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A Difficulty in Characterising Mixed Nash Equilibria in a Strategic Market Game*

Ralph W. Bailey[†], Maria Kozlovskaya[‡] and Indrajit Ray[§]

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Abstract

We analyse the conditions for a particular type of mixed strategy profile to be an equilibrium in a specific buy and sell strategic market game *à la* Shapley and Shubik (1977), with two goods, using best responses of a player against random bids by the other players. We first identify a difficulty in characterising mixed Nash equilibria; we show that the expected utility is not quasiconcave in strategies. We still prove that any mixed strategy Nash equilibrium profile in which every player faces a mix over only two positive bids is *purifiable*, that is, is a mixture of some pure strategy Nash equilibrium profiles. Moreover, we prove that the outcome (the price and the allocations) is deterministic in any such equilibrium.

Keywords: Mixed bids, Mixed strategy Nash equilibrium, Strategic market games.

JEL Classification Numbers: C72.

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1 INTRODUCTION

Randomness in economic outcomes (prices and net allocations) has been extensively studied both in competitive and strategic models. Some of these studies are based on intrinsic uncertainty while many examine extrinsic uncertainty. In the competitive framework, the concept of sunspot equilibrium (Cass and Shell, 1983) models extrinsic uncertainty; in non-cooperative games, the notion of correlated equilibrium (Aumann, 1974, 1987) has been widely studied.

Strategic market games, introduced by Shapley (1976), Shapley and Shubik (1977), Dubey and Shubik (1978, 1980) among others, develop a method of constructing a non-cooperative game from the general equilibrium framework (for surveys, see Dubey, 1994; Ali Khan and Sun, 2002; Giraud, 2003).

In strategic market games, players can influence the prices through their buy and sell orders. In the Shapley and Shubik (1977) version of the game, one good is specified as the monetary medium used for buying and selling all other commodities.¹ The price at each single-good trading post is simply the money / good ratio at that post.

The definition of non-cooperative equilibrium in this context is simply a Nash equilibrium in pure strategies, where no trader, given the bids of others, can improve by deviating unilaterally. There is a rich literature on randomness in terms of both correlated and sunspot equilibria within this framework (see for example, Aumann, Peck and Shell, 1988; Davila, 1999; Forges, 1991; Forges and Peck, 1995; Maskin and Tirole, 1987; Peck, 1994; Peck and Shell, 1991; Polemarchakis and Ray, 2006). Surprisingly however, random moves of individuals in this game are still left to be analysed. To the best of our knowledge, the first and perhaps only work that considers mixed strategies in market games is by Levando (2012); Levando and Sakharov (2018) analyse a market game with two goods and two players with a specific utility function, however they do not explicitly characterise the structure of the mixed equilibria in their game. Voliotis (2006) analyses boundedly rational traders who act strategically in a “sell all” version of strategic market game. All strategic market games *à la* Shapley and Shubik (1977) have a no-trade (“trivial”) equilibrium, where nothing is offered on either side of each trading post. The existence of equilibria with positive bids is still an open question (Busetto and Codognato, 2006). Here we do not contribute to the existence question, but merely characterise a special case of mixed strategy Nash equilibria, if they exist. Mitra, Ray and Roy (2024) characterise “trading” equilibria that exist in buy-and-sell games.

Shapley and Shubik, in their paper (1977, p. 948, footnote 17) remark: “The definition of Nash

¹There are possible variations (“buy or sell” or “sell all”) of this game. Also, there are other ways to model trading posts in strategic market games. In Sahi and Yao (1989) and Amir *et al* (1990), all goods can be exchanged directly; hence a trader’s bid for a good specifies the amounts of all other goods she wants to sacrifice in exchange for it. In these complete-market models, a price vector (one price for each good as in Sahi and Yao, 1989) or a price matrix (one price for each pair of goods as in Amir *et al*, 1990) equate the supply and the demand for each good.

includes the possibility of mixed strategies. These have no plausible interpretation in our present model so we shall be searching only for Nash equilibrium in pure strategies.” Shapley himself however, seems to have a contrary view when he claims (1976, p. 158): “... a well-defined game for which other types of solution may be attempted (including the Nash equilibrium in mixed strategies).” We agree with the view expressed in Shapley (1976) above. The strategic market game is indeed a well-defined non-cooperative game and therefore one should consider mixed strategies. The purpose of this paper is to study the structure of possible mixed strategy Nash equilibria in such games.

To characterise the set of mixed Nash equilibria, we consider a simple framework for a buy-and-sell game, with *two* goods only (a commodity and a “money”) and n players. We apply the concept of mixed strategies directly to the strategy sets of the players: the players are allowed to randomise over pure strategies in their respective strategy sets. The outcome (the prices and the allocations) is obtained on the basis of the realisations of the mixed strategies. The payoff for a player is the expected payoff over the outcomes from the realised strategies.

The intellectual appeal of strategic market games lies in their explicit pricing mechanism, where each trader can single-handedly influence the price. We show that this analytical completeness comes at the cost of equilibrium under-determinacy. In particular, the explicit pricing mechanism means the players’ payoff functions are not concave and their best responses are not necessarily unique. The intuition behind the above observation is as follows. Each player can affect the market price by varying the amounts she sells and buys. Since these two variables affect the price, they do not linearly cancel out. This means the player’s payoff is not concave in strategies even if it is concave in final allocations of commodities.²

Moreover, expected utility need not even be quasiconcave in strategies, as we show in an example. Hence, the best response to a mixed profile of other player’s strategies might not be unique. This means strategic market games have a higher level of equilibrium under-determinacy than previously thought. Not only do they have multiple pure strategy equilibria, fully mixed-strategy equilibria also cannot be ruled out. The presence of fully mixed equilibria would have important implications in the real-life counterparts of strategic market games, suggesting price instability in commodity markets.

We first analyse best responses for a player in our set-up. Our first result shows that the best response for a player against a mixed strategy profile is unique, apart from some degenerate cases. We prove that the outcome (price and allocation) from a mixed strategy equilibrium profile in such a market is deterministic. However, we can fully characterise the set of mixed Nash equilibria just for the special case in which the player who plays a mixed strategy is mixing over only *two* positive bids. Although this profile is very restrictive, we can prove a conclusive result and show that there

²Concavity of a utility function in the final holdings of commodities is a standard assumption which reflects the decreasing marginal utility of those commodities.

is no effect of mixing in this game. We find that the mixed Nash equilibria are *purifiable*, as they are simply mixtures of pure equilibria. This perhaps justifies the dismissive judgment of Shapley and Shubik mentioned above, at any rate as regards this special case.

It must be stressed that in strategic market games, a player does not usually have a unique best response. Indeed this is true when only pure strategies are considered, as noted by Shapley (1976). Therefore, the very problem of existence of a Nash equilibrium in pure strategies has not been solved so far (see Busetto and Codognato, 2006). Consequently, the characterisation of the set of MSNE for this game seems to be a problematic task, given that even in the case of pure strategies we cannot rule out that the set of equilibria is empty. Our work is particularly relevant for this issue, since our characterisation shows that MSNEs are purifiable.

2 MODEL

Consider an exchange economy with n (≥ 2) agents (indexed by $i = 1, 2, \dots, n$) and two commodities, labelled 1 and 2, whose consumption quantities are denoted respectively by x and y . Commodity 2 is money (the *numéraire* good). Each agent i is endowed with a positive vector of goods, $(w_{i1}, w_{i2}) \gg 0$ and has a concave, strictly increasing³ and differentiable utility function over her final allocation of the two commodities, $u_i(x_i, y_i) : R_+^2 \rightarrow R$. The final allocations are determined in a strategic market game.

Each agent in the economy is a player in the game: $N = \{1, \dots, n\}$. In a buy-and-sell game, players announce the amount of good 1 they want to sell, denoted by q_i , and also the amount of money they want to spend on buying back the same good (good 1), denoted by b_i . Hence, (q_i, b_i) is player i 's pure strategy, or bid. Let $(q, b) = (q_i, b_i)_{i=1, \dots, n}$ denote the profile of pure strategies. A player cannot bid more than her endowment; hence, her pure strategy set is given by $S_i = \{(q_i, b_i) : 0 \leq q_i \leq w_{i1}, 0 \leq b_i \leq w_{i2}\}$, $i = 1, \dots, n$. Given a strategy profile (q, b) , the market-clearing price $p(q, b)$ is formed as a ratio of total bid to total supply (if positive):

$$\begin{aligned} p &= \sum_{i=1}^n b_i / \sum_{i=1}^n q_i && \text{if } \sum_{i=1}^n q_i > 0; \\ &= 0 && \text{if } \sum_{i=1}^n q_i = 0. \end{aligned} \tag{1}$$

A player's final allocation is determined from her initial allocation (w_{i1}, w_{i2}) , her bid (q_i, b_i) and the price p (of good 1):

³The assumption of strict monotonicity was first introduced in Dubey and Shubik (1978), however, it was not present in the Shapley and Shubik (1977) paper. Busetto and Codognato (2006) used it to prove their existence results.

$$\begin{cases} x_i(q, b) &= w_{i1} - q_i + b_i/p & \text{if } p > 0, \\ &= w_{i1} - q_i & \text{if } p = 0 \\ y_i(q, b) &= w_{i2} - b_i + q_i p. \end{cases} \quad (2)$$

Player i 's payoff from her final allocations of the two goods is given by $u_i(q, b) = u_i(x_i(q, b), y_i(q, b))$, $i = 1, \dots, n$.

