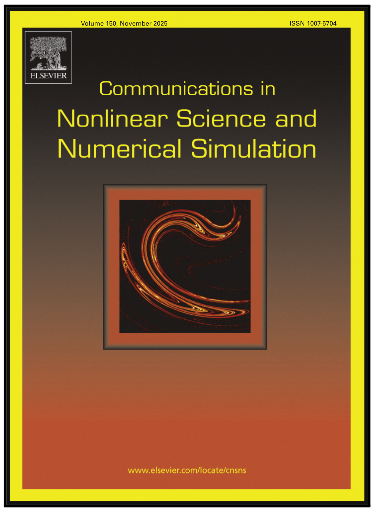

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Highlights

We revised the manuscript “Singular properties of high-order spectral densities of supOU processes” according to all comments of the reviewer and prepared the detailed response letter.

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Singular properties of high-order spectral densities of supOU processes

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Abstract

This paper investigates the properties of high-order spectral densities and high-order cumulants of superpositions of Ornstein-Uhlenbeck (supOU) processes. As is well known, the long-range dependence of a stochastic process is typically characterized by the second-order spectral density or correlation function. We introduce the property of high-order long-range dependence using high-order spectral densities and cumulants. Sufficient conditions are derived for the supOU process to exhibit high-order long-range dependence. A numerical study of supOU processes with Dickman and gamma marginals is presented to illustrate the theoretical findings.

Keywords: supOU process, spectral density, long-range dependence, bispectrum, high-order spectrum

1. Introduction

Superposition of Ornstein-Uhlenbeck type processes (supOU process) is a strictly stationary process that was proposed in [3] as a generalization of the Ornstein-Uhlenbeck (OU) process. The supOU process has a self-decomposable stationary distribution and a flexible covariance structure, including (second-order) long-range dependence; see [3, 4, 20] and references therein.

The supOU processes have been used in various applications, mainly in financial econometrics [5, 6, 17], but also in other areas, such as astrophysics [26]. The limit theorems for supOU processes were studied in [20], where it was also established that integrated supOU processes might exhibit an unusual limiting behaviour of moments and cumulants called intermittency.

High-order spectral analysis has been proposed for studying non-linearity and non-Gaussian phenomena, see the pioneering work [10] by Blanc-Lapierre and Fortet. Further developments were made by Kolmogorov, Brillinger and Shiryaev, see [27, 12, 41]. High-order spectral analysis has been applied in boundary layer dynamics [21], acoustics [14], seismic analysis [23], turbulence, economics [9], testing for Gaussianity and linearity [22, 36, 45, 42],

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deconvolution and phase reconstruction for non-Gaussian linear processes [30], refined parameter estimation [13], and signal processing [11]. It was also used in [29] for proving a CLT-type theorem and in [1] for quasi-likelihood estimation of random fields. We refer to [33] and [39] for a review of possible applications of the higher-order spectral analysis in signal processing and communication systems.

Let us recall that the superposition of Ornstein-Uhlenbeck processes (supOU process) can be defined as an integral with respect to a homogeneous infinitely divisible independently scattered random measure. Specifically, we denote the set of positive real numbers as $\mathbb{R}_+ = (0, \infty)$, the set of all Borel subsets of $\mathbb{R}$ as $\mathcal{B}(\mathbb{R})$, the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ as $Leb(\cdot)$, a random measure on $\mathbb{R}_+ \times \mathbb{R}$ as Λ. We assume that

(1) there exists a Lévy-Khintchine triplet (b, σ^2, μ) of some infinitely divisible distribution such that

$$\int_{|x|>1} \log(1 + |x|)\, \mu(dx) < \infty, \tag{1}$$

and let $\kappa(z)$ denote the corresponding Lévy exponent given by Lévy-Khintchine formula,

$$\kappa(z) = izb - \frac{1}{2}z^2\sigma^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbb{1}_{[-1,1]}(x))\mu(dx),$$

(2) there exists a measure ν on $\mathbb{R}_+$ such that

$$\eta^{-1} := \int_{\mathbb{R}_+} \frac{\nu(d\xi)}{\xi} < \infty, \tag{2}$$

and the equality

$$\kappa_{\Lambda(A)}(z) = \log \mathbb{E} e^{iz\Lambda(A)} = (\nu \times Leb)(A) \cdot \kappa(z) \tag{3}$$

holds for any $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $(\nu \times Leb)(A) < \infty$.

The measure Λ on $\mathbb{R}_+ \times \mathbb{R}$ is then called an infinitely divisible independently scattered random measure and (b, σ^2, μ, ν) is called the generating quadruple of Λ. Moreover, the strictly stationary stochastic process $X(t)$, $t \in \mathbb{R}$, given by

$$X(t) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} e^{-\xi(t-s)} \mathbb{1}_{[0,\infty)}(t-s) \Lambda(d\xi, \eta ds), \tag{4}$$

is well-defined and referred to as a supOU process, see [3, 4, 5, 17, 40]. Note that the generating quadruple (b, σ^2, μ, ν) of the measure Λ completely determines the finite-dimensional distributions of the supOU process $X(t)$.

The notion of long-range dependence has emerged in relation to data patterns observed in the fields of economics, telecommunications, and hydrology. Its origins can be traced back to empirical observations by Hurst, who studied the flow of the Nile River. Several models were subsequently proposed to explain this phenomenon, one of which involved stationary processes with slowly decaying correlation functions [37], see also [38]. As a result, long-range dependence is most commonly — though not exclusively — defined through the second-order characteristics of a process: the non-integrability of the correlation function or, equivalently, the unboundedness of the spectral density at zero. We refer to [7, 8] for an extensive study of the theory of second-order long-range dependence and its statistical implications. This definition is supported primarily by the limiting behaviour of the process. For example, it has been shown that for stationary Gaussian processes, continuity of the spectral density is a sufficient condition for the Central Limit Theorem (CLT) to hold [28], making these processes, in some sense, similar to models with independent increments. In contrast, for models with unbounded spectral density, non-central limit theorems frequently arise [35]. This dichotomy reinforces the importance of second-order characteristics in analysing the behaviour of stochastic processes. However, for non-Gaussian processes (such as supOU processes), the dependence structure cannot be fully described by second-order characteristics alone. In such cases, the consideration of higher-order cumulants or higher-order spectral densities is required to refine the characterization of their dependence structure, nonlinearity and phase relationships.

The third- and fourth-order spectral densities are particularly useful for practical interpretations. Specifically, the third-order spectral density shows asymmetry and nonlinear interactions such as quadratic phase coupling between two frequencies [33]. In addition, the departure of the third-order spectral density from zero indicates non-Gaussianity [33, 39]. The third-order spectral density shows cubic interactions among frequency components and is linked to heavy tails and clustering of extremes, relevant in finance and turbulence [33, 38].

In the present paper, using an analogy to how the singularity of the second-order spectral density characterizes the long-memory property of a stationary process, we investigate the conditions under which a non-Gaussian supOU process exhibits either the singularity of its high-order spectral density or the non-integrability of its high-order cumulant function. In particular, we propose to define the property of high-order long-range dependence in terms of high-order spectral densities and high-order cumulants. Similar problems for non-Gaussian stochastic processes with linear structures and the superpositions of diffusions with linear generators have been studied in [24, 43, 44], see also [2] for high-order spectral analysis of Burgers and KPZ turbulence.

The paper is structured as follows. In Section 2, we recall the definitions of the cumulants and spectral densities of a stochastic process. We also derive expressions for the high-order cumulants and spectral density of non-Gaussian supOU processes. In Section 3, we first review the known characterisation of (second-order) long-range dependence and then define third- and high-order long-range dependence and establish sufficient conditions for a supOU process to exhibit this property. The main results are presented in Theorems 3.5, 3.7, 3.10, and 3.14. In Section 4, we illustrate the theoretical results of Section 3 by numerical simulations of Dickman and gamma supOU processes.

2. Preliminaries

Let $X(t)$, $t \in \mathbb{R}$, be the stochastic process. If $\mathbb{E}|X(t)|^m < \infty$ for any $t \in \mathbb{R}$ we can define the mth-order cumulant of a process $X(t)$ as a function $c_m : \mathbb{R}^m \to \mathbb{R}$ such that

$$\begin{aligned} c_m(t, \tau_1, \ldots, \tau_{m-1}) &= \frac{1}{i^m} \left(\frac{\partial^m}{\partial z_1 \ldots \partial z_m} \kappa_{X(t), X(t+\tau_1), \ldots, X(t+\tau_{m-1})}(z_1, \ldots, z_m) \right) \Bigg|_{z_1 = \ldots = z_m = 0} \\ &= \frac{1}{i^m} \left(\frac{\partial^m}{\partial z_1 \ldots \partial z_m} \log \mathbb{E} e^{i(z_1 X(t) + z_2 X(t+\tau_1) \ldots + z_m X(t+\tau_{m-1}))} \right) \Bigg|_{z_1 = \ldots = z_m = 0}. \end{aligned} \tag{5}$$

Note that in the case of a strictly stationary process $X(t)$ the cumulant $c_m(t, \tau_1, \ldots, \tau_{m-1})$ does not depend on t. Thus, for any stationary process, the cumulant $c_m(\cdot)$ can be defined as a function operating on $\mathbb{R}^{m-1}$ and we further denote the mth-order cumulant of a stationary process by $c_m(\tau_1, \ldots, \tau_{m-1})$ instead of $c_m(t, \tau_1, \ldots, \tau_{m-1})$.

Moreover, assume that there exists a complex-valued function $f_m(\omega_1, \ldots, \omega_{m-1})$ such that for any $\tau_1, \ldots, \tau_{m-1} \in \mathbb{R}$

$$c_m(\tau_1, \ldots, \tau_{m-1}) = \int\limits_{\mathbb{R}^{m-1}} \exp\left\{ i \sum_{j=1}^{m-1} \omega_j \tau_j \right\} f_m(\omega_1, \ldots, \omega_{m-1}) d\omega_1 \ldots d\omega_{m-1},$$

i.e. the cumulant $c_m(\cdot)$ is the Fourier transform of $f_m(\cdot)$. We call the function $f_m(\omega_1, \ldots, \omega_{m-1})$ the mth-order spectral density of the process $X(t)$, $t \in \mathbb{R}$.

Let further $X(t)$, $t \in \mathbb{R}$, be a supOU process defined in (4). Recall that in the case of a supOU process we can give an easy condition to make sure that $\mathbb{E}|X(t)|^m < \infty$ holds.

