Synthesis of Dynamic Responses of Redundant Robot Manipulators

Josue Moreno[1], José J. Patiño[2], Carlos G. Helguero[1] and Carlos Saldarriaga[1]

***Abstract*— In this paper, we present and validate a novel methodology that synthesizes the dynamic response of robotic manipulators performing Cartesian impedance-related tasks. By leveraging linear system theory and addressing the kinematic redundancies inherent in the system, we derive and directly solve equations to compute the damping parameters based on desired control criteria (damping ratios). The proposed equations correspond to the complex eigenvalues governing the system's dynamics. Unlike existing approaches, this method bypasses complex optimization or iterative processes, providing direct solutions to damping or stiffness parameters. Our approach applies broadly, irrespective of specific redundancy conditions or Cartesian task coordinates. Using a 7-DoF robot, we demonstrate that multiple solutions can impose similar dynamic responses. We further show that the system's natural frequencies must align with defined criteria, and while imposing damping ratios may suggest infinite possible frequency values, physically meaningful natural frequencies are calculated based on robot geometry, stiffness, and mass matrices. This ensures that these values are not arbitrary. This methodology contributes significantly to the field by simplifying implementation and improving system stability and predictability.**

I. Introduction

In modern robotics, impedance control is a well-established technique for managing the interaction between robots and their environment, particularly in tasks that require precise torque management. Impedance control adjusts the system's stiffness and damping to ensure safe and effective interactions. One key capability of impedance control in robotic manipulators, particularly redundant ones with more degrees of freedom (DoF) than those needed for a task, is the ability to modulate their dynamic response. These robots offer flexibility and robustness, but selecting the appropriate parameters for control poses challenges.

Several recent studies propose diverse methods to determine stiffness and damping parameters by different analytical or data-based methods, with the goal of suppressing unwanted vibrations. However, these methods often require iterative optimization processes or trial-and-error techniques.

In this article, we introduce an innovative methodology that directly calculates damping parameters from specific desired damping ratios, bypassing the need for complex optimization and considering the configuration dependent dynamics of the coupled robotic system. By addressing and mathematically handling redundancies, we derive and solve equations that directly provide the damping parameters for a multi-dimensional, highly coupled, and positive semidefinite robotic system. This approach does not assume or limit redundancy conditions, making it broadly applicable. Additionally, while imposing damping ratios can yield multiple possible frequency values, by utilizing the robot's geometry, stiffness, and mass matrices, we ensure that the obtained natural frequencies are theoretically correct, grounded in the robot's physical properties, avoiding arbitrary values. This methodology simplifies the implementation process and enhances system stability and predictability. We demonstrate the efficacy of our approach using a 7-DoF Panda robotic arm in two specific cases: Cartesian impedance control with 3 task coordinates (x, y, z directions) and with 5 task coordinates (x, y, z, α, β directions). This work contributes significantly to the field by providing a straightforward and reliable method to modulate dynamic responses in redundant robotic systems.

II. Related Work

In classical control theory, the analysis of the damping ratio is fundamental in characterizing the dynamic behavior of mechanical systems. In particular, for a two-DoF translational system, the damping ratio dictates key response characteristics such as overshoot, settling time, and stability. The mathematical modeling of such systems is typically based on mass-spring-damper equations, where the damping ratio (ζ) is defined as $\zeta = \frac{c}{2\sqrt{km}}$, with c representing the damping coefficient, k the stiffness, and m the mass [1].

A detailed discussion of these principles is presented in Ogata's Modern Control Engineering [1], which explores various analytical and design methodologies based on damping ratio considerations. This theoretical foundation is particularly relevant for extending impedance control strategies to robotic applications, ensuring stable and predictable interactions in dynamic environments.

Impedance control is a well-established technique in the field of robotics for managing a robot's interaction with its environment, including handling fragile or dangerous objects, people, or other robots. This is achieved by modifying the mechanical impedance, adjusting parameters such as the stiffness and damping of the system. Hogan's work introduced the concept of impedance control, laying the theoretical foundation that many researchers have since expanded to address challenges in dynamic and unstable environments [2], [3], [4], [5].

[1]Josue Moreno, Carlos G. Helguero and Carlos Saldarriaga are with the Facultad de Ingeniería en Mecánica y Ciencias de la Producción, Escuela Superior Politécnica del Litoral, ESPOL, Campus Gustavo Galindo Km 30.5 Vía Perimetral, P.O. Box 09-01-5863, Guayaquil, Ecuador. josrmore@espol.edu.ec, chelguer@espol.edu.ec, cxsaldar@espol.edu.ec.