Suppose all players apart from i are playing a pure strategy. Let (Q_{-i}, B_{-i}) denote the other players' total bid: $Q_{-i} \equiv \sum_j q_j$, $j \in N \setminus \{i\}$; $B_{-i} \equiv \sum_j b_j$, $j \in N \setminus \{i\}$.

Suppose $(Q_{-i}, B_{-i}) \gg 0$, meaning $Q_{-i} > 0$ and $B_{-i} > 0$. We maintain this assumption throughout.

Then the price can be written as

$$p = \frac{B_{-i} + b_i}{Q_{-i} + q_i} > 0 \quad (3)$$

and system (2) rearranged as follows:

$$\begin{cases} x_i &= w_{i1} + Q_{-i} - B_{-i}/p; \\ y_i &= w_{i2} + B_{-i} - Q_{-i}p. \end{cases} \quad (4)$$

In words, given the other players' strategy the final allocation and payoff of player i depends only on the price.

Importantly, for a fixed (Q_{-i}, B_{-i}) , player i 's utility is strictly concave in price, which we show in the following lemma.

Lemma 1 *Let $u_i(x_i, y_i) : R_+^2 \rightarrow R$ be strictly increasing in each of its arguments, and concave. Let $x_i = x_i(p) : R_+ \rightarrow R_+$ be a strictly concave function and $y_i = y_i(p) : R_+ \rightarrow R_+$ be a weakly concave function. Then the composition $u_i = u_i(x_i(p), y_i(p)) : R_+ \rightarrow R$ is strictly concave.*

Proof. Strict concavity of $x_i(p)$, with weak concavity of $y_i(p)$, implies that $x_i(\lambda p + \lambda' p') > \lambda x_i(p) + \lambda' x_i(p')$ and $y_i(\lambda p + \lambda' p') \geq \lambda y_i(p) + \lambda' y_i(p')$. The strict monotonicity and the concavity of $u_i(x_i, y_i)$ imply that $u(x_i(\lambda p + \lambda' p'), y_i(\lambda p + \lambda' p')) > u(\lambda x_i(p) + \lambda' x_i(p'), \lambda y_i(p) + \lambda' y_i(p')) \geq \lambda u(x_i(p), y_i(p)) + \lambda' u(x_i(p'), y_i(p'))$. Thus, we have shown, with p and p' distinct, $\lambda \in (0, 1)$, $\lambda' = 1 - \lambda$, and $p'' = \lambda p + \lambda' p'$, $u(x_i(p''), y_i(p'')) > \lambda u(x_i(p), y_i(p)) + \lambda' u(x_i(p'), y_i(p'))$. ■

2.1 Mixed Strategy

When it comes to mixed strategies in this game, we consider mixing over finitely many positive points only. By a positive point, we mean a strategy where both dimensions are strictly positive: $(q_i, b_i) \gg 0$ if $q_i > 0$ and $b_i > 0$. Let $K \in N$ denote the number of points player i is mixing over.

Definition 1 A mixed strategy of player i is a probability distribution μ over finitely many pure positive strategies $(q_i^k, b_i^k) \gg 0$, $k = 1, 2, \dots, K$ (where $K \geq 2$) with respective probabilities μ_k such that $\mu_k > 0$ for all $k = 1, 2, \dots, K$ and $\sum_{k=1}^K \mu_k = 1$.

Player i 's payoff from a mixed strategy profile is the usual expected payoff.

First, we present an important negative result. In a strategic market game with a concave (in fact, strictly concave) utility over allocations, the resulting expected utility over strategies need not be quasiconcave.

Example 1. Consider a player i with the initial endowment of $(w_{i1}, w_{i2}) = (1, 1)$ and the following utility function over the allocation of the two commodities: $u_i(x_i, y_i) = x_i^\alpha + y_i^\alpha$, where $\alpha \in (0, 1]$. It is easy to verify that this utility function is concave, strictly increasing and differentiable.

Let the player face two different total bids $(Q_{-i}^1, B_{-i}^1) = (0.9, 0.37)$ and $(Q_{-i}^2, B_{-i}^2) = (0.01, 0.1)$, with probability 0.5 each.

Now consider the following two strategies of the player: $(q_i, b_i) = (0.05, 0.04)$ and $(q'_i, b'_i) = (1, 0.64)$, with respective price realisations as follows:

$$(q_i, b_i) = (0.05, 0.04) \text{ with price realisations } p_1 = \frac{0.04+0.37}{0.05+0.9} = \frac{41}{95}; p_2 = \frac{0.04+0.1}{0.05+0.01} = \frac{7}{3},$$

$$(q'_i, b'_i) = (1, 0.64) \text{ with price realisations } p'_1 = \frac{0.64+0.37}{1+0.9} = \frac{101}{190}; p'_2 = \frac{0.64+0.1}{1+0.01} = \frac{74}{101}.$$

Now take the $\frac{1}{2}$ - $\frac{1}{2}$ convex combination (the average) of these two strategies:

$$(q''_i, b''_i) = (0.525, 0.34) \text{ with price realisations } p''_1 = \frac{0.34+0.37}{0.525+0.9} = \frac{142}{285}; p''_2 = \frac{0.34+0.1}{0.525+0.01} = \frac{88}{107}.$$

With a linear utility function for this player: $u_i(x_i, y_i) = x_i + y_i$, the player's expected utility is not quasiconcave in her strategy, since the expected utility of a convex combination of strategies is less than the smaller of the two endpoint expected utilities, as shown below:

$$E(u_i(q''_i, b''_i)) = 2.0296 < \min\{2.0340, 2.0309\} = \min\{E(u_i(q_i, b_i)), E(u_i(q'_i, b'_i))\}.$$

The same is true for a strictly concave utility function $u_i(x_i, y_i) = x_i^{0.9} + y_i^{0.9}$ as well, as shown below.

$$E(u_i(q_i, b_i)) = \frac{1}{2} \left[\left(1.9 - \frac{0.37*95}{41}\right)^{0.9} + \left(1.37 - \frac{0.9*41}{95}\right)^{0.9} + \left(1.01 - \frac{0.1*3}{7}\right)^{0.9} + \left(1.1 - \frac{0.01*7}{3}\right)^{0.9} \right] = 2.0304.$$

$$E(u_i(q'_i, b'_i)) = \frac{1}{2} \left[\left(1.9 - \frac{0.37*190}{101}\right)^{0.9} + \left(1.37 - \frac{0.9*101}{190}\right)^{0.9} + \left(1.01 - \frac{0.1*101}{74}\right)^{0.9} + \left(1.1 - \frac{0.01*74}{101}\right)^{0.9} \right] = 2.0261.$$

$$E(u_i(q''_i, b''_i)) = \frac{1}{2} \left[\left(1.9 - \frac{0.37*285}{142}\right)^{0.9} + \left(1.37 - \frac{0.9*142}{285}\right)^{0.9} + \left(1.01 - \frac{0.1*107}{88}\right)^{0.9} + \left(1.1 - \frac{0.01*88}{107}\right)^{0.9} \right] = 2.0255.$$

$$E(u_i(q''_i, b''_i)) = 2.0255 < \min\{2.0304, 2.0261\} = \min\{E(u_i(q_i, b_i)), E(u_i(q'_i, b'_i))\}.$$

It therefore remains an open question whether the best response set for a player against a mixed strategy profile (of others) must always be a singleton, or even be a convex set. We are, nevertheless, able to characterise mixed strategy Nash equilibria in a special case in the next section.

In particular, we deal with a specific mixed strategy profile, as below.

Definition 2 (i) For a two-player game ($n = 2$), a **2-point mixed strategy profile** is a profile in which at least one player plays a mixed strategy (as in Definition 1) however no player mixes over more than two pure strategies (that is, for any mixing player, $K = 2$ as in Definition 1).

(ii) For $n > 2$, a **2-point mixed strategy profile** is a profile in which only one player j mixes over two pure strategies (that is, in player j 's mixed strategy, $K = 2$ as in Definition 1), while all other players play pure strategies.

In accordance with Definition 2, in a 2-point mixed strategy profile any player faces *either* a deterministic (pure) total bid by the other players (if none of them are mixing) *or* a probability distribution over two pure positive total bids by the other players (if only one of the other players is mixing, say playing $(q_j^1, b_j^1) \gg 0$ with probability σ_{-i}^1 and a bid $(q_j^2, b_j^2) \gg 0$ with probability $\sigma_{-i}^2 = 1 - \sigma_{-i}^1$). In the latter case the resulting mixed bid, denoted by σ_{-i} , is a probability distribution over two strictly positive total bids: $(Q_{-i}^1, B_{-i}^1) \equiv (q_j^1, b_j^1) + \sum_{l \neq i, j}^N (q_l, b_l)$ with probability σ_{-i}^1 and a bid $(Q_{-i}^2, B_{-i}^2) \equiv (q_j^2, b_j^2) + \sum_{l \neq i, j}^N (q_l, b_l)$ with probability $\sigma_{-i}^2 = 1 - \sigma_{-i}^1$. In words, each total bid in the mix is derived by adding the respective strategy of the mixing player to the deterministic sum of strategies of other (non-mixing) players.