Condition 1. *Assume that for some $m \in \mathbb{N}$ the Lévy measure μ satisfies*

$$\int_{|x|>1} |x|^m \, \mu(dx) < \infty.$$

Condition 1 implies that $\frac{\partial^m}{\partial z^m}\kappa(0) < \infty$ and $\mathbb{E}|X(t)|^m < \infty$ (see [6]) and whenever it holds with some $m \geq 2$ we denote

$$\varkappa_m = \frac{1}{i^m}\frac{\partial^m}{\partial z^m}\kappa(z)\bigg|_{z=0}.$$

Remark 2.1. Note that the assumption (1) is weaker than Condition 1 and is sufficient for a supOU process to be well-defined and stationary. However, we need to impose this stronger condition to ensure that the mth moment of $X(t)$ exists. Note also that the case where $\varkappa_m = 0$ for $m \geq 3$ corresponds to the case of Gaussian processes and will not be of our interest.

Recall now that for any supOU process $X(t)$ the cumulant generating function of the vector $(X(t_1), \ldots, X(t_m))$ is given by

$$\kappa_{X(t_1),\ldots,X(t_m)}(z_1, \ldots, z_m) = \int_0^\infty \int_{\mathbb{R}} \eta\kappa\left(\sum_{j=1}^m z_j e^{-\xi(t_j - s)} \mathbb{1}_{[0,\infty)}(t_j - s)\right) \nu(d\xi) ds, \tag{6}$$

where $t_1, \ldots, t_m \in \mathbb{R}$, $\kappa(\cdot)$ and $\nu(\cdot)$ are as in (3), see [3] for details. It is easy to show that the correlation function of the supOU process $X(t)$, if it is square-integrable, is given by

$$r(\tau) = \operatorname{corr}(X(t), X(t+\tau)) = \eta \int_0^\infty e^{-|\tau|\xi} \frac{\nu(d\xi)}{\xi}, \ \tau \in \mathbb{R}. \tag{7}$$

Having the expression for the correlation function of the supOU process we can easily obtain the expression for the mth-order cumulant, assuming it exists, presented in the following theorem.

Theorem 2.2. *Let $X(t)$, $t \in \mathbb{R}$, be a supOU process defined in* (4) *and assume that Condition 1 holds for some $m \geq 2$. Then the mth-order cumulant of $X(t)$ can be given by*

$$c_m(\tau_1, \ldots, \tau_{m-1}) = \frac{\varkappa_m}{m} r\left(\sum_{j=1}^{m-1} \tau_j - m \cdot \min\{\tau_1, \ldots, \tau_{m-1}, 0\}\right), \ \tau_1, \ldots, \tau_{m-1} \in \mathbb{R}, \tag{8}$$

where $r(\cdot)$ is the correlation function given by (7).

Proof. Let $\tau_0 = 0$. Using (5) and (6), for any $\tau_1, \ldots, \tau_{m-1} \in \mathbb{R}$ we obtain

$$
\begin{aligned}
c_m(\tau_1, \ldots, \tau_{m-1}) &= \frac{1}{i^m} \left(\frac{\partial^m}{\partial z_1 \ldots \partial z_m} \kappa_{X(t+\tau_0), X(t+\tau_1), \ldots, X(t+\tau_{m-1})}(z_1, \ldots, z_m) \right) \Bigg|_{z_1 = \ldots = z_m = 0} \\
&= \frac{1}{i^m} \int_0^\infty \int_{\mathbb{R}} \eta \prod_{j=0}^{m-1} \left(e^{-\xi(t+\tau_j - s)} \mathbb{1}_{[0,\infty)}(t + \tau_j - s) \right) \frac{\partial^m}{\partial z^m} \kappa(z) \Big|_{z=0} \nu(d\xi) ds \\
&= \frac{1}{i^m} \int_0^\infty \int_{\mathbb{R}} \eta \prod_{j=0}^{m-1} \left(e^{-\xi(\tau_j - s)} \mathbb{1}_{[0,\infty)}(\tau_j - s) \right) \frac{\partial^m}{\partial z^m} \kappa(z) \Big|_{z=0} \nu(d\xi) ds \\
&= \varkappa_m \int_0^\infty \int_{\mathbb{R}} \eta e^{-\xi \sum_{j=0}^{m-1} (\tau_j - s)} \mathbb{1}_{[0,\infty)}(\min\{\tau_0, \ldots, \tau_{m-1}\} - s) \nu(d\xi) ds \\
&= \varkappa_m \int_0^\infty e^{-\xi \sum_{j=0}^{m-1} \tau_j} \left(\int_{-\infty}^{\min\{\tau_0, \ldots, \tau_{m-1}\}} \eta e^{m\xi s} ds \right) \nu(d\xi) \\
&= \frac{\varkappa_m}{m} \int_0^\infty \eta e^{-\xi \left(\sum_{j=0}^{m-1} \tau_j - m \cdot \min\{\tau_0, \ldots, \tau_{m-1}\} \right)} \frac{\nu(d\xi)}{\xi} \\
&= \frac{\varkappa_m}{m} r \left(\sum_{j=0}^{m-1} \tau_j - m \cdot \min\{\tau_0, \ldots, \tau_{m-1}\} \right),
\end{aligned}
$$

that completes the proof. □

Remark 2.3. Formula (8) can also be obtained from [1, Theorem 2.1].

If we assume that Condition 1 holds for some $m \geq 2$, then the mth-order spectral density of the supOU process (4) with generating quadruple (b, σ^2, μ, ν) exists and is of the form

$$
f_m(\omega_1, \ldots, \omega_{m-1}) = \int_0^\infty f_{m,\xi}(\omega_1, \ldots, \omega_{m-1}) \frac{\nu(d\xi)}{\xi}, \tag{9}
$$

see [24], with $f_{m,\xi}(\omega_1, \ldots, \omega_{m-1})$ being the mth-order spectral density of the supOU process (4) with generating quadruple $(a, \sigma^2, \mu, \delta_\xi)$, which can be given by

$$
f_{m,\xi}(\omega_1, \ldots, \omega_{m-1}) = \varrho_m \frac{\xi}{(\xi + i\omega_1) \ldots (\xi + i\omega_{m-1})(\xi - i(\omega_1 \ldots + \omega_{m-1}))}, \tag{10}
$$

where $\varrho_m = \frac{\eta \varkappa_m}{(2\pi)^{m-1}}$, $m \geq 2$ and δ_ξ is the Dirac measure concentrated at the point ξ, see Example 3 in [12].

3. Long-range dependence

In this section, we investigate the behaviour of the mth-order spectral density of the supOU process $X(t)$ individually for each m. The case $m = 2$ is discussed in Section 3.1, $m = 3$ in Section 3.2, and $m \geq 4$ in Section 3.3.

3.1. Second-order long-range dependence

Let us first recall some known results about the second-order spectrum as well as the definition of the long-range dependence via second-order characteristics. For more detailed discussions of different notions and interpretations of long-range dependence we refer to [37]. The second-order long-range dependence of the supOU is studied in [4].

Consider the supOU process $X(t)$ as given in (4). In this subsection, we assume that Condition 1 holds with $m = 2$ and that $\varkappa_2 \neq 0$. From the formulas (9) and (10) with $m = 2$ we obtain that the second-order spectral density of the supOU process is given by

$$f_2(\omega) = \varrho_2 \int_0^\infty \frac{\nu(d\xi)}{\xi^2 + \omega^2},$$

while its correlation function can be represented as

$$r(\tau) = \eta \int_0^\infty e^{-|\tau|\xi} \frac{\nu(d\xi)}{\xi} = \int_{\mathbb{R}} \left(\int_0^\infty e^{i\omega\tau} \frac{\eta\varkappa_2}{2\pi(\xi^2 + \omega^2)} \nu(d\xi) \right) d\omega.$$

Let us formulate an additional condition on the measure ν.

Condition 2. *Assume that*

(i) the measure ν has a density $q(\xi)$, $\xi > 0$, i.e. $\nu(d\xi) = q(\xi)d\xi$,

(ii) there exist $\alpha > 0$ and a slowly varying at zero function $\ell(\cdot)$ (see [19, p.445]) such that

$$q(\xi) = \xi^\alpha \ell(\xi), \ \xi > 0.$$

Remark 3.1. Condition 2 characterises the behaviour of the measure ν near zero and as it will be shown later, plays a crucial role in the behaviour of the cumulant functions at infinity and the spectral densities near the origin. Consequently, this condition will be repeatedly referenced throughout this work.

Since the correlation function (7) is given as the Laplace transform of the measure $\frac{\nu(d\xi)}{\xi}$ we can apply the Tauberian-Abelian theorem (see e.g. [18]), to obtain that the property given by Condition 2 is equivalent to

$$r(\tau) = \eta\Gamma(\alpha) \frac{\ell(1/|\tau|)}{|\tau|^\alpha}, \ \tau \to \infty. \tag{11}$$

The property (11) means that $r(\tau)$ is $(-\alpha)$-regularly varying function as $\tau \to \infty$. In turn, if $\alpha \in (0, 1)$ we can deduce from (11) that the correlation function is not integrable as $\tau \to \infty$, i.e.

$$\int_{\mathbb{R}} r(\tau) d\tau = \infty, \tag{12}$$

and that the second-order spectral density has an integrable singularity at zero, namely

$$f_2(\omega) = \frac{\eta}{2\cos(\alpha\pi/2)} \frac{\ell(\omega)}{|\omega|^{1-\alpha}}, \ \omega \to 0. \tag{13}$$

Properties (12) and (13) are commonly used as the definition of second-order long-range dependence.

Example 3.2. Let ν be the measure with *Gamma*$(\alpha, 1 + \alpha)$ density

$$q(\xi) = \frac{\alpha^{\alpha+1}}{\Gamma(\alpha + 1)} \xi^\alpha e^{-\alpha\xi}, \ \xi \geq 0.$$

In this case $\eta = 1$ and

$$r(\tau) = \frac{1}{(1 + |\tau|/\alpha)^\alpha}, \ \tau \in \mathbb{R}.$$

For $0 < \alpha < 1$ it holds that

$$f_2(\omega) = \frac{\alpha^\alpha}{2\cos(\alpha\pi/2)\Gamma(\alpha)} \cdot \frac{1}{|\omega|^{1-\alpha}}(1 + o(1)),\ \omega \to 0,$$

whereas for $\alpha > 1$ we have

$$f_2(\omega) = \frac{\alpha}{\pi(\alpha-1)}(1 + o(1)),\ \omega \to 0,$$

see [16]. We will use the *Gamma* measure in our simulation study in Section 4.

Other examples of the measure ν can be found in [4].

