ 

$e\alpha\beta e =$  .

(2) Upwards strands are coloured blue, downwards strands red and counterclockwise turns green (clockwise turns will not appear in the argument):

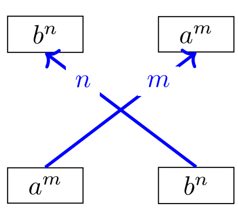


This colouring is solely for the convenience of the reader and does not have any additional meaning.

(3) We denote multiple unadorned parallel strands all starting at one box and ending at another box by a single thick strand labelled with the strand multiplicity.

Thus, an upward braid of thick strands of multiplicities $n_1, \ldots, n_m$ permuting m boxes $a_1^{n_1}$, ..., $a_m^{n_m}$ defines a 2-isomorphism between the corresponding 1-compositions of $\mathsf{P}_{a_i}^{n_i}$. It depends only on the permutation type $\sigma \in S_m$ of the braid. Moreover, as seen in Remark 6.7, it commutes with the symmetrisers and thus defines the 2-isomorphism $\widetilde{\phi(\sigma)}$ of (6.3) between the corresponding 1-compositions of $\mathsf{P}_{a_i}^{(n_i)}$.

For example, the 2-isomorphism $\mathsf{P}_a^{(m)}\mathsf{P}_b^{(n)} \to \mathsf{P}_b^{(n)}\mathsf{P}_a^{(m)}$ of (6.4) is



(4) A thick strand from box a^n to box b^n labelled with an element

$$\alpha \in \operatorname{Sym}^n \operatorname{Hom}_{\mathcal{V}}(a, b)$$

denotes the 2-morphism

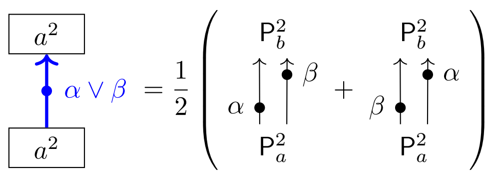
$$\Xi^{\mathsf{P}}(\psi(\alpha))$$

of Remark 6.8. As it commutes with the symmetrisers, it defines a 2-morphism $\widetilde{\psi(\alpha)}$ between the corresponding symmetric powers of Ps or Qs.

For example, suppose that α is an elementary symmetric tensor

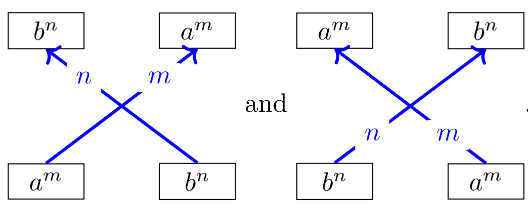
$$\alpha = \alpha_1 \vee \cdots \vee \alpha_n := \frac{1}{n!} \sum_{\sigma \in S_n} \alpha_{\sigma(1)} \cdots \alpha_{\sigma(n)}.$$

A thick strand labelled α is the sum of all permutations of n parallel strands adorned with the α_is. In particular, for even degree α and β we have



With the above notation in mind, we have immediately:

Proof of Theorem 6.3(1). We claim that the 2-morphisms



are inverse to each other. Indeed, as $\widetilde{(-)}$ is compatible with the compositions, we can vertically concatenate the diagrams and then apply Lemma 5.5 (3) multiple times to get the claim. Thus we have the relation $\mathsf{P}_a^{(m)}\mathsf{P}_b^{(n)} \cong \mathsf{P}_b^{(n)}\mathsf{P}_a^{(m)}$. The second relation $\mathsf{Q}_a^{(m)}\mathsf{Q}_b^{(n)} \cong \mathsf{Q}_b^{(n)}\mathsf{Q}_a^{(m)}$ is implied by a similar pair of diagrams but involving downward strands. □

6.2. The Heisenberg commutation relations: homotopy level

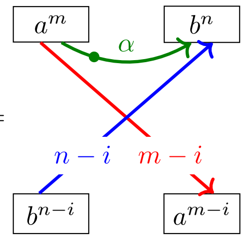
Next, let us construct the 2-morphism giving (6.1). Defining a map

$$g_i\colon \operatorname{Sym}^i \operatorname{Hom}_{\mathcal{V}}(a,b) \otimes_{\Bbbk} \mathsf{P}_b^{(n-i)}\mathsf{Q}_a^{(m-i)} \to \mathsf{Q}_a^{(m)}\mathsf{P}_b^{(n)}.$$

is equivalent to defining a map

$$\tilde{g}_i\colon \operatorname{Sym}^i \operatorname{Hom}_{\mathcal{V}}(a,b) \to \operatorname{Hom}_{\mathbf{H}_{\mathcal{V}}}\big(\mathsf{P}_b^{(n-i)}\mathsf{Q}_a^{(m-i)},\, \mathsf{Q}_a^{(m)}\mathsf{P}_b^{(n)}\big).$$

For any $\alpha \in \operatorname{Sym}^i \operatorname{Hom}(a, b)$, define

$\tilde{g}_i(\alpha) =$ 

Now, define $g := \sum_i g_i$. The map g does in general not have an inverse on the DG level. However, we can define an inverse map f in $H^*(\mathbf{H}_\mathcal{V})$. To define a map

$$f_i \colon \mathsf{Q}_a^{(m)}\mathsf{P}_b^{(n)} \to \operatorname{Sym}^i \operatorname{Hom}_{H^*(\mathcal{V})}(a, b) \otimes_\Bbbk \mathsf{P}_b^{(n-i)}\mathsf{Q}_a^{(m-i)},$$

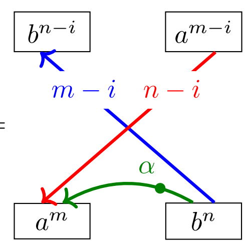
we define a map

$$\tilde{f}_i \colon \big(\operatorname{Sym}^i \operatorname{Hom}_{H^*(\mathcal{V})}(a, b)\big)^* \to \operatorname{Hom}_{\mathbf{H}_\mathcal{V}}\big(\mathsf{Q}_a^{(m)}\mathsf{P}_b^{(n)}, \, \mathsf{P}_b^{(n-i)}\mathsf{Q}_a^{(m-i)}\big),$$

or equivalently a map

$$\tilde{f}'_i \colon \operatorname{Sym}^i \operatorname{Hom}_{H^*(\mathcal{V})}(b, Sa) \to \operatorname{Hom}_{H^*(\mathbf{H}_\mathcal{V})}\big(\mathsf{Q}_a^{(m)}\mathsf{P}_b^{(n)}, \, \mathsf{P}_b^{(n-i)}\mathsf{Q}_a^{(m-i)}\big),$$

where we use the identification $\operatorname{Hom}(a, \, b)^* = \operatorname{Hom}(b, \, Sa)$ in $H^*(\mathcal{V})$. Set

$\tilde{f}'_i(\alpha) =$ 

Finally, set

$$f = \sum_i i! \binom{m}{i} \binom{n}{i} f_i.$$

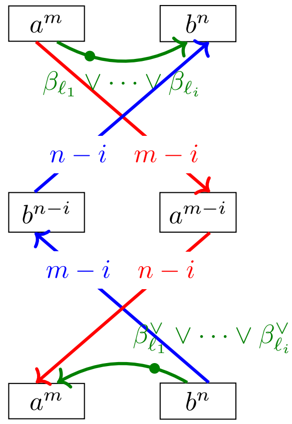
We now show that f and g are inverse isomorphisms in $H^*(\mathbf{H}_\mathcal{V})$. The proof is entirely combinatorial: one composition follows from repeated application of the second relation in (3.10), which holds in $H^*(\mathbf{H}_\mathcal{V})$ by Lemma 5.7. The other follows from relations (5.11). The reader uninterested in combinatorics may want to skip ahead to Section 6.3.

6.2.1. The composition $g \circ f$ is the identity. For simplicity, in this section we denote the image of any closed degree zero 2-morphism of $\mathbf{H}_\mathcal{V}$ in $H^*(\mathbf{H}_\mathcal{V})$ by the same symbol as the original 2-morphism.

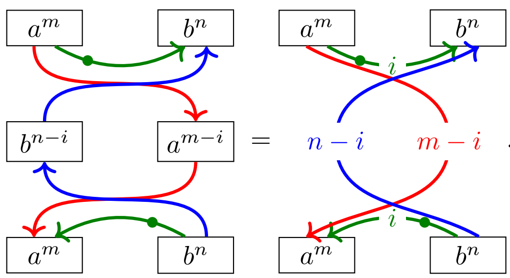
Remark 6.9. Choose a basis $\{\beta_\ell\}$ for $H^*(\operatorname{Hom}_\mathcal{V}(a, \, b))$ with dual basis $\{\beta_\ell^\vee\}$ of $H^*(\operatorname{Hom}_\mathcal{V}(b, \, Sa))$. Let $I = (\ell_1, \ldots, \ell_i)$ be a multi-index. Then the dual to $\beta_{\ell_1} \vee \cdots \vee \beta_{\ell_i} \in \operatorname{Sym}^i H^*(\operatorname{Hom}_\mathcal{V}(a, \, b))$ is $\frac{1}{m(I)} \beta_{\ell_1}^\vee \vee \cdots \vee \beta_{\ell_i}^\vee \in \operatorname{Sym}^i H^*(\operatorname{Hom}_\mathcal{V}(b, \, Sa))$, where $m(I) = \prod m_j(I)!$ with $m_j(I)$ the number of times the index j appears in I.

Let $\phi^{0,0}_{m-i,n-i,i}$ denote the 2-morphism $g_i \circ f_i$. We have

$$\phi^{0,0}_{m-i,n-i,i} = \sum_{\ell_1,\ldots,\ell_i}$$



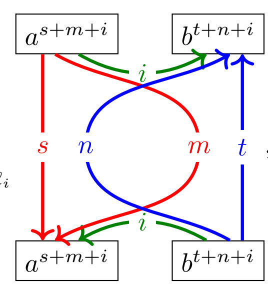
We first verify that the composition condition (6.5) holds:



It does because we can absorb the middle idempotents into the top or bottom ones. We can move elements of (or their sums) of S_{n-i} (resp. S_{m-i}) in the middle idempotents all the way up or down their strands where they can be viewed as elements of the corresponding subgroup $S_{n_i} < S_n$ (resp. $S_{m-i} < S_m$). Pre- or postcomposing with these does not change the symmetriser idempotent of S_n (resp. S_m).

Adding s downward strands on the left and t upward strands on the right, we denote the resulting 2-endomorphism of $\mathsf{Q}_a^{(m+s)}\mathsf{P}_b^{(n+t)}$ by $\phi^{s,t}_{m-i,n-i,i}$. Relabeling slightly, with a choice of basis as in Remark 6.9 this gives

$$\phi^{s,t}_{m,n,i} = \sum_{\ell_1,\ldots,\ell_i}$$



where the arcs are labeled by $\beta_{\ell_1} \vee \cdots \vee \beta_{\ell_i}$ and $\beta^\vee_{\ell_1} \vee \cdots \vee \beta^\vee_{\ell_i}$ respectively. One notes that

$$\phi^{s,t}_{m,0,i} = \phi^{s+m,t}_{0,0,i} \quad \text{and} \quad \phi^{s,t}_{0,n,i} = \phi^{s,t+n}_{0,0,i}.$$

To simplify notation, write $\psi^s_{m,i} \coloneqq \phi^{s,s}_{m,m,i}$ for the symmetric situation.

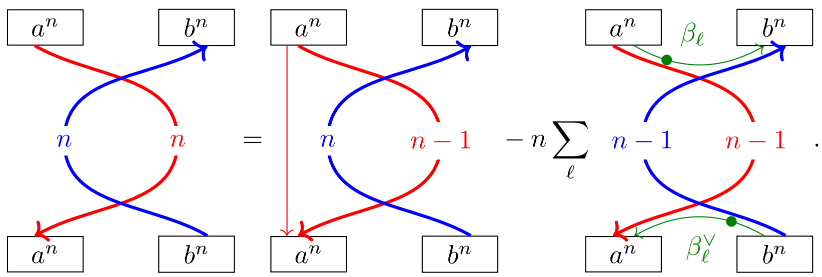
Lemma 6.10.

$$\psi^0_{n,0} = -n\psi^0_{n-1,1} + \sum_{i=0}^{n-1}(-1)^i \frac{(n-1)!}{(n-1-i)!}\psi^1_{n-1-i,i}.$$

Proof. First move the left-most downwards strand of $\psi^0_{n,0}$ all the way to the left. To do so, one has to untwist n down-up double crossings, introducing n terms of the from $-\psi^0_{n-1,1}$ via relation (3.10):

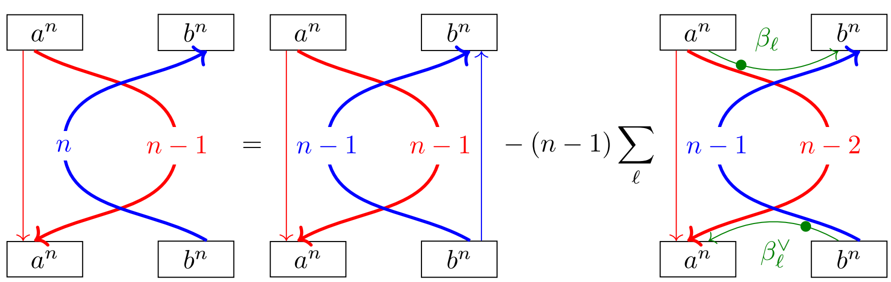
$$\psi^0_{n,0} = \phi^{1,0}_{n-1,n,0} - n\psi^0_{n-1,1},$$

or graphically,

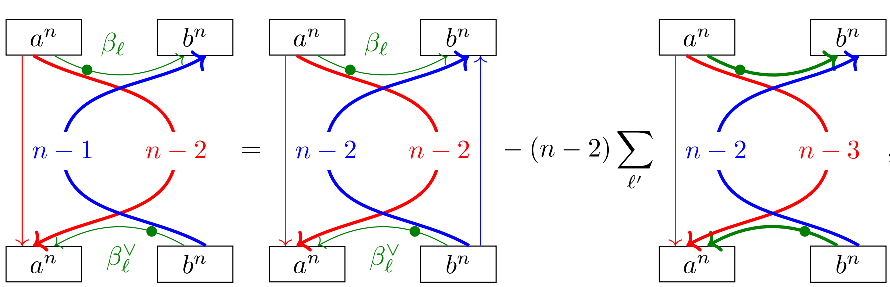


Now move the rightmost upward strand of $\phi^{1,0}_{n-1,n,0}$ all the way to the right. To do so, one has to untwist with $n-1$ downwards strands, introducing $n-1$ terms of the form $-\phi^{1,0}_{n-2,n-1,1}$:

$$\phi^{1,0}_{n-1,n,0} = \psi^1_{n-1,0} - (n-1)\phi^{1,0}_{n-2,n-1,1}.$$



Repeat the last step for $-(n-1)\phi^{1,0}_{n-2,n-1,1}$, obtaining $-(n-1)\psi^1_{n-2,1}$ and $(n-1)(n-2)$ terms of the form $\phi^{1,0}_{n-3,n-2,2}$:



where the dots are marked with $\beta_\ell \vee \beta_{\ell'}$ and $\beta_\ell^\vee \vee \beta_{\ell'}^\vee$ respectively. Recursive application of this procedure yields the desired formula. □

Rearranging and changing indices by 1, we obtain

$$\psi^1_{n,0} = \psi^0_{n+1,0} + (n+1)\psi^0_{n,1} + \sum_{i=1}^{n}(-1)^{i+1}\frac{n!}{(n-i)!}\psi^1_{n-i,i}. \tag{6.6}$$

The remainder of the argument is now just repeated application of this formula.

Lemma 6.11.

$$\psi^1_{n,0} = \psi^0_{n+1,0} + (2n+1)\psi^0_{n,1} + n^2\psi^0_{n-1,2}.$$

Proof. For $n = 0$, this is just equation (6.6). Using induction and (6.6) we get

$$\begin{aligned}\psi^1_{n+1,0} &= \psi^0_{n+1,0} + (n+2)\psi^0_{n+1,1} + \sum_{i=1}^{n+1}(-1)^{i+1}\frac{(n+1)!}{(n+1-i)!}\psi^1_{n+1-i,i} \\ &= \psi^0_{n+1,0} + (n+2)\psi^0_{n+1,1} + \sum_{i=1}^{n+1}(-1)^{i+1}\frac{(n+1)!}{(n+1-i)!}\Big(\psi^0_{n+2-i,i} + \\ &\qquad + (2(n+1-i)+1)\psi^0_{n+1-i,i+1} + (n+1-i)^2\psi^0_{n-i,i+2}\Big).\end{aligned}$$

Carefully rearranging terms, one obtains

$$\begin{aligned}&\psi^0_{n+1,0} + (n+2)\psi^0_{n+1,1} + \frac{(n+1)!}{n!}\psi^0_{n+1,1} \\ &\qquad + \left(-\frac{(n+1)!}{(n-1)!} + (2n+1)\frac{(n+1)!}{n!}\right)\psi^0_{n,2} + \\ &+ \sum_{\ell=3}^{n+2}\Bigg((-1)^{\ell+1}\frac{(n+1)!}{(n+1-\ell)!} + (-1)^{\ell}\frac{(n+1)!}{(n+2-\ell)!}(2(n+2-\ell)+1) + \\ &\qquad (-1)^{\ell-1}\frac{(n+1)!}{(n+3-\ell)!}(n+3-\ell)^2\Bigg)\psi^0_{n+2-\ell,\ell} \\ &\qquad\qquad + \left(\frac{(n+1)!}{0!}(-1)^{n+2} + \frac{(n+1)!}{1!}(-1)^{n+1}1^2\right)\psi^0_{0,n+2},\end{aligned}$$

which one easily checks to be equal to

$$\psi^0_{n+2,0} + (2n+3)\psi^0_{n+1,1} + (n+1)^2\psi^0_{n,2}. \qquad \square$$

Lemma 6.12.

$$\psi^k_{0,0} = \sum_{i=0}^{k} i!\binom{k}{i}^2\psi^0_{k-i,i}.$$

Proof. For $k = 1$ this is immediate from (6.6). Assume that the identity holds for some integer k. Then

$$\psi^{k+1}_{0,0} = \sum_{i=0}^{k} i!\binom{k}{i}^2\psi^1_{k-i,i}.$$

We can now substitute in the identity of Lemma 6.11.

$$\psi^{k+1}_{0,0} = \sum_{i=0}^{k} i!\binom{k}{i}^2\left(\psi^0_{k-i+1,i} + (2(k-i)+1)\psi^0_{k-i,i+1} + (k-i)^2\psi^0_{k-i-1,i+2}\right)$$

Rearranging gives

$$\psi^0_{k+1,0} + \big(k^2 + (2k+1)\big)\psi^0_{k,1} + \sum_{\ell=2}^{k}\Bigg(\ell!\binom{k}{\ell}^2 + (\ell-1)!\binom{k}{\ell-1}^2(2(k-\ell+1)+1) + (\ell-2)!\binom{k}{\ell-2}^2(k-\ell+2)^2\Bigg)\psi^0_{k+1-\ell,\ell} + \Bigg(k!\binom{k}{k}^2 + (k-1)!\binom{k}{k-1}^2 1^2\Bigg)\psi^0_{0,k+1}.$$

Again, one easily checks this to be equal to

$$\sum_{i=0}^{k+1} i!\binom{k+1}{i}^2 \psi^0_{k+1-i,i}. \qquad \square$$

COROLLARY 6.13.

$$\phi^{m,n}_{0,0,0} = \sum_{i=0}^{\min(m,n)} i!\binom{m}{i}\binom{n}{i}\phi^{0,0}_{m-i,n-i,i}.$$

In other words $g \circ f = 1$.

PROOF. Without loss of generality we can assume that $m \geq n$, say $m = n + j$. We will induct on j. We already considered the case that $j = 0$.

We ignore the left-most string and use the induction hypothesis to obtain

$$\phi^{n+j+1,n}_{0,0,0} = \sum_{i=0}^{n} i!\binom{n+j}{i}\binom{n}{i}\phi^{1,0}_{n+j-i,n-i,i}.$$

As in the first step of the proof of Lemma 6.11, we have

$$\phi^{1,0}_{n+j-i,n-i,i} = \phi^{0,0}_{n+j-i+1,n-i,i} + (n-i)\phi^{0,0}_{n+j-i,n-i-1,i+1}.$$

Thus,

$$\phi^{n+j+1,n}_{0,0,0} = \sum_{i=0}^{n} i!\binom{n+j}{i}\binom{n}{i}(\phi^{0,0}_{n+j-i+1,n-i,i} + (n-i)\phi^{0,0}_{n+j-i,n-i-1,i+1}).$$

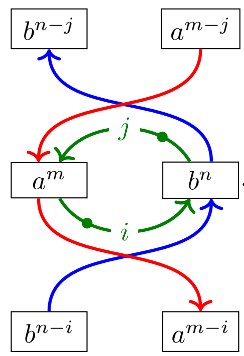
Grouping terms, this is

$$0!\binom{n+j}{0}\binom{n}{0}\phi^{0,0}_{n+j+1,n,0} + \sum_{\ell=1}^{n}\Bigg(\ell!\binom{n+j}{\ell}\binom{n}{\ell} + (\ell-1)!\binom{n+j}{\ell-1}\binom{n}{\ell-1}(n-\ell+1)\Bigg)\phi^{0,0}_{n+j+1-\ell,n-\ell,\ell}.$$

This is easily shown to be equal to the desired expression

$$\sum_{\ell=0}^{n} \ell!\binom{n+j+1}{\ell}\binom{n}{\ell}\phi^{0,0}_{n+j+1-\ell,n-\ell,\ell}. \qquad \square$$

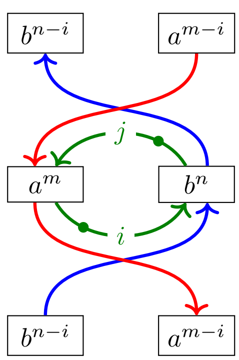
6.2.2. The composition $f \circ g$ is the identity. We have

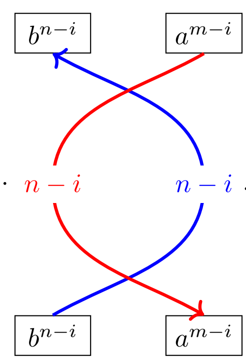
$f_j \circ g_i \; =$ 

.

When $i \neq j$, every combination of summands of the middle symmetriser idempotents produces a diagram which contains a left curl and hence vanishes. Thus $f_j \circ g_i = 0$ if $i \neq j$.

When $i = j$, we claim that the composition condition (6.5) holds up to a coefficient:

(6.7) 

$= \dfrac{1}{i!\binom{m}{i}\binom{n}{i}} \cdot$ 

.