Definition 3 A 2-point mixed total bid faced by player i (denoted by σ_{-i}) is a probability distribution over two strictly positive total bids of other players $(Q_{-i}^k, B_{-i}^k) \gg 0$, $k = 1, 2$ with respective probabilities σ_{-i}^k such that $\sigma_{-i}^k > 0$ for $k = 1, 2$ and $\sum_{k=1}^K \sigma_{-i}^k = 1$.

In other words, in a 2-point mixed strategy profile each player is facing at most one other player who is mixing. In a game with more than two players, we are looking at strategy profiles where only one player is mixing (over two positive points), while in a game with two players, such profiles include both players mixing over two positive bids.

The definition of equilibrium in this game is standard Nash equilibrium (either in pure or in mixed strategies). A (pure or mixed) strategy profile is said to be a (Nash) equilibrium if every player is playing a best response against other players' strategies in the profile.

3 BEST RESPONSE ANALYSIS

Before we find the best response of player i against $(Q_{-i}, B_{-i}) \gg 0$, we observe the following.

If $B_{-i} = Q_{-i} = 0$, the set of allocations achievable by player i consists of her initial endowment, and all the points where she has less of one of the goods than she was endowed with: $\{(w_{i1} - q_i, w_{i2}) \cup (w_{i1}, w_{i2} - b_i) \mid q_i \in [0, w_{i1}], b_i \in [0, w_{i2}]\}$. By component-wise monotonicity of $u_i(x_i, y_i)$, the unique

maximum is the initial endowment, which can be achieved by either abstaining from trade: $(q_i, b_i) = (0, 0)$, or sending positive quantities of both goods, which return to the player: $(q_i, b_i) \gg 0$. If she bids a positive amount of only one good, she generates a zero price and “burns” her bid.

If $B_{-i} > 0$ and $Q_{-i} = 0$, the set of allocations achievable by player i consists of two line segments: (i) an open set where she acquires all B_{-i} by bidding any amount $q_i > 0$ and generating a positive price $p = (B_{-i} + b_i)/q_i$, and (ii) a closed set where she does not bid any $q_i = 0$, generating a zero price; any b_i she bids is “burned”. Mathematically, her set of achievable allocations is $\{(w_{i1} - q_i + b_i q_i / (B_{-i} + b_i), w_{i2} + B_{-i}) \mid q_i \in (0, w_{i1}], b_i \in [0, w_{i2}]\} \cup \{(w_{i1}, w_{i2} - b_i) \mid b_i \in [0, w_{i2}]\}$. To obtain player i 's best response set, first observe that for any $q_i > 0$ and any b_i she can increase her utility by bidding a fraction of q_i instead (for example, $q_i/2$), because she would capture the same amount of the second good in exchange for a smaller amount of the first good: $u(w_{i1} - q_i/2 + b_i q_i/2 / (B_{-i} + b_i), w_{i2} + B_{-i}) > u(w_{i1} - q_i + b_i q_i / (B_{-i} + b_i), w_{i2} + B_{-i})$ for all $q_i > 0$. Second, for $q_i = 0$ and any $b_i > 0$ player i can increase her utility by not burning money: $u(w_{i1}, w_{i2}) > u(w_{i1}, w_{i2} - b_i)$. Finally, if $q_i = b_i = 0$ player i can increase her utility by sending a sufficiently small amount of the first good to the market, to capture the entire supply of the second good. Indeed, by continuity of $u_i(x_i, y_i)$ there exists a q_i^* sufficiently small that she would be happy to trade it off for the whole of B_{-i} , i.e. such that $u(w_{i1}, w_{i2}) < u(w_{i1} - q_i^*, w_{i2} + B_{-i})$. In other words, for any strategy of player i , a different strategy can be found resulting in greater utility. Hence player i 's best response set to $B_{-i} > 0$ and $Q_{-i} = 0$ is empty.

By symmetry, the same argument applies to the case where $Q_{-i} > 0$ and $B_{-i} = 0$. Player i 's best response set in that case is also empty. It follows that, if a strategy profile is such that any player i is facing either $B_{-i} > 0$ and $Q_{-i} = 0$ or $Q_{-i} > 0$ and $B_{-i} = 0$, that strategy profile cannot be a Nash equilibrium (a mutual best response).

Now let $A_i(Q_{-i}, B_{-i})$ denote the set of allocations achievable by player i , given $(Q_{-i}, B_{-i}) \gg 0$. The set can be characterised by solving one of the equations in system (4) for p and substituting into the other:

$$A_i(Q_{-i}, B_{-i}) = \{(x_i, y_i) \in R_+ \mid x_i y_i - (w_{i2} + B_{-i})x_i - (w_{i1} + Q_{-i})y_i + w_{i1}w_{i2} + w_{i1}B_{-i} + w_{i2}Q_{-i} = 0\} \quad (5)$$

Observe that i 's set of achievable allocations $A_i(Q_{-i}, B_{-i})$ is the graph of the strictly concave continuous function $y_i(x_i) = w_{i2} + B_{-i} - Q_{-i}B_{-i} / (w_{i1} + Q_{-i} - x_i)$ with the domain $x \in [0, w_{i1} + w_{i2}Q_{-i} / (w_{i2} + B_{-i})]$, which is a hyperbola segment joining the axes at allocations achievable respectively by the strategies $(q_i, b_i) = (0, w_{i2})$ and $(q_i, b_i) = (w_{i1}, 0)$.

The set $A_i(Q_{-i}, B_{-i})$ is thereby closed and bounded. Hence by the Extreme Value Theorem a continuous function $u_i(x_i, y_i)$ attains a global maximum (x_i^*, y_i^*) over the set.

Suppose two distinct points (x_i, y_i) and (x'_i, y'_i) maximise the concave $u_i(x_i, y_i)$, then so would their convex combination (x''_i, y''_i) . However, the vertical projection of that point onto the graph of $y_i(x_i)$ would attain a strictly greater utility $u_i(x''_i, y_i(x''_i)) > u_i(x''_i, y''_i) = u_i(x_i, y_i) = u_i(x'_i, y'_i)$, by strict monotonicity of $u_i(x_i, y_i)$ and strict concavity of $y_i(x_i)$ – a contradiction; hence the maximum (x_i^*, y_i^*) is unique.

Player i can achieve (x_i^*, y_i^*) by a continuum of strategies (q_i, b_i) satisfying equation (6):

$$b_i = q_i \frac{B_{-i}}{w_{i1} - x_i^* + Q_{-i}} + \frac{B_{-i}(x_i^* - w_{i1})}{w_{i1} - x_i^* + Q_{-i}} \quad (6)$$

Equation (6) implicitly characterises player i 's best response set to an aggregate bid of the other players, (Q_{-i}, B_{-i}) .

We now analyse best responses of a player against a mixed total bid from others (a non-degenerate probability distribution over the sum of bids and offers of players other than i).

Our main theoretical contribution is the extension of the best response analysis of a player i to a 2-point mixed strategy profile. The best response here cannot be easily characterised by closed-form equations such as (5) and (6). However, the best response to a mixed total bid has an important property, which we derive below.

Proposition 1 *The best response to a 2-point mixed total bid is unique, apart from one degenerate case when the best response lines to the realisations of the mixed bid coincide. In that case, any point in that joint best response set is a best response to the mixed strategy.*

The detailed proof of Proposition 1 is relegated to Appendix. In the proof, we show that the best response falls into one of the three cases:

- Case 1. Best response lines to realisations of the mixed bid coincide. In this case, any point on that line is a best response against the mixed bid (the degenerate case).
- Case 2. Best response lines to realisations of the mixed bid have an intersection in the player's strategy set. In this case, this intersection is the unique best response against the mixed bid.
- Case 3. Best response lines to realisations of the mixed bid do not have an intersection in the player's strategy set. In this case, a point on the boundary of the player's strategy set is the unique best response against the mixed bid.

We provide examples, one each, of these three cases in the Appendix.

4 MIXED STRATEGY NASH EQUILIBRIUM

The definition of Nash equilibrium (in pure or mixed strategies) is standard for such a game.

Definition 4 *A 2-point mixed strategy Nash equilibrium (hereafter a **2-point MSNE**) is a 2-point mixed strategy profile such that no player can strictly increase their expected utility by playing a different strategy.*

We now proceed to characterise the 2-point MSNE of our market game. We start by showing that the outcome of 2-point MSNE is deterministic.

Proposition 2 *If a 2-point mixed strategy profile is a 2-point MSNE, then the realised outcome (price and final allocations) is deterministic.*

Proof. We invoke the proof of Proposition 1 to show that the best response to a mixed strategy is either is a unique (pure) bid (Cases 2 and 3), or a continuum of bids generating the same price (Case 1).

Consider a 2-point MSNE in a game with more than two players (that is, $n > 2$). By Definition 2(ii), only one player is mixing. Then that player is doing so in the manner described in the previous step, resulting in deterministic price and outcome for all players.

Now consider a 2-point MSNE in a game with two players (that is, $n = 2$); by Definition 2(i), two players are mixing. Then each mixing player has to be best-responding to a mixed strategy of the other player. A mixed strategy can only be a best response to a mixed total bid in Case 1 (coinciding best-response lines) described in the proof of Proposition 1. This case entails the player mixing between strategies collinear with the realisations of the other players' total bids, hence entailing the same price, and the same outcome for both the players. ■

Before characterising 2-point MSNEs further, we introduce a useful concept.

