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Strain-limiting viscoelasticity with stress rate dependence via Edelen's theory of primitive thermodynamics

M. Ostoja-Starzewski and Y. Şengül

Abstract. The overarching aim of this paper is to investigate the viscoelastic material response within the context of strain-limiting theory of elasticity by following Edelen's approach of primitive thermodynamics. To achieve this, first, a thermodynamic derivation of strain-limiting viscoelasticity is given, where the constitutive relation is expressed in terms of the strain, stress, and stress rate, based on Edeelen's representation theory. This also shows that it is possible to obtain a geometrically linear and physically nonlinear viscoelastic model including a non-dissipative contribution. Secondly, a parallelism of the model under consideration with Maxwell-Cattaneo heat conduction is shown. This is the first time in the literature that a viscoelastic material model without the presence of the rate of the strain is derived in connection with the well-known linear models of viscoelasticity. Third, the equation of motion representing the dynamics of the viscoelastic material is derived where the elastic part of the stress tensor is assumed to be linear and basic solutions to this equation are given. Finally, we find that, in the planar case under quasi-static traction boundary conditions, a shift in material properties is possible such that the stress and stress rate fields remain unchanged.

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1. Introduction

During the last two decades, a new framework for the modelling and mathematical analysis of material behaviour, the so-called strain-limiting theories, has attracted significant attention. Taking advantage of the idea of generalizing hyperelasticity as well as the causality assumption on strain and stress [38], as a result of the use of implicit constitutive theory [27], it has been possible to obtain a geometrically linear and physically nonlinear theory both in elasticity [28,29] and viscoelasticity [11–13,31,35]. The novelty of this approach is that it allows one to obtain a nonlinear constitutive relation including the linearized strain and the stress in the case of elasticity, and additionally their rates in the case of viscoelasticity. This is not possible in Cauchy elasticity. To see this, assume that the stress $\boldsymbol{\sigma}$ is explicitly given as a function of the left Cauchy-Green strain tensor $\mathbf{B}$ as $\boldsymbol{\sigma} = f(\mathbf{B})$ for a nonlinear function f. Under the smallness assumption of the displacement gradient $\max_{\mathbf{x},t}|\nabla \mathbf{u}| \ll 1$, where $|\cdot|$ stands for the trace norm, one can only obtain a linear relation between the stress and the linearized strain of the form $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$, where $:$ stands for the matrix product, $\boldsymbol{\varepsilon}$ is the linearized strain, and $\mathbb{C}$ is the fourth-order elasticity tensor. Therefore, starting with Cauchy elasticity and linearizing the strain, one can only obtain a linear relationship between the linearized strain and the stress, whereas the new framework generalizes this to geometrically linear and physically nonlinear relations (see [36] for an extensive review of this theory).

As another novel contribution of the current work, we adopt the viewpoint of Edelen for dissipation associated with the viscoelastic response. Edelen [9] talks about primitive thermodynamics in his 1974 article as follows:

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"*New answers to an old question in a new context provide the basis for constructing a thermodynamics which is surprisingly simple and yet capable of modelling viscous fluids, viscoelastic bodies, elastic bodies, and heat conduction*".

This sentence, among other things, explains the importance and contribution of the theory developed by Edelen for the viscoelastic response of materials. As he argues in his paper, the conclusions drawn from the Clasius-Duhem inequality heavily depend on the choice of the dependent and independent quantities. Moreover, the change in dependent variables makes a little effect on the results while the change of independent quantities as he puts it "*will provide new theories and new results which may yield simpler and more direct means of modelling known physical phenomena*". As he suggested, in order to incorporate the stress rate into our constitutive relation, we choose the independent variables in Clasius-Duhem inequality appropriately together with incorporating Edelen's idea of primitive thermodynamics leading to a nonlinear relationship between the linearized strain, the stress and the stress rate, which is the main contribution of this work.

There are various studies in the literature employing and/or generalizing Edelen's approach, which also provide some justification for the usage of his theory as well as applications. For example, in [21], by applying Edelen's decomposition theorem, the plastic quantities were determined from the dissipation potential, while the elastic quantities were specified by the internal energy, in effect the derivation of a continuous cooling transformation diagram and the isothermal time-temperature-transformation diagram was derived. In relation to viscoelasticity, in [18], constitutive relationships for thermo-diffusive flows in viscoelastic medium determined by Kelvin-Voigt model have been presented. Authors of [19] used Edelen's decomposition theorem for the development of models of single-frequency vibrations and dissipative heating of inelastic piezoelectric solids and methods for solving the associated nonlinear boundary-value problems. More related to the current work, in [23], a thermodynamic derivation of the Maxwell–Cattaneo equation of heat conduction was derived via Edelen's primitive thermodynamics. More recently, in [25], it was shown that the dissipation function of linear processes in continuum thermomechanics may be treated as the average of the statistically fluctuating dissipation rate on small spatial scales, coinciding with a general solution of the Clausius–Duhem inequality according to Edelen. In [14], an alternative to Edelen's derivation of a formula connecting the dissipation potential to the dissipation rate was given, whereas in [15] the importance of Edelen's work for the development of continuum models of viscoplasticity and granular media was highlighted. Finally, in [34], by generalizing Edelen's primitive thermodynamics, a new thermodynamic framework was proposed for the design of new macroscopic models, as well as the combination of existing models, e.g. heat-conducting non-Newtonian fluids.

For the strain-limiting approach in viscoelasticity, in [12], a thermodynamical approach was adopted to derive a constitutive relation including the stress rate in one space dimension. The novelty in the present study, on the other hand, is the contribution of Edelen's primitive thermodynamics complemented with the assumptions of strain-limiting theory, which has not been investigated before, as well as the fact that the analysis is carried out in a three-dimensional space.

The outline of the paper is as follows. In Sect. 2, starting with the Clausius-Duhem inequality we derive our geometrically linear and physically nonlinear constitutive relation (2.14) including the linearized strain, the stress and the stress rate by following the approach of Edelen within strain-limiting theory. We end up with the equation of motion given in terms of stress (see (2.16)). In Sect. 3, we show a parallelism between the constitutive relation (2.14) and the Maxwell-Cattaneo heat conduction in terms of the presence of respective time rates. In Sect. 4, we give some simple solutions to the model we study, and finally, in Sect. 5, we show that the stress fields remain invariant under the conditions on the compliance tensor as in the case of classical planar elasticity.

TABLE 1. *The relations of material derivatives to* $\mathcal{L}_v$

Scalar f	$\mathcal{L}_v f = v_k f,_k$	$\dot{f} = \partial_t f + \mathcal{L}_v f$
Vector f_i	$\mathcal{L}_v f_i = f_i,_k v_k + v_k,_i f_k$	$\dot{f}_i = \partial_t f_i + \mathcal{L}_v f_i - v_k,_i f_k = \partial_t f_i + f_i,_k v_k$
Tensor f_{ij}	$\mathcal{L}_v f_{ij} = v_k f_{ij},_k + v_k,_i f_{kj} + f_{ik} v_k,_j$	$\dot{f}_{ij} = \partial_t f_{ij} + \mathcal{L}_v f_{ij} - v_k,_i f_{kj} - f_{ik} v_k,_j = \partial_t f_{ij} + v_k f_{ij},_k$

2. Derivation of the model using primitive thermodynamics

To derive a strain-limiting viscoelastic model depending on the rate of stress, we proceed with the representation theory of Edelen [7, 8, 9] as it provides the most general solution of the Clausius-Duhem inequality. In particular, two functionals representing the free energy ψ and the dissipation functional ϕ are used. As it will become clear later, we will use the Almansi strain tensor and the Cauchy stress tensor as the conjugate pair, and in order to be able to do this we will make use of the Lie derivative as the objective notion of derivative (Table 1).

We begin with the Clausius-Duhem inequality in spatial form

$$-\rho\dot{\psi} + \sigma_{ij} d_{ij} - \rho s \dot{T} - q_i \frac{T,_i}{T} \geq 0, \tag{2.1}$$

where $d_{ij} := v_{(i},_{j)}$ is the deformation rate, T is the absolute temperature, σ_{ij} is the Cauchy stress, q_i is the heat flux, and ρ is the mass density. The overdot denotes the material time derivative: $\dot{f}_{kl\ldots} \equiv Df_{kl\ldots}/Dt = \partial_t f_{kl\ldots} + v_j f_{kl\ldots},_j$ for a tensor $f_{kl\ldots}$ of any rank. Noting that d_{ij} is the Lie derivative of the (Euler-)Almansi tensor $A_{ij} = \frac{1}{2}(\delta_{ij} - F_{iA}F_{jA})$, where $F_{iA} = \partial x_i / \partial X_A$ is the deformation gradient, we can equivalently rewrite (2.1) in the Lie form

$$-\rho(\partial_t \psi + \mathcal{L}_v \psi) + \sigma_{ij}\mathcal{L}_v A_{ij} - \rho s(\partial_t T + \mathcal{L}_v T) - q_i \frac{T,_i}{T} \geq 0. \tag{2.2}$$

By $\mathcal{L}_v$ we denote the Lie derivative (from differential geometry) of a scalar, covariant vector, and covariant rank-2 tensor fields, taking these forms, respectively;
With A_{ij} and σ_{ij} covariant tensors, we have

$$\mathcal{L}_v A_{ij} = d_{ij}, \mathcal{L}_v \sigma_{ij} = v_k \sigma_{ij},_k + v_k,_i \sigma_{kj} + \sigma_{ik} v_k,_j .$$

Hereinafter, we employ two tensor notations, indicial and symbolic (both Cartesian), whichever one is more convenient.

To see the conjugacy of $\mathbf{A}$ to $\boldsymbol{\sigma}$ begin with the definition: "A stress–strain pair $(\boldsymbol{\tau}, \boldsymbol{\varepsilon})$ is said to be work-conjugate if, for every admissible virtual velocity field, $\delta W = \int \boldsymbol{\tau} : \delta\boldsymbol{\varepsilon}\, dv$ represents the internal virtual power density". Now, for a continuum that has undergone a motion $\mathbf{x}$, superpose an infinitesimal virtual motion on the current configuration: $\mathbf{x}_\varepsilon(\mathbf{x}) = \mathbf{x} + \varepsilon\boldsymbol{\xi}(\mathbf{x})$, where $\mathbf{H} := \nabla\boldsymbol{\xi}(\mathbf{x})$. Its incremental deformation gradient (relative to the current configuration) and the left Cauchy-Green (Finger) tensor are

$$\hat{\mathbf{F}} = \mathbf{I} + \varepsilon\mathbf{H} + o(\varepsilon), \ \hat{\mathbf{b}} = \hat{\mathbf{F}}\hat{\mathbf{F}}^T = \mathbf{I} + \varepsilon\left(\mathbf{H} + \mathbf{H}^T\right) + o(\varepsilon).$$

Hence, for this increment,

$$\hat{\mathbf{b}} = \mathbf{I} - \varepsilon\left(\mathbf{H} + \mathbf{H}^T\right) + o(\varepsilon), \hat{\mathbf{A}} = \frac{1}{2}\left(\mathbf{I} - \hat{\mathbf{b}}^{-1}\right) = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) + o(\varepsilon).$$

Therefore the first variation (Gâteaux derivative) of the Almansi strain in the direction of ξ is

$$\delta\mathbf{A} = \left.\frac{\hat{\mathbf{A}}}{d\mathbf{A}}\right|_{\varepsilon=0} = \mathrm{sym}(\nabla\boldsymbol{\xi}).$$

For a classical (Cauchy) continuum, the internal virtual power corresponding to the virtual velocity field $\boldsymbol{\xi}$ is

$$\delta W_{\text{int}} = \int \boldsymbol{\tau} : \text{sym}\left(\nabla \boldsymbol{\xi}\right) \; dv,$$

which shows that $\boldsymbol{\tau}$ is the Cauchy stress $\boldsymbol{\sigma}$ energetically conjugate to the Almansi strain $\mathbf{A}$.

The conjugacy of $\mathbf{A}$ to $\boldsymbol{\sigma}$ is also noted in [4, pg. 515], [1, Table 7]). Here, one may note that two other well-known conjugate pairs, dictated as special cases of Seth-Hill family of material and spatial strain measures, are the (Green-Lagrange strain and the second Piola-Kirchhoff stress) and (logarithmic Hencky strain and the Kirchhoff stress)(see e.g. [20]). However, for our purposes, the Cauchy stress and the Almansi strain pair is preferable. The main reason for this is that we would like to use the strain-limiting assumption rather than the conjugacy to obtain a constitutive relation in terms of the linearized strain.

Restricting ourselves to materials described by $\psi\left(\mathbf{A}, T\right)$, introduce the Legendre transform from ψ to the complementary energy $\psi^*\left(\boldsymbol{\sigma}, T\right)$ (Gibbs potential per unit mass)

$$\psi\left(\mathbf{A}, T\right) + \psi^*\left(\boldsymbol{\sigma}, T\right) = \frac{1}{\rho}\boldsymbol{\sigma} : \mathbf{A}, \tag{2.3}$$

which gives

$$A_{ij} = \rho \frac{\partial \psi^*}{\partial \sigma_{ij}}, \;\; s = -\frac{\partial \psi^*}{\partial T}.$$

Hence, the Lie derivative of (2.3) is (in symbolic notation)

$$\mathcal{L}_v \psi\left(\mathbf{A}, T\right) + \mathcal{L}_v \psi^*\left(\boldsymbol{\sigma}, T\right) = \frac{1}{\rho}\left[\left(\mathcal{L}_v \boldsymbol{\sigma}\right) : \mathbf{A} + \boldsymbol{\sigma} : \left(\mathcal{L}_v \mathbf{A}\right)\right],$$

which ensures that we work with the objective stress rate. Substituting into (2.2), we obtain the Clausius-Duhem inequality in the Lie form

$$\partial_t \psi + \rho\left(\mathcal{L}_v \psi^*\right) - \left(\mathcal{L}_v \sigma_{ij}\right) A_{ij} - \rho s\left(\partial_t T + \mathcal{L}_v T\right) - q_i \frac{T_{,i}}{T} \geq 0,$$

and, employing the chain rule for $\mathcal{L}_v \psi^* = \left(\partial \psi^* / \partial \sigma_{ij}\right) \mathcal{L}_v \sigma_{ij}$,

$$\partial_t \psi + \rho \frac{\partial \psi^*}{\partial \sigma_{ij}} \mathcal{L}_v \sigma_{ij} - \mathcal{L}_v \sigma_{ij} A_{ij} - \rho s\left(\partial_t T + \mathcal{L}_v T\right) - q_i \frac{T_{,i}}{T} \geq 0.$$

$$\text{or } \partial_t \psi + \rho \frac{\partial \psi^*}{\partial \sigma} \mathcal{L}_v \boldsymbol{\sigma} - \mathcal{L}_v \boldsymbol{\sigma} : \mathbf{A} - \rho s \dot{T} - q \frac{\nabla T}{T} \geq 0.$$

Making the assumptions of

1. ψ and ψ^* independent of time,
2. isothermal processes ($T =$ constant),
3. incompressible media (so we can write ψ^* in place of $\rho\psi^*$),

we arrive at

$$\mathcal{L}_v \sigma_{ij}\left(\frac{\partial \psi^*}{\partial \sigma_{ij}} - A_{ij}\right) \geq 0.$$

This inequality may be written compactly in a canonical form as

$$\mathbf{V} \cdot \mathbf{Y}\left(\mathbf{V}\right) \geq 0, \tag{2.4}$$

where $\mathbf{V}$ and $\mathbf{Y}$ are vectors of *thermodynamic velocities* and *thermodynamic forces*, respectively. That is, $\mathbf{V}$ consists of the stress matrix

$$\mathbf{V} = \left[\mathcal{L}_v \sigma_{ij}\right], \tag{2.5}$$

while $\mathbf{Y}$ is identified as

$$\mathbf{Y} = \left[\frac{\partial \psi^*}{\partial \sigma_{ij}} - A_{ij}\right]. \tag{2.6}$$

The key problem concerns the finding of a most general solution of (2.4), i.e. a constitutive relation of $\mathbf{Y}$ in terms of $\mathbf{V}$. In the vein of the primitive thermodynamics of Edelen, the most general form of that relation is given as

$$\mathbf{Y} = \nabla_{\mathbf{V}}\phi^* (\mathbf{Y};\mathbf{w}) + \mathbf{U}(\mathbf{Y};\mathbf{w}), \text{ or } Y_i = \frac{\partial \phi^* (\mathbf{Y};\mathbf{w})}{\partial V_i} + U_i (\mathbf{Y};\mathbf{w}), \tag{2.7}$$

where $\phi^* (\mathbf{Y};\mathbf{w})$ is the dissipation (potential) function and the non-dissipative vector $\mathbf{U}$ does not contribute to the entropy production and it satisfies

$$\mathbf{U}\cdot\mathbf{V} = 0. \tag{2.8}$$

Also, the vector of thermostatic variables is

$$\mathbf{w} = [\sigma_{ij}].$$

On account of (2.5), (2.6) and (2.7), we find

$$\frac{\partial\psi^*}{\partial\sigma_{ij}} - A_{ij} = \frac{\partial\phi^*}{\partial\dot{\sigma}_{ij}} + U_{ij} \implies A_{ij} = \frac{\partial\psi^*}{\partial\sigma_{ij}} - \frac{\partial\phi^*}{\partial\dot{\sigma}_{ij}} - U_{ij}. \tag{2.9}$$

Under the assumption of the smallness of the deformation (or displacement) gradient as $\max\limits_{\mathbf{x},t}|\nabla\mathbf{u}| \ll 1$, where $|\cdot|$ stands for the trace norm, we know that A_{ij} reduces to the linearized strain tensor $A_{ij} \to \varepsilon_{ij} = u_{(i,j)}$. This assumption places the theory to be developed under the context of geometric linearity. Denoting the nonlinear elastic and viscoelastic parts of the strain in (2.9) as $\nabla\psi^*(\mathbf{w})$ and $\nabla\phi^*(\dot{\mathbf{w}})$, we obtain the geometrically linear and physically nonlinear constitutive relation as

$$\boldsymbol{\varepsilon} = \nabla\psi^*(\mathbf{w}) - \nabla\phi^*(\dot{\mathbf{w}}) - \mathbf{U}, \tag{2.10}$$

which is given in terms of the linearized strain, the stress, the stress rate and the non-dissipative vector $\mathbf{U}$, which is coming from Edelen's primitive thermodynamics theory. This is the first time in the literature that such a constitutive relation has been derived starting with Clausius-Duhem inequality in connection with viscoelasticity. Clearly, constitutive relation (2.10) gives a potentially nonlinear relation between the linearized strain $\boldsymbol{\varepsilon}$, the stress tensor $\mathbf{w}$, the rate of the stress tensor $\dot{\mathbf{w}}$ and the non-dissipative vector $\mathbf{U}$, the nonlinearity being dependent on the forms of ψ^* and ϕ^*. The choices of these functions depend on the actual material being modelled, and they can be highly nonlinear and complex. As mentioned earlier, a one-dimensional model was obtained in [12] including the stress rate. However, the model studied in that paper was linear in the stress rate. A more general constitutive relation including the stress rate was also considered in [5] where there is no separation for the stress and stress rate dependence. Our subsequent studies will include some suitable nonlinear choices for these functions allowing for the modelling of experimentally observed phenomena in connection with strain-limiting material response including some metal alloys and biological fibres.

Before trying to understand the fully nonlinear model, it is reasonable to completely study the corresponding linear one. We will do so in the remaining of this work. To this end, we will consider the complementary energy of the form

$$\psi^*(\mathbf{w}) = \frac{1}{2}C^{-1}_{ij} w_i w_j \equiv \psi^*(\sigma_{ij}) = \frac{1}{2}\mathbb{C}^{-1}_{ijkl}\sigma_{ij}\sigma_{kl},$$

where $\mathbb{C}^{-1}_{ijkl}$ is the stiffness (elasticity) tensor of classical (small strain) hyperelasticity; it has one major and two minor symmetries. Note: $\partial\psi^*/\partial\sigma_{ij} = \mathbb{C}^{-1}_{ijkl}\sigma_{kl}$. Similarly, we consider the linear dissipative part in (2.10), that is, we consider dissipation functional given as

$$\phi^* := \int_0^1 \mathbf{V}\cdot\mathbf{Y}(\lambda\mathbf{V})d\lambda = \int_0^1 D_{ij} w_i w_j d\lambda,$$

or equivalently written as the quadratic given by

$$\phi^* := \frac{1}{2}\mathbb{D}_{ijkl}\dot{\sigma}_{ij}\dot{\sigma}_{kl}, \tag{2.11}$$

where $\mathbb{D}_{ijkl}$ is a viscosity-type tensor of this hyper-dissipative model; it has one major and two minor symmetries. Note that $\partial\phi^*/\partial\dot{\sigma}_{ij} = \mathbb{D}_{ijkl}\dot{\sigma}_{kl}$.

To satisfy the orthogonality condition of Edelen, the admissible non-dissipative components of $\mathbf{U}$ are set as

$$U_{ij} = \mathbb{B}_{ijkl}\dot{\sigma}_{kl} \text{ with } \mathbb{B}_{ijkl} = -\mathbb{B}_{klij}, \tag{2.12}$$

where the major anti-symmetry of the tensor $\mathbb{B}_{ijkl}$ assures the satisfaction of (2.8); the minor symmetries $\mathbb{B}_{ijkl} = \mathbb{B}_{jikl} = \mathbb{B}_{ijlk}$ hold since the Cauchy stress and strain tensors are symmetric. By (2.11) and (2.12), the Almansi strain tensor in (2.9) can be written as

$$A_{ij} = \frac{\partial\psi^*}{\partial\sigma_{ij}} - \frac{\partial\phi^*}{\partial\dot{\sigma}_{ij}} - U_{ij} = \mathbb{C}^{-1}_{ijkl}\sigma_{kl} - \mathbb{D}_{ijkl}\dot{\sigma}_{kl} - \mathbb{B}_{ijkl}\dot{\sigma}_{kl}, \tag{2.13}$$

which may be interpreted as an additive decomposition into quasi-conservative $^{(q)}$ and dissipative $^{(d)}$ contributions as

$$\begin{gathered} A_{ij} = A^{(q)}_{ij} - A^{(d)}_{ij}, \\ A^{(q)}_{ij} = \mathbb{C}^{-1}_{ijkl}\sigma_{kl},\ A^{(d)}_{ij} = -\mathbb{H}_{ijkl}\dot{\sigma}_{kl} \\ \mathbb{H}_{ijkl} = \mathbb{D}_{ijkl} + \mathbb{B}_{ijkl}. \end{gathered}$$

Rewriting (2.13) in terms of the linearized strain using $A_{ij} = \varepsilon_{ij}$, we obtain the linear version of (2.10) as

$$\varepsilon_{ij} = \mathbb{C}^{-1}_{ijkl}\sigma_{kl} - \mathbb{H}_{ijkl}\dot{\sigma}_{kl}. \tag{2.14}$$

With reference to Edelen's theory, the classical Onsager reciprocity conditions

$$\frac{\partial Y_i(\lambda\mathbf{V})}{\partial V_j} = \frac{\partial Y_j(\lambda\mathbf{V})}{\partial V_i}$$

are recovered if and only if $\mathbf{U} = \mathbf{0}$. This indeed is seen from

$$\frac{\partial}{\partial\dot{\sigma}_{kl}}\left[\frac{\partial\phi^*}{\partial\dot{\sigma}_{ij}} + U_{ij}\right] = \frac{\partial}{\partial\dot{\sigma}_{ij}}\left[\frac{\partial\phi^*}{\partial\dot{\sigma}_{kl}} + U_{kl}\right].$$

Now, consider the balance of linear momentum given as

$$\sigma_{ij,j} + F_i = \rho\,\ddot{u}_i, \tag{2.15}$$

where $\sigma_{ij} = \sigma_{ji}$ and F_i is the body force. Taking the gradient of both sides of (2.15) with respect to i and j, and adding the resulting equations, we obtain

$$\rho\ddot{\varepsilon}_{ij} = \frac{1}{2}(F_{i,j} + F_{j,i}) + \frac{1}{2}\Big((\sigma_{ik,k})_{,j} + (\sigma_{jk,k})_{,i}\Big).$$

Rewriting this equation using the symmetry notation gives

$$\rho\ddot{\varepsilon}_{ij} = \frac{1}{2}F_{(i,j)} + \frac{1}{2}\sigma_{(ik,k),j}.$$

Substituting the constitutive relation (2.14) into this equation, we finally obtain the equation for the stress given as

$$\rho\,\mathbb{C}^{-1}_{ijkl}\ddot{\sigma}_{kl} - \rho\mathbb{H}_{ijkl}\dddot{\sigma}_{kl} = \sigma_{(ik,k),j} + F_{(i,j)}. \tag{2.16}$$

Equation (2.16) is a linear equation for stress including Edelen's primitive thermodynamics term. At the same time, this is also a generalization of the *Ignaczak equation of elastodynamics*; we return to it in the next section. It is clear that this equation can easily be generalized to the nonlinear case by assuming a nonlinear free energy density, in which case a more realistic capture of physics would be possible.

3. A parallelism to Maxwell-Cattaneo heat conduction

In this section, we provide an analogy between the stress rate constitutive relation (2.14) and Maxwell-Cattaneo equation. The motivation behind this is that it is not possible to derive such a stress rate type relation by means of spring-dashpot systems as in the cases of Zener model, Kelvin-Voigt model and Maxwell model. More precisely, as a result of using spring-dashpot systems, it is not possible to eliminate the strain-rate term $\dot{\varepsilon}$ due to its coefficient being strictly positive, and hence it is never possible to derive a constitutive relation as (2.14). Here, we show that, in fact, such a system can be shown to be valid in analogy to the derivation of Maxwell-Cattaneo equation, which is the first time in the literature to be realized.

We first recall from (2.14) that

$$\varepsilon_{ij} = \mathbb{C}^{-1}_{ijkl}\sigma_{kl} - \mathbb{H}_{ijkl}\dot{\sigma}_{kl} \Rightarrow \mathbb{C}_{prij}\varepsilon_{ij} = \mathbb{C}_{prij}\mathbb{C}^{-1}_{ijkl}\sigma_{kl} - \mathbb{C}_{prij}\mathbb{H}_{ijkl}\dot{\sigma}_{kl} = \sigma_{pr} - \mathbb{G}_{prkl}\dot{\sigma}_{kl}.$$

The 3d model given in (2.13) as

$$\sigma_{pr} - \mathbb{G}_{prkl}\dot{\sigma}_{kl} = \mathbb{C}_{prij}\varepsilon_{ij}, \tag{3.1}$$

where $\mathbb{G}_{ijkl} = \mathbb{C}_{ijkl}(\mathbb{D}_{ijkl} + \mathbb{B}_{ijkl})$, reduces in 1d to

$$\sigma + p_1\dot{\sigma} = q_0\varepsilon, \tag{3.2}$$

where p_1 and q_0 are the standard notations for material parameters in the differential description of viscoelasticity [41]. Thus, the strain-limiting viscoelasticity of (3.1) and (3.2) appears to be a special case of the Zener model

$$\sigma + p_1\dot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon},$$

where p_1, q_0, and q_1 are the standard notations for material parameters in the differential description of viscoelasticity, e.g. [41]. In fact, the Zener model can be constructed from an arrangement of two springs and a dashpot (indeed, in two alternative ways). However, there exists no arrangement of springs and dashpots that would result in (3.2)! Thus, a question arises: How can the strain-limiting viscoelasticity be justified?

Consider the constitutive equation of Maxwell-Cattaneo heat conductivity in a 3d anisotropic material

$$\tau\dot{q}_i + q_i = -\kappa_{ij}T_{,j} \tag{3.3}$$

where q_i is the heat flux vector, T is the absolute temperature, τ is the relaxation time, and κ_{ij} is the thermal conductivity; see [26] for a brief historical review of this equation. As before, the overdot denotes the material time derivative, in which, in the case of finite strains, the convective derivative term would have to be augmented by the Lie derivative of heat flux [3]. In 1d (on the x axis) (3.3) simplifies to

$$\tau\dot{q} + q = -\kappa T_x \tag{3.4}$$

where the subscript x indicates the spatial derivative with respect to x. Now, recall the energy balance

$$\rho\dot{e} = \rho c_v\dot{T} = -q_x \tag{3.5}$$

where ρ is the mass density, e the internal energy, and c_v the specific heat at fixed volume (as required by the rigid conductor setting). Combining (3.4) with (3.5), we derive the governing/field (telegraph) equation of the Maxwell-Cattaneo heat conduction

$$\ddot{T} + \frac{1}{\tau}\dot{T} = c^2T_{xx} \text{ or } \tau\ddot{T} + \dot{T} = \tau c^2T_{xx} = \frac{\kappa}{\rho c_v}T_{xx} \tag{3.6}$$

where $c = \sqrt{\kappa/\rho c_v\tau}$ is the speed of second sound. The heat flux satisfies the same differential equation

$$\ddot{q} + \frac{1}{\tau}\dot{q} = c^2q_{xx} \text{ or } \tau\ddot{q} + \dot{q} = \tau c^2q_{xx} = \frac{\kappa}{\rho c_v}q_{xx} \tag{3.7}$$

Note that in the limit $\tau \to 0$ we recover the Fourier law and the diffusion equation, whereas the limit $\tau \to \infty$ has no physical meaning.

Table 2. *Comparing the Maxwell-Cattaneo heat conduction with strain-limiting viscoelasticity*

Heat conductivity	mechanics
Heat flux	Stress
Temperature gradient	Strain
M-C model (3.3)	Viscoelastic model (3.2)
Heat energy balance (3.3)	Linear momentum balance (3.8)
LHS in (3.6)	LHS in (3.9)
LHS in (3.7)	LHS (3.10)

Now, recall the linear momentum balance of continuum mechanics 1d (only with the d'Alembert force)

$$\sigma_x = \rho \ddot{u} \tag{3.8}$$

and, combining with (3.2), derive the 1d Navier equation of elastodynamics

$$\tau \rho \dddot{u} + \rho \ddot{u} = C u_{xx} \text{ or } \tau \dddot{u} + \ddot{u} = c_L^2 u_{xx} \tag{3.9}$$

and the 1d the stress equation of elastodynamics

$$\tau \rho \dddot{\sigma} + \ddot{\sigma} = C \sigma_{xx} \text{ or } \tau \dddot{\sigma} + \ddot{\sigma} = c_L^2 \sigma_{xx}, \tag{3.10}$$

where, with reference to Table 2, τ plays the role of a relaxation time ($= p_1$ in (3.2)) and $c = \sqrt{C/\rho}$ the speed of propagation of damped longitudinal (pressure) waves. If we consider the anti-derivatives of $\dot{u}$ and $\dot{\sigma}$, our viscoelasticity is seen as a mechanics analogue of the Maxwell-Cattaneo heat propagation. Alternatively, one might consider time rates of T and q, which would result in third- and second-order time derivatives on the left-hand sides of equations (3.6) and (3.7). Thus, there is a parallelism (not a perfect analogy) between the heat conductivity and the strain-limiting viscoelasticity in 1d (say, in a thin rod).

Note that just like in the case of telegraph equation [40], two different wave motions can be studied: temporally attenuated and spatially periodic (TASP) and spatially attenuated and temporally periodic (SATP) for $\dot{u}$ and $\dot{\sigma}$. Moreover, in the limit $\tau \to 0$ we recover the Hooke law and the wave equation, whereas the limit $\tau \to \infty$ has no physical meaning.

Returning to 3d mechanics, we give the generalization of (3.10) to an inhomogeneous and anisotropic body

$$\left(\rho^{-1} \sigma_{(ik,k}\right)_{,j)} + \left(\rho^{-1} F_{(i}\right)_{,j)} = \mathbb{C}_{ijkl}^{-1} \ddot{\sigma}_{kl} + \left(\mathbb{D}_{ijkl} + \mathbb{B}_{ijkl}\right) \dddot{\sigma}_{kl}, \tag{3.11}$$

where

$$F_{ij} = F_{i,j} + F_{j,i} + \frac{\lambda}{\lambda + 2\mu} F_{k,k}\, \delta_{ij}.$$

This is a generalization of the *Ignaczak equation of elastodynamics* [16,17,24] to strain-limiting viscoelasticity with stress rate dependence. Note that, in contradistinction to the displacement formulation, generalizing the Navier equation of elastodynamics

$$\left(C_{ijkl} u_{(k,l)}\right)_{,j} + f_i = \rho \ddot{u}_i,$$

the stress formulation (3.11) avoids gradients of compliance but introduces gradients of mass density. Among the advantages of the formulation in terms of (3.11) is the possibility of statement of boundary conditions directly in terms of stress tractions, as will be demonstrated in a subsequent paper.

4. Special solutions

A similar model to (2.16) was derived by Erbay and Şengül [12] in one-dimensional setting without the contribution of Edelen's part. In that paper, the constitutive relation obtained by the authors was composed of a nonlinear elastic part and a linear dissipative part, and using the equation of motion in the one-dimensional setting together with the derived constitutive relation, it was possible to obtain a single partial differential equation in terms of the stress. The rest of the work was dedicated to showing that the ordinary differential equation corresponding to the travelling wave solutions of this model was the same as the one corresponding to a strain-rate model, and hence they share the same travelling wave profiles. In this section, rather than studying travelling wave solutions by reducing the partial differential equation into an ordinary differential equation, we want to investigate the partial differential equation and its solutions. Therefore, we study (2.16) as a partial differential equation in some special cases. Since we have a linear model, it is relatively easy to find solutions in these special cases which we believe are the first steps to be taken towards a more general nonlinear theory for (2.10).

4.1. A 1D special solution

Firstly, we consider the one-dimensional case with $\sigma(x,t) = X(x)\tau(t)$ for some functions X and τ, corresponding to separating the space and time variables. One-dimensional version of (2.16) when there is no body force reads

$$\rho\, C^{-1}\ddot{\sigma} - \rho\, H\dddot{\sigma} = \sigma_{xx}, \tag{4.1}$$

where C and H denote the scalar functions corresponding to the one-dimensional versions of $\mathbb{C}$ and $\mathbb{H}$. Now, substituting $\sigma(x,t) = X(x)\tau(t)$ and taking derivatives we obtain two ordinary differential equations given as

$$\begin{aligned} \rho\, C^{-1}\ddot{\tau} - \rho\, H\dddot{\tau} - \gamma\tau = 0, \\ X'' - \gamma X = 0, \end{aligned} \tag{4.2}$$

for a constant γ. The second equation in (4.2) immediately gives the solution $X(x)$ as an exponential type function in x, whereas the first equation leads to a solution in the form of exponentials with real and complex conjugate powers.

4.2. Wave propagation in a 1d rod versus a plane wave in 3d

In a very thin 1d rod, (3.9) and (3.10) are the governing equations. No effect due to B_{ijkl} can arise.

Considering a plane wave in 3d, we begin with σ_{11} satisfying (3.11) and having the form

$$\sigma_{11}(\mathbf{x},t) = \mathbf{d} f(\mathbf{x}\cdot\mathbf{p} - \omega t),$$

where the unit vector $\mathbf{d} = \mathbf{e}_1$ is the basis vector of axis x_1, $\mathbf{p}$ is the unit vector in the direction of propagation (i.e. also x_1), and ω is the frequency. As a result, one obtains

$$\rho^{-1}\sigma_{11,11} = C^{-1}_{ijkl}\ddot{\sigma}_{kl} + (D_{ijkl} + B_{ijkl})\,\dddot{\sigma}_{kl}. \tag{4.3}$$

Equation (4.3) can be seen as the 2d version of (4.1) for which we expect to obtain solutions in the form of exponentials.

5. Invariance of the stress and the stress rate fields

We can show that the stress and stress rate fields are invariant under the same (i.e. CLM-type) conditions on the compliance tensor $\mathbb{D}_{ijkl}$ as on $\mathbb{C}^{-1}_{ijkl}$ in the classical case of planar elasticity. However, there is no shift on $\mathbb{B}_{ijkl}$ because there is only one material constant "inside" that tensor like K^0. We proceed in the vein of [2]; see also [6,37].

Consider a planar viscoelastic solid occupying a simply-connected domain B in $\mathbb{R}^2$ (i.e. (x_1, x_2)-plane). The compliance tensors $\mathbb{S}_{ijkl}= \mathbb{C}^{-1}_{ijkl}$ and $\mathbb{H}_{ijkl} = \mathbb{D}_{ijkl} + \mathbb{B}_{ijkl}$ are isotropic and assumed to be twice-differentiable. We write the compliance-form viscoelastic relation as

$$4\varepsilon_{ij} = 2S\sigma_{ij} + (A - S)\,\sigma_{kk}\delta_{ij} + 2S^{(\mathbb{H})}\dot{\sigma}_{ij} + \left(A^{(\mathbb{H})} - S^{(\mathbb{H})}\right)\dot{\sigma}_{kk}\delta_{ij}, \quad i, j, k = 1, 2, \tag{5.1}$$

involving two planar elastic compliances: bulk compliance A and shear compliance S

$$A = \frac{1}{\kappa_{2d}} = \frac{\varkappa - 1}{2\mu_{2d}}\ S = \frac{1}{\mu_{2d}},$$

and two planar viscous compliances: bulk-type $A^{(\mathbb{H})}$ and shear-type $S^{(\mathbb{H})}$

$$A^{(\mathbb{H})} = \frac{1}{\kappa^{(\mathbb{H})}_{2d}} = \frac{\varkappa - 1}{2\mu^{(\mathbb{H})}_{2d}}\ S^{(\mathbb{H})} = \frac{1}{\mu^{(\mathbb{H})}_{2d}}.$$

In other words, A and S specify the $\mathbb{S}$ tensor, while $A^{(\mathbb{H})}$ and $S^{(\mathbb{H})}$ specify the $\mathbb{D}$ tensor. No components of $\mathbb{B}$ are present in (5.1).

Also, $\nu_{2d} \in [-1, 1]$ is the planar Poisson ratio, while $\varkappa = (3 - \nu_{2d}) / (1 + \nu_{2d})$ is the Kolosov constant for plane strain as well plane stress; $\varkappa$ is taken the same for elastic and viscous properties. It is convenient to introduce two uniaxial compliances, one for elastic and another for viscous responses

$$C = \frac{1}{E_{2d}} = \frac{\varkappa + 1}{8\mu_{2d}}, C^{(\mathbb{H})} = \frac{1}{E^{(\mathbb{H})}_{2d}} = \frac{\varkappa + 1}{8\mu^{(\mathbb{H})}_{2d}}.$$

Hence,

$$A + S = 4C, A^{(\mathbb{H})} + S^{(\mathbb{H})} = 4C^{(\mathbb{H})}. \tag{5.2}$$

Now, we consider a quasi-static equilibrium

$$\sigma_{ij},_j(\mathbf{x}) = 0 \text{ or } \nabla \cdot \sigma(\mathbf{x}) = \mathbf{0},$$

with the solid subjected to quasi-static traction boundary conditions

$$\sigma_{ji}(\mathbf{x})\, n_j = t^{(n)}_i(\mathbf{x}) \text{ and } \dot{\sigma}_{ji}(\mathbf{x})\, n_j = \dot{t}^{(n)}_i(\mathbf{x})\ \forall \mathbf{x} \in \partial B,$$

in such a way that for all times the global equilibrium is satisfied

$$\int_{\partial B} \mathbf{t}^{(n)}(\mathbf{x})\, dS = \mathbf{0} \quad \texttt{and} \quad \int_{\partial \mathtt{B}} \mathbf{x} \times \mathbf{t}^{(\mathrm{n})}(\mathbf{x})\, \mathtt{dS} = \mathbf{0}. \tag{5.3}$$

While time derivatives appear in (5.1), it is understood throughout, on account of the assumption of quasi-statics, that the inertia effects are absent.

Now, substituting (5.1) into the compatibility condition $\varepsilon_{ji} = (u_{i,j} + u_{j,i})/2$, and using (5.3), we obtain, after some manipulations,

$$\begin{aligned} &\nabla^2[\tfrac{1}{2}(A + S)(\sigma_{11} + \sigma_{22})] - (S_{,11}\sigma_{11} + 2S_{,12}\sigma_{12} + S_{,22}\sigma_{22}) \\ &+\nabla^2[\tfrac{1}{2}\left(A^{(\mathbb{H})} + S^{(\mathbb{H})}\right)(\dot{\sigma}_{11} + \dot{\sigma}_{22})] - \left(S^{(\mathbb{H})}_{,11}\dot{\sigma}_{11} + 2S^{(\mathbb{H})}_{,12}\dot{\sigma}_{12} + S^{(\mathbb{H})}_{,22}\dot{\sigma}_{22}\right) = 0. \end{aligned} \tag{5.4}$$

Looking at (5.4) (see e.g. [22, Section 5.2]), the following question may now be asked: "Supposing that the pairs (A, S) and $\left(A^{(\mathbb{H})}, S^{(\mathbb{H})}\right)$ are changed to some pairs $\left(\widehat{A}, \widehat{S}\right)$ and $\left(\widehat{A^{(\mathbb{H})}}, \widehat{S^{(\mathbb{H})}}\right)$, then under what restrictions would the original stress $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ and stress rate $(\dot{\sigma}_{11}, \dot{\sigma}_{22}, \dot{\sigma}_{12})$ fields remain unchanged?"

Denoting by a hat the quantities pertaining to the new material, the required *invariance of the stress and stress rate fields* is written (in symbolic notation) as

$$\sigma(\mathbf{x}) = \widehat{\sigma}(\mathbf{x}), \quad \dot{\sigma}(\mathbf{x}) = \dot{\widehat{\sigma}}(\mathbf{x}). \tag{5.5}$$

Examining (5.4) we see that these relations need to hold

$$\begin{gathered}\widehat{A} + \widehat{S} = m(A + S) \;\; \widehat{S}_{,11} = mS_{,11} \;\; \widehat{S}_{,22} = mS_{,22} \;\; \widehat{S}_{,12} = mS_{,12} \\ \widehat{A^{(\mathbb{H})}} + \widehat{S^{(\mathbb{H})}} = m\left(A^{(\mathbb{H})} + S^{(\mathbb{H})}\right) \;\; \widehat{S^{(\mathbb{H})}}_{,11} = mS^{(\mathbb{H})}{}_{,11} \\ \widehat{S^{(\mathbb{H})}}_{,22} = mS^{(\mathbb{H})}{}_{,22} \;\; \widehat{S^{(\mathbb{H})}}_{,12} = mS^{(\mathbb{H})}{}_{,12}\,,\end{gathered}$$

where m is an arbitrary scalar. Note that this means

$$\begin{gathered}\widehat{A} = mA + a + bx_1 + cx_2 \;\; \widehat{S} = mS - a - bx_1 - cx_2 \;\; \widehat{C} = mC \\ \widehat{A^{(\mathbb{H})}} = mA^{(\mathbb{H})} + a + bx_1 + cx_2 \;\; \widehat{S^{(\mathbb{H})}} = mS^{(\mathbb{H})} - a - bx_1 - cx_2 \;\; \widehat{C^{(\mathbb{H})}} = mC^{(\mathbb{H})},\end{gathered} \tag{5.6}$$

where the third and sixth equalities come from (5.2). The constants m, a, b, and c are subject to restrictions dictating that the compliances be nonnegative.

Since the $\mathbb{B}$ tensor does not appear in (5.1), the non-hyperelastic properties cannot be shifted in such a way as to ensure the stress invariance (5.5).

The result that the stress field is unchanged (invariant) under such a shift of compliances is called the *CLM stress invariance* or *transformation* after the authors [2]; see also [6,37]. Effectively, this represents one more generalization of the CLM transformation besides a range of other results; see the review in Chapter 5 of [22].

The pair of materials satisfying the CLM shift (5.2) is called *equivalent materials*. Quoting from the latter book, "*In effect, one obtains a so-called reduced parameter dependence, which is important in parametric studies of composites, both experimental and theoretical. It can be used as a check for numerical and analytical calculations, it reduces the number of output parameters and facilitates the presentation of results, leading to savings in time and space resources*".

6. Conclusion

The focus of this work is to combine the novel approaches of strain-limiting theory and Edelen's theory to introduce a new model for the viscoelastic response of materials including the linearized strain, the stress and the stress rate. A thermodynamical derivation of constitutive equations is developed on the basis of Edelen's approach of primitive thermodynamics, where the non-dissipative forces represent the viscoelastic response that cannot be grasped by a dissipation potential. Also, an analogy of the model under consideration with Maxwell-Cattaneo heat conduction is shown. This is a new approach that a viscoelastic material model without the presence of the rate of the strain is derived from thermodynamics in connection with the well-known, linear models of viscoelasticity. Moreover, Edelen's part in the constitutive relation is included in a geometrically linear and physically nonlinear model which has not been obtained before. Using this constitutive relation, we are able to derive an equation of motion in terms of the stress field and relate it to the well-known Ignaczak equation of elastodynamics. The equation of motion representing the dynamics of the viscoelastic material is derived where the elastic part of the stress tensor is assumed to be linear, and basic solutions to this equation are given.

We hope that many open questions of existence and uniqueness of solutions in 3D as well as the investigation of wave propagation, etc. concerning this new model as well as its nonlinear generalizations will be answered in future studies leading to a complete understanding of such viscoelastic response which has also been observed in many experiments.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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M. Ostoja-Starzewski
Department of Mechanical Science and Engineering, Beckman Institute
University of Illinois at Urbana-Champaign
Urbana IL61801
USA
e-mail: martinos@illinois.edu

Y. Şengül
School of Mathematics
Cardiff University
Cardiff CF24 4AG
UK
e-mail: sengultezely@cardiff.ac.uk

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