

Topological invariants for bundles of C^* -algebras with symmetries

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Abstract

By a classical result of Dixmier and Douady, one can classify locally trivial bundles of C^* -algebras with the compact operators as fibres by methods from algebraic topology. Building on work by Dadarlat, Evans and Pennig in this thesis we develop an equivariant generalisation of this theory to a wider class of C^* -algebras starting with a concrete setting. More precisely, we consider infinite tensor product actions of $G = \mathbb{Z}/p\mathbb{Z}$ on the UHF-algebra $D = \text{End}(V)^{\otimes \infty}$ for a finite-dimensional unitary G -representation V and determine the equivariant homotopy type of the group $\text{Aut}(D \otimes \mathbb{K})$, where \mathbb{K} are the compact operators on $\ell^2(G) \otimes H_0$ for a separable Hilbert space H_0 with $\dim(H_0) = \infty$. We show that this group carries an equivariant infinite loop space structure revealing it as the first space of a G -spectrum, which we prove to be equivalent to the positive units $gl_1(KU^D)_+$ of equivariant KU^D -theory. Here, KU^D is a G -spectrum representing $X \mapsto K_*^G(C(X) \otimes D)$. As a consequence we are able to classify equivariant $(D \otimes \mathbb{K})$ -bundles over finite CW-complexes by the first group of the cohomology theory associated to $gl_1(KU^D)_+$. In doing so, we propose an equivariant infinite loop spaces machine which may be of independent interest.

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Introduction

This thesis explores connections between two seemingly disparate research areas: operator algebras and equivariant stable homotopy theory. These arise by looking at equivariant bundles of G - C^* -algebras: families of operator algebras with symmetries (encoded by the action of a group G) that are continuously and locally trivially parametrised by a topological space. From a homotopy-theoretic standpoint, such bundles admit a complete classification in terms of the C^* -algebra automorphisms. These hold a physical significance: they can be interpreted as symmetries of the quantum system associated to the operator algebra. It turns out that there is a class of C^* -algebras whose automorphism groups also carry stably homotopical information, namely they underlie a spectrum. This allows us to completely classify the equivariant C^* -algebra bundles via an algebro-topological invariant, which is obtained by manipulating ring objects in the category of spectra.

Units of ring spectra. Let R be a commutative ring. Forgetting its additive structure we may focus on the multiplication in R and consider its group of units $GL_1(R)$. The analogue of commutative rings in stable homotopy theory are commutative ring spectra. Famous examples here include the ones representing ordinary cohomology or topological K -theory. Similarly to the algebraic setting an associative ring spectrum E has a space of units $GL_1(E)$, which turns out to be the zeroth space of a spectrum $gl_1(E)$ of units in case E is an E_∞ -ring spectrum. This theory of units originated in Sullivan's MIT notes [52], where they were used to study obstructions to the orientability of vector bundles and spherical fibrations.

Each model of ring spectra has its own definition of the corresponding units. The definition for E_∞ -ring spectra goes back to May, Quinn, Ray and Tornehave [31]. The units of a commutative symmetric ring spectrum were initially defined in work by Schlichtkrull [47]. It was shown by Lind in [28] that the comparison functors between the various models of ring spectra give rise to weak equivalences between their respective units.

Bundles of stabilised strongly self-absorbing C^* -algebras. The work in this thesis stems from an observation that interlinks the unit spectrum $gl_1(KU^{\mathcal{D}})$ of a variant of complex topological K -theory with the theory of C^* -algebras. Here, \mathcal{D} denotes a strongly self-absorbing C^* -algebra. A unital C^* -algebra \mathcal{D} belongs to this class if there is an isomorphism $\varphi: \mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{D}$ that is approximately unitarily equivalent to the left tensor embedding. These algebras play a cornerstone role in the classification programme of separable nuclear simple C^* -algebras. Examples include infinite tensor products of matrix algebras (i.e. infinite UHF-algebras), the Jiang-Su algebra \mathcal{Z} and the infinite Cuntz algebra \mathcal{O}_∞ . To each such algebra \mathcal{D} one can associate a commutative symmetric ring spectrum $KU_*^{\mathcal{D}}$ that represents the cohomology theory

$$X \mapsto K_*(C(X) \otimes \mathcal{D})$$

on finite CW-complexes [6, Sec. 4.1], i.e. the operator-algebraic K -theory of the C^* -algebra of continuous \mathcal{D} -valued functions on X . In particular, we have weak equivalences $KU^{\mathbb{C}} \simeq KU^{\mathcal{Z}} \simeq KU^{\mathcal{O}_\infty}$ induced by the unit maps $\mathbb{C} \rightarrow \mathcal{Z}$ and $\mathbb{C} \rightarrow \mathcal{O}_\infty$, and these three spectra represent complex topological K -theory $X \mapsto K^*(X)$ by the Serre-Swan theorem.

Let \mathbb{K} denote the C^* -algebra of compact operators on a separable infinite-dimensional Hilbert space. Dadarlat and Pennig constructed a map $\text{Aut}(\mathcal{D} \otimes \mathbb{K}) \rightarrow \Omega KU_1^{\mathcal{D}}$ in [6], which factors through

$$\text{Aut}(\mathcal{D} \otimes \mathbb{K}) \rightarrow GL_1(KU^{\mathcal{D}}) \tag{1.1}$$

and extends to a map of infinite loop spaces. The abelian group $K_0(\mathcal{D})$ has

a natural order structure and the spectrum $gl_1(KU^{\mathcal{D}})$ restricts to the spectrum of positive units $gl_1(KU^{\mathcal{D}})_+$ by pullback via the group homomorphism $GL_1(K_0(\mathcal{D}))_+ \rightarrow GL_1(K_0(\mathcal{D}))$. The map (1.1) gives a weak equivalence of infinite loop spaces $\text{Aut}(\mathcal{D} \otimes \mathbb{K}) \simeq \Omega^\infty gl_1(KU^{\mathcal{D}})_+$. In case \mathcal{D} is purely infinite the order structure on $K_0(\mathcal{D})$ is trivial (i.e. $K_0(\mathcal{D})_+ = K_0(\mathcal{D})$) and we have $\text{Aut}(\mathcal{D} \otimes \mathbb{K}) \simeq \Omega^\infty gl_1(KU^{\mathcal{D}})$ [6, Thm. 1.1].

Important consequences of this result arise for twisted K -theory, which was initially developed in [9], see [23] for a survey. The infinite loop map from the last paragraph gives rise to an equivalence

$$B\text{Aut}(\mathcal{D} \otimes \mathbb{K}) \simeq BGL_1(KU^{\mathcal{D}})_+.$$

Since the right hand side is the classifying space of (positive) twists of $KU^{\mathcal{D}}$, we obtain a description of twisted $KU^{\mathcal{D}}$ -theory in terms of bundles of strongly self-absorbing C^* -algebras. It includes the geometric twists classified by $K(\mathbb{Z}, 3) \simeq B\text{Aut}(\mathbb{K})$, i.e. the group $[X, K(\mathbb{Z}, 3)] \cong H^3(X, \mathbb{Z})$, considered in [2, 42] as a special case.

The quest for an equivariant theory. Equivariant twisted K -theory (with an equivariant geometric twist) has found applications in mathematical physics through a deep theorem by Freed, Hopkins and Teleman [16]: for a compact, simple and simply-connected Lie group G and $k \in \mathbb{Z}$ the isomorphism classes of positive energy representations of the loop group LG generate the Verlinde ring $\text{Ver}_k(G)$, which features in conformal and topological field theories. By [16, Thm. 1] there is a ring isomorphism

$$\text{Ver}_k(G) \cong K_G^{\dim(G), \tau(k)}(G),$$

with the fusion product on the Verlinde ring and the Pontrjagin product on twisted K -theory. The equivariant twist $\tau(k) \in H_G^3(G, \mathbb{Z}) \cong \mathbb{Z}$ corresponds to $k + h^\vee$, where h^\vee denotes the dual Coxeter number of G .

While it is not too difficult to see that the equivariant geometric twists correspond to cocycles representing classes in $H_G^3(G, \mathbb{Z})$, a classification of the most general

equivariant twists requires the construction of a G -space $BGL_1(KU)$ and has not been established. At the point of writing the only results in the literature covering units of equivariant ring spectra are to the best of our knowledge [45] and [49, Ex. 5.1.17]. Nevertheless, candidates for equivariant higher twists over $SU(n)$ in terms of C^* -algebra bundles have been constructed in [15].

Apart from the applications in mathematical physics highlighted above, our operator-algebraic model for the units of equivariant K -theory also provides an equivariant refinement of the higher Dixmier-Douady theory in [6] with applications in the classification of group actions on C^* -algebras. To see why, let H be a countable discrete group and suppose that it acts on $\mathcal{D} \otimes \mathbb{K}$ via $\alpha: H \rightarrow \text{Aut}(\mathcal{D} \otimes \mathbb{K})$. Now assume that we also have a finite group G that acts through automorphisms on H via $\hat{\gamma}$ and in addition on $\mathcal{D} \otimes \mathbb{K}$ in such a way that $\alpha_{\hat{\gamma}_g(h)} = \gamma_g \alpha_h \gamma_g^{-1}$. If we have chosen functorial models for EH and BH , then the associated bundle

$$EH \times_{\alpha} \text{Aut}(\mathcal{D} \otimes \mathbb{K}) \rightarrow BH$$

comes equipped with a natural G -action. Denoting the set of all these H -actions with compatible G -actions by $\text{Act}_G(H, \mathcal{D} \otimes \mathbb{K})$ we obtain a map

$$\text{Act}_G(H, \mathcal{D} \otimes \mathbb{K}) \rightarrow [BH, B\text{Aut}(\mathcal{D} \otimes \mathbb{K})]^G. \tag{1.2}$$

If \mathcal{D} is a Kirchberg algebra in the UCT-class, $G = \{e\}$ and H is amenable and torsion-free, then the above map induces a bijection between cocycle-conjugacy classes of outer actions and the homotopy set on the right hand side by deep results of Meyer [33], Gabe and Szabó [17]. If BG is in addition a finite CW-complex, then this homotopy set is the first group of a cohomology theory making it accessible to computations via algebraic topology. In the equivariant case the right hand side should evaluate to the equivariant cohomology theory associated to the positive unit spectrum of equivariant $KU^{\mathcal{D}}$ -theory. In fact, (1.2) makes sense with much weaker assumptions on G . However, equivariant stable homotopy theory becomes more intricate for more general groups.

Main results. Developing these ideas requires combining methods from equivariant stable homotopy theory with topological considerations about automorphism groups of C^* -algebras. In this thesis we present an important first advance in this direction. As a starting point we picked a setting that is very explicit: we fix the group G to be $\mathbb{Z}/p\mathbb{Z}$ for a prime p , and we consider a UHF-algebra D given by the infinite tensor product of a unitary finite-dimensional G -representation V , i.e.

$$D = \text{End}(V)^{\otimes \infty}.$$

This is a G - C^* -algebra in a natural way when equipped with the infinite tensor product of the action by conjugation. The restriction to prime order cyclic groups has two benefits: there are only two subgroups to consider, and some of the results obtained in [14] for circle actions on UHF-algebras carry over immediately to the case of $\mathbb{Z}/p\mathbb{Z}$ -actions. We then determine the G -equivariant homotopy type of $\text{Aut}(D \otimes \mathbb{K})$. In particular, we show that the automorphism group is an equivariant infinite loop space associated to a naive G -spectrum, which we identify in our main result.

Theorem (Theorem 3.4.7). *Let $G = \mathbb{Z}/p\mathbb{Z}$ for a prime $p \in \mathbb{N}$. Let V be a finite-dimensional unitary G -representation and let*

$$D = \text{End}(V)^{\otimes \infty}$$

be the associated G - C^ -algebra. The group $\text{Aut}(D \otimes \mathbb{K})$ is a G -equivariant infinite loop space underlying a naive G -spectrum $EH_{\mathcal{I}}\text{Aut}(D \otimes \mathbb{K})$ and we have an equivalence of naive G -spectra*

$$EH_{\mathcal{I}}\text{Aut}(D \otimes \mathbb{K}) \simeq gl_1(KU^D)_+.$$

The group $\text{Aut}(D \otimes \mathbb{K})$ has two equivariant deloopings: one arising from the group structure, the other one from the tensor product. Just as in the non-equivariant setting these two agree. As a result the first group of the cohomology theory associated to $gl_1(KU^D)_+$ classifies equivariant C^* -algebra bundles with fibre $D \otimes \mathbb{K}$ in the following sense:

Corollary (Corollary 3.4.8). *Let $E_D^*(X) = gl_1(KU^D)_+^*(X)$ be the $\mathbb{Z}/p\mathbb{Z}$ -equivariant cohomology theory associated to the spectrum $gl_1(KU^D)_+$. Let X be a finite CW-complex. Then*

$$E_D^0(X) \cong [X, \text{Aut}(D \otimes \mathbb{K})]^{\mathbb{Z}/p\mathbb{Z}} \quad \text{and} \quad E_D^1(X) \cong [X, \text{BAut}(D \otimes \mathbb{K})]^{\mathbb{Z}/p\mathbb{Z}}.$$

In particular, isomorphism classes of $\mathbb{Z}/p\mathbb{Z}$ -equivariant C^ -algebra bundles with fibres isomorphic to the $\mathbb{Z}/p\mathbb{Z}$ -algebra $D \otimes \mathbb{K}$ over X form a group with respect to the fibrewise tensor product, which is isomorphic to $E_D^1(X)$.*

Looking ahead. While this thesis focuses primarily on the specific example described above, the techniques we employ are designed to extend to a much wider class of G - C^* -algebras. Moreover, one can expect to obtain richer structure on the equivariant C^* -algebra bundles by constructing genuine G -spectra rather than naive ones. For instance, the equivariant cohomology theories associated to genuine G -spectra are $RO(G)$ -graded [26] (as opposed to \mathbb{Z} -graded) and come equipped with transfer maps, endowing them with the structure of a Mackey functor. The final chapter presents preliminary work towards both these directions.

Structure. In Chapter 2 we recall some preliminaries from operator algebras, K -theory and (equivariant) stable homotopy theory.

The core of this thesis is contained in Chapter 3, which is based on joint work with Ulrich Pennig available on the preprint server arXiv as arXiv:2410.06947. In Section 3.1 we describe an equivariant infinite loop space machine for finite groups based on a category of diagram spaces called commutative \mathcal{I} - G -monoids, whose output is a naive positive Ω - G -spectrum. We use this machine to construct two such spectra. In Section 3.2 we construct a G -spectrum of units for a localisation of equivariant K -theory. This is done in two steps: in Section 3.2.1 we refine the construction of a commutative symmetric ring spectrum KU^D associated to a strongly self-absorbing C^* -algebra D to the equivariant case (Definition 3.2.3). Then we construct its equivariant units in Section 3.2.2 at the level of diagram spaces, and finally lift them to a naive G -spectrum of units (Definition 3.2.8).

Our goal is to compare this spectrum to the one associated to the automorphism group of the stabilisation of $D = \text{End}(V)^{\otimes \infty}$ for an action of $G = \mathbb{Z}/p\mathbb{Z}$. To do that, in Section 3.3.1 we interpret $\text{Aut}(D \otimes \mathbb{K})$ as the first space $\mathcal{G}_D(\mathbf{1})$ in a commutative \mathcal{I} - G -monoid \mathcal{G}_D . In Section 3.3.2 we then collect a series of results which provide the equivariant homotopy type of this group, mainly using techniques from [14]. We also compute the homotopy groups of the equivariant automorphisms in Corollary 3.3.15 and show that $\text{Aut}(D \otimes \mathbb{K})$ has the G -homotopy type of a CW-complex (Lemma 3.3.16). The main result is contained in Section 3.4, where we explicitly describe a map between the $\mathbb{Z}/p\mathbb{Z}$ -spectrum coming from the equivariant infinite loop space structure of $\text{Aut}(D \otimes \mathbb{K})$ and the $\mathbb{Z}/p\mathbb{Z}$ -spectrum of equivariant units of KU^D , and prove that it is an equivalence onto the positive units (Theorem 3.4.7). In particular, this provides a classification of $\mathbb{Z}/p\mathbb{Z}$ -equivariant $\text{Aut}(D \otimes \mathbb{K})$ -bundles in terms of an equivariant cohomology theory (Corollary 3.4.8). We discuss a model for the classifying space of such equivariantly locally trivial bundles in Section 3.4.1, and we state and prove the main theorems in Section 3.4.2.

Chapter 4 consists of two independent parts, which lay the groundwork for extending the results of Chapter 3 to a far broader setting. In Section 4.1, we provide a partial generalisation of the tools needed to determine the equivariant homotopy type of $\text{Aut}(D \otimes \mathbb{K})$ which applies to strongly self-absorbing C^* -dynamical systems for compact groups actions. In Section 4.2, we try to address a fundamental limitation of the equivariant infinite loop space machine described in Section 3.1, whose output only admits deloopings with respect to trivial G -spheres (as opposed to all representation spheres). Our proposal will appear in an article in preparation.

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Preliminaries

Given the interdisciplinary nature of this thesis, this chapter aims to make it accessible to readers with little or no background in either operator algebras or (equivariant) stable homotopy theory. We also recall the basics of K -theory in both its operator and topological flavours.

2.1 Operator algebras

The theory of operator algebras originated in the 1930s and the 1940s, when von Neumann and Gelfand initiated respectively the theories of von Neumann algebras and C^* -algebras. Good analogies to have in mind are that von Neumann algebras provide a noncommutative generalisation of measure theory, whereas the theory of C^* -algebras can be interpreted as noncommutative topology, as we will see. In this section we go over the basics of C^* -algebras and give hints about their classification programme, which was started by work of Elliott [11] and is currently a very active area of research. In particular we describe the class of strongly self-absorbing C^* -algebras, which play a cornerstone role in this programme. A standard reference for most material in this section is [35].

2.1.1 C^* -algebras

Definition 2.1.1. A C^* -algebra A is an algebra over \mathbb{C} with a norm $a \mapsto \|a\|$ and an antilinear involution $a \mapsto a^*$, $a \in A$, such that A is complete with respect

to the norm, and

$$\|ab\| \leq \|a\|\|b\| \text{ and } \|a^*a\| = \|a\|^2$$

for every $a, b \in A$.

In other words, a C^* -algebra is a Banach $*$ -algebra such that $\|a^*a\| = \|a\|^2$ for all $a \in A$. A $*$ -homomorphism $\varphi : A \rightarrow B$ between C^* -algebras is an algebra homomorphism which satisfies $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. It is easy to check that a $*$ -homomorphism is necessarily norm-decreasing, i.e., $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$.

We say that a C^* -algebra A is unital if it has a multiplicative unit, and that A is simple if its only two-sided closed ideals are the trivial ideals 0 and A (note that every ideal in a C^* -algebra is automatically a C^* -algebra itself).

Example 2.1.2. $A = \mathbb{C}$ is a unital C^* -algebra with involution given by complex conjugation.

Example 2.1.3. If H is an Hilbert space, then its bounded operators $\mathcal{B}(H)$ form a C^* -algebra.

Example 2.1.4. Let X be a locally compact Hausdorff space and consider the algebra $C_0(X)$ of all continuous functions $f : X \rightarrow \mathbb{C}$ such that for each $\varepsilon > 0$ there is a compact subset $K \subseteq X$ with $|f(x)| \leq \varepsilon$ for all $x \in X - K$, equipped with the supremum norm. Then $C_0(X)$ is a C^* -algebra with involution $f \mapsto \bar{f}$.

At closer inspection, the last two examples turn out to be far more general: every C^* -algebra can be thought of as a subalgebra of some $\mathcal{B}(H)$, and all commutative ones are isomorphic to some $C_0(X)$.

Theorem 2.1.5 (Gelfand-Naimark). *For each C^* -algebra A there exists a Hilbert space H and an isometric $*$ -homomorphism $A \rightarrow \mathcal{B}(H)$.*

The Hilbert space H is obtained by regarding A as a vector space, equipping it with an inner product using a state, and forming its completion.

Theorem 2.1.6 (Gelfand). *Every abelian C^* -algebra is isometrically $*$ -isomorphic to the C^* -algebra $C_0(X)$ for some locally compact Hausdorff space X .*

Here given a C^* -algebra A , one defines $X = \text{hom}(A, \mathbb{C})$ and gives it the weakest topology such that $\phi \mapsto \phi(a)$ is continuous. The evaluation map

$$A \rightarrow C_0(X), \quad a \mapsto (\phi \mapsto \phi(a))$$

is the required $*$ -isomorphism.

This result motivates the slogan that the study of (non-commutative) C^* -algebras can be interpreted as non-commutative topology. This point of view has proven fruitful, by bringing ideas and techniques from topology into operator algebras and vice versa. We will discuss a successful example of this process in Section 2.2.

Let A be a unital C^* -algebra. An element $a \in A$ is called positive if $aa^* = a^*a$ and $\text{sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda \cdot 1 \text{ is not invertible}\} \subseteq \mathbb{R}^+$. A is said to be purely infinite if for every non-zero positive element $a \in A$ there is an element $x \in A$ such that $x^*ax = 1$.

There is a number of techniques for constructing new C^* -algebras from old ones. Here we discuss two, namely direct limits and tensor products. Let $\{A_n\}_{n=1}^\infty$ be a sequence of C^* -algebras and suppose that for each n we are given a $*$ -homomorphism $\varphi_n : A_n \rightarrow A_{n+1}$. The product $\prod_{k=1}^\infty A_k$ is a $*$ -algebra with operations defined pointwise, and we can consider its $*$ -subalgebra

$$A' := \{a = (a_k)_k \in \prod_{k=1}^\infty A_k : \exists N \in \mathbb{N} \text{ for which } a_{k+1} = \varphi_k(a_k) \text{ for all } k \geq N\}.$$

Then $p(a) = \lim_{k \rightarrow \infty} \|a_k\|$ is a C^* -seminorm on A' , i.e., a seminorm on A' satisfying

$$p(ab) \leq p(a)p(b), \quad p(a^*) = p(a), \quad p(a^*a) = p(a)^2$$

for all $a, b \in A$. For any such a $*$ -seminorm q , the set $N = q^{-1}(0)$ is a self-adjoint ideal of A' and we get a C^* -norm (a C^* -seminorm which is also a norm) on the quotient algebra A'/N by $q(a) = \|a + N\|$. One can check that the Banach space completion of A'/N with this norm is a C^* -algebra, which we call the enveloping C^* -algebra of (A', q) .

Definition 2.1.7. The direct limit of the sequence $\{(A_n, \varphi_n)\}_{n=1}^\infty$, denoted by $\lim A_n$, is the enveloping C^* -algebra of (A', p) .

If the A_n are finite-dimensional C^* -algebras, then their direct limit is called an AF-algebra. Such algebras form an interesting class to analyse, since they can be highly non-trivial while still being closely related to finite-dimensional C^* -algebras. A notable example is given by UHF-algebras, which we will introduce in Section 2.1.2.

We now discuss tensor products of C^* -algebras. Note that if A and B are C^* -algebras there are in general multiple C^* -norms on their algebraic tensor product $A \otimes B$. The completion with respect to any of such norms is a C^* -algebra which contains $A \otimes B$ as a dense subalgebra.

Definition 2.1.8. The minimal (also called spatial) norm $\|\cdot\|_{min}$ is defined by the inclusion

$$A \otimes B \subseteq \mathcal{B}(H) \otimes \mathcal{B}(K) \subseteq \mathcal{B}(H \otimes_{std} K),$$

where \otimes_{std} is the standard tensor product of Hilbert spaces. We denote by $A \otimes_{min} B$ the C^* -completion of $A \otimes B$ under this norm, and call it the minimal tensor product of A and B .

Definition 2.1.9. The maximal norm $\|\cdot\|_{max}$ is defined by

$$\|c\|_{max} = \sup\{\|c\|_\gamma : \|\cdot\|_\gamma \text{ is a } C^*\text{-norm on } A \otimes B\}.$$

We denote by $A \otimes_{max} B$ the C^* -completion of $A \otimes B$ under this norm, and call it the maximal tensor product of A and B .

One can show that for every other C^* -norm $\|\cdot\|$, for all C^* -algebras A and B and all $c \in A \otimes B$ one has

$$\|c\|_{min} \leq \|c\| \leq \|c\|_{max}.$$

There are C^* -algebras A for which these two norms coincide for every C^* -algebra B ; this implies in particular that they define a unique norm on $A \otimes B$.

Definition 2.1.10. A C^* -algebra A is nuclear if, for each C^* -algebra B , there is only one C^* -norm on $A \otimes B$.

Example 2.1.11. For each $n \geq 1$, the C^* -algebra $M_n(\mathbb{C})$ is nuclear. One can show that every finite-dimensional C^* -algebra is $*$ -isomorphic to a direct sum of matrix algebras. As a consequence, all finite dimensional C^* -algebras are nuclear.

Other notable examples are given by the C^* -algebra of compact operators on a separable Hilbert space, all commutative C^* -algebras, and inductive limits of nuclear C^* -algebras.

In recent years experts have been able to classify a large class of C^* -algebras satisfying the Universal Coefficient Theorem (UCT). This is a condition of topological flavour, which relates Kasparov's bivariant KK -theory to homomorphisms between K -groups and goes back to Rosenberg and Schochet [43]. We will say more about this in Section 2.2.1.

2.1.2 Strongly self-absorbing C^* -algebras

The notion of strongly self-absorbing C^* -algebra was introduced by Toms and Winter in [56]. Here we give an equivalent definition with a more topological flavour to it.

Definition 2.1.12. A unital C^* -algebra D is called strongly self-absorbing if it is separable and there exists a $*$ -isomorphism $\psi : D \rightarrow D \otimes D$ and a path of

unitaries $u : [0, 1] \rightarrow U(D \otimes D)$ such that for all $d \in D$

$$\lim_{t \rightarrow 1} \|\psi(d) - u_t \cdot (d \otimes 1) \cdot u_t^*\| = 0.$$

To put this class into context note that some properties that are essential in the classification programme of separable, simple, nuclear C^* -algebras are related to tensorial absorption of strongly self-absorbing ones:

- (i) If A is separable, simple and nuclear, then A is purely infinite if and only if $A \otimes \mathcal{O}_\infty \cong A$ [25, Thm. 3.15], [40, Thm. 7.2.6 (ii)]. Here, \mathcal{O}_∞ denotes the infinite Cuntz algebra, which is strongly self-absorbing (and therefore purely infinite).
- (ii) Separable, simple, unital, nuclear C^* -algebras in the UCT class with at most one trace are classified up to \mathcal{Z} -stability by their ordered K -theory [54, Cor. E]. Here, \mathcal{Z} denotes the Jiang-Su algebra, which is strongly self-absorbing and “up to \mathcal{Z} -stability” means after taking a tensor product with \mathcal{Z} .
- (iii) Let A be a separable, simple, unital and nuclear C^* -algebra. Then $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ [25, Cor. 3.8]. Here, \mathcal{O}_2 is the Cuntz algebra on two generators, which is also strongly self-absorbing.

Apart from \mathbb{C} , \mathcal{Z} , \mathcal{O}_∞ and \mathcal{O}_2 other examples of strongly self-absorbing C^* -algebras arise from UHF-algebras, which are infinite tensor products of matrix algebras: let $n \in \mathbb{N}$ and consider the sequence

$$\mathbb{C} \rightarrow \dots \rightarrow M_n(\mathbb{C})^{\otimes k} \rightarrow M_n(\mathbb{C})^{\otimes(k+1)} \rightarrow M_n(\mathbb{C})^{\otimes(k+2)} \rightarrow \dots,$$

where the connecting homomorphisms are given by $T \mapsto T \otimes 1_n$. Denote the colimit of this sequence in the category of unital C^* -algebras by $M_n^{\otimes \infty}$. If p_1, \dots, p_r are the prime factors of n , then we have

$$M_n^{\otimes \infty} \cong M_{p_1}^{\otimes \infty} \otimes \dots \otimes M_{p_r}^{\otimes \infty}.$$

For an arbitrary subset P of the prime numbers we therefore define (by slight abuse of notation)

$$M_P^{\otimes\infty} = \bigotimes_{p \in P} M_p^{\otimes\infty}.$$

If P is the set of all prime numbers, the resulting algebra is called the universal UHF-algebra, usually denoted by \mathcal{Q} , because $K_0(\mathcal{Q}) \cong \mathbb{Q}$. A list of all strongly self-absorbing C^* -algebras in the UCT class can be found in Figure 2.1.

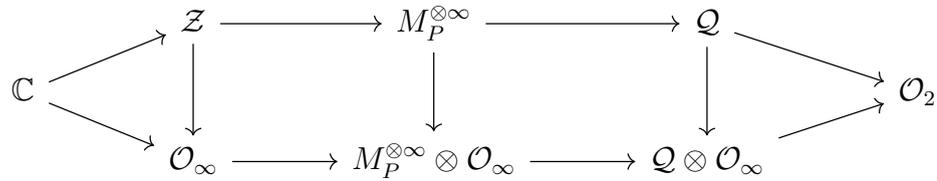


Figure 2.1: The list of strongly self-absorbing C^* -algebras in the UCT class. An arrow indicates tensorial absorption (e.g. $\mathcal{O}_\infty \otimes \mathcal{Z} \cong \mathcal{O}_\infty$).

Because of its infinite tensor product structure it is easy to construct interesting group actions on UHF-algebras. Let G be a discrete group and let $\rho: G \rightarrow U(V)$ be a finite-dimensional unitary G -representation, then G acts by conjugation on $\text{End}(V)$. This G -action extends to the UHF-algebra

$$D = \text{End}(V)^{\otimes\infty}.$$

Note that while the action of G on $\text{End}(V)$ is inner, this is no longer the case for the induced action on D , where the action is in general only approximately inner.

2.2 *K*-theory

Topological K -theory was developed by Atiyah and Hirzebruch in 1961 based on Grothendieck's work on algebraic varieties. The idea is to capture the topology of a space X by studying vector bundles over it. It is possible to define K -theory directly from X , or in terms of matrices of continuous complex-valued functions on X . This provides a more algebraic approach which extends to matrices over

general Banach algebras. In fact, in the early 1970s *K*-theory was introduced as a tool in C^* -algebra theory, where it has been shown to be a powerful invariant (in some cases, a complete one). The standard references for this section are [41] for operator *K*-theory, [37] for topological *K*-theory, and [38] for equivariant *K*-theory.

2.2.1 Operator *K*-theory

Let A be a unital C^* -algebra.

Definition 2.2.1. An element $p \in A$ is called a projection if $p = p^2 = p^*$. The set of all projections in A is denoted by $\mathcal{P}(A)$.

Let

$$\mathcal{P}_n(A) = \mathcal{P}(M_n(A)), \quad \mathcal{P}_\infty(A) = \prod_{n=1}^{\infty} \mathcal{P}_n(A)$$

for n a positive integer. Define an equivalence relation \sim on $\mathcal{P}_\infty(A)$ as follows: for p a projection in $\mathcal{P}_n(A)$ and q a projection in $\mathcal{P}_m(A)$, we say $p \sim q$ if there is a matrix $v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$. This allows us to consider the quotient

$$\mathcal{D}(A) = \mathcal{P}_\infty(A) / \sim,$$

which is an abelian monoid with addition

$$[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right].$$

To any such an algebraic object one can always associate an abelian group by formally introducing inverses. If M is an abelian monoid, define its Grothendieck group $\text{Gr}(M) := M \times M / \sim$, where $(m_1, m_2) \sim (n_1, n_2)$ if there exists $k \in M$ such that $m_1 + n_2 + k = m_2 + n_1 + k$.

Definition 2.2.2. The K_0 -group of A is the Grothendieck group of $\mathcal{D}(A)$:

$$K_0(A) = \text{Gr}(\mathcal{D}(A)).$$

Example 2.2.3. Let $A = \mathbb{C}$. Then $\mathcal{P}_n(\mathbb{C}) \xleftrightarrow{1:1} \{\text{subspaces of } \mathbb{C}^n\}$ and $[p] = [q]$ in $\mathcal{D}(\mathbb{C})$ if and only if $\text{tr}(p) = \text{tr}(q)$. Hence $\mathcal{D}(\mathbb{C}) \cong \mathbb{N}$ and

$$K_0(\mathbb{C}) = \text{Gr}(\mathbb{N}) \cong \mathbb{Z}.$$

If g is an element in $K_0(A)$, then $g = [p] - [q]$ for some $p \in \mathcal{P}_k(A)$ and $q \in \mathcal{P}_l(A)$. Choose n greater than k, l and set $p' = p \oplus 0_{n-k}$ and $q' = q \oplus 0_{n-l}$. Then $p \sim p'$ and $q \sim q'$, hence $g = [p'] - [q']$. Therefore we get the following description, known as the standard picture of K_0 :

$$K_0(A) = \{[p] - [q] : p, q \in \mathcal{P}_n(A), n \in \mathbb{N}\}.$$

Note that we can add extra structure to $K_0(A)$ by specifying its positive cone

$$K_0(A)^+ = \{[p] : p \in \mathcal{P}_\infty(A)\} \subseteq K_0(A).$$

The pair $(K_0(A), K_0(A)^+)$ is called the ordered K -theory of A .

Definition 2.2.4. An element $u \in A$ is a unitary if $uu^* = u^*u = 1$. The set of all unitaries in A is denoted $\mathcal{U}(A)$.

Let

$$\mathcal{U}_n(A) = \mathcal{U}(M_n(A)), \quad \mathcal{U}_\infty(A) = \prod_{n=1}^{\infty} \mathcal{U}_n(A)$$

for n a positive integer. Define an equivalence relation \sim on $\mathcal{U}_\infty(A)$ as follows: for $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, we say $u \sim v$ if there is a natural number $k \geq n, m$ such that

$$\begin{pmatrix} u & 0 \\ 0 & 1_{k-n} \end{pmatrix} \text{ is homotopic to } \begin{pmatrix} v & 0 \\ 0 & 1_{k-m} \end{pmatrix}.$$

Again, the quotient $\mathcal{U}_\infty(A)/\sim$ is an abelian semigroup with addition

$$[u] + [v] = \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right].$$

In fact, it turns out to be a group, with inverse of $[u]$ given by $[u^*]$.

Definition 2.2.5. $K_1(A) = \mathcal{U}_\infty(A) / \sim$.

Example 2.2.6. $K_1(\mathbb{C}) = 0$ since $\mathcal{U}_n(\mathbb{C})$ is path connected for any n .

Let $\varphi : A \rightarrow B$ be a $*$ -homomorphism. Since φ maps projections to projections, we can define a group homomorphism $K_0(\varphi) : K_0(A) \rightarrow K_0(B)$ given by

$$K_0(\varphi)([p]) = [\varphi(p)],$$

and another group homomorphism $K_1(\varphi) : K_1(A) \rightarrow K_1(B)$ given by

$$K_1(\varphi)([u]) = [\varphi(u)].$$

Proposition 2.2.7. K_0 and K_1 are covariant functors from the category of unital C^* -algebras to the category of abelian groups.

If A is a non-unital C^* -algebra, consider its unitization $A^+ := A \oplus \mathbb{C}$ (with multiplication $(a, z_1)(b, z_2) = (ab + z_1b + z_2a, z_1z_2)$ and unit $(0, 1)$) and define

$$K_0(A) = \text{Ker}(K_0(A^+) \rightarrow K_0(\mathbb{C})) \text{ and } K_1(A) = K_1(A^+).$$

The K_0 and K_1 -groups of a C^* -algebra A are related as follows. Define the suspension of A by

$$SA := \{f \in C(\mathbb{T}, A) : f(1) = 0\} \cong C_0(\mathbb{R}) \otimes A$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Proposition 2.2.8. $K_1(A) \cong K_0(SA)$.

This allows to inductively define higher K -groups by

$$K_n(A) = K_{n-1}(SA).$$

What happens is that in fact, $K_{n+2}(A)$ is naturally isomorphic to $K_n(A)$ for every $n \in \mathbb{N}$. This result goes by the name of Bott periodicity [41, Thm. 11.1.2].

Theorem 2.2.9. *There is a natural isomorphism*

$$\beta_A : K_0(A) \rightarrow K_1(SA),$$

called the Bott map.

The Bott map is constructed as follows. For every $n \in \mathbb{N}$ and every projection $p \in \mathcal{P}_n(A)$, define the projection loop

$$f_p(z) = zp + (1 - p), \quad z \in \mathbb{T}.$$

Note that f_p is a unitary in $C(\mathbb{T}, A)$ with $f_p(1) = 0$, hence it gives rise to an element in $K_1(SA)$. The Bott map is defined as the induced map

$$\beta_A : K_0(A) \rightarrow K_1(SA), \quad [p] \mapsto [f_p].$$

The K -groups enjoy some important properties, such as homotopy invariance, stability and continuity. Two $*$ -homomorphisms are said to be homotopic if there is an homotopy made up of $*$ -homomorphisms between them. Let $i \in \{0, 1\}$.

Proposition 2.2.10 (Homotopy invariance). *If $\varphi, \psi : A \rightarrow B$ are homotopic $*$ -homomorphisms, then $K_i(\varphi) = K_i(\psi)$.*

Proposition 2.2.11 (Stability). *Let \mathbb{K} be the C^* -algebra of all compact operators on a separable Hilbert space. For any C^* -algebra A , the canonical $*$ -homomorphism $A \rightarrow A \otimes \mathbb{K}$, $a \mapsto a \otimes p$ induces isomorphisms $K_i(A) \cong K_i(A \otimes \mathbb{K})$ for any choice of a rank one projection $p \in A$.*

Proposition 2.2.12 (Continuity). *For each sequence of C^* -algebras $\{(A_n, \varphi_n)\}_{n=1}^{\infty}$, we have $K_i(\lim A_n) \cong \lim K_i(A_n)$.*

In [24], Kasparov introduced a bivariant KK -theory which originates from the observation that families of elliptic operators give rise to maps between K -groups.

This consists of groups $KK(A, B)$ for two C^* -algebras A, B , which generalise K -theory (in the sense that $KK(\mathbb{C}, A) \cong K(A)$) and whose elements are generalised homomorphisms from A to B . These groups form the morphisms in a category KK whose objects are separable C^* -algebras. One can show that the category KK is completely determined by three universal properties, namely homotopy invariance, stability and split exactness [20]. In fact, it is possible to construct a $(\infty, 1)$ -category representing KK -theory in a purely algebraic way by forcing such properties [5].

K -theory and KK -theory play a crucial role in the classification programme for simple, separable, nuclear C^* -algebras. It is often assumed that these algebras satisfy the UCT, which is equivalent to say that they are KK -equivalent to a commutative C^* -algebra.

Theorem 2.2.13 (Kirchberg-Phillips [39]). *Simple, separable, nuclear, purely infinite C^* -algebras are classified up to stable equivalence by their KK -theory. If they are also unital and in the UCT class, then they are classified by their K -theory and the class of the unit in K_0 .*

The following classification result is the final step in a long-standing program. Elliott, Gong, Lin, and Niu [12], building on work by many hands, had previously established this theorem under the additional assumption that all traces are quasidiagonal; [54, Thm. A] removed this assumption and completed the classification.

Theorem 2.2.14 ([54, Cor. D]). *Simple, separable, nuclear, infinite dimensional C^* -algebras A in the UCT class with finite nuclear dimension (a generalisation of topological covering dimension) are classified by their ordered K -theory, the class of unit in K_0 and trace data.*

2.2.2 Topological K -theory

Let X be a compact Hausdorff space. Denote by $\text{Vect}(X)$ the collection of isomorphism classes $[V]$ of finite dimensional complex vector bundles over X . This is an abelian monoid under the internal Whitney sum, defined as

$$(V_1, \pi_1) + (V_2, \pi_2) = (V_1 \oplus V_2, \pi_{\oplus})$$

where $V_1 \oplus V_2 = \{(v_1, v_2) : \pi_1(v_1) = \pi_2(v_2)\}$ and $\pi_{\oplus}(v_1, v_2) = \pi_1(v_1)$.

Definition 2.2.15. The Grothendieck completion of $\text{Vect}(X)$ is denoted $K^0(X)$.

Example 2.2.16. Let $X = \{*\}$. A vector bundle over a point is just a vector space. Since these are classified by rank, we get $K^0(*) \cong \mathbb{Z}$.

Consider

$$\widehat{\text{GL}}(C(X)) = \bigcup_{n \in \mathbb{N}} \text{GL}_n(C(X))$$

and define on it an equivalence relation by

$$S \sim \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}$$

for all $n \in \mathbb{N}$ and matrices $S \in \text{GL}_n(C(X))$. Denote the set of equivalence classes by $\text{GL}(C(X)) = \widehat{\text{GL}}(C(X))/\sim$. Similarly, denote by $\text{GL}_n(C(X))_0$ the connected component of the identity matrix in $\text{GL}_n(C(X))$ and define

$$\widehat{\text{GL}}(C(X))_0 = \bigcup_{n \in \mathbb{N}} \text{GL}_n(C(X))_0.$$

The equivalence relation \sim restricts to $\widehat{\text{GL}}(C(X))_0$ and we write $\text{GL}(C(X))_0 = \widehat{\text{GL}}(C(X))_0/\sim$.

Proposition 2.2.17. $\text{GL}(C(X))$ is a group under matrix multiplication, and $\text{GL}(C(X))_0$ is a normal subgroup.

Definition 2.2.18. $K^1(X)$ is the quotient group $\mathrm{GL}(C(X))/\mathrm{GL}(C(X))_0$.

Example 2.2.19. Let $X = \{*\}$. We have $\mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})_0$ for every $n \in \mathbb{N}$, hence $K^1(*) = 0$.

Let $\varphi : X \rightarrow Y$ be a continuous map between compact Hausdorff spaces. We get a map $\mathrm{Vect}(Y) \rightarrow \mathrm{Vect}(X)$ by pullback of vector bundles, and a group homomorphism $\mathrm{GL}(C(Y)) \rightarrow \mathrm{GL}(C(X))$ by applying the induced morphism $\varphi^* : C(Y) \rightarrow C(X)$ entrywise.

Proposition 2.2.20. K^0 and K^1 are contravariant functors from the category of compact Hausdorff spaces to the category of abelian groups.

An important property of topological *K*-theory is homotopy invariance.

Proposition 2.2.21. Suppose φ and ψ are homotopic maps from X to Y . Then $K^i(\varphi) = K^i(\psi)$ for $i = 0, 1$.

We can extend *K*-theory to locally compact Hausdorff spaces by declaring

$$K^i(X) = \mathrm{Ker}(K^i(X^+) \rightarrow K^i(*)).$$

Then in analogy to Proposition 2.2.8 we have the following relation for a locally compact Hausdorff space X .

Proposition 2.2.22. $K^1(X) \cong K^0(X \times \mathbb{R})$.

Again, we can use it to define higher topological *K*-groups of locally compact Hausdorff spaces by

$$K^n(X) = K^{n-1}(X \times \mathbb{R}^n).$$

Theorem 2.2.23 (Bott periodicity, [37, Thm. 2.7]). *There is a natural isomorphism*

$$K^0(X) \xrightarrow{\cong} K^2(X).$$

The following result provides a precise correspondence between operator and topological K -theory.

Theorem 2.2.24 (Serre-Swan). *If $V \rightarrow X$ is a complex vector bundle over a compact Hausdorff space X , then there exists a vector bundle $V' \rightarrow X$ such that*

$$V \oplus V' \cong X \times \mathbb{C}^n$$

for some $n \in \mathbb{N}$.

This theorem establishes a bijection between vector bundles over X and finitely generated projective $C(X)$ -modules. These are in turn determined by projections $p : X \rightarrow \text{End}(X \otimes \mathbb{C}^n) \cong M_n(C(X))$. Therefore we get

$$K^0(X) \cong K_0(C(X)),$$

hence, since $S(C(X)) \cong C(X \times \mathbb{R})$, by Proposition 2.2.22 and Theorem 2.2.23

$$K^n(X) \cong K_n(C(X))$$

for all $n \in \mathbb{Z}$ and for any compact Hausdorff space X .

There is also a version of the Serre-Swan theorem in K -theory with coefficients in a C^* -algebra A , which we will use later. Recall that to any projection valued function in $C(X, M_n(A))$ one can associate a finitely generated projective Hilbert A -module bundle, and vice versa (see [46, Thm. 3.14]). If we denote by $K(X, A)$ the Grothendieck group of isomorphism classes of such bundles over a compact Hausdorff space X , we have $K(X, A) \cong K_0(C(X) \otimes A)$ [46, Prop. 3.17]. As an example, we illustrate how this correspondence allows for a concrete description of the index map in K -theory.

Example 2.2.25. Let A be a unital C^* -algebra. By Bott periodicity $K_1(A) \cong K_0(SA)$. This isomorphism has the following explicit description (see for instance [41, Thm. 10.1.3]): let $u \in M_{2n}(A)$ and choose a path

$$\gamma : [0, 1] \rightarrow U(M_{2n}(A)) \quad \text{with } \gamma(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \gamma(1) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}.$$

The image of $[u] \in K_1(A)$ is the formal difference of projections

$$\left[\gamma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \gamma^* \right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right],$$

which defines an element in $K_0(SA)$. This element corresponds to the formal difference $[\mathcal{H}_\gamma] - [\underline{\mathcal{H}}]$ of Hilbert A -module bundles over S^1 , where

$$\mathcal{H}_\gamma = \{([t], v) \in S^1 \times \mathbb{C}^2 \otimes A \mid \gamma(t)^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \gamma(t)v = v\}$$

and $\underline{\mathcal{H}}$ is the trivial bundle. To the same unitary $u \in U(A)$ we can also associate another Hilbert A -module bundle over S^1 , defined by

$$\mathcal{H}_u = (0, 1) \times A / \sim \quad \text{where } (1, a) \sim (0, ua).$$

The upshot is that this simpler construction already encodes the suspension isomorphism (up to a basepoint shift). More precisely, there is an isomorphism of Hilbert A -module bundles

$$\mathcal{H}_u \xrightarrow{\cong} \mathcal{H}_\gamma,$$

defined by $(t, a) \mapsto ([t], \gamma(t)^*(e_1 \otimes a))$.

2.2.3 Equivariant *K*-theory

Both operator and topological *K*-theory have an equivariant version. Let G be a compact group.

Definition 2.2.26. A G - C^* -algebra is a C^* -algebra A together with a group homomorphism $\alpha : G \rightarrow \text{Aut}(A)$ such that $g \mapsto \alpha_g(a)$ is a continuous function from G to A for all $a \in A$.

Definition 2.2.27. Let A be a unital G - C^* -algebra. A finitely generated projective (G, A, α) -module is a pair (E, λ) , where E is a finitely generated projective A -module and $\lambda : G \rightarrow \mathcal{B}(E)$ is a continuous representation of G on E , such that

$$\lambda_g(ea) = \lambda_g(e)\alpha_g(a)$$

for all $g \in G$, $e \in E$ and $a \in A$.

Denote by Proj_G the set of isomorphism classes of finitely generated projective (G, A, α) -modules. This is an abelian monoid under \oplus , where the direct sum gets an obvious G -action.

Definition 2.2.28. The Grothendieck completion of Proj_G is denoted $K_0^G(A)$.

$K_0^G(A)$ has a $R(G)$ -module structure via $[V] \cdot [E] = [V \otimes E]$ with the diagonal action and $(v \otimes e)a = v \otimes ea$ for all $v \in V$, $e \in E$ and $a \in A$.

Just as non-equivariant K -theory we can extend the definition to non-unital C^* -algebras via $K_0^G(A) = \text{Ker}(K_0^G(A^+) \rightarrow K_0^G(\mathbb{C}))$ (where the action on \mathbb{C} is trivial). We also have analogs of the most important properties, such as homotopy invariance, stability, continuity and Bott periodicity. K_1^G can be defined in terms of invertible elements, but in view of the periodicity theorem we will define it by $K_1^G(A) := K_0^G(SA)$.

When $A = C(X)$ is commutative, the G -action on $C(X)$ corresponds to a continuous G -action on the space X . Just as in the non-equivariant case here the topological and operator theories agree. Let X be a G -space, i.e. a space with a continuous G -action. A G -vector bundle is a vector bundle E over X with a G -action by vector bundle automorphisms such that the projection map $p : E \rightarrow X$ is G -equivariant. The Whitney sum of two G -vector bundles gets an obvious G -action, hence isomorphism classes of finite dimensional complex G -vector bundles form an abelian monoid $\text{Vect}_G(X)$.

Definition 2.2.29. The Grothendieck completion of $\text{Vect}_G(X)$ is denoted $K_G^0(X)$.

$K_G^0(X)$ is a commutative ring under the tensor product of vector bundles with diagonal G -action, with identity given by the class of the trivial one-dimensional

bundle $X \times \mathbb{C}$ with the G -action $g \cdot (x, \lambda) = (gx, \lambda)$. An equivariant version of Bott periodicity holds and we can define

$$K_G^{2n}(X) = K_G^0(X) \text{ and } K_G^{2n+1}(X) = K_G(S^1 \wedge X) \text{ for all } n.$$

Proposition 2.2.30. K_0^G and K_1^G (resp. K_G^0 and K_G^1) are covariant (resp. contravariant) functors from the category of unital G - C^* -algebras (resp. compact Hausdorff G -spaces) to the category of $R(G)$ -modules.

The space of continuous sections of a G -vector bundle has a natural G -action making it into a finitely generated projective $(C(X), G, \alpha)$ -module and we have the following equivariant version of the Serre-Swan's theorem.

Theorem 2.2.31. Let X be a compact G -space with action α . Then isomorphism classes of G -bundles over X are in bijection with isomorphism classes of finitely generated projective $(C(X), G, \alpha)$ -modules. Hence $K_G^0(X) \cong K_0^G(C(X))$.

Example 2.2.32. $K_G^0(*) \cong K_0^G(\mathbb{C}) \cong R(G)$.

Finally, we note that the construction of KK -groups can be adapted to define equivariant KK_G -groups for all locally compact groups G , and again we have $KK_G(\mathbb{C}, A) \cong K_G(A)$ for all G - C^* -algebras A .

2.3 Stable homotopy theory

The goal of algebraic topology is to classify topological spaces via algebraic invariants. Examples of such invariants are given by the homotopy and (co)homology groups of a space. The founding result of stable homotopy theory is the Freudenthal theorem, which states that the suspension homomorphism on homotopy groups of spheres

$$\pi_{n+k}(S^n) \xrightarrow{\cong} \pi_{n+k+1}(S^{n+1})$$

is an isomorphism for $n > k + 1$. This motivated the introduction of the stable homotopy groups of spheres $\pi_k^{st}(S) := \pi_{n+k}(S^n)$, which for $n > k + 1$ are thus independent of n . Similar phenomena which occur in essentially the same way for any sufficiently large dimension are called stable too. Cohomology theories are closely related to stable objects, called spectra.

2.3.1 Spectra and cohomology theories

Topological K -theory is an example of a cohomology theory.

Definition 2.3.1. A (reduced) cohomology theory is a collection of functors

$$E^n : \text{CW}_*^{\text{op}} \rightarrow \text{Ab}$$

for $n \in \mathbb{Z}$, equipped with natural transformations (called suspension isomorphisms)

$$\sigma^n : E^n \xrightarrow{\sim} E^{n+1} \circ \Sigma,$$

where $\Sigma(X) := S^1 \wedge X$ is the reduced suspension, satisfying:

- (i) (homotopy invariance) If $f \simeq g : X \rightarrow Y$, then $f^* = g^* : E^n(Y) \rightarrow E^n(X)$;
- (ii) (exactness) Each cofiber sequence

$$A \hookrightarrow X \rightarrow X/A$$

gives an exact sequence

$$E^n(X/A) \rightarrow E^n(X) \rightarrow E^n(A).$$

A cohomology theory is called multiplicative if each graded cohomology group $E^*(X)$ comes equipped with a graded ring structure.

Example 2.3.2. Let D be a strongly self-absorbing C^* -algebra and consider the functor $X \mapsto K_*(C(X) \otimes D)$. The tensor product induces a ring structure on

$K_*(C(X) \otimes D)$, given by

$$K_*(C(X) \otimes D) \times K_*(C(X) \otimes D) \longrightarrow K_*(C(X \times X) \otimes D) \xrightarrow{\Delta^*} K_*(C(X) \otimes D)$$

where the first arrow comes from the external tensor product $K_m(A) \times K_n(B) \rightarrow K_{m+n}(A \otimes B)$ and applying the inverse isomorphism $\psi^{-1}: D \otimes D \xrightarrow{\cong} D$, and $\Delta: X \rightarrow X \times X$ denotes the diagonal map. The multiplication does not depend on the choice of ψ because $\text{Aut}(D)$ is contractible [7, Thm. 2.3] and K -theory is homotopy invariant. Thus, the assignment $X \rightarrow K_*(C(X) \otimes D)$ defines a multiplicative cohomology theory on finite CW-complexes.

Let $\{E^n\}$ be a cohomology theory. By Brown representability theorem each functor E^n is representable, in the sense that there exists a CW-complex X_n^E such that

$$E^n(Y) \cong [Y, X_n^E].$$

Applying the suspension isomorphism $E^n(Y) \cong E^{n+1}(\Sigma Y)$ we also get

$$[Y, X_n^E] \cong [\Sigma Y, X_{n+1}^E] \cong [Y, \Omega X_{n+1}^E]$$

for each CW-complex Y , hence we get

$$E_n \simeq \Omega E_{n+1} \simeq \cdots \simeq \Omega^k E_{n+k}$$

for each k , which is what is called an infinite loop space. This motivates the following definition.

Definition 2.3.3. An Ω -spectrum X is a sequence of based spaces X_n equipped with homotopy equivalences $X_n \simeq \Omega X_{n+1}$ for each $n \geq 0$.

We discussed how every cohomology theory produces an Ω -spectrum. The converse is also true: every Ω -spectrum X defines a cohomology theory by $E_X^n(X) := [Y, X_n]$.

Definition 2.3.4. A spectrum X is a sequence of based spaces X_n equipped with maps $\Sigma X_n \rightarrow X_{n+1}$ for each $n \geq 0$.

A spectrum is an Ω -spectrum if the adjoints of its structure maps are homotopy equivalences. A spectrum X defines a cohomology theory by

$$E_X^n(Y) := \operatorname{colim}_k [Y, \Omega^k X_{n+k}].$$

Definition 2.3.5. The n -th homotopy group of a spectrum X is

$$\pi_n X := \operatorname{colim}_k \pi_{n+k} X_k.$$

Two spectra are said to be stably weak equivalent if there is a map between them inducing isomorphisms on all homotopy groups. The category obtained by inverting stable weak equivalences of spectra is called the stable homotopy category. In [1], Adams showed that this is a symmetric monoidal category by constructing a well-behaved smash product. Unfortunately, no such structure can exist in the ordinary category of spectra [27]. Various alternative categories have been proposed in order to solve this problem; here we have chosen to work with Schwede's symmetric spectra [48].

Definition 2.3.6. A symmetric spectrum is a sequence of pointed spaces X_n equipped with a basepoint preserving continuous left Σ_n -action and structure maps $\sigma_n : X_n \wedge S^1 \rightarrow X_{n+1}$, for each $n \geq 0$. This data is subject to the condition that the composite

$$X_n \wedge S^m \xrightarrow{\sigma_n \wedge \operatorname{Id}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge \operatorname{Id}} \dots \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$ -equivariant for all $n, m \geq 0$. Here Σ_m acts on the source by permuting the sphere coordinates, and $\Sigma_n \times \Sigma_m$ acts on the target by restriction of the Σ_{n+m} -action.

A morphism of symmetric spectra $f : X \rightarrow Y$ consists of Σ_n -equivariant based maps $f_n : X_n \rightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps.

Example 2.3.7. The symmetric sphere spectrum \mathbb{S} is given by $\mathbb{S}_n = S^n$, where the symmetric group acts by permuting the coordinates, and $\sigma_n : S^n \wedge S^1 \rightarrow S^{n+1}$ is the canonical homeomorphism.

The key feature of symmetric spectra is that they form a symmetric monoidal category with respect to their smash product \wedge [48, Thm. 5.10]. A symmetric ring spectrum is a symmetric spectrum R together with a multiplication morphism $\mu : R \wedge R \rightarrow R$ and a unit morphism $\iota : \mathbb{S} \rightarrow R$ which are suitably associative and unital. The sphere spectrum is a symmetric ring spectrum, with multiplication given by the canonical homeomorphism $S^n \wedge S^m \rightarrow S^{n+m}$ and the identity map as the unit.

The homotopy theory of symmetric spectra presents an issue, namely the category obtained by inverting weak equivalences with respect to the homotopy groups of the underlying ordinary spectra (which we will call naive homotopy groups of a symmetric spectrum) is not equivalent to the stable homotopy category. Therefore one has to define the correct homotopy groups of a symmetric spectrum X in a more sophisticated way, namely by $\pi_k(X) := \pi_k^{naive}(QX)$ where QX is an Ω -spectrum which is stably equivalent to X [48, Def. 6.1]. However, there is a large class of symmetric spectra, called semistable, for which these two notions agree [48, Prop. 6.3]. All of our examples belong to this class, so we will not make distinctions between the two kinds of homotopy groups in the rest of the thesis.

Definition 2.3.8. A morphism of symmetric spectra $f : X \rightarrow Y$ is a stable equivalence if the induced map $\pi_k(f) : \pi_k(X) \rightarrow \pi_k(Y)$ is an isomorphism for all $k \in \mathbb{Z}$.

2.3.2 Equivariant stable homotopy theory

Let $G\mathcal{T}op$ be the category of based compactly generated weakly Hausdorff spaces equipped with an action of a finite group G which fixes the basepoint, and based equivariant maps. Weak equivalences in $G\mathcal{T}op$ are defined to be morphisms $f : X \rightarrow Y$ such that $f^H : X^H \rightarrow Y^H$ is a weak equivalence in the underlying category $\mathcal{T}op$, for all subgroups $H \leq G$.

The orbits G/H form a full subcategory of $G\mathcal{T}op$ which we denote by O_G , the orbit category. Note that for subgroups $K \leq H \leq G$, we have a natural map of G -sets $G/K \rightarrow G/H$, and a corresponding inclusion map of fixed points $X^H \rightarrow X^K$ in the opposite direction for a G -space X . This means we have a functor

$$O_G^{\text{op}} \rightarrow \mathcal{T}op \quad \text{given by} \quad G/H \mapsto X^H.$$

Definition 2.3.9. An O_G -space is a functor $O_G^{\text{op}} \rightarrow \mathcal{T}op$. Denote by O_G -spaces the category of O_G -spaces and continuous natural transformations.

As described above there is a fixed point functor $G\mathcal{T}op \rightarrow O_G$ -spaces defined on objects by

$$X \mapsto ((G/H) \mapsto X^H).$$

By a celebrated theorem by Elmendorf [13] this functor is an equivalence on homotopy categories. Its homotopy inverse can be constructed explicitly as the geometric realization of the simplicial bar construction:

$$\Phi: O_G\text{-spaces} \rightarrow G\mathcal{T}op, \quad X \mapsto |B(X, O_G, M)|, \quad (2.1)$$

where M is the forgetful functor realising orbits as spaces (see [22, Sec. 8.8] for details).

The category of symmetric spectra easily adapts to the equivariant case [19].

Definition 2.3.10. A G -symmetric spectrum is a symmetric spectrum X together with a G -action on each level X_n which commutes with the Σ_n -action and for which all structure maps $\sigma_n : X_n \wedge S^1 \rightarrow X_{n+1}$ are G -equivariant for the trivial action on S^1 .

A crucial feature of the homotopy theory of G -symmetric spectra is that they can be evaluated on finite G -sets as follows. Let M be a finite G -set of order m and denote by $\text{Bij}(\mathbf{m}, M)$ the discrete space of bijections between the sets $\mathbf{m} = \{1, \dots, m\}$ and M . We can evaluate a G -symmetric spectrum X on M by

$$X(M) := X_m \wedge \text{Bij}(\mathbf{m}, M)_+ / \{(\sigma x \wedge f) \sim (x \wedge f\sigma), \sigma \in \Sigma_m\}$$

with diagonal G -action $g[x \wedge f] := [gx \wedge gf]$.

Definition 2.3.11. Let n be an integer and $H \leq G$ a subgroup. The n -th H -equivariant homotopy group $\pi_n^{H, \mathcal{U}}$ of a G -symmetric spectrum X (with respect to a universe \mathcal{U}) is

$$\pi_n^{H, \mathcal{U}} X := \operatorname{colim}_{M \in s_G(\mathcal{U})} [S^{n \sqcup M}, X(M)]^H,$$

where $s_G(\mathcal{U})$ is the poset of finite G -subsets of \mathcal{U} .

Similarly to Definition 2.3.8, maps which induce isomorphisms on all equivariant homotopy groups are called stable equivalences. Therefore the choice of a universe in the definition above corresponds to the choice of a homotopical structure on the category of G -symmetric spectra. The two extreme choices are deserving of a name. The stable homotopical structure on the category of G -symmetric spectra corresponding to the choice of the trivial G -universe is called the naive homotopical structure, and the one corresponding to a complete G -universe is called the genuine homotopical structure. The latter has more weak equivalences than the former: all representation spheres S^M are inverted, rather than just the trivial ones S^m .

For the purposes of this thesis, the naive setting is good enough, and we will state our results in this generality. There are, however, some features of the genuine theory that could prove useful to establish additional structure on the C^* -algebraic side. We will make a first step in this direction in Section 4.2.

Equivariant higher Dixmier-Douady theory

In this chapter, which constitutes the heart of the thesis, we build to our main results about the classification of $\mathbb{Z}/p\mathbb{Z}$ -equivariant $(D \otimes \mathbb{K})$ -bundles, where D is the C^* -algebra defined in 3.3. Using the machine described in Section 3.1 we produce two naive positive Ω - G -spectra, namely the G -spectrum of positive units of KU^D -theory (Section 3.2) and a G -spectrum associated to the automorphism group of $D \otimes \mathbb{K}$ (Section 3.3). Eventually, in Section 3.4 these are shown to be equivalent. We explain how this provides a full classification of the equivariant bundles.

3.1 An equivariant infinite loop space machine

We saw in Section 2.3.1 that cohomology theories can be defined by mapping into an infinite loop space, i.e., a space which admits a k -th delooping for all $k \in \mathbb{N}$. Any topological abelian monoid M has such a structure: one can form its classifying space BM and iterate the construction to get a sequence of spaces $B^k M$ with the property that $B^{k-1} M \rightarrow \Omega B^k M$ is a weak equivalence for $k \geq 2$. In fact, one can build a much larger class of examples by looking at topological spaces which are commutative monoids only “up to homotopy”. These are called Γ -spaces and were introduced by Segal in [50].

This demonstrates that the notion of a strictly commutative monoid in topological spaces is not suitable for doing homotopy theory, as it does not model enough

homotopy types. However, one can define a category of diagram spaces, called \mathcal{I} -spaces [44], in which strict commutative monoids model infinite loop spaces. In this thesis, we will use this technology to construct Γ -spaces out of commutative monoids in \mathcal{I} -spaces. Finally, we will turn them into positive Ω -spectra. All of these constructions are easily generalised to the naive equivariant setting, in which we will spell out the definitions. We defer a preliminary discussion about the genuine theory of equivariant infinite loop spaces to Section 4.2.

Let \mathcal{I} be the category of finite sets $\mathbf{n} = \{1, \dots, n\}$ (including $\mathbf{0} = \emptyset$) and injective functions. It is a symmetric monoidal category via concatenation of finite ordered sets $\mathbf{m} \sqcup \mathbf{n}$ along with symmetric structure maps $\mathbf{m} \sqcup \mathbf{n} \rightarrow \mathbf{n} \sqcup \mathbf{m}$ given by the obvious shuffles.

Definition 3.1.1. An \mathcal{I} - G -space is a functor $X: \mathcal{I} \rightarrow G\mathcal{T}op$. An \mathcal{I} - G -space X is called a (commutative) \mathcal{I} - G -monoid if it comes equipped with a natural transformation $\mu: X \times X \rightarrow X \circ \sqcup$ which is associative, unital (and commutative) in the sense of [6, Def. 2.2].

Let Γ^{op} be the category of finite based sets $\mathbf{n}_+ = \{0, 1, \dots, n\}$ with basepoint 0 and based maps.

Definition 3.1.2. A Γ - G -space is a functor $\Gamma^{op} \rightarrow G\mathcal{T}op$. A Γ - G -space A is called *special* if for every n the based map

$$A(\mathbf{n}_+) \rightarrow \left(\prod_{i=1}^n A(\mathbf{1}_+) \right),$$

induced by projections $\delta_i: \mathbf{n}_+ \rightarrow \mathbf{1}_+$ sending i to 1 and all other elements to the basepoint, is a weak equivalence. We refer to this map as the equivariant Segal map.

The notions of \mathcal{I} - O_G -spaces and (special) Γ - O_G -spaces are defined analogously by replacing the target category by O_G -spaces.

There is a way to associate a special Γ - G -space to any commutative \mathcal{I} - G -monoid. This is completely analogous to the non-equivariant case [28, Construction 12.1] with some minor adjustments; for the sake of completeness, we recall how it works.

Let X be a commutative \mathcal{I} - G -monoid. Let $P(\mathbf{n}_+)$ be the category of finite sets $0 \notin A \subset \mathbf{n}_+$ and define $\mathcal{I}(\mathbf{n}_+)$ to be the category of functors $\theta : P(\mathbf{n}_+) \rightarrow \mathcal{I}$ with the property that for every pair of disjoint subsets $A, B \in P(\mathbf{n}_+)$ the diagram

$$\theta(A) \rightarrow \theta(A \cup B) \leftarrow \theta(B)$$

(induced by inclusions) is a coproduct diagram in \mathcal{I} . Morphisms in $\mathcal{I}(\mathbf{n}_+)$ are natural transformations. A functor $\theta \in \mathcal{I}(\mathbf{n}_+)$ is given by the data of a collection of objects $\theta_i := \theta(\{i\}) \in \mathcal{I}$ for each $1 \leq i \leq n$ (by definition, $\theta(\emptyset) = \mathbf{0}$), plus morphisms $\theta_i \rightarrow \theta(A)$ for each $A \subset \{1, \dots, n\}$ which assemble into a canonical isomorphism

$$\bigoplus_{i \in A} \theta_i \cong \theta(A). \tag{3.1}$$

Consider the forgetful functor $F : \mathcal{I}(\mathbf{n}_+) \rightarrow \mathcal{I}^n$ that sends θ to $(\theta_1, \dots, \theta_n)$. We get a functor $X(\mathbf{n}_+) : \mathcal{I}(\mathbf{n}_+) \rightarrow \mathcal{GT}op$ given by the composite

$$\begin{aligned} X(\mathbf{n}_+) : \mathcal{I}(\mathbf{n}_+) &\xrightarrow{F} \mathcal{I}^n \xrightarrow{X^n} \mathcal{GT}op \\ \theta &\mapsto \prod_{i=1}^n X(\theta_i). \end{aligned}$$

Finally, define a functor $H_{\mathcal{I}}X : \Gamma^{\text{op}} \rightarrow \mathcal{GT}op$ on objects as the homotopy colimit (in the sense of [22, Def. 5.8.1])

$$H_{\mathcal{I}}X(\mathbf{n}_+) = \text{hocolim}_{\mathcal{I}(\mathbf{n}_+)} X(\mathbf{n}_+)$$

and set $H_{\mathcal{I}}X(\mathbf{0}_+) = *$. Note that $H_{\mathcal{I}}X(\mathbf{1}_+) = \text{hocolim}_{\mathcal{I}} X =: X_{h\mathcal{I}}$.

In order to make $H_{\mathcal{I}}X$ into a Γ - G -space we need to have functoriality; this is only possible because X has the structure of an commutative \mathcal{I} - G -monoid. Let $\alpha : \mathbf{m}_+ \rightarrow \mathbf{n}_+$ be a map of finite based sets. There is a functor $\alpha_* : \mathcal{I}(\mathbf{m}_+) \rightarrow$

$\mathcal{I}(\mathbf{n}_+)$ given by precomposition with α^{-1} . We have a natural transformation $X(\alpha) : X(\mathbf{m}_+) \rightarrow X(\mathbf{n}_+) \circ \alpha_*$ defined as

$$\begin{aligned} X(\alpha)_\theta : \prod_{i=1}^m X(\theta_i) &\xrightarrow{\pi} \prod_{j=1}^n \prod_{i \in \alpha^{-1}(j)} X(\theta_i) \xrightarrow{\mu} \prod_{j=1}^n X(\bigoplus_{i \in \alpha^{-1}(j)} \theta_i) \\ &\cong \prod_{j=1}^n X(\theta(\alpha^{-1}(j))) \end{aligned}$$

where π is the projection away from the factors indexed by elements $i \in \mathbf{m}_+$ that are mapped to the basepoint by α , and the last isomorphism is induced by the canonical isomorphism (3.1). In this definition a choice is apparently involved when ordering the product indexed by $i \in \alpha^{-1}(j)$, but commutativity of μ ensures that it does not actually matter. Finally, we lift the natural transformation $X(\alpha)$ to the first map of homotopy colimits below:

$$\mathrm{hocolim}_{\mathcal{I}(\mathbf{m}_+)} X(\mathbf{m}_+) \rightarrow \mathrm{hocolim}_{\mathcal{I}(\mathbf{m}_+)} X(\mathbf{n}_+) \circ \alpha_* \xrightarrow{\alpha_*} \mathrm{hocolim}_{\mathcal{I}(\mathbf{n}_+)} X(\mathbf{n}_+).$$

This composition defines $H_{\mathcal{I}}X(\alpha)$.

Proposition 3.1.3. *The functor $H_{\mathcal{I}}X$ is a special Γ - G -space.*

Proof. We need to check that the equivariant Segal map

$$H_{\mathcal{I}}X(\mathbf{n}_+) \rightarrow \left(\prod_{i=1}^n H_{\mathcal{I}}X(\mathbf{1}_+) \right)$$

is a weak equivalence. Using the canonical isomorphism

$$(\mathrm{hocolim}_{\mathcal{D}} X)^n \cong \mathrm{hocolim}_{\mathcal{D}^n} X^n$$

this boils down to checking that the map of homotopy colimits induced by the forgetful functor F

$$\mathrm{hocolim}_{\mathcal{I}(\mathbf{n}_+)} X(\mathbf{n}_+) \rightarrow \mathrm{hocolim}_{\mathcal{I}^n} X^n$$

is a weak equivalence, which in turn follows from the Bousfield-Kan cofinality criterion [22, Thm. 5.8.15] applied to the functor $F : \mathcal{I}(\mathbf{n}_+) \rightarrow \mathcal{I}^n$, provided we can show that F is homotopy final in the sense of [22, Def. 5.8.13]. This reduces

to verify that for any $v = (v_1, \dots, v_n) \in \mathcal{I}^n$, the comma category (v/F) has an initial object. This is in fact the case, and such an initial object is given by the triple

$$(v, \theta, \text{id} : v \rightarrow F(\theta)) \in (v/F) \tag{3.2}$$

where θ is the functor $P(\mathbf{n}_+) \rightarrow \mathcal{I}$ given by

$$\theta(A) = \bigoplus_{i \in A} v_i. \quad \square$$

Remark 3.1.4. In the rest of the thesis we will often work with O_G -spaces instead of G -spaces. The construction outlined above can be repeated verbatim when X is a commutative \mathcal{I} - O_G -monoid. The output in this case is a special Γ - O_G -space, which we will always turn into a special Γ - G -space again using Φ from (2.1). Hence, we will consider $H_{\mathcal{I}}$ as a functor from commutative \mathcal{I} - O_G -monoids to Γ - G -spaces. Here a key observation is that the functor $G\mathcal{T}op \rightarrow O_G$ -spaces preserves products on the nose, hence allowing us to compare the output of this machinery to the one that we get by starting with the corresponding commutative \mathcal{I} - G -monoid. As one would expect the two resulting objects are homotopy equivalent, because of Elmendorf's theorem and the fact that the homotopy colimit construction commutes with taking fixed points.

There are several ways to get a G -spectrum out of a special Γ - G -space. All of these constructions require some sort of G -cofibrancy condition on the Γ - G -space and differ by the flavour of the G -spectra they produce. As we mentioned, for our purposes it is enough to get a naive G -spectrum; in this case, the cofibrancy condition can always be assumed to hold thanks to a functorial cofibrant replacement theorem [32, Prop. 2.12]. The following is a straightforward generalization of the Segal machine to G -spaces [32, Def. 2.22].

Definition 3.1.5. Let A be a special Γ - G -space. For each p , let $A[\mathbf{p}_+]$ be the Γ - G -space that sends \mathbf{q}_+ to $A(\mathbf{p}_+ \wedge \mathbf{q}_+)$, where $\mathbf{p}_+ \wedge \mathbf{q}_+ := \mathbf{p}\mathbf{q}_+$. Define the classifying Γ - G -space $\mathbb{B}A$ to be the Γ - G -space whose p -th G -space is the geometric realization

$|A[\mathbf{p}_+]|$. Now let $\mathbb{B}^0 A = A$ and iteratively define $\mathbb{B}^n A = \mathbb{B}(\mathbb{B}^{n-1} A)$. We denote by EA the resulting naive G -spectrum with n -th G -space

$$(EA)_n = (\mathbb{B}^n A)(\mathbf{1}_+).$$

Note that this machine actually produces positive Ω - G -spectra, which means that the structure maps $(EA)_n \rightarrow \Omega(EA)_{n+1}$ for $n \geq 1$ are G -weak equivalences, while the first structure map $(EA)_0 = A(\mathbf{1}_+) \rightarrow \Omega(EA)_1$ is a group completion [32, Prop. 2.18].

So far we have shown how to construct a positive Ω - G -spectrum out of a commutative \mathcal{I} - O_G -monoid. Our goal for the rest of the subsection is to show that for a stable \mathcal{I} - O_G -monoid X (see Definition 3.1.8), all of the information about the homotopy type of such a spectrum (except in level 0) is contained in the O_G -space $X(\mathbf{1})$.

The naive G -spectra we obtain as the output of the equivariant Segal machine (using Γ - G -spaces) are spectra objects in G -spaces. Therefore homotopy groups and equivalences are defined as follows.

Definition 3.1.6. Let n be an integer and $H \leq G$ be a subgroup. The n -th equivariant homotopy group $\pi_n^H X$ of a naive G -spectrum X is defined as

$$\pi_n^H(X) = \operatorname{colim}_{k \rightarrow \infty} [S^{n+k}, X_k]^H$$

where H acts trivially on S^{n+k} . We say a morphism $f : X \rightarrow Y$ of naive G -spectra is an equivalence if the induced map $\pi_n^H(f) : \pi_n^H(X) \rightarrow \pi_n^H(Y)$ is an isomorphism for all integers n and all subgroups $H \leq G$.

Lemma 3.1.7. For every positive Ω - G -spectrum X , every $n \geq 0$ and every subgroup $H \leq G$ the map

$$\pi_n(\Omega X_1^H) \rightarrow \pi_n^H(X)$$

is an isomorphism.

Proof. Note that the right hand side is $\pi_n^H(\Omega^k X_k)$ and by the definition of a positive Ω - G -spectrum we have that $\Omega X_1 \rightarrow \Omega^k X_k$ is an equivariant weak equivalence for $k \geq 1$. □

Definition 3.1.8. An \mathcal{I} - O_G -space is called stable if all morphisms $\mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{I} with $\mathbf{m} \neq \mathbf{0}$ are mapped to weak equivalences for all cosets G/H .

Remark 3.1.9. The category \mathcal{I} is filtered and the homotopy group functor π_* preserves filtered colimits. Moreover, taking fixed points commutes with homotopy colimits for actions of finite groups. This implies that for a stable \mathcal{I} - O_G -space Z we have

$$Z(\mathbf{1})(G/H) \simeq \Phi(Z(\mathbf{1}))^H \simeq ((\Phi \circ Z)_{h\mathcal{I}})^H \simeq Z(G/H)_{h\mathcal{I}}.$$

Lemma 3.1.10. Let X, Y be two stable commutative \mathcal{I} - O_G -monoids such that $\pi_0(X(\mathbf{1})(G/H))$ and $\pi_0(Y(\mathbf{1})(G/H))$ are groups for all subgroups $H \leq G$ and let $f : X \rightarrow Y$. We have the following commutative diagram

$$\begin{array}{ccc} \pi_n^H(EH_{\mathcal{I}}X) & \longrightarrow & \pi_n^H(EH_{\mathcal{I}}Y) \\ \simeq \uparrow & & \uparrow \simeq \\ \pi_{n+1}(\left((EH_{\mathcal{I}}X)_1\right)^H) & \longrightarrow & \pi_{n+1}(\left((EH_{\mathcal{I}}Y)_1\right)^H) \\ \simeq \uparrow & & \uparrow \simeq \\ \pi_n(X(\mathbf{1})(G/H)) & \longrightarrow & \pi_n(Y(\mathbf{1})(G/H)) \end{array}$$

In particular, f induces an isomorphism on π_n^H of the corresponding G -spectra if and only if the map

$$\pi_n(X(\mathbf{1})(G/H)) \rightarrow \pi_n(Y(\mathbf{1})(G/H))$$

is an isomorphism.

Proof. The two upper vertical arrows are isomorphisms by Lemma 3.1.7. Now let $Z \in \{X, Y\}$. Note that $\pi_0^H(Z(\mathbf{1})) \cong \pi_0^H(Z_{h\mathcal{I}}) \cong \pi_0^H((EH_{\mathcal{I}}Z)_0)$ is a group by

assumption. Observe that $((EH_{\mathcal{I}}Z)_0)^H \simeq (\Omega(EH_{\mathcal{I}}Z)_1)^H$ by the group completion theorem and

$$((EH_{\mathcal{I}}Z)_0)^H \simeq (H_{\mathcal{I}}Z(\mathbf{1}_+))^H \simeq ((\Phi \circ Z)_{h\mathcal{I}})^H.$$

Thus the two lower vertical arrows are isomorphisms as well by Remark 3.1.9. \square

3.2 The G -spectrum of units of KU^D -theory

In this section we construct a commutative G -symmetric ring spectrum KU^D for $G = \mathbb{Z}/p\mathbb{Z}$ associated to a functor taking values in graded G - C^* -algebras, i.e. $\mathbb{Z}/2\mathbb{Z}$ -graded C^* -algebras together with an action by the group G that preserves the grading. Using the infinite loop space machine described in the previous section we define its G -spectrum of units.

3.2.1 The spectrum KU^D

Definition 3.2.1. Let Σ be the category of finite sets and bijective maps. This is a symmetric monoidal category with respect to the coproduct \sqcup . Let G - C^* Alg be the category of graded G - C^* -algebras and equivariant grading-preserving $*$ -homomorphisms. This is also a symmetric monoidal category with respect to the graded minimal tensor product \otimes . A *coefficient system* is a symmetric monoidal functor

$$C: (\Sigma, \sqcup) \rightarrow (G\text{-}C^*\text{Alg}, \otimes).$$

The tensor product of two coefficient systems C_1 and C_2 is defined to be

$$(C_1 \otimes C_2)(S) = C_1(S) \otimes C_2(S)$$

for a finite set S . Let $\eta_{S,T}^i: C_i(S \sqcup T) \rightarrow C_i(S) \otimes C_i(T)$. The corresponding natural isomorphism for $C_1 \otimes C_2$ is

$$\begin{aligned}
 (C_1 \otimes C_2)(S \sqcup T) &= C_1(S \sqcup T) \otimes C_2(S \sqcup T) \\
 &\quad \downarrow \eta_{S,T}^1 \otimes \eta_{S,T}^2 \\
 C_1(S) \otimes C_1(T) \otimes C_2(S) \otimes C_2(T) \\
 &\quad \downarrow \text{id} \otimes \text{flip} \otimes \text{id} \\
 (C_1 \otimes C_2)(S) \otimes (C_1 \otimes C_2)(T) &= C_1(S) \otimes C_2(S) \otimes C_1(T) \otimes C_2(T)
 \end{aligned}$$

where flip denotes the symmetry of G - C^* Alg (taking the grading into account).

Our main example of a coefficient system is constructed as follows: let V be a finite-dimensional complex inner product space and let $\rho: G \rightarrow U(V)$ be a unitary representation of G . Let

$$D = \text{End}(V)^{\otimes \infty} \tag{3.3}$$

be the infinite UHF-algebra associated to V . The infinite tensor product of the adjoint action of G on $\text{End}(V)$ turns this into a G - C^* -algebra, which we consider to be trivially graded.

To define the stabilisation of D , we need the compact operators, which we turn into a G - C^* -algebra as follows: let H_0 be an infinite-dimensional separable Hilbert space and define $H_G = \ell^2(G) \otimes H_0$, where $\ell^2(G)$ is the finite-dimensional Hilbert space given by the direct sum over all irreducible representations of G . Let $\mathbb{K} = \mathbb{K}(H_G)$ be the compact operators on H_G . This is a G - C^* -algebra where G acts on $\mathbb{K}(\ell^2(G) \otimes H_0)$ via the adjoint action. We consider $D \otimes \mathbb{K}$ equipped with the tensor product action of G , which we denote by α , and the trivial grading.

Definition 3.2.2. Let C be a coefficient system. We call it *stable* if for every finite set S we have $C(S) \otimes \mathbb{K} \cong C(S)$ as graded G - C^* -algebras.

To define the ring spectrum we will need two non-trivially graded C^* -algebras that feature in the definition of KU^D . The first one is the graded suspension algebra $\widehat{\mathcal{S}} = C_0(\mathbb{R})$ equipped with the grading by odd and even functions. This is

not only an algebra, but also a coassociative and cocommutative coalgebra with respect to the comultiplication $\Delta: \widehat{\mathcal{S}} \rightarrow \widehat{\mathcal{S}} \otimes \widehat{\mathcal{S}}$, where the tensor product in the codomain is the graded one.

The second $\mathbb{Z}/2\mathbb{Z}$ -graded C^* -algebra needed in the construction is the complex Clifford algebra. Let $S \in \Sigma$ be a finite set and let $\mathbb{C}[S]$ be the finite-dimensional complex inner product space equipped with the symmetric bilinear form

$$b(v, w) = \sum_{s \in S} v_s w_s,$$

where v_s, w_s denote the components with respect to the orthonormal basis S . The Clifford algebra $\mathcal{Cl}(S)$ is defined to be the quotient

$$\mathcal{Cl}(S) = T(\mathbb{C}[S])/I(\mathbb{C}[S], b)$$

where $T(W) = \bigoplus_{n \in \mathbb{N}_0} W^{\otimes n}$ is the tensor algebra and $I(W, b)$ is the ideal generated by $v \otimes w - w \otimes v - 2b(v, w)1$ for $v, w \in W$ where b is a bilinear form on W . To define the grading we declare the elements $w \in W$ to be odd. There is a natural isomorphism of graded algebras

$$\mathcal{Cl}(S \sqcup T) \cong \mathcal{Cl}(S) \otimes \mathcal{Cl}(T),$$

which turns $S \mapsto \mathcal{Cl}(S)$ into a coefficient system with trivial G -action. For $n \in \mathbb{N}$ let $\mathcal{Cl}_n = \mathcal{Cl}(\{1, \dots, n\})$. We also define

$$(D \otimes \mathbb{K})(S) = \bigotimes_{s \in S} D \otimes \mathbb{K}$$

and note the natural isomorphism

$$(D \otimes \mathbb{K})(S \sqcup T) = \bigotimes_{s \in S \sqcup T} D \otimes \mathbb{K} \cong \left(\bigotimes_{s \in S} D \otimes \mathbb{K} \right) \otimes \left(\bigotimes_{s \in T} D \otimes \mathbb{K} \right)$$

given by reordering the tensor factors. Equipped with this natural isomorphism $S \mapsto (D \otimes \mathbb{K})(S)$ is a trivially graded stable coefficient system, where the group acts on D and \mathbb{K} as described above.

In order to describe Bott periodicity in the spectral picture we define $\hat{\eta}_1 \in \text{hom}_{\text{gr}}(\widehat{\mathcal{S}}, C_0(\mathbb{R}) \otimes \mathbb{C}\ell_1)$ to be the Bott element as in [21, Def. 1.26]. Taking the adjoint map we can view this element as a base-point preserving continuous map

$$\tilde{\eta}_1: S^1 \rightarrow \text{hom}_{\text{gr}}(\widehat{\mathcal{S}}, \mathbb{C}\ell_1)$$

and extend it to

$$\eta_1: S^1 \rightarrow \text{hom}_{\text{gr}}(\widehat{\mathcal{S}}, \mathbb{C}\ell_1 \otimes D \otimes \mathbb{K}), \quad z \mapsto \tilde{\eta}_1(z) \otimes 1_D \otimes e.$$

Note that $1_D \otimes e$ is fixed by the group action on $D \otimes \mathbb{K}$. Now we define the spectrum KU^D as follows:

Definition 3.2.3. Let G be a finite group, V be a unitary G -representation and let D be the associated UHF G - C^* -algebra as defined in (3.3). Let $(KU_n^D)_{n \in \mathbb{N}_0}$ denote the sequence of G -spaces

$$KU_n^D = \text{hom}_{\text{gr}}(\widehat{\mathcal{S}}, (\mathbb{C}\ell \otimes D \otimes \mathbb{K})(\{1, \dots, n\})),$$

where $S \mapsto (\mathbb{C}\ell \otimes D \otimes \mathbb{K})(S)$ is the coefficient system defined above. Notice that there is a natural $(\Sigma_n \times G)$ -action on $(\mathbb{C}\ell \otimes D \otimes \mathbb{K})(\{1, \dots, n\})$ by functoriality. The spaces KU_n^D can be equipped with a multiplication as follows

$$\mu_{m,n}: KU_m^D \wedge KU_n^D \rightarrow KU_{m+n}^D, \quad \varphi \wedge \psi \mapsto (\varphi \otimes \psi) \circ \Delta.$$

There are corresponding unit maps $\eta_n: S^n \rightarrow KU_n^D$ defined inductively by $\eta_n = (\eta_{n-1} \otimes \eta_1) \circ \Delta$ with η_1 as above.

Remark 3.2.4. Non-equivariant versions of the spectrum KU^D have been defined in [6] for all strongly self-absorbing C^* -algebras \mathcal{D} . In this case the spectrum KU^D represents topological K -theory with coefficients in $K_0(\mathcal{D})$, i.e. a localisation of K -theory. For instance,

$$KU^{\mathbb{C}} \simeq KU^{\mathbb{Z}} \simeq KU^{\mathcal{O}_\infty} \simeq KU \quad \text{and} \quad KU^{M_n^\infty} \simeq KU \left[\frac{1}{n} \right]$$

where KU denotes the usual complex topological K -theory spectrum.

Given two graded G - C^* -algebras A and B denote by $[A, B]_G$ the homotopy classes of grading preserving $*$ -homomorphisms $A \rightarrow B$. Let $\hat{\mathbb{C}}^2$ be the graded Hilbert space with $\hat{\mathbb{C}}_+^2 = \mathbb{C} = \hat{\mathbb{C}}_-^2$ and let $\hat{H}_G = \hat{\mathbb{C}}^2 \otimes H_G = \hat{\mathbb{C}}^2 \otimes L^2(G) \otimes H_0$. Let $\hat{\mathbb{K}} = \mathbb{K}(\hat{H}_G)$. This is also a graded G - C^* -algebra. The following lemma is an equivariant generalisation of the converse functional calculus in [57]. In fact, the equivariant case can be reduced to the non-equivariant one via the Green-Julg theorem. A proof along these lines (which works for the much more general setting of proper locally compact groupoids with Haar system) can be found for example in [8, Prop. 3.1].

Lemma 3.2.5. *Let G be a finite group and let B be a graded G - C^* -algebra. Then the converse functional calculus explained in [57, Thm. 4.7] provides an isomorphism of groups*

$$[\hat{\mathcal{S}}, B \otimes \hat{\mathbb{K}}]^G \cong KK_G(\mathbb{C}, B),$$

which is natural in B .

We are now in the position to prove the main theorem of this section.

Theorem 3.2.6. *The spaces KU_*^D together with the maps $\mu_{*,*}$ and η_* form a commutative G -symmetric ring spectrum (in the sense of [19, Def. 2.2] for the trivial universe), with coefficients (for $H \leq G$)*

$$\pi_n^H(KU^D) \cong K_n^H(D).$$

Proof. The proof that the sequence of spaces $(KU_n^D)_{n \in \mathbb{N}_0}$ forms a commutative G -symmetric ring spectrum is very similar to the non-equivariant case discussed in [6, Thm. 4.2]. Therefore we omit the details and just point out that KU_n^D comes equipped with a G -action, which is trivial everywhere except on $D \otimes \mathbb{K}$. In particular, it commutes with the Σ_n -action.

To compute $\pi_n^H(KU^D)$ we need to determine $[S^{n+k}, KU_k^D]^H$ for $k \in \mathbb{N}_0$. Observe that

$$KU_k^D \cong \text{hom}_{\text{gr}}\left(\widehat{\mathcal{S}}, \mathcal{C}l_k \otimes (D \otimes \mathbb{K})^{\otimes k}\right)$$

and therefore

$$[S^{n+k}, KU_k^D]^H \cong \left[\widehat{\mathcal{S}}, C_0(\mathbb{R}^n) \otimes C_0(\mathbb{R}^k, \mathcal{C}l_k) \otimes (D \otimes \mathbb{K})^{\otimes k}\right]^H. \quad (3.4)$$

From [3, Cor. 14.5.3] we deduce the following isomorphisms of graded G - C^* -algebras:

$$\mathcal{C}l_k \otimes \widehat{\mathbb{K}} \cong \mathcal{C}l_{k-1} \otimes \mathcal{C}l_1 \otimes \widehat{\mathbb{K}} \cong \mathcal{C}l_{k-1} \otimes \mathcal{C}l_1 \otimes \mathbb{K} \cong \mathcal{C}l_k \otimes \mathbb{K},$$

where \mathbb{K} on the right hand side denotes the trivially graded compact operators. By Lemma 3.2.5 we therefore can express the right hand side of (3.4) as equivariant K -theory groups as follows:

$$\begin{aligned} & KK_H(\mathbb{C}, C_0(\mathbb{R}^n) \otimes C_0(\mathbb{R}^k, \mathcal{C}l_k) \otimes (D \otimes \mathbb{K})^{\otimes k}) \\ & \cong KK_H(\mathbb{C}, C_0(\mathbb{R}^n) \otimes (D \otimes \mathbb{K})^{\otimes k}) \cong K_n^H((D \otimes \mathbb{K})^{\otimes k}), \end{aligned}$$

where the first isomorphism is a consequence of Bott periodicity in equivariant KK -theory. For $m \geq k$ the above identification fits into the following commutative diagram

$$\begin{array}{ccc} [S^{n+k}, KU_k^D]^H & \xrightarrow{\cong} & K_n^H((D \otimes \mathbb{K})^{\otimes k}) \\ \downarrow & & \downarrow \\ [S^{n+m}, KU_m^D]^H & \xrightarrow{\cong} & K_n^H((D \otimes \mathbb{K})^{\otimes m}) \end{array}$$

where the right hand vertical arrow is induced by inclusions $(D \otimes \mathbb{K})^{\otimes k} \rightarrow (D \otimes \mathbb{K})^{\otimes m}$ that introduce tensor factors $1 \otimes e$. All of these maps are isomorphisms in K -theory. Hence,

$$\pi_n^H(KU^D) = \text{colim}_{k \rightarrow \infty} [S^{n+k}, KU_k^D]^H \cong K_n^H(D). \quad \square$$

Remark 3.2.7. With only minor modifications one can adapt the above proof to see that KU_n^D extends to a commutative G -symmetric ring spectrum for an arbitrary choice of G -universe in the sense of [19, Def. 2.2]. We omit the details here, since we will only make use of KU^D as a naive G -spectrum.