[2] José J. Patiño is with the School of Computer Science & Informatics at Cardiff University, UK. patinominanjj@cardiff.ac.uk.

An important advance in this field is the work described in [6], where an analytical method is presented to modulate the dynamic response of robotic manipulators by selecting suitable stiffness and damping parameters. The approach focuses on mapping these parameters (matrices) into the joint space and analyzing the vibration dynamics to suppress undesirable vibration modes, thus avoiding the use of commonly used trial and error methods. This closed-form approach is particularly beneficial for redundant robots, as demonstrated in their experimental validation on 7-DoF robotic manipulators [6]. However, while the presented methodology provides valuable insights into how damping parameter selection affects the system's behavior, it does not directly impose a desired dynamic behavior on the system. This process involves multiple iterations and trade-offs between damping and control criteria.

A complementary study expanded on the application of damping ratio prediction for redundant Cartesian impedance-controlled robots using machine learning techniques. This research leveraged large datasets and advanced computational models to predict appropriate damping ratios, optimizing control parameters for better dynamic response. Thus, the combination of analytical methods and machine learning provides a robust framework to improve robot performance in complex interaction tasks [7].

Building upon these fundamental studies, we introduce a novel analytical methodology to directly calculate the values of the damping matrix from specified damping ratios, via the complex eigenvalues of the system, which describe the dynamic response of the system through their linear combination. While previous research focuses mainly on mapping, predicting, or learning parameters using various analytical and computational methods, our novel analytical approach offers a straightforward and theoretically sound computational route that considers any number of redundancies and simplifies the process. Unlike other existing methods, our approach does not impose restrictions on redundancy conditions, making it widely applicable in robotics.

This direct calculation methodology ensures precise control over the damping characteristics of the system, imposing theoretically sound parameters that consider the dynamics. By directly addressing the calculation of damping coefficients from given damping ratios, our research contributes significantly to the field of impedance control in robotics, improving both the accuracy of dynamic response modulation and simplifying the implementation process for practical applications.

III. Theoretical Background

A. General equations

A robotic arm with n degrees of freedom is represented by its general equation of motion

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}(\mathbf{t}) + \mathbf{G}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}(\mathbf{t}) + \mathbf{g}(\mathbf{q}) = \tau + \tau_{\mathbf{ext}} \tag{1}$$

where $\mathbf{q}$ corresponds to the vector of n joint angles , $\mathbf{M}$ the mass matrix, $\mathbf{G}$ contains all the Coriolis and centrifugal terms, $\mathbf{g}$ the gravity compensation term and τ_{ext} the external torques. This arm operates in an m-dimensional task-space coordinate system, limited to at most 6 task coordinates. In case n is greater than m, the resulting robotic system is kinematically redundant [8]. Implementing an impedance control law, the applied motor torques τ can be selected so that the system in (1) becomes

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}(\mathbf{t}) + \mathbf{C}\dot{\mathbf{q}}(\mathbf{t}) + \mathbf{K}\mathbf{q}(\mathbf{t}) = \tau_{\mathbf{ext}} \tag{2}$$

where $\mathbf{C}$ is the damping matrix and $\mathbf{K}$ is the stiffness matrix, both in the joint space. However, for most robotic applications, the (usually diagonal) stiffness and damping matrices are defined in the Cartesian space. Using the configuration dependent Jacobian matrix of the robot, $\mathbf{J}_{m\times n}$, the Cartesian-space ($m \times m$) terms $\mathbf{K}_C = \text{diag}\left(\begin{bmatrix} k_x & k_y & \dots & k_m \end{bmatrix}\right)$, and $\mathbf{C}_C = \text{diag}\left(\begin{bmatrix} c_x & c_y & \dots & c_m \end{bmatrix}\right)$ can be mapped into the joint space of the robot [9], [10]

$$\mathbf{K} = \mathbf{J}^T\mathbf{K}_C\mathbf{J} + \mathbf{K}_g + \mathbf{J}^T\mathbf{C}_C\dot{\mathbf{J}}, \quad \mathbf{C} = \mathbf{J}^T\mathbf{C}_C\mathbf{J} \tag{3}$$

where $\mathbf{K}_g = \left[\left(\frac{\partial \mathbf{J}^T}{\partial q_1}\mathbf{f}\right)\left(\frac{\partial \mathbf{J}^T}{\partial q_2}\mathbf{f}\right)\cdots\left(\frac{\partial \mathbf{J}^T}{\partial q_n}\mathbf{f}\right)\right]$, and $\mathbf{f}$ is the external forces vector.

B. Methodology for synthesizing the dynamic response

Once the system is mapped into the joint space of the robot, we can proceed to describe the methodology to establish a direct relationship between the damping matrix values and the desired dynamic characteristics of the (positive semi-definite in case of redundancies) system, and obtain a solution to the synthesis problem.

The process begins by establishing the rigid body or zero-potential energy (ZP) mode of the system through the null space definition of the stiffness matrix $\mathbf{K}$, which asserts that the rigid body mode $\mathbf{u_0}$ is an element of the null space of matrix $\mathbf{K}$ [11].

$$\mathbf{K}\mathbf{u_0} = \mathbf{0} \tag{4}$$

Note that for the intended robot configuration, the values of the stiffness matrix are constant; the only variables are the values of the diagonal damping matrix that we will try to obtain so as to match with the desired damping ratios ζ's of the system. Next, the computed $\mathbf{u_0}$ vector is normalized with respect to the mass matrix using the relation $\mathbf{u_0^T M u_0} = 1$. Since it corresponds to the ZP or non-oscillatory mode, its assigned frequency is 0, $\omega_0 = 0$.

$$\mathbf{u_{N0}} = \frac{\mathbf{u_0}}{\sqrt{\mathbf{u_0^T M u_0}}} \tag{5}$$

Here, $\mathbf{u_{N0}}$ represents the normalized vector obtained by scaling $\mathbf{u}_0$ with respect to its weighted magnitude defined by the matrix $\mathbf{M}$. This normalized vector is then used to decouple and isolate the zero-potential energy (ZP) mode from the system

$$\mathbf{u_0^T M q} = \mathbf{0} \tag{6}$$

If defined as $\mathbf{u_0^T M} = [s_1\, s_2\, \dots\, s_n]^T$, then the previous equation becomes:

$$\sum_{i=1}^{n} s_i q_i = s_1 q_1 + s_2 q_2 + \ldots + s_n q_n = 0 \tag{7}$$

Thus, solving for q_i we get:

$$q_i = -\left(\frac{s_1}{s_i}q_1 + \frac{s_2}{s_i}q_2 + \ldots + \frac{s_{i-1}}{s_i}q_{i-1} + \frac{s_{i+1}}{s_i}q_{i+1} + \ldots + \frac{s_n}{s_i}q_n\right) \tag{8}$$

and we can obtain a constraint matrix $\mathbf{S}$. This way, a reduced space $\mathbf{q}'$ can be obtained, which is free from the ZP or the effect of the redundant DoF of the robot [6].

$$\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -\frac{s_1}{s_n} & -\frac{s_2}{s_n} & \ldots & -\frac{s_{n-1}}{s_n} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_{n-1} \end{pmatrix} \tag{9}$$

$$\mathbf{q} = \mathbf{S}\mathbf{q}' \tag{10}$$

Using this expression, the system dynamic equation is simplified and reformulated as a positive definite system.

$$\mathbf{M}'\ddot{\mathbf{q}}' + \mathbf{C}'\dot{\mathbf{q}}' + \mathbf{K}'\mathbf{q}' = \mathbf{S}^{\mathbf{T}}\tau_{\mathbf{ext}} \tag{11}$$

$$\mathbf{M}' = \mathbf{S}^{\mathbf{T}}\mathbf{M}\mathbf{S}, \quad \mathbf{C}' = \mathbf{S}^{\mathbf{T}}\mathbf{C}\mathbf{S}, \quad \mathbf{K}' = \mathbf{S}^{\mathbf{T}}\mathbf{K}\mathbf{S} \tag{12}$$

If more than one redundant DoF is present, this process of redundancy (or ZP) elimination described in equations (4) to (12) is repeated $n - m$ times using the matrices of the newly reduced system and the subsequent constraint matrices as necessary until the system becomes positive definite, otherwise the system would not be analytically solvable. From this formulation, and using linear system theory, it is possible to determine the state matrix $\mathbf{A}$. Therefore, the eigenvalues of matrix $\mathbf{A}$ can be directly related to the dynamic characteristics of the system, which is the main objective of the paper.

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -(\mathbf{M}')^{-\mathbf{1}}\mathbf{K}' & -(\mathbf{M}')^{-\mathbf{1}}\mathbf{C}' \end{bmatrix}_{2m\times 2m} \tag{13}$$

Since the only unknowns in matrix $\mathbf{A}$ are contained in matrix $\mathbf{C}'$, which is a linear combination of the damping coefficients c_x, c_y, ... ,c_m, the eigenvalues of $\mathbf{A}$ are used to establish a relationship with the desired damping ratios, as they encapsulate the system's dynamic behavior. Given an eigenvalue λ_i of the matrix $\mathbf{A}$, the characteristic polynomial can be expressed as [12]:

$$|\mathbf{A} - \lambda_i \mathbf{I}| = 0 \tag{14}$$

$$\lambda_i^{2m} + a_1\lambda_i^{2m-1} + \ldots + a_{2m-1}\lambda_i^1 + a_{2m} = 0 \tag{15}$$

where i refers to each of the $2m$ modes of the reduced positive definite system. To facilitate the interpretation and handling of the system dynamics, the eigenvalues are expressed in polar form:

$$\lambda_i = r_i\left(\cos(\theta_i) + \sin(\theta_i)\, j\right), \quad \text{for } i = 1, \ldots, 2m \tag{16}$$

where j is the imaginary unit, defined as $j^2 = -1$. In this context, r_i represents the system's natural frequency ($r_i = \omega_i$), which can be analytically determined from the actual reduced inertia and stiffness matrices using mechanical vibration theory [13]:

$$\mathbf{D_{in}} = (\mathbf{K}')^{-\mathbf{1}}\mathbf{M}' \tag{17}$$

$$\omega_p = \sqrt{\frac{1}{\xi_p}}, \quad \text{for } p = 1, \ldots, m \tag{18}$$

Where ξ_p corresponds to an eigenvalue of matrix $\mathbf{D_{in}}$ in Equation (17). Furthermore, θ_i represents the angle in polar form (Equation 16), which, from a geometric perspective, is directly related to the desired (given) damping ratios ζ_i:

$$\cos^{-1}(\zeta_i) = \tan^{-1}\left(\frac{Im(\lambda_i)}{Re(\lambda_i)}\right) = \theta_i \tag{19}$$

Substituting this expression into Equation (16), and separating real and imaginary components leads to the following system of equations:

$$r_i^{2m}\cos((2m)\theta_i) + a_1 r_i^{2m-1}\cos((2m-1)\theta_i) + a_2 r_i^{2m-2}\cos((2m-2)\theta_i) + \ldots + a_{2m-1}r_i^1\cos(\theta_i) + a_{2m} = 0 \tag{20}$$

$$r_i^{2m}\sin((2m)\theta_i) + a_1 r_i^{2m-1}\sin((2m-1)\theta_i) + a_2 r_i^{2m-2}\sin((2m-2)\theta_i) + \ldots + a_{2m-1}r_i^1\sin(\theta_i) = 0 \tag{21}$$

In this way, the desired damping ratios of the system are directly related to the equations to be solved for the elements of the $\mathbf{C}_c$ matrix. It is important to note that for a system with m task coordinates, a total of $2m$ equations with m unknowns are obtained, leading to a wide variety of $\frac{(2m)!}{(m!)^2}$ possible solutions, of which we always choose those ensuring the positive definiteness and stability of the system (eigenvalues on the left-hand side of the complex plane). Figure 1 shows a summary of the methodology presented in this Section.

IV. Results

In order to validate the proposed methodology, we provide a set of simulation results with different $(n - m)$ number of redundant DoF's. The synthesis of dynamics is performed on a 7 DoF Franka Panda robotic arm. A Cartesian impedance control task with m task (Cartesian) coordinates is utilized to demonstrate the handling of multiple redundancies and subsequent synthesis of dynamic responses. The robotic arm has the following initial configuration: $\mathbf{q}_0 = [\pi/4, \pi/6, -\pi/4, -\pi/2, \pi/3, \pi/2, \pi/6]$ rad, and the Cartesian stiffness matrices are shown in the Appendix.

A. Example for 4 redundant DoF's ($m = 3$)

In this example, Cartesian impedance control with 3 Cartesian coordinates ($m = 3$) is considered, resulting in a redundancy of 4 DoF. This redundancy setup comprises 3 task coordinates for the translational directions (x, y, z). The desired damping ratios to be imposed are: $\zeta = \begin{bmatrix} 0.98 & 0.69 & 0.59 \end{bmatrix}$.

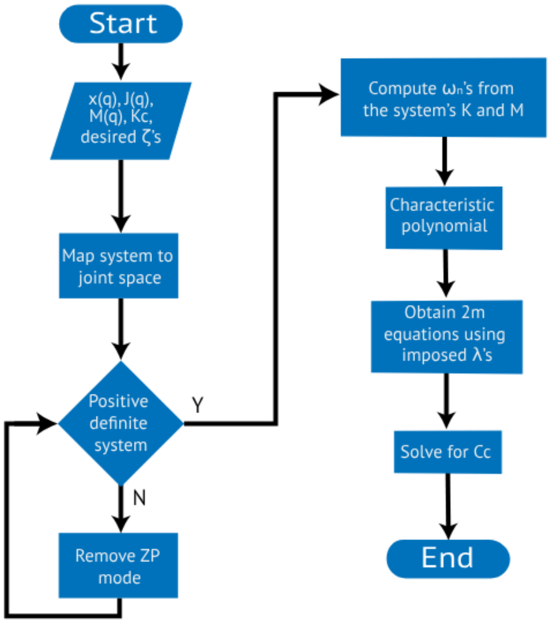

Fig. 1. Flowchart describing the proposed methodology

The corresponding $(m \times n)$ $\mathbf{J}(\mathbf{q})$ Jacobian matrix is obtained, and the desired diagonal stiffness, and unknown damping matrices are mapped to the joint space:

$$\mathbf{K} = \begin{bmatrix} 1261.0 & 77.03 & 1037.0 & 272.8 & 147.0 & -160.4 & 0 \\ 77.03 & 335.8 & 158.4 & -199.4 & -19.73 & -12.87 & 0 \\ 1037.0 & 158.4 & 879.6 & 162.2 & 112.6 & -132.7 & 0 \\ 272.8 & -199.4 & 162.2 & 300.2 & 75.73 & 13.34 & 0 \\ 147.0 & -19.73 & 112.6 & 75.73 & 25.97 & -6.855 & 0 \\ -160.4 & -12.87 & -132.7 & 13.34 & -6.855 & 41.58 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = function(c_x, c_y, c_z)_{7\times 7}$$

After mapping the system to the joint space and reducing it to a positive definite system by successively removing the $(n - m = 4)$ ZP modes, we derive the characteristic polynomial and solve for the eigenvalues, which govern the dynamic response of the system. This process results in a set of six equations with three unknowns, leading to 20 sets of possible solutions for the damping parameters. Below are the three most representative Cartesian damping matrix solutions with their corresponding errors presented in Table I.
$\mathbf{C_{C1}} = \text{diag}(\begin{bmatrix}78.3 & 227.0 & 76.3\end{bmatrix})$;
$\mathbf{C_{C2}} = \text{diag}(\begin{bmatrix}72.1 & 80.8 & 121.0\end{bmatrix})$;
$\mathbf{C_{C3}} = \text{diag}(\begin{bmatrix}157.0 & 75.5 & 95.0\end{bmatrix})$.

Solution $\mathbf{C}_{C1}$ generates the smallest error in all damping ratios, making it the most suitable solution. This highlights the precision of the proposed methodology in the synthesis of dynamic responses through the imposed damping ratios.

TABLE I
COMPARISON OF SOLUTION FOR 3 CARTESIAN COORDINATES

Solution	ζ Required	ζ Obtained	Error
C_{C1}	0.980	1.000	0.0200
	0.690	0.698	0.0079
	0.590	0.588	0.0019
C_{C2}	0.980	0.990	0.0102
	0.690	0.663	0.0272
	0.590	0.585	0.0049
C_{C3}	0.980	1.000	0.0200
	0.690	0.735	0.0445
	0.590	0.556	0.0341

B. Example for two redundant DoF's (m=5)

In this example, a Cartesian impedance control with 5 Cartesian coordinates $(m = 5)$ is considered, with a redundancy of 2 DoF. This redundancy setup comprises 3 coordinates for the translational directions (x, y, z) and 2 coordinates for the orientation (α, β). The desired damping ratios to be imposed are: $\zeta = \begin{bmatrix}0.99 & 0.89 & 0.78 & 0.66 & 0.57\end{bmatrix}$. The Jacobian matrix is adjusted and the stiffness and damping matrices are mapped to the joint space of the robot.

$$\mathbf{K} = \begin{bmatrix} 1261.0 & 77.03 & 1037.0 & 272.8 & 147.0 & -160.4 & 0 \\ 77.03 & 345.8 & 158.4 & -206.5 & -26.8 & -16.4 & 6.124 \\ 1037.0 & 158.4 & 882.1 & 159.1 & 115.7 & -136.4 & 1.402 \\ 272.8 & -206.5 & 159.1 & 308.9 & 76.98 & 20.36 & -6.047 \\ 147.0 & -26.8 & 115.7 & 76.98 & 34.72 & -8.881 & -2.613 \\ -160.4 & -16.4 & -136.4 & 20.36 & -8.881 & 48.29 & -4.237 \\ 0 & 6.124 & 1.402 & -6.047 & -2.613 & -4.237 & 4.536 \end{bmatrix}$$

$$\mathbf{C} = function(c_x, c_y, c_z, c_\alpha, c_\beta)_{7\times 7}$$

The methodology follows the same approach as in the previous example but requires only two steps for the reduction to a positive definite system. Although there are more than 200 combinations (sets of possible solutions), some of them generate lower errors than others. The obtained values for the best three sets of damping coefficients are as follows:
$\mathbf{C_{C1}} = \text{diag}(\begin{bmatrix}192.41 & 101.26 & 63.21 & 1.01 & 12.73\end{bmatrix})$;
$\mathbf{C_{C2}} = \text{diag}(\begin{bmatrix}151.89 & 105.26 & 83.04 & 1.59 & 6.04\end{bmatrix})$;
$\mathbf{C_{C3}} = \text{diag}(\begin{bmatrix}154.34 & 110.01 & 80.93 & 1.72 & 5.95\end{bmatrix})$. Similarly, a comparison of the errors is given in Table II.

V. DISCUSSION

The proposed methodology focuses on formulating a positive definite system by eliminating the redundancies in order to establish a direct relationship between the damping matrix values and the system's dynamic characteristics. This approach offers significant advantages over traditional methods, which often rely on iterative optimization processes or trial-and-error practices. By not assuming or limiting redundancy conditions, our methodology allows for broader applicability

TABLE II
COMPARISON OF SOLUTION FOR 5 CARTESIAN COORDINATES

Solution	ζ Required	ζ Obtained	Error
C_{C1}	0.99	1.000	0.0100
	0.89	0.873	0.0170
	0.78	0.810	0.0300
	0.66	0.659	0.0001
	0.57	0.566	0.0004
C_{C2}	0.99	1.000	0.0100
	0.89	0.879	0.0110
	0.78	0.809	0.0290
	0.66	0.649	0.0110
	0.57	0.564	0.0060
C_{C3}	0.99	1.000	0.0100
	0.89	0.921	0.0310
	0.78	0.777	0.0030
	0.66	0.638	0.0220
	0.57	0.565	0.0050

across various robotic systems, particularly in systems with different degrees of freedom.

To demonstrate our methodology, we applied it to a 7-degree-of-freedom (DoF) Panda robotic arm. Starting with the Jacobian and mass matrices, we reduced the system by analytically eliminating the $n-m$ ZP modes of motion. This involved normalizing the rigid modes vector and formulating a positive definite system, which simplified the dynamic equation. The eigenvalues from the resulting matrix were used to calculate natural frequencies and establish a direct relationship with the desired damping ratios, allowing us to derive the required damping parameters.

For the 3 Cartesian coordinates case, where 4 redundant DoFs were present, we needed to map the stiffness and damping matrices to the joint space and performed ZP redundancy reduction in four steps. With desired damping ratios $\zeta = [0.98, 0.69, 0.59]$, the method provided damping parameters $\mathbf{C}_c = diag([78.3, 227.0, 76.3])$ generating a system with minimal error with respect to the desired damping ratios of the system.

In the 5 Cartesian coordinates case, including x, y, z, α, and β in the task space, we mapped the matrices to the joint space and reduced redundancy in two steps. For the desired damping ratios $\zeta = [0.99, 0.89, 0.78, 0.66, 0.57]$, the method yielded damping parameters $\mathbf{C}_c = diag([192.41, 101.26, 63.21, 1.01, 12.73])$, achieving high precision in damping with minimal error.

While imposing damping ratios might suggest an infinite range of possible frequency values due to the nature of complex eigenvalues, the actual natural frequencies are derived from the robot's stiffness and mass matrices, ensuring that these frequencies are physically meaningful and not arbitrary. This result confirms the practical viability of the proposed methodology, which simplifies dynamic response synthesis and ensures system stability, since our methodology allows us to directly impose proper eigenvalues on the system. A detailed comparison was performed with an alternative method presented in [14], where the Cartesian damping matrix is computed as:

$$\mathbf{D_d} = 2\mathbf{Q}(\mathbf{q})\mathbf{D}_{\xi_\mathbf{i}}\mathbf{K}_{\mathbf{d0}}^{\mathbf{1/2}}\mathbf{Q}^{\mathbf{T}}(\mathbf{q}) \tag{22}$$

In this equation, $\mathbf{D_d}$ represents the damping matrix, $\mathbf{Q}(\mathbf{q})$ is assumed to be the identity matrix (as chosen in [14], $\mathbf{D}_{\xi_\mathbf{i}}$ is the diagonal matrix of the desired damping ratios for each direction (assuming a decoupled system), and $\mathbf{K_{d0}}$ is the diagonal Cartesian stiffness matrix. Using the stiffness values $\mathbf{K_{d0}} = \text{diag}([2000, 3000, 1000])$ and the desired damping ratios $\mathbf{D}_{\xi_\mathbf{i}} = \text{diag}([0.98, 0.69, 0.59])$, the resulting damping matrix is: $\mathbf{D_d} = diag(\begin{bmatrix}87.65 & 75.59 & 37.31\end{bmatrix})$. Using these values, the system's state matrix $\mathbf{A}$ achieved high damping ratios errors: of 0.22, 0.17, and 0.30 compared to the desired values. This reveals a significant discrepancy compared to the required values of 0.98, 0.69 and 0.59. It should also be noted that the expression in equation (22) does not consider neither the mass matrix nor the robot configuration.

In contrast, the proposed method achieved significantly lower errors. For instance, solution $\mathbf{C}_{C1}$ achieved damping ratios $\zeta = \begin{bmatrix}1 & 0.698 & 0.588\end{bmatrix}$ with errors 0.02, 0.0079, and 0.0019, respectively.

Unlike the approach in [14], which depends only on the stiffness matrix $\mathbf{K_{d0}}$ and remains configuration-independent, the proposed approach incorporates the stiffness matrix, Jacobian, and inertia matrix, dynamically adapting to the robot's pose. This adaptability is crucial for maintaining optimal performance across configurations.

Additionally, the proposed method consistently outperforms the alternative, with errors below 0.1 and often as low as 0.01, ensuring better control precision. Figure 2 shows simulation results that compare the dynamic responses using three different approaches: the proposed methodology, the modal iterative method as shown in [6], and the arbitrary method from equation (22), in blue, red, and yellow colors, respectively. As we can see, the proposed methodology and the iterative one are somewhat comparable, while the arbitrary one had significantly more oscillations and higher settling time and overshoot, as expected from theory. Future work may explore hybrid strategies that combine configuration-dependent adaptability with computational efficiency, including experimental validation in unstructured environments.

The proposed methodology offers a more precise, direct and adaptable solution to synthesize the dynamic response of robotic manipulators. Its ability to adjust damping parameters based on the robot's configuration ensures better performance and stability, making it a more versatile and effective approach for modern robotic applications.

VI. CONCLUSION

We have presented and validated an innovative methodology for synthesizing the dynamic response of redundant robots performing Cartesian impedance control. We developed an approach that allows for direct calculation of damping parameters from specific damping ratios, bypassing complex iterative processes, and trial-and-error practices. Notably, our methodology does not impose or limit the redundancy conditions, making it broadly applicable to various robotic systems.

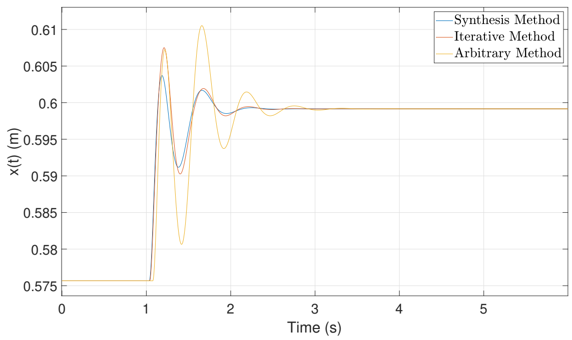

Fig. 2. Dynamic response comparison of a Panda 7 DoF robot subject to an initial external perturbation using different damping matrices.

This approach ensures that, while imposing damping ratios can suggest infinite associated frequency values, natural frequencies are calculated based on the actual physical characteristics of the robot. Our method has proven to be precise, efficient, and flexible, adapting to different Cartesian degrees of freedom while significantly enhancing the stability and predictability of the system. By simplifying implementation, this approach facilitates innovation in the design and control of advanced robots, opening new possibilities in various robotic applications and future research directions.

This methodology could be applied to collaborative robots in manufacturing or autonomous vehicles, where adaptive control is important. Potential future research includes integrating this methodology with machine learning algorithms to enhance adaptability in dynamic environments, further improving computational efficiency and robotic performance in real-world applications.

APPENDIX

A. Matrices from the example of impedance control for 3 cartesian DoF (all in SI units)

$$\mathbf{C_C} = \text{diag}\left(\begin{bmatrix} c_x & c_y & c_z \end{bmatrix}\right);$$
$$\mathbf{K_C} = \text{diag}\left(\begin{bmatrix} 2000 & 3000 & 1000 \end{bmatrix}\right)$$

$$M^{iv} = \begin{bmatrix} 11400.0 & 13411.0 & 4445.0 \\ 13411.0 & 15855.0 & 5231.0 \\ 4445.0 & 5231.0 & 1734.0 \end{bmatrix}$$

$$K^{iv} = \begin{bmatrix} 9.801 \times 10^6 & 1.171 \times 10^7 & 3.786 \times 10^6 \\ 1.171 \times 10^7 & 1.407 \times 10^7 & 4.524 \times 10^6 \\ 3.786 \times 10^6 & 4.524 \times 10^6 & 1.463 \times 10^6 \end{bmatrix}$$

B. Matrices from the example of impedance control for 5 cartesian DoF

$$\mathbf{C_C} = \text{diag}\left(\begin{bmatrix} c_x & c_y & c_z & c_\alpha & c_\beta \end{bmatrix}\right);$$
$$\mathbf{K_C} = \text{diag}\left(\begin{bmatrix} 2000 & 3000 & 1000 & 10 & 10 \end{bmatrix}\right)$$

$$M'' = \begin{bmatrix} 2.135 & 0.393 & 1.434 & 0.3642 & -0.005979 \\ 0.393 & 2.027 & 0.7275 & -0.8317 & -0.06497 \\ 1.434 & 0.7275 & 1.16 & -0.00223 & -0.01013 \\ 0.3642 & -0.8317 & -0.00223 & 0.8003 & 0.04175 \\ -0.005979 & -0.06497 & -0.01013 & 0.04175 & 0.02876 \end{bmatrix}$$

$$K'' = \begin{bmatrix} 1456.0 & 134.7 & 1130.0 & 262.9 & 151.3 \\ 134.7 & 354.3 & 198.9 & -210.5 & -23.84 \\ 1130.0 & 198.9 & 907.4 & 155.6 & 114.7 \\ 262.9 & -210.5 & 155.6 & 309.4 & 77.24 \\ 151.3 & -23.84 & 114.7 & 77.24 & 34.59 \end{bmatrix}$$

REFERENCES

[1] K. Ogata, *Modern Control Engineering*, 4th ed. USA: Prentice Hall PTR, 2001.

[2] N. Hogan, "Impedance control: An approach to manipulation: part i - theory, part ii - implementation, part iii - applications." *Journ. of Dyn. Systems, Measurement and Control*, vol. 107, no. 1, pp. 1–24, 1985.

[3] O. Khatib, "A unified approach for motion and force control of robot manipulators: The operational space formulation," *IEEE Journal on Robotics and Automation*, vol. 3, no. 1, pp. 43–53, February 1987.

[4] L. Villani and J. De Schutter, "Force control," in *Springer Handbook of Robotics*, B. Siciliano and O. Khatib, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008, pp. 161–185.

[5] N. Brissonneau, B. He, G. C. Thomas, and L. Sentis, "Biologically-inspired impedance control with hysteretic damping," *IEEE Control Systems Letters*, vol. 5, no. 5, pp. 1717–1722, 2021.

[6] C. Saldarriaga, N. Chakraborty, and I. Kao, "Damping selection for cartesian impedance control with dynamic response modulation," *IEEE Transactions on Robotics*, vol. 38, no. 3, pp. 1915–1924, 2022.

[7] J. Patino, A. Encalada-Davila, J. Sampietro, C. Tutiven, C. Saldarriaga, and I. Kao, "Damping ratio prediction for redundant cartesian impedance-controlled robots using machine learning techniques," *Mathematics*, vol. 11, no. 4, 2023. [Online]. Available: https://www.mdpi.com/2227-7390/11/4/1021

[8] B. Siciliano, L. Sciavicco, L. Villani, and G. Oriolo, *Robotics: Modelling, Planning and Control*, ser. Adv Textbooks in Cont and Signal Proc. Springer London, 2010.

[9] S.-F. Chen and I. Kao, "Conservative congruence transformation for joint and cartesian stiffness matrices of robotic hands and fingers," *The International Journal of Robotics Research*, vol. 19, no. 9, pp. 835–847, 2000.

[10] J. J. Patiño and C. Saldarriaga, "On mapping stiffness and damping in robotic impedance control: a spatial validation of coupling," in *2023 American Control Conference (ACC)*, 2023, pp. 774–779.

[11] C. Saldarriaga and I. Kao, "Impedance Control on Redundant Manipulators With Zero-Potential-Energy Motions: Theory and Experimental Validation," *Journal of Mechanisms and Robotics*, vol. 15, no. 5, p. 051010, 11 2022. [Online]. Available: https://doi.org/10.1115/1.4055966

[12] G. Strang, *Introduction to Linear Algebra*, 4th ed. Wellesley-Cambridge Press, 2009.

[13] L. Meirovitch, *Fundamentals of Vibrations*. McGraw-Hill, 2001.

[14] A. Albu-Schaffer, C. Ott, U. Frese, and G. Hirzinger, "Cartesian impedance control of redundant robots: recent results with the dlr-light-weight-arms," in *2003 IEEE International Conference on Robotics and Automation (Cat. No.03CH37422)*, vol. 3, Sep. 2003, pp. 3704–3709 vol.3.