3.2. Third-order long-range dependence

In this section, drawing an analogy from the definition of second-order long-range dependence, we define the property of third-order long-range dependence. We also provide sufficient conditions for the supOU process to exhibit this property.

Let us further denote $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{m-1}) \in \mathbb{R}^{m-1}$ and $\boldsymbol{\tau} = (\tau_1, \dots, \tau_{m-1}) \in \mathbb{R}^{m-1}$ for $m \geq 2$. Let $\|\mathbf{x}\| = \sqrt{x_1^2 + \ldots + x_{m-1}^2}$ be the Euclidean norm of $\mathbf{x} = (x_1, \dots, x_{m-1}) \in \mathbb{R}^{m-1}$ and let $S^{m-2} = \{\mathbf{x} \in \mathbb{R}^{m-1} : \|\mathbf{x}\| = 1\}$ denote a unit sphere in $\mathbb{R}^{m-1}$.

Definition 3.3. We will say that a strictly stationary process $X(t)$, $t \in \mathbb{R}$, such that $\mathbb{E}|X(t)|^3 < \infty$ possesses the property of third-order long-range dependence if either of the two following conditions holds.

(a) The third-order spectral density $f_3(\boldsymbol{\omega})$ of $X(t)$ exists and there exists a slowly varying at 0 function $\ell(\cdot)$ and $H > 0$ such that

$$\lim_{\|\boldsymbol{\omega}\|\to 0} \frac{f_3(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|^{-H}\ell(\|\boldsymbol{\omega}\|)} C\left(\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}\right) = 1, \tag{14}$$

where $C : S^1 \to \mathbb{C}$ is some bounded and bounded away from zero function.

(b) The third-order cumulant of $X(t)$ satisfies

$$\iint_{\mathbb{R}^2} |c_3(\tau_1, \tau_2)| d\tau_1 d\tau_2 = \infty.$$

Remark 3.4. Note that the equivalence of (11) and (13) for the second-order characteristics follows from the Tauberian-Abelian theorem and, since (12) follows from (11) for $\alpha \in (0, 1)$, it provides the duality of conditions for second-order long-range dependence in time- and frequency-domains (see Section 3.1). However, there is no such result for the high-order characteristics. Thus, the high-order long-range dependence property can be defined in different ways: in time domain using the cumulant function or in the frequency domain through the spectral density. As for the case of the second-order characteristics, where the definitions in real and spectral domains were initially given separately, we follow the same approach and require only one of the corresponding conditions to hold for third-order long-range dependence. For a slightly different definition that involves both the spectral density and the third-order cumulant (in the case of a discrete-time stochastic process), see [44].

The following theorem provides conditions for a supOU process to exhibit third-order long-range dependence.

Theorem 3.5. *Let $X(t)$ be the supOU process given in* (4)*. Assume that Condition 1 holds with $m = 3$, $\varkappa_3 \neq 0$, and Condition 2 holds for some $0 < \alpha < 2$ with function $\ell(x)$ being bounded, bounded away from zero in the neighbourhood of* 0*. Then both parts* (*a*) *and* (*b*) *of Definition 3.3 hold and thus $X(t)$ exhibits third-order long-range dependence. Moreover, the convergence is uniform in* (14) *with $H = 2 - \alpha$.*

Proof. To prove (14) recall that due to (9) and (10) (with $m = 3$) the spectral density of the supOU process can be given as

$$f_3(\omega_1, \omega_2) = \varrho_3 \int_0^\infty \frac{\nu(d\xi)}{(\xi + i\omega_1)(\xi + i\omega_2)(\xi - i(\omega_1 + \omega_2))},$$

or, if we use polar coordinates (ρ, ϕ) with $\omega_1 = \rho \cdot \cos(\phi), \omega_2 = \rho \cdot \sin(\phi)$, it becomes

$$f_3(\rho, \phi) = \varrho_3 \int_0^\infty \rho^{-3} \frac{1}{(\xi + i\cos\phi)(\xi + i\sin\phi)(\xi - i(\sin\phi + \cos\phi))} \nu(\rho d\xi).$$

Under the assumption that Condition 2 holds we obtain

$$f_3(\rho, \phi) = \varrho_3 \int_0^\infty \frac{\ell(\rho\xi)}{\rho^{2-\alpha}} h_\phi(\xi) d\xi.$$

where

$$h_\phi(\xi) = \frac{\xi^\alpha}{(\xi + i\cos\phi)(\xi + i\sin\phi)(\xi - i(\sin\phi + \cos\phi))}, \ \xi > 0, \ \phi \in [0, 2\pi).$$

Note that $\sup_{\phi\in[0,2\pi]} |h_\phi(\xi)| \le 3\xi^{\alpha-1}(1 \wedge \xi^{-2})$, where $\wedge$ stands for the minimum, and the inequality

$$\int_0^\infty \sup_{\phi\in[0,2\pi)} |h_\phi(\xi)| d\xi < \infty \tag{15}$$

holds for $\alpha \in (0, 2)$. Moreover, we get

$$\left| \int_0^\infty h_\phi(\xi) d\xi \right| \ge \left| \int_0^\infty \Re(h_\phi(\xi)) d\xi \right| \ge \int_0^\infty \frac{\xi^3}{(\xi^2 + 1)^2} d\xi > 0,$$

where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$. Since $\ell(\rho)$ is bounded away from zero for ρ small enough, we obtain

$$\sup_{\phi\in[0,2\pi]} \left| \int_0^\infty \frac{\ell(\rho\xi)}{\ell(\rho)} h_\phi(\xi) d\xi - \int_0^\infty h_\phi(\xi) d\xi \right| \le \int_0^\infty \sup_{\phi\in[0,2\pi]} |h_\phi(\xi)| \left| \frac{\ell(\rho\xi)}{\ell(\rho)} - 1 \right| d\xi.$$

Using (15), the definition of the slowly varying function and Lebesgue-dominated convergence theorem, we obtain

$$\sup_{\phi\in[0,2\pi]} \left| \int_0^\infty \frac{\ell(\rho\xi)}{\ell(\rho)} h_\phi(\xi) d\xi - \int_0^\infty h_\phi(\xi) d\xi \right| \to 0 \text{ as } \rho \to 0.$$

Therefore

$$\lim_{\rho\to 0} \frac{f_3(\rho, \phi)}{\rho^{\alpha-2}\ell(\rho)} C_\alpha(\phi) = 1, \quad C_\alpha^{-1}(\phi) = \varrho_3 \int_0^\infty h_\phi(\xi) d\xi,$$

where $C_\alpha : [0, 2\pi] \to \mathbb{C}$ is the uniformly bounded, bounded away from zero function and the convergence is uniform for $\phi \in [0, 2\pi]$, which concludes the proof of the first part.

Let us now consider the cumulant of the third order. Using (7) and (8), we obtain

$$\iint_{\mathbb{R}^2} |c_3(\tau_1, \tau_2)| d\tau_1 d\tau_2 = \frac{\eta|\varkappa_3|}{3} \int_0^\infty \left(\iint_{\mathbb{R}^2} e^{-\xi(\tau_1+\tau_2-3\min\{\tau_1,\tau_2,0\})} d\tau_1 d\tau_2 \right) \frac{\nu(d\xi)}{\xi}$$

and

$$\begin{aligned}\iint\limits_{\mathbb{R}^2} e^{-\xi(\tau_1+\tau_2-3\min\{\tau_1,\tau_2,0\})}d\tau_1 d\tau_2 &= 2\int\limits_0^\infty\int\limits_0^{\tau_1} e^{-\xi(\tau_1+\tau_2)}d\tau_1 d\tau_2 \\ +2\int\limits_0^\infty\int\limits_{-\infty}^0 e^{-\xi(\tau_1-2\tau_2)}d\tau_1 d\tau_2 &+ 2\int\limits_{-\infty}^0\int\limits_{-\infty}^{\tau_1} e^{-\xi(\tau_1-2\tau_2)}d\tau_1 d\tau_2 = \frac{3}{\xi^2}.\end{aligned} \tag{16}$$

Therefore, due to the assumption that Condition 2 holds, we have

$$\iint\limits_{\mathbb{R}^2} |c_3(\tau_1,\tau_2)|d\tau_1 d\tau_2 = \eta|\varkappa_3|\int\limits_0^\infty \frac{\nu(d\xi)}{\xi^3} = \eta|\varkappa_3|\int\limits_0^\infty \xi^{\alpha-3}\ell(\xi)d\xi = \eta|\varkappa_3|\left(\int\limits_0^1 \xi^{\alpha-3}\ell(\xi)d\xi + \int\limits_1^\infty \xi^{\alpha-3}\ell(\xi)d\xi\right).$$

Note that for $\alpha < 2$ the second integral in the above formula is bounded by $\int_0^\infty \xi^{-1}\ell(\xi)d\xi < \infty$ due to (2) and the first one is infinite, that completes the proof. □

3.3. High-order long-range dependence

Similarly to Definition 3.3 we define mth-order long-range dependence for any $m \geq 2$.

Definition 3.6. We will say that a strictly stationary process $X(t)$, $t \in \mathbb{R}$, such that $\mathbb{E}|X(t)|^m < \infty$ possesses the property of mth-order long-range dependence if either of the two following conditions holds.

(a) The mth-order spectral density $f_m(\boldsymbol{\omega})$, $\boldsymbol{\omega} \in \mathbb{R}^{m-1}$, of $X(t)$ exists and has a singularity at $\mathbf{0}$ such that

$$\lim_{\|\boldsymbol{\omega}\|\to 0} \frac{f_m(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|^{-H}\ell(\|\boldsymbol{\omega}\|)} C\left(\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}\right) = 1 \tag{17}$$

where $H > 0$, $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ is a slowly-varying at zero function, $C : S^{m-2} \to \mathbb{C}$ is some bounded, bounded away from zero function away from some set of Lebesgue measure zero on S^{m-2}.

(b) The mth-order cumulant of $X(t)$ satisfies

$$\int\limits_{\mathbb{R}^{m-1}} |c_m(\boldsymbol{\tau})|d\boldsymbol{\tau} = \infty. \tag{18}$$

It turns out that the behaviour of the mth-order spectral density resembles that described in Theorem 3.5 for the third-order spectral density, under the assumption that Condition 2 holds with $\alpha \in (m-3, m-1)$; in particular, in this case the convergence in (17) is uniform.