3.2.2 Equivariant units of KU^D

Given a (commutative) G -symmetric ring spectrum R with unit maps η_n and multiplication maps $\mu_{m,n}$, the assignment

$$\mathbf{n} \mapsto (G/H \mapsto (\Omega^n R_n)^H)$$

defines on objects a (commutative) \mathcal{I} - O_G -monoid $\Omega^\bullet R$. A morphism $\alpha : \mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{I} induces a unique permutation $\bar{\alpha} : \mathbf{n} = \mathbf{l} \sqcup \mathbf{m} \rightarrow \mathbf{n}$, which acts like α on the last m entries and is monotonic on the first l entries. This allows to define a map $(\Omega^m R_m)^H \rightarrow (\Omega^n R_n)^H$ by sending $f \in (\Omega^m R_m)^H$ to the composition

$$S^m \xrightarrow{\bar{\alpha}^{-1}} S^m = S^l \wedge S^m \xrightarrow{\eta_l \wedge f} R_l \wedge R_m \xrightarrow{\mu_{l,m}} R_n \xrightarrow{\bar{\alpha}} R_n,$$

thus yielding functoriality of $\Omega^\bullet R$ with respect to injective maps.

Non-equivariantly, the group of units of R is classically defined by the following pullback (see [28, Def. 11.3]):

$$\begin{array}{ccc} \mathrm{GL}_1 R & \hookrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \mathrm{GL}_1(\pi_0(R)) & \hookrightarrow & \pi_0(R) \end{array} \quad (3.5)$$

In the equivariant setting we want this condition to be realised for all fixed points at the same time. Hence we define the \mathcal{I} - O_G -space of units of R (which we denote by $\mathrm{GL}_1^\bullet R$ again) by the same formal pullback (3.5) as above, but in the category of \mathcal{I} - O_G -spaces. Unraveling the definition shows that it encodes the information about invertible elements for all fixed points, namely $(\mathrm{GL}_1^\bullet R(\mathbf{n}))(G/H)$ is made up of those elements of $(\Omega^n R_n)^H$ which are sent to units on π_0 , where π_0 is a O_G -set, defined by $\pi_0(X)(G/H) = \pi_0^H(X)$. For semistable commutative G -symmetric ring spectra this has the correct homotopy type (see [28, Rem. 11.4]), so in particular for KU^D . The machinery illustrated in Section 3.1 allows us to define the special Γ - G -space of units $H_{\mathcal{I}}\mathrm{GL}_1^\bullet R$ and its associated spectrum.

Definition 3.2.8. The G -spectrum of units of a commutative G -symmetric ring spectrum R is

$$gl_1 R = EH_{\mathcal{I}}GL_1^{\bullet}R.$$

Remark 3.2.9. A similar approach to units of equivariant ring spectra has been worked out in [45, Sec. 9], with the difference that the author uses special Γ - G -spaces (actually a slight variation, namely Γ_G -spaces) directly. Let us show how for a special Γ - O_G -space arising from a stable commutative \mathcal{I} - O_G -monoid $\Omega^{\bullet}R$ for a G -symmetric commutative ring spectrum R the two machines define weakly equivalent objects. We will denote the Γ - O_G -space of units defined in [45] by UR . It is defined by the pullback

$$\begin{array}{ccc} UR & \longleftarrow & H_{\mathcal{I}}\Omega^{\bullet}R \\ \downarrow & & \downarrow \\ GL_1(\pi_0(H_{\mathcal{I}}\Omega^{\bullet}R)) & \longleftarrow & \pi_0(H_{\mathcal{I}}\Omega^{\bullet}R) \end{array}$$

where $H_{\mathcal{I}}\Omega^{\bullet}R$ is the special Γ - O_G -space associated to $\Omega^{\bullet}R$. Note that $H_{\mathcal{I}}GL_1^{\bullet}R$ also fits into a homotopy pullback diagram

$$\begin{array}{ccc} H_{\mathcal{I}}GL_1^{\bullet}R & \longrightarrow & H_{\mathcal{I}}\Omega^{\bullet}R \\ \downarrow & & \downarrow \\ \pi_0(H_{\mathcal{I}}GL_1^{\bullet}R) & \longrightarrow & \pi_0(H_{\mathcal{I}}\Omega^{\bullet}R) \end{array}$$

The map of Γ - G -spaces $H_{\mathcal{I}}GL_1^{\bullet}R \rightarrow H_{\mathcal{I}}\Omega^{\bullet}R$ is an isomorphism on all higher homotopy groups. Hence, it suffices to show that

$$\pi_0(UR(\mathbf{1}_+)) \cong \pi_0((H_{\mathcal{I}}GL_1^{\bullet}R)(\mathbf{1}_+)).$$

By stability we have $(H_{\mathcal{I}}GL_1^{\bullet}R)(\mathbf{1}_+) = GL_1^{\bullet}R_{h\mathcal{I}} \simeq GL_1^{\bullet}R(\mathbf{1})$ and therefore

$$\pi_0((H_{\mathcal{I}}GL_1^{\bullet}R)(\mathbf{1}_+)) \cong \pi_0(GL_1^{\bullet}R(\mathbf{1})) \cong GL_1(\pi_0(\Omega^{\infty}R)).$$

On the other hand, since $(H_{\mathcal{I}}\Omega^{\bullet}R)(\mathbf{1}_+) = \Omega^{\bullet}R_{h\mathcal{I}} \simeq \Omega^{\infty}R$ we have that $UR(\mathbf{1}_+)$ is defined by

$$\begin{array}{ccc} UR(\mathbf{1}_+) & \longleftarrow & \Omega^{\bullet}R_{h\mathcal{I}} \\ \downarrow & & \downarrow \\ GL_1(\pi_0(\Omega^{\infty}R)) & \longleftarrow & \pi_0(\Omega^{\infty}R) \end{array}$$

Hence by definition

$$\pi_0(UR(\mathbf{1}_+)) \cong \mathrm{GL}_1(\pi_0(\Omega^\infty R)) \cong \pi_0(H_{\mathcal{I}}\mathrm{GL}_1^\bullet R(\mathbf{1}_+)).$$

We will focus on the equivariant units of the G -symmetric ring spectrum $R = KU^D$ constructed in Section 3.2.1. The following result is basically a consequence of Bott periodicity.

Lemma 3.2.10. *The \mathcal{I} - O_G -space $\mathrm{GL}_1^\bullet KU^D$ is stable.*

Proof. We need to check that all morphisms $\mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{I} with $\mathbf{m} \neq \mathbf{0}$ are mapped to weak equivalences for all cosets G/H . Let $H \leq G$. Note that for π_0 we have

$$\pi_0((\mathrm{GL}_1^\bullet KU^D)(\mathbf{m}))(G/H) \cong \pi_0((\mathrm{GL}_1^\bullet KU^D)(\mathbf{n}))(G/H) \cong \mathrm{GL}_1(K_0^H(D))$$

for all \mathbf{m}, \mathbf{n} , whereas for $k > 0$ we have the following commutative diagram

$$\begin{array}{ccc} \pi_k((\mathrm{GL}_1^\bullet KU^D)(\mathbf{m}))(G/H) & \longrightarrow & \pi_k((\mathrm{GL}_1^\bullet KU^D)(\mathbf{n}))(G/H) \\ \cong \downarrow & & \downarrow \cong \\ \pi_k(\Omega^m KU_m^D) & \longrightarrow & \pi_k(\Omega^n KU_n^D) \end{array}$$

where the bottom horizontal arrow is an isomorphism when $m > 0$. □

3.3 Automorphisms of $D \otimes \mathbb{K}$

The automorphism group of a separable C^* -algebra, equipped with the point-norm topology, is a separable and metrisable topological group. If D is strongly self-absorbing, then $\mathrm{Aut}(D)$ is contractible, and $\mathrm{Aut}(D \otimes \mathbb{K})$ is well-pointed and has the homotopy type of a CW-complex [7]. In this section we extend the non-equivariant results in [7] and the \mathbb{T} -equivariant results in [14] to the $\mathbb{Z}/p\mathbb{Z}$ -equivariant setting. This allows us to determine the $\mathbb{Z}/p\mathbb{Z}$ -equivariant homotopy type of $\mathrm{Aut}(D \otimes \mathbb{K})$, which we show to be an equivariant infinite loop space using the machinery from Section 3.1.

3.3.1 Infinite loop space structure

The C^* -algebra $D = \text{End}(V)^\infty$ is unital, separable and strongly self-absorbing. It has been shown in [7] that the automorphism group of the stabilisation of such algebras carries an infinite loop space structure. In order to extend this to the equivariant case we consider $D \otimes \mathbb{K}$ as a G - C^* -algebra for a finite group G (which we will later choose to be $\mathbb{Z}/p\mathbb{Z}$) with the action on D defined in (3.3).

Let $\mathcal{G}_D(\mathbf{n}) = \text{Aut}((D \otimes \mathbb{K})^{\otimes n})$ (with $(D \otimes \mathbb{K})^{\otimes 0} := \mathbb{C}$, so that $\mathcal{G}_D(\mathbf{0})$ is the trivial group). The functor \mathcal{G}_D is defined on a morphism $\alpha: \mathbf{m} \rightarrow \mathbf{n}$ in \mathcal{I} as $\mathcal{G}_D(\alpha)(g) = \bar{\alpha} \circ (\text{id}_{(D \otimes \mathbb{K})^{\otimes(n-m)}} \otimes g) \circ \bar{\alpha}^{-1}$, with $g \in \mathcal{G}_D(\mathbf{m})$ and $\bar{\alpha}$ as explained at the beginning of Section 3.2.2. The tensor product equips $\mathcal{G}_D(\mathbf{n})$ with the structure of a commutative \mathcal{I} - G -monoid, via

$$\mu_{m,n} : \mathcal{G}_D(\mathbf{m}) \times \mathcal{G}_D(\mathbf{n}) \rightarrow \mathcal{G}_D(\mathbf{m} \sqcup \mathbf{n}), \quad (g, h) \mapsto g \otimes h.$$

Note that in this case $\mathcal{G}_D(\mathbf{n})$ has both a group multiplication and a monoid structure and they satisfy the Eckmann-Hilton condition from [6, Def. 3.1]. It will prove useful to have a compatibility condition between the two.

Definition 3.3.1. Let X be a \mathcal{I} - G -monoid such that $X(\mathbf{n})$ is also a group for each \mathbf{n} . Let $\iota_m : \mathbf{0} \rightarrow \mathbf{m}$ be the unique such morphism in \mathcal{I} . We say that X has *compatible inverses* if there is a path from $(\iota_m \sqcup \text{id}_{\mathbf{m}})_*(g) \in X(\mathbf{m} \sqcup \mathbf{m})$ to $(\text{id}_{\mathbf{m}} \sqcup \iota_m)_*(g)$ for all m and $g \in X(\mathbf{m})$.

If X has compatible inverses, then for each $g \in X(\mathbf{m})$ there is a path connecting $\mu_{m,m}(g, g^{-1}) \in X(\mathbf{m} \sqcup \mathbf{m})$ and $1_{m \sqcup m} \in X(\mathbf{m} \sqcup \mathbf{m})$ [6, Lem. 3.3].

From now on we restrict to $G = \mathbb{Z}/p\mathbb{Z}$ for a prime $p \in \mathbb{N}$.

Lemma 3.3.2. \mathcal{G}_D is a stable commutative \mathcal{I} - $\mathbb{Z}/p\mathbb{Z}$ -monoid with compatible inverses.

Proof. See [6, Thm. 4.5] for $H = \{e\}$. The proof for $H = \mathbb{Z}/p\mathbb{Z}$ is an easy adaptation of [14, Lem. 4.2] where this is proven for $H = \mathbb{T}$. \square

Recall we can always regard an \mathcal{I} - G -monoid as an \mathcal{I} - O_G -monoid by taking fixed points, since this preserves monoidality. By slight abuse of notation we will refer to the latter again as \mathcal{G}_D in Section 3.4.2, where we will compare it with the \mathcal{I} - O_G -space of units.

3.3.2 Equivariant homotopy type of $\text{Aut}(D \otimes \mathbb{K})$

Let $G = \mathbb{Z}/p\mathbb{Z}$. Since the G - C^* -algebra D defined in (3.3) is an infinite tensor product of a single representation, one can compute its equivariant K -theory using continuity. Note that the map $T \mapsto T \otimes 1_D$ induces multiplication by V on K -theory. Therefore,

$$K_0^G(\mathbb{C}) \cong K_0^G(\text{End}(V)^{\otimes n}) \cong \mathbb{Z}[t]/(t^p - 1) =: R \quad \text{and} \quad K_0^G(D) \cong R[p_V^{-1}],$$

where $p_V \in R$ is the polynomial corresponding to the G -representation V . The group $K_0^G(D)$ is in fact a ring. But it also has the structure of an ordered group. The positive cone of $K_0^G(D)$ corresponds under this isomorphism to $R[p_V^{-1}]_+$, which is defined for $q \in R$ and $k \in \mathbb{N}_0$ by

$$\frac{q}{p_V^k} \in R[p_V^{-1}]_+ \Leftrightarrow q \cdot p_V^l \text{ has non-negative coefficients for some } l \in \mathbb{N}_0.$$

Taking this order structure into account we need to consider the subgroup of positive units: Let $R_+ \subset R$ be the semiring of positive elements contained in a commutative unital ring R with an ordered group structure and define

$$GL_1(R)_+ = \{x \in GL_1(R) \cap R_+ \mid x^{-1} \in R_+\},$$

i.e. those elements of R_+ which are invertible as elements of R and whose inverse is in R_+ (see Remark 3.3.6 for the reason why this definition is different from the one in [14, p. 17]).

As we have already seen above, some of the results from [14] are easily adapted to the subgroup $\mathbb{Z}/p\mathbb{Z} \leq \mathbb{T}$. For example, determining the equivariant homotopy type of $\text{Aut}(D \otimes \mathbb{K})$ can be achieved using the same approach as in [14].

Lemma 3.3.3. *Let $e \in \mathbb{K}$ be a projection of rank one that is fixed by the $\mathbb{Z}/p\mathbb{Z}$ -action. There are two continuous paths*

$$\begin{aligned}\gamma &: [0, 1] \rightarrow \text{hom}_{\mathbb{Z}/p\mathbb{Z}}(D, D \otimes D), \\ \beta &: [0, 1] \rightarrow \text{hom}_{\mathbb{Z}/p\mathbb{Z}}(D \otimes \mathbb{K}, (D \otimes \mathbb{K})^{\otimes 2})\end{aligned}$$

(where the equivariant homomorphisms are equipped with the point-norm topology) with the following properties:

- i) $\gamma(0)(d) = d \otimes 1_D$, $\gamma(1)(d) = 1_D \otimes d$, $\beta(0)(a) = a \otimes (1_D \otimes e)$ and $\beta(1)(a) = (1_D \otimes e) \otimes a$,
- ii) γ restricts to $(0, 1) \rightarrow \text{Iso}_{\mathbb{Z}/p\mathbb{Z}}(D, D \otimes D)$ and β restricts to $(0, 1) \rightarrow \text{Iso}_{\mathbb{Z}/p\mathbb{Z}}(D \otimes \mathbb{K}, (D \otimes \mathbb{K})^{\otimes 2})$,
- iii) $\beta(t)(1_D \otimes e) = (1_D \otimes e)^{\otimes 2}$ for all $t \in [0, 1]$.

Proof. The path γ is the result of patching together two paths that are constructed exactly as in [14, Lem. 2.3] (see also [14, p. 11]). Since the analysis in [14, Thm. 2.4] carries over to $\mathbb{Z}/p\mathbb{Z}$, the stabiliser group $\text{Aut}_{\mathbb{Z}/p\mathbb{Z}, e}(\mathbb{K})$ is contractible in the point-norm topology (so in particular path-connected). Therefore the argument in [14, Lem. 2.5] works for $\mathbb{Z}/p\mathbb{Z}$ as well giving a continuous path

$$\bar{\beta}: [0, 1] \rightarrow \text{hom}_{\mathbb{Z}/p\mathbb{Z}}(\mathbb{K}, \mathbb{K} \otimes \mathbb{K})$$

with properties analogous to the ones listed in the lemma. The path β is then obtained by combining γ and $\bar{\beta}$. □

Corollary 3.3.4. *Let $e \in \mathbb{K}$ be a projection of rank one fixed by $\mathbb{Z}/p\mathbb{Z}$. The group $\text{Aut}_{\mathbb{Z}/p\mathbb{Z}}(D)$ and the stabiliser subgroup $\text{Aut}_{\mathbb{Z}/p\mathbb{Z}, 1_D \otimes e}(D \otimes \mathbb{K})$ fixing the projection $1_D \otimes e$ are both contractible.*

Proof. This is proven exactly as in [14, Thm. 2.6]. The contracting homotopies are constructed from the paths γ and β , respectively, given in Lemma 3.3.3. \square

The above corollary allows us to determine the equivariant homotopy type of $\text{Aut}(D \otimes \mathbb{K})$ using the evaluation map

$$\text{Aut}_H(D \otimes \mathbb{K}) \rightarrow \text{Proj}((D \otimes \mathbb{K})^H), \quad \beta \mapsto \beta(1_D \otimes e) \quad (3.6)$$

suitably restricted in the codomain for $H \leq \mathbb{Z}/p\mathbb{Z}$. As in [14, Lem. 2.8] Rieffel's criterion applies to the $\mathbb{Z}/p\mathbb{Z}$ -action on \mathbb{K} and since colimits of saturated actions are saturated, it follows that the inclusion map induces an isomorphism

$$K_0((D \otimes \mathbb{K})^{\mathbb{Z}/p\mathbb{Z}}) \rightarrow K_0^{\mathbb{Z}/p\mathbb{Z}}(D \otimes \mathbb{K}) \cong K_0^{\mathbb{Z}/p\mathbb{Z}}(D).$$

By stability we also have $K_0(D \otimes \mathbb{K}) \cong K_0(D)$. Combining (3.6) with the natural map $\pi_0(\text{Proj}((D \otimes \mathbb{K})^H)) \rightarrow K_0((D \otimes \mathbb{K})^H)$ and the above isomorphisms therefore gives a map

$$\kappa: \pi_0(\text{Aut}_H(D \otimes \mathbb{K})) \rightarrow K_0^H(D).$$

Lemma 3.3.5. *The map κ restricts to a group isomorphism*

$$\pi_0(\text{Aut}_H(D \otimes \mathbb{K})) \xrightarrow{\cong} GL_1(K_0^H(D))_+.$$

Proof. The case $H = \{e\}$ is proven in [7, Cor. 2.17]. That κ is multiplicative in the case $H = \mathbb{Z}/p\mathbb{Z}$ is shown as in the proof of [14, Lem. 2.9] and is based on the paths constructed in Lemma 3.3.3. The argument also proves that $[\alpha^{-1}] \in \pi_0(\text{Aut}_{\mathbb{Z}/p\mathbb{Z}}(D \otimes \mathbb{K}))$ is the inverse of $[\alpha]$. Hence, $\kappa([\alpha^{-1}])$ is the inverse of $\kappa([\alpha])$ and in particular positive. Thus, the restriction gives a well-defined group homomorphism to $GL_1(K_0^H(D))_+$.

Showing its injectivity can be done using the same argument as in the proof of [14, Lem. 2.9]. For reasons that will become clear in Remark 3.3.6 we spell out the proof of surjectivity explicitly. Recall that

$$K_0^{\mathbb{Z}/p\mathbb{Z}}(\mathbb{C}) \cong \mathbb{Z}[t]/(t^p - 1) =: R \quad \text{and} \quad K_0^{\mathbb{Z}/p\mathbb{Z}}(D) \cong R[p_V^{-1}],$$

where $p_V(t)$ denotes the polynomial corresponding to the defining representation of $D = \text{End}(V)^{\otimes \infty}$. Let $q \in GL_1(R[p_V^{-1}])_+$. Then

$$q = \frac{r}{p_V^k}$$

for some $r \in R_+$, $k \in \mathbb{N}_0$ and by our definition of $GL_1(R[p_V^{-1}])_+$ there is $s \in R_+$ such that $r \cdot s = p_V^l$ for some $l \in \mathbb{N}_0$. Since r and s are both positive, this corresponds to a tensor product decomposition of the $\mathbb{Z}/p\mathbb{Z}$ -representation $V^{\otimes l}$, i.e. we have two representations V_r and V_s corresponding to r and s , respectively, such that

$$V_r \otimes V_s \cong V^{\otimes l} \quad \text{and} \quad \text{End}(V_r) \otimes \text{End}(V_s) \cong \text{End}(V)^{\otimes l}. \quad (3.7)$$

First assume that V_s contains a copy of the trivial representation. Then V_r is a subrepresentation of $V^{\otimes l}$. Observe that because of (3.7) the G - C^* -algebra D tensorially absorbs $\text{End}(V^r)$. Let

$$\psi_r: \text{End}(V_r) \otimes D \cong \text{End}(V_r) \otimes (\text{End}(V_r) \otimes \text{End}(V_s))^{\otimes \infty} \rightarrow D.$$

be the equivariant $*$ -isomorphism that shifts the tensor factors and identifies the result with D using (3.7). Since $V_r \otimes H \cong H$ for $H = \ell^2(\mathbb{Z}/p\mathbb{Z}) \otimes H_0$, we can also choose an isomorphism

$$\phi_r: \mathbb{K}(H) \otimes \text{End}(V_r) \rightarrow \mathbb{K}(H \otimes V_r) \rightarrow \mathbb{K}(H).$$

The composition $(\phi_r \otimes \text{id}_D) \circ (\text{id}_{\mathbb{K}} \otimes \psi_r^{-1})$ gives $\alpha_r \in \text{Aut}_{\mathbb{Z}/p\mathbb{Z}}(D \otimes \mathbb{K})$ after swapping D and \mathbb{K} . By construction, $\alpha_r(1_D \otimes e)$ is an invariant projection, whose image is the subrepresentation V_r , i.e. $[\alpha_r(1_D \otimes e)] = r \in R_+$. Similarly, we can construct $\alpha_{p_V} \in \text{Aut}_{\mathbb{Z}/p\mathbb{Z}}(D \otimes \mathbb{K})$ with $[\alpha_{p_V}(1_D \otimes e)] = p_V$. Since κ is a group homomorphism, we have $\kappa([\alpha_{p_V}^{-1}]^k \circ \alpha_r) = q$.

If V_s does not contain the trivial representation, then there is $m \in \mathbb{Z}/p\mathbb{Z}$ such that $\mathbb{C}_m \otimes V_s$ does, where \mathbb{C}_m is the character corresponding to m . The above argument then shows that $t^{-m}q(t)$ is represented by an automorphism β . As described in the proof of [14, Lem. 2.9] there is an equivariant automorphism $\alpha_t \in \text{Aut}_{\mathbb{Z}/p\mathbb{Z}}(\mathbb{K})$ with the property $[\alpha_t(e)] = t \in R_+$ and we have $\kappa([\text{id}_D \otimes \alpha_t^m] \circ \beta) = q$. \square

Remark 3.3.6. There is a small mistake in the proof of [14, Lem. 2.9]: The definition of $GL_1(R_+)$, the surjectivity part of the proof of the lemma and the computation of the generators of $GL_1(R_+)$ are actually only correct under the additional assumption that the prime factors of p_V are positive. This is not automatic as the example

$$p_V(t) = 1 + t^3 = (1 - t + t^2)(1 + t)$$

shows. The above proof of surjectivity also works in the case $G = \mathbb{T}$ and fixes this gap. Note that there is another reason why the proof of [14, Lem. 2.9] does not carry over verbatim: The ring $\mathbb{Z}[t]/(t^p - 1)$ contains zero divisors and is therefore not a unique factorization domain.

Proposition 3.3.7. *There is a homotopy equivalence*

$$\mathrm{Proj}_{1 \otimes e}((D \otimes \mathbb{K})^H) \simeq BU(D^H).$$

Proof. See [7, Cor. 2.9] for $H = \{e\}$. The proof of the case $H = \mathbb{Z}/p\mathbb{Z}$ is the same as the one of [14, Prop. 3.2]. \square

Remark 3.3.8. Let V be a right Hilbert A -module and note Hilbert A -module bundles with fibre isomorphic to V over X are up to isomorphism in bijection with $[X, BU(V)]$, where $U(V)$ denotes the unitary group of V . This observation allows us to reinterpret Proposition 3.3.7 in terms of a statement about right Hilbert $D^{\mathbb{Z}/p\mathbb{Z}}$ -module bundles. In fact, the proof works by constructing a principal $U(D^{\mathbb{Z}/p\mathbb{Z}})$ -bundle

$$EU(D^{\mathbb{Z}/p\mathbb{Z}}) = U(M((D \otimes \mathbb{K})^{\mathbb{Z}/p\mathbb{Z}})) / (U((1 - p_0)M((D \otimes \mathbb{K})^{\mathbb{Z}/p\mathbb{Z}})(1 - p_0)))$$

over $\mathrm{Proj}_{1 \otimes e}((D \otimes \mathbb{K})^{\mathbb{Z}/p\mathbb{Z}})$ (where we have set $p_0 = e \otimes 1 \in H_G \otimes D^{\mathbb{Z}/p\mathbb{Z}}$) and showing that it is contractible. The bundle $EU(D^{\mathbb{Z}/p\mathbb{Z}})$ defines in turn a right Hilbert $D^{\mathbb{Z}/p\mathbb{Z}}$ -module bundle

$$\mathcal{H}_{\mathrm{univ}} = EU(D^{\mathbb{Z}/p\mathbb{Z}}) \times_{U(D^{\mathbb{Z}/p\mathbb{Z}})} D^{\mathbb{Z}/p\mathbb{Z}}$$

over $\text{Proj}_{p_0}((D \otimes \mathbb{K})^{\mathbb{Z}/p\mathbb{Z}})$. We can also define naturally another right Hilbert $D^{\mathbb{Z}/p\mathbb{Z}}$ -module bundle over the same base space, namely

$$\tilde{\mathcal{H}}_{\text{univ}} = \{(p, \xi) \in \text{Proj}_{p_0}((D \otimes \mathbb{K})^{\mathbb{Z}/p\mathbb{Z}}) \times (H_G \otimes D^{\mathbb{Z}/p\mathbb{Z}}) : p \cdot \xi = \xi\}.$$

Then the map

$$\alpha : \mathcal{H}_{\text{univ}} \rightarrow \tilde{\mathcal{H}}_{\text{univ}}$$

defined by $\alpha([u, b]) = (up_0u^*, u \cdot (e_0 \otimes b))$, where $e_0 \in H$ is a unit vector in the image of p_0 , is a morphism of right Hilbert $D^{\mathbb{Z}/p\mathbb{Z}}$ -module bundles, with the properties of being an isomorphism on the fibres and making the triangle

$$\begin{array}{ccc} \mathcal{H}_{\text{univ}} & \xrightarrow{\alpha} & \tilde{\mathcal{H}}_{\text{univ}} \\ & \searrow & \swarrow \\ & \text{Proj}_{p_0}((D \otimes K)^{\mathbb{Z}/p\mathbb{Z}}) & \end{array}$$

commute. Thus it is an isomorphism of right Hilbert $D^{\mathbb{Z}/p\mathbb{Z}}$ -module bundles.

Proposition 3.3.9. *There is a homotopy equivalence*

$$\text{Aut}_{H,0}(D \otimes \mathbb{K}) \simeq \text{Proj}_{1 \otimes e}((D \otimes \mathbb{K})^H)$$

for $H \leq \mathbb{Z}/p\mathbb{Z}$.

Proof. The proof is based on the observation that the evaluation map

$$\text{Aut}_{H,0}(D \otimes \mathbb{K}) \rightarrow \text{Proj}_{1 \otimes e}((D \otimes \mathbb{K})^H), \quad \beta \mapsto \beta(1 \otimes e)$$

is a principal $\text{Aut}_{\mathbb{Z}/p\mathbb{Z}, 1 \otimes e}(D \otimes \mathbb{K})$ -bundle with contractible fibre by Corollary 3.3.4. For further details see [7, Cor. 2.9] for $H = \{e\}$, and [14, Prop. 3.3] where this is discussed for the circle group \mathbb{T} with a proof that carries over to the case $H = \mathbb{Z}/p\mathbb{Z}$. □

The following elementary consideration will be needed in order to compute the K_0 -group of the fixed point algebra of D .

Lemma 3.3.10. *Let V be a $\mathbb{Z}/p\mathbb{Z}$ -representation which contains at least two non-isomorphic irreducible subrepresentations, and let $p_V = \sum_{i=0}^{p-1} a_i t^i$ be the associated character polynomial. Then there exists $N \in \mathbb{N}$ such that the N -th power*

$$p_V^N = \sum_{j=0}^{p-1} b_j t^j$$

has coefficients $b_j \geq 2$ for all $j \in \{0, \dots, p-1\}$.

Proof. We know that the character polynomial $p_V(t) = \sum_{i=0}^d a_i t^i \in \frac{\mathbb{Z}[t]}{(t^p-1)}$ has at least two nonzero coefficients. First, assume one of them is a_1 and let the other one be a_l , $l \in \{0, 2, \dots, p-1\}$. By taking the square p_V^2 we get $a_2 > 0$ and the coefficient of t^{l+1} increases by at least 2, and by iterating the process we get $N \in \mathbb{N}$ satisfying the claim.

If $a_1 = 0$, by assumption there exists $k \in \{2, \dots, p-1\}$ such that $a_k \neq 0$. Let r be the multiplicative inverse of $k \bmod p$. Then the first order coefficient of p_V^r is nonzero and the above argument applies. \square

Remark 3.3.11. The above lemma is actually the only reason why we work with a prime number p . In fact the results in this chapter hold for cyclic groups of any order, provided that a suitable representation V (i.e., such that p_V satisfies the claim of Lemma 3.3.10) is chosen.

Lemma 3.3.12. *The K -groups of the fixed-point algebra $D^{\mathbb{Z}/p\mathbb{Z}}$ are given by*

$$K_0(D^{\mathbb{Z}/p\mathbb{Z}}) \cong R(\mathbb{Z}/p\mathbb{Z})[p_V^{-1}] \cong K_0^{\mathbb{Z}/p\mathbb{Z}}(D) \text{ and } K_1(D^{\mathbb{Z}/p\mathbb{Z}}) = 0.$$

Proof. The proof works similarly as in [14, Lem. 3.5]. First note that because the group acts on each factor separately, the fixed-point algebra $D^{\mathbb{Z}/p\mathbb{Z}}$ is isomorphic to the direct limit

$$\text{End}(V)^{\mathbb{Z}/p\mathbb{Z}} \rightarrow \dots \rightarrow (\text{End}(V)^{\otimes n})^{\mathbb{Z}/p\mathbb{Z}} \rightarrow (\text{End}(V)^{\otimes (n+1)})^{\mathbb{Z}/p\mathbb{Z}} \rightarrow \dots,$$

where the connecting $*$ -homomorphisms are given by $T \mapsto T \otimes 1$. For each factor we have

$$(\text{End}(V)^{\otimes n})^{\mathbb{Z}/p\mathbb{Z}} \cong \text{hom}_{\mathbb{Z}/p\mathbb{Z}}(V^{\otimes n}, V^{\otimes n}),$$

which is in turn isomorphic to a direct sum of matrix algebras. Combining the decomposition of V into character subspaces

$$V = \bigoplus_{k \in \mathbb{Z}} V_k,$$

where $V_k = \{v \in V \mid \rho(z)v = z^k v \ \forall z \in \mathbb{Z}/p\mathbb{Z}\}$, with Schur's lemma this turns out to be

$$\bigoplus_{k \in \mathbb{Z}} \text{hom}_{\mathbb{Z}/p\mathbb{Z}}(V_k, V_k) \cong \bigoplus_i M_{n_i}(\mathbb{C}) \otimes \text{hom}_{\mathbb{Z}/p\mathbb{Z}}(\mathbb{C}_{m_i}, \mathbb{C}_{m_i}) \cong \bigoplus_i M_{n_i}(\mathbb{C})$$

where $V_{m_i} \cong \mathbb{C}^{n_i} \otimes \mathbb{C}_{m_i}$. This implies that $K_0(\text{End}(V)^{\otimes n})^{\mathbb{Z}/p\mathbb{Z}}$ is the free abelian group generated by the irreducible subrepresentations of $V^{\otimes n}$. If we apply K_0 to each term in the direct limit above and identify a representation with its character polynomial, the connecting isomorphism becomes multiplication by p_V . Moreover, Lemma 3.3.10 implies in particular that $p_V^{\mathbb{N}}$ contains all powers of t with positive coefficients. The claim for K_0 follows by combining these two considerations and continuity of K -theory. Since $D^{\mathbb{Z}/p\mathbb{Z}}$ is an AF-algebra we also have $K_1(D^{\mathbb{Z}/p\mathbb{Z}}) = 0$.

For equivariant K -theory, note that

$$K_0^{\mathbb{Z}/p\mathbb{Z}}(\text{End}(V)^{\otimes n}) \cong K_0^{\mathbb{Z}/p\mathbb{Z}}(\mathbb{C}) \cong R(\mathbb{Z}/p\mathbb{Z})$$

by stability, and then again by continuity $K_0^{\mathbb{Z}/p\mathbb{Z}}(D) \cong R(\mathbb{Z}/p\mathbb{Z})[p_V^{-1}]$. The description as a direct limit also shows that this is in fact a ring isomorphism. \square

We can now compute the homotopy groups of $\text{Aut}_{\mathbb{Z}/p\mathbb{Z}}(D \otimes \mathbb{K})$. By Proposition 3.3.7 and Proposition 3.3.9, this boils down to computing the ones of $U(D^{\mathbb{Z}/p\mathbb{Z}})$. Here the fact that the representation ring of $\mathbb{Z}/p\mathbb{Z}$ has finite rank over \mathbb{Z} plays a fundamental role.

Theorem 3.3.13. *We have $\pi_{2k}(U(D^{\mathbb{Z}/p\mathbb{Z}})) = 0$ and*

$$\pi_{2k+1}(U(D^{\mathbb{Z}/p\mathbb{Z}})) \cong R(\mathbb{Z}/p\mathbb{Z})[p_V^{-1}] \cong K_0^{\mathbb{Z}/p\mathbb{Z}}(D)$$

for $k \in \mathbb{N}_0$.

Proof. The proof works similarly as in [14, Thm. 3.7]. Let $n \in \mathbb{N}_0$. The group $\pi_n(U(D^{\mathbb{Z}/p\mathbb{Z}}))$ is the direct limit of the sequence

$$\cdots \rightarrow \pi_n(U(\text{End}(V^{\otimes k})^{\mathbb{Z}/p\mathbb{Z}})) \rightarrow \pi_n(U(\text{End}(V^{\otimes k+1})^{\mathbb{Z}/p\mathbb{Z}})) \rightarrow \cdots$$

where the connecting homomorphism is induced by the map $T \mapsto T \otimes 1$.

Consider the representing polynomial for V

$$p_V = \sum_{i=0}^{p-1} a_i t^i.$$

The coefficients a_i can be interpreted as the multiplicities of the character subspaces V_i of V , i.e., $a_i = \dim(V_i)$. Hence, if

$$p_V^k = \sum_{j=0}^{p-1} b_j t^j,$$

then by Schur's lemma (compare Lemma 3.3.12)

$$\text{End}(V^{\otimes k})^{\mathbb{Z}/p\mathbb{Z}} \cong \bigoplus_{j=0}^{p-1} M_{b_j}(\mathbb{C}) \otimes \text{End}(W_j),$$

where $W_j \subset V^{\otimes k}$ are the irreducible subrepresentations, which appear with multiplicity b_j . Hence,

$$U(\text{End}(V^{\otimes k})^{\mathbb{Z}/p\mathbb{Z}}) \cong \prod_{j=0}^{p-1} U(b_j).$$

By Lemma 3.3.10 there exists $N \in \mathbb{N}$ such that p_V^N has all coefficients ≥ 2 . Since the sequence

$$\text{End}(V^{\otimes 0})^{\mathbb{Z}/p\mathbb{Z}} \rightarrow \text{End}(V^{\otimes N})^{\mathbb{Z}/p\mathbb{Z}} \rightarrow \cdots \rightarrow \text{End}(V^{\otimes mN})^{\mathbb{Z}/p\mathbb{Z}} \rightarrow \cdots$$

also has direct limit $D^{\mathbb{Z}/p\mathbb{Z}}$, we can work with $V^{\otimes N}$ instead of V and assume without loss of generality that p_V has all coefficients ≥ 2 . It is easily seen that

the coefficients b_j of p_V^k then satisfy $b_j > k$ for $k \in \mathbb{N}$. This implies that for $k > \frac{n}{2}$ the unitary groups $U(b_j)$ will all have dimension $b_j > \frac{n}{2}$. Now let U_∞ be the colimit over the inclusions $U(n) \hookrightarrow U(n+1)$ that add a 1 in the lower right corner. In the above situation n falls into the stable range for the homotopy groups, which implies that $\pi_n(U(b_j)) \cong \pi_n(U_\infty)$, which is isomorphic to \mathbb{Z} if n is odd and vanishes if n is even.

Therefore for $k > \frac{n}{2}$ each term in the sequence which computes the group $\pi_n(U(D^{\mathbb{Z}/p\mathbb{Z}}))$ is isomorphic to the direct sum of one copy of \mathbb{Z} for every irreducible $\mathbb{Z}/p\mathbb{Z}$ -representation, i.e., is isomorphic to the representation ring $R(\mathbb{Z}/p\mathbb{Z})$ as $R(\mathbb{Z}/p\mathbb{Z})$ -modules. Since the connecting homomorphism is given by multiplication by p_V , the result follows after taking the direct limit. \square

Remark 3.3.14. The proof of Theorem 3.3.13 shows in particular that the natural map

$$\pi_n(U(D^{\mathbb{Z}/p\mathbb{Z}})) \rightarrow K_1(C(S^n, D^{\mathbb{Z}/p\mathbb{Z}}))$$

given by mapping the class of $\gamma : S^n \rightarrow U(D^{\mathbb{Z}/p\mathbb{Z}})$ to the class of the corresponding unitary $u_\gamma \in U(C(S^n, D^{\mathbb{Z}/p\mathbb{Z}}))$ is an isomorphism.

Corollary 3.3.15. *We have*

$$\pi_n(\text{Aut}_{\mathbb{Z}/p\mathbb{Z}}(D \otimes \mathbb{K})) \cong \begin{cases} GL_1(R(\mathbb{Z}/p\mathbb{Z})[p_V^{-1}])_+ & \text{if } n = 0, \\ R(\mathbb{Z}/p\mathbb{Z})[p_V^{-1}] & \text{if } n > 0 \text{ and } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This follows by combining Lemma 3.3.5, Proposition 3.3.7, Proposition 3.3.9, and Theorem 3.3.13. \square

Finally, we prove two technical results which we will need later but might be of independent interest.

Lemma 3.3.16. *The topological group $\text{Aut}(D \otimes \mathbb{K})$ equipped with the conjugation action of G has the G -homotopy type of a G -CW-complex.*

Proof. Pick a projection $e \in \mathbb{K}$ whose image is the one-dimensional trivial G -representation. Let $q_0 = 1 \otimes e \in D \otimes \mathbb{K}$. Let $P_D: O_G^{\text{op}} \rightarrow \mathcal{T}op$ be the O_G -space defined by

$$P_D(G/H) = \text{Proj}_{I_H}((D \otimes \mathbb{K})^H),$$

where $I_H = GL_1(K_0^H(D)_+)$. Let $A_D: O_G^{\text{op}} \rightarrow \mathcal{T}op$ be the O_G -space associated to the G -space $\text{Aut}(D \otimes \mathbb{K})$ equipped with the conjugation action, i.e.

$$A_D(G/H) = (\text{Aut}(D \otimes \mathbb{K}))^H = \text{Aut}_H(D \otimes \mathbb{K}).$$

Let $\pi: A_D \rightarrow P_D$ be the map of O_G -spaces given by $\pi(\beta) = \beta(q_0)$ and note that π provides a homotopy equivalence $A_D(G/H) \rightarrow P_D(G/H)$ for each subgroup $H \subseteq G$. For $H = \{e\}$ this is proven in [7, Lem. 2.16 and Thm. 2.5], for $H = \mathbb{Z}/p\mathbb{Z}$ the proofs of [14, Prop. 3.1, Lem. 2.9 and Thm. 2.6] for circle actions carry over to the case of $\mathbb{Z}/p\mathbb{Z}$.

Let $\Phi: \text{Fun}(O_G^{\text{op}}, \mathcal{T}op) \rightarrow G\mathcal{T}op$ be the homotopy inverse of the fixed-point functor defined in (2.1). Note that $\Phi(X)$ can be written as the geometric realisation of a simplicial space obtained from the bar construction. The fact that it provides a homotopy inverse is witnessed by a simplicial homotopy equivalence. If we therefore use the fat geometric realisation (i.e. ignoring the degeneracy maps), the resulting Φ is still a homotopy inverse. Moreover, π provides a level-wise G -homotopy equivalence of simplicial spaces

$$B_\bullet(A_D, O_G, M) \rightarrow B_\bullet(P_D, O_G, M).$$

The fat geometric realisation turns this into a G -homotopy equivalence. This is well-known in the non-equivariant setting (see [50, Prop. A.1 (ii)]) and can be deduced from the fact that pushouts along cofibrations preserve homotopy equivalences and that colimits over cofibrations are homotopy invariant. Both of these statements are still true in the G -equivariant setting (see [58, Thm. 1.1 and Thm. 1.2]). Each of the spaces $P_D(G/H)$ has the homotopy type of a CW-complex by [7, Lem. 2.7]. Therefore $\Phi(P_D)$ has the homotopy type of a G -CW-complex by

[34, Prop. 13.2] and we have the following sequence of G -homotopy equivalences

$$\mathrm{Aut}(D \otimes \mathbb{K}) \xleftarrow{\cong} \Phi(A_D) \xrightarrow{\cong} \Phi(P_D). \quad \square$$

Lemma 3.3.17. *The topological group $\mathrm{Aut}(D \otimes \mathbb{K})$ equipped with the conjugation action of G is equivariantly well-pointed in the sense that the inclusion $\{\mathrm{id}\} \rightarrow \mathrm{Aut}(D \otimes \mathbb{K})$ is a G -cofibration.*

Proof. By a straightforward G -equivariant generalisation of [51, Thm. 2] (see also [18, Lem. 2.3]) it suffices to construct a G -invariant continuous map $v: \mathrm{Aut}_0(D \otimes \mathbb{K}) \rightarrow [0, 1]$ such that

1. $v^{-1}(0) = \{\mathrm{id}\}$,
2. $U_v = v^{-1}([0, 1]) = \{\beta \in \mathrm{Aut}_0(D \otimes \mathbb{K}) \mid v(\beta) < 1\}$ deformation retracts G -equivariantly to id .

Pick an arbitrary countable dense subset $S = \{a_k \in D \otimes \mathbb{K} \mid k \in \mathbb{N}\}$. The point-norm topology on $\mathrm{Aut}(D \otimes \mathbb{K})$ is the metric topology associated to the metric

$$d(\beta_1, \beta_2) = \sum_{k \in \mathbb{N}} \sum_{g \in G} \frac{\|\beta_1(\alpha_g(a_k)) - \beta_2(\alpha_g(a_k))\|}{2^k \|\alpha_g(a_k)\|}.$$

Since each α_g gives a bijection $S \rightarrow S$ and $\|\alpha_g(b)\| = \|b\|$ for all $b \in D \otimes \mathbb{K}$, this metric is G -invariant with respect to the conjugation action. Let $q_0 = 1 \otimes e \in D \otimes \mathbb{K}$ be as in the proof of Lemma 3.3.16 and let

$$\Xi: \mathrm{Aut}(D \otimes \mathbb{K}) \rightarrow \mathrm{Proj}_I(D \otimes \mathbb{K}), \quad \beta \mapsto \beta(q_0)$$

where $I \subseteq \pi_0(\mathrm{Proj}(D \otimes \mathbb{K}))$ is the image of Ξ_* on π_0 . We may now proceed as in [7, Prop. 2.26]. Let

$$v(\beta) = \max\{\min\{d(\beta, \mathrm{id}), \frac{1}{2}\}, \min\{1, 2\|\beta(q_0) - q_0\|\}\},$$

which is G -invariant, continuous and satisfies $v^{-1}(0) = \{\mathrm{id}\}$. Note that

$$U_v = \Xi^{-1}(W) \quad \text{with} \quad W = \{q \in \mathrm{Proj}_{q_0}(D \otimes \mathbb{K}) \mid \|q - q_0\| < \frac{1}{2}\}.$$

By [4, Prop. II.3.3.4] there is a continuous map $u_{q_0}: W \rightarrow U(M(D \otimes \mathbb{K}))$ with the properties

- (i) $u_{q_0}(q) q_0 u_{q_0}(q)^* = q$,
- (ii) $g \cdot (u_{p_0}(q)) = u_{p_0}(g \cdot q)$ for all $g \in G$.

Now consider the continuous map

$$\kappa: W \times \text{Aut}_{q_0}(D \otimes \mathbb{K}) \rightarrow \Xi^{-1}(W)$$

given by $\kappa(q, \beta) = \text{Ad}_{u_{q_0}(q)} \circ \beta$. Note κ is well-defined by (i) and is G -equivariant by (ii). Define the inverse map

$$\theta: \Xi^{-1}(W) \rightarrow W \times \text{Aut}_{q_0}(D \otimes \mathbb{K})$$

by $\theta(\alpha) = (\alpha(q_0), \text{Ad}_{u_{q_0}(\alpha(q_0))^*} \circ \alpha)$. A straightforward computation shows that $\theta \circ \kappa = \text{id}$ and $\kappa \circ \theta = \text{id}$, which implies that θ is G -equivariant as well. Therefore κ is a G -equivariant homeomorphism. Hence, it suffices to prove that the domain equivariantly deformation retracts to (q_0, id) . Let χ be the characteristic function of $(\frac{1}{2}, 1]$. The G -equivariance of functional calculus and the choice of W ensure that $h(q, t) = \chi((1-t)q + tq_0)$ is a well-defined equivariant retraction of W to q_0 . The homotopy described in [14, Thm. 2.6] uses the path [14, eqn. (9)]

$$\gamma: [0, 1] \rightarrow \text{hom}_G(D \otimes \mathbb{K}, D \otimes \mathbb{K} \otimes D \otimes \mathbb{K})$$

connecting $a \otimes q_0$ to $q_0 \otimes a$ through isomorphisms in the interior of the interval. This carries over to $G = \mathbb{Z}/p\mathbb{Z}$ and shows that $\text{Aut}_{q_0}(D \otimes \mathbb{K})$ equivariantly deformation retracts to id . Combining both retractions gives the result. \square

3.4 Bundles of UHF-algebras with a $\mathbb{Z}/p\mathbb{Z}$ -action

In this section we will take a closer look at equivariant bundles of C^* -algebras with fibre given by the $\mathbb{Z}/p\mathbb{Z}$ - C^* -algebra $D \otimes \mathbb{K}$. We will assume that they are

equivariantly locally trivial in a “strong” sense (Definition 3.4.1). As in the non-equivariant case one can construct a classifying space for such bundles and show that it is G -equivalent to the Γ - G -space delooping $B_{\otimes} \text{Aut}(D \otimes \mathbb{K}) := (EH_{\mathcal{T}}\mathcal{G}_D)_1$. In particular, this means that $\mathbb{Z}/p\mathbb{Z}$ -equivariant $D \otimes \mathbb{K}$ -bundles are classified by the first group of the cohomology theory defined by $EH_{\mathcal{T}}\mathcal{G}_D$, which we are able to identify with the positive units of KU^D -theory in Theorem 3.4.7.

3.4.1 The classifying space of equivariant

Aut($D \otimes \mathbb{K}$)-bundles

Definition 3.4.1. Let G be a finite group. Let A be a G - C^* -algebra and denote the action by α . Let X be a topological G -space and let $\pi: \mathcal{A} \rightarrow X$ be a locally trivial C^* -algebra bundle with fibre A over X . We will say that \mathcal{A} is a *locally trivial (G, A, α) -bundle* (or a *locally trivial (G, A) -bundle* if the action is clear) if

- a) G acts from the left on the total space \mathcal{A} in such a way that π is G -equivariant,
- b) for every $x \in X$ there exists a G -invariant open neighbourhood $U \subseteq X$ and a G -equivariant homeomorphism

$$\varphi_U: U \times A \rightarrow \mathcal{A}|_U,$$

where $\mathcal{A}|_U = \pi^{-1}(U)$ and G acts on $U \times A$ via $g \cdot (x, a) = (g \cdot x, \alpha_g(a))$.

To save space we will drop the adjective “locally trivial” from the notation in the following and instead mention the cases explicitly where we do not assume it. By elementary bundle theory each C^* -algebra bundle $\mathcal{A} \rightarrow X$ with fibre A gives rise to a principal $\text{Aut}(A)$ -bundle $\mathcal{P} \rightarrow X$ such that there is a bundle isomorphism $\mathcal{A} \cong \mathcal{P} \times_{\text{Aut}(A)} A$. In fact, this construction induces a bijection between isomorphism classes of C^* -algebra bundles with fibre A and principal $\text{Aut}(A)$ -bundles.

We outline a few details of the construction of \mathcal{P} : Let $\mathcal{A}_x = \pi^{-1}(\{x\})$. The fibre \mathcal{P}_x of \mathcal{P} over $x \in X$ is given by

$$\mathcal{P}_x = \text{Iso}(A, \mathcal{A}_x), \tag{3.8}$$

which has a canonical right $\text{Aut}(A)$ -action, and the topology on \mathcal{P} is fixed by the property that a local trivialisation $\varphi_U: U \times A \rightarrow \mathcal{A}|_U$ over U gives rise to a homeomorphism

$$U \times \text{Aut}(A) \rightarrow \mathcal{P}|_U.$$

Now assume that $\mathcal{A} \rightarrow X$ is a (G, A) -bundle. We will see that the associated principal $\text{Aut}(A)$ -bundle is equivariant in the following sense.

Definition 3.4.2. Let G be a finite group, let X be a topological space and let A be a G - C^* -algebra. A $(G, \text{Aut}(A))$ -bundle is a locally trivial principal $\text{Aut}(A)$ -bundle $\mathcal{P} \rightarrow X$ together with a left G -action by bundle maps such that each $x \in X$ has a G -invariant trivialising neighbourhood $U \subseteq X$ and a G -equivariant trivialisation

$$\mathcal{P}|_U \rightarrow U \times \text{Aut}(A)$$

where G acts from the left on $\text{Aut}(A)$ via the action on A and diagonally on $U \times \text{Aut}(A)$.

Note that $(G, \text{Aut}(A))$ -bundles are locally trivial equivariant bundles in the sense of [55, Def. 2.3] for the local object $\text{Aut}(A) \rightarrow *$.

Lemma 3.4.3. *Let G be a finite group and let A be a G - C^* -algebra. The principal $\text{Aut}(A)$ -bundle associated to a (G, A) -bundle is a $(G, \text{Aut}(A))$ -bundle. Conversely, given a $(G, \text{Aut}(A))$ -bundle $\mathcal{P} \rightarrow X$, then the associated C^* -algebra bundle is a (G, A) -bundle over X .*

Proof. The action of G on \mathcal{A} induces a corresponding action on \mathcal{P} by post-composition (see (3.8)) such that G acts by bundle maps. Fix $x \in X$. Choose a

G -invariant trivialising neighbourhood $U \subseteq X$ and a G -equivariant homeomorphism

$$\varphi_U: U \times A \rightarrow \mathcal{A}|_U.$$

The map φ_U induces a G -equivariant trivialisation

$$\psi_U: U \times \text{Aut}(A) \rightarrow \mathcal{P}|_U, \quad (y, \beta) \mapsto \varphi_{U,y} \circ \beta.$$

Conversely, assume that $\mathcal{P} \rightarrow X$ is a $(G, \text{Aut}(A))$ -bundle. A G -equivariant trivialisation $\psi_U: U \times \text{Aut}(A) \rightarrow \mathcal{P}|_U$ gives rise to a corresponding one for $\mathcal{A} = \mathcal{P} \times_{\text{Aut}(A)} A$ defined by

$$\varphi_U: U \times A \rightarrow \mathcal{A}|_U, \quad (y, a) \mapsto [\psi_U(y, \text{id}), a]. \quad \square$$

Following [30, 55] we will now construct a universal $(G, \text{Aut}(A))$ -bundle $E\text{Aut}(A) \rightarrow B\text{Aut}(A)$ over the classifying space $B\text{Aut}(A)$. Let

$$\mathcal{E}_n = \text{Aut}(A)^{n+1} \quad \text{and} \quad \mathcal{B}_n = \text{Aut}(A)^n$$

The face maps $d_i: \mathcal{E}_n \rightarrow \mathcal{E}_{n-1}$ are given by

$$d_i(\beta_0, \dots, \beta_n) = (\beta_0, \dots, \beta_{i-1} \circ \beta_i, \dots, \beta_n) \quad \text{for } i > 0$$

and $d_0(\beta_0, \dots, \beta_n) = (\beta_1, \dots, \beta_n)$. The degeneracy maps $s_i: \mathcal{E}_n \rightarrow \mathcal{E}_{n+1}$ are

$$s_i(\beta_0, \dots, \beta_n) = (\beta_0, \dots, \beta_{i-1}, \text{id}, \beta_i, \dots, \beta_n).$$

The face and degeneracy maps for \mathcal{B}_n are defined similarly except for d_n , which drops the last component instead of composing. Both of these are simplicial G -spaces with respect to the action that is given by conjugation on the first n factors of \mathcal{E}_n and \mathcal{B}_n and by the left action on the last factor of \mathcal{E}_n . In the case of \mathcal{E}_n this is

$$g \cdot (\beta_0, \dots, \beta_n) = (\alpha_g \beta_0 \alpha_g^{-1}, \dots, \alpha_g \beta_{n-1} \alpha_g^{-1}, \alpha_g \beta_n).$$

Note that this action is compatible with the structure maps and that the obvious projection map $\mathcal{E}_\bullet \rightarrow \mathcal{B}_\bullet$ onto the first n factors is G -equivariant.

The two simplicial spaces \mathcal{E}_\bullet and \mathcal{B}_\bullet agree with the bar constructions $B(*, \text{Aut}(A)_c, \text{Aut}(A)_l)$ and $B(*, \text{Aut}(A)_c, *)$ in the category of G -spaces, respectively, where $\text{Aut}(A)_c$ is $\text{Aut}(A)$ equipped with the conjugation action and $\text{Aut}(A)_l$ is the same space equipped with the left action. Alternatively, the space $B\text{Aut}(A)$ can be also viewed as the geometric realisation of the topological category associated to the group $\text{Aut}(A)$. Let

$$E\text{Aut}(A) = |\mathcal{E}_\bullet|, \quad B\text{Aut}(A) = |\mathcal{B}_\bullet|.$$

The projection map $\mathcal{E}_\bullet \rightarrow \mathcal{B}_\bullet$ induces a continuous G -equivariant map $\pi: E\text{Aut}(A) \rightarrow B\text{Aut}(A)$ on the geometric realisation.

Proposition 3.4.4. *Let $G = \mathbb{Z}/p\mathbb{Z}$. Let V be a finite-dimensional unitary G -representation. Let $D = \text{End}(V)^{\otimes \infty}$ and $\mathbb{K} = \mathbb{K}(\ell^2(G) \otimes H_0)$. Then*

$$\pi: E\text{Aut}(D \otimes \mathbb{K}) \rightarrow B\text{Aut}(D \otimes \mathbb{K})$$

is a universal $(G, \text{Aut}(D \otimes \mathbb{K}))$ -bundle.

Proof. The inclusion $\{\text{id}\} \rightarrow \text{Aut}(D \otimes \mathbb{K})$ is a G -equivariant cofibration by Lemma 3.3.17. By [30, Thm. 8.2] the map $\pi: E\text{Aut}(D \otimes \mathbb{K}) \rightarrow B\text{Aut}(D \otimes \mathbb{K})$ is thus a principal $\text{Aut}(D \otimes \mathbb{K})$ -bundle. The construction of the local trivialisations in [30, Thm. 8.2] is based on the observation that the n th simplicial filtration step included in the full space $E\text{Aut}(D \otimes \mathbb{K})$ is a cofibration (see also [30, Thm. 7.6]). Since this is true G -equivariantly here, the trivialisation described at the end of the proof of [30, Thm. 8.2] is in fact G -equivariant in the sense of Definition 3.4.2. Therefore $E\text{Aut}(D \otimes \mathbb{K}) \rightarrow B\text{Aut}(D \otimes \mathbb{K})$ is a $(G, \text{Aut}(D \otimes \mathbb{K}))$ -bundle.

To see that it is universal we may follow [55, Thm. 5.1]: Recall that a local object $X \rightarrow G/H$ is a $(G, \text{Aut}(A))$ -bundle over the G -space G/H for a subgroup $H \leq G$. We have to show that for every local object $X \rightarrow G/H$ featuring in our local triviality condition the associated bundle

$$(E\text{Aut}(D \otimes \mathbb{K}) \times X)/\text{Aut}(D \otimes \mathbb{K}) \rightarrow G/H \tag{3.9}$$

is G -shrinkable. In our case there is only one local object to check, namely $X = \text{Aut}(D \otimes \mathbb{K}) \rightarrow * = G/G$, i.e. the right $\text{Aut}(D \otimes \mathbb{K})$ -bundle over the point with the given left action of G on $\text{Aut}(D \otimes \mathbb{K})$. In this case the bundle in (3.9) is homeomorphic to $E\text{Aut}(D \otimes \mathbb{K}) \rightarrow *$ with the G -action given by conjugation, i.e. the G -space given by $B(*, \text{Aut}(D \otimes \mathbb{K})_c, \text{Aut}(D \otimes \mathbb{K})_c)$. Let $H \subseteq G$ be a subgroup. Taking fixed points is a finite limit and therefore commutes with geometric realisation. Combining these observations we have

$$E\text{Aut}(D \otimes \mathbb{K})^H \cong E\text{Aut}_H(D \otimes \mathbb{K}),$$

which is contractible. Hence, $E\text{Aut}(D \otimes \mathbb{K})$ is weakly G -contractible and the result follows from Lemma 3.3.16 and [34, Prop. 13.2]. \square

Remark 3.4.5. The proof of universality shows that $E\text{Aut}(D \otimes \mathbb{K})$ is in fact a model for $E_{\mathcal{F}(\mathcal{R})}\text{Aut}(D \otimes \mathbb{K})$ for the family \mathcal{R} of local representations in the sense of [29, Def. 3.4] that consists of all conjugates of the action $\alpha: G \rightarrow \text{Aut}(D \otimes \mathbb{K})$ and the trivial homomorphism $\{\text{id}\} \rightarrow \text{Aut}(D \otimes \mathbb{K})$ (see [29, Thm. 11.5]). Note, however, that the equivariant principal bundles considered in [29] are less restrictive than our definition. In particular, [29, Def. 2.1] only demands that local triviality holds non-equivariantly, whereas Definition 3.4.2 asks for G -equivariant local trivialisations (as in [55]). With the more flexible notion, Condition (H) (see [29, Def. 6.1]) is needed to ensure homotopy invariance. It is not clear to us, if Condition (H) holds in our case.

Note that the G -space $\text{Aut}(D \otimes \mathbb{K})$ has two equivariant deloopings:

1. The classifying space $B\text{Aut}(D \otimes \mathbb{K})$ equipped with the G -action induced by the conjugation action of G on $\text{Aut}(D \otimes \mathbb{K})$ defined above.
2. The Γ - G -space delooping given by

$$B_{\otimes}\text{Aut}(D \otimes \mathbb{K}) = (EH_{\mathbb{Z}}\mathcal{G}_D)_1$$

induced by the tensor product structure on automorphisms.

Lemma 3.4.6. *Let $G = \mathbb{Z}/p\mathbb{Z}$ and let D and \mathbb{K} be the G - C^* -algebras defined above. Then there is a weak G -equivalence*

$$BAut(D \otimes \mathbb{K}) \simeq B_{\otimes}Aut(D \otimes \mathbb{K}).$$

As a consequence, $B_{\otimes}Aut(D \otimes \mathbb{K})$ is a classifying space for G -equivariant $D \otimes \mathbb{K}$ -bundles over finite CW-complexes.

Proof. The proof is an equivariant version of [6, Thm. 3.6] and can be reduced to that statement: With $\mu = \otimes$ induced by the tensor product and $\nu = \circ$ given by composition there is an intermediate space $B_{\otimes}BAut(D \otimes \mathbb{K})$ constructed in the proof of [6, Thm. 3.6]. We can carry out this construction in the category of G -spaces. Then it suffices to check that for a subgroup $H \leq G$ the induced maps on fixed points

$$B_{\otimes}Aut(D \otimes \mathbb{K})^H \longrightarrow \Omega B_{\otimes}BAut(D \otimes \mathbb{K})^H \longleftarrow BAut(D \otimes \mathbb{K})^H$$

is a weak equivalence. But because fixed points commute with limits and the group action is by conjugation the above boils down to the following sequence of maps

$$B_{\otimes}Aut_H(D \otimes \mathbb{K}) \xrightarrow{\simeq} \Omega B_{\otimes}BAut_H(D \otimes \mathbb{K}) \xleftarrow{\simeq} BAut_H(D \otimes \mathbb{K}),$$

all of which are weak equivalences by the original [6, Thm. 3.6]. □

3.4.2 Homotopy classification of $\mathbb{Z}/p\mathbb{Z}$ bundles of UHF-algebras

Let $e \in \mathbb{K}$ be a projection of rank one fixed by G . We can naturally define a collection of maps

$$\theta_n^H: Aut_H((D \otimes \mathbb{K})^{\otimes n}) \rightarrow (\Omega^n KU_n^D)^H = \text{hom}_{\text{gr}}(\widehat{\mathcal{S}}, C_0(\mathbb{R}^n) \otimes \mathcal{C}\ell_1 \otimes (D \otimes \mathbb{K})^H)$$

$$\alpha \mapsto (f \mapsto \hat{\eta}_n(f) \otimes \alpha(1 \otimes e))$$

for all $H \leq G$. Since \mathcal{G}_D has compatible inverses (see Lemma 3.3.2), these maps factor over a morphism

$$\theta : \mathcal{G}_D \rightarrow \mathrm{GL}_1^\bullet \mathrm{KU}^D$$

of commutative \mathcal{I} - O_G -monoids. Corollary 3.3.15 suggests it might be worth to investigate whether θ is a weak equivalence when $G = \mathbb{Z}/p\mathbb{Z}$. This turns out to be true if one takes the order structure on $\pi_0(\mathrm{KU}^D) \cong K_0(D)$ into account. Assuming that $\pi_0(R)$ has an order structure we define $\mathrm{GL}_1^\bullet(R)_+$ by a pullback diagram analogous to the one in (3.5) with $\mathrm{GL}_1(\pi_0(R))$ replaced by $\mathrm{GL}_1(\pi_0(R))_+$. Let $gl_1(R)_+ = EH_{\mathcal{I}}\mathrm{GL}_1^\bullet(R)_+$ be the associated G -spectrum.

Theorem 3.4.7. *There is a map of $\mathbb{Z}/p\mathbb{Z}$ -spectra*

$$EH_{\mathcal{I}}\mathrm{Aut}(D \otimes \mathbb{K}) \rightarrow gl_1(\mathrm{KU}^D)$$

which is an isomorphism on all higher equivariant homotopy groups π_n^H with $n > 0$, and the inclusion $\mathrm{GL}_1(K_0^H(D))_+ \hookrightarrow \mathrm{GL}_1(K_0^H(D))$ on π_0^H , for $H \leq \mathbb{Z}/p\mathbb{Z}$. In particular, we have an equivalence of $\mathbb{Z}/p\mathbb{Z}$ -spectra

$$EH_{\mathcal{I}}\mathrm{Aut}(D \otimes \mathbb{K}) \simeq gl_1(\mathrm{KU}^D)_+.$$

Proof. By Lemma 3.1.10 it is enough to check that the maps

$$\theta_1^H : \mathrm{Aut}_H(D \otimes \mathbb{K}) \rightarrow (\Omega \mathrm{KU}_1^D)^H$$

defined above enjoy the prescribed properties, for $H \in \{\{e\}, \mathbb{Z}/p\mathbb{Z}\}$. Observe that θ_1^H factors through

$$\begin{array}{ccc} \mathrm{Aut}_H(D \otimes \mathbb{K}) & \xrightarrow{\theta_1^H} & \mathrm{hom}_{\mathrm{gr}}(\widehat{\mathcal{S}}, C_0(\mathbb{R}) \otimes \mathbb{C}\ell_1 \otimes (D \otimes \mathbb{K})^H) \\ \Phi \downarrow & \nearrow \Psi & \\ \mathrm{Proj}((D \otimes \mathbb{K})^H) & & \end{array}$$

where $\Psi(p) = (f \mapsto \eta_1(f) \otimes p)$. By Proposition 3.3.9 the map Φ is a homotopy equivalence when the domain is restricted to $\mathrm{Aut}_{H,0}(D \otimes \mathbb{K})$ and the codomain

to the component of $1 \otimes e$. Recall that $\hat{\eta}_1 \in \text{hom}_{\text{gr}}(\widehat{\mathcal{S}}, C_0(\mathbb{R}, \mathbb{C}\ell_1))$ is the Bott element. Note that Ψ factors in turn as

$$\text{Proj}((D \otimes \mathbb{K})^H) \longrightarrow \text{hom}_{\text{gr}}(\widehat{\mathcal{S}}, (D \otimes \mathbb{K})^H) \longrightarrow \text{hom}_{\text{gr}}(\widehat{\mathcal{S}}, C_0(\mathbb{R}, \mathbb{C}\ell_1) \otimes (D \otimes \mathbb{K})^H).$$

Here the first map sends a projection p to $\epsilon \cdot \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$. The second map sends φ to $\varphi \otimes \hat{\eta}_1 \circ \Delta$ and shifts the grading to $C_0(\mathbb{R}) \otimes \mathbb{C}\ell_1$, and it is an isomorphism on π_0 (since it induces multiplication by the Bott element). Therefore the discussion after [57, Thm. 4.7], plus Lemma 3.3.5 show that θ_1^H induces the inclusion $\text{GL}_1(K_0^H(D))_+ \hookrightarrow \text{GL}_1(K_0^H(D))$ on π_0 .

To check π_n for $n > 0$ we may restrict Φ to the equivalence

$$\Phi: \text{Aut}_{H,0}(D \otimes \mathbb{K}) \rightarrow \text{Proj}_{1 \otimes e}((D \otimes \mathbb{K})^H)$$

and Ψ accordingly. Consider the following commutative square

$$\begin{array}{ccc} \pi_n(\text{Proj}_{1 \otimes e}((D \otimes \mathbb{K})^H)) & \xrightarrow{\Psi_*} & \pi_n((\Omega KU_1^D)^H) \\ \beta \downarrow & & \downarrow \cong \\ K_0(C_0(S^n, *) \otimes D^H) & \xrightarrow{\cong} & K'(C_0(\mathbb{R}) \otimes \mathbb{C}\ell_1 \otimes C(S^n) \otimes D^H) \end{array}$$

where we are using the notation $K'(A) = \pi_0(\text{hom}_{\text{gr}}(\widehat{\mathcal{S}}, A \otimes \mathbb{K}))$ introduced in [57]. Note that any element $\gamma : S^n \rightarrow \text{Proj}_{1 \otimes e}((D \otimes \mathbb{K})^H)$ induces a projection $p_\gamma \in C(S^n) \otimes D^H$, and set $\beta(\gamma) = [p_\gamma] - [1_{C(S^n)} \otimes 1 \otimes e]$. The bottom horizontal map sends $[p] - [q]$ to $[(f \mapsto \hat{\eta}_1 \otimes \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix})]$ and is an isomorphism by Bott periodicity. Finally, note that any element in $\pi_n((\Omega KU_1^D)^H)$ defines a map $\varphi \in \text{hom}_{\text{gr}}(\widehat{\mathcal{S}}, C_0(\mathbb{R}) \otimes \mathbb{C}\ell_1 \otimes C(S^n) \otimes (D \otimes \mathbb{K})^H)$. The right hand vertical arrow maps $[\varphi]$ to $\left[\begin{pmatrix} \varphi & 0 \\ 0 & 1_{C(S^n) \otimes \hat{\eta}_1} \end{pmatrix} \right]$ and has an inverse given by $\psi \mapsto \psi \oplus \begin{pmatrix} 1_{C(S^n) \otimes \hat{\eta}_1} & 0 \\ 0 & 0 \end{pmatrix}$, where \oplus is the addition operation described in [57]. Note that this is a basepoint correction, because in Ψ_* we use a basepoint for π_n in the $[1 \otimes e]$ -component on both sides instead of the usual 0-component.

The last step is to prove that β is an isomorphism as well. It will follow that Ψ induces isomorphisms on all higher homotopy groups, hence so does θ . For this

purpose, let us consider the following square:

$$\begin{array}{ccc}
 \pi_{n-1}(U(D^H)) & \xrightarrow{\cong} & \pi_n(\mathrm{Proj}_{1 \otimes e}((D \otimes \mathbb{K})^H)) \\
 \cong \downarrow & & \downarrow \beta \\
 K_1(C_0(S^{n-1}, *) \otimes D^H) & \xrightarrow{\cong} & K_0(C_0(S^n, *) \otimes D^H)
 \end{array}$$

Here the top horizontal arrow is the inverse of the isomorphism described in Remark 3.3.8 composed with the canonical isomorphism $\pi_n(BG) \cong \pi_{n-1}(G)$. This map is defined by constructing a principal G -bundle over S^n with the prescribed transition map $\psi \in \pi_{n-1}(G)$, explicitly given by $P_\psi = S^{n-1} \times I \times G$ modulo the equivalence relation \sim generated by

$$(z, 1, g) \sim (z, 0, \psi(z) \cdot g) \text{ and } (z_0, t, g) \sim (z_0, 0, g), \quad (3.10)$$

with the quotient map to $\Sigma S^{n-1} \cong S^n$ as projection map (see [36, Sec. 4.4]). Recall β maps $[\gamma]$ to $[p_\gamma] - [1_{C(S^n)} \otimes 1 \otimes e]$ as in the diagram above. We can interpret the composition of these two arrows as the map sending $\gamma \in \pi_{n-1}(U(D^H))$ to the formal difference $[\mathcal{H}_\gamma] - [\underline{\mathcal{H}}]$ of right Hilbert D^H -module bundles over S^n , where

$$\mathcal{H}_\gamma = (S^{n-1} \times I \times U(D^H)) / \sim,$$

and the equivalence relation is as in (3.10) with $\psi = \gamma$. On the other hand, if we start from the left hand vertical arrow we encounter the isomorphism described in Remark 3.3.14, and the bottom horizontal arrow is the suspension isomorphism in K -theory (see Example 2.2.25). It is easy to check that this composite also maps γ to $[\mathcal{H}_\gamma] - [\underline{\mathcal{H}}]$. Therefore the diagram commutes, implying that the map β is an isomorphism, thus concluding the proof. \square

Our final result combines Lemma 3.4.6 with Theorem 3.4.7.

Corollary 3.4.8. *$\mathrm{Aut}(D \otimes \mathbb{K})$ is an equivariant infinite loop space with associated $\mathbb{Z}/p\mathbb{Z}$ -equivariant cohomology theory $E_D^*(X) = gl_1(KU^D)_+^*(X)$, and*

$$E_D^0(X) = [X, \mathrm{Aut}(D \otimes \mathbb{K})]^{\mathbb{Z}/p\mathbb{Z}} \quad \text{and} \quad E_D^1(X) = [X, B\mathrm{Aut}(D \otimes \mathbb{K})]^{\mathbb{Z}/p\mathbb{Z}}.$$

In particular, isomorphism classes of $\mathbb{Z}/p\mathbb{Z}$ -equivariant C^ -algebra bundles with fibres isomorphic to the $\mathbb{Z}/p\mathbb{Z}$ -algebra $D \otimes \mathbb{K}$ over the finite CW-complex X form a group with respect to the fibrewise tensor product that is isomorphic to $E_D^1(X)$.*

Outlook

We conclude by collecting some ingredients needed to extend the results of this thesis in a twofold direction: first, to more general classes of groups and G - C^* -algebras; second, to the genuine equivariant setting, in which one can hope to establish further structure on the C^* -algebra bundles. Section 4.1 is based on a note that Taro Sogabe shared with me during my visit to Kyoto, for which I thank him. In Section 4.2 we make a proposal for a genuine equivariant infinite loop space machine which could be used to construct a genuine G -spectrum of units (as opposed to a naive one), but is certainly of independent interest.

4.1 Contractibility of $\text{Aut}_{G,1D \otimes e}(D \otimes \mathbb{K})$

The goal of this short section is to generalise Corollary 3.3.4 to the case of a compact group G and a strongly self-absorbing C^* -dynamical system (D, σ) such that $\text{Aut}_G(D)$ is path-connected.

Lemma 4.1.1. *There exists a sequence of unitaries $U_t : \ell^2\mathbb{N} \xrightarrow{\cong} \ell^2\mathbb{N}$ such that*

$$U_t \xrightarrow[t \rightarrow \infty]{} S$$

in the strong operator topology, where $S : \xi_l \mapsto \xi_{l+1}$ is the shift operator.

Proof. For $t \in [n, n+1)$, define

$$U_t(\xi_l) = \begin{cases} \xi_{l+1} & l \leq n-1 \\ \cos(\frac{\pi}{2}(t-n))\xi_1 + \sin(\frac{\pi}{2}(t-n))\xi_{n+1} & l = n \\ \sin(\frac{\pi}{2}(t-n))\xi_1 + \cos(\frac{\pi}{2}(t-n))\xi_{n+1} & l = n+1 \\ \xi_l & l \geq n+2 \end{cases}.$$

□

Lemma 4.1.2. *Let Γ_1, Γ_2 be two discrete countably infinite sets. There exists a continuous family of unitaries*

$$V_t \in \mathcal{B}(\mathbb{C}\xi \oplus \ell^2\Gamma_1 \oplus \ell^2\Gamma_2)$$

for $t \in [0, \infty)$, such that:

(i) $V_t\xi = V_t^*\xi = \xi$ for all $\xi \in \mathbb{C}\xi \oplus \ell^2\Gamma_1 \oplus \ell^2\Gamma_2$,

(ii) V_t converges in the strong operator topology to an isometry

$$S : \mathbb{C}\xi \oplus \ell^2\Gamma_1 \oplus \ell^2\Gamma_2 \rightarrow \mathbb{C}\xi \oplus \ell^2\Gamma_1.$$

Proof. Let U_t be as in Lemma 4.1.1 and define

$$U'_t := \sum_{k=1}^{\infty} U_t \otimes e_k : \ell^2\mathbb{N} \otimes \ell^2\mathbb{N} \xrightarrow{\cong} \ell^2\mathbb{N} \otimes \ell^2\mathbb{N},$$

so that U'_t converges to an isometry $S' : \ell^2\mathbb{N} \otimes \ell^2\mathbb{N} \rightarrow \ell^2\mathbb{N}_{\geq 2} \otimes \ell^2\mathbb{N}$. Then by identifying

$$\ell^2\Gamma_1 \cong \ell^2\mathbb{N}_{\geq 2} \otimes \ell^2\mathbb{N}, \quad \ell^2\Gamma_2 \cong \mathbb{C}\xi_1 \oplus \ell^2\mathbb{N}$$

we are able to define

$$V_t := \xi \otimes \xi^* + U'_t : \mathbb{C}\xi \oplus \ell^2\Gamma_1 \oplus \ell^2\Gamma_2 \rightarrow \mathbb{C}\xi \oplus \ell^2\Gamma_1 \oplus \ell^2\Gamma_2$$

and

$$S := \xi \otimes \xi^* + S' : \mathbb{C}\xi \oplus \ell^2\Gamma_1 \oplus \ell^2\Gamma_2 \rightarrow \mathbb{C}\xi \oplus \ell^2\Gamma_1$$

satisfying (i) and (ii). □

Let G be a compact group and consider the compact operators \mathbb{K} as a G -algebra with respect to an action that has every character appearing with infinite multiplicity: so $\mathbb{K} = \mathbb{K}(H)$ where $H = \bigoplus_{\pi \in \text{Irr}(G)} V_\pi \otimes \ell^2 \mathbb{N}$.

Proposition 4.1.3. *Let \mathbb{K} be as above and let $e \in \mathbb{K}$ be a projection of rank one. There exists a point-norm continuous path*

$$\gamma : [0, 1] \rightarrow \text{hom}_{G,e}(\mathbb{K}, \mathbb{K} \otimes \mathbb{K})$$

into the G -equivariant homomorphisms fixing e , such that

- (i) $\gamma(0)(T) = T \otimes e$ and $\gamma(1)(T) = e \otimes T$,
- (ii) γ restricts to $(0, 1) \rightarrow \text{Iso}_{G,e}(\mathbb{K}, \mathbb{K} \otimes \mathbb{K})$.

Proof. Write $e = \xi \otimes \xi^*$ and

$$H \otimes H \cong \mathbb{C} \otimes \ell^2 \mathbb{N} \oplus \bigoplus_{\pi \in \text{Irr}(G), \pi \neq \text{triv}} V_\pi \otimes \ell^2 \mathbb{N},$$

then pick Γ_1, Γ_2 discrete countably infinite sets such that

$$\ell^2 \mathbb{N} \cong \mathbb{C}\xi \oplus \ell^2 \Gamma_1 \oplus \ell^2 \Gamma_2$$

and

$$H \otimes \xi \cong \mathbb{C} \otimes (\mathbb{C}\xi \oplus \ell^2 \Gamma_1) \oplus \bigoplus_{\pi \in \text{Irr}(G), \pi \neq \text{triv}} V_\pi \otimes (\mathbb{C}\xi \oplus \ell^2 \Gamma_1).$$

We can apply Lemma 4.1.2 and get a G -equivariant path of unitaries

$$\hat{V}_t := 1_{\mathbb{C}} \otimes V_t \oplus \bigoplus_{\pi} 1_{V_\pi} \otimes V_t$$

converging in the strong operator topology to the isometry

$$\hat{S} := 1_{\mathbb{C}} \otimes S \oplus \bigoplus_{\pi} 1_{V_\pi} \otimes S.$$

Then $\text{Ad } \hat{V}_t : \mathbb{K} \otimes \mathbb{K} \rightarrow \mathbb{K} \otimes \mathbb{K}$, $\text{Ad } \hat{S} : \mathbb{K} \otimes \mathbb{K} \rightarrow \mathbb{K} \otimes e$ are G -equivariant and fix e , and $\text{Ad } \hat{V}_t \rightarrow \text{Ad } \hat{S}$ in the point-norm topology. We can construct a half-sided path in $\text{hom}_{G,e}(\mathbb{K}, \mathbb{K} \otimes \mathbb{K})$ by

$$\gamma(t)(T) := \hat{V}_t \hat{S}^* T \hat{S} \hat{V}_t^*$$

where we have identified $\mathbb{K} \cong \mathbb{K} \otimes e$. The full path is finally assembled by constructing the other half-sided path in the same way (where we define $U'_t := \sum_{k=1}^{\infty} e_k \otimes V_t$ in the proof of Lemma 4.1.2) and using path-connectedness of $U(H)$. \square

We want to use Proposition 4.1.3 to generalise our result about the contractibility of the stabiliser group $\text{Aut}_{G,1_D \otimes e}(D \otimes \mathbb{K})$ to a more general class of G - C^* -algebras D . The natural candidate are strongly self-absorbing C^* -dynamical systems [53].

Definition 4.1.4. Let D be a separable, unital C^* -algebra and G a second-countable, locally compact group. Let $\sigma : G \curvearrowright D$ be a point-norm continuous action. We say that (D, σ) is a strongly self-absorbing C^* -dynamical system if the equivariant first-factor embedding

$$\text{id}_D \otimes 1_D : (D, \sigma) \rightarrow (D \otimes D, \sigma \otimes \sigma)$$

is approximately G -unitarily equivalent to an isomorphism.

Example 4.1.5. Let $G = \mathbb{Z}/p\mathbb{Z}$. The pair (D, σ) defined in 3.3 is a strongly self-absorbing C^* -dynamical system.

Using Proposition 4.1.3, the proofs of Lemma 3.3.3 and Corollary 3.3.4 apply verbatim to this setting and we get the following.

Proposition 4.1.6. *Let G be a compact group and let (D, σ) be a strongly self-absorbing C^* -dynamical system such that $\text{Aut}_G(D)$ is path-connected. Let $e \in \mathbb{K}$ be a projection of rank one fixed by G . The group $\text{Aut}_G(D)$ and the stabiliser subgroup $\text{Aut}_{G,1_D \otimes e}(D \otimes \mathbb{K})$ fixing the projection $1_D \otimes e$ are both contractible.*

4.2 \mathcal{I}_G -spaces

Let G be a finite group. Throughout, we will understand finite G -sets to be of the form \mathbf{n}^α , where $\mathbf{n} = \{1, \dots, n\}$ and G acts by $g \cdot i = \alpha(g)(i)$ for a group

homomorphism $\alpha : G \rightarrow \Sigma_n$. Analogously, we will denote based finite G -sets by \mathbf{n}_+^α where \mathbf{n}_+ is the based set $\{0, 1, \dots, n\}$ with basepoint 0.

Let \mathcal{I}_G be the G -category of finite G -sets \mathbf{n}^α (including $\mathbf{0} = \emptyset$) and injective functions, with G acting by conjugation on morphisms. It is a symmetric monoidal category via concatenation of finite ordered sets $\mathbf{n}^\alpha \sqcup \mathbf{m}^\beta =: (\mathbf{n} \sqcup \mathbf{m})^{(\alpha, \beta)}$ where $(\alpha, \beta) \in \Sigma_n \times \Sigma_m \subset \Sigma_{n+m}$. Let \mathcal{Top}_G be the G -category of G -spaces and all continuous maps, with G acting by conjugation on morphisms.

Definition 4.2.1. An \mathcal{I}_G -space is a functor $X : \mathcal{I}_G \rightarrow \mathcal{Top}_G$ such that for each pair $(\mathbf{n}^\alpha, \mathbf{m}^\beta)$ of objects in \mathcal{I}_G , the map of morphism spaces

$$\mathrm{hom}_{\mathcal{I}_G}(\mathbf{n}^\alpha, \mathbf{m}^\beta) \rightarrow \mathrm{hom}_{\mathcal{Top}_G}(X(\mathbf{n}^\alpha), X(\mathbf{m}^\beta))$$

given by $f \mapsto X(f)$ is G -equivariant (in other words, X is a G -functor). An \mathcal{I}_G -space X is called a (commutative) \mathcal{I}_G -monoid if it comes equipped with a natural transformation $\mu : X \times X \rightarrow X \circ \sqcup$ which is equivariant, associative, unital (and commutative) in the sense of [6, Def. 2.2].

While \mathcal{I}_G -spaces do not seem to appear in the literature, they generate an equivariant analogue of Γ -spaces which has been extensively studied in [32]. Let \mathcal{F}_G be the G -category of based G -sets and all based functions, with G acting by conjugation on morphisms. Note that by restricting to the trivial homomorphisms $\varepsilon_n : G \rightarrow \Sigma_n$ we get an inclusion $\Gamma^{\mathrm{op}} \subset \mathcal{F}_G$.

Definition 4.2.2. A \mathcal{F}_G -space is a functor $Y : \mathcal{F}_G \rightarrow \mathcal{Top}_G$ such that for each pair $(\mathbf{n}_+^\alpha, \mathbf{m}_+^\beta)$ of objects in \mathcal{F}_G , the map of morphism spaces

$$\mathrm{hom}_{\mathcal{F}_G}(\mathbf{n}_+^\alpha, \mathbf{m}_+^\beta) \rightarrow \mathrm{hom}_{\mathcal{Top}_G}(Y(\mathbf{n}_+^\alpha), Y(\mathbf{m}_+^\beta))$$

given by $\phi \mapsto Y(\phi)$ is G -equivariant (in other words, Y is a G -functor). We write Y_n for $Y(\mathbf{n}_+^{\varepsilon_n})$.

Definition 4.2.3. Let Y be a \mathcal{F}_G -space. Define a G -map

$$\partial_{\mathbf{n}_+^\alpha} : \mathbf{n}_+^\alpha \wedge Y(\mathbf{n}_+^\alpha) \rightarrow Y_1$$

by $\partial_{\mathbf{n}_+^\alpha}(j, y) = (\delta_j)_*(y)$ for $1 \leq j \leq n$, where δ_j is induced by the j -th projection $\mathbf{n}_+^\alpha \rightarrow \mathbf{1}_+^{\varepsilon_1}$. The Segal G -map

$$\delta_{\mathbf{n}_+^\alpha} : Y(\mathbf{n}_+^\alpha) \rightarrow Y_1^{\mathbf{n}_+^\alpha}$$

is the adjoint of $\partial_{\mathbf{n}_+^\alpha}$. A \mathcal{F}_G -space Y is called *special* if the Segal G -maps $\delta_{\mathbf{n}_+^\alpha}$ are G -weak equivalences for all finite based G -sets \mathbf{n}_+^α .

4.2.1 From commutative \mathcal{I}_G -monoids to genuine G -spectra

Let X be a commutative \mathcal{I}_G -monoid. Let $P(\mathbf{n}_+^\alpha)$ be the G -poset of finite G -sets $0 \notin A \subset \mathbf{n}_+^\alpha$ and define $\mathcal{I}_G(\mathbf{n}_+^\alpha)$ to be the G -category of functors $\theta : P(\mathbf{n}_+^\alpha) \rightarrow \mathcal{I}_G$ with the property that for every pair of disjoint subsets $A, B \in P(\mathbf{n}_+^\alpha)$ the diagram

$$\theta(A) \rightarrow \theta(A \cup B) \leftarrow \theta(B)$$

(induced by inclusions) is a coproduct diagram in \mathcal{I}_G . Morphisms in $\mathcal{I}_G(\mathbf{n}_+^\alpha)$ are natural transformations η . The group G acts by $g \cdot \theta(A) = \theta(g^{-1}A)$ on objects, and by $(g \cdot \eta)_A = g\eta_{g^{-1}(A)}g^{-1}$ on morphisms. We write $\theta_i := \theta(\{i\}) \in \mathcal{I}_G$ for $1 \leq i \leq n$ (by definition, $\theta(\emptyset) = \mathbf{0}$).

Let $\mathcal{I}_G^{\mathbf{n}_+^\alpha}$ be a G -category with G -action given by $g \cdot (k_1, \dots, k_n) = (k_{\alpha_g^{-1}(1)}, \dots, k_{\alpha_g^{-1}(n)})$ on objects and by $g \cdot (f_1, \dots, f_n) = (gf_{\alpha_g^{-1}(1)}g^{-1}, \dots, gf_{\alpha_g^{-1}(n)}g^{-1})$ on morphisms. Consider the forgetful G -functor $F : \mathcal{I}_G(\mathbf{n}_+^\alpha) \rightarrow \mathcal{I}_G^{\mathbf{n}_+^\alpha}$ that sends θ to $(\theta_1, \dots, \theta_n)$. We get a functor $X(\mathbf{n}_+^\alpha) : \mathcal{I}_G(\mathbf{n}_+^\alpha) \rightarrow \mathcal{T}op_G$ given by the composite

$$X(\mathbf{n}_+^\alpha) : \mathcal{I}_G(\mathbf{n}_+^\alpha) \xrightarrow{F} \mathcal{I}_G^{\mathbf{n}_+^\alpha} \xrightarrow{X^{\mathbf{n}_+^\alpha}} \mathcal{T}op_G$$

$$\theta \mapsto \prod_{i=1}^n X(\theta_i).$$

The functor $X(\mathbf{n}_+^\alpha)$ can be equipped with the structure of an $\mathcal{I}_G(\mathbf{n}_+^\alpha)$ -shaped G -diagram in $\mathcal{T}op_G$ in the sense of [10, Def.1.1] (though we do not spell out the

details here). Hence we can define a functor $H_{\mathcal{I}_G}X : \mathcal{F}_G \rightarrow \mathcal{T}op_G$ on objects as the homotopy colimit (in the sense of [10, Def. 1.16])

$$H_{\mathcal{I}_G}X(\mathbf{n}_+^\alpha) = \text{hocolim}_{\mathcal{I}_G(\mathbf{n}_+^\alpha)} X(\mathbf{n}_+^\alpha)$$

where we set $H_{\mathcal{I}_G}X(\mathbf{0}_+) = *$. The value of $H_{\mathcal{I}_G}X$ on morphisms is defined as follows. Let $\phi : \mathbf{n}_+^\alpha \rightarrow \mathbf{m}_+^\beta$ be a map in \mathcal{F}_G . There is a functor $\phi_* : \mathcal{I}_G(\mathbf{n}_+^\alpha) \rightarrow \mathcal{I}_G(\mathbf{m}_+^\beta)$ given by precomposition with ϕ^{-1} . We have a natural transformation $X(\phi) : X(\mathbf{n}_+^\alpha) \rightarrow X(\mathbf{m}_+^\beta) \circ \phi_*$ defined as

$$\begin{aligned} X(\phi)_\theta : \prod_{i=1}^n X(\theta_i) &\xrightarrow{\pi} \prod_{j=1}^m \prod_{i \in \phi^{-1}(j)} X(\theta_i) \xrightarrow{\mu} \prod_{j=1}^m X(\bigoplus_{i \in \phi^{-1}(j)} \theta_i) \\ &\cong \prod_{j=1}^m X(\theta(\phi^{-1}(j))) \end{aligned}$$

where π is the projection away from the factors indexed by elements $i \in \mathbf{n}_+^\alpha$ that are mapped to the basepoint by ϕ , and the last isomorphism is induced by the canonical isomorphism (3.1). In this definition a choice is apparently involved when ordering the product indexed by $i \in \phi^{-1}(j)$, but commutativity of μ ensures that it does not actually matter. Finally, we lift the natural transformation $X(\phi)$ to the first map of homotopy colimits below:

$$\text{hocolim}_{\mathcal{I}_G(\mathbf{n}_+^\alpha)} X(\mathbf{n}_+^\alpha) \rightarrow \text{hocolim}_{\mathcal{I}_G(\mathbf{n}_+^\alpha)} X(\mathbf{m}_+^\beta) \circ \phi_* \xrightarrow{\phi_*} \text{hocolim}_{\mathcal{I}_G(\mathbf{m}_+^\beta)} X(\mathbf{m}_+^\beta).$$

This composition defines $H_{\mathcal{I}_G}X(\phi)$. It remains to verify that $H_{\mathcal{I}_G}X$ is a G -functor to conclude it defines a \mathcal{F}_G -space.

We give a sketch of the proof of specialness of $H_{\mathcal{I}_G}X$. We need to show that the Segal G -maps

$$\delta_{\mathbf{n}_+^\alpha} : H_{\mathcal{I}_G}X(\mathbf{n}_+^\alpha) \rightarrow (H_{\mathcal{I}_G}X)_1^{\mathbf{n}_+^\alpha}$$

are G -weak equivalences for all finite based G -sets \mathbf{n}_+^α . Using the canonical isomorphism

$$(\text{hocolim}_{\mathcal{I}_G} X)^{\mathbf{n}_+^\alpha} \cong \text{hocolim}_{\mathcal{I}_G^{\mathbf{n}_+^\alpha}} X^{\mathbf{n}_+^\alpha}$$

this boils down to checking that the map of homotopy colimits induced by F

$$\text{hocolim}_{\mathcal{I}_G(\mathbf{n}_+^\alpha)} X(\mathbf{n}_+^\alpha) \rightarrow \text{hocolim}_{\mathcal{I}_G^{\mathbf{n}_+^\alpha}} X^{\mathbf{n}_+^\alpha}$$

are G -weak equivalences. By the equivariant cofinality criterion in [10, Thm.2.25] this is the case if we can show that the forgetful G -functor $F : \mathcal{I}_G(\mathbf{n}_+^\alpha) \rightarrow \mathcal{I}_G^{\mathbf{n}_+^\alpha}$ is homotopy right cofinal in the sense of [10, Def.2.24]. This reduces to check that for any $k = (k_1, \dots, k_n) \in \mathcal{I}_G^{\mathbf{n}_+^\alpha}$, the fixed comma categories $(k/F)^H$ are contractible for all $H \leq \text{Stab}_G(j)$. This can be proven, for instance, by constructing an initial object as in (3.2).

The upshot of this section is that one can associate to any commutative \mathcal{I}_G -monoid X a genuine positive Ω - G -spectrum $\mathbb{S}_G^{\mathcal{F}G} H_{\mathcal{I}_G} X$, where $\mathbb{S}_G^{\mathcal{F}G}$ is the genuine homotopical Segal machine defined in [32, Def. 3.24].

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