Definition 5 *A 2-point MSNE is called **purifiable** if any realisation of the players' strategies under that 2-point MSNE forms a pure strategy Nash equilibrium.*

Clearly, in a game with more than two players, any 2-point MSNE has exactly two pure strategy realisations while in a game with two players, there may be either two or four possible realisations. "Purifiability" in Definition 4 implies a mixture of pure strategy Nash equilibria. It turns out that "non-purifiable" 2-point MSNE do not exist, as the theorem below demonstrates. In other words, in every such 2-point MSNE, any of its constituent pure-strategy profiles is by itself a pure Nash equilibrium.

Theorem 1 *Consider a 2-good, n -player strategic market game. Any 2-point MSNE of this game is purifiable.*

Proof. First note that from the proof of Proposition 2, in any 2-point MSNE, the price is deterministic. This means that all players best-responding to a mixed total bid are playing a strategy giving rise to the same price under both total bids in the mix. Geometrically, their strategy is thus necessarily lying on the line connecting the two pure total bids of the other players (implying also that this line intersects the strategy set). Since all strategies on this line give rise to the same expected utility, they are also best responses.

Recall that, according to Proposition 1, the best response to a mixed strategy is either a unique point in the pure strategy space, or, in the extreme case where the best response lines to constituent pure profiles in that mixed strategy coincide, this whole line is the best response. However we just argued above that a unique best response cannot be part of 2-point MSNE. The only remaining possibility is a continuum of best responses, which is possible only in Case 1, where any point in the best-response continuum is also a best response against either constituent pure bid of the other players.

Theorem 1 now follows from this observation. It follows that in any 2-point MSNE, all players facing a mixed total bid are playing a strategy which would have been a best response against all constituent pure bids within this mixed bid, which proves the Theorem. ■

An important implication of Theorem 1 is that Cases 2 and 3 from the proof of Proposition 1 cannot be part of our 2-point MSNE (the cases where the best response is unique). The only remaining case is Case 1 (coinciding BR lines; hence any point on that line is a best response). Section 7.1 of the Appendix presents an example of a mixed strategy Nash equilibrium in a game with two players, who are both mixing in accordance with Case 1. That equilibrium is purifiable, illustrating Theorem 1.

5 FURTHER ANALYSIS

5.1 More than 2 goods

One may ask if our analysis could be generalised by considering n commodities. This would also raise the question of which kind of strategic market game should be used with more than 2 commodities; for instance, one may use decentralised trading posts or centralised consistent market clearing prices as in the Sahi and Yao (1989) model. Consideration of n -commodity strategic market games opens the general question of the interplay between price mechanisms and Nash equilibria in such games.

The type of market price mechanism we have in mind, if the number of commodities generalises from 2 of the paper to g , is as follows. Commodity g is the numéraire ('money'). There are $g - 1$ trading posts. Trader i leaves quantity q_{ij} of commodity j and quantity b_{ij} of money at post j . Trader i has strictly concave utility $u_i : R_+^g \rightarrow R$. The market-clearing price of commodity j is $p_j = \frac{\sum_i b_{ij}}{\sum_i q_{ij}}$, $1 \leq j \leq g - 1$. For such a market game, the only significant result we have relates to pure-strategy

equilibria. In any pure-strategy equilibrium in which player i faces plays (q_i, b_i) when facing the aggregate action (q_{-i}, b_{-i}) by the other players, leading to price $p = \frac{b_i + b_{-i}}{q_i + q_{-i}}$, player i could equally well play $(q'_i, b'_i) \equiv (q_i, b_i) + \lambda(1, p)$ for any λ for which (q'_i, b'_i) is a feasible action. Such an action would leave p unaltered, would also be a best response, and would define an alternative pure equilibrium.

Lemma 1 for the 2-good case, in this paper, that a convex combinations of actions leads to a convex combination of prices, seems not to generalise for more than two goods in the above set up. Although the price of each good is a convex combination of its two prices, the same is not true of the price vector as a whole. So, it might be hard to show that a player's utility function, facing random strategies of the other players, over actions is concave, for more than two goods; indeed, this result might not even be true. Therefore, the possibility that there are "interesting" 2-point MSNE, with more than two commodities, remains open.

5.2 Corner endowments

One may also wonder about mixed strategy Nash equilibria in a two-commodity version with corner endowments. With corner endowments, our model would be akin to a bilateral oligopoly model, as the one introduced by Gabszewicz and Michel (1997), in which case, all three kinds of strategic market games, namely those of Shapley and Shubik (1977), Sahi and Yao (1989), and Amir *et al* (1990), coincide, offering a possible comparison with different market price mechanisms. Let us therefore analyse our market game with corner endowments.

Suppose each agent i is endowed with a positive amount of only one of the goods, and none of the other (corner endowments): $w_{i1} \cdot w_{i2} = 0$ and $w_{i1} + w_{i2} > 0$ for all $i \in N$. Player i 's strategy (q_i, b_i) is now effectively unidimensional, as she can only offer a zero amount of the good she was not endowed with: $q_i \cdot b_i = 0$.

If the game has only two players (endowed with different goods), no trade is its only equilibrium.⁴ To see why, suppose player 1 submits a positive amount of their good on the market. Player 2 can capture the whole supply by bidding an infinitesimal amount of the other good. The less she bids, the higher her utility, as long as she bids a positive amount. Hence, the best response to a positive bid of the other player does not exist. If the other player bids nothing, then bidding any amount will be a net loss, hence bidding nothing in return is the best response.

Now suppose there are $n > 2$ players in the game. Consider player 1 and let her be endowed with good 1, without loss of generality: $w_{11} > w_{12} = 0$. The player's strategy is hence $(q_1, 0)$, where $q_1 \in [0, w_{11}]$. Her final allocation is determined as follows:

⁴If both players are endowed with the same good, the price is always zero and whatever amounts they send to the market are lost, hence sending nothing is a strictly dominant strategy.

$$\begin{cases} x_1(q_1, p) &= w_{11} - q_1 \\ y_1(q_1, p) &= q_1 p, \end{cases} \quad (7)$$

where p , as before, denotes the market-clearing price (i.e. the ratio of total bid to total supply, if positive):

$$\begin{aligned} p &= \sum_{i=2}^n b_i / \sum_{i=1}^n q_i && \text{if } \sum_{i=1}^n q_i > 0; \\ &= 0 && \text{if } \sum_{i=1}^n q_i = 0. \end{aligned} \quad (8)$$

We first note that a no-trade equilibrium always exists with $Q_{-1} = B_{-1} = 0$, where player 1's best response is $q_1 = 0$ since any amount she sends to the market is lost. By symmetry, all other players in this profile are also best-responding. Moreover, no equilibrium exists where any player faces $Q_{-i} > B_{-i} = 0$ or $B_{-i} > Q_{-i} = 0$, because any of the players endowed with the zero-supply good could increase their utility by submitting an infinitesimal amount to the trading post. Hence we consider only equilibria where player 1 faces a positive total bid of other players, that is, $(Q_{-1}, B_{-1}) > 0$. Note that, as shown by Busetto and Codognato (2006), the game might not contain any equilibria with active trading posts.

When $(Q_{-1}, B_{-1}) > 0$, the market-clearing price can be written down as $\frac{B_{-1}}{q_1 + Q_{-1}}$, which means $y_1(q_1, Q_{-1}, B_{-1})$ is strictly concave in q_1 , while $x_1(q_1, Q_{-1}, B_{-1})$ is linear in q_1 :

$$\begin{cases} x_1(q_1, Q_{-1}, B_{-1}) &= w_{11} - q_1 \\ y_1(q_1, Q_{-1}, B_{-1}) &= \frac{q_1 B_{-1}}{q_1 + Q_{-1}}, \end{cases} \quad (9)$$

By Lemma 1, the composition $u = u_i(x_1(q_1, Q_{-1}, B_{-1}), y_1(q_1, Q_{-1}, B_{-1})) : R_+ \rightarrow R$ is strictly concave in q_1 , hence it attains a unique maximum on the player's compact convex strategy set $[0, w_{11}]$. In other words, player 1 always has a unique best response to any $(Q_{-1}, B_{-1}) \gg 0$. Hence she will never best-respond to a pure total bid with a mixed strategy.

Now suppose player 1 is facing a mixed total bid, i.e., a probability distribution μ over finitely many pure total bids (Q_{-1}^k, B_{-1}^k) , $k = 1, 2, \dots, K$ (where $K \geq 2$) with respective probabilities μ_k such that $\mu_k > 0$ for all $k = 1, 2, \dots, K$ and $\sum_{k=1}^K \mu_k = 1$. Her payoff in this case is the standard expected utility – a convex combination of utilities in each realisation of the mixed total bid: $U = \sum_{k=1}^K \mu_k u_1(x_1(q_1, Q_{-1}^k, B_{-1}^k), y_1(q_1, Q_{-1}^k, B_{-1}^k))$. As a convex combination of strictly concave functions, the expected utility is also strictly concave in q_1 ; hence, the player has a unique best response to the mixed total bid μ . By symmetry, the same argument applies to all other players. To summarise, there are no mixed equilibria in the game with corner endowments, since every player has a unique best response to any strategy profile of the other players, pure or mixed.