Theorem 3.7. *Assume that $X(t)$, $t \in \mathbb{R}$, is a supOU process* (4) *such that Condition 1 holds with some $m \geq 3$ and $\varkappa_m \neq 0$. Assume also that Condition 2 holds with $m-3 < \alpha < m-1$ and the function $\ell(\cdot)$ being bounded, bounded away from zero in the neighbourhood of zero. Then* (17) *holds with $H = m-\alpha-1$ and thus the supOU process exhibits mth-order long-range dependence. Moreover, the convergence in* (17) *is uniform.*

Proof. Let us denote

$$g_m(\boldsymbol{\omega};\xi) = (\xi + i\omega_1)\dots(\xi + i\omega_{m-1})(\xi - i(\omega_1 + \dots + \omega_{m-1})), \tag{19}$$

then the function $f_{m,\xi}(\boldsymbol{\omega})$ given by (10) can be written as

$$f_{m,\xi}(\boldsymbol{\omega}) = \varrho_m \frac{\xi}{g_m(\boldsymbol{\omega};\xi)}.$$

Let $\hat{\omega} = \frac{1}{\|\omega\|}(\omega_1, \ldots, \omega_{m-1}) \in S^{m-2}$. Note that $\sup\limits_{\hat{\omega} \in S^{m-2}} \frac{1}{|g_m(\hat{\omega};\xi)|} \leq M\xi^{2-m}(1 \wedge \xi^{-2})$ for some positive constant M. Therefore, for any $\alpha \in (m-3, m-1)$ we obtain

$$\int_0^\infty \sup_{\hat{\omega} \in S^{m-2}} \frac{\xi^\alpha}{|g_m(\hat{\omega};\xi)|} d\xi < \infty.$$

Under the assumption that Condition 2 holds, we obtain

$$f_m(\omega) = \int_0^\infty f_{m,\xi}(\omega) \frac{\nu(d\xi)}{\xi} = \varrho_m \int_0^\infty \frac{\xi^\alpha \ell(\xi)}{g_m(\omega;\xi)} d\xi = \varrho_m \int_0^\infty \|\omega\|^{1+\alpha-m} \frac{\xi^\alpha \ell(\|\omega\|\xi)}{g_m(\hat{\omega};\xi)} d\xi,$$

and, arguing similarly as for the third-order spectral density, we can prove that, whenever the slowly varying function $\ell(\|\omega\|)$ is bounded and bounded away from zero for small values of $\|\omega\|$, the relation (17) holds for any $m-3 < \alpha < m-1$ with $H = m - \alpha - 1$. Moreover, the convergence is uniform with respect to $\hat{\omega} \in S^{m-2}$. □

While Theorem 3.7 describes the behaviour of the mth-order spectral density of the supOU process in the case where Condition 2 holds with $\alpha \in (m-1, m-3)$, the situation becomes more complicated for $\alpha < m-3$. In particular, the spectral density may be unbounded even if $\|\omega\| \not\to 0$. Before presenting the next theorem, we first need to describe the symmetries of the spectral density.

Lemma 3.8. *Let Π_{m-1} be a set of all permutations of the set $\{1, \ldots, m-1\}$. Then for any $\omega \in \mathbb{R}^{m-1}$ and any $\pi \in \Pi_{m-1}$ we have*

$$f_m(\omega_1, \ldots, \omega_{m-1}) = f_m(\omega_{\pi(1)}, \ldots, \omega_{\pi(m-1)}), \tag{20}$$

that is, $f_m(\omega)$ is invariant under permutation of variables, and, furthermore,

$$f_m(\omega_1, \ldots, \omega_{m-1}) = f_m(-\sum_{j=1}^{m-1} \omega_j, \omega_2 \ldots, \omega_{m-1}). \tag{21}$$

Using notations from [44] we will further denote $\omega_{1:n} = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ the extraction of n coordinates from $\omega \in \mathbb{R}^d$ and $\|\omega_{1:n}\| = \sqrt{\omega_1^2 + \ldots + \omega_n^2}$, where $n \leq d$. Denote $\omega_m = -(\omega_1 + \ldots + \omega_{m-1})$ for simplicity, then for any $\pi \in \Pi_m$ we can rewrite (20) and (21) as $f_m(\omega_{1:m}) = f_m(\omega_{\pi(1:m)})$, where $\omega_{\pi(1:m)} = (\omega_{\pi(1)}, \ldots, \omega_{\pi(m)})$.

To study the behaviour of $f_m(\omega)$ along axes and hyperplanes spanned by some axes, for any $\pi \in \Pi_m$ we introduce the region $\Omega_\pi \subset \mathbb{R}^{m-1}$ such that

$$\Omega_\pi = \{\, \omega \in \mathbb{R}^{m-1} : |\omega_{\pi(1)}| \leq \ldots \leq |\omega_{\pi(m)}| \,\}. \tag{22}$$

Remark 3.9. The regions Ω_π, $\pi \in \Pi_m$, are not optimal (minimal) for the mth-order spectral densities. The derivation of the principal domains of the high-order spectral density is discussed in [15].

Introduce also the regions $\mathcal{I}_k = \cup_{\pi \in \Pi_m} \{\omega \in \mathbb{R}^{m-1} : \omega_{\pi(1:k+1)} = \mathbf{0}\}$, $k \in \mathbb{Z}$, $1 \leq k \leq m-3$; note that each such region $\mathcal{I}_k$ is a set of Lebesgue measure zero in $\mathbb{R}^{m-1}$.

Theorem 3.10. *Assume that $X(t)$, $t \in \mathbb{R}$, is a supOU process* (4) *such that Condition 1 holds for $m \geq 3$, $\varkappa_m \neq 0$ and Condition 2 holds for some $\alpha > 0$ with the function $\ell(\cdot)$ being bounded, bounded away from zero in the neighbourhood of* 0*. Then the following statements hold.*

(i) For $\alpha > m$ the function $f_m(\omega)$ is bounded for all $\omega \in \mathbb{R}^{m-1}$;

(ii) *For $m-3<\alpha<m-1$ the function $f_m(\boldsymbol{\omega})$ is unbounded when $\|\boldsymbol{\omega}\|\to 0$ and* (17) *holds with convergence being uniform, $H=m-1-\alpha$ and some bounded, bounded away from zero function $C: S^{m-2}\to\mathbb{C}$;*

(iii) *Let $\pi\in\Pi_m$ be an arbitrary permutation and Ω_π its associated region given by* (22). *If $\alpha\in(k-1,k)$ with $1\le k\le m-3$ being integer, the function $f_m(\boldsymbol{\omega})$ is unbounded when $\|\boldsymbol{\omega}_{\pi(1:(k+1))}\|\to 0$. Moreover, for $\boldsymbol{\omega}\in\Omega_\pi$ we have*

$$\lim_{\|\boldsymbol{\omega}_{\pi(1:(k+1))}\|\to 0}\frac{f_m(\boldsymbol{\omega})\cdot\|\boldsymbol{\omega}\|^2\|\boldsymbol{\omega}_{\pi(1:m-2)}\|\ldots\|\boldsymbol{\omega}_{\pi(1:k+2)}\|}{\|\boldsymbol{\omega}_{\pi(1:k+1)}\|^{\alpha-k}\ell(\|\boldsymbol{\omega}_{\pi(1:k+1)}\|)}K_\pi\left(\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}\right)\overset{\boldsymbol{\omega}\in\Omega_\pi}{=}1. \tag{23}$$

where $K_\pi: S^{m-2}\cap\Omega_\pi\to\mathbb{C}$ is some bounded, bounded away from zero function.

Proof. The part (i) is obvious, while (ii) follows from Theorem 3.7. Thus, we need to prove the part (iii) only. Without loss of generality, we can consider the case of $\boldsymbol{\omega}\in\Omega_{\pi_{tr}}$, where π_{tr} is a trivial permutation, that is, $\boldsymbol{\omega}_{\pi_{tr}(1:m-1)}=\boldsymbol{\omega}_{1:(m-1)}=\boldsymbol{\omega}$.

Recall that under the assumption that Condition 2 holds the mth-order spectral density can be written as

$$f_m(\boldsymbol{\omega})=\varrho_m\int_0^\infty\frac{\xi^\alpha\ell(\xi)}{g_m(\boldsymbol{\omega},\xi)}d\xi,$$

where $g_m(\boldsymbol{\omega},\xi)$ is given by (19). Let us now rewrite it for $\|\boldsymbol{\omega}\|>0$ as

$$f_m(\boldsymbol{\omega})=\frac{\varrho_m}{\|\boldsymbol{\omega}\|^{m-\alpha-1}}\int_0^\infty\frac{\xi^\alpha\ell(\|\boldsymbol{\omega}\|\xi)}{g_m(\hat{\boldsymbol{\omega}},\xi)}d\xi, \tag{24}$$

where $\hat{\boldsymbol{\omega}}=\frac{1}{\|\boldsymbol{\omega}\|}(\omega_1,\ldots,\omega_{m-1})\in S^{m-2}$.

Notice now that it holds for any $\boldsymbol{\omega}\in\Omega_{\pi_{tr}}$ that $(m-1)^{-\frac{1}{2}}\le|\omega_{m-1}|/\|\boldsymbol{\omega}\|\le 1$ and $1\le|\omega_m|/\|\boldsymbol{\omega}\|\le(m-1)$, where $\omega_m=-(\omega_1+\ldots+\omega_{m-1})$ as denoted earlier. Therefore, intuitively speaking, the contribution to the singularity in this region is made only by the components of $\hat{\omega}_{(1:m-2)}=(|\omega_1|/\|\boldsymbol{\omega}\|,\ldots,|\omega_{m-2}|/\|\boldsymbol{\omega}\|)\in\mathbb{R}^{m-2}$.