Indeed, the pair of the middle idempotents in the LHS of (6.7) are a 2-morphism

$$\frac{1}{m!}\frac{1}{n!}\sum_{\sigma \in S_m, \tau \in S_n} \sigma \circ_1 \tau \tag{6.8}$$

where $\circ_1$ denotes 1-composition. We first observe that if $\sigma \notin S_{m-i} \times S_i < S_m$ or $\tau \notin S_i \times S_{n-i}$ the resulting diagram contains a left curl and hence vanishes. Let

$$\sigma = (\sigma_{m-i}, \sigma_i) \in S_{m-i} \times S_i,$$
$$\tau = (\tau_i, \tau_{n-i}) \in S_i \times S_{n-i}.$$

On the diagram coresponding to this summand, we can slide σ_i along the central bubble and compose it with τ_i. We obtain a counterclockwise bubble of i parallel strands with a single element $\tau_i\sigma_i \in S_i$ inserted into it. Unless this element is id_{S_i}, the resulting diagram contains a left curl. When it is id_{S_i}, we get an unmarked i-stranded counterclockwise bubble which is the identity endomorphism of $\mathbb{1}$ and hence can be erased. On the remaining diagram, we can absorb σ_{m-i} and τ_{n-i} into the top or bottom idempotents and thus obtain the diagram on the RHS of (6.7).

Thus when expanding the middle idempotents in the LHS of (6.7) the non-vanishing diagrams are given by the summands

$$\sigma_{m-i} \circ_1 \upsilon_i \circ_1 \upsilon_i^{-1} \circ_1, \tau_{n-i} \qquad \sigma_{m-i} \in S_{m-i}, \upsilon_i \in S_i, \tau_{n-i} \in S_{n-i}$$

of (6.8). There are $(m-i)!i!(n-i)!$ of them and each produces the diagram on the RHS of (6.7), whence the equality in (6.7) holds.

By the left relation in (3.10) the RHS of (6.7) is

$$\frac{1}{i!\binom{m}{i}\binom{n}{i}} \operatorname{id}_{\mathsf{P}_b^{(n-i)}\mathsf{Q}_a^{(m-i)}} .$$

Since $g = \sum g_i$ and $f = \sum_i i!\binom{m}{i}\binom{n}{i} f_i$, it follows that $f \circ g = \operatorname{id}$. This finishes the proof of Theorem 6.3.

6.3. The transposed generators

Given any partition λ of n write $e_\lambda \in \Bbbk[S_n]$ for the corresponding Young symmetriser. It is a minimal idempotent of $\Bbbk[S_n]$. Thus, similar to the definition of the 1-morphisms $\mathsf{P}_a^{(n)}$ and $\mathsf{Q}_a^{(n)}$, it induces 1-morphisms P_a^λ and Q_a^λ in $\mathbf{H}_\mathcal{V}$.

Recall the transposed generators $p_a^{(1^n)}$ and $q_a^{(1^n)}$, $n \in \mathbb{Z}_{>0}$ from Section 2.2.2. We have the antisymmetriser idempotent corresponding to the partition (1^n)

$$e_{\text{sign}} = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma \in \Bbbk[S_n]$$

on which S_n acts by the sign character. Let $\mathsf{P}_a^{(1^n)}$ and $\mathsf{Q}_a^{(1^n)}$ be the corresponding 1-morphisms defined analogously to Definition 6.2.

Arguing as in Remark 6.8, we see that elements of $\operatorname{Sym}^n \operatorname{Hom}(a, b)$ define morphisms from $\mathsf{P}_a^{(1^n)}$ to $\mathsf{P}_b^{(1^n)}$, while those of $\bigwedge^n \operatorname{Hom}(a, b)$ define morphisms from $\mathsf{P}_a^{(n)}$ to $\mathsf{P}_b^{(1^n)}$.

The category $\mathbf{H}'_\mathcal{V}$ has a covariant autoequivalence F which

- is identity on objects and 1-morphisms,
- on 2-morphisms it multiplies the crossings by -1, while preserving all other generating diagrams.

The induced autoequivalence F of $\mathbf{H}_\mathcal{V}$ swaps the 1-morphisms above with those of Section 6.1:

$$F(\mathsf{P}_a^{(n)}) = \mathsf{P}_a^{(1^n)}, \quad F(\mathsf{P}_a^{(1^n)}) = \mathsf{P}_a^{(n)}, \quad F(\mathsf{Q}_a^{(n)}) = \mathsf{Q}_a^{(1^n)}, \quad F(\mathsf{Q}_a^{(1^n)}) = \mathsf{Q}_a^{(n)}.$$

Thus the relations of Theorem 6.3 also hold for the transposed 1-morphisms.

Lemma 6.14. *If $\mathcal{V}$ is pretriangulated, then for any $a \in \mathcal{V}$ we have in $\mathbf{H}_\mathcal{V}$ isomorphisms*

$$\mathsf{P}_{a[1]} \cong \mathsf{P}_a[1] \text{ and } \mathsf{Q}_{a[1]} \cong \mathsf{Q}_a[-1],$$

and isomorphisms

$$\mathsf{P}_{a[1]}^{(n)} \cong \mathsf{P}_a^{(1^n)}[n] \quad \text{and} \quad \mathsf{Q}_{a[1]}^{(n)} \cong \mathsf{Q}_a^{(1^n)}[-n].$$

Proof. We prove the statements about Ps. Those about Qs are proved similarly with a twist in the sign; see the end of the proof below.

Let $i\colon a[1] \to a$ be the degree -1 morphism in $\mathcal{V}$ defined by id_a. Let $\iota'\colon \mathsf{P}_{a[1]} \to \mathsf{P}_a$ be the corresponding morphism $\uparrow\!\bullet\, i$ in $\mathbf{H}'_\mathcal{V}$. Finally, let $\iota\colon \mathsf{P}_{a[1]} \to \mathsf{P}_a[1]$ be the degree zero morphism in $\mathbf{H}'_\mathcal{V}$ defined by ι'. It is an isomorphism as it has an inverse $\iota^{-1} : \mathsf{P}_a[1] \to \mathsf{P}_{a[1]}$ which is similarly defined by id_a.

By definition, $\mathsf{P}^{(n)}_{a[1]}$ is the convolution of the twisted complex

$$\dots \xrightarrow{e_{\mathrm{triv}}} \mathsf{P}^n_{a[1]} \xrightarrow{1-e_{\mathrm{triv}}} \mathsf{P}^n_{a[1]} \xrightarrow{e_{\mathrm{triv}}} \mathsf{P}^n_{a[1]} \xrightarrow{1-e_{\mathrm{triv}}} \underset{\deg.\,0}{\mathsf{P}^n_{a[1]}},$$

while is $\mathsf{P}^{(1^n)}_a[n]$ the convolution of

$$\dots \xrightarrow{e_{\mathrm{sign}}} \mathsf{P}^n_a[n] \xrightarrow{1-e_{\mathrm{sign}}} \mathsf{P}^n_a[n] \xrightarrow{e_{\mathrm{sign}}} \mathsf{P}^n_a[n] \xrightarrow{1-e_{\mathrm{sign}}} \underset{\deg.\,0}{\mathsf{P}^n_a[n]}.$$

Consider the following map of twisted complexes

$$\tilde{\iota}^n := \begin{array}{ccccccccc} \dots & \xrightarrow{e_{\mathrm{triv}}} & \mathsf{P}^n_{a[1]} & \xrightarrow{1-e_{\mathrm{triv}}} & \mathsf{P}^n_{a[1]} & \xrightarrow{e_{\mathrm{triv}}} & \mathsf{P}^n_{a[1]} & \xrightarrow{1-e_{\mathrm{triv}}} & \mathsf{P}^n_{a[1]} \\ & & \downarrow{\iota^n} & & \downarrow{\iota^n} & & \downarrow{\iota^n} & & \downarrow{\iota^n} \\ \dots & \xrightarrow{e_{\mathrm{sign}}} & \mathsf{P}^n_a[n] & \xrightarrow{1-e_{\mathrm{sign}}} & \mathsf{P}^n_a[n] & \xrightarrow{e_{\mathrm{sign}}} & \mathsf{P}^n_a[n] & \xrightarrow{1-e_{\mathrm{sign}}} & \mathsf{P}^n_a[n]. \end{array}$$

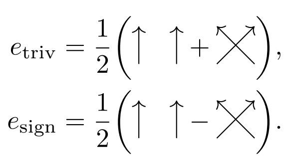
We claim that $\iota^n \colon \mathsf{P}^n_{a[1]} \to \mathsf{P}^n_a[n]$ intertwines the idempotents e_{triv} and e_{sign}:

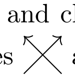
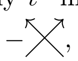
$$\iota^n e_{\mathrm{triv}} = e_{\mathrm{sign}} \iota^n.$$

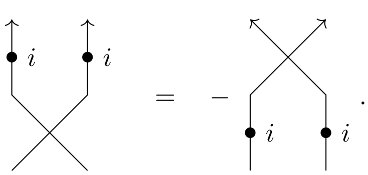
It follows that $\tilde{\iota}^n$ is closed of degree 0. We conclude that it is an isomorphism, as ι^n is one.

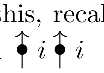
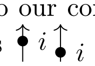
To prove the claim, it suffices to show that degree $-n$ map $\iota^n \colon \mathsf{P}^n_{a[1]} \to \mathsf{P}^n_a$ intertwines e_{triv} and e_{sign}. This is a straightforward verification in $\mathbf{H}'_{\mathcal{V}}$. We give the details for $n = 2$; the general case follows in the same manner.

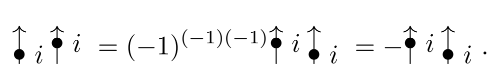
When $n = 2$, we have



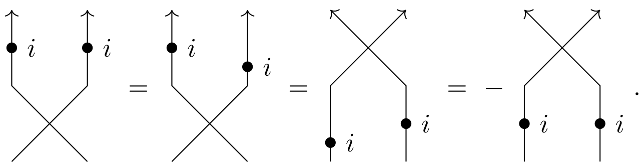
The 2-morphism $\uparrow\ \uparrow$ is the identity map, and clearly ι^2 intertwines the identity maps. It remains to show that it intertwines  and $-$, that is:



To see this, recall that according to our convention explained in Remark 3.1, the diagram  should be read as . Since i has degree -1, the graded interchange law states



Consequently:



For Q, let $i\colon a \to a[1]$ be the degree 1 morphism in $\mathcal{V}$ defined by id_a. Let $\iota'\colon \mathsf{Q}_{a[1]} \to \mathsf{Q}_a$ be the corresponding morphism $\downarrow\!\bullet\, i$ in $\mathbf{H}'_{\mathcal{V}}$. Moreover, let $\iota\colon \mathsf{Q}_{a[1]} \to \mathsf{Q}_a[-1]$ be the degree zero morphism in $\mathbf{H}'_{\mathcal{V}}$ defined by ι'. Again, it is an isomorphism with inverse $\iota^{-1} : \mathsf{Q}_a[-1] \to \mathsf{Q}_{a[1]}$ defined similarly by id_a. The rest of the proof is similar. ☐

This result affords us the following further relations:

PROPOSITION 6.15.

(1) For any $a, b \in \mathcal{V}$ and $n, m \in \mathbb{N}$ the following holds in $\mathbf{H}_{\mathcal{V}}$:

$$\mathsf{P}_a^{(1^m)}\mathsf{P}_b^{(n)} \cong \mathsf{P}_b^{(n)}\mathsf{P}_a^{(1^m)}, \quad \mathsf{Q}_a^{(1^m)}\mathsf{Q}_b^{(n)} \cong \mathsf{Q}_b^{(n)}\mathsf{Q}_a^{(1^m)},$$

(2) For any $a, b \in \mathcal{V}$ and $n, m \in \mathbb{N}$ we have a homotopy equivalence in $\mathbf{H}_{\mathcal{V}}$:

$$\bigoplus_{i=0}^{\min(m,n)} \bigwedge^{i} \mathrm{Hom}_{\mathcal{V}}(a, b) \otimes_{\Bbbk} \mathsf{P}_b^{(n-i)}\mathsf{Q}_a^{(1^{m-i})} \to \mathsf{Q}_a^{(1^m)}\mathsf{P}_b^{(n)}.$$

and thus the following holds in $H^*(\mathbf{H}_{\mathcal{V}})$

$$\mathsf{Q}_a^{(1^m)}\mathsf{P}_b^{(n)} \cong \bigoplus_{i=0}^{\min(m,n)} \bigwedge^{i} \mathrm{Hom}_{H^*(\mathcal{V})}(a, b) \otimes_{\Bbbk} \mathsf{P}_b^{(n-i)}\mathsf{Q}_a^{(1^{m-i})}$$

The above also holds with the roles of (1^m) and (n) interchanged.

PROOF. Replace a with $b[-1]$, resp. with $b[1]$ in Lemma 6.14 to get (up to a shift) claim (1) from Theorem 6.3 (1). Claim (2) follows similarly from Theorem 6.3 (2) using the identification

$$\mathrm{Sym}^i \mathrm{Hom}(a[1], b) \cong \mathrm{Sym}^i(\mathrm{Hom}(a, b)[-1]) \cong \bigwedge^{i}(\mathrm{Hom}(a, b))[-1]$$

of graded symmetric powers. For the final statement, apply the automorphism F. ☐

EXAMPLE 6.16. Let $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup. In Example 5.10 we defined the 1-morphisms $P_i = \mathsf{P}_{\mathcal{E}_i}$ and $Q_i = \mathsf{Q}_{\mathcal{E}_i}[1]$ for each $i \in I_\Gamma$. Thus the 1-morphism $Q_i^{(n)}$ of $\mathcal{H}^\Gamma$ in **[13]** corresponds to the 1-morphism $\mathsf{Q}_{\mathcal{E}_i[-1]}^{(1^n)}$. From (5.17) one obtains

$$\mathrm{Hom}^*(\mathcal{E}_i[-1], \mathcal{E}_j) = \begin{cases} \mathbb{C}[1] \oplus \mathbb{C}[-1], & i = j \\ \mathbb{C}, & \langle i, j\rangle = -1 \\ 0, & \text{otherwise.} \end{cases}$$

The k-th exterior power of $\mathbb{C}[1] \oplus \mathbb{C}[-1]$ is $\bigoplus_{j=0}^{k} \mathbb{C}[k-2j]$. Identifying it with $H^*(\mathbb{P}^k)[k]$, we see that Proposition 6.15 agrees with [**13**, Proposition 2]:

$$P_i^{(m)} P_j^{(n)} \cong P_j^{(n)} P_i^{(m)}, \quad Q_i^{(m)} \mathsf{Q}_j^{(n)} \cong Q_j^{(n)} Q_i^{(m)},$$

$$Q_i^{(m)} P_j^{(n)} \cong \begin{cases} \bigoplus_{k=0}^{\min(m,n)} H^*(\mathbb{P}^k)[k] \otimes_{\Bbbk} P_j^{(n-k)} Q_i^{(m-k)} & \text{if } i = j \in I_\Gamma, \\ P_j^{(n)} Q_i^{(m)} \oplus P_j^{(n-1)} Q_i^{(m-1)} & \text{if } \langle i, j \rangle = -1, \\ P_j^{(n)} Q_i^{(m)} & \text{if } \langle i, j \rangle = 0. \end{cases}$$

6.4. Grothendieck groups

Recall the definition of the numerical Grothendieck group $\mathrm{K}_0^{\mathrm{num}}$ of a DG category given in Section 4.9. It is the quotient of the usual Grothendieck group by the kernel of the Euler pairing. Recall from Section 2.2.1 that we write $H_{\mathcal{V}}$ for the idempotent modified Heisenberg algebra of the lattice $(\mathrm{K}_0^{\mathrm{num}}(\mathcal{V}), \chi)$. We note again that we use the numerical Grothendieck group to ensure that this algebra has trivial centre.

In this section we compare $H_{\mathcal{V}}$ to the Grothendieck group of the Heisenberg category $\mathbf{H}_{\mathcal{V}}$. Let $\mathrm{K}_0(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ be the $\Bbbk$-linear category with the same objects as $\mathbf{H}_{\mathcal{V}}$ and morphism spaces

$$\mathrm{Hom}_{\mathrm{K}_0(\mathbf{H}_{\mathcal{V}}, \Bbbk)}(N, N') = \mathrm{K}_0\big(\mathrm{Hom}_{\mathbf{H}_{\mathcal{V}}}(N, N'), \Bbbk\big),$$

where for any DG category $\mathcal{A}$ we set $\mathrm{K}_0(\mathcal{A}, \Bbbk) = \mathrm{K}_0(\mathcal{A}) \otimes_{\mathbb{Z}} \Bbbk$. As forming Grothendieck groups is functorial, the 1-composition of $\mathbf{H}_{\mathcal{V}}$ induces the composition on $\mathrm{K}_0(\mathbf{H}_{\mathcal{V}}, \Bbbk)$.

A closed string diagram defines an endomorphism of $\mathbb{1}$. Some of these endomorphisms are non-trivial and are not subject to any relations. For example, those defined by clockwise bubbles, the compositions of clockwise cups followed by clockwise caps. Thus the categories $\mathrm{Hom}_{\mathbf{H}_{\mathcal{V}}}(N, N')$ are not Hom-finite. Thus we cannot use the Euler pairing to obtain the corresponding numerical Grothendieck groups.

REMARK 6.17. For $\mathcal{V} = \text{dg-Vect}_{\Bbbk}^{\mathrm{f}}$ the Hom-spaces of $\mathbf{H}_{\mathcal{V}}$, while infinite-dimensional, are controlled by $\mathrm{End}(\mathbb{1})$ and the degenerate affine Hecke algebra [**31**, Proposition 4]. Some version of this observation is expected to hold more generally, see for example [**13**, Conjecture 2]. It is not however clear how to define the degenerate affine Hecke algebra in our generality. We intend to return to this question in future work. Instead, we use an ad-hoc definition of the numerical Grothendieck group given in Definition 6.18 below.

To kill the centre, we need to at least quotient each $\mathrm{K}_0\big(\mathrm{Hom}_{\mathbf{H}_{\mathcal{V}}}(N, N'), \Bbbk\big)$ by the classes $[\mathsf{P}_a]$ and $[\mathsf{Q}_a]$ for $[a]$ in the kernel of the Euler pairing on $\mathrm{K}_0(\mathcal{V})$, as well as by any direct summands of these coming from the symmetric group action on parallel strands.

To formulate this, recall the functors of Remark 6.1:

$$\Xi'^{\mathsf{P}}_{N,N+n} \colon \mathcal{S}^n\mathcal{V} \to \mathrm{Hom}_{\mathbf{H}_{\mathcal{V}}}(N, N+n).$$

Taking h-perfect hulls we obtain functors

$$\Xi^{\mathsf{P}}_{N,N+n} \colon \mathcal{H}perf(\mathcal{S}^n\mathcal{V}) \to \mathrm{Hom}_{\mathbf{H}_{\mathcal{V}}}(N, N+n),$$

and similarly contravariant functors $\Xi^{\mathsf{Q}}_{N,N+n}$. These further package up into 2-functors

$$\Xi^{\mathsf{P}}, \Xi^{\mathsf{Q}} \colon \mathbf{Hperf}(\mathbf{Sym}_{\mathcal{V}}) \to \mathbf{H}_{\mathcal{V}}.$$

As these are integral parts of the structure of $\mathbf{H}_{\mathcal{V}}$, we expect them to descend to the numerical Grothendieck groups. We thus make the following definition:

Definition 6.18. Let I be the two-sided ideal of $\mathrm{K}_0(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ generated by the images under Ξ^{P} and Ξ^{Q} of the kernels of the Euler pairings on $\mathrm{K}_0(\mathbf{Hperf}(\mathbf{Sym}_{\mathcal{V}}), \Bbbk)$. The 1-category $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ is the quotient of $\mathrm{K}_0(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ by I.

Remark 6.19. Recall the 1-morphisms P^{λ}_a and Q^{λ}_a defined in Section 6.3. The ideal I contains the classes $[\mathsf{P}^{\lambda}_a]$ and $[\mathsf{Q}^{\lambda}_a]$ for all $a \in \mathcal{V}$ with $[a]$ in the kernel of the Euler pairing and all Young diagrams λ.

If I is generated by these classes, then using the Giambelli identity, I is in this case equivalently generated by classes of the form $[\mathsf{P}^{(n)}_a]$ and $[\mathsf{Q}^{(n)}_a]$, see for example [**13**, Remark 6]. This is exactly the minimal ideal one needs to quotient out in order for the Heisenberg algebra to have no centre.

In general, however, there may exist images of additional homotopy idempotents in the kernel of the Euler pairing on $\mathrm{K}_0(\mathcal{S}^n\mathcal{V}, \Bbbk)$. In order to catch these and to obtain the expected natural morphisms $\mathrm{K}_0^{\mathrm{num}}(\mathbf{Sym}_{\mathcal{V}}, \Bbbk) \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ one needs to use the less intuitive definition of I given above.

At the outset, we completed $\mathcal{V}$ to $\mathcal{H}perf\,\mathcal{V}$, see the introduction to Chapter 5. We can therefore choose a basis of $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V})$ consisting of the classes of objects of $\mathcal{V}$. The elements $p^{(n)}_a$ and $q^{(n)}_a$ indexed by the objects a in this basis generate the Heisenberg algebra $H_{\mathcal{V}}$. Theorem 6.3 implies that there is a canonical morphism of $\Bbbk$-algebras

$$\pi \colon H_{\mathcal{V}} \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$$

sending the generators $p^{(n)}_a$ to the class of $\mathsf{P}^{(n)}_a$ and $q^{(n)}_a$ to the class of $\mathsf{Q}^{(n)}_a$.

Theorem 6.20. *The map* $\pi \colon H_{\mathcal{V}} \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ *is an injective map of* $\Bbbk$-*algebras.*

Proof. In Chapter 7 we construct a categorical analogue of the Fock space together with a 2-representation of $\mathbf{H}_{\mathcal{V}}$ on it. By Corollary 8.5, this 2-representation induces on the level of K-groups a homomorphism of algebras

$$H_{\mathcal{V}} \xrightarrow{\pi} \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk) \to \mathrm{End}\left(\bigoplus_{N\geq 0} \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)\right). \tag{6.9}$$

As $1 \in \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^0\mathcal{V}, \Bbbk) \cong \Bbbk$ is annihilated by $H^-_{\mathcal{V}} \setminus \{1_0\}$ and is fixed by 1_0, Lemma 2.9 produces an embedding

$$F_{\mathcal{V}} \to \bigoplus_{N\geq 0} \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk) \tag{6.10}$$

of the classical Fock space. Hence the representation (6.9) of $H_{\mathcal{V}}$ on

$$\bigoplus_{N\geq 0} \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)$$

is faithful. Therefore π is necessarily injective. □

Surjectivity of π is a considerably subtler question, due to the possible appearance of additional homotopy idempotents when taking the perfect hull $\mathbf{H}'_{\mathcal{V}}$. This is closely related to the question of whether $\mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)$ and

$$F_{\mathcal{V}}^N = \bigoplus_{k_1+2k_2+\cdots=N} \bigotimes_i \mathrm{Sym}^{k_i}(\mathrm{K}_0^{\mathrm{num}}(\mathcal{V}, \Bbbk)),$$

the degree N part of the Fock space are isomorphic. To the authors' knowledge, there exists no general criterion for this, cf. the remarks in Section 8.2.

Conjecture 6.21. *If the canonical morphism $F_{\mathcal{V}}^N \to \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V})$ is an isomorphism, then so is π.*

The main content of the conjecture is that on the level of Grothendieck groups the operation of taking perfect hulls only adds the classes $[\mathsf{P}_a^{(n)}]$ and $[\mathsf{Q}_a^{(n)}]$ as additional generators. On the homotopy categories, taking the perfect hull corresponds to taking the triangulated hull and Karoubi completion. Thus, alternatively, the statement is that the only relevant idempotents in the homotopy category are those arising from the action of the symmetric groups on upward or downward strands. We prove a converse to Conjecture 6.21 in Section 8.3.

We want to stress that a 2-representation of $\mathbf{H}_{\mathcal{V}}$ is completely determined by the images of P_a, Q_a, R_a, and the generating 2-morphisms. Thus the possible appearance of additional idempotents in $\mathbf{H}_{\mathcal{V}}$ (i.e., π being possibly non-surjective) does not complicate the construction of categorical Heisenberg actions.

Example 6.22. Taking $\mathcal{V} = \Bbbk$, the 1-morphisms in $\mathbf{H}_{\mathcal{V}}$ are homotopy direct summands of one-sided twisted complexes of direct sums of $\mathsf{P}_{\Bbbk}$ and $\mathsf{Q}_{\Bbbk}$. As $\mathrm{Hom}_{\mathcal{V}}(\Bbbk, \Bbbk) = \Bbbk$, such one-sided complexes are actual complexes and their morphisms are morphisms of complexes. Idempotents of such complexes must be idempotent in each degree. It follows that $\mathrm{K}_0(\mathbf{H}_{\mathcal{V}}, \Bbbk) = \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ coincides with the Grothendieck group of Khovanov's category [**31**]. By the main result of [**10**] this further coincides with the infinite Heisenberg algebra.

In general, 1-morphisms in $\mathbf{H}_{\mathcal{V}}$ may be one-sided twisted complexes with nontrivial higher differentials. One cannot then simply take idempotents in each degree. The conjecture says that the situation is however no worse than in $\mathcal{S}^n\mathcal{V}$.

6.5. Quantum enhancement

Several previous works on Heisenberg categorification, like [**13**], use a quantum deformation of the Heisenberg algebra. This *quantum Heisenberg algebra* $H_{\mathcal{V}}^t$ has coefficients taken from $\Bbbk[t, t^{-1}]$, where t is a formal variable.

For a graded vector space V define

$$[V] := \sum_{n\in\mathbb{Z}} \dim V_n t^n.$$

Using this, the unital algebra $\underline{H}_{\mathcal{V}}^t$ is defined by the same generators and relations as $\underline{H}_{\mathcal{V}}$ except that relation (2.5) is replaced by

$$q_a^{(n)} p_b^{(m)} = \sum_{k=0}^{\min(m,n)} [\mathrm{Sym}^k H^* \mathrm{Hom}(a,b)]\, p_b^{(m-k)} q_a^{(n-k)}.$$

Its idempotent modification $H_{\mathcal{V}}^q$ is then obtained exactly as in Section 2.2.1.

EXAMPLE 6.23. For $n \in \mathbb{N}$, let $[n]$ denote the quantum integer

$$[n] \coloneqq \frac{t^{-n} - t^{n}}{t^{-1} - t} = t^{-n+1} + t^{-n+3} + \cdots + t^{n-3} + t^{n-1}.$$

Note that when setting $t = 1$ in the last expression, one gets $[n] = n$. Define moreover $[n] := [-n]$ for $n \in \mathbb{Z}_{<0}$. Suppose that there is a set of generating objects of $\mathcal{V}$ such that Hom-spaces between these objects satisfy

$$[H^* \operatorname{Hom}(a, b)] = [\langle a, b \rangle_\chi].$$

If moreover the form χ is symmetric, then our definition coincides with [**45**, Definition 5.1] (see also [**13**, Equation (6)]). These conditions hold e.g. in Example 6.16.

Since $\mathbf{H}_\mathcal{V}$ is graded, $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_\mathcal{V}, \Bbbk)$ is an algebra over $\Bbbk[t, t^{-1}]$, where t acts via the shift. Similarly, $\mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)$ is naturally a $\Bbbk[t, t^{-1}]$-module, such that $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_\mathcal{V}, \Bbbk)$ acts $\Bbbk[t, t^{-1}]$-linearly on it. Hence, there is a $\Bbbk[t, t^{-1}]$-algebra homomorphism

$$\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_\mathcal{V}, \Bbbk) \to \operatorname{End}_{\Bbbk[t,t^{-1}]}\left(\bigoplus_{N\geq 0} \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)\right).$$

PROPOSITION 6.24. *The morphism π extends to an injective map of $\Bbbk[t, t^{-1}]$-algebras*

$$\pi : H_\mathcal{V}^t \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_\mathcal{V}, \Bbbk).$$

PROOF. We need only show that π is a map of $\Bbbk[t, t^{-1}]$-algebras, that is, π is compatible with the t-action on the source and the target. This is straigthforward from the definitions. $\square$

Letting $H_\mathcal{V}^{t-} \subset H_\mathcal{V}^t$ denote again the subalgebra generated by the set

$$\left\{q_a^{(n)} 1_k : a \in M,\, k \leq 0,\, n \geq 0\right\},$$

the quantum Fock space is obtained as the induced representation

$$F_\mathcal{V}^t = \operatorname{Ind}_{H_\mathcal{V}^{t-}}^{H_\mathcal{V}^t}(\mathrm{triv}_0) \cong H_\mathcal{V}^t \otimes_{H_\mathcal{V}^{t-}} \Bbbk[t, t^{-1}].$$

The embedding (6.10) is also compatible with the shift, so it can be enhanced to

$$F_\mathcal{V}^t \to \bigoplus_{N\geq 0} \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk).$$

CHAPTER 7

The Categorical Fock Space

As in the additive case, we construct a category called the categorical Fock space from the symmetric powers of the DG category $\mathcal{V}$. We show that the Heisenberg category $\mathbf{H}_{\mathcal{V}}$ acts on this categorical Fock space. The relation between this representation and the classical Fock space representation is explored in the next section.

7.1. Symmetric powers of DG categories

Recall from Definition 4.42 that the Nth symmetric power of $\mathcal{V}$ is defined as $\mathcal{S}^N\mathcal{V} = \mathcal{V}^{\otimes N} \rtimes S_N$.

EXAMPLE 7.1. If X is a scheme, then $\mathcal{S}^N\mathcal{I}(X) \cong \mathcal{I}(X^N)^{S_N}$ is Morita equivalent to the standard DG enhancement $\mathcal{I}([X^N/S_N])$ of the N-th symmetric quotient stack of X. We thus have $\mathrm{D_c}(\mathcal{S}^N\mathcal{I}(X)) \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}([X^N/S_N])$, the derived category of S_N-equivariant perfect complexes on X^N [**21**, Example 2.2.8(a)].

DEFINITION 7.2. For any $1 \leq k \leq N$ define the group monomorphism

$$\iota_k \colon S_{N-1} \hookrightarrow S_N,$$

by identifying S_{N-1} with the subgroup of S_N consisting of permutations which keep k fixed.

LEMMA 7.3. *The group S_N admits the following decomposition into S_{N-1}-cosets:*

$$S_N = \sum_{i=1}^{N} (1i)\iota_1(S_{N-1}) = \sum_{i=1}^{N} \iota_1(S_{N-1})(1i)$$

This observation can be used to rearrange the complete decomposition (4.8) of the diagonal bimodule of $\mathcal{S}^N\mathcal{V}$ as follows.

COROLLARY 7.4. *There is the following direct sum decompositions of the diagonal bimodule:*

$$\mathcal{S}^N\mathcal{V} \cong \bigoplus_{i=1}^{N} {}_i\mathcal{V}_1 \otimes {}_{\hat{1}\circ(1i)} \left(\mathcal{S}^{N-1}\mathcal{V}\right)_{\hat{1}} \cong \bigoplus_{i=1}^{N} {}_1\mathcal{V}_i \otimes {}_{\hat{1}} \left(\mathcal{S}^{N-1}\mathcal{V}\right)_{\hat{1}\circ(1i)}$$

where the left and right indices denote taking the left and right arguments of the bimodule $\mathcal{S}^N\mathcal{V}$ and applying the following:

- *for i in $\{1,\ldots,n\}$, the map $i \colon \mathrm{Ob}(\mathcal{S}^N\mathcal{V}) \to \mathrm{Ob}(\mathcal{V})$ projects to the i-th factor,*
- *for i in $\{1,\ldots,n\}$, the map $\hat{i} \colon \mathrm{Ob}(\mathcal{S}^N\mathcal{V}) \to \mathrm{Ob}(\mathcal{V}^{\otimes(N-1)})$ projects to all factors but i-th,*

- *for i, j in $\{1,\dots,n\}$, the map $(ij)\colon \operatorname{Ob}(\mathcal{S}^N\mathcal{V}) \to \operatorname{Ob}(\mathcal{S}^N\mathcal{V})$ transposes i-th and j-th factors.*

We illustrate this notation. Let $\underline{a} = a_1 \otimes \cdots \otimes a_N$, $\underline{b} = b_1 \otimes \cdots \otimes b_N \in \mathcal{S}^N\mathcal{V}$. Then

$$ {}_{\underline{b}}\left(\mathcal{S}^N\mathcal{V}\right)_{\underline{a}} = \operatorname{Hom}_{\mathcal{S}^N\mathcal{V}}\left(a_1 \otimes \cdots \otimes a_N,\, b_1 \otimes \cdots \otimes b_N\right). $$

Our notation gives

$$ {}_{\underline{b}}\left({}_i\mathcal{V}_1\right)_{\underline{a}} = \operatorname{Hom}_{\mathcal{V}}(a_1, b_i), $$

and

$$ {}_{\underline{b}}\left({}_{\hat{1}\circ(1i)}\left(\mathcal{S}^{N-1}\mathcal{V}\right)_{\hat{1}}\right)_{\underline{a}} = \\ \operatorname{Hom}_{\mathcal{S}^{N-1}\mathcal{V}}\left(a_2 \otimes \cdots \otimes a_N,\, b_2 \otimes \cdots \otimes b_{i-1} \otimes b_1 \otimes b_{i+1} \otimes \cdots \otimes b_N\right). $$

It is clear that there is natural inclusion of DG $\Bbbk$-modules

$$ \begin{array}{c} \operatorname{Hom}_{\mathcal{V}}(a_1, b_i) \otimes \operatorname{Hom}_{\mathcal{S}^{N-1}\mathcal{V}}\left(a_2 \otimes \cdots \otimes a_N,\, b_2 \otimes \cdots \otimes b_{i-1} \otimes b_1 \otimes b_{i+1} \otimes \cdots \otimes b_N\right) \\ \big\downarrow \\ \operatorname{Hom}_{\mathcal{S}^N\mathcal{V}}\left(a_1 \otimes \cdots \otimes a_N,\, b_1 \otimes \cdots \otimes b_N\right), \end{array} $$

and the proof below demonstrates that summing this over all $i \in \{1,\dots,N\}$ gives a complete decomposition of the diagonal bimodule.

Let us stress that the index maps i, $\hat{i}$ and (ij) are maps of sets and are not functorial. Thus the expressions like ${}_1\mathcal{V}_1$ in Corollary 7.4 are not $\mathcal{S}^N\mathcal{V}$-bimodules by themselves: while

$$ {}_{\underline{b}}({}_i\mathcal{V}_1)_{\underline{a}} = \operatorname{Hom}_{\mathcal{V}}(a_1, b_i) $$

is perfectly well-defined, one cannot uniquely pick out the first factor in some

$$ \alpha \in \operatorname{Hom}_{\mathcal{S}^N\mathcal{V}}(\underline{b},\, \underline{b}') $$

to act with it on $\operatorname{Hom}_{\mathcal{V}}(a_1, b_i)$. Nonetheless, if we use Lemma 7.3 to decompose α with respect to the permutation type into $\sum \alpha_i$, then each α_i does act naturally on the summand ${}_i\mathcal{V}_1 \otimes {}_{\hat{1}\circ(1i)}\left(\mathcal{S}^{N-1}\mathcal{V}\right)_{\hat{1}}$. Thus we can view Corollary 7.4 as an isomorphism of $\mathcal{S}^N\mathcal{V}$-bimodules, with the index maps indicating the left and right actions of $\mathcal{S}^N\mathcal{V}$ on the decompositions.

Proof of Corollary 7.4. First, by the decomposition (4.8) we have:

$$ \mathcal{S}^N\mathcal{V} \cong \bigoplus_{\sigma \in S_N} \left(\mathcal{V}^{\otimes N}\right)_{\sigma}. $$

We then use the decomposition $S_N = \sum_{i=1}^N (1i)\iota_1(S_{N-1})$ from Lemma 7.3 to obtain

$$ \bigoplus_{\sigma \in S_N} \left(\mathcal{V}^{\otimes N}\right)_{\sigma} \cong \bigoplus_{i=1}^N \bigoplus_{\sigma \in S_{N-1}} \left(\mathcal{V}^{\otimes N}\right)_{(1i)\iota_1(\sigma)}. $$

The $\mathcal{V}^{\otimes N}$-bimodule isomorphism $\left(\mathcal{V}^{\otimes N}\right)_{(1i)\iota_1(\sigma)} \cong {}_{(1i)}\left(\mathcal{V}^{\otimes N}\right)_{\iota_1(\sigma)}$ given by $\alpha \mapsto (1i)\cdot\alpha$ implies that

$$ \bigoplus_{i=1}^N \bigoplus_{\sigma \in S_{N-1}} \left(\mathcal{V}^{\otimes N}\right)_{(1i)\iota_1(\sigma)} \cong \bigoplus_{i=1}^N \bigoplus_{\sigma \in S_{N-1}} {}_{(1i)}\left(\mathcal{V}^{\otimes N}\right)_{\iota_1(\sigma)}. $$

Now we can decompose $\mathcal{V}^{\otimes N}$ into ${}_1\mathcal{V}_1 \otimes {}_{\hat{1}} \left(\mathcal{V}^{\otimes(N-1)}\right)_{\hat{1}}$, which further gives us

$$\bigoplus_{i=1}^{N} \bigoplus_{\sigma\in S_{N-1}} {}_{(1i)} \left(\mathcal{V}^{\otimes N}\right)_{\iota_1(\sigma)} = \bigoplus_{i=1}^{N} \bigoplus_{\sigma\in S_{N-1}} {}_i\mathcal{V}_1 \otimes {}_{\hat{1}\circ(1i)} \left(\mathcal{V}^{\otimes(N-1)}\right)_{\sigma\circ\hat{1}} .$$

Finally, by (4.8) we have $\mathcal{S}^{N-1}\mathcal{V} = \bigoplus_{\sigma\in S_{N-1}} \left(\mathcal{V}^{\otimes(N-1)}\right)_{\sigma}$ and therefore

$$\bigoplus_{i=1}^{N} \bigoplus_{\sigma\in S_{N-1}} {}_i\mathcal{V}_1 \otimes {}_{\hat{1}\circ(1i)} \left(\mathcal{V}^{\otimes(N-1)}\right)_{\sigma\circ\hat{1}} \cong \bigoplus_{i=1}^{N} {}_i\mathcal{V}_1 \otimes {}_{\hat{1}\circ(1i)} \left(\mathcal{S}^{N-1}\mathcal{V}\right)_{\hat{1}} .$$

This establishes the first decomposition. The second decomposition is proved similarly. □

Recall from Section 4.8 that $\mathcal{S}^N\mathcal{V}$ and $\mathcal{V}^{\otimes N}$ have the same objects, while the morphisms of $\mathcal{S}^N\mathcal{V}$ are generated under composition by those of $\mathcal{V}^{\otimes N}$ plus the formal isomorphisms corresponding to the elements of S_N. Thus the data of a DG functor from $\mathcal{S}^N\mathcal{V}$ to some DG category $\mathcal{B}$ is the data of a functor $\mathcal{V}^{\otimes N} \to \mathcal{B}$ plus the data of where the formal isomorphisms go.

Definition 7.5. Let $a \in \mathcal{V}$. Define the functor

$$\phi_a \colon \mathcal{S}^{N-1}\mathcal{V} \to \mathcal{S}^N\mathcal{V}$$

to be the extension of the functor

$$\mathcal{V}^{\otimes(N-1)} \xrightarrow{a\otimes \mathrm{id}} \mathcal{V}^{\otimes N}$$

which sends the formal isomorphisms of S_{N-1} to those of S_N via

$$\iota_1 \colon S_{N-1} \hookrightarrow S_N,$$

the embedding as the subgroup of permutations which are trivial on the first element.

As explained in Section 4.2.3, we have three induced functors

$$\phi_a^* \colon \mathcal{M}od\text{-}\mathcal{S}^{N-1}\mathcal{V} \to \mathcal{M}od\text{-}\mathcal{S}^N\mathcal{V},$$
$$\phi_{a*} \colon \mathcal{M}od\text{-}\mathcal{S}^N\mathcal{V} \to \mathcal{M}od\text{-}\mathcal{S}^{N-1}\mathcal{V},$$
$$\phi_a^! \colon \mathcal{M}od\text{-}\mathcal{S}^{N-1}\mathcal{V} \to \mathcal{M}od\text{-}\mathcal{S}^N\mathcal{V},$$

which form two adjoint pairs (ϕ_a^*, ϕ_{a*}) and $(\phi_{a*}, \phi_a^!)$. The action of the first two functors on representable objects can be described as follows.

Lemma 7.6. *Let h^r denote right representable modules, as per Section 4.2.1. Then:*

(1) *For any $a_1 \otimes \cdots \otimes a_{N-1} \in \mathcal{S}^{N-1}\mathcal{V}$ we have*

$$\phi_a^*\big(h^r(a_1 \otimes \cdots \otimes a_{N-1})\big) \cong h^r(a \otimes a_1 \otimes \cdots \otimes a_{N-1}).$$

(2) *For any $a_1 \otimes \cdots \otimes a_N \in \mathcal{S}^N\mathcal{V}$ we have*

$$\phi_{a*}\big(h^r(a_1 \otimes \cdots \otimes a_N)\big) \cong \bigoplus_{i=1}^{N} \mathrm{Hom}_{\mathcal{V}}(a, a_i) \otimes h^r(a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N)$$

PROOF. For Part (1), we have:

$$\phi_a^*\big(h^r(a_1\otimes\cdots\otimes a_{N-1})\big) := h^r(a_1\otimes\cdots\otimes a_{N-1})\otimes_{\mathcal{S}^{N-1}\mathcal{V}}\ {}_{\phi_a}\mathcal{S}^N\mathcal{V} \cong h^r(a\otimes a_1\otimes\cdots\otimes a_{N-1}).$$

For Part (2), we have

$$\phi_{a*}\big(h^r(a_1\otimes\cdots\otimes a_N)\big) := h^r(a_1\otimes\cdots\otimes a_N)\otimes_{\mathcal{S}^N\mathcal{V}}\mathcal{S}^N\mathcal{V}_{\phi_a}$$

By Corollary 7.4 we have

$$\mathcal{S}^N\mathcal{V}_{\phi_a}\cong\bigoplus_{i=1}^N {}_i\mathcal{V}_a\otimes{}_{\hat{1}\circ(1i)}\mathcal{S}^{N-1}\mathcal{V}$$

and hence

$$h^r(a_1\otimes\cdots\otimes a_N)\otimes_{\mathcal{S}^N\mathcal{V}}\mathcal{S}^N\mathcal{V}_{\phi_a} \cong\bigoplus_{i=1}^N {}_{a_i}\mathcal{V}_a\otimes{}_{a_2\otimes\cdots\otimes a_{i-1}\otimes a_1\otimes a_{i+1}\otimes\cdots\otimes a_N}(\mathcal{V}^{\otimes(N-1)}\rtimes S_{N-1}).$$

Since in $\mathcal{S}^{N-1}\mathcal{V}$ we have

$$a_2\otimes\cdots\otimes a_{i-1}\otimes a_1\otimes a_{i+1}\otimes\dots a_N\cong a_1\otimes\cdots\widehat{a_i}\cdots\otimes a_N,$$

we have

$$\phi_{a*}\big(h^r(a_1\otimes\cdots\otimes a_N)\big)\cong\bigoplus_{i=1}^N\operatorname{Hom}_{\mathcal{V}}(a,a_i)\otimes h^r(a_1\otimes\cdots\widehat{a_i}\cdots\otimes a_N). \qquad\square$$

Lemma 7.6 (1) shows that the bimodule ${}_{\phi_a}\mathcal{S}^N\mathcal{V}$ defining ϕ_a^* is always right-representable. Thus it is always right-perfect and right-h-projective. On the other hand, by Lemma 7.6 (2) the bimodule $\mathcal{S}^N\mathcal{V}_{\phi_a}$ defining ϕ_{a*} is always right-h-flat, but is right-perfect and right h-projective if and only if $\mathcal{V}$ is proper. Similarly, $\mathcal{S}^N\mathcal{V}_{\phi_a}$ is always left representable, while ${}_{\phi_a}\mathcal{S}^N\mathcal{V}$ is always left-h-flat, but is left-perfect and left-h-projective if and only if $\mathcal{V}$ is proper. We conclude that when $\mathcal{V}$ is proper both ${}_{\phi_a}\mathcal{S}^N\mathcal{V}$ and $\mathcal{S}^N\mathcal{V}_{\phi_a}$ are left- and right-perfect and left- and right-h-projective. In particular, they define 1-morphisms in $\mathbf{EnhCat}_{\mathrm{kc}}^{\mathrm{dg}}$ and, by abuse of notation, we denote these again by ϕ_a^* and ϕ_{a*}, respectively.

The twisted inverse image functor $\phi_a^!$ is not a priori a functor of tensoring with a bimodule. However, in presence of a homotopy Serre functor, it is quasi-isomorphic to one:

PROPOSITION 7.7. *Let $\mathcal{V}$ be proper and assume that $\mathcal{V}$ admits a homotopy Serre functor S. Then there is a quasi-isomorphism of DG functors*

$$\star_a\colon\phi_{Sa}^*\to\phi_a^!.$$

PROOF. Let $E\in\mathcal{M}od\text{-}\mathcal{S}^{N-1}\mathcal{V}$. By Corollary 7.4 we have

$$\phi_{Sa}^*E = E\otimes_{\mathcal{S}^{N-1}\mathcal{V}}\ {}_{\phi_{Sa}}\mathcal{S}^N\mathcal{V}\cong E\otimes_{\mathcal{S}^{N-1}\mathcal{V}}\left(\bigoplus_{i=1}^N {}_{Sa}\mathcal{V}_i\otimes_{\Bbbk}\mathcal{S}^{N-1}\mathcal{V}_{\hat{1}\circ(1i)}\right).$$

The homotopy Serre functor S on $\mathcal{V}$ comes with a quasi-isomorphism

$$\eta\colon\mathcal{V}\to({}_S\mathcal{V})^*.$$

Since $\mathcal{V}$ is proper, $\eta^*\colon {}_S\mathcal{V} \to \mathcal{V}^*$ is also a quasi-isomorphism. Hence so is

$$\begin{aligned} &E \otimes_{\mathcal{S}^{N-1}\mathcal{V}} \left(\bigoplus_{i=1}^{N} {}_{Sa}\mathcal{V}_i \otimes_{\Bbbk} \mathcal{S}^{N-1}\mathcal{V}_{\hat{1}\circ(1i)} \right) \\ &\qquad \to E \otimes_{\mathcal{S}^{N-1}\mathcal{V}} \left(\bigoplus_{i=1}^{N} ({}_i\mathcal{V}_a)^* \otimes_{\Bbbk} \mathcal{S}^{N-1}\mathcal{V}_{\hat{1}\circ(1i)} \right). \end{aligned} \tag{7.1}$$

Since ${}_{\hat{1}\circ(1i)}\mathcal{S}^{N-1}\mathcal{V}$ are representables, we have

$$\mathrm{Hom}_{\mathcal{S}^{N-1}\mathcal{V}}({}_{\hat{1}\circ(1i)}\mathcal{S}^{N-1}\mathcal{V},\, E) \cong E_{\hat{1}\circ(1i)} \cong E \otimes_{\mathcal{S}^{N-1}\mathcal{V}} \mathcal{S}^{N-1}\mathcal{V}_{\hat{1}\circ(1i)}$$

and therefore

$$\begin{aligned} &E \otimes_{\mathcal{S}^{N-1}\mathcal{V}} \left(\bigoplus_{i=1}^{N} ({}_i\mathcal{V}_a)^* \otimes \mathcal{S}^{N-1}\mathcal{V}_{\hat{1}\circ(1i)} \right) \\ &\qquad \cong \bigoplus_{i=1}^{N} ({}_i\mathcal{V}_a)^* \otimes \mathrm{Hom}_{\mathcal{S}^{N-1}\mathcal{V}} \left({}_{\hat{1}\circ(1i)}\mathcal{S}^{N-1}\mathcal{V},\, E \right). \end{aligned}$$

Since $\mathcal{V}$ is proper, ${}_i\mathcal{V}_a$ are perfect as $\Bbbk$-modules. Thus the natural map

$$\begin{aligned} &\bigoplus_{i=1}^{N} ({}_i\mathcal{V}_a)^* \otimes \mathrm{Hom}_{\mathcal{S}^{N-1}\mathcal{V}} \left({}_{\hat{1}\circ(1i)}\mathcal{S}^{N-1}\mathcal{V},\, E \right) \\ &\qquad \longrightarrow \bigoplus_{i=1}^{N} \mathrm{Hom}_{\mathcal{S}^{N-1}\mathcal{V}} \left({}_i\mathcal{V}_a \otimes {}_{\hat{1}\circ(1i)}\mathcal{S}^{N-1}\mathcal{V},\, E \right), \end{aligned}$$

is a quasi-isomorphism. Finally, by Corollary 7.4 again, we have

$$\bigoplus_{i=1}^{N} \mathrm{Hom}_{\mathcal{S}^{N-1}\mathcal{V}} \left({}_i\mathcal{V}_a \otimes {}_{\hat{1}\circ(1i)}\mathcal{S}^{N-1}\mathcal{V},\, E \right) \cong \mathrm{Hom}_{\mathcal{S}^{N-1}\mathcal{V}} \left(\mathcal{S}^N\mathcal{V}_{\phi_a},\, E \right) = \phi_a^! E.$$

□

COROLLARY 7.8. *Let $\mathcal{V}$ be proper and assume it admits a homotopy Serre functor S. The bimodule approximation $\underline{\mathrm{Apx}}(\phi_a^!)$ is a right- and left-perfect and left-h-projective $\mathcal{S}^{N-1}\mathcal{V}$-$\mathcal{S}^N\mathcal{V}$-bimodule.*

PROOF. By the definition of the bimodule approximation functor in Section 4.3, for any $b \in \mathcal{S}^{N-1}\mathcal{V}$, the fibre ${}_b\underline{\mathrm{Apx}}(\phi_a^!)$ is the $\mathcal{S}^N\mathcal{V}$-module $\phi_a^!(h^r(b))$. By Proposition 7.7, $\phi_a^!(h^r(b))$ is quasi-isomorphic to $\phi_{Sa}^*(h^r(b))$. Since the latter is the representable object $h^r(\phi_a(b))$, we conclude that the former is perfect.

Now let $c \in \mathcal{S}^N\mathcal{V}$. We have

$$\begin{aligned} \underline{\mathrm{Apx}}(\phi_a^!)_c = \phi_a^!(\mathcal{S}^{N-1}\mathcal{V})_c &\cong \mathrm{Hom}_{\mathcal{S}^{N-1}\mathcal{V}} \left({}_c(\mathcal{S}^N\mathcal{V})_{\phi_a},\, \mathcal{S}^{N-1}\mathcal{V} \right) \\ &= \mathrm{Hom}_{\mathcal{S}^{N-1}\mathcal{V}} \left(\phi_{a*}(c),\, \mathcal{S}^{N-1}\mathcal{V} \right). \end{aligned}$$

It is well known that the dualisation functor sends h-projective and perfect modules to h-projective and perfect modules [**1**, Section 2.2]. By Lemma 7.6 (2) and properness of $\mathcal{V}$, the $\mathcal{S}^{N-1}\mathcal{V}$-module $\phi_{a*}(c)$ is h-projective and perfect, hence so is its dual $\mathrm{Hom}_{\mathcal{S}^{N-1}\mathcal{V}} \left(\phi_{a*}(c),\, \mathcal{S}^{N-1}\mathcal{V} \right)$. □

7.2. The categorical Fock space $\mathbf{F}_{\mathcal{V}}$

In Chapter 5 we fixed a smooth and proper enhanced triangulated category $\mathcal{V} \in \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$. That is, $\mathcal{V}$ is a smooth and proper DG category considered as a Morita enhancement of the triangulated category $\mathrm{D_c}(\mathcal{V}) = \mathrm{H}^0(\mathcal{H}perf\, \mathcal{V})$. As $\mathcal{V}$ is smooth and proper, it admits an enhanced Serre functor S given by the bimodule $\mathcal{V}^*$ [**43**], and in Section 4.7 we proved that it lifts to a homotopy Serre functor S on $\mathcal{H}perf\, \mathcal{V}$. Replacing $\mathcal{V}$ by $\mathcal{H}perf\, \mathcal{V}$ if necessary, we can assume that $\mathcal{V}$ itself admits a homotopy Serre functor S.

We then defined the Heisenberg 2-category $\mathbf{H}_{\mathcal{V}}$ of $\mathcal{V}$. It was constructed in two steps:

(1) First, we defined in Sections 5.1 and 5.2 a strict DG 2-category $\mathbf{H}'_{\mathcal{V}}$. Its object set is $\mathbb{Z}$, its 1-morphisms are freely generated by formal symbols $\mathsf{P}_a, \mathsf{R}_a \colon N \to N+1$ and $\mathsf{Q}_a \colon N \to N-1$ for $a \in \mathcal{V}$, and its 2-morphisms are certain string diagrams connecting up the endpoints which correspond to Ps, Qs, and Rs of the source and target 1-morphisms.
(2) Next, in Section 5.4 we took the perfect hull (see Section 4.5) of $\mathbf{H}'_{\mathcal{V}}$ and then a monoidal Drinfeld quotient (see Section 4.6)of $\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})$ by a certain 2-sided ideal $I_{\mathcal{V}}$ of 1-morphisms. This was to make each R_a homotopy equivalent to P_{Sa} and impose a certain homotopy relation on Ps and Qs. The resulting $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory is the Heisenberg 2-category $\mathbf{H}_{\mathcal{V}}$.

Our next aim is to construct a 2-representation $\mathbf{F}_{\mathcal{V}}$ of $\mathbf{H}_{\mathcal{V}}$ analogous to the Fock space representation of a Heisenberg algebra.

LEMMA 7.9. *The Yoneda embedding of* $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ *into* $\mathbf{Hperf}(\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}})$ *is a quasi-equivalence. In particular, both of these are DG enhancements of the strict* 2*-category* $\mathbf{EnhCat}_{\mathrm{kc}}$ *of enhanced triangulated categories.*

PROOF. The procedure of taking the perfect hull does not change the Morita equivalence class of a DG category and, if the DG category is pre-triangulated and its homotopy category is Karoubi-complete, it does not change the homotopy category either. The 1-morphism categories $\mathrm{Hom}_{\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}}(\mathcal{A}, \mathcal{B})$ of $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ are defined so that their homotopy categories are $D_{\mathcal{B}\text{-}\mathcal{P}erf}(\mathcal{A}\text{-}\mathcal{B})$ In particular, they are triangulated and Karoubi-complete. We conclude that taking the perfect hull of $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ does not change its homotopy 2-category. □

We therefore make the following definition.

DEFINITION 7.10.

(1) The strict DG 2-category $\mathbf{F}'_{\mathcal{V}}$ is the 1-full subcategory of $\mathbf{dgModCat}$ (see Section 4.3) whose objects are symmetric powers $\mathcal{S}^N\mathcal{V}$ with $N \in \mathbb{Z}$. By convention, $\mathcal{S}^N\mathcal{V}$ is the zero category if $N < 0$ and is the unit object $\Bbbk$ of $\mathbf{dgModCat}$ if $N = 0$.
(2) The *categorical Fock space* $\mathbf{F}_{\mathcal{V}}$ of $\mathcal{V}$ is the $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory which is the perfect hull of the 1-full subcategory of $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ whose objects are the symmetric powers $\mathcal{S}^N\mathcal{V}$ with $N \in \mathbb{Z}$.

7.3. The representation $\Phi'_{\mathcal{V}}$: the generators

In this and the next section we carry out the first step of the construction outlined in Section 7.2 and define a strict DG 2-functor

$$\Phi'_{\mathcal{V}}\colon \mathbf{H}'_{\mathcal{V}} \to \mathbf{F}'_{\mathcal{V}}.$$

Objects: For any object $N \in \mathbb{Z}$ of $\mathbf{H}'_{\mathcal{V}}$ we define

$$\Phi'_{\mathcal{V}}(N) = \mathcal{S}^N\mathcal{V}.$$

1-*morphisms:* The 1-morphisms of $\mathbf{H}'_{\mathcal{V}}$ are freely generated by $\mathsf{P}_a, \mathsf{R}_a\colon N \to N+1$ and $\mathsf{Q}_a\colon N \to N-1$ for all $a \in \mathcal{V}$ and $N \in \mathbb{Z}$. We define $\Phi'_{\mathcal{V}}$ on morphisms by setting

$$\begin{aligned}\Phi'_{\mathcal{V}}(\mathsf{P}_a) &= \phi_a^*,\\ \Phi'_{\mathcal{V}}(\mathsf{Q}_a) &= \phi_{a*},\\ \Phi'_{\mathcal{V}}(\mathsf{R}_a) &= \phi_a^!\end{aligned}$$

where ϕ_a^*, ϕ_{a*}, and $\phi_a^!$ are the DG functors we constructed in Section 7.1 for any $a \in \mathcal{V}$ and $N \in \mathbb{Z}$.

For clarity, we write P_a (respectively, Q_a, R_a) for ϕ_a^*, (respectively, ϕ_{a*} , $\phi_a^!$), when considered as the image of P_a (respectively, Q_a, R_a) under $\Phi'_{\mathcal{V}}$.

EXAMPLE 7.11. Let X be a smooth projective variety and $\mathcal{V} = \mathcal{I}(X)$ be the standard enhancement of $D^b_{\mathrm{coh}}(X)$. As per Example 7.1, the symmetric powers $\mathcal{S}^N\mathcal{V}$ of $\mathcal{V}$ are Morita enhancements of the derived categories $\mathcal{I}([X^N/S_N])$ of the symmetric quotient stacks of X. Functors P_a and Q_a are the DG enhancements of functors $P_a^{(1)}$ and $Q_a^{(1)}$ defined by Krug in [**33**, Section 2.4], while $Q_{S^{-1}a}$ correspond to the left adjoints considered in [**33**, Section 3.2]. The higher powers $P_a^{(n)}$ and $Q_a^{(n)}$ will arise automatically from our calculus, cf. Example 8.3.

EXAMPLE 7.12. Let $a, b \in \mathcal{V}$ and let $a_1 \otimes \dots \otimes a_N \in \mathcal{S}^N\mathcal{V}$. Let $h^r(a_1 \otimes \dots \otimes a_N)$ be the corresponding representable module in $\mathcal{H}perf(\mathcal{S}^N\mathcal{V})$.

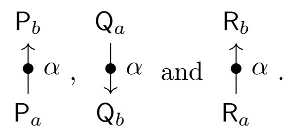
(1) We have

$$\begin{aligned}Q_b P_a h^r(a_1 \otimes \dots \otimes a_N) &\cong \phi_{b*}\phi_a^* h^r(a_1 \otimes \dots \otimes a_N)\\ &\cong \phi_{b*} h^r(a \otimes a_1 \otimes \dots \otimes a_N)\\ &\cong \operatorname{Hom}_{\mathcal{V}}(b,a) \otimes h^r(a_1 \otimes \dots \otimes a_N) \oplus\\ &\quad \oplus \left(\bigoplus_{i=1}^N \operatorname{Hom}_{\mathcal{V}}(b, a_i) \otimes h^r(a \otimes a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N)\right).\end{aligned}$$

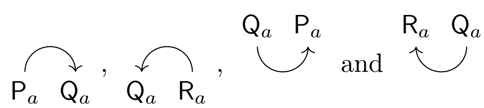
(2) On the other hand,

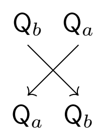
$$\begin{aligned}P_aQ_bh^r(a_1\otimes\cdots\otimes a_N) &\cong \phi_a^*\phi_{b*}h^r(a_1\otimes\cdots\otimes a_N)\\ &\cong \phi_a^*\left(\bigoplus_{i=1}^N \mathrm{Hom}_{\mathcal{V}}(b,a_i)\otimes h^r(a_1\otimes\cdots\widehat{a_i}\cdots\otimes a_N)\right)\\ &\cong \bigoplus_{i=1}^N \mathrm{Hom}_{\mathcal{V}}(b,a_i)\otimes \phi_a^*h^r(a_1\otimes\cdots\widehat{a_i}\cdots\otimes a_N)\\ &\cong \bigoplus_{i=1}^N \mathrm{Hom}_{\mathcal{V}}(b,a_i)\otimes h^r(a\otimes a_1\otimes\cdots\widehat{a_i}\cdots\otimes a_N).\end{aligned}$$

2-*morphisms:* The 2-morphisms of $\mathbf{H}'_{\mathcal{V}}$ are generated, subject to relations, by four sets of generating 2-morphisms, cf. Section 5.1:

(1) The marked arrows 

(2) The Serre relation 

(3) The cups and caps 

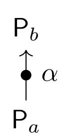
(4) The crossing 

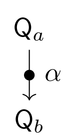
We define $\Phi'_{\mathcal{V}}$ on these generating 2-morphisms as follows:

(1) Given $\alpha\in\mathrm{Hom}_{\mathcal{V}}(a,b)$, we have a natural transformation of functors $\mathcal{S}^{N-1}\mathcal{V}\to\mathcal{S}^N\mathcal{V}$:

$$\alpha\otimes\mathrm{id}\colon\quad \phi_a=a\otimes\mathrm{id}\longrightarrow\phi_b=b\otimes\mathrm{id}\,.$$

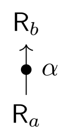
We set

$\Phi'_{\mathcal{V}}$ (

) $=(\alpha\otimes\mathrm{id})^*$, $\Phi'_{\mathcal{V}}$ (

) $=(\alpha\otimes\mathrm{id})_*$

and

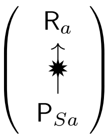
$\Phi'_{\mathcal{V}}$ (

) $=(\alpha\otimes\mathrm{id})^!$.

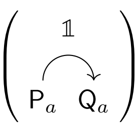
We denote these natural transformations by P_α, Q_α and R_α respectively.

(2) With $\star_a \colon \phi^*_{Sa} \to \phi^!_a$ as in Proposition 7.7, we set

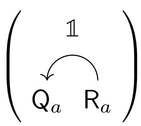
$$\Phi'_{\mathcal{V}}\left(\text{diagram}\right) = \star_a,$$



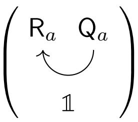
(3) As seen in Section 4.2.3, we have adjunctions $(\phi^*_a \dashv \phi_{a*})$ and $(\phi_{a*} \dashv \phi^!_a)$. We set:



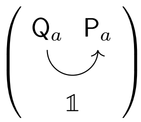
$$\Phi'_{\mathcal{V}}(\ldots) = \left[\phi^*_a \phi_{a*} \xrightarrow{\text{counit}} \text{id}\right],$$



$$\Phi'_{\mathcal{V}}(\ldots) = \left[\phi_{a*} \phi^!_a \xrightarrow{\text{counit}} \text{id}\right],$$



$$\Phi'_{\mathcal{V}}(\ldots) = \left[\text{id} \xrightarrow{\text{unit}} \phi^!_a \phi_{a*}\right],$$

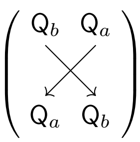


$$\Phi'_{\mathcal{V}}(\ldots) = \left[\text{id} \xrightarrow{\text{unit}} \phi_{a*} \phi^*_a\right].$$

(4) We have an isomorphism of functors $\mathcal{S}^{N-2}\mathcal{V} \to \mathcal{S}^N\mathcal{V}$

$$(12) : \phi_a \circ \phi_b \cong \phi_b \circ \phi_a$$

given objectwise by the transposition $(12) \in S_n$. We set



$$\Phi'_{\mathcal{V}}(\ldots) = (12)_*.$$

REMARK 7.13. The differentials on natural transformations in $\mathbf{F}'_{\mathcal{V}}$ match those in $\mathbf{H}'_{\mathcal{V}}$. For the dots this follows from $d(\alpha \otimes \text{id}) = d(\alpha) \otimes \text{id}$, while all the other defining transformations (the Serre map η, adjunctions and the transposition) are closed.

EXAMPLE 7.14. In the notation of Example 7.12, the adjunction unit $\text{id} \to Q_a P_a$ is given on representables by embedding $h^r(a_1 \otimes \cdots \otimes a_N)$ as $\text{id}_a \otimes h^r(a_1 \otimes \cdots \otimes a_N)$ into the first summand. The adjunction counit $P_a Q_a \to \text{id}$ is induced by the evaluation maps $\text{Hom}_{\mathcal{V}}(a, a_i) \otimes a \to a_i$, followed by the transposition $(1i)$ and the universal morphism out of the direct sum.

EXAMPLE 7.15. Using the decomposition of Corollary 7.4, we have for any $\mathcal{S}^N\mathcal{V}$-module E

$$\begin{aligned} Q_b R_a(E) = \phi_{b*}\phi^!_a(E) &\cong \text{Hom}_{\mathcal{S}^N\mathcal{V}}\big({}_{\phi_b}(\mathcal{S}^{N+1}\mathcal{V})_{\phi_a}, E\big) \\ &\cong \text{Hom}_{\mathcal{S}^N\mathcal{V}}({}_b\mathcal{V}_a \otimes \mathcal{S}^N\mathcal{V}, E) \oplus \bigoplus_{i=1}^{N} \text{Hom}_{\mathcal{S}^N\mathcal{V}}({}_i\mathcal{V}_a \otimes {}_{\hat{1}\circ(1i)}\mathcal{S}^N\mathcal{V}, E). \end{aligned}$$

The adjunction counit $Q_a R_a \to \text{id}$ is given by projecting onto

$$\text{Hom}_{\mathcal{S}^N\mathcal{V}}({}_a\mathcal{V}_a \otimes \mathcal{S}^N\mathcal{V}, E)$$

followed by the morphism induced by the map $\Bbbk \to {}_a\mathcal{V}_a$ sending $1 \mapsto \text{id}_a$:

$$\text{Hom}_{\mathcal{S}^N\mathcal{V}}({}_a\mathcal{V}_a \otimes \mathcal{S}^N\mathcal{V}, E) \to \text{Hom}_{\mathcal{S}^N\mathcal{V}}(\mathcal{S}^N\mathcal{V}, E) \cong E.$$

To see this, note that by the description of adjunction units and counits for Tensor-Hom adjunction [**2**, Section 2.1] our adjunction counit comes from the natural evaluation map

$$\mathrm{Hom}_{\mathcal{S}^N\mathcal{V}}(\mathcal{S}^{N+1}\mathcal{V}_{\phi_a}, E) \otimes_{\mathcal{S}^{N+1}\mathcal{V}} \mathcal{S}^{N+1}\mathcal{V}_{\phi_a} \to E, \quad \sum f \otimes g \mapsto \sum f(g)$$

via the identification of the left-hand side with $\mathrm{Hom}_{\mathcal{S}^N\mathcal{V}}({}_{\phi_a}\mathcal{S}^{N+1}\mathcal{V}_{\phi_a}, E)$ via the isomorphism $f \mapsto f \otimes 1$. Thus our counit is the map

$$\mathrm{Hom}_{\mathcal{S}^N\mathcal{V}}({}_{\phi_a}\mathcal{S}^{N+1}\mathcal{V}_{\phi_a}, E) \to E$$

given by $f \mapsto f(1)$. Since $1 \in {}_{\phi_a}\mathcal{S}^{N+1}\mathcal{V}_{\phi_a}$ lies in the component

$$\mathrm{Hom}(a, a) \otimes \mathcal{S}^N\mathcal{V},$$

we can project to that. Then evaluating at $1_a \otimes 1_{\mathcal{S}^N\mathcal{V}}$ is first mapping $\mathrm{Hom}_{\mathcal{S}^N\mathcal{V}}(\mathrm{Hom}(a,a)\otimes\mathcal{S}^N\mathcal{V}, E)$ to $\mathrm{Hom}_{\mathcal{S}^N\mathcal{V}}(\mathcal{S}^N\mathcal{V}, E)$ and then identifying this with E. This gives the claim.

Example 7.16. Let $\Gamma \leq \mathrm{SL}(2,\mathbb{C})$ be finite and $\mathcal{V}$ as in Examples 5.10 and 6.16. Let A_1^Γ denote $\mathbb{C}[x,y] \rtimes \Gamma$, the skew group algebra. Its abelian category of modules Mod-A_1^Γ is equivalent to $\mathrm{Coh}([\mathbb{C}^2/\Gamma])$, the abelian category of coherent sheaves on the quotient stack. We can therefore view the algebra A_1^Γ as a Morita DG enhancement of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}([\mathbb{C}^2/\Gamma])$ and view $\mathcal{S}^N A_1^\Gamma$ as a Morita DG enhancement of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Sym}^N[\mathbb{C}^2/\Gamma])$. In $\mathcal{H}perf\, \mathcal{S}^N A_1^\Gamma$ take the full subcategory corresponding to the sheaves supported at the origin $(0,\dots,0) \in \mathrm{Sym}^N[\mathbb{C}^2/\Gamma]$ where 0 is the origin of $\mathbb{C}^2$. Its homotopy category is the target of the 2-representation considered in [**13**, Section 4]. The functors P_i and Q_i representing P_i and Q_i from Example 5.10 as well as the natural transformations defined above are the same as those constructed in [**13**, Section 4.3]. Again, the higher powers $P_i^{(n)}$ and $Q_i^{(n)}$ arise automatically from our calculus (see Example 6.16 and Section 8.1).

7.4. The representation $\Phi'_\mathcal{V}$: the Heisenberg 2-relations

We now prove the following:

Theorem 7.17. *The images assigned in Section 7.3 to the generating 2-morphisms of $\mathbf{H}'_\mathcal{V}$ satisfy the Heisenberg 2-relations of Section 5.2. We thus have a strict DG 2-functor*

$$\Phi'_\mathcal{V} \colon \mathbf{H}'_\mathcal{V} \to \mathbf{F}'_\mathcal{V}.$$

We verify the Heisenberg 2-relations of Section 5.2 in a series of lemmas.

Lemma 7.18. *Let α be a 2-morphism in $\mathbf{H}'_\mathcal{V}$ between 1-morphisms $N \to N'$ which only involve Ps and Qs. The natural transformation $\Phi'_\mathcal{V}(\alpha)$ of DG functors $\mathcal{M}od\text{-}\mathcal{S}^N\mathcal{V} \to \mathcal{M}od\text{-}\mathcal{S}^{N'}\mathcal{V}$ is completely determined by its action on representable modules $h^r(a_1 \otimes \cdots \otimes a_N)$.*

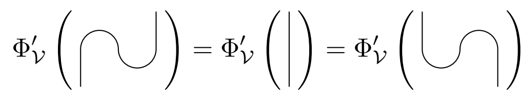
This Lemma means that any relation in $\mathbf{H}'_\mathcal{V}$ whose source and target only involve Ps and Qs can be verified in $\mathbf{F}'_\mathcal{V}$ by checking it on the representable modules.

Proof. By definition, $\Phi'_\mathcal{V}$ maps Ps and Qs to the functors of extension and restriction of scalars. These are tensor functors – they are given by tensoring with a bimodule. In other words, they lie in the image of the fully faithful functor

$$\underline{\otimes} \colon \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B} \to \mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{A},\, \mathcal{M}od\text{-}\mathcal{B}),$$

described in the section Section 4.3. Its right adjoint is the bimodule approximation functor $\underline{\mathrm{Apx}}$ and the fully faithfullness of $\underline{\otimes}$ implies that a natural transformations of tensor functors is completely determined by its image under $\underline{\mathrm{Apx}}$. The claim now follows, since $\underline{\mathrm{Apx}}$ is the restriction to the diagonal bimodule, i.e. to the representables. $\square$

Lemma 7.19. *The straightening relation* (5.9) *is satisfied in* $\mathbf{F}'_{\mathcal{V}}$*:*



for any allowed orientation and labeling of the strands.

Proof. Caps and cups are sent to the unit and counit morphisms of adjoint pairs of functors. The claim now follows from the standard relations

$$\big(F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F\big) = \mathrm{id}_F \qquad \text{and} \qquad \big(G \xrightarrow{\varepsilon G} GFG \xrightarrow{G\eta} G\big) = \mathrm{id}_G$$

satisfied by any adjunction $(F \dashv G)$ with unit η and counit ϵ. $\square$

Lemma 7.20. *Relation* (5.7) *is satisfied is* $\mathbf{F}'_{\mathcal{V}}$*: dots may slide through cups and caps.*

Proof. We need to show that the following pairs of maps are equal for any $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a,b)$:

(1) $P_aQ_b \xrightarrow{P_\alpha Q_{\mathrm{id}_b}} P_bQ_b \xrightarrow{\text{counit}} \mathrm{id}$ and $P_aQ_b \xrightarrow{P_{\mathrm{id}_a} Q_\alpha} P_aQ_a \xrightarrow{\text{counit}} \mathrm{id}$;
(2) $\mathrm{id} \xrightarrow{\text{unit}} Q_aP_a \xrightarrow{Q_{\mathrm{id}_a} P_\alpha} Q_aP_b$ and $\mathrm{id} \xrightarrow{\text{unit}} Q_bP_b \xrightarrow{Q_\alpha P_{\mathrm{id}_b}} Q_aP_b$.
(3) $Q_bR_a \xrightarrow{Q_{\mathrm{id}_b} R_\alpha} Q_bR_b \xrightarrow{\text{counit}} \mathrm{id}$ and $Q_bR_a \xrightarrow{Q_\alpha R_{\mathrm{id}_a}} Q_aR_a \xrightarrow{\text{counit}} \mathrm{id}$;
(4) $\mathrm{id} \xrightarrow{\text{unit}} R_aQ_a \xrightarrow{R_\alpha Q_{\mathrm{id}_a}} R_bQ_a$ and $\mathrm{id} \xrightarrow{\text{unit}} R_bQ_b \xrightarrow{R_{\mathrm{id}_b} Q_\alpha} R_bQ_a$;

By adjunction, (1) and (2) are equivalent, as are (3) and (4). We will show (1). The proof of (3) is similar, using the description of Example 7.15.

From Example 7.12 (2) it follows that

$$P_aQ_bh^r(a_1 \otimes \cdots \otimes a_N) = \bigoplus_{i=1}^{N} \mathrm{Hom}_{\mathcal{V}}(b, a_i) \otimes h^r(a \otimes a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N). \tag{7.2}$$

The map $P_\alpha Q_{\mathrm{id}_b}$ is given on each summand by applying α to the second factor. It lands in

$$\bigoplus_{i=1}^{N} \mathrm{Hom}_{\mathcal{V}}(b, a_i) \otimes h^r(b \otimes a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N).$$

The counit map takes each summand and evaluates the first factor on the second factor:

$$\bigoplus_{i=1}^{N} \mathrm{Hom}_{\mathcal{V}}(b, a_i) \otimes h^r(b \otimes a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N) \to h^r(a_1 \otimes \cdots \otimes a_N).$$

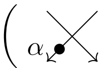
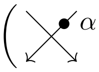
Computing the second composition in a similar way, we see that the equality of these compositions is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}(b, a_i) \otimes h^r(a \otimes a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N) & \xrightarrow{\mathrm{id} \otimes \alpha} & \mathrm{Hom}(b, a_i) \otimes h^r(b \otimes a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N) \\ \big\downarrow{\scriptstyle \alpha \otimes \mathrm{id}} & & \big\downarrow \\ \mathrm{Hom}(a, a_i) \otimes h^r(a \otimes a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N) & \longrightarrow & h^r(a_1 \otimes \cdots \otimes a_N) \end{array}.$$

This diagram commutes by the functoriality of tensor product. □

The next observation is immediate from the construction.

LEMMA 7.21. *Relation* (5.8) *is satisfied in* $\mathbf{F}'_{\mathcal{V}}$. *That is, dots move freely through crossings:*

$\Phi'_{\mathcal{V}}\Bigg($  $\Bigg) = \Phi'_{\mathcal{V}}\Bigg($  $\Bigg).$

LEMMA 7.22. *The symmetric group relations* (5.10) *hold in* $\mathbf{F}'_{\mathcal{V}}$.

PROOF. For the double crossing, the identity $((12)_*)^2 = \mathrm{id}$ follows from the fact that $(12)^2 = \mathrm{id}$ in $\mathcal{S}^N\mathcal{V}$. The triple move similarly follows by splitting the steps as

$$(12) \circ (23) \circ (12) = (23) \circ (12) \circ (23).$$ □

LEMMA 7.23.

(1) *The composition relation* (5.5) *holds in* $\mathbf{F}'_{\mathcal{V}}$. *Namely,* $(Q_b \xRightarrow{Q_\alpha} Q_a) \circ (Q_c \xRightarrow{Q_\beta} Q_b)$ *is equal to* $(-1)^{|\alpha||\beta|} \cdot Q_c \xRightarrow{Q_{\beta\circ\alpha}} Q_a$.

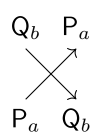
(2) *Relation* (5.6) *holds in* $\mathbf{F}'_{\mathcal{V}}$. *Namely,* $(P_{Sb} \xRightarrow{\star_b} R_b) \circ (P_{Sa} \xRightarrow{P_{S\alpha}} P_{Sb})$ *is equal to* $(R_a \xRightarrow{R_\alpha} R_b) \circ (P_{Sa} \xRightarrow{\star_A} R_a)$.

PROOF. Part (1) is clear from $(\beta \otimes \mathrm{id}) \circ (\alpha \otimes \mathrm{id}) = (\beta \circ \alpha) \otimes \mathrm{id}$, taking the sign rules for contravariant DG functors into account. Part (2) is a consequence of naturality of the Serre morphism η^*. □

LEMMA 7.24. *For every* $a, b \in \mathrm{Ob}(\mathcal{V})$ *and* $a_1 \otimes \cdots \otimes a_N \in \mathcal{S}^N\mathcal{V}$ *there exists a natural isomorphism on representable objects*

$$Q_bP_a(h^r(a_1\otimes\cdots\otimes a_N)) \cong \big(\mathrm{Hom}_{\mathcal{V}}(b,a)\otimes h^r(a_1\otimes\cdots\otimes a_N)\big) \oplus P_aQ_b(h^r(a_1\otimes\cdots\otimes a_N)).$$

The image of



under $\Phi'_{\mathcal{V}}$ *embeds* $P_aQ_b(h^r(a_1 \otimes \cdots \otimes a_N))$ *as the second summand.*

PROOF. The first assertion follows from Example 7.12. The image of the crossing under $\Phi'_{\mathcal{V}}$ is:

$$\phi_a^*\phi_{b*} \xrightarrow{\text{unit}} \phi_a^*\phi_{b*}\phi_{a*}\phi_a^* \cong \phi_a^*\phi_{a*}\phi_{b*}\phi_a^* \xrightarrow{\text{counit}} \phi_{b*}\phi_a^*.$$

Here we used that the commutativity of the the tensor product implies that

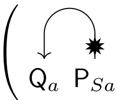
$$\phi_{b*}\phi_{a*} \cong \phi_{b\otimes a*} \cong \phi_{a\otimes b*} \cong \phi_{a*}\phi_{b*}.$$

The second assertion follows from the description of unit and counit maps in Example 7.14. □

The following gives a description of the image of the "starred cup."

LEMMA 7.25. *The natural transformation*

$$\zeta = \Phi'_{\mathcal{V}}\left(\quad \right)$$



is given by the bimodule map

$$\zeta\colon {}_{\phi_{Sa}}\mathcal{S}^N\mathcal{V} \otimes_{\mathcal{S}^N\mathcal{V}} \mathcal{S}^N\mathcal{V}_{\phi_a} \to \mathcal{S}^{N-1}\mathcal{V}$$
$$f \otimes h \mapsto \operatorname{Tr}\left({}_{Sa}(f\circ h)_a\right)\ {}_{\widehat{Sa}}(f\circ h)_{\widehat{a}},$$

where the notation indicates that we take the first summand in terms of the decomposition of ${}_{\phi_{Sa}}\mathcal{S}^N\mathcal{V}_{\phi_a}$ provided by Corollary 7.4. In terms of Example 7.12, ζ maps

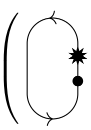
$$Q_aP_{Sa}(h^r(a_1\otimes\cdots\otimes a_{N-1})) \cong \operatorname{Hom}_{\mathcal{V}}(a,Sa)\otimes h^r(a_1\otimes\cdots\otimes a_{N-1}) \oplus$$
$$\oplus \left(\bigoplus_{i=1}^{N-1} \operatorname{Hom}_{\mathcal{V}}(a,a_i)\otimes h^r(a\otimes a_1\otimes\cdots\widehat{a_i}\cdots\otimes a_{N-1})\right)$$

onto the first summand, followed by applying the Serre trace map Tr.

PROOF. Proposition 7.7 gives the star quasi-isomorphism on a. Then, similarly as in Example 3.19, the counit is a projection onto the first summand from Corollary 7.4 followed by the Serre trace applied to $\operatorname{Hom}(a,Sa)$. □

LEMMA 7.26. *The Serre trace relation on the right hand side of* (5.11) *holds in* $\mathbf{F}'_{\mathcal{V}}$*:*

$$\Phi'_{\mathcal{V}}\left(\quad \right) = \operatorname{Tr}(\alpha),$$



PROOF. Assume first that $N=0$. Then we need to compute the image of $h^r(1)$ for $1\in\mathcal{S}^0\mathcal{V}=\Bbbk$. By Example 7.14, the unit corresponding to the cup at the bottom sends this to

$$\mathrm{id}_a \in Q_aP_a(h^r(1)) \cong \phi_{a*}\phi_a^*h^r(1) \cong \operatorname{Hom}_{\mathcal{V}}(a,a).$$

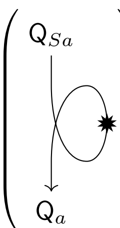
Composing with $Q_{\mathrm{id}_a}P_\alpha$ sends this to $\alpha\in\operatorname{Hom}(a,Sa)$. Finally, the starred cup

$$\zeta = \operatorname{counit}\circ(\phi_{A,*}\star_A)$$

sends α to $\operatorname{Tr}(\alpha)$ by Lemma 7.25. For general N, we need to compute the image of $h^r(a_1\otimes\cdots\otimes a_N)$ for $a_1\otimes\cdots\otimes a_N\in\mathcal{S}^N\mathcal{V}$. We get the same computation as above but tensored over $\Bbbk$ with the identity morphism of $a_1\otimes\cdots\otimes a_N$. □

LEMMA 7.27. *The left curl on the left side of* (5.11) *vanishes in* $\mathbf{F}'_{\mathcal{V}}$*:*

$$\Phi'_{\mathcal{V}}\left(\quad \right) = 0.$$



PROOF. This diagram decomposes as

$$Q_a \xrightarrow{Q_{\mathrm{id}_a}\mathrm{unit}} Q_a Q_{Sa} P_{Sa} \xrightarrow{(12)_*} Q_{Sa} Q_a R_a \xrightarrow{Q_{\mathrm{id}_{Sa}}\zeta} Q_{Sa},$$

where ζ is as in Lemma 7.25. Using the notation of Example 7.12, the first step embeds

$$Q_a(h^r(a_1 \otimes \dots a_N)) \cong \bigoplus_{i=1}^{N} \mathrm{Hom}(a, a_i) \otimes h^r(a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N)$$

into the first factor of

$$\begin{aligned} &Q_a Q_{Sa} P_{Sa}(h^r(a_1 \otimes \dots a_N)) \\ &\quad\cong \left(\bigoplus_{i=1}^{N} \mathrm{Hom}(a, a_i) \otimes \mathrm{Hom}(Sa, Sa) \otimes h^r(a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N)\right) \oplus \\ &\quad\quad \oplus \bigoplus_{j=1}^{N} \Bigg(\mathrm{Hom}(a, Sa) \otimes \mathrm{Hom}(Sa, a_j) \otimes h^r(a_1 \otimes \cdots \widehat{a_j} \cdots \otimes a_N) \\ &\quad\quad \oplus \bigoplus_{\substack{i=1 \\ i \neq j}}^{N} \mathrm{Hom}(a, a_i) \otimes \mathrm{Hom}(Sa, a_j) \otimes h^r(Sa \otimes a_1 \otimes \cdots \widehat{a_i} \cdots \widehat{a_j} \cdots \otimes a_N) \Bigg) \end{aligned}$$

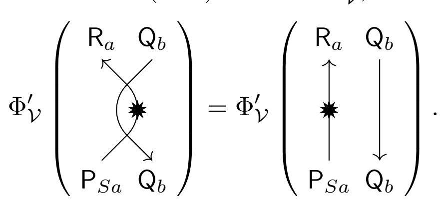
by tensoring with $\mathrm{id}_{Sa} \in \mathrm{Hom}(Sa, Sa)$. The crossing changes the order of the summands, and the starred cap ζ projects onto the summand

$$\bigoplus_{i=1}^{N} \mathrm{Hom}(a, Sa) \otimes \mathrm{Hom}(Sa, a_i) \otimes h^r(a_1 \otimes \cdots \widehat{a_i} \cdots \otimes a_N)$$

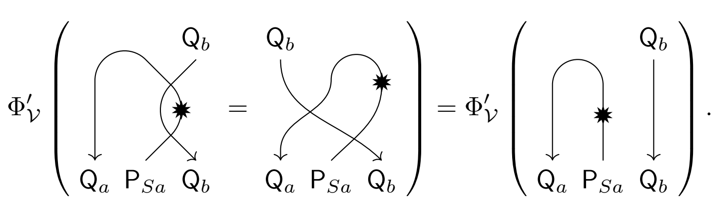
followed by the Serre trace applied to $\mathrm{Hom}(a, Sa)$. As the component corresponding to this summand is zero after the first step, the whole composition vanishes. □

REMARK 7.28. The proof of Lemma 7.27 also explains why the right curls in $\mathbf{H}_{\mathcal{V}}$ are not required to vanish. Therein the unit at the first step and the counit at the last step are both given by diagonal maps, and hence they do not automatically compose to zero.

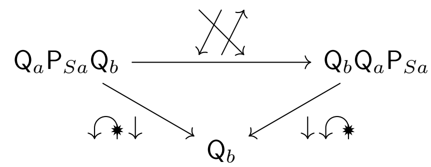
LEMMA 7.29. *The relation in* (5.12) *holds in* $\mathbf{F}'_{\mathcal{V}}$, *i.e.*



PROOF. To use Lemma 7.18, we prove the statement which is equivalent by adjunction:



In other words, $\Phi'_{\mathcal{V}}$ preserves the commutativity of the diagram



By Lemma 7.24 the first crossing in $\Phi'_{\mathcal{V}}(\ldots)$ embeds $Q_aP_{Sa}Q_b(h^r(a_1\otimes\cdots\otimes a_N))$, that is

$$\bigoplus_{i=1}^{N}\Big(\mathrm{Hom}(a,Sa)\otimes\mathrm{Hom}(b,a_i)\otimes h^r(a_1\otimes\cdots\widehat{a_i}\cdots\otimes a_N)\oplus \oplus\bigoplus_{\substack{j=1\\ j\neq i}}^{N}\mathrm{Hom}(a,a_j)\otimes\mathrm{Hom}(b,a_i)\otimes h^r(Sa\otimes a_1\otimes\cdots\widehat{a_i}\cdots\widehat{a_j}\cdots\otimes a_N)\Big), \tag{7.3}$$

into $Q_aQ_bP_{Sa}(h^r(a_1\otimes\cdots\otimes a_N))$, that is

$$Q_a\big(\mathrm{Hom}(b,Sa)\otimes h^r(a_1\otimes\cdots\otimes a_N)\big)\oplus Q_aP_{Sa}Q_b\big(h^r(a_1\otimes\cdots\otimes a_N)\big)=$$
$$\bigoplus_{j=1}^{N}\mathrm{Hom}(a,a_i)\otimes\mathrm{Hom}(b,Sa)\otimes h^r(a_1\otimes\cdots\widehat{a_i}\cdots\otimes a_N)\oplus Q_aP_{Sa}Q_b\big(h^r(a_1\otimes\cdots\otimes a_N)\big).$$

The second crossing changes the summand order. By Lemma 7.25 $\Psi'_{\mathcal{V}}(\ldots)$ projects onto

$$\bigoplus_{i=1}^{N}\mathrm{Hom}(a,Sa)\otimes\mathrm{Hom}(b,a_i)\otimes h^r(a_1\otimes\cdots\widehat{a_i}\cdots\otimes a_N), \tag{7.4}$$

followed by $\mathrm{Tr}\colon\mathrm{Hom}(a,Sa)\to\Bbbk$. On the other hand, $\Psi'_{\mathcal{V}}(\ldots)$ projects (7.3) directly onto (7.4), followed by the Serre trace. Thus the two sides are the same natural transformation. □

7.5. From $\Phi'_{\mathcal{V}}$ to $\Phi_{\mathcal{V}}$

In the previous two sections, we constructed a strict 2-functor

$$\Phi'_{\mathcal{V}}\colon\mathbf{H}'_{\mathcal{V}}\to\mathbf{F}'_{\mathcal{V}}$$

of strict DG 2-categories. Recall that $\mathbf{F}'_{\mathcal{V}}$ is a 1-full subcategory of **dgModCat**, the strict DG 2-category whose objects are small DG categories, and whose 1-morphisms are DG functors between their module categories. We next apply the lax 2-functor of bimodule approximation defined in Section 4.3:

$$\underline{\mathrm{Apx}}\colon\mathbf{dgModCat}\to\mathbf{dgMor}.$$

Its target is the DG bicategory **dgMor** whose objects are small DG categories and whose 1-morphisms are their DG bimodule categories. On objects, $\underline{\mathrm{Apx}}$ is the identity map. On 1-morphisms, for any small DG categories $\mathcal{A}$ and $\mathcal{B}$ it is the DG functor

$$\underline{\mathrm{Apx}}\colon\mathcal{DGF}un(\mathcal{M}od\text{-}\mathcal{A},\,\mathcal{M}od\text{-}\mathcal{B})\to\mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B},$$

defined by $F\mapsto F(\mathcal{A})$.

The 1-morphisms of $\mathbf{H}'_{\mathcal{V}}$ are generated by P_a, Q_a, and R_a for $a \in \mathcal{V}$. 2-functor $\Phi'_{\mathcal{V}}$ sends these to DG functors ϕ^*_a, ϕ_{a*}, and $\phi^!_a$. In Section 7.1 we proved that the images of these under $\underline{\mathrm{Apx}}$ are left h-projective and right-perfect bimodules. We thus obtain a composition

$$\mathbf{H}'_{\mathcal{V}} \xrightarrow{\Phi'_{\mathcal{V}}} \mathbf{F}'_{\mathcal{V}} \xrightarrow{\underline{\mathrm{Apx}}} \mathbf{dgMor},$$

whose image is contained in the 2-full subcategory $\mathbf{dgMor}_{\mathrm{lfrp}}$ of $\mathbf{dgMor}$ consisting of the left-h-flat and right-perfect bimodules.

We remark that the 2-functor $\underline{\mathrm{Apx}}$ does not send all 1-morphisms of $\mathbf{F}'_{\mathcal{V}}$ to $\mathbf{dgMor}_{\mathrm{lfrp}}$. Indeed, by definition $\mathrm{Hom}_{\mathbf{F}'_{\mathcal{V}}}(0,1)$ consists of all DG functors $\mathcal{M}od\text{-}\Bbbk \to \mathcal{M}od\text{-}\mathcal{V}$. For any $E \in \mathcal{M}od\text{-}\mathcal{V}$ we have the functor $(-)\otimes E$ which $\underline{\mathrm{Apx}}$ sends to E considered as $\Bbbk$-$\mathcal{V}$-bimodule. Thus for any non-perfect E the corresponding tensor functor $(-)\otimes E$ is a 1-morphism of $\mathbf{F}'_{\mathcal{V}}$ whose image under $\underline{\mathrm{Apx}}$ isn't right-perfect.

Recall the $\mathbf{Ho}(\mathbf{dgCat})$-enriched bicategory $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ of enhanced triangulated categories defined in Section 4.4. We next apply a strict 2-functor

$$L\colon \mathbf{dgMor}_{\mathrm{lfrp}} \to \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}.$$

On objects, L is the identity map. On 1-morphisms, depending on which of the two definitions of $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ one uses, L is either the natural embedding

$$\mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B}_{\mathrm{lfrp}} \hookrightarrow \mathcal{A}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{B}_{\mathrm{lfrp}},$$

into the bar category of bimodules, or the natural embedding

$$\mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B}_{\mathrm{lfrp}} \hookrightarrow \mathcal{A}\text{-}\mathcal{M}od\text{-}\mathcal{B}_{\mathrm{lfrp}}/\,\mathcal{A}c,$$

into the Drinfeld quotient by acyclics. On the level of homotopy categories, both are just the standard localisation of DG bimodules by quasi-isomorphisms.

We thus obtain a composition

$$\mathbf{H}'_{\mathcal{V}} \xrightarrow{\Phi'_{\mathcal{V}}} \mathbf{F}'_{\mathcal{V}} \xrightarrow{\underline{\mathrm{Apx}}} \mathbf{dgMor}_{\mathrm{lfrp}} \xrightarrow{L} \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}. \tag{7.5}$$

The 2-functors $\Phi'_{\mathcal{V}}$ and L are strict. In general, the 2-functor $\underline{\mathrm{Apx}}$ is lax, but it follows from Proposition 4.13 that on the DG functors ϕ^*_a, ϕ_{a*}, and $\phi^!_a$ its coherence morphisms are quasi-isomorphisms. Since L sends quasi-isomorphisms to homotopy equivalences, it follows that the composition (7.5) is a homotopy strong 2-functor.

Next, we take perfect hulls as per Section 4.22. By definition, $\mathbf{F}_{\mathcal{V}}$ is the perfect hull of the 1-full subcategory of $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ comprising the symmetric powers $\mathcal{S}^N\mathcal{V}$. Thus it contains the perfect hull of the image of (7.5). We thereby obtain a homotopy strong 2-functor

$$\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}}) \xrightarrow{\mathcal{H}perf(L\circ\underline{\mathrm{Apx}}\circ\Phi'_{\mathcal{V}})} \mathbf{F}_{\mathcal{V}}. \tag{7.6}$$

The Heisenberg 2-category $\mathbf{H}_{\mathcal{V}}$ is the monoidal Drinfeld quotient of $\mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})$ by the two-sided 1-morphism ideal $\mathcal{I}_{\mathcal{V}}$ generated by the following two classes of 1-morphisms:

(1) For each $a \in \mathcal{V}$, the cone of the Serre relation 2-morphism

$$\mathsf{P}_{Sa} \xrightarrow{\;\;} \mathsf{R}_a, \tag{7.7}$$

(2) For each $a, b \in \mathcal{V}$, the cone of the 2-morphism

$$\mathsf{P}_b\mathsf{Q}_a \oplus (\mathbb{1} \otimes \operatorname{Hom}(a,b)) \xrightarrow{\left[\times,\ \psi_2\right]} \mathsf{Q}_a\mathsf{P}_b. \tag{7.8}$$

We claim that (7.6) sends these to null-homotopic 1-morphisms in $\mathbf{F}_{\mathcal{V}}$. It suffices to check that (7.6) sends the 2-morphisms (7.7) and (7.8) to homotopy equivalences. Recall that in both definitions of $\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ in Section 4.4 its 1-morphisms are DG bimodules and its 2-morphisms are defined in terms of morphisms of DG bimodules. In first definition we take bar morphisms and in the second we take the Drinfeld quotient of the usual bimodule category by acyclics. In both cases, all usual morphisms of DG bimodules are valid 2-morphisms. We say that a 2-morphism is a quasi-isomorphism if it is an usual morphism of DG bimodules which is a quasi-isomorphism. All such 2-morphisms are homotopy equivalences: for bar morphisms this is shown in [**2**, Cor. 3.8], while in the Drinfeld quotient by acyclics the cone of a quasi-isomorphism is null-homotopic because it is acyclic. It thus suffices to check that (7.6) sends (7.7) and (7.8) to quasi-isomorphisms. For the former this follows by Lemma 7.7, and for the latter by Example 7.12.

We conclude that (7.6) sends all the 1-morphisms in $\mathcal{I}_{\mathcal{V}}$ to null-homotopic ones. By the universal property of the Drinfeld quotient, (7.6) lifts to a homotopy-lax 2-functor

$$\Phi_{\mathcal{V}} \colon \mathbf{H}_{\mathcal{V}} = \mathbf{Hperf}(\mathbf{H}'_{\mathcal{V}})/\mathcal{I}_{\mathcal{V}} \to \mathbf{F}_{\mathcal{V}}.$$

This homotopy strong 2-functor gives our categorical Fock space $\mathbf{F}_{\mathcal{V}}$ the structure of a representation of the Heisenberg 2-category $\mathbf{H}_{\mathcal{V}}$:

THEOREM 7.30. *The constructions above give a homotopy strong 2-functor*

$$\Phi_{\mathcal{V}} \colon \mathbf{H}_{\mathcal{V}} \to \mathbf{F}_{\mathcal{V}},$$

that is, a 2-categorical representation of $\mathbf{H}_{\mathcal{V}}$ on $\mathbf{F}_{\mathcal{V}}$.

COROLLARY 7.31. *There exists a 2-categorical representation of $\mathbf{H}_{H^*(\mathcal{V})}$ on the categories $H^*(\mathcal{H}perf\, \mathcal{S}^N\mathcal{V})$.*

PROOF. This follows immediately by combining Theorem 7.30 with Corollary 5.8. ☐

If one is only interested in the action of homotopy categories, the functor $L \circ \underline{\mathrm{Apx}}$ above can safely be ignored. More precisely, on homotopy categories one has a canonical isomorphism $\phi_{Sa} \cong \phi^{!}_{a}$ and hence one only needs to understand the functors ϕ^*_a and $\phi_{a,*}$. As these functors are already given by bimodules, the functor $L \circ \underline{\mathrm{Apx}}$ simply restricts them to $\mathcal{H}perf\, \mathcal{S}^N\mathcal{V}$.

CHAPTER 8

Structure of the Categorical Fock Space

8.1. The symmetrised operators

As described in Section 6.1, the 1-morphisms P_a, Q_a and R_a induce 1-morphisms $\mathsf{P}_a^{(n)}$, $\mathsf{Q}_a^{(n)}$ and $\mathsf{R}_a^{(n)}$ of $\mathbf{H}_\mathcal{V}$ for $n \geq 0$ via symmetrisers. These are represented by operators $P_a^{(n)}$, $Q_a^{(n)}$ and $R_a^{(n)}$ on $\mathbf{F}_\mathcal{V}$. In order to explicitly describe the effect of these operators on $\mathbf{F}_\mathcal{V}$, we consider the functor

$$\phi_{a^n} \colon \mathcal{S}^N\mathcal{V} \to \mathcal{S}^{N+n}\mathcal{V}, \qquad a_1 \otimes \cdots \otimes a_N \mapsto a \otimes \cdots \otimes a \otimes a_1 \otimes \ldots a_N.$$

The 1-morphisms P_a^n, Q_a^n and R_a^n are the images of the functors

$$\phi_{a^n}^* \colon \mathcal{M}od\text{-}\mathcal{S}^N\mathcal{V} \to \mathcal{M}od\text{-}\mathcal{S}^{N+n}\mathcal{V},$$
$$\phi_{a^n,*} \colon \mathcal{M}od\text{-}\mathcal{S}^{N+n}\mathcal{V} \to \mathcal{M}od\text{-}\mathcal{S}^N\mathcal{V},$$

and

$$\phi_{a^n}^! \colon \mathcal{M}od\text{-}\mathcal{S}^N\mathcal{V} \to \mathcal{M}od\text{-}\mathcal{S}^{N_n}\mathcal{V},$$

under the functor $L \circ \underline{\mathrm{Apx}}$ of Section 7.5, with $P_a^{(0)} = Q_a^{(0)} = R_a^{(0)} = \mathrm{id}$. Recall that in Definition 7.10 we defined the Fock space $\mathbf{F}_\mathcal{V}$ as a 1-full subcategory of the 2-category $\mathbf{Hperf}(\mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}})$ which by Lemma 7.9 is a DG enhancement of the strict 2-category $\mathbf{EnhCat}_{\mathrm{kc}}$ of enhanced triangulated categories. Thus 1-morphisms in the Fock space are enhanced functors between enhanced triangulated categories. The underlying exact functors have the following explicit description:

LEMMA 8.1. *Let a be an object of $\mathcal{V}$. Then:*

(1) The exact functor

$$p_a^n \colon \mathrm{D_c}(\mathcal{S}^N\mathcal{V}) \to \mathrm{D_c}(\mathcal{S}^{N+n}\mathcal{V})$$

underlying the enhanced functor P_a^n is isomorphic to the composition

$$\mathrm{D_c}(\mathcal{S}^N\mathcal{V}) \xrightarrow{h^r(a^n)\otimes(-)} \mathrm{D_c}(\mathcal{S}^n\mathcal{V} \otimes \mathcal{S}^N\mathcal{V}) \xrightarrow{\mathrm{Ind}^{S_{N+n}}_{S_n \times S_N}} \mathrm{D_c}(\mathcal{S}^{N+n}\mathcal{V}),$$

where $h^r(a^n) \otimes (-)$ is the evaluation of (4.5) at $h^r(a^n)$. This DG functor sends any $E \in \mathcal{M}od\text{-}\mathcal{S}^N\mathcal{V}$ to the module over $\mathcal{S}^N\mathcal{V} \otimes \mathcal{S}^n\mathcal{V}$ whose fibers are given by the tensor product over $\Bbbk$ of the fibers of $h^r(a^n)$ and the fibers of E. As it sends acyclics to acyclics, it descends to the derived categories as-is.

(2) The exact functor

$$q_a^n \colon \mathrm{D_c}(\mathcal{S}^N\mathcal{V}) \to \mathrm{D_c}(\mathcal{S}^{N+n}\mathcal{V})$$

underlying the enhanced functor Q_a^n is isomorphic to the composition

$$\mathrm{D_c}(\mathcal{S}^{N+n}\mathcal{V}) \xrightarrow{\mathrm{Res}^{S_n \times S_N}_{S_{N+n}}} \mathrm{D_c}(\mathcal{S}^N\mathcal{V} \otimes \mathcal{S}^n\mathcal{V}) \xrightarrow{\mathrm{Hom}_{\mathcal{S}^n\mathcal{V}}(h^r(a^n),-)} \mathrm{D_c}(\mathcal{S}^N\mathcal{V}),$$

where $\operatorname{Hom}_{\mathcal{S}^n\mathcal{V}}(h^r(a^n),-)$ *is the right adjoint of* $h^r(a^n)\otimes(-)$*. It is the DG functor of taking* Hom*-spaces as* $\mathcal{S}^N\mathcal{V}$ *modules. With* $h^r(a^n)$ *in the first argument, it is isomorphic to the functor* $(-)_{a^n}$ *of taking fibers over* $a^n \in \mathcal{S}^n\mathcal{V}$*. As it sends acyclics to acyclics, it descends to the derived categories as-is.*

Proof. Since Q^n_a is the 2-categorical right adjoint of P^n_a and $\Phi_{\mathcal{V}}$ is homotopy monoidal, Q^n_a is a homotopy right adjoint of P^n_a. Therefore q^n_a is the right adjoint of p^n_a. We thus only prove (1), as (2) follows by adjunction.

By definition, P^n_a is the image of $\phi^*_{a^n}$ under the functor $L\circ\underline{\operatorname{Apx}}$ of taking bimodule approximation, and then projecting to the derived category of bimodules. Since $\phi^*_{a^n}$ is already a tensor functor, it restricts to

$$\phi^*_{a^n}\colon\ \mathcal{H}perf(\mathcal{S}^N\mathcal{V})\to\mathcal{H}perf(\mathcal{S}^{N+n}\mathcal{V}),$$

and the corresponding exact functor p^n_a is the H^0-truncation of this restriction.

We can view ϕ_{a^n} as the image of $a^n\in\mathcal{S}^n\mathcal{V}$ under the DG functor

$$\begin{aligned}\phi\colon\quad & \mathcal{S}^n\mathcal{V}\xrightarrow{\text{unit}}\mathcal{DGF}un(\mathcal{S}^N\mathcal{V},\,\mathcal{S}^n\mathcal{V}\otimes\mathcal{S}^N\mathcal{V})\xrightarrow{i^{n+N}_{n,N}\circ(-)}\\ & \to\mathcal{DGF}un(\mathcal{S}^N\mathcal{V},\,\mathcal{S}^{N+n}\mathcal{V})\xrightarrow{(-)^*}\mathcal{DGF}un\left(\mathcal{H}perf(\mathcal{S}^N\mathcal{V}),\,\mathcal{H}perf(\mathcal{S}^{N+n}\mathcal{V})\right),\end{aligned}$$

where $i^{n+N}_{n,N}\colon\mathcal{S}^n\mathcal{V}\otimes\mathcal{S}^N\mathcal{V}\to\mathcal{S}^{N+n}\mathcal{V}$ is the natural inclusion. It can now be readily verified that ϕ is isomorphic to the composition of the Yoneda embedding $\mathcal{S}^n\mathcal{V}\hookrightarrow\mathcal{H}perf(\mathcal{S}^n\mathcal{V})$ with

$$\begin{array}{c}\mathcal{H}perf(\mathcal{S}^n\mathcal{V})\\ \big\downarrow{\scriptstyle\text{unit}}\\ \mathcal{DGF}un\left(\mathcal{H}perf(\mathcal{S}^N\mathcal{V}),\,\mathcal{H}perf(\mathcal{S}^n\mathcal{V})\otimes\mathcal{H}perf(\mathcal{S}^N\mathcal{V})\right)\\ \big\downarrow{\scriptstyle(4.5)\circ(-)}\\ \mathcal{DGF}un\left(\mathcal{H}perf(\mathcal{S}^N\mathcal{V}),\,\mathcal{H}perf(\mathcal{S}^n\mathcal{V}\otimes\mathcal{S}^N\mathcal{V})\right)\\ \big\downarrow{\scriptstyle(i^{n+N}_{n,N})^*\circ(-)}\\ \mathcal{DGF}un\left(\mathcal{H}perf(\mathcal{S}^N\mathcal{V}),\,\mathcal{H}perf(\mathcal{S}^{N+n}\mathcal{V})\right).\end{array}\tag{8.1}$$

Since the DG category isomorphism

$$\mathcal{S}^n\mathcal{V}\otimes\mathcal{S}^N\mathcal{V}\cong(S_n\times S_N)\rtimes\mathcal{V}^{N+n},$$

and the equivalence (4.9) identify

$$(i^{n+N}_{n,N})^*\colon\ \mathcal{H}perf(\mathcal{S}^n\mathcal{V}\otimes\mathcal{S}^N\mathcal{V})\to\mathcal{H}perf(\mathcal{S}^{N+n}\mathcal{V})$$

with the induction functor

$$\operatorname{Ind}^{S_{N+n}}_{S_n\times S_N}\colon\ \mathcal{H}perf^{S_n\times S_N}(\mathcal{V}^{N+n})\to\mathcal{H}perf^{S_{N+n}}(\mathcal{V}^{N+n}),$$

the desired claim follows. □

Let $\phi^*_{e_{\text{triv}}}$, $\phi_{e_{\text{triv}},*}$, and $\phi^!_{e_{\text{triv}}}$ be the images of the idempotent $\phi_{e_{\text{triv}}}\colon\phi_{a^n}\to\phi_{a^n}$ under the functors $(-)^*$, $(-)_*$, and $(-)^!$, respectively. While the idempotents e_{triv} and $\phi_{e_{\text{triv}}}$ are not apriori split, the idempotents $\phi^*_{e_{\text{triv}}}$, $\phi_{e_{\text{triv}},*}$, and $\phi^!_{e_{\text{triv}}}$ always are. The splitting is obtained by taking all elements invariant under the action of S_n on a^n. For example, given any module $E\in\mathcal{M}od\text{-}\mathcal{S}^N\mathcal{V}$ we consider the elements of $\phi^*_{a^n}(E)$ which are invariant under the endomorphisms induced by $\sigma\colon a^n\to a^n$ for all $\sigma \in S_n$. These form a submodule which splits the idempotent $\phi^*_{e_{\text{triv}}}$ on E. In fact, $\phi^*_{a^n}$ is the functor of tensoring with the bimodule

$$_{\phi_{a^n}}\mathcal{S}^N\mathcal{V} := \operatorname{Hom}_{\mathcal{S}^N\mathcal{V}}(-, a^n \otimes -),$$

and $\phi^*_{e_{\text{triv}}}$ is split by its submodule of elements invariant under the action of S_n on a^n.

By construction, the 1-morphisms $P^{(n)}_a$, $Q^{(n)}_a$ and $R^{(n)}_a$ are the images under projection $L \circ \underline{\text{Apx}}$ to $\mathbf{F}_{\mathcal{V}}$ of the homotopical splittings of idempotents $\phi^*_{e_{\text{triv}}}$, $\phi_{e_{\text{triv}},*}$, and $\phi^!_{e_{\text{triv}}}$ given by the construction in Section 6.1. Thus $P^{(n)}_a$, $Q^{(n)}_a$ and $R^{(n)}_a$ are homotopy equivalent to the images under $L \circ \underline{\text{Apx}}$ of the genuine splittings of these idempotents. We thus have:

COROLLARY 8.2. *Let a be an object of $\mathcal{V}$. Let $h^r(a^n)^{S_n} \in \mathcal{H}perf\, \mathcal{S}^n\mathcal{V}$ be the submodule of $h^r(a^n)$ consisting of S_n-invariant elements. Then:*

(1) The exact functor

$$p^{(n)}_a \colon \mathrm{D_c}(\mathcal{S}^N\mathcal{V}) \to \mathrm{D_c}(\mathcal{S}^{N+n}\mathcal{V})$$

underlying the enhanced functor $P^{(n)}_a$ is isomorphic to the composition

$$\mathrm{D_c}(\mathcal{S}^N\mathcal{V}) \xrightarrow{h^r(a^n)^{S_n}\otimes(-)} \mathrm{D_c}(\mathcal{S}^n\mathcal{V}\otimes\mathcal{S}^N\mathcal{V}) \xrightarrow{\operatorname{Ind}^{S_{N+n}}_{S_n\times S_N}} \mathrm{D_c}(\mathcal{S}^{N+n}\mathcal{V}).$$

(2) The exact functor

$$q^{(n)}_a \colon \mathrm{D_c}(\mathcal{S}^N\mathcal{V}) \to \mathrm{D_c}(\mathcal{S}^{N+n}\mathcal{V})$$

underlying enhanced functor $Q^{(n)}_a$ is isomorphic to the composition

$$\mathrm{D_c}(\mathcal{S}^{N+n}\mathcal{V}) \xrightarrow{\operatorname{Res}^{S_n\times S_N}_{S_{N+n}}} \mathrm{D_c}(\mathcal{S}^N\mathcal{V}\otimes\mathcal{S}^n\mathcal{V}) \xrightarrow{\operatorname{Hom}_{\mathcal{S}^n\mathcal{V}}(h^r(a^n)^{S_n},-)} \mathrm{D_c}(\mathcal{S}^N\mathcal{V}).$$

PROOF. As before, (2) follows by adjunction from (1).

To prove the latter, recall that in the proof of Lemma 8.1 we have established that p^n_a is isomorphic to the H^0-truncation of $\phi(a^n)$ where ϕ is isomorphic to the composition of the Yoneda embedding and (8.1). The idempotent $e_{\text{triv}} : a^n \to a^n$ becomes split once we apply the Yoneda embedding $h^r(-)$ and $h^r(a^n)^{S_n}$ is the corresponding direct summand. Therefore the idempotent $\phi(e_{\text{triv}})$ is split and the corresponding direct summand of $\phi(a_n)$ is given by the image of $h^r(a^n)^{S_n}$ under (8.1). Since $p^{(n)}_a$ is isomorphic to this direct summand of p^n_a, the claim folllows. □

The 1-morphisms $\mathsf{P}^{(n)}_a$, $\mathsf{Q}^{(n)}_a$ and $\mathsf{R}^{(n)}_a$ satisfy a number of relations arising from the relations between in $\mathbf{H}_{\mathcal{V}}$. For example:

(1) There are adjunctions $P^{(n)}_a \dashv Q^{(n)}_a$ and $Q^{(n)}_a \dashv R^{(n)}_a$.
(2) By Remark 6.5, for every $\alpha \in \operatorname{Sym}^n(\operatorname{Hom}(a,b))$ there are natural transformations $P^{(n)}_a \overset{\alpha}{\Rightarrow} P^{(n)}_b$ and $Q^{(n)}_b \overset{\alpha}{\Rightarrow} Q^{(n)}_a$.
(3) By Theorem 6.3, for any a, $b \in \mathcal{V}$ and n, $m \in \mathbb{N}$ we have natural isomorphisms

$$P^{(m)}_a P^{(n)}_b \cong P^{(n)}_b P^{(m)}_a, \quad Q^{(m)}_a Q^{(n)}_b \cong Q^{(n)}_b Q^{(m)}_a, \tag{8.2}$$

and a homotopy isomorphism

$$\bigoplus_{i=0}^{\min(m,n)} \operatorname{Sym}^i(\operatorname{Hom}_{\mathcal{V}}(a,b)) \otimes_{\Bbbk} P^{(n-i)}_b Q^{(m-i)}_a \to Q^{(m)}_a P^{(n)}_b. \tag{8.3}$$

Example 8.3. Let X be a smooth and projective variety. Continuing Example 7.11, we obtain the symmetrised operators $P_a^{(n)}$ and $Q_a^{(n)}$. These reproduce the remaining functors from [**33**, Section 2.4] for the DG derived categories. Thus Corollary 7.31 enhances the representation defined by Krug to a 2-categorical action of $H^*(\mathbf{H}_{\mathcal{I}(X)})$. On the homotopy categories, (8.2) and (8.3) become

$$P_a^{(m)}P_b^{(n)} \cong P_b^{(n)}P_a^{(m)}, \quad Q_a^{(m)}Q_b^{(n)} \cong Q_b^{(n)}Q_a^{(m)},$$
$$Q_a^{(m)}P_b^{(n)} \cong \bigoplus_{i=0}^{\min(m,n)} \operatorname{Sym}^i \operatorname{Hom}^*(a,b) \otimes_{\Bbbk} P_b^{(n-i)}Q_a^{(m-i)}.$$

This provides a new proof of [**33**, Theorem 1.4].

8.2. Grothendieck groups and the classical Fock space

8.2.1. Constructing a representation of the Heisenberg algebra. In Section 4.9 we defined the numerical Grothendieck group $\mathrm{K}_0^{\mathrm{num}}(\mathcal{V}, \Bbbk)$ of a smooth and proper DG category $\mathcal{V}$. As finite tensor products of DG categories preserve both of these properties, $\mathcal{V}^{\otimes N}$ is smooth and proper. It is then evident from the decomposition (4.8) of the diagonal bimodule, that $\mathcal{S}^N\mathcal{V}$ is smooth and proper as well. Thus its numerical Grothendieck group is well-defined.