6 CONCLUDING REMARKS

Our result shows that there are no “interesting” mixed strategy equilibria in a market game, where some players are made indifferent between their strategies by a specific mixing probability of the other players (at least for the case where each player is facing a mixed bid over at most two positive profiles). In every realisation of a 2-point MSNE the outcome (price and allocations) of the game and utilities of all players are exactly the same. This result provides theoretical support to the intuitive dismissal by Shapley and Shubik (1977) of mixed equilibria case as uninteresting.

We admit that our characterisation of MSNE is indeed valid only in a very restrictive set-up. In particular, we avoid the problem of payoff discontinuity at $q_i = 0$ (respectively, $b_i = 0$) when $Q_{-i} = 0$ (respectively, $B_{-i} = 0$) by restricting our analysis to strictly positive bids in the mixed strategy. However, given the difficulty (posed by the fact that the expected utility may not be quasi-concave), this is the best result we could achieve. We have not managed to generalise these results, despite significant effort. We recognise that the most important open question in the literature is that of equilibrium existence (Busetto and Codognato, 2006); however, here we do not contribute to this debate, but merely characterise mixed strategy equilibria if they exist..

One may ask whether the restriction to mixed strategies over two actions is justified. Perhaps for instance a more natural arena for analysis would be games in which each player mixes over a convex set of actions. This, we think, is a speculation naturally arising from the limited nature of the results we have been able to attain, and therefore to some extent dependent on them. We postpone exploring other avenues to future research.

Some further comments are in order.

First, that our analysis might not be robust against small perturbations of the game, at any rate if we try to extend it to the case where a player can face a mixture of more than two aggregate bids by the other players. This is because in such a game we cannot guarantee uniqueness of a player’s best response (which remains an open question). As far as we know, a small increase in a player’s endowments might cause a switch to a quite different best response and a quite different equilibrium. Hence, in such an extended game, our purifiable equilibria are not “nice” in the sense of Cordella and Gabszewicz (1998) and Busetto and Codognato (2006).

Second, payoff concavity can be restored by restricting the strategy space to only one dimension for each trading post (“buy or sell” or “sell all” variations of the game). However, such restrictions can be circumvented in real-life markets. Characterising equilibria in the natural unrestricted formulation of a buy-and-sell strategic market game can shed light on price stability in such economies. If one does consider either the buy-or-sell or the sell-all version of the market game model, the best response of any player against a mixed strategy profile will be unique, and no mixed equilibria will exist, similarly

to the analysis in Section 5.2 above.

Third, it is interesting to know whether market games have mixed strategy Nash equilibria that are non-purifiable (in the sense of this paper). The question provides an insight into such games: if ‘non-trivial’ equilibria in this sense exist, then market games become both relatively harder and relatively more interesting to analyze. If they do not, which we might take to be the null conjecture about such games, then they are in a certain sense simple. We cannot claim to have resolved the existence question we set ourselves, but along with the purifiability results we have obtained, we have found an obstacle for future analysts to consider, namely that best response functions against mixed strategies are not necessarily quasiconcave.

In market game models, pure Nash equilibria are in general, (Pareto-) inefficient (Dubey 1980, Dubey and Shubik, 1980; Dubey and Rogawski, 1990). Our paper indicates that mixed strategies in strategic market games might not generate new equilibrium outcomes, even if we allow mixed strategies, in a restricted sense. It is now interesting to know whether mixed strategy Nash equilibria are efficient or not in a more general construct.

One could consider the effect of a replication of our basic game, i.e., about what would happen in case the number of traders increased, possibly without limit. Though the replicated game is simpler than a general more than 2-player game, in that the possible actions are more highly structured, the additional structure does not help us, as far as we can see. This is not a situation we have so far been able to analyse.

7 APPENDIX 1: PROOF OF PROPOSITION 1

Proof of Proposition 1. Consider player i and let one of the other players ($j \neq i$) play a mixture over two positive pure strategy profiles: $(Q, B) > 0$ and $(Q', B') > 0$. The best response sets to each of these profiles are denoted by BR and BR' respectively. As shown in the main text of the paper, BR and BR' are straight upward sloping lines in (q_i, b_i) space, characterised by equation (6). Finally, the player's payoff in a realisation of the mixed bid is denoted $u_{Q,B}(q_i, b_i)$ and $u_{Q',B'}(q_i, b_i)$ for the cases when the total bid is (Q, B) and (Q', B') respectively.

Player i 's strategy set S_i is a rectangle in (q_i, b_i) space. Fix other players' total bid (Q, B) and observe that player i 's best response line passes through the point $(-Q, -B)$, if extended to the third quadrant.⁵ In the analysis below, we consider these extended best response lines defined on R^2 , keeping in mind that only the line segments within S_i contains feasible strategies: $b_i = BR(q_i) : [0, w_{i1}] \rightarrow [0, w_{i2}]$.

Depending on the relative position of the two extended best response lines to realisations of the mixed bid, three cases can be considered:

Case 1. Best response lines coincide.

Case 2. Best response lines intersect in the player's strategy set.

Case 3. Best response lines intersect outside the player's strategy set.

We consider the three options one by one and prove that, in Case 1, the whole line is the best response, while in Cases 2 and 3, the best response is a unique point (q_i^*, b_i^*) in player i 's strategy set.

We also provide examples of best responses in each case for a player with the utility function $u_i = x_i y_i$ and an endowment $w_{i1} = w_{i2} = 3$, unless specified otherwise.⁶

For a player with the utility $u_i = x_i y_i$ the best response to total bid (Q, B) can be characterised as follows:

$$b_i = q_i \sqrt{\frac{B(w_{i2} + B)}{Q(w_{i1} + Q)}} + \sqrt{QB} \sqrt{\frac{w_{i2} + B}{w_{i1} + Q}} - B. \quad (10)$$

7.1 Case 1. Coinciding Best Response Lines

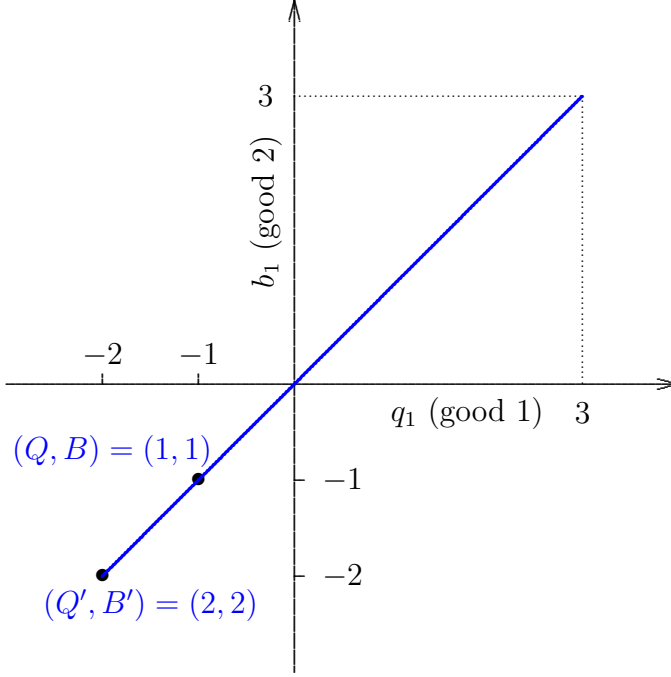
Let $BR = BR'$. In this case, any point on the line is a best response, generating the same price and final allocation for player i . Indeed, a point on the line maximises player i 's utility under either realisation of the other players' mixed bid, and hence also maximises the expected utility.

⁵Indeed, note that $(-Q, -B)$ satisfies the best response equation (6).

⁶The results in this paper hold for all admissible utility functions and endowments, rather than that specific player only.

Result: The best response coincides with the best response line to the realisations of the mixed bid.

Figure 1: Example 2. Best Response Line to (1,1) and (2,2)



Example 2. Consider $(Q, B) = (1, 1)$ and $(Q', B') = (2, 2)$.

Using the formula (10) and substituting $w_{i1} = w_{i2} = 3$, we find that player i 's best response lines to $(1, 1)$ and $(2, 2)$ coincide (Figure 1):

$$b_i = q_i. \quad (11)$$

Any point on the line $b_i = q_i$ maximises both $u_{Q,B}(q_i, b_i)$ and $u_{Q',B'}(q_i, b_i)$, and hence also their convex combination $U = \mu u_{Q,B}(q_i, b_i) + (1 - \mu) u_{Q',B'}(q_i, b_i)$.

Case 1 can be part of a mixed strategy Nash Equilibrium, as Example 2a below shows.

Example 2a. Consider player j who has the same utility function and endowments as player i .⁷ In a game between these two players, a strategy profile where both mix between $(q_i, b_i) = (q_j, b_j) = (1, 1)$ and $(q_i, b_i) = (q_j, b_j) = (2, 2)$ with any probabilities $p_i, p_j \in [0, 1]$ is a mutual best response. Indeed, any strategy such that $b_i = q_i$ (respectively, $b_j = q_j$) is a best response of i (respectively, j) to any

⁷The result will still hold if the players have different utility functions.

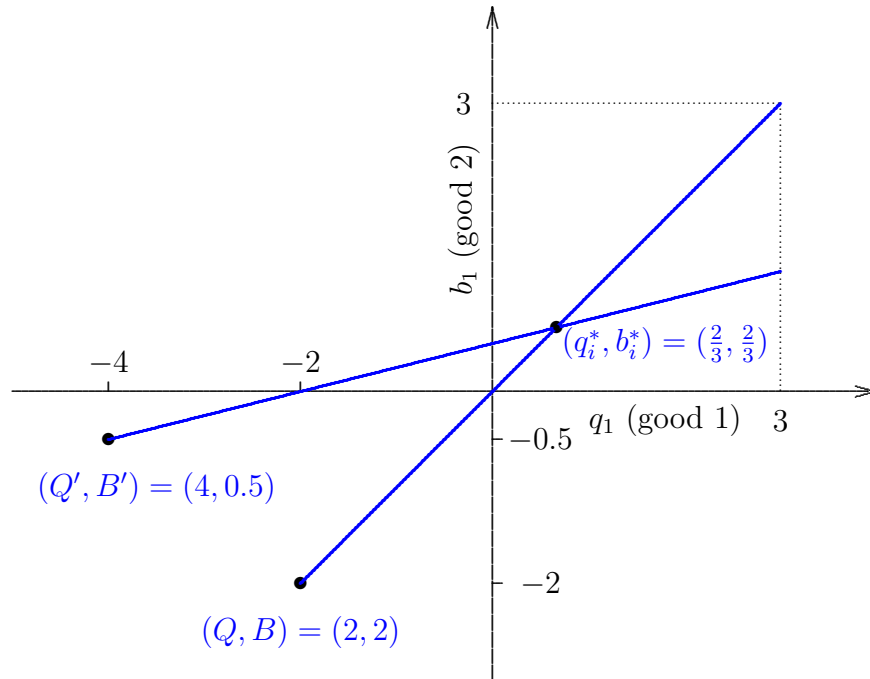
mixed total bid such that $Q_{-i} = B_{-i}$ (respectively, $Q_{-j} = B_{-j}$), and hence to any mixture between such total bids, as shown in equation (10) and Example 2.

7.2 Case 2. Intersecting Best Response Lines

Let $BR \cap BR' = (q_i^*, b_i^*) \in S_i$ (that is, the best-response lines intersect within S_i). By definition of the best response, $(q_i^*, b_i^*) = \arg \max u_{Q,B}(q_i, b_i) = \arg \max u_{Q',B'}(q_i, b_i)$ and hence, $(q_i^*, b_i^*) = \arg \max \{\mu u_{Q,B}(q_i, b_i) + (1 - \mu) u_{Q',B'}(q_i, b_i)\}$. Moreover, (q_i^*, b_i^*) is the unique best response, since any other point in S_i lies outside either BR or BR' , hence generating a strictly lower utility in at least one realisation of the other player's strategies.

Result: The best response is a unique point (q_i^*, b_i^*) .

Figure 2: Example 3. Best Response Lines to (2,2) and (4,0.5)



Example 3. Consider $(Q, B) = (2, 2)$ and $(Q', B') = (4, 0.5)$.

It is easy to show using equation (10) that player i 's best response to $(2, 2)$ is $b_i = q_i$, whereas her best response to $(4, 0.5)$ is $b_i = 0.25q_i + 0.5$. These lines intersect at the point $(q_i, b_i) = (\frac{2}{3}, \frac{2}{3})$, which is the best response to both $(2, 2)$ and $(4, 0.5)$ and hence also to any mixture between them (Figure 2).

In order to exhaust possible best response cases, we now consider a situation when best response lines to (Q, B) and (Q', B') do not intersect in player i 's strategy set.

7.3 Case 3. Best Response Lines Intersecting Outside the Strategy Set

Let $BR \cap BR' = (q_i^*, b_i^*) \notin S_i$ (in words, BR lines intersect outside S_i). If the best response lines do not cross in S_i , one of them passes through S_i to the left of the other. Without loss of generality, let BR denote the left best response line and BR' the right best response line. As we show below, the best response in this case is unique and lies on the boundary of S_i .

First, we show that a player's best response to such mixed bid lies on the boundary of her strategy set. Second, we show that it is unique.

Claim 1 *Consider a mixed total bid $(\mu(Q, B), (1 - \mu)(Q', B'))$ such that the best response lines to (Q, B) and (Q', B') are distinct and intersect outside S_i . A player's best response to the bid is either a unique point on the boundary of the strategy set S_i , or all the points within S_i on the line collinear with $(-Q, -B)$, $(-Q', -B')$.*

Proof: See Section 7.4.

Claim 1 asserts that a best response in Case 3 lies on the boundary of the strategy set. Moreover, it is either unique, or belongs to the best response set which is a line segment collinear with the realisations of a mixed total bid. In the next claim we rule out the latter possibility.

Claim 2 *Suppose the mixed total bid of all players apart from player i is $(\mu(Q, B); (1 - \mu)(Q', B'))$, such that the best response lines to (Q, B) and (Q', B') are distinct and intersect outside S_i . Then the points within S_i on the line collinear with $(-Q, -B)$ and $(-Q', -B')$ cannot all be best responses.*

Proof: See Section 7.6.

As shown above, the best response in Case 3 is a unique point at the boundary. It can also be shown that the best response lies strictly between BR and BR' (proof available on demand).

Note that the best response can belong to either the inner or the outer boundary. Section 7.5 below collects examples of boundary best responses to a mixed total bid (Case 3).

Result: The best response is a unique point (q_i^*, b_i^*) .

7.4 Proof of Claim 1

First, observe that a function $U_i(q_i, b_i, \sigma_{-i}) \equiv \mu u_{Q,B}(q_i, b_i) + (1 - \mu)u_{Q',B'}(q_i, b_i)$ is continuous. Hence, by the extreme value theorem, it attains a maximum on a closed bounded set S_i , which is a best response.

To show that a best response lies on the boundary, consider a point (q_i^*, b_i^*) which lies in the interior of S_i .

We show that such (q_i^*, b_i^*) cannot be a best response. Consider contour lines of $u_{Q,B}$ and $u_{Q',B'}$ passing through (q_i^*, b_i^*) , denoted L and L' respectively. As demonstrated in the paper, given the

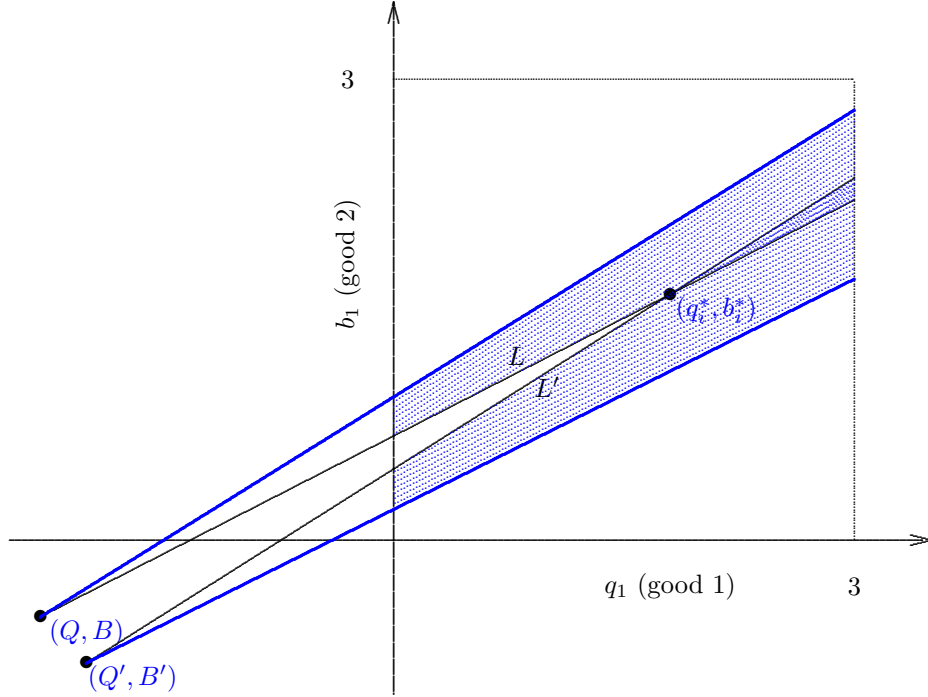
others' total bid (Q, B) , a player's utility is completely determined by the price p . Fixing (Q, B) , player i 's bids in S_i resulting in price p satisfy the following:

$$b_i = pq_i + pQ - B. \quad (12)$$

Hence, contour lines of $u_{Q,B}$ are straight lines passing through $(-Q, -B)$. Moreover, since utility is strictly concave in price, a contour line $u_{Q,B}$ separates the plane into lower and upper contour sets of the points on the line.

Note that, unless L and L' (i.e. the contour lines of $u_{Q,B}$ and $u_{Q',B'}$) coincide, they separate R_+^2 into four areas. One of these areas is the intersection of upper contour sets of (q_i^*, b_i^*) with respect to $u_{Q,B}$ and $u_{Q',B'}$ (a double shaded triangular area in Figure 3).