Denoting $q_2(\xi,\hat{\boldsymbol{\omega}})=(\|\hat{\boldsymbol{\omega}}_{1:(m-2)}\|\xi+i\hat{\omega}_{m-1})(\|\hat{\boldsymbol{\omega}}_{1:(m-2)}\|\xi+i\hat{\omega}_m)$, we rewrite (24) as

$$\begin{aligned}f_m(\boldsymbol{\omega})&=\frac{1}{\|\boldsymbol{\omega}\|^{m-\alpha-1}\|\hat{\boldsymbol{\omega}}_{1:(m-2)}\|^{m-3-\alpha}}\int_0^\infty\frac{\xi^\alpha\cdot\ell(\|\boldsymbol{\omega}\|\cdot\|\hat{\boldsymbol{\omega}}_{1:(m-2)}\|\xi)}{q_2(\xi,\hat{\boldsymbol{\omega}})\prod_{j=1}^{m-2}(\xi+i\hat{\omega}_j)}d\xi\\&=\frac{1}{\|\boldsymbol{\omega}\|^2\|\boldsymbol{\omega}_{1:(m-2)}\|^{m-3-\alpha}}\int_0^\infty\frac{\xi^\alpha\cdot\ell(\|\boldsymbol{\omega}_{1:(m-2)}\|\xi)}{q_2(\xi,\hat{\boldsymbol{\omega}})\prod_{j=1}^{m-2}(\xi+i\hat{\omega}_j)}d\xi,\end{aligned}$$

where $\hat{\omega}_j=\frac{\hat{\omega}_j}{\|\hat{\boldsymbol{\omega}}_{1:(m-2)}\|}$, $j=1,\ldots,m-2$, $\hat{\boldsymbol{\omega}}_{1:(m-2)}\in S^{m-3}$ and we have used the relation $\|\boldsymbol{\omega}\|\cdot\|\hat{\boldsymbol{\omega}}_{1:(m-2)}\|=\|\boldsymbol{\omega}_{1:(m-2)}\|$. Let us denote $q_3(\xi,\hat{\boldsymbol{\omega}})=(\|\hat{\boldsymbol{\omega}}_{1:(m-3)}\|\xi+i\hat{\boldsymbol{\omega}}_{m-2})(\|\hat{\boldsymbol{\omega}}_{1:(m-3)}\|\xi+i\hat{\boldsymbol{\omega}}_{m-1})(\|\hat{\boldsymbol{\omega}}_{1:(m-3)}\|\xi+i\hat{\boldsymbol{\omega}}_m)$. We note that

$$q_3(\xi,\hat{\boldsymbol{\omega}})=(\|\hat{\boldsymbol{\omega}}_{1:(m-3)}\|\xi+i\hat{\boldsymbol{\omega}}_{m-2})(\|\hat{\boldsymbol{\omega}}_{1:(m-3)}\|\cdot\|\hat{\boldsymbol{\omega}}_{1:(m-2)}\|\xi+i\hat{\boldsymbol{\omega}}_{m-1})(\|\hat{\boldsymbol{\omega}}_{1:(m-3)}\|\cdot\|\hat{\boldsymbol{\omega}}_{1:(m-2)}\|\xi+i\hat{\boldsymbol{\omega}}_m).$$

In a similar manner, we define $q_4(\xi,\hat{\boldsymbol{\omega}}),\ldots,q_{m-k-1}(\xi,\hat{\boldsymbol{\omega}})$. We note that the function $|q_j(\xi,\hat{\boldsymbol{\omega}})|^{-1}$ is bounded uniformly in $\xi>0$ whenever $\hat{\boldsymbol{\omega}}\in\Omega_{\pi_{tr}}\cap S^{m-2}$, $j=2,\ldots,m-k-1$.

Thus, repeating the procedure above until we get elements $\tilde{\omega}_{1:(k+1)}$ belonging to the $(k+1)$-dimensional sphere S^{k+1}, after all the simplifications we obtain

$$f_m(\boldsymbol{\omega})=\frac{1}{\|\boldsymbol{\omega}\|^2\|\boldsymbol{\omega}_{1:(m-2)}\|\ldots\|\boldsymbol{\omega}_{1:(k+2)}\|\,\|\boldsymbol{\omega}_{1:(k+1)}\|^{k-\alpha}}\int_0^\infty\frac{\xi^\alpha\cdot\ell(\|\boldsymbol{\omega}_{1:k}\|\xi)}{q_{m-k-1}(\xi,\hat{\boldsymbol{\omega}})\prod_{j=1}^{k+1}(\xi+i\tilde{\omega}_j)}d\xi,$$

where $\tilde{\omega}_{1:(k+1)} \in S^k$. Since $k-1 < \alpha < k$, and q_{m-k-1} is uniformly bounded in $\xi > 0$, similarly to Theorem 3.7, we get

$$\lim_{\|\omega_{1:(k+1)}\|\to 0} \frac{f_m(\omega)\cdot\|\omega\|^2\|\omega_{1:(m-2)}\|\dots\|\omega_{1:(k+2)}\|}{\|\omega_{(1:k+1)}\|^{\alpha-k}\ell(\|\omega_{\pi(1:k+1)}\|)}\mathcal{K}(\hat{\omega}) \overset{\omega\in\Omega_{\pi_{tr}}}{=} 1,$$

where $\mathcal{K} = K_{\pi_{tr}}$ is a bounded, bounded away from zero function away from $\mathcal{I}_k$, and the proof is completed. □

Remark 3.11. Note that the equation (23) provides information about the singularities of $f_m(\omega)$ along curves where $\|\omega\| \to 0$ may or may not hold. The next example illustrates how the spectral density behaves differently depending on a curve approaching the origin. The following example and corollary focus specifically on the behaviour near the origin and, in particular, we demonstrate that the equality (17) holds and thus that the part (a) of Definition 3.6 holds for a supOU process whenever Condition 2 holds with $0 < \alpha < m-1$ being non-integer.

We further use the notation $g(x) \sim f(x)$ as $x \to x_0$ if $g(x)/f(x) \to c \in (0,\infty)$ as $x \to x_0$.

Example 3.12. Let us consider the case of $m = 4$ for simplicity. Assume also that the conditions of Theorem 3.10 (iii) hold with $k = 1$, i.e., $\alpha \in (0,1)$.

(a) Consider the curve generated by $\omega = (u,u,1)$ as $u \to 0+$. Then $\|\omega\| \to 1$, but $\|\omega_{1:2}\| \sim u$ as $u \to 0+$ and

$$\lim_{u\to 0+} \frac{f_4(\omega)\big|_{\omega=(u,u,1)}}{u^{\alpha-1}\ell(u)} = C_1$$

for some finite constant C_1.

(b) Consider the curve generated by $\omega = (u,u,u)$ as $u \to 0+$. Then $\|\omega\| \sim u$, $\|\omega_{1:2}\| \sim u$ as $u \to 0+$ and

$$\lim_{u\to 0+} \frac{f_4(\omega)\big|_{\omega=(u,u,u)}}{u^{\alpha-3}\ell(u)} = C_2$$

for some finite constant C_2.

(c) Consider the curve generated by $\omega = (u^2,u^2,u)$ as $u \to 0+$. Then $\|\omega\| \sim u$, $\|\omega_{1:2}\| \sim u^2$ as $u \to 0+$ and

$$\lim_{u\to 0+} \frac{f_4(\omega)\big|_{\omega=(u^2,u^2,u)}}{u^{2\alpha-4}\ell(u^2)} = C_3$$

for some finite constant C_3.

From Example 3.12 (parts (b) and (c)) it becomes clear that the statement of Theorem 3.7 does not hold if $\alpha < m-3$, i.e. there is no bounded, bounded away from zero on S^{m-2} function $C(\cdot)$ such that the equality (17) holds for a fixed $H > 0$. However, the phenomena of $2\alpha-4 \neq \alpha-3$ in the denominator in Example 3.12 (c) is due to the fact that $\hat{\omega}$ approaches the point $(0,0,1)$ on the axis, where the 4th-order spectral density is not defined. However, the next result shows that this situation can be avoided if we exclude some regions of Lebesgue measure zero from consideration.

Corollary 3.13. *Assume that the assumptions of Theorem 3.10 (iii) hold. Then the spectral density $f_m(\omega)$ is well defined on $\mathbb{R}^{m-1}\setminus\mathcal{I}_k$ and (17) holds with $H = m-\alpha-1$ for any $\hat{\omega} \in S^{m-2}\setminus\mathcal{I}_k$; moreover, the convergence is uniform on any compact subset of $S^{m-2}\setminus\mathcal{I}_k$.*

Proof. Notice that on any compact subset of $S^{m-1}\setminus\mathcal{I}_k$ it holds that $\|\hat{\omega}_{\pi(1:d)}\|$, $k+1 \le d \le m-1$, is bounded away from zero for any $\pi \in \Pi_m$, where $\hat{\omega}_{\pi(1:d)} = \omega_{\pi(1:d)}/\|\omega\|$, $d \le m-1$. The rest of the proof is the same as for Theorem 3.10 (iii). □

Thus, whenever $\alpha \in ((m-3, m-1) \cup ((0, m-3) \setminus \mathbb{Z}))$, we see that the part ($a$) of Definition 3.6 holds under assumptions of Theorem 3.10. To give a simple demonstration of the behaviour of the spectral density, similarly to Example 3.12 (b), we consider curves $\boldsymbol{\omega}_u = u\hat{\boldsymbol{\omega}}$, where $\hat{\boldsymbol{\omega}} \in S^{m-2} \setminus \mathcal{I}_k$ is fixed. In this case, under the assumptions of Theorem 3.10 (iii), it follows from Corollary 3.13 that there exists a finite constant $C_{\hat{\boldsymbol{\omega}}}$ such that

$$\lim_{u\to 0} \frac{f_m(\boldsymbol{\omega})\big|_{\boldsymbol{\omega}=u\hat{\boldsymbol{\omega}}}}{u^{\alpha+1-m}\ell(u)} = C_{\hat{\boldsymbol{\omega}}}.$$

In the following theorem, we will also prove that the part (b) of Definition 3.6 holds under some assumptions.