With this in mind, define the $\Bbbk$-linear 1-category

$$\operatorname{End}\left(\bigoplus_N \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)\right)$$

to have as objects $\mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk) := \Bbbk \otimes_{\mathbb{Z}} \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V})$, and as morphisms the $\Bbbk$-linear maps between these vector spaces. Thus a $\Bbbk$-linear functor into this category is an idempotent-modified version of a representation on the vector space $\bigoplus_N \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)$.

We next use the 2-functor $\Phi_{\mathcal{V}}\colon \mathbf{H}_{\mathcal{V}} \to \mathbf{F}_{\mathcal{V}}$ to define a 1-functor from $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ to $\bigoplus_N \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)$. Recall our definition of $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$: it is the quotient of $\mathrm{K}_0(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ by the two-sided ideal generated by the images under $\Xi^P, \Xi^Q : \mathbf{Hperf}(\mathbf{Sym}_{\mathcal{V}}) \to \mathbf{H}_{\mathcal{V}}$ of the kernel of the Euler pairing, see Section 6.4. To show that this two-sided ideal gets sent by $\Phi_{\mathcal{V}}$ to the kernel of the Euler pairing on $\bigoplus_N \mathrm{K}_0(\mathcal{S}^N\mathcal{V}, \Bbbk)$, we need the following lemma:

Lemma 8.4. *The composition*

$$\Phi_{\mathcal{V}} \circ \Xi^P : \mathbf{Hperf}(\mathbf{Sym}_{\mathcal{V}}) \to \mathbf{F}_{\mathcal{V}},$$

is the following 2*-functor. On the object sets, it is* $\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$*. On the* 1*-morphism categories* $N \to N+n$*, it is homotopy equivalent to the composition of* $L \circ \underline{\mathrm{Apx}}$ *with the DG functor*

$$\mathcal{H}perf\,(\mathcal{S}^n\mathcal{V}) \xrightarrow{(8.1)} \mathcal{DGF}un\left(\mathcal{H}perf(\mathcal{S}^N\mathcal{V}),\, \mathcal{H}perf\left(\mathcal{S}^{N+n}\mathcal{V}\right)\right).$$

Proof. By definition, $\Phi_{\mathcal{V}} \circ \Xi^P$ is the perfect hull of the 2-functor

$$\mathbf{Sym}_{\mathcal{V}} \xrightarrow{L\circ\underline{\mathrm{Apx}}\circ\Phi'_{\mathcal{V}}\circ\Xi^{P'}_{\mathcal{V}}} \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}.$$

On the 1-morphism categories $N \to N+n$, the composition $\Phi'_{\mathcal{V}} \circ \Xi^{P'}_{\mathcal{V}}$ is the functor

$$\phi\colon \mathcal{S}^n\mathcal{V} \to \mathcal{DGF}un\left(\mathcal{H}perf(\mathcal{S}^N\mathcal{V}),\, \mathcal{H}perf(\mathcal{S}^{N+n}\mathcal{V})\right),$$

defined in the proof of Lemma 8.1. Therefore, on the 1-morphism categories $N \to N+n$, the composition $\Phi_{\mathcal{V}} \circ \Xi^P$ is the functor $(L \circ \underline{\mathrm{Apx}} \circ \phi)^*$.

Since ϕ is isomorphic to

$$\mathcal{S}^n\mathcal{V} \xrightarrow{\text{Yoneda}} \mathcal{H}perf\,(\mathcal{S}^n\mathcal{V}) \xrightarrow{(8.1)} \mathcal{DGF}un\left(\mathcal{H}perf(\mathcal{S}^N\mathcal{V}),\, \mathcal{H}perf\left(\mathcal{S}^{N+n}\mathcal{V}\right)\right)$$

the desired assertion now follows from the following fundamental fact. Let $\mathcal{A}$ and $\mathcal{B}$ be any DG categories and $F\colon \mathcal{A} \to \mathcal{B}$ any DG functor. If F decomposes as

$$\mathcal{A} \xrightarrow{\text{Yoneda}} \mathcal{H}perf(\mathcal{A}) \xrightarrow{G} \mathcal{B},$$

for some DG functor G, then F^* is homotopy equivalent to

$$\mathcal{H}perf(\mathcal{A}) \xrightarrow{G} \mathcal{B} \xrightarrow{\text{Yoneda}} \mathcal{H}perf(\mathcal{B}).$$

To see this, consider the commutative square

$$\begin{array}{ccc} \mathcal{H}perf(\mathcal{A}) & \xrightarrow{\quad G \quad} & \mathcal{B} \\ \big\downarrow \text{Yoneda} & & \big\downarrow \text{Yoneda} \\ \mathcal{H}perf(\mathcal{H}perf(\mathcal{A})) & \xrightarrow{G^*} & \mathcal{H}perf(\mathcal{B}), \end{array}$$

and observe that the DG functor

$$\text{Yoneda}\colon \mathcal{H}perf(\mathcal{A}) \to \mathcal{H}perf(\mathcal{H}perf(\mathcal{A}))$$

is homotopy equivalent to the DG functor

$$\text{Yoneda}^*\colon \mathcal{H}perf(\mathcal{A}) \to \mathcal{H}perf(\mathcal{H}perf(\mathcal{A})).$$ □

We can now construct the desired 1-functor:

COROLLARY 8.5. *The 2-functor $\Phi_{\mathcal{V}}\colon \mathbf{H}_{\mathcal{V}} \to \mathbf{F}_{\mathcal{V}}$ from Theorem 7.30 induces a 1-functor*

$$\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}},\, \Bbbk) \to \mathrm{End}\left(\bigoplus_N \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V},\, \Bbbk)\right).$$

In other words one obtains a representation of $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}},\, \Bbbk)$ on $\bigoplus \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V},\, \Bbbk)$.

PROOF. Functoriality of Grothendieck groups gives a 1-functor

$$\Phi_{\mathcal{V}}\colon \mathrm{K}_0(\mathbf{H}_{\mathcal{V}},\, \Bbbk) \to \mathrm{End}\left(\bigoplus_N \mathrm{K}_0(\mathcal{S}^N\mathcal{V},\, \Bbbk)\right).$$

We claim that any morphism $\mathrm{K}_0(\mathcal{S}^N\mathcal{V},\, \Bbbk) \to K^0(\mathcal{S}^M\mathcal{V},\, \Bbbk)$ in its image takes the kernel of the Euler pairing χ on $\mathrm{K}_0(\mathcal{S}^N\mathcal{V},\, \Bbbk)$ to its kernel on $K^0(\mathcal{S}^M\mathcal{V},\, \Bbbk)$. As per Section 7.1, the 1-morphisms to which $\Phi_{\mathcal{V}}$ maps generating 1-morphisms Ps, Qs, and Rs of $\mathbf{H}'_{\mathcal{V}}$ are left- and right-perfect bimodules. Hence the same is true of all 1-morphisms in $\Phi_{\mathcal{V}}(\mathbf{H}'_{\mathcal{V}})$. By construction, $\Phi_{\mathcal{V}}(\mathbf{H}_{\mathcal{V}})$ lies in the $\mathcal{H}perf$-hull of $\Phi_{\mathcal{V}}(\mathbf{H}'_{\mathcal{V}})$, and thus the 1-morphisms in $\Phi_{\mathcal{V}}(\mathbf{H}_{\mathcal{V}})$ are also left- and right-perfect bimodules. By [**2**, Theorem 4.1] the corresponding exact functors $\mathrm{D}_{\mathrm{c}}(\mathcal{S}^N\mathcal{V}) \to \mathrm{D}_{\mathrm{c}}(\mathcal{S}^M\mathcal{V})$ have left adjoints. Arguing as in Lemma 4.49, we see that the induced maps $\mathrm{K}_0(\mathcal{S}^N\mathcal{V},\, \Bbbk) \to K^0(\mathcal{S}^M\mathcal{V},\, \Bbbk)$ take $\ker\chi$ to $\ker\chi$.

We thus have a 1-functor

$$\Phi_{\mathcal{V}}\colon \mathrm{K}_0(\mathbf{H}_{\mathcal{V}},\, \Bbbk) \to \mathrm{End}\left(\bigoplus_N \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V},\, \Bbbk)\right).$$

It remains to show that this functor descends to $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$. For the definition of the latter, see Section 6.4.

Let $E \in \mathcal{H}perf(\mathcal{S}^n\mathcal{V})$ be in the kernel of the Euler pairing. Let us view E as a 1-morphism $N \to N+n$ in $\mathbf{Hperf}(\mathbf{Sym}_{\mathcal{V}})$. By Lemma 8.4, $\Phi_{\mathcal{V}} \circ \Xi^P(E)$ is an enhanced functor whose underlying exact functor is

$$\mathrm{D_c}(\mathcal{S}^N\mathcal{V}) \xrightarrow{E\otimes(-)} \mathrm{D_c}(\mathcal{S}^n\mathcal{V} \otimes \mathcal{S}^N\mathcal{V}) \xrightarrow{\mathrm{Ind}_{S_n\times S_N}^{S_{N+n}}} \mathrm{D_c}(\mathcal{S}^{N+n}\mathcal{V}).$$

We have to show that its image lies in $\ker\chi$, and thus the induced map of K_0^{num} is zero. Let $F \in \mathcal{H}perf(\mathcal{S}^N\mathcal{V})$ and observe that

$$\Phi_{\mathcal{V}} \circ \Xi^P(E)(F) \cong \mathrm{Ind}_{S_n\times S_N}^{S_{N+n}}(E\otimes F) \cong \mathrm{Ind}_{S_N\times S_n}^{S_{N+n}}(F\otimes E) \cong \Phi_{\mathcal{V}} \circ \Xi^P(F)(E).$$

Above we already established that the underlying exact functor of any 1-morphism in the image of $\Phi_{\mathcal{V}}$ takes $\ker\chi$ to $\ker\chi$. Thus $\Phi_{\mathcal{V}} \circ \Xi^P(F)(E)$ lies in $\ker\chi$, and hence so does $\Phi_{\mathcal{V}} \circ \Xi^P(E)(F)$. By adjunction, $\Phi_{\mathcal{V}} \circ \Xi^Q(E)(F)$ lies in $\ker\chi$ as well.

We have now established that on the level of Grothendieck groups $\Phi_{\mathcal{V}}$ kills the image under Ξ^P and Ξ^Q of the kernel of the Euler pairing on $\mathrm{K}_0(\mathbf{Hperf}(\mathbf{Sym}_{\mathcal{V}}), \Bbbk)$. Since $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ is the quotient of $\mathrm{K}_0(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ by the two-sided ideal generated by this image, we conclude that $\Phi_{\mathcal{V}}$ descends to a functor $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk) \to \mathrm{End}\left(\bigoplus_N \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)\right)$, as desired. □

8.2.2. Genuine categorification. Consider $\Phi_{\mathcal{V}}$ as homomorphism of algebras and compose it with the algebra homomorphism $\pi\colon H_{\mathcal{V}} \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ of Section 6.4 to obtain a homomorphism

$$H_{\mathcal{V}} \to \mathrm{End}\left(\bigoplus_N \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)\right).$$

The vector $1 \in \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^0\mathcal{V}, \Bbbk) \cong \Bbbk$ is annihilated by all elements of $H_{\mathcal{V}}^- \setminus \{1_0\}$ and is kept invariant by 1_0. Lemma 2.9 then implies that there is a graded $H_{\mathcal{V}}$-module embedding

$$\phi\colon F_{\mathcal{V}} = \bigoplus_N F_{\mathcal{V}}^N \hookrightarrow \bigoplus_N \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk) \tag{8.4}$$

of the appropriate classical Fock space.

The following is a generalisation of [**33**, Section 3.1]. For a partition λ, write $r(\lambda)_i$ for the number of parts of λ of size i.

Corollary 8.6. *Suppose that the following dimension formula holds:*

$$\dim \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk) = \sum_{\lambda\dashv N} \prod_i \dim \mathrm{Sym}^{r(\lambda)_i} \mathrm{K}_0^{\mathrm{num}}(\mathcal{V}, \Bbbk)$$

where the sum runs over all partitions λ of N and the product over all sizes i of parts of λ. Then (8.4) is an $H_{\mathcal{V}}$-module isomorphism. That is, $\mathbf{F}_{\mathcal{V}}$ categorifies $F_{\mathcal{V}}$.

Proof. The assumption and (2.8) implies that the dimensions of the graded vector spaces $F_{\mathcal{V}}$ and $\bigoplus_N \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)$ agree in each degree. Hence, these graded spaces must be isomorphic. □

Example 8.7. The assumption of Corollary 8.6 is satisfied in the following cases:

(1) Let X be a smooth projective variety and $\mathcal{V} = \mathcal{I}(X)$ as in Examples 7.11 and 8.3. Assume moreover that the numerical Grothendieck group satisfies a Künneth formula:

$$\mathrm{K}_0^{\mathrm{num}}(\mathcal{V}^{\otimes N}) \cong (\mathrm{K}_0^{\mathrm{num}}(\mathcal{V}))^{\otimes N}.$$

This is the case, by Example 4.46, if the Chow groups of X tensored with $\mathbb{Q}$ satisfy the Künneth formula. A sufficient condition for this is that the Chow motive of X is a summand of a direct sum of Tate motives [**54**]. Note that this is a very strong assumption which is closely related to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ having a full exceptional collection. It is already false for elliptic curves as we see in the counterexample in §8.2.3.

As the Chern character is additive on disjoint varieties, by Hirzebruch—Riemann-–Roch we can replace K_0 with $\mathrm{K}_0^{\mathrm{num}}$ in [**55**, Theorem 1] to get a direct sum decomposition:

$$\mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk) \cong \bigoplus_{\lambda\dashv N} \bigotimes_i \mathrm{Sym}^{r(\lambda)_i} \mathrm{K}_0^{\mathrm{num}}(\mathcal{V}, \Bbbk)$$

where the sum runs over all partitions λ of N and the product over all sizes i of parts of λ.

Hence, $\mathcal{V}$ satisfies the assumption of Corollary 8.6.

(2) Let $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup and $\mathcal{V}$ as in Examples 5.10, 6.16 and 7.16. Then the dimension assumption for the usual K-groups follows from the combination of [**56**, Proposition 5], Göttsche's formula for the Betti numbers of Hilbert schemes and the fact the topological and algebraic K-theories agree on the minimal resolution of the quotient variety $\mathbb{C}^2/\Gamma$ (as both are described by the representation theory of G [**38**, Chapter 4]). The Euler form equals the intersection form on the resolution, which is given by the appropriate finite type Cartan matrix. This is known to be non-degenerate. Hence, the kernel of χ is trivial in each case, and the dimension assumption descends to the numerical K-groups.

Remark 8.8. An alternative way to obtain Example 8.7 (1) in many cases is to combine the main result of [**10**] proving that the map

$$\pi : H_{\mathcal{V}} \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$$

is an isomorphism when $X = \mathrm{Spec}(\Bbbk)$ is a point (and hence also when $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ has a full exceptional collection) with our Theorem 8.13 below.

8.2.3. A counterexample. We now give an example of π not being an isomorphism. Let X be a smooth projective curve and $n \in \mathbb{Z}_{>0}$. Denote by

$$X^{(N)} = X^N/S_N$$

the N-th symmetric power of X. This is a smooth projective variety of dimension N.

Let λ be any partition of N. Write r_i for the number of parts of size i in λ. Define the closed subvariety

$$X[\lambda] \subset X^N$$

to be the fixed point locus of some $\sigma \in S_N$ of cycle type λ. Different choices of σ produce canonically isomorphic $X[\lambda]$. Explicitly, $X[\lambda]$ consists of $(x_1, \ldots, x_N)$ where $x_i = x_{\sigma(i)}$ for all $i \in 1, \ldots, N$. Thus $X[\lambda] \cong X^k$ where k is the total number of parts in λ. The action of the centraliser $C(\sigma) \subset S_N$ on X^N restricts to $X[\lambda]$ as the action of $S_\lambda = \prod_i S_{r_i}$ which permutes the factors of X^k which correspond to the parts of the same size in λ. The quotient variety is

$$X[\lambda]/S_\lambda = \prod_i X^{(r_i)}.$$

For any ordering $\lambda^1, \ldots, \lambda^p$ of partitions of N refining the dominance order, there is a semiorthogonal decomposition

$$\begin{aligned} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}([X^N/S_N]) &= \left\langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X[\lambda^1]/S_{\lambda^1}), \ldots, \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X[\lambda^p]/S_{\lambda^p}) \right\rangle \\ &= \left\langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}\left(\prod_i X^{(r(\lambda^1)_i)}\right), \ldots, \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}\left(\prod_i X^{(r(\lambda^p)_i)}\right)\right\rangle \end{aligned}$$

by [**40**]. As $N = 2$ has two partitions ($\lambda^1 = (2)$ and $\lambda^2 = (1,1)$), we have for the second symmetric quotient stack the semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}([X^2/S_2]) = \left\langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X), \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{(2)}) \right\rangle. \tag{8.5}$$

Let now X be an elliptic curve. It is known that for each $N > 1$ the Abel-Jacobi map realizes $X^{(N)}$ as a $\mathbb{P}^{N-1}$-bundle over X, see [**12**, Section 1.1]. Hence, by [**39**] there is a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{(N)}) = \langle \underbrace{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X), \ldots, \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)}_{N \text{ times}} \rangle.$$

Combining this for $N = 2$ with (8.5), we obtain a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}([X^2/S_2]) = \left\langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X), \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X), \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \right\rangle. \tag{8.6}$$

Recall that

$$\begin{aligned} \mathrm{K}_0(X) &\xrightarrow{\sim} \quad \mathbb{Z} \oplus \mathrm{Pic}(X) \\ [F] &\mapsto \quad (\mathrm{rk} F, \det F) \end{aligned}$$

is an isomorphism [**28**, Exercise II.6.11]. From this we get that

$$\begin{aligned} \mathrm{K}_0^{\mathrm{num}}(X) &\xrightarrow{\sim} \quad \mathbb{Z} \oplus \mathrm{Pic}(X)/\mathrm{Pic}^0(X) \cong \mathbb{Z} \oplus \mathbb{Z} \\ [F] &\mapsto \quad (\mathrm{rk} F, \deg \det F) \end{aligned}$$

is also an isomorphism. Here $\mathrm{Pic}(X)/\mathrm{Pic}^0(X) = NS(X)$ is the Neron-Severi group of X. Hence, by (8.6)

$$\dim \mathrm{K}_0^{\mathrm{num}}([X^2/S_2], \Bbbk) = 6.$$

On the other hand, the dimension of the degree 2 part of the classical Fock space of X by (2.8) is

$$\dim \mathrm{Sym}^1 \mathrm{K}_0^{\mathrm{num}}(X, \Bbbk) + \dim \mathrm{Sym}^2 \mathrm{K}_0^{\mathrm{num}}(X, \Bbbk) = \binom{2+1-1}{1} + \binom{2+2-1}{2} = 5.$$

Therefore, ϕ cannot be an isomorphism when X is an elliptic curve. By Theorem 8.13 below the same holds for π.

8.3. The Fock space as a quotient

The following constructions are easier to express in a monoidal setting, rather than in the 2-categorical setting we worked in so far. Thus, let $\underline{\mathbf{H}}_{\mathcal{V}}$ be the $\mathbf{Ho}(\mathbf{dgCat}^1)$-monoidal DG 1-category obtained from $\mathbf{H}_{\mathcal{V}}$ by identifying all objects and all 1-morphism categories $\operatorname{Hom}(N, N+n)$ for fixed $n \in \mathbb{Z}$. Concretely, set

$$\underline{\mathbf{H}}_{\mathcal{V}} = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n)$$

with the monoidal structure given by the horizontal composition in $\mathbf{H}_{\mathcal{V}}$ via the identification

$$\operatorname{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n_1) \otimes \operatorname{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n_2) \cong \operatorname{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n_1) \otimes \operatorname{Hom}_{\mathbf{H}_{\mathcal{V}}}(n_1, n_1+n_2) \\ \to \operatorname{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n_1+n_2).$$

Applying the same flattening procedure to $\mathbf{F}_{\mathcal{V}}$, we obtain a DG category

$$\underline{\mathbf{F}}_{\mathcal{V}} = \bigoplus_{n \geq 0} \mathcal{H}perf\, \overline{\mathcal{P}erf}(\mathcal{S}^n \mathcal{V}). \tag{8.7}$$

In $\mathbf{F}_{\mathcal{V}}$ we do not have a Hom-category isomorphism

$$\operatorname{Hom}_{\mathbf{F}_{\mathcal{V}}}(0, n_2) \cong \operatorname{Hom}_{\mathbf{F}_{\mathcal{V}}}(n_1, n_1+n_2).$$

However, there is a natural functor between the two:

$$\operatorname{Hom}_{\mathbf{F}_{\mathcal{V}}}(0, n_2) = \mathcal{H}perf\, \overline{\mathcal{P}erf}(\mathcal{S}^{n_2}\mathcal{V}) \\ \xrightarrow{(\mathcal{S}^{n_1}\mathcal{V} \otimes_k (-))^*} \mathcal{H}perf\big(\mathcal{S}^{n_1}\mathcal{V}\text{-}\overline{\mathcal{M}od}\text{-}(\mathcal{S}^{n_1}\mathcal{V} \otimes_k \mathcal{S}^{n_2}\mathcal{V})\big)_{\mathrm{lfrp}} \\ \xrightarrow{\operatorname{Ind}^{S_{n_1+n_2}}_{S_{n_1}, S_{n_2}}} \mathcal{H}perf\big(\mathcal{S}^{n_1}\mathcal{V}\text{-}\overline{\mathcal{M}od}\text{-}\mathcal{S}^{n_1+n_2}\mathcal{V}\big)_{\mathrm{lfrp}} = \operatorname{Hom}_{\mathbf{F}_{\mathcal{V}}}(n_1, n_1+n_2).$$

Using it, we obtain from the 1-composition of $\mathbf{F}_{\mathcal{V}}$ a monoidal structure on $\underline{\mathbf{F}}_{\mathcal{V}}$ given by

$$\mathcal{H}perf\, \overline{\mathcal{P}erf}(\mathcal{S}^{n_1}\mathcal{V}) \otimes \mathcal{H}perf\, \overline{\mathcal{P}erf}(\mathcal{S}^{n_2}\mathcal{V}) \to \mathcal{H}perf\, \overline{\mathcal{P}erf}(\mathcal{S}^{n_1+n_2}\mathcal{V}),$$

induced by the natural inclusion $\mathcal{S}^{n_1}\mathcal{V} \otimes \mathcal{S}^{n_2}\mathcal{V} \hookrightarrow \mathcal{S}^{n_1+n_2}\mathcal{V}$.

Applying the flattening to the 2-functor $\Phi_{\mathcal{V}} \colon \mathbf{H}_{\mathcal{V}} \to \mathbf{F}_{\mathcal{V}}$ constructed in Chapter 7 we obtain a DG functor $\Phi_{\mathcal{V}} \colon \underline{\mathbf{H}}_{\mathcal{V}} \to \underline{\mathbf{F}}_{\mathcal{V}}$. One can readily check that it is homotopy monoidal with respect to the monoidal structures on $\underline{\mathbf{H}}_{\mathcal{V}}$ and $\underline{\mathbf{F}}_{\mathcal{V}}$ described above.

Next, take the functor $L \colon \mathbf{dgMor}_{\mathrm{lfrp}} \to \mathbf{EnhCat}^{\mathrm{dg}}_{\mathrm{kc}}$ defined in Section 7.5, restrict it to the categories $\mathcal{S}^n\mathcal{V}$ and apply the flattening. We obtain a DG functor

$$\bigoplus_n \mathcal{P}erf\, \mathcal{S}^n\mathcal{V} \xrightarrow{L} \bigoplus_n \overline{\mathcal{P}erf}\mathcal{S}^n\mathcal{V}.$$

Precomposing it with the inclusions $\mathcal{H}perf\, \mathcal{S}^n\mathcal{V} \hookrightarrow \mathcal{P}erf\, \mathcal{S}^n\mathcal{V}$ we obtain the quasi-equivalence

$$\bigoplus_n \mathcal{H}perf\, \mathcal{S}^n\mathcal{V} \xrightarrow{L} \bigoplus_n \overline{\mathcal{P}erf}\mathcal{S}^n\mathcal{V}.$$

Since for any DG category $\mathcal{A}$ the Yoneda embedding $\overline{\mathcal{P}erf}\mathcal{A} \to \mathcal{H}perf\, \overline{\mathcal{P}erf}\mathcal{A}$ is a quasi-equivalence, we further obtain a quasi-equivalence

$$\bigoplus_n \mathcal{H}perf\, \mathcal{S}^n\mathcal{V} \xrightarrow{L} \bigoplus_n \overline{\mathcal{P}erf}\mathcal{S}^n\mathcal{V} \xrightarrow{\text{Yoneda}} \bigoplus_n \mathcal{H}perf\, \overline{\mathcal{P}erf}\mathcal{S}^n\mathcal{V} = \underline{\mathbf{F}}_{\mathcal{V}}. \tag{8.8}$$

Corollary 8.9. *The DG functor*

$$\bigoplus_n \mathcal{H}perf\, \mathcal{S}^n\mathcal{V} \xrightarrow{\Xi^P} \underline{\mathbf{H}}_{\mathcal{V}} \xrightarrow{\Phi_{\mathcal{V}}} \underline{\mathbf{F}}_{\mathcal{V}},$$

filters through the quasi-quivalence (8.8) *as a functor homotopic to*

$$\mathrm{id}_{\bigoplus_n \mathcal{H}perf\, \mathcal{S}^n\mathcal{V}}.$$

Proof. This follows from the proof of Lemma 8.4 and the fact that the composition

$$\mathcal{H}perf\,(\mathcal{S}^n\mathcal{V}) \xrightarrow{(8.1)} \mathcal{DGF}un\,(\mathcal{H}perf(\Bbbk),\, \mathcal{H}perf\,(\mathcal{S}^n\mathcal{V})) \xrightarrow{\mathrm{Apx}} \mathcal{H}perf\,(\mathcal{S}^n\mathcal{V})$$

is the identity functor. □

Remark 8.10. Corollary 8.9 implies, in particular, that the DG-category $\bigoplus \mathcal{H}perf\, \mathcal{S}^n\mathcal{V}$ is a homotopy retract of $\underline{\mathbf{H}}_{\mathcal{V}}$, that is — a retract in the category $\mathbf{Ho}(\mathbf{dgCat}^1)$.

In particular, $\mathcal{H}perf\, \mathcal{V}$ itself is a homotopy retract of $\underline{\mathbf{H}}_{\mathcal{V}}$. Thus, on the level of underlying triangulated categories, we have a faithful embedding $\mathrm{D_c}(\mathcal{V}) \hookrightarrow \mathrm{D_c}(\underline{\mathbf{H}}_{\mathcal{V}})$.

As explained in Section 2.2, the classical Fock space $\underline{F}_{\mathcal{V}}$ is isomorphic to $\underline{H}_{\mathcal{V}}/I$ where I is the left ideal generated by the $q^{(n)}_{[a]}$ for $[a] \in \mathrm{K}_0^{\mathrm{num}}(\mathcal{V}, \Bbbk)$ and $n > 0$. Moreover, as seen in Section 8.2, we have an embedding $\phi\colon \underline{F}_{\mathcal{V}} \hookrightarrow \bigoplus_N \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^N\mathcal{V}, \Bbbk)$.

Motivated by this, we define

$$\widetilde{\mathbf{F}_{\mathcal{V}}} \coloneqq \underline{\mathbf{H}}_{\mathcal{V}}/\mathcal{I},$$

where $\mathcal{I}$ is the left ideal generated by objects Q_a for $a \in \mathcal{V}$.

Lemma 8.11. *The DG category* $\bigoplus \mathcal{H}perf\, \mathcal{S}^n\mathcal{V}$ *is a homotopy retract of* $\widetilde{\mathbf{F}_{\mathcal{V}}}$. *Specifically, the following composition is a homotopy retraction:*

$$\bigoplus \mathcal{H}perf\, \mathcal{S}^n\mathcal{V} \xrightarrow{\Xi^P} \underline{\mathbf{H}}_{\mathcal{V}} \xrightarrow{\mathrm{Drinfeld}} \widetilde{\mathbf{F}_{\mathcal{V}}}. \tag{8.9}$$

Moreover, this composition is quasi-essentially surjective on objects.

Proof. In view of Corollary 8.9 it suffices to prove that the homotopy retraction

$$\Phi_{\mathcal{V}}\colon \underline{\mathbf{H}}_{\mathcal{V}} \to \underline{\mathbf{F}}_{\mathcal{V}}$$

filters in $\mathbf{Ho}(\mathbf{dgCat}^1)$ through the Drinfeld quotient functor

$$\underline{\mathbf{H}}_{\mathcal{V}} \to \underline{\mathbf{H}}_{\mathcal{V}}/\mathcal{I} = \widetilde{\mathbf{F}_{\mathcal{V}}}.$$

By the universal property of Drinfeld quotient (Theorem 4.27), it suffices to prove that $\Phi_{\mathcal{V}}$ sends all objects of $\mathcal{I}$ to null-homotopic ones. By the definition of the monoidal structure on $\underline{\mathbf{H}}_{\mathcal{V}}$, it suffices to check this on objects Q_a for $a \in \mathcal{V}$ which generate $\mathcal{I}$ as a left ideal.

In fact, $\Phi_{\mathcal{V}}$ sends all of these to zero. Indeed, we compute $\Phi_{\mathcal{V}}(\mathsf{Q}_a)$ by evaluating the corresponding 2-functor on 1-morphisms $\mathsf{Q}_a \in \mathrm{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, -1)$. By construction, the 2-functor $\Phi_{\mathcal{V}}$ sends all objects $n \in \mathbb{Z}_{<0}$ to zero, and hence for any $n < 0$ it sends the whole 1-morphism category $\mathrm{Hom}_{\mathbf{H}_{\mathcal{V}}}(0, n)$ to zero.

For the final claim, recall that 1-morphism categories of $\mathbf{H}_{\mathcal{V}}$ are Drinfeld quotients of the perfect hulls of those of $\mathbf{H}'_{\mathcal{V}}$. We then take a further Drinfeld quotient to obtain $\widetilde{\mathbf{F}_{\mathcal{V}}}$. As taking Drinfeld quotient doesn't change the objects of a category, the objects of $\widetilde{\mathbf{F}_{\mathcal{V}}}$ are perfect modules over the 1-morphism categories $\mathrm{Hom}_{\mathbf{H}'_{\mathcal{V}}}(0, n)$. We can therefore view them as homotopy idempotents of twisted complexes over $\mathrm{Hom}_{\mathbf{H}'_{\mathcal{V}}}(0, n)$.

The objects of $\mathrm{Hom}_{\mathbf{H}'_{\mathcal{V}}}(0, n)$ are words on P, Q, and Rs. In $\mathbf{H}_{\mathcal{V}}$, P and R become homotopy equivalent. Furthermore, the homotopy equivalence (6.1) in $\mathbf{H}_{\mathcal{V}}$ allows us to turn any subword QP into a direct sum of PQs and $\mathbb{1}$s. Since any word ending in Q is null-homotopic in $\widetilde{\mathbf{F}_{\mathcal{V}}}$, we conclude that all objects of $\mathrm{Hom}_{\mathbf{H}'_{\mathcal{V}}}(0, n)$ are homotopy equivalent in $\widetilde{\mathbf{F}_{\mathcal{V}}}$ to direct sums of words on just Ps.

It remains to show that any morphism between words on Ps in $\mathrm{Hom}_{\mathbf{H}'_{\mathcal{V}}}(0, n)$ becomes homotopic in $\widetilde{\mathbf{F}_{\mathcal{V}}}$ to something that lies in the image of Ξ^P. In other words, homotopic to a diagram containing just the crossings. Since there are no Qs involved, we only need to show that we can get rid of curls and of bubbles. The relations (5.11) imply that counterlclockwise curls are homotopic to zero, while counterclockwise bubbles are homotopic to scalar multiples of identity maps.

Suppose we have a clockwise bubble. If there is no vertical string to the right of it, the diagram can be written as a 2-composition filtering through a word ending in Q and hence vanishes. If there is a vertical string to the right of the bubble, we use the homotopy relations in Lemma 5.7 to replace a downward string in the bubble and the (upward) vertical string by a cup and a cap plus a double crossing. The replacement by a cup and a cap absorbs the bubble into the vertical string and gets rid of it. The replacement by a crossing makes the vertical string cross the bubble. We then use the symmetric group relations on upward strands to move the bubble completely to right of the vertical string. If there are any more vertical strings to the right of the bubble, we repeat this procedure.

A similar argument works for clockwise curls. If there are no vertical strings to the right of it, the diagram passes through a word ending in Q and hence vanishes. If there are, we can similarly move the curl to the right of string: the replacement by a cup and a cap turns the curl into a crossing and gets rid of it, while the replacement by a double crossing makes the vertical string cross the curl, and we can then use a triple move to finish moving the curl completely to the right of the vertical string. $\square$

We have shown above that $\Phi_{\mathcal{V}}$ filters through the Drinfeld quotient $\underline{\mathbf{H}}_{\mathcal{V}} \to \widetilde{\mathbf{F}_{\mathcal{V}}}$. Let

$$\tilde{\Phi}_{\mathcal{V}} \colon \widetilde{\mathbf{F}_{\mathcal{V}}} \to \underline{\mathbf{F}}_{\mathcal{V}}$$

be the corresponding quasi-functor. On the other hand, let the quasi-functor

$$\tilde{\Xi}^P \colon \underline{\mathbf{F}}_{\mathcal{V}} \to \widetilde{\mathbf{F}_{\mathcal{V}}}$$

be the composition of (8.9) with the formal inverse of the quasi-equivalence (8.8). By Corollary 8.9, $\tilde{\Phi}_{\mathcal{V}}$ is a homotopy left inverse of $\tilde{\Xi}^P$.

Define the numerical Grothendieck groups $\mathrm{K}_0^{\mathrm{num}}(\underline{\mathbf{H}}_{\mathcal{V}}, \Bbbk)$ and $\mathrm{K}_0^{\mathrm{num}}(\widetilde{\mathbf{F}_{\mathcal{V}}}, \Bbbk)$ similarly to the definition of $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ in Section 6.4. Namely, they are the quotients of $\mathrm{K}_0(\underline{\mathbf{H}}_{\mathcal{V}}, \Bbbk)$ and $\mathrm{K}_0(\widetilde{\mathbf{F}_{\mathcal{V}}}, \Bbbk)$ under the images of the kernel of the Euler form on $\bigoplus_n \mathrm{K}_0(\mathcal{S}^n\mathcal{V}, \Bbbk)$ under Ξ_P and (8.9), respectively. Then $\mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ is the idempotent modification of $\mathrm{K}_0^{\mathrm{num}}(\underline{\mathbf{H}}_{\mathcal{V}}, \Bbbk)$.

By Lemma 8.11, the map of K-groups induced by $\tilde{\Xi}^P$ is injective. By our definitions, it descends to an injective map of numerical K-groups, and so does any left inverse of it. We thus obtain:

COROLLARY 8.12. *The following composition is the identity map:*

$$\mathrm{K}_0^{\mathrm{num}}(\underline{\mathbf{F}}_{\mathcal{V}}, \Bbbk) \xrightarrow{\tilde{\Xi}^P} \mathrm{K}_0^{\mathrm{num}}(\widetilde{\mathbf{F}_{\mathcal{V}}}, \Bbbk) \xrightarrow{\tilde{\Phi}_{\mathcal{V}}} \mathrm{K}_0^{\mathrm{num}}(\underline{\mathbf{F}}_{\mathcal{V}}, \Bbbk).$$

Corollary 8.12 together with the morphism ϕ of (8.4) gives an embedding of the classical Fock space into the numerical Grothendieck group of the category $\widetilde{\mathbf{F}_{\mathcal{V}}}$:

$$\underline{F}_{\mathcal{V}} = \bigoplus_n \underline{F}_{\mathcal{V}}^n \xrightarrow{\phi} \bigoplus_n \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^n\mathcal{V}, \Bbbk) \cong \mathrm{K}_0^{\mathrm{num}}(\underline{\mathbf{F}}_{\mathcal{V}}, \Bbbk) \xrightarrow{\tilde{\Xi}^P} \mathrm{K}_0^{\mathrm{num}}(\widetilde{\mathbf{F}_{\mathcal{V}}}, \Bbbk)$$

where

$$\bigoplus_n \underline{F}_{\mathcal{V}}^n \cong \bigoplus_{n\geq 0} \bigoplus_{\lambda\dashv n} \bigotimes_i \mathrm{Sym}^{r(\lambda)_i} \mathrm{K}_0^{\mathrm{num}}(\mathcal{V}, \Bbbk)$$

and $r(\lambda)_i$ is the number of parts of size i in λ.

We now prove a converse to Conjecture 6.21.

THEOREM 8.13. *If $\pi\colon H_{\mathcal{V}} \to \mathrm{K}_0^{\mathrm{num}}(\mathbf{H}_{\mathcal{V}}, \Bbbk)$ is an isomorphism, then so are ϕ and $\tilde{\Xi}^P$:*

$$\bigoplus_{n\geq 0} \bigoplus_{\lambda\dashv n} \bigotimes_i \mathrm{Sym}^{r(\lambda)_i} \mathrm{K}_0^{\mathrm{num}}(\mathcal{V}, \Bbbk) \cong \bigoplus_{n\geq 0} \mathrm{K}_0^{\mathrm{num}}(\mathcal{S}^n\mathcal{V}, \Bbbk) \cong \mathrm{K}_0^{\mathrm{num}}(\widetilde{\mathbf{F}_{\mathcal{V}}}, \Bbbk).$$

PROOF. Let I be the left ideal of $\underline{H}_{\mathcal{V}}$ generated by $q_{[a]}^{(n)}$ with $n > 0$ and $a \in \mathcal{V}$. We have

$$\begin{array}{ccc} \underline{H}_{\mathcal{V}} & \xrightarrow{\text{quotient by } I} & \underline{F}_{\mathcal{V}} \\ {\scriptstyle \underline{\pi}}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathrm{K}_0^{\mathrm{num}}(\underline{\mathbf{H}}_{\mathcal{V}}, \Bbbk) & \xrightarrow{\Phi_{\mathcal{V}}} & \mathrm{K}_0^{\mathrm{num}}(\underline{\mathbf{F}}_{\mathcal{V}}, \Bbbk) \\ & \searrow^{\text{Drinfeld}} & \uparrow{\scriptstyle \tilde{\Phi}_{\mathcal{V}}} \\ & & \mathrm{K}_0^{\mathrm{num}}(\widetilde{\mathbf{F}_{\mathcal{V}}}, \Bbbk). \end{array} \tag{8.10}$$

The Drinfeld quotient induces a surjective map of the K-groups by [**25**, Proposition VIII.3.1]. By our definitions, this descends to the surjective map of the numerical K-groups in (8.10).

By assumption of the Theorem, the map $\underline{\pi}$ is an isomorphism. By (8.10), the injective map ϕ is then surjective, and thus an isomorphism. Now observe that the Drinfeld quotient map kills $\underline{\pi}(I)$. The surjective map $\tilde{\Phi}$ is therefore injective and thus an isomorphism. Its right inverse $\tilde{\Xi}^P$ is then also an isomorphism. □

8.4. Reconstruction of the base category

It is natural to ask to what extent we can recover the base category $\mathcal{V}$ from its Heisenberg category $\mathbf{H}_{\mathcal{V}}$. Given the nature of our construction, the best we can hope for is to recover $\mathcal{V}$ up to Morita equivalence. This recovers $\mathcal{H}perf\,\mathcal{V}$, that is – the compact derived category $\mathrm{D}_c(\mathcal{V})$.

We are not allowed to use our categorical Fock space $\mathbf{F}_{\mathcal{V}}$ in this reconstruction as it is not built from $\mathbf{H}_{\mathcal{V}}$, but directly from $\mathcal{V}$. In particular, $\mathbf{F}_{\mathcal{V}}$ contains $\mathcal{H}perf\,\mathcal{V}$ as the 1-morphism category $\mathrm{Hom}_{\mathbf{F}_{\mathcal{V}}}(0, 1)$. However, this gives us our strategy: we obtain our categorical Fock space quotient $\widetilde{\mathbf{F}_{\mathcal{V}}}$ intrinsically from $\mathbf{H}_{\mathcal{V}}$ together with $\mathbb{Z}$-grading which remembers the flattening

$$\underline{\mathbf{H}}_{\mathcal{V}} = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{H}_{\mathcal{V}}}(0,n).$$

If we could show that the natural functor of Lemma 8.11

$$\bigoplus \mathcal{H}perf\, \mathcal{S}^n\mathcal{V} \xrightarrow{(8.9)} \widetilde{\mathbf{F}_{\mathcal{V}}}.$$

is a quasi-equivalence, we could recover $\mathcal{H}perf\, \mathcal{V}$ as 1-graded part $\widetilde{\mathbf{F}^1_{\mathcal{V}}}$ of $\widetilde{\mathbf{F}_{\mathcal{V}}}$. In Lemma 8.11 we come tantalisingly close: we show (8.9) to be quasi-faithful and quasi-essentially surjective on objects. In fact, in the proof of Lemma 8.11 we show that it is also quasi-full on those morphisms in $\widetilde{\mathbf{F}_{\mathcal{V}}}$ which come from the perfect hull of $\mathbf{H}'_{\mathcal{V}}$. The only morphisms we can't get so far are those added by taking the two Drinfeld quotients – the first one to get $\mathbf{H}_{\mathcal{V}}$ and the second one to get $\widetilde{\mathbf{F}_{\mathcal{V}}}$.

We conjecture that one can get even these and thus (8.9) is a quasi-equivalence. This would allow one to recover $\mathcal{H}perf\, \mathcal{V}$ as $\widetilde{\mathbf{F}^1_{\mathcal{V}}}$. For the moment, however, we only have:

LEMMA 8.14. *Let $\mathcal{V}$ and $\mathcal{W}$ be smooth and proper DG categories and assume that there is a quasi-equivalence of* $\mathbf{Ho}(\mathbf{dgCat})$*-enriched bicategories which is the identity on objects:*

$$\mathbf{H}_{\mathcal{V}} \cong \mathbf{H}_{\mathcal{W}}.$$

Then:

(1) *There is a $\mathbb{Z}$-graded quasi-equivalence*

$$\widetilde{\mathbf{F}_{\mathcal{V}}} \cong \widetilde{\mathbf{F}_{\mathcal{W}}}.$$

(2) *There are quasi-faithful quasi-essentially surjective functors*

$$\mathcal{H}perf\, \mathcal{V} \to \widetilde{\mathbf{F}^1_{\mathcal{V}}} \cong \widetilde{\mathbf{F}^1_{\mathcal{W}}} \leftarrow \mathcal{H}perf\, \mathcal{W}.$$

PROOF. For the first claim, recall that we construct the categorical Fock space quotient $\widetilde{\mathbf{F}_{\mathcal{V}}}$ as the Drinfeld quotient of $\mathbf{H}_{\mathcal{V}}$ by the left ideal I generated by objects Q_a for $a \in \mathcal{V}$. We can equivalently take I to be the left ideal generated by all 1-morphisms in $\operatorname{Hom}_{\mathbf{H}_{\mathcal{V}}}(n, n-k)$ for $k > 0$. Since the quasi-equivalence $\mathbf{H}_{\mathcal{V}} \cong \mathbf{H}_{\mathcal{W}}$ is identity on the objects $n \in \mathbb{Z}$ it preserves this ideal and hence descends to a quasi-equivalence $\widetilde{\mathbf{F}_{\mathcal{V}}} \cong \widetilde{\mathbf{F}_{\mathcal{W}}}$.

The second claim follows directly from Lemma 8.11. □

This is enough to show that the Heisenberg categories of $\mathcal{I}(\mathbb{P}^1)$ and $\mathcal{I}(\mathrm{pt} \sqcup \mathrm{pt})$ are distinct:

EXAMPLE 8.15. For the categories $\mathcal{I}(\mathbb{P}^1)$ and $\mathcal{I}(\mathrm{pt} \sqcup \mathrm{pt})$ of Example 4.48 our Lemma 8.14 is still enough to see that

$$\mathbf{H}_{\mathcal{I}(\mathbb{P}^1)} \ncong \mathbf{H}_{\mathcal{I}(\mathrm{pt}\sqcup\mathrm{pt})}.$$

The decomposition $\mathcal{I}(\mathrm{pt} \sqcup \mathrm{pt}) = \mathcal{I}(\mathrm{pt}) \oplus \mathcal{I}(\mathrm{pt})$ induces a decomposition

$$\operatorname{Hom}_{\mathbf{H}'_{\mathcal{I}(\mathrm{pt}\sqcup\mathrm{pt})}}(0,1) = \operatorname{Hom}_{\mathbf{H}'_{\mathcal{I}(\mathrm{pt})}}(0,1) \oplus \operatorname{Hom}_{\mathbf{H}'_{\mathcal{I}(\mathrm{pt})}}(0,1)$$

as follows. The objects of the Hom-space on the LHS consists of words with one more P than Q. There is no morphism between two such words if the difference of P's and Q's indexed by *one* of the two generating objects is positive in one of the words but nonpositive in the other word. This decomposition then induces an orthogonal decomposition

$$\widetilde{\mathbf{F}^1_{\mathcal{I}(\mathrm{pt})}} \oplus \widetilde{\mathbf{F}^1_{\mathcal{I}(\mathrm{pt})}}.$$

By Lemma 8.14 (2), there exists a faithful and essentially surjective functor

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^1) = \mathrm{H}^0(\mathcal{I}(\mathbb{P}^1)) \to \mathrm{H}^0(\widetilde{\mathbf{F}_{\mathcal{I}(\mathbb{P}^1)}}).$$

Therefore, if $\mathrm{H}^0(\widetilde{\mathbf{F}_{\mathcal{I}(\mathbb{P}^1)}})$ had an orthogonal decomposition, so would have $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^1)$. But this would imply that $\mathbb{P}^1$ is disconnected.

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Index