Figure 3: Upper Contour Sets of (q_i^*, b_i^*) w.r.t $u_{Q,B}$ and $u_{Q',B'}$



It follows that, unless (q_i^*, b_i^*) is on the boundary, the intersection of the strict upper contour sets of (q_i^*, b_i^*) with respect to $u_{Q,B}$ and $u_{Q',B'}$ in S_i is non-empty:

$$\{(q_i, b_i) : u_{Q,B}(q_i, b_i) > u_{Q,B}(q_i^*, b_i^*)\} \cup \{(q_i, b_i) : u_{Q',B'}(q_i, b_i) > u_{Q',B'}(q_i^*, b_i^*)\} \cup S_i \neq \emptyset \quad (13)$$

At any point in that intersection, the expected utility is strictly greater than at (q_i^*, b_i^*) . Hence, (q_i^*, b_i^*) cannot be a best response.

Third, on any boundary, $U_i(q_i, b_i)$ becomes a one-dimensional function (either $U_{q_i}(b_i)$ or $U_{b_i}(q_i)$) of the strategic variable which is not fixed on that boundary. Moreover, $U_i(\cdot)$ is strictly concave in that variable, hence a best response which lies on the boundary is unique. To see why the one-dimensional restriction of expected utility is strictly concave, fix other players' total bid (Q, B) and one dimension of player i 's strategy (q_i) . Observe that i 's utility $u_{Q,B}(b_i) = x_i(b_i)y_i(b_i)$ is a strictly concave function. This follows from Lemma 1 and the fact that $x_i(b_i) = w_{i1} - q_i + b_i(q_i + Q)/(b_i + B)$ is strictly concave while $y_i(b_i) = w_{i1} - b_i + q_i(b_i + B)/(q_i + Q)$ is weakly concave. Similarly, $u_{Q',B'}(b_i) = x_i(b_i)y_i(b_i)$ is strictly concave. Hence, i 's expected utility $U_{q_i}(b_i) = \mu u_{Q,B}(b_i) + (1 - \mu)u_{Q',B'}(b_i)$ is also strictly concave as a convex combination of strictly concave functions (the same holds for $U_{b_i}(q_i)$).

Fourth, if $L = L'$, we cannot rule out the case that a point on the line is the best response (the upper contour sets of that point with respect to the two realised utilities do not intersect). Moreover, by definition, both $(-Q, -B)$ and $(-Q', -B')$ lie on this line; hence, any point on the line generates the same price and the same outcome for player i under (Q, B) and (Q', B') . It follows that the whole line $L = L'$ is the best response.

Summing up the third and the fourth points, the best response is either unique and lies on the boundary, or is the whole line collinear with (Q, B) and (Q', B') . Q.E.D.

7.5 Examples of Case 3.

When best-response lines intersect outside S_i , there are three possible options for the unique best response to the mixed bid, illustrated by Examples 4, 5 and 6 below.

Example 4. Converging BR lines; unique best response on the outer boundary.

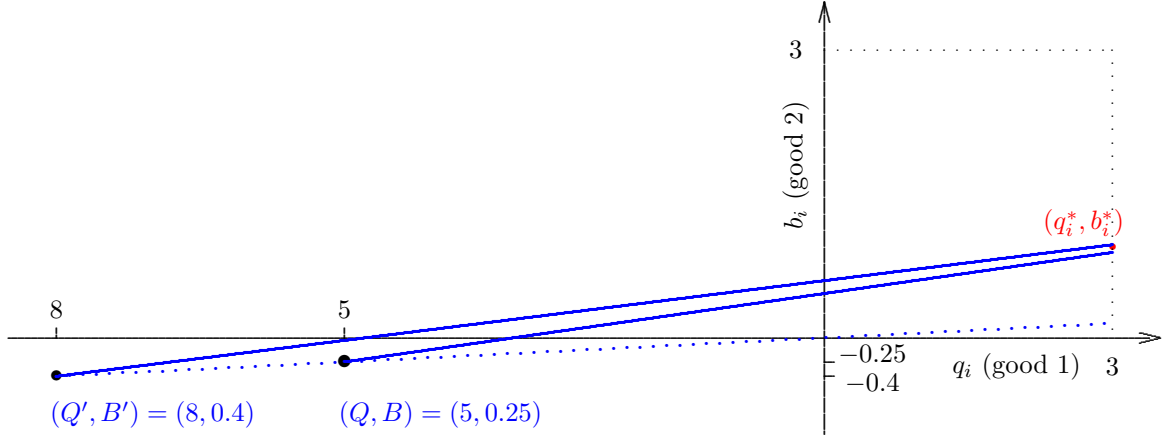
Let other players' total bids be $(Q, B) = (5, 0.25)$ and $(Q', B') = (8, 0.4)$. The best response to these total bids are $b_i = q_i\sqrt{1.3}/8 + \sqrt{6.5}/8 - 0.25$. and $b_i = \sqrt{1.87}q_i + 8\sqrt{1.87} - 0.4$ respectively. Figure 4 shows that these BR lines cross in the first quadrant outside S_i ; hence, they are converging.

Also note that when the best-response lines are converging the best response always lies on the *outer* boundary. This is because, for any point on the inner boundary, the intersection of upper contour sets of $u_{Q,B}$ and $u_{Q',B'}$ lies *above and to the right* of this point (inside the strategy set) and hence be achievable. In particular, in our example the unique best response to a mixed strategy can be determined from the following equation:

$$\mu \frac{13/2 - 5(b_i^* + 0.25)^2}{(b_i^* + 0.25)^2} = (1 - \mu) \frac{8(b_i^* + 0.4)^2 - 374/25}{(b_i^* + 0.4)^2}, \quad (14)$$

e.g., if the other player is mixing with probability $\mu = 1216/4051$, then player i 's best response is

Figure 4: Example 4. Best Response Lines to $(5, 0.25)$ and $(8, 0.4)$

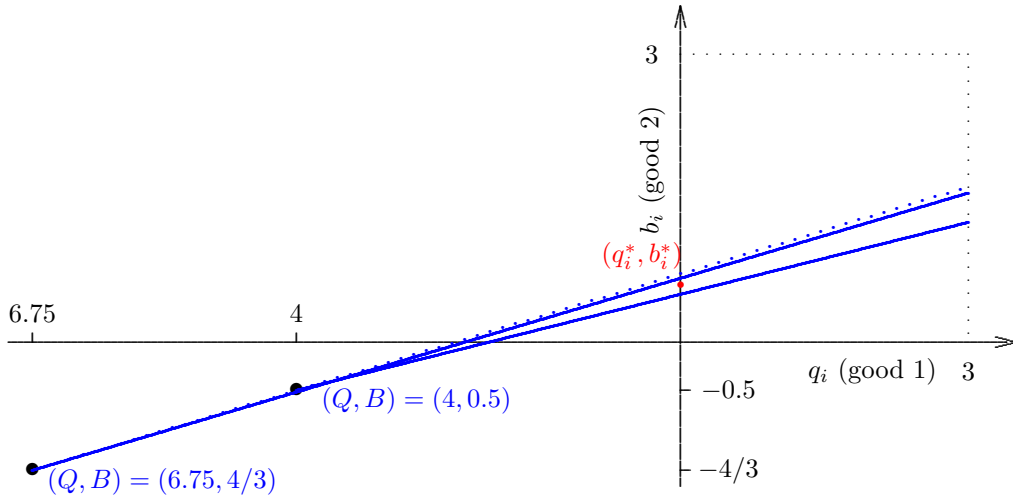


$$(q_i^*, b_i^*) = (3, 0.95).$$

Example 5. Diverging BR lines; unique best response on the inner boundary.

Let the other players' total bids be $(Q, B) = (4, 0.5)$ and $(Q', B') = (6.75, 4/3)$. The best responses to these total bids are $b_i = 0.25q_i + 0.5$ and $b_i = 8/27q_i + 2/35$ respectively. As shown in Figure 5, these best-response lines cross in the third quadrant; hence, they are diverging.

Figure 5: Example 5. Best Response Lines to $(4, 0.5)$ and $(6.75, 4/3)$



Note that the best response to this mixed strategy lies on the inner boundary. This is because, for any point on the outer boundary, the intersection of upper contour sets of $u_{Q,B}$ and $u_{Q',B'}$ lies *below*

and to the left of this point (inside the strategy set) and hence be achievable. In particular, in our example the unique best response to a mixed strategy can be determined from the following equation:

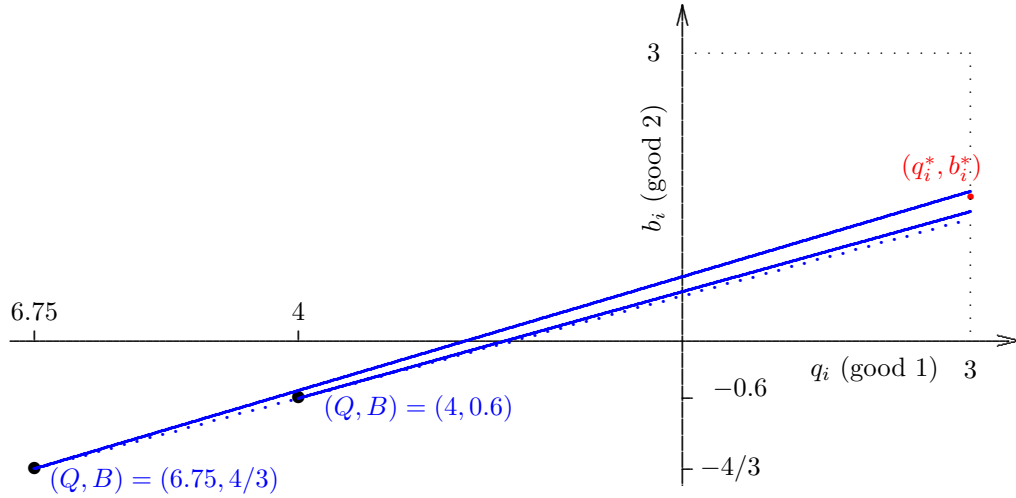
$$\mu \frac{-7(b_i^* + 1.5)(b_i^* - 0.5)}{(b_i^* + 0.5)^2} = (1 - \mu) \frac{39(b_i^* + 10/3)(b_i^* - 2/3)}{4(b_i^* + 4/3)^2}. \quad (15)$$

For example, if $\mu = 92807/257643$, then player i 's best response is $(q_i^*, b_i^*) = (0, 0.6)$.

Example 6. Diverging BR lines; unique best response on the outer boundary.

Let the other players' total bids be $(Q, B) = (4, 0.6)$ and $(Q', B') = (6.75, 4/3)$. The best responses to these total bids are $b_i = 0.3q_i\sqrt{6/7} + 1.2\sqrt{6/7} - 0.6$ and $b_i = 8/27q_i + 2/35$ respectively. These best-response lines cross in the third quadrant; hence, they are diverging.

Figure 6: Example 6. Best Response Lines to $(4, 0.6)$ and $(6.75, 4/3)$



Unlike the previous example, the unique optimum lies on the outer boundary, by the logic similar to that of Example 4. The unique best response to a mixed strategy can be determined from the following equation:

$$\mu \frac{4(b_i^* + 0.6)^2 - 15.12}{(b_i^* + 0.6)^2} = (1 - \mu) \frac{169/3 - 27/4(b_i^* + 4/3)^2}{(b_i^* + 4/3)^2}. \quad (16)$$

For example, if $\mu = 2163/6787$, then player i 's best response is $(q_i^*, b_i^*) = (3, 1.5)$.

7.6 Proof of Claim 2

Let $b_i^*(q_i)$ denote the function whose graph is a straight line connecting $(-Q, -B)$ and $(-Q', -B')$ in R^2 . It is easy to derive the formula for $b_i^*(q_i)$:

$$b_i^* = q_i \frac{B' - B}{Q' - Q} + \frac{B'Q - BQ'}{Q' - Q}. \quad (17)$$

The set of i 's strategies on the line is denoted $S_i^* \equiv \{(q_i, b_i) : b_i = b_i^*(q_i)\} \cup S_i$. If all points in the set S_i^* were best responses, then any point $(q_i, b_i^*(q_i))$ in the set would need to be the maximiser of $U_{q_i}(b_i)$. Since $U_{q_i}(b_i)$ is strictly concave (as shown in the proof of Claim 1), any point $(q_i, b_i^*(q_i))$ on the line would need to satisfy the first-order condition:

$$\mu \frac{\partial u_{Q,B}}{\partial b_i}(q_i, b_i^*(q_i)) + (1 - \mu) \frac{\partial u_{Q',B'}}{\partial b_i}(q_i, b_i^*(q_i)) = 0 \quad (18)$$

Applying the chain rule to (18) obtains:

$$\begin{aligned} & \mu \left[\frac{\partial u_i(x,y)}{\partial x}(q_i, b_i^*(q_i)) \frac{\partial x_{Q,B}}{\partial b_i}(q_i, b_i^*(q_i)) + \frac{\partial u_i(x,y)}{\partial y}(q_i, b_i^*(q_i)) \frac{\partial y_{Q,B}}{\partial b_i}(q_i, b_i^*(q_i)) \right] \\ & + (1 - \mu) \left[\frac{\partial u_i(x,y)}{\partial x}(q_i, b_i^*(q_i)) \frac{\partial x_{Q',B'}}{\partial b_i}(q_i, b_i^*(q_i)) + \frac{\partial u_i(x,y)}{\partial y}(q_i, b_i^*(q_i)) \frac{\partial y_{Q',B'}}{\partial b_i}(q_i, b_i^*(q_i)) \right] = 0 \end{aligned} \quad (19)$$

Note that, at all points along S_i^* the player's final allocation of x_i and y_i is the same (under either realisation of mixed total bid). Hence, her utility $u_i(x_i, y_i)$ is the same, and, most importantly, partial derivatives of utility with respect to x_i and y_i are the same. Rewrite (19) denoting $u_x \equiv \frac{\partial u_i(x_i, y_i)}{\partial x_i}(q_i, b_i^*(q_i))$ and $u_y \equiv \frac{\partial u_i(x_i, y_i)}{\partial y_i}(q_i, b_i^*(q_i))$:

$$\mu \left[u_x \frac{\partial x_{Q,B}}{\partial b_i}(q_i, b_i^*(q_i)) + u_y \frac{\partial y_{Q,B}}{\partial b_i}(q_i, b_i^*(q_i)) \right] + (1 - \mu) \left[u_x \frac{\partial x_{Q',B'}}{\partial b_i}(q_i, b_i^*(q_i)) + u_y \frac{\partial y_{Q',B'}}{\partial b_i}(q_i, b_i^*(q_i)) \right] = 0$$

The derivatives of x_i and y_i w.r.t. to b_i can be easily calculated, and rearranged using the formula for $b_i^*(q_i)$:

$$\frac{\partial x_{Q,B}}{\partial b_i} = \frac{B(q_1 + Q)}{(b_1 + B)^2} = \frac{B(Q' - Q)^2}{(B' - B)^2(q_1 + Q)}; \quad (20)$$

$$\frac{\partial x_{Q',B'}}{\partial b_i} = \frac{B'(q_1 + Q')}{(b_1 + B')^2} = \frac{B'(Q' - Q)^2}{(B' - B)^2(q_1 + Q')}; \quad (21)$$

$$\frac{\partial y_{Q,B}}{\partial b_i} = \frac{-Q}{q_1 + Q}; \quad (22)$$

$$\frac{\partial y_{Q',B'}}{\partial b_i} = \frac{-Q'}{q_1 + Q'}. \quad (23)$$

The first-order condition can then be rewritten as follows:

$$\mu \left[u_x \frac{B(Q' - Q)^2}{(B' - B)^2(q_1 + Q)} - u_y \frac{Q}{q_1 + Q} \right] = (1 - \mu) \left[u_x \frac{B'(Q' - Q)^2}{(B' - B)^2(q_1 + Q')} - u_y \frac{Q'}{q_1 + Q'} \right]. \quad (24)$$

The expression in the square brackets on the right-hand-side of (24) is the value of the partial derivative of $u_{Q',B'}$ with respect to b_i at point $(q_i, b_i^*(q_i))$. It equals zero iff $(q_i, b_i^*(q_i))$ lies on player

i 's best response line to (Q', B') . However, recall that $(q_i, b_i^*(q_i))$ is a point on the line connecting $(-Q', -B')$ and $(-Q, -B)$. This line intersects the best response to (Q', B') at $(-Q', -B')$ and not at $(q_i, b_i^*(q_i))$.

Hence the expression in the square bracket is non-zero, and we can divide both sides of (24) by it. Rearranging (24), we thereby obtain

$$\frac{\left[u_x B (Q' - Q)^2 - u_y Q (B' - B)^2 \right] (q_1 + Q')}{\left[u_x B' (Q' - Q)^2 - u_y Q' (B' - B)^2 \right] (q_1 + Q)} = \frac{\mu - 1}{\mu}. \quad (25)$$

Denote $A = \frac{\left[u_x B (Q' - Q)^2 - u_y Q (B' - B)^2 \right] (q_1 + Q')}{\left[u_x B' (Q' - Q)^2 - u_y Q' (B' - B)^2 \right] (q_1 + Q)}$. Observe that A is constant w.r.t. q_i (indeed, as argued above, u_x and u_y are constant w.r.t. q_i). Equation (25) becomes

$$A \frac{q_1 + Q'}{q_1 + Q} = \frac{\mu - 1}{\mu}. \quad (26)$$

Denote $f(q_i) = A \frac{q_1 + Q'}{q_1 + Q}$. The derivation above implies that all points in S_i^* are best responses iff $f(q_i) = (\mu - 1)/\mu$ for all q_i . In other words, $f(q_i)$ needs to be constant with respect to q_i , i.e., $\partial f(q_i)/\partial q_i = 0$:

$$A \frac{Q - Q'}{(q_1 + Q)^2} = 0. \quad (27)$$

Expression (27) holds if either (i) $Q = Q'$, in which case the line connecting $(-Q, -B)$ and $(-Q', -B')$ does not intersect S_i and hence the points on the line cannot be best responses; or (ii) $A = 0$. However, if $A = 0$, then $f(q_i) = 0 \neq (\mu - 1)/\mu$, i.e. the first-order condition does not hold, implying that points in S_i^* are not best responses. In either case, the points in S_i collinear with $(-Q, -B)$ and $(-Q', -B')$ cannot all be best responses. Q.E.D.

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