Theorem 3.14. *Assume that* $X(t)$, $t \in \mathbb{R}$, *is a supOU process given in* (4) *such that Condition 1 holds for* $m \geq 2$, $\varkappa_m \neq 0$ *and Condition 2 holds for some* $0 < \alpha < m-1$. *Then* (18) *holds, and thus the supOU process exhibits mth-order long-range dependence.*

Proof. First, we will prove by induction that

$$\int_{\mathbb{R}^{m-1}} e^{-\xi(\tau_1+\ldots+\tau_{m-1}-m\min\{\tau_1,\ldots,\tau_{m-1},0\})} d\boldsymbol{\tau} = \frac{m}{\xi^{m-1}}, \quad m \geq 3. \tag{25}$$

The base step is valid due to (16). Assume now that (25) holds for some $m \geq 3$. Then we get

$$\begin{aligned}
&\int_{\mathbb{R}^m} e^{-\xi(\tau_1+\ldots+\tau_{m-1}+\tau_m-(m+1)\min\{\tau_1,\ldots,\tau_m-1,\tau_m,0\})} d\tau_1\ldots d\tau_m = \\
&= \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^{\min\{\tau_1,\ldots,\tau_{m-1},0\}} e^{-\xi(\tau_1+\ldots+\tau_{m-1}-m\tau_m)} d\tau_m \right) d\tau_1\ldots d\tau_{m-1} \\
&+ \int_{\mathbb{R}^{m-1}} \left(\int_{\min\{\tau_1,\ldots,\tau_{m-1},0\}}^{\infty} e^{-\xi(\tau_1+\ldots+\tau_{m-1}+\tau_m-(m+1)\min\{\tau_1,\ldots,\tau_{m-1},0\})} d\tau_m \right) d\tau_1\ldots d\tau_{m-1} \\
&= \left(1+\frac{1}{m}\right) \int_{\mathbb{R}^{m-1}} \frac{1}{\xi} e^{-\xi(\tau_1+\ldots+\tau_{m-1}-m\min\{\tau_1,\ldots,\tau_{m-1},0\})} d\tau_1\ldots d\tau_{m-1} = \frac{m+1}{\xi^m},
\end{aligned}$$

which proves (25) for any $m \geq 2$. Therefore, under assumption that Condition 2 holds we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{m-1}} |c_m(\boldsymbol{\tau})| d\boldsymbol{\tau} &= \int_{\mathbb{R}^{m-1}} \left(\frac{\eta|\varkappa_m|}{m} \int_0^{\infty} e^{-\xi(\tau_1+\ldots+\tau_{m-1}+\tau_m-(m+1)\min\{\tau_1,\ldots,\tau_{m-1},\tau_m,0\})} \frac{\nu(d\xi)}{\xi} \right) d\boldsymbol{\tau} \\
&= \eta|\varkappa_m| \int_0^{\infty} \frac{\nu(d\xi)}{\xi^m} = \eta|\varkappa_m| \int_0^{\infty} \xi^{\alpha-m}\ell(\xi) d\xi,
\end{aligned}$$

that completes the proof. □

Thus, continuing Example 3.2 it follows from Theorem 3.10, Corollary 3.13 and Theorem 3.14 that the supOU process with the measure ν given by the $Gamma(\alpha, 1+\alpha)$ density exhibits mth-order long-range dependence, $m \geq 2$, whenever Condition 1 holds and $0 < \alpha < m-1$ is non-integer. In addition, we give a few more examples of the measure ν satisfying the conditions of Theorem 3.10.

Example 3.15. Let us consider the measure $\nu_{a,\beta}$ with density $q_{a,\beta}(x)$, $a \in (0, 1]$, $\beta > 0$, such that

$$q_{a,\beta}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\beta+k)x^{a(\beta+k)}}{k!\Gamma(\beta)\Gamma(a(\beta+k))}, \ x > 0.$$

Note that $\frac{q_{a,\beta}(x)}{x}$ is the density of the two-parametric Mittag-Leffler distribution, thus $\eta_{a,\beta} = (\int_{\mathbb{R}_+} \xi^{-1} q_{a,\beta}(\xi) d\xi)^{-1} = 1$, see [31]. Moreover, the correlation function of the supOU process having characteristics $(b, \sigma^2, \mu, \nu_{a,\beta})$ can be expressed as

$$r_{a,\beta}(t) = \frac{1}{(1 + |t|^a)^\beta}$$

and it is clear that

$$r_{a,\beta}(t) = \frac{\ell(1/|t|)}{|t|^{a\beta}}$$

with $\ell(x) = \frac{1}{(|x|^a+1)^\beta}$ being a slowly varying function near zero which is also bounded, bounded away from zero in the neighbourhood of zero. From Tauberian-Abelian theorems we obtain

$$\nu_{a,\beta}(d\xi) = \frac{1}{\Gamma(a\beta)} \xi^{a\beta} \ell(\xi) d\xi, \ \xi > 0$$

and thus Condition 2 holds with $\alpha = a\beta$. From Theorem 3.14 we have that the supOU process having characteristics $(b, \sigma^2, \mu, \nu_{a,\beta})$ exhibits mth-order long-range dependence whenever $a\beta < m - 1$. Moreover, from Theorem 3.10 we have that for non-integer $a\beta < m - 1$ the mth-order spectral density satisfies (17).

Example 3.16. Let the measure ν_α have the density

$$q_\alpha(\xi) = \frac{\sin(\pi\alpha)}{\pi} \frac{\xi^\alpha}{1 + 2\cos(\pi\alpha)\xi^\alpha + \xi^{2\alpha}}, \ \alpha \in (0, 1),$$

which clearly satisfies Conditions of Theorem 3.10. Then the supOU process with characteristics $(b, \sigma^2, \mu, \nu_\alpha)$ exhibits mth-order long-range dependence, $m \geq 2$ whenever Condition 1 holds and $\alpha < m - 1$ is non-integer. Moreover, $\int_0^\infty \xi^{-1} q_\alpha(\xi)\, d\xi = 1$ and the correlation function is given by

$$r(t) = E_{\alpha,1}(-|t|^\alpha), \ t \in \mathbb{R},$$

where $E_{\alpha,1}(\cdot)$ is the Mittag-Leffler function, see [25]. We note that $r(t)$ has the asymptotic representation

$$r(t) = \frac{1}{|t|^\alpha \Gamma(1 - \alpha)} + O\left(\frac{1}{|t|^{2\alpha}}\right), \ t \to \infty,$$

and satisfies the double inequality

$$\frac{1}{1 + \Gamma(1 - \alpha)|t|^\alpha} \leq r(t) \leq \frac{1}{1 + (\Gamma(1 + \alpha))^{-1}|t|^\alpha}.$$

We refer to [4] for more details and examples of the measure ν.

4. Simulation study

To demonstrate the properties of the high-order spectral density, we perform a simulation study, where we depict the empirical second- and third-order spectral densities (further referred to as the spectrum and bispectrum, respectively) as well as the autocorrelation function (acf) for supOU processes with Dickman marginal distribution (shortly, supDOU processes) and gamma marginal distribution (shortly, supΓOU processes).

4.1. High-order long-range dependence in supDOU processes

We begin the introduction of the supDOU process with the definition of the Dickman distribution. We present one possible definition; see [32, 34] for alternative definitions and further details.

Definition 4.1. A random variable D_θ has the Dickman distribution with parameter $\theta > 0$, shortly, $D_\theta \sim GD(\theta)$, if it satisfies the distributional fixed-point equation

$$D_\theta \stackrel{d}{=} U^{1/\theta}(1 + D_\theta),$$

where " $\stackrel{d}{=}$ " denotes the equality in distribution, U is independent of D_θ and has the uniform distribution on $(0, 1]$.

In the next proposition, we recall some properties of the Dickman distribution.

Proposition 4.2. *[34]*

(i) The Laplace transform of $D_\theta \sim GD(\theta)$ is given by

$$\psi(s) = \mathbb{E}e^{-sD_\theta} = \exp\left\{-\theta \int_0^1 (1 - e^{-su})\frac{du}{u}\right\}.$$

(ii) For $\theta_1, \theta_2 > 0$, if $D_{\theta_1} \sim GD(\theta_1)$ and $D_{\theta_2} \sim GD(\theta_2)$ are independent, then $D_{\theta_1} + D_{\theta_2} \sim GD(\theta_1 + \theta_2)$.

(iii) All the moments of a Dickman random variable are finite, and its k-th-order cumulant equals θ/k. In particular

$$\mathbb{E}D_\theta = \theta, \quad \mathrm{Var}D_\theta = \frac{\theta}{2}.$$

Let us consider a supOU process $X_D(t)$, $t \geq 0$, which can be obtained from formula (4) by letting

$$b = \theta, \quad \sigma^2 = 0, \quad \mu = \theta\delta_1,$$

where δ_1 is the Dirac measure concentrated at 1. The supOU process $X_D(t)$ has the marginal distribution $GD(\theta)$ and the correlation function (7). We will refer to this class of supOU processes as *Dickman supOU processes* (*supDOU processes*). This class has been recently proposed in [19].

As in the case of supDOU processes, Condition 1 holds for any $m \in \mathbb{N}$ we can further consider the mth-order cumulant and the mth-order spectral density for any $m \geq 2$. For simulations, we choose $m = 2$ and $m = 3$ and the measure ν to have the $Gamma(\alpha, 1+\alpha)$ density (see Example 3.2) with different values of α corresponding to different dependence structures.

Notice that the choice of values of α comes from the theoretical findings. Indeed, from the discussion at the end of Section 3.1 follows that the spectrum is unbounded near zero when $\alpha < 1$, but stays uniformly bounded if $\alpha > 1$. Moreover, due to Theorem 3.5 the bispectrum exhibits singularity at zero for $\alpha < 2$ with the rate of growth depending on α value, whereas the bispectrum is uniformly bounded in the case $\alpha > 2$. Thus, our choice is aimed to demonstrate all possible situations. At the same time the chosen value of the parameter θ is arbitrary as it does not influence the dependence structure of the process.

In Figure 1 we show the $Gamma(\alpha, 1 + \alpha)$ density for various α. We can see that these densities are considerably overlapped and the left tail of the density influences the long-range dependence property. Direct calculation shows that only 0.5% of the gamma density for $\alpha = 0.5$ yields the long-range dependence property. This means that long-range dependence is simulated by a few events over long periods of time.

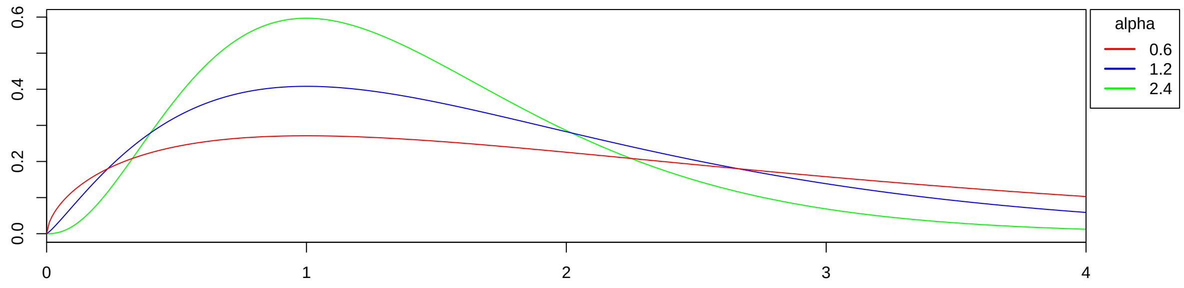

Figure 1: The $Gamma(\alpha, 1 + \alpha)$ density for $\alpha = 0.6, 1.2, 2.4$.

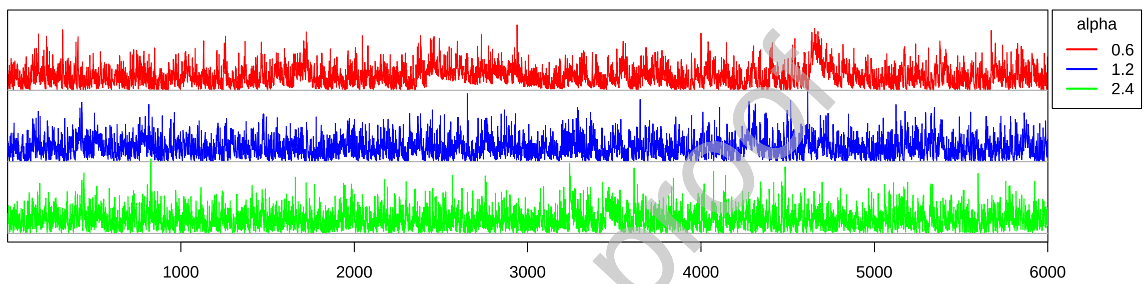

Figure 2: Realisations of the supDOU process for $\alpha = 0.6, 1.2, 2.4$ and $\theta = 1$, $\Delta = 1$.

In Figure 2 we show realisations of the supDOU process for various α, $\theta = 1$, discretization $\Delta = 1$, obtained by the algorithm given in the Appendix.

We see that all realisations could have a non-stationary shape on some short intervals and the level of local non-stationarity increases as α decreases.

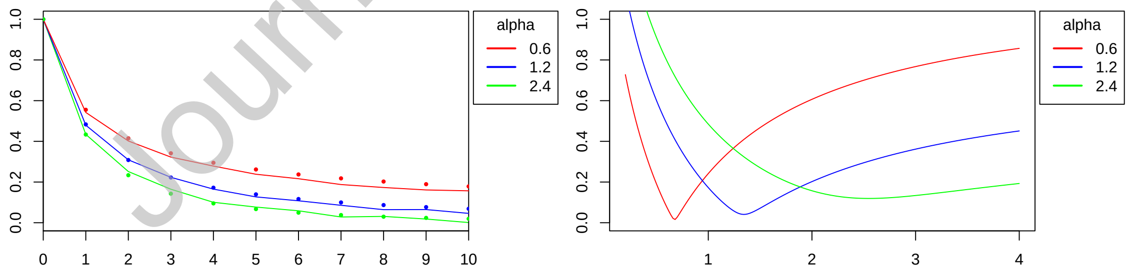

Figure 3: Left: The true acf (dotted) and empirical acf (solid line) of realisations of the supDOU process for $\alpha = 0.6, 1.2, 2.4$ and $\theta = 1$, $\Delta = 1$. Right: The Hellinger distance between the true acf and the empirical acf as a function of α.

In Figure 3 we show the true and empirical acfs of the supDOU process for various α and $\theta = 1$, and the Hellinger distance

$$D_H(\alpha) = \sqrt{\sum_{j=0}^{10} \left(\sqrt{\hat{r}_j} - \sqrt{r_\alpha(j)} \right)^2},$$

between the true acf $r_\alpha(t) = (1 + |t|/\alpha)^{-\alpha}$ and the empirical acf $\hat{r}_j$. We can see that the empirical acf could be close to the true acf for some random realisations but the empirical acf usually has a noticeable discrepancy from the true acf. We also see that the Hellinger distance could be used for the estimation of α.

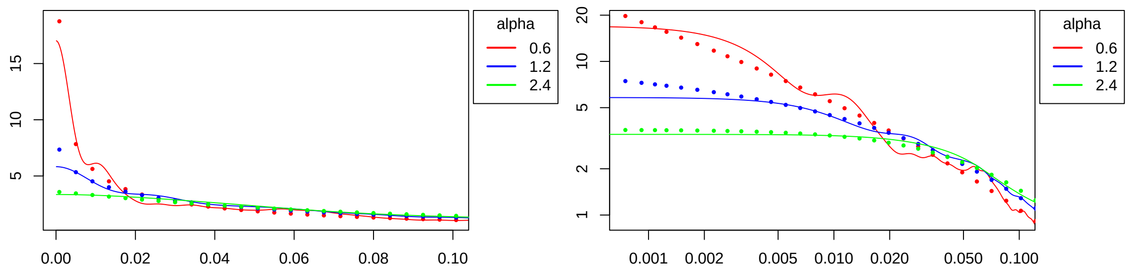

Figure 4: Left: The true spectrum (dotted) and smoothed empirical spectrum (solid line) of realisations of the supDOU process for $\alpha = 0.6, 1.2, 2.4$ and $\theta = 1$, $\Delta = 1$. Right: The same plot in log-scales.

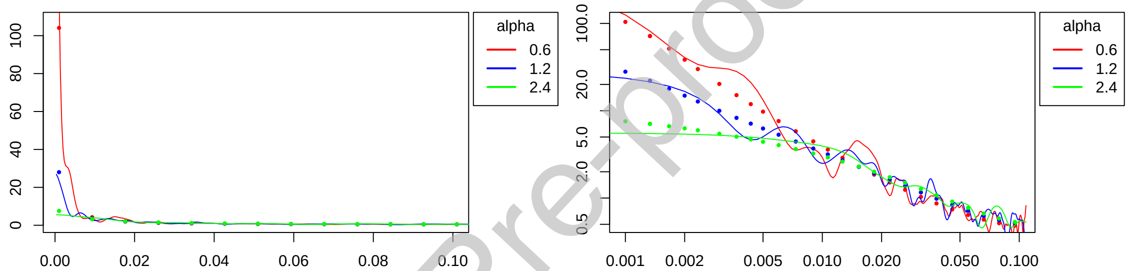

Figure 5: Left: The true bispectrum (dotted) and smoothed empirical bispectrum (solid line) with $\omega_1 = \omega_2$ of realisations of the supDOU process for $\alpha = 0.6, 1.2, 2.4$ and $\theta = 1$, $\Delta = 1$. Right: The same plot in log-scales.

In Figure 4 we show the true spectrum and the smoothed empirical spectrum of realisations of the supDOU process for various α and $\theta = 1$. We can see that the spectrum is linear near zero with slope $1 - \alpha$ in logarithmic coordinates when $0 < \alpha < 1$. Also, the spectrum does not have a singularity at zero when $\alpha > 1$.

In Figure 5 we show the true bispectrum and the smoothed empirical bispectrum with $\omega_1 = \omega_2$ of realisations of the supDOU process for various α and $\theta = 1$. The empirical bispectrum was obtained by the two-dimensional FFT of the Hankel matrix

$$\hat{C} = (\hat{c}_3(t_i, t_j))_{i,j=0}^n = \frac{\varkappa_3}{3} \begin{pmatrix} \hat{r}(0) & \hat{r}(\Delta) & \hat{r}(2\Delta) & \hat{r}(3\Delta) & \dots \\ \hat{r}(\Delta) & \hat{r}(2\Delta) & \hat{r}(3\Delta) & \hat{r}(4\Delta) & \dots \\ \hat{r}(2\Delta) & \hat{r}(3\Delta) & \hat{r}(4\Delta) & \hat{r}(5\Delta) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

for the points $t_i = i\Delta$, $i = 0, 1, \dots, n$, see Theorem 6. We can see that the bispectrum with $\omega_1 = \omega_2$ is linear at zero with slope $2 - \alpha$ in logarithmic coordinates when $0 < \alpha < 2$. We also observe that the bispectrum on the diagonal does not have a singularity at zero when $\alpha > 2$.

In Figure 6 we show the smoothed empirical bispectrum of a realisation of the supDOU process for $\alpha = 1.2$ and $\theta = 1$. We can see that the bispectrum is nearly linear in radial directions at the origin with slope $2 - \alpha$ in logarithmic coordinates when $0 < \alpha < 2$.

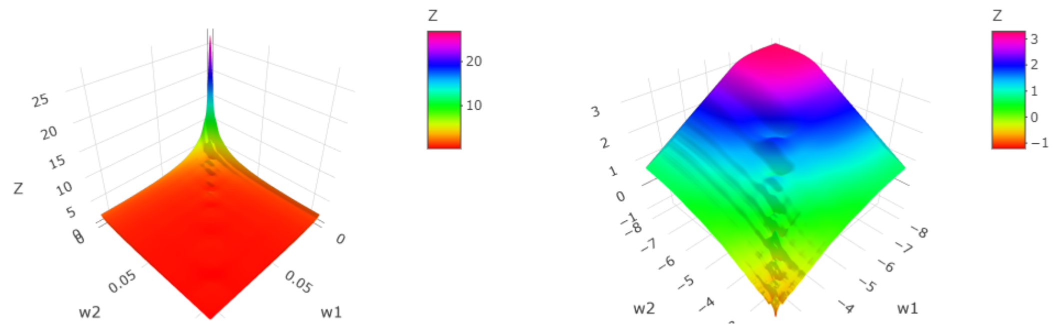

Figure 6: Left: The smoothed empirical bispectrum of a realisation of the supDOU process for $\alpha = 1.2$ and $\theta = 1$, $\Delta = 1$. Right: The same plot in log-scales.

4.2. High-order long-range dependence in supΓOU processes

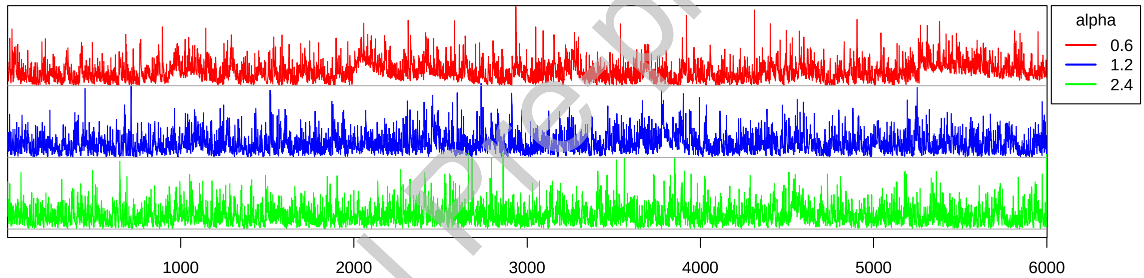

Figure 7: Realisations of the supΓOU process for $\alpha = 0.6, 1.2, 2.4$, $\theta = 2$, $\rho = 2$, $\Delta = 1$.

In Figure 7 we show realisations of the supΓOU process for various α, where the marginal distribution is the gamma distribution with parameters $\theta = 2$ and $\rho = 2$, discretization $\Delta = 1$. Here again the choice of values of α is aimed at demonstration of all possible situations regarding second- and third-order dependence structure of the process, whereas the choice of θ and ρ is rather arbitrary. Realisations were obtained by the algorithm given in Appendix.

In Figure 8 we show the true and empirical acfs of the supΓOU process for various α and the Hellinger distance defined earlier. We can see that the empirical acf is rather close to the true acf.

In Figure 9 we show the true spectrum and the smoothed empirical spectrum of realisations of the supΓOU process for various α. We can see that the spectrum is linear with slope $1 - \alpha$ in logarithmic coordinates when $0 < \alpha < 1$. In addition, the spectrum does not have a singularity at zero when $\alpha > 1$.

In Figure 10 we show the true bispectrum and the smoothed empirical bispectrum with $\omega_1 = \omega_2$ of realisations of the supΓOU process for various α. We can observe that the bispectrum with $\omega_1 = \omega_2$ is linear at zero with slope $2 - \alpha$ in logarithmic coordinates when $0 < \alpha < 2$. We also observe that the bispectrum on the diagonal does not have a singularity at zero when $\alpha > 2$.

In the case of supOU processes with the Mittag-Leffler correlation function, we apply a simulation algorithm

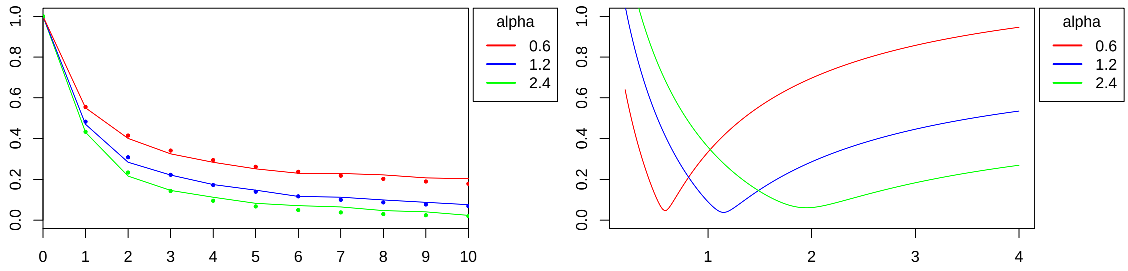

Figure 8: Left: The true acf (dotted) and empirical acf (solid line) of realisations of the supΓOU process for $\alpha = 0.6, 1.2, 2.4$ and $\theta = 2$, $\rho = 2$, $\Delta = 1$. Right: The Hellinger distance between the true acf and the empirical acf as a function of α.

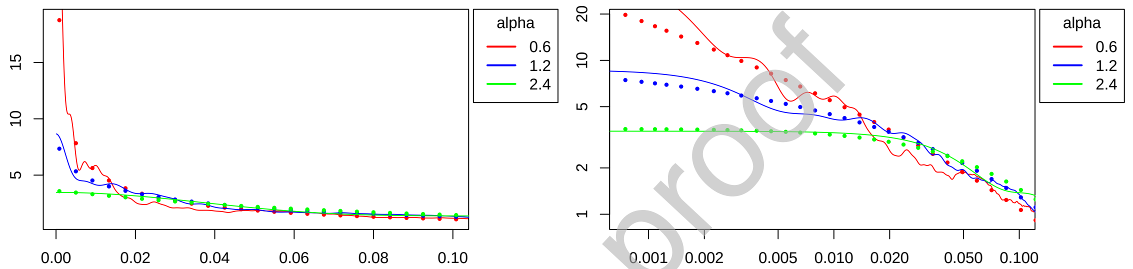

Figure 9: Left: The true spectrum (dotted) and smoothed empirical spectrum (solid line) of realisations of the supΓOU process for $\alpha = 0.6, 1.2, 2.4$ and $\theta = 2$, $\rho = 2$, $\Delta = 1$. Right: The same plot in log-scales.

based on a direct superposition of OU processes because the shot-noise representation is not possible for the measure ν_α, see [17] and details in Appendix. The numerical study shows the same empirical behaviour of the second-order and high-order characteristics but we omit pictures for the sake of brevity.

5. Conclusions

In the present paper, we investigated the properties of high-order spectral densities and cumulants of supOU processes. In particular, we discussed the distinction between short-range and long-range dependence using various characteristics of the process.

The importance of the results presented lies primarily in the various applications of higher-order characteristics, as described in the Introduction; see also [8]. Although the direct consequences of singular behaviour in spectral densities (or the slow decay of cumulants) on the short-term behaviour of the process are not easily described, there is evidence that these phenomena convey important structural information about the process. Notable examples include the emergence of nonCLT-type limit theorems and their implications, such as intermittency and fractal properties in complex physical systems, which are commonly characterized by non-Gaussianity, non-linear interactions, extreme-value clustering, and phase coupling.

Finally, we conducted a large simulation study of supOU processes with Dickman and gamma marginals, which supports the theoretical findings. We demonstrated that the singularity of the spectrum and bispectrum depends on the tail parameter α of the correlation function. The estimation theory for the parameters of supOU processes remains a subject for future research.

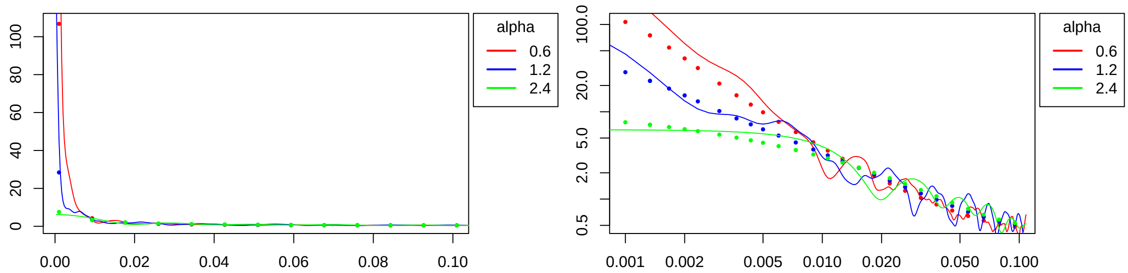

Figure 10: Left: The true bispectrum (dotted) and smoothed empirical bispectrum (solid line) with $\omega_1 = \omega_2$ of realisations of the supΓOU process for $\alpha = 0.6, 1.2, 2.4$ and $\theta = 2, \rho = 2, \Delta = 1$. Right: The same plot in log-scales.

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Appendix

The R code for simulation of the supDOU process with the correlation function $r(t) = 1/(1+|t|/\alpha)^{\alpha}$ is given below and based on the algorithm introduced in [19]. The marginal distribution of the supDOU process is the Dickman distribution with parameter θ.

```
r_supDOU <- function(alpha, theta, dt, T, T_min) {
  t <- seq(0, T, dt) # Vector of times
  n <- length(t)     # Number of time steps.
  X_sup <- rep(0, n) # Initialization for cumulative summation
  arrival = T_min+rexp(1, rate = theta)
  while(arrival<t[n])
  {
    R_K <- rgamma(1, shape = alpha + 1, rate = alpha)
    ind = floor(arrival/dt)+2
    if(arrival<0)
      X_sup = X_sup+exp(R_K*(arrival-t))
    else
      X_sup[ind:n] = X_sup[ind:n]+exp(R_K*(arrival-t[ind:n]))
    arrival = arrival+rexp(1, rate = theta)
  }
  return(X_sup)
}
x = r_supDOU(alpha=1.2, theta=1, dt=1, T=6000, T_min=-100)
```

The R code for simulation of the supΓOU process with the correlation function $r(t) = 1/(1+|t|/\alpha)^{\alpha}$ is given below and based on the algorithm described in (Qu, Dassiios, Zhao, 2019), $\alpha > 0$. The marginal distribution is the gamma distribution with parameters θ and ρ.

```
r_supGammaOU <- function(alpha, theta, rho, dt, T, T_min) {
  t <- seq(0, T, dt) # Vector of times
  n <- length(t)  # Number of time steps.
  X_sup <- rep(0, n) #Initialization of supOU vector
  arrival = T_min+rexp(1, rate = rho)
  while(arrival<t[n])
  {
    R_K <- rgamma(1, shape = alpha + 1, rate = alpha)
    Z = rexp(1, rate= theta)
    ind = floor(arrival/dt)+2
    if(arrival<0)
      X_sup = X_sup+Z*exp(R_K*(arrival-t))
    else
      X_sup[ind:n] = X_sup[ind:n]+Z*exp(R_K*(arrival-t[ind:n]))
    arrival = arrival+rexp(1, rate = rho)
  }
  return(X_sup)
}
x = r_supGammaOU(alpha=1.2, theta=3, rho=2, dt=1, T=6000, T_min=-100)
```

The R code for simulation of the supΓOU process with the correlation function $r(t) = E_{\alpha,1}(-|t|^{\alpha})$ is given below and based on the direct superposition of OU processes, $\alpha \in (0,1)$. The previous algorithm cannot be used because $\int_0^\infty q_\alpha(\xi)d\xi = \infty$, where $q_\alpha(\xi)$ is defined in Example 3.16. The marginal distribution of the supΓOU process is the gamma distribution with parameters θ and ρ.

```
r_GammaOU <- function(R, theta, rho, dt, T, T_min) {
  t <- seq(0, T, dt) # Vector of times
```

```
  n <- length(t)  # Number of time steps.
  OU <- rep(0, n) # Initialization of OU vector
  arrival = T_min+rexp(1, rate = rho)/R # rho - shape of gamma
  while(arrival<t[n])
  {
    Z = rexp(1, rate= theta) # theta - rate of gamma
    ind = floor(arrival/dt)+2
    if(arrival<0)
      OU = OU+Z*exp(R*(arrival-t))
    else
      OU[ind:n] = OU[ind:n]+Z*exp(R*(arrival-t[ind:n]))
    arrival = arrival+rexp(1, rate = rho)/R
  }
  return(OU)
}
r_supGammaOU_MLcor <- function(alpha, theta, rho, dt, T, T_min) {
  M = 20 # alpha must belong to the interval (0,1)
  p = (c(1:M)-0.5)/M # uniform points on [0,1]
  b = rep(1/M, M) # weights of points
  R = (sin(p*alpha*pi)/sin((1-p)*alpha*pi))^(1/alpha) # OU rates
  R[R>10] = 10
  supOU = r_GammaOU(R[1], theta, rho*b[1], dt, T, T_min)
  for(k in 2:M)
  {
    ou = r_GammaOU(R[k], theta, rho*b[k], dt, T, T_min)
    supOU = supOU+ou
  }
  return(supOU)
}
x=r_supGammaOU_MLcor(alpha=0.6, theta=3, rho=2, dt=0.5, T=6000, T_min=-100)
```

CRediT author statement

Anastasiia Kovtun: Methodology, Software, Original draft preparation Nikolai Leonenko: Supervision, Methodology, Reviewing and Editing Andrey Pepelyshev: Methodology, Software, Visualization, Reviewing and Editing

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: