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Axially symmetric nonlinear wave propagation in elastic rods made of strain-limiting materials☆

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This paper is dedicated to the memory of K.R. Rajagopal.

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ABSTRACT

We study the propagation of nonlinear longitudinal waves in circular cylindrical elastic rods. We assume that the material is homogeneous, isotropic and is a strain-limiting elastic material with an asymptotic parameter that limits the overall range of strains. Starting from the three-dimensional equations of strain-limiting elasticity and using the Bernoulli–Navier hypothesis, we derive one-dimensional nonlinear equations governing the dynamics of the rod for two different material models proposed by Rajagopal and Rodriguez. Apart from this fully nonlinear case, the rod equations corresponding to linear and weakly nonlinear regimes are also derived. Two important differences between linear and weakly nonlinear cases are observed for the corresponding rod equations for two material models. The first one is that the equations describing the axial and radial dynamics are coupled for one material model and uncoupled for the other. The second one is that longitudinal wave propagation is dispersive for one of the material models but not for the other one. We also observe that when the asymptotic parameter is equal to 1, linear rod equations corresponding to one of these two models reduce to the well-known Mindlin–Herrmann equations.

1. Introduction

For over two hundred years, especially after the emergence of Green elasticity in the 1940s, Cauchy elasticity has been the main tool for modelling of nonlinearly elastic materials. In Cauchy elasticity, the relationship between stress and strain for a material undergoing a deformation is expressed in terms of stress as an explicit function of strain. As a result, under the assumption of the smallness of the displacement gradient, one can only obtain a linear relationship between the Cauchy stress and the linearized strain, which is the symmetric part of the displacement gradient. However, this is not what is observed in experiments including some metal alloys, polymers, and many other materials, where a nonlinear response is observed even for small strains. Rajagopal came up with the idea that in order to model the nonlinear phenomena observed in the small strain regime, one should not consider Cauchy elasticity, but an implicit relationship between strain and stress, instead. This allows one to express the linearized strain as a nonlinear function of the Cauchy stress when the magnitudes of the displacement gradients are small (Rajagopal, 2003, 2007, 2010, 2011a, 2011b, 2014, 2018). This framework has been adopted in many studies in the context of elasticity (Huang et al., 2017; Rajagopal & Saccomandi, 2014; Rajagopal & Şengül, 2024), viscoelasticity (Erbay & Şengül, 2015; Muliana et al., 2013; Rajagopal & Srinivasa, 2011; Şengül, 2021) and fracture (Gou et al., 2015; Itou et al., 2018, 2019).

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In their recently published paper, Rajagopal and Rodriguez (Rajagopal & Rodriguez, 2024) have rigorously demonstrated that a nonlinear constitutive relation between linearized strain and stress arises as the leading-order approximation of an asymptotic expansion. The interesting point is that the content of Rajagopal and Rodriguez (2024) is based on an asymptotic parameter $\delta \ll 1$ that limits the overall range of strain, not the magnitude of the displacement gradients. Based on this parameter, Rajagopal and Rodriguez (2024) have introduced a one-parameter family of strain-limiting functions as implicit constitutive relations in terms of the second Piola–Kirchhoff stress tensor and Green-Saint Venant strain tensor. In addition to their general convergence results, they also discuss certain well-known nonlinear constitutive relations between stress and linearized strain. Examples of these popular nonlinear constitutive relations include the ones having density-dependent Young's moduli and the ones derived from strain energies beyond quadratic forms of linearized strain.

In this study, we derive the one-dimensional nonlinear dynamic rod equations, based on the Bernoulli–Navier hypothesis, describing axial wave propagation in circular cylindrical elastic rods made of homogeneous, isotropic, strain-limiting materials. The material response of the rod is described by taking into account the constitutive relations of two different material models, which were proposed in Rajagopal and Rodriguez (2024) and are similar to some used in many studies on strain-limiting elasticity. The interesting feature of these two constitutive relations is that they are both reversible allowing one to express stress in terms of strain. In addition to the dynamic rod equations corresponding to the fully nonlinear case, the equations corresponding to linear and weakly nonlinear regimes are also presented using an asymptotic expansion technique based on the slenderness parameter of the rod. We observe significant differences between the rod equations corresponding to these two different material models. More explicitly, for one of the material models, axial wave propagation is dispersive and is affected by radial strain, whereas, for the other one, axial wave propagation is non-dispersive and independent of radial strain. We point out that the most important reason for these differences stems from the structural differences in the constitutive relations where strain is expressed as a nonlinear function of stress. More precisely, the fact that one of the nonlinear constitutive relations contains only the first invariant of the stress tensor and the other one contains only the second invariant, forms the basis of this behavioural difference.

The nonlinear dynamic equations governing longitudinal wave propagation in circular cylindrical rods have been discussed in many studies in the literature within the framework of classical elasticity, some of which can be listed as Clarkson et al. (1986), Cohen and Dai (1993), Coleman and Newman (1990), Dai (1998), Dai and Fan (2004), Nariboli (1970), Ostrovskii and Sutin (1977), Soerensen et al. (1984), Wright (1982) and Wright (1985). Within the same context, the most recent two studies (Nobili & Saccomandi, 2024) and Nobili (2024) provide some insights into the range of validity of the Love hypothesis in nonlinear elastic rods.

The rest of this work is organized as follows. Section 2 presents the basic equations of strain-limiting elasticity and introduces the constitutive relations for two different specific materials. In Section 3, the equations governing the axisymmetric dynamic deformations of the rod composed of a homogeneous, isotropic, strain-limiting elastic material are derived under the Bernoulli–Navier hypothesis and written in dimensionless form. Section 4 presents the rod equations obtained from the first of the two material models for linear, weakly nonlinear, and fully nonlinear regimes. In Section 5, a similar study is conducted for the second material model, and the differences between the equations obtained from these two models are discussed. Finally, in Section 6, we finish with a conclusion.

2. Basic equations

We denote by $\mathbf{u}(\mathbf{x}, t)$ the displacement field of the body at time t and at the current position $\mathbf{x} \in \mathbb{R}^3$ of a particle $\mathbf{X}$ in the reference configuration. That is, $\mathbf{u} = \mathbf{x} - \mathbf{X}$. We assume that the body begins in a stress-free configuration and the deformation of the body is denoted by $\chi(\mathbf{X}, t)$. Then, the deformation gradient is defined to be $\mathbf{F} = \mathrm{Grad}\, \mathbf{x}(\mathbf{X}, t) = \partial \chi / \partial \mathbf{X}$. The right Cauchy–Green deformation tensor is given by $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$. The Green-Saint Venant strain tensor $\mathbf{E}$ is defined as $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ where $\mathbf{I}$ is the identity tensor. We assume that the Jacobian $J = \det \mathbf{F} > 0$ due to non-penetration of the body.

In the absence of the body forces, the equation of motion takes the form

$$\mathrm{Div}\, \mathbf{T} = \rho_0 \ddot{\chi}, \tag{2.1}$$

where $\mathbf{T}$ is the nominal stress tensor and defined as the transpose of the first Piola–Kirchhoff stress tensor $\mathbf{P}$ through $\mathbf{T} = \mathbf{P}^{\mathrm{T}}$, and the constant ρ_0 is the reference density. We complement the equation of motion with the traction boundary condition on a part of the boundary which can be written as

$$\mathbf{T}^{\mathrm{T}}\mathbf{N} = \hat{\mathbf{t}} \quad \text{on} \quad \partial\Sigma, \tag{2.2}$$

where $\mathbf{N}$ is the unit outward normal to the boundary $\partial\Sigma$. The vector $\hat{\mathbf{t}}$ depends only on $\mathbf{X}$, and it is the load (or traction) per unit area of $\partial\Sigma$.

Following Rajagopal and Rodriguez (2024), we consider implicit constitutive relations of the form

$$\mathbf{E} = \mathbf{f}_\delta(\mathbf{E}, \mathbf{S}), \tag{2.3}$$

where $\mathbf{f}_\delta$ is (possibly) nonlinear, Lipschitz continuous function defined on an open set and bounded by the parameter $\delta \ll 1$ on its domain, and $\mathbf{S} = \mathbf{T}\mathbf{F}^{-\mathrm{T}} = \mathbf{F}^{-1}\mathbf{P}$ is the second Piola–Kirchhoff stress tensor. Assuming that the parameter $\delta > 0$ is physically interpreted as the maximum possible strain the material can undergo, such a relationship is defined in Rajagopal and Rodriguez

(2024) through functions $\mathbf{f}_\delta$ belonging to a family of *strain-limiting functions with limiting small strains*. These functions must satisfy the following properties; there exists $C_0, C_1 > 0$ and $D_0 > 0$, independent of δ, such that for all δ,

$$\begin{aligned}
&\forall\ \mathbf{E},\mathbf{S},\quad |\mathbf{f}_\delta(\mathbf{E},\mathbf{S})| \le C_0\,\delta,\\
&\forall\ \mathbf{E}_1 \neq \mathbf{E}_2,\mathbf{S},\quad \frac{|\mathbf{f}_\delta(\mathbf{E}_2,\mathbf{S}) - \mathbf{f}_\delta(\mathbf{E}_1,\mathbf{S})|}{|\mathbf{E}_2 - \mathbf{E}_1|} \le C_1,\\
&\forall\ \mathbf{E},\mathbf{S}_1 \neq \mathbf{S}_2,\quad \frac{|\mathbf{f}_\delta(\mathbf{E},\mathbf{S}_2) - \mathbf{f}_\delta(\mathbf{E},\mathbf{S}_1)|}{|\mathbf{S}_2 - \mathbf{S}_1|} \le D_0\delta,
\end{aligned} \tag{2.4}$$

where $|\cdot|$ denotes the Frobenius norm $|\mathbf{A}| = (\mathrm{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}}))^{1/2}$ for a tensor $\mathbf{A}$. Rajagopal and Rodriguez (2024) discuss various three-dimensional examples of constitutive relations satisfying (2.4) in their paper and conclude that an arbitrary bounded Lipschitz continuous function also generates such a family of strain-limiting functions.

In this work, we study two different constitutive relations suggested in Rajagopal and Rodriguez (2024) as explicit examples of families of strain-limiting functions with limiting small strains. For the proofs showing that both families satisfy (2.4) we refer the reader to Rajagopal and Rodriguez (2024). One of these two models is given by

$$\begin{aligned}
&\mathbf{E} = \frac{1+\nu}{E_\delta}\mathbf{S} - \frac{\nu}{E_\delta}(\mathrm{tr}\,\mathbf{S})\,\mathbf{I},\\
&E_\delta(\mathbf{E}) = \frac{E_0}{\delta}\left[1 + \frac{a}{\delta}\left(\frac{1}{(\det(\mathbf{I}+2\mathbf{E}))^{1/2}} - 1\right)\right],
\end{aligned} \tag{2.5}$$

(see Rajagopal and Rodriguez (2024, eqn. (3.10))), where E_0 is the Young's modulus, ν is the Poisson's ratio, and a is a nonnegative constant. As mentioned in Rajagopal and Rodriguez (2024), one can see (2.5) as a generalization of the classical linear constitutive relation for an isotropic solid with a generalized Young's modulus depending on the density. When $a = 0$, (2.5) contains the geometrical nonlinearity only, which implies that a represents the contribution of the physical nonlinearity. Note that in the stress free configuration $E_\delta(\mathbf{0}) = E_0/\delta$.

The second model to be considered in this study is given by

$$\mathbf{E} = \frac{\delta\,\mathbf{S}}{(E_0^2 + |\mathbf{S}|^2)^{1/2}}, \tag{2.6}$$

(see Rajagopal and Rodriguez (2024), eqn. (3.17)), where $|\mathbf{S}|$ is the Frobenius norm of $\mathbf{S}$. At this point, it is worth noting that similar constitutive relations have been presented in the literature (see, for instance, Rajagopal (2010, 2011b)) but they are written in terms of the Cauchy stress and the linearized strain.

It is worth to mention here that the constitutive relation (2.5) involves $\mathrm{tr}\,\mathbf{S}$ together with three material constants E_0, ν and a, excluding δ, whereas (2.6) involves $|\mathbf{S}|$ and only one material constant E_0. We will return to this point later in Section 5.

3. Axially symmetric dynamic deformations

In this section we define the geometry of the problem. We are considering a homogeneous, isotropic cylindrical rod of finite length L with cross section of constant radius R_0 in its undistorted reference configuration, and we investigate the axial wave propagation in this rod. We use the cylindrical coordinates $\{R,\Theta,Z\}$ and $\{r,\theta,z\}$ in the reference and current configurations, respectively. The corresponding unit basis vectors are denoted by $\{\mathbf{E}_R,\mathbf{E}_\Theta,\mathbf{E}_Z\}$ and $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$. We assume that the lateral surface of the circular cylindrical rod is taken to be free of traction.

Assuming the Bernoulli–Navier hypothesis, we write the coordinates r,θ,z at time t of the material point that has coordinates R,Θ,Z in the reference configuration in the form

$$r = R + R\,U(Z,t), \qquad \theta = \Theta, \qquad z = Z + W(Z,t), \tag{3.1}$$

where U is a dimensionless function and it is a measure of radial strain. As a result we have the radial displacement $V := r - R$ given as $V = V(R,Z,t) = RU(Z,t)$, and the axial displacement $W := z - Z$ given as $W = W(Z,t)$. As explained in Coleman and Newman (1990) the first equation in (3.1) states that the motion in cross-sectional planes is radially symmetric and affine, the second equation states that the rod is not twisted, the third one expresses the hypothesis that each normal cross-sectional material plane remains planar and normal during the motion.

As a result of (3.1), we find the deformation gradient tensor as

$$\mathbf{F} = (1+U)\mathbf{e}_r\otimes\mathbf{E}_R + R\frac{\partial U}{\partial Z}\mathbf{e}_r\otimes\mathbf{E}_Z + (1+U)\mathbf{e}_\theta\otimes\mathbf{E}_\Theta + \left(1+\frac{\partial W}{\partial Z}\right)\mathbf{e}_z\otimes\mathbf{E}_Z, \tag{3.2}$$

where the symbol $\otimes$ denotes the tensor product. Similarly, we have

$$J = (1+U)^2\left(1+\frac{\partial W}{\partial Z}\right) > 0. \tag{3.3}$$

One can also calculate the components of the Green-Saint Venant strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})$ as

$$\begin{aligned}
&E_{RR} = U + \frac{1}{2}U^2 = E_{\Theta\Theta}, \quad E_{RZ} = \frac{1}{2}R(1+U)\frac{\partial U}{\partial Z},\\
&E_{ZZ} = \frac{\partial W}{\partial Z} + \frac{1}{2}\left(\frac{\partial W}{\partial Z}\right)^2 + \frac{1}{2}R^2\left(\frac{\partial U}{\partial Z}\right)^2,\\
&E_{R\Theta} = E_{\Theta R} = E_{\Theta Z} = E_{Z\Theta} = 0,
\end{aligned} \tag{3.4}$$

which are functions of axial strain $\frac{\partial W}{\partial Z}$, radial strain U, and the axial derivative of the radial strain, $\frac{\partial U}{\partial Z}$. From the constitutive relations (2.5)–(2.6) and the identity $E_{RR} = E_{\Theta\Theta}$ obtained above, we get $S_{RR} = S_{\Theta\Theta}$ for the components of the second Piola–Kirchhoff stress tensor. Similarly, from (2.5)–(2.6) we obtain

$$S_{R\Theta} = S_{\Theta R} = S_{\Theta Z} = S_{Z\Theta} = 0, \tag{3.5}$$

using $E_{R\Theta} = E_{\Theta R} = E_{\Theta Z} = E_{Z\Theta} = 0$, respectively. Also, using the relation $\mathbf{T} = \mathbf{S}\mathbf{F}^{\mathrm{T}}$ we get the components of the nominal stress tensor in terms of $U, \frac{\partial U}{\partial Z}, \frac{\partial W}{\partial Z}$ and the components of $\mathbf{S}$ as

$$\begin{aligned}
&T_{Rr} = (1+U)S_{RR} + R\frac{\partial U}{\partial Z}S_{RZ}, \quad T_{Rz} = \left(1+\frac{\partial W}{\partial Z}\right)S_{RZ},\\
&T_{Zr} = (1+U)S_{ZR} + R\frac{\partial U}{\partial Z}S_{ZZ}, \quad T_{Zz} = \left(1+\frac{\partial W}{\partial Z}\right)S_{ZZ},\\
&T_{\Theta\theta} = (1+U)S_{\Theta\Theta}.
\end{aligned} \tag{3.6}$$

Furthermore, using (3.2) and (3.5) in $\mathbf{T} = \mathbf{S}\mathbf{F}^{\mathrm{T}}$, we obtain

$$T_{R\theta} = T_{\Theta r} = T_{\Theta z} = T_{Z\theta} = 0. \tag{3.7}$$

In cylindrical coordinates, the equations of motion (2.1) corresponding to the deformation field (3.1) become

$$\frac{\partial T_{Rr}}{\partial R} + \frac{1}{R}(T_{Rr} - T_{\Theta\theta}) + \frac{\partial T_{Zr}}{\partial Z} = \rho_0\frac{\partial^2 r}{\partial t^2}, \tag{3.8}$$

$$\frac{\partial T_{Rz}}{\partial R} + \frac{1}{R}T_{Rz} + \frac{\partial T_{Zz}}{\partial Z} = \rho_0\frac{\partial^2 z}{\partial t^2}, \tag{3.9}$$

where the equation of motion in the circumferential direction is identically satisfied. Since the lateral surface of the cylindrical rod is assumed to be stress free, the boundary conditions reduce to

$$T_{Rr} = 0 \quad \text{and} \quad T_{Rz} = 0 \quad \text{at} \quad R = R_0. \tag{3.10}$$

Since the constitutive relations (2.5)–(2.6) are given in terms of $\mathbf{S}$, it is suitable to write the equations of motion and the boundary conditions in terms of the components of $\mathbf{S}$. Using (3.6) we rewrite (3.8)–(3.9) as

$$(1+U)\frac{\partial S_{RR}}{\partial R} + R\frac{\partial U}{\partial Z}\left(\frac{\partial}{\partial R} + \frac{2}{R}\right)S_{RZ} + \frac{\partial}{\partial Z}\Big((1+U)S_{ZR}\Big) + \frac{\partial}{\partial Z}\left(R\frac{\partial U}{\partial Z}S_{ZZ}\right) = \rho_0 R\frac{\partial^2 U}{\partial t^2}, \tag{3.11}$$

$$\left(1+\frac{\partial W}{\partial Z}\right)\left(\frac{\partial}{\partial R} + \frac{1}{R}\right)S_{RZ} + \frac{\partial}{\partial Z}\left(\left(1+\frac{\partial W}{\partial Z}\right)S_{ZZ}\right) = \rho_0\frac{\partial^2 W}{\partial t^2}, \tag{3.12}$$

where we used the fact that $S_{RR} = S_{\Theta\Theta}$. Similarly, if we use (3.6) and the fact that the Jacobian $J > 0$, the boundary conditions (3.10) take the form

$$S_{RR} = 0 \quad \text{and} \quad S_{RZ} = 0 \quad \text{at} \quad R = R_0.$$

3.1. Non-dimensionalization

We define the non-dimensional quantities through

$$\mathbf{S} = E_0\bar{\mathbf{S}}, \quad R = R_0\bar{R}, \quad Z = L\bar{Z}, \quad E_\delta = E_0\bar{E}_\delta, \quad r = R_0\bar{r}, \quad z = L\bar{z}, \quad t = \frac{L}{c_0}\bar{t}, \quad U = \bar{U}, \quad W = L\bar{W}, \tag{3.13}$$

where $c_0 = \left(\frac{E_0}{\rho_0}\right)^{1/2}$ is the one-dimensional bar speed, and the overbar indicates the non-dimensional quantity. Dropping the overbars for convenience, we can rewrite the equations of motion (3.11)–(3.12) as

$$(1+U)\frac{\partial S_{RR}}{\partial R} + \gamma R\frac{\partial U}{\partial Z}\left(\frac{\partial}{\partial R} + \frac{2}{R}\right)S_{RZ} + \gamma\frac{\partial}{\partial Z}\Big((1+U)S_{ZR}\Big) + \gamma^2\frac{\partial}{\partial Z}\left(R\frac{\partial U}{\partial Z}S_{ZZ}\right) = \gamma^2 R\frac{\partial^2 U}{\partial t^2}, \tag{3.14}$$

$$\left(1+\frac{\partial W}{\partial Z}\right)\left(\frac{\partial}{\partial R} + \frac{1}{R}\right)S_{RZ} + \gamma\frac{\partial}{\partial Z}\left(\left(1+\frac{\partial W}{\partial Z}\right)S_{ZZ}\right) = \gamma\frac{\partial^2 W}{\partial t^2}, \tag{3.15}$$

where $\gamma = \frac{R_0}{L}$ is the non-dimensional small parameter reflecting the slenderness of the rod. The boundary conditions become

$$S_{RR} = 0 \quad \text{and} \quad S_{RZ} = 0 \quad \text{at} \quad R = 1. \tag{3.16}$$

Finally, the non-zero components of the Green-Saint Venant strain tensor take the form

$$\begin{aligned}
&E_{RR} = U + \frac{1}{2}U^2 = E_{\Theta\Theta},\\
&E_{RZ} = \frac{1}{2}\gamma R(1+U)\frac{\partial U}{\partial Z},\\
&E_{ZZ} = \frac{\partial W}{\partial Z} + \frac{1}{2}\left(\frac{\partial W}{\partial Z}\right)^2 + \frac{1}{2}\gamma^2 R^2\left(\frac{\partial U}{\partial Z}\right)^2.
\end{aligned} \tag{3.17}$$

3.2. Nonlinear rod equations

Following Wright (1982), we multiply (3.14) by R and integrate both the resulting equation and (3.15) over the cross-section of the rod to get

$$-2(1+U)\Lambda_2+(1+U)\frac{\partial\Lambda_3}{\partial Z}+\frac{\partial^2 U}{\partial Z^2}\Lambda_4+\frac{\partial U}{\partial Z}\frac{\partial\Lambda_4}{\partial Z}=\frac{\gamma^2}{2}\frac{\partial^2 U}{\partial t^2}, \tag{3.18}$$

$$\frac{\partial^2 W}{\partial Z^2}\Lambda_1+\left(1+\frac{\partial W}{\partial Z}\right)\frac{\partial\Lambda_1}{\partial Z}=\frac{\partial^2 W}{\partial t^2}, \tag{3.19}$$

where we have used the boundary conditions (3.16) and the following definitions

$$\Lambda_1=2\int_0^1 S_{ZZ}R\,dR,\qquad \Lambda_2=2\int_0^1 S_{RR}R\,dR,\qquad \Lambda_3=2\gamma\int_0^1 S_{RZ}R^2\,dR,\qquad \Lambda_4=2\gamma^2\int_0^1 S_{ZZ}R^3\,dR. \tag{3.20}$$

Here Λ_1 and Λ_2 are the tensions in the axial and radial directions, respectively, while Λ_3 and Λ_4 are the higher order moments. It is worth mentioning here that if (3.20) were to be rewritten in the dimensional form, the total forces and couples acting across the cross section would be averaged over the cross section. The explicit evaluations of $\Lambda_1,\Lambda_2,\Lambda_3$ and Λ_4 will be accomplished in the next two sections after specifying the constitutive relations. We observe that while (3.18) describes the motion in the radial direction, (3.19) describes the motion in the axial direction.

4. Material model defined in Eq. (2.5)

In this section, we consider the material model defined in (2.5). Using (2.5), (3.3) and (3.13), we write

$$E_\delta=\frac{1}{\delta}\left[1+\frac{a}{\delta}\left(\frac{1}{(1+U)^2\left(1+\frac{\partial W}{\partial Z}\right)}-1\right)\right]. \tag{4.1}$$

Inverting the constitutive relation (2.5), we can write the nonzero components of the stress $\mathbf{S}$ in terms of the strain $\mathbf{E}$ as

$$\mathbf{S}=\frac{E_\delta}{1+\nu}\left[\mathbf{E}+\frac{\nu}{1-2\nu}(\mathrm{tr}\,\mathbf{E})\mathbf{I}\right], \tag{4.2}$$

where $E_\delta(U,\frac{\partial W}{\partial Z})$ is given by (4.1). Using (4.2) in (3.20) and performing integration by parts, we obtain

$$\begin{aligned}
\Lambda_1&=\frac{E_\delta(U,\frac{\partial W}{\partial Z})}{1+\nu}\left(\frac{2\nu}{1-2\nu}\phi+\frac{1-\nu}{1-2\nu}\psi+\frac{(1-\nu)\gamma^2}{4(1-2\nu)}\left(\frac{\partial U}{\partial Z}\right)^2\right),\\
\Lambda_2&=\frac{E_\delta(U,\frac{\partial W}{\partial Z})}{1+\nu}\left(\frac{1}{1-2\nu}\phi+\frac{\nu}{1-2\nu}\psi+\frac{\nu\gamma^2}{4(1-2\nu)}\left(\frac{\partial U}{\partial Z}\right)^2\right),\\
\Lambda_3&=\frac{\gamma^2\,E_\delta(U,\frac{\partial W}{\partial Z})}{4(1+\nu)}(1+U)\frac{\partial U}{\partial Z},\\
\Lambda_4&=\frac{\gamma^2\,E_\delta(U,\frac{\partial W}{\partial Z})}{1+\nu}\left(\frac{\nu}{1-2\nu}\phi+\frac{1-\nu}{2(1-2\nu)}\psi+\frac{(1-\nu)\gamma^2}{6(1-2\nu)}\left(\frac{\partial U}{\partial Z}\right)^2\right),
\end{aligned} \tag{4.3}$$

where we use the notation

$$\phi=U+\frac{1}{2}U^2,\quad \psi=\frac{\partial W}{\partial Z}+\frac{1}{2}\left(\frac{\partial W}{\partial Z}\right)^2. \tag{4.4}$$

Eqs. (3.18)–(3.19), (4.1), (4.3) and (4.4) govern the dynamics of the elastic rod made of strain-limiting material described by (2.5). Substituting (4.1) and (4.3) into (3.18)–(3.19) we may get the explicit forms of the rod equations corresponding to the fully nonlinear case but the resulting equations are cumbersome and will not be presented here. There is a need for an intermediate approximation which provides a more tractable set of equations but retains the physical nature of the nonlinear model. So far the resulting equations are exact, and in the next two subsections we consider the linear and the weakly nonlinear cases derived using approximations based on the smallness of U and $\frac{\partial W}{\partial Z}$.

4.1. Linear case

Before considering the weakly nonlinear case, it will be constructive to study the linearized equations. We assume that both U and $\frac{\partial W}{\partial Z}$ and their derivatives are small enough to justify the linear approximation. In such a case, we can approximate E_δ as

$$E_{\delta\ell}=\frac{1}{\delta}-\frac{a}{\delta^2}\left(2U+\frac{\partial W}{\partial Z}\right). \tag{4.5}$$

Using this, the linearized forms of $\Lambda_1, \Lambda_2, \Lambda_3$ and Λ_4 given in (4.3) can be calculated as

$$\begin{aligned} \Lambda_{1\ell} &= \frac{1}{\delta(1+\nu)}\left(\frac{2\nu}{1-2\nu}U + \frac{1-\nu}{1-2\nu}\frac{\partial W}{\partial Z}\right), \\ \Lambda_{2\ell} &= \frac{1}{\delta(1+\nu)}\left(\frac{1}{1-2\nu}U + \frac{\nu}{1-2\nu}\frac{\partial W}{\partial Z}\right), \\ \Lambda_{3\ell} &= \frac{\gamma^2}{4\,\delta(1+\nu)}\frac{\partial U}{\partial Z}, \\ \Lambda_{4\ell} &= \frac{\gamma^2}{\delta(1+\nu)}\left(\frac{\nu}{1-2\nu}U + \frac{1-\nu}{2(1-2\nu)}\frac{\partial W}{\partial Z}\right). \end{aligned} \tag{4.6}$$

As expected, the constant a, which demonstrates the effect of physical nonlinearity, does not appear in these equations. Similarly, we obtain the linearized form of the equations of motion given in (3.18)–(3.19) as follows:

$$-2\Lambda_{2\ell} + \frac{\partial \Lambda_{3\ell}}{\partial Z} = \frac{\gamma^2}{2}\frac{\partial^2 U}{\partial t^2}, \tag{4.7}$$

$$\frac{\partial \Lambda_{1\ell}}{\partial Z} = \frac{\partial^2 W}{\partial t^2}. \tag{4.8}$$

If we substitute (4.6) into (4.7)–(4.8), we obtain two coupled linear equations of motion as

$$\frac{\gamma^2}{2(1+\nu)}\frac{\partial^2 U}{\partial Z^2} - \frac{4}{(1+\nu)(1-2\nu)}U - \frac{4\nu}{(1+\nu)(1-2\nu)}\frac{\partial W}{\partial Z} = \delta\,\gamma^2\frac{\partial^2 U}{\partial t^2}, \tag{4.9}$$

$$\frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{\partial^2 W}{\partial Z^2} + \frac{2\nu}{(1+\nu)(1-2\nu)}\frac{\partial U}{\partial Z} = \delta\,\frac{\partial^2 W}{\partial t^2} \tag{4.10}$$

for U and W. Using Lame constants, non-dimensionalized by E_0,

$$\lambda = \frac{\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{1}{2(1+\nu)},$$

we can rewrite (4.9)–(4.10) in the form

$$\gamma^2\mu\,\frac{\partial^2 U}{\partial Z^2} - 8\,(\lambda+\mu)U - 4\,\lambda\frac{\partial W}{\partial Z} = \delta\,\gamma^2\frac{\partial^2 U}{\partial t^2}, \tag{4.11}$$

$$(\lambda+2\mu)\frac{\partial^2 W}{\partial Z^2} + 2\,\lambda\frac{\partial U}{\partial Z} = \delta\,\frac{\partial^2 W}{\partial t^2}. \tag{4.12}$$

If we set $\delta = 1$, (4.11)–(4.12) reduce to the Mindlin–Herrmann equations (Mindlin & Herrmann, 1950) that has been widely studied in the literature. We refer the reader to Nobili and Saccomandi (2024) and Nobili (2024) for a discussion on how the Mindlin–Herrmann equations relate to the Love hypothesis and the Pochhammer–Chree solutions within the context of classical elasticity. An interesting point here is that the effects of the inertia terms decrease for decaying values of δ. Regarding wave propagation, a primary feature of (4.11)–(4.12) is dispersion. Taking U and W to be proportional to $e^{i(kx-\omega t)}$ in (4.11)–(4.12), we find the dispersion relation in the form

$$\gamma^2(\delta\omega^2 - (\lambda+2\mu)k^2)(\delta\omega^2 - \mu k^2) - 8(\lambda+\mu)(\delta\omega^2 - k^2) = 0, \tag{4.13}$$

where k and ω denote the dimensionless wavenumber and the dimensionless frequency, respectively. As expected, the dispersion relation (4.13) is equivalent to the dispersion relation of the Mindlin–Herrmann equations for $\delta = 1$. The dispersion curve (4.13) in the plane (k,ω) consists of two branches, the upper branch and the lower branch called the optical branch and the acoustic branch, respectively. It is interesting to note that neglecting the quadratic term in δ in (4.13) for small δ leads to the dispersion relation

$$\frac{\omega^2}{k^2} = \frac{8(\lambda+\mu) + \mu(\lambda+2\mu)\gamma^2k^2}{\delta[8(\lambda+\mu) + (\lambda+3\mu)\gamma^2k^2]},$$

which consists of the acoustic branch only.

4.2. Weakly nonlinear case

In the previous section, we found that when $\delta = 1$ the linearized equations are in fact the Mindlin-Herrmann equations. Here, we derive the nonlinear equations which are correct up to the order of γ^2, where γ is the slenderness parameter. We now define a new set of dependent variables as

$$U = \gamma^2\widetilde{U}, \qquad \frac{\partial W}{\partial Z} = \gamma^2\frac{\partial \widetilde{W}}{\partial Z}. \tag{4.14}$$

Now, we will use these in the nonlinear Eqs. (3.18)–(3.19), (4.1),(4.3) and (4.4), and keep the first two orders of approximation with respect to γ^2. Substituting (4.14) into (4.1) we obtain

$$E_\delta = \frac{1}{\delta} - \frac{a\gamma^2}{\delta^2}\left(2\widetilde{U} + \frac{\partial \widetilde{W}}{\partial Z}\right) + \cdots. \tag{4.15}$$

Substituting (4.14) into (4.3)–(4.4) and expanding the resulting equations in Taylor series in γ^2 we get

$$\begin{aligned} \Lambda_1 &= \Lambda_1^{(1)}\gamma^2 + \Lambda_1^{(2)}\gamma^4 + \cdots, \\ \Lambda_2 &= \Lambda_2^{(1)}\gamma^2 + \Lambda_2^{(2)}\gamma^4 + \cdots, \\ \Lambda_3 &= \Lambda_3^{(2)}\gamma^4 + \cdots, \\ \Lambda_4 &= \Lambda_4^{(2)}\gamma^4 + \cdots. \end{aligned} \tag{4.16}$$

As expected, the functions $\Lambda_1^{(1)}, \Lambda_2^{(1)}, \Lambda_3^{(2)}, \Lambda_4^{(2)}$ in the above expansions correspond to the linear terms. If $\widetilde{U}$ is replaced for U and $\widetilde{W}$ for W in (4.6), then $\Lambda_{1\ell}, \Lambda_{2\ell}, \Lambda_{3\ell}, \Lambda_{4\ell}$ are replaced by $\Lambda_1^{(1)}, \Lambda_2^{(1)}, \gamma^2\Lambda_3^{(2)}, \gamma^2\Lambda_4^{(2)}$, respectively. Therefore, we do not write them explicitly here. Also, $\Lambda_1^{(2)}$ and $\Lambda_2^{(2)}$ which contain only quadratic nonlinearities are obtained as

$$\begin{aligned} \Lambda_1^{(2)} &= \frac{1}{(1+\nu)\delta}\left\{\frac{\nu}{1-2\nu}\widetilde{U}^2 + \frac{1-\nu}{2(1-2\nu)}\left(\frac{\partial\widetilde{W}}{\partial Z}\right)^2 - \frac{2a}{\delta}\left(2\widetilde{U} + \frac{\partial\widetilde{W}}{\partial Z}\right)\left[\frac{\nu}{1-2\nu}\widetilde{U} + \frac{1-\nu}{2(1-2\nu)}\frac{\partial\widetilde{W}}{\partial Z}\right]\right\}, \\ \Lambda_2^{(2)} &= \frac{1}{(1+\nu)\delta}\left\{\frac{1-\nu}{2(1-2\nu)}\widetilde{U}^2 + \frac{\nu}{2(1-2\nu)}\left(\frac{\partial\widetilde{W}}{\partial Z}\right)^2 - \frac{2a}{\delta}\left(2\widetilde{U} + \frac{\partial\widetilde{W}}{\partial Z}\right)\left[\frac{1-\nu}{2(1-2\nu)}\widetilde{U} + \frac{\nu}{2(1-2\nu)}\frac{\partial\widetilde{W}}{\partial Z}\right]\right\}. \end{aligned} \tag{4.17}$$

Using (4.14) and (4.16) in (3.18)–(3.19) and neglecting the higher-order terms we get

$$-2\Lambda_2^{(1)} + \gamma^2\left(-2\Lambda_2^{(2)} - 2\widetilde{U}\Lambda_2^{(1)} + \frac{\partial\Lambda_3^{(2)}}{\partial Z}\right) = \frac{\gamma^2}{2}\frac{\partial^2\widetilde{U}}{\partial t^2}, \tag{4.18}$$

$$\frac{\partial\Lambda_1^{(1)}}{\partial Z} + \gamma^2\left(\frac{\partial^2\widetilde{W}}{\partial Z^2}\Lambda_1^{(1)} + \frac{\partial\Lambda_1^{(2)}}{\partial Z} + \frac{\partial\widetilde{W}}{\partial Z}\frac{\partial\Lambda_1^{(1)}}{\partial Z}\right) = \frac{\partial^2\widetilde{W}}{\partial t^2}, \tag{4.19}$$

Substituting (4.17) into (4.18)–(4.19) we obtain

$$\frac{\gamma^2}{2(1+\nu)}\frac{\partial^2\widetilde{U}}{\partial Z^2} - \frac{4}{(1+\nu)(1-2\nu)}\widetilde{U} - \frac{4\nu}{(1+\nu)(1-2\nu)}\frac{\partial\widetilde{W}}{\partial Z} + \gamma^2 N_1(\widetilde{U},\widetilde{W}) = \delta\gamma^2\frac{\partial^2\widetilde{U}}{\partial t^2}, \tag{4.20}$$

$$\frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{\partial^2\widetilde{W}}{\partial Z^2} + \frac{2\nu}{(1+\nu)(1-2\nu)}\frac{\partial\widetilde{U}}{\partial Z} + \gamma^2 N_2(\widetilde{U},\widetilde{W}) = \delta\frac{\partial^2\widetilde{W}}{\partial t^2}, \tag{4.21}$$

where the quadratic nonlinear terms N_1 and N_2 are given by

$$N_1(\widetilde{U},\widetilde{W}) = -\frac{2}{(1+\nu)(1-2\nu)}\left\{\left(3-4\frac{a}{\delta}\right)\widetilde{U}^2 + 2\left(\nu-(1+2\nu)\frac{a}{\delta}\right)\widetilde{U}\frac{\partial\widetilde{W}}{\partial Z} + \nu\left(1-2\frac{a}{\delta}\right)\left(\frac{\partial\widetilde{W}}{\partial Z}\right)^2\right\},$$

and

$$N_2(\widetilde{U},\widetilde{W}) = \frac{1}{(1+\nu)(1-2\nu)}\left\{2\nu\left(1-4\frac{a}{\delta}\right)\widetilde{U}\frac{\partial\widetilde{U}}{\partial Z} + 2\left(\nu-\frac{a}{\delta}\right)\left(\frac{\partial\widetilde{U}}{\partial Z}\frac{\partial\widetilde{W}}{\partial Z} + \widetilde{U}\frac{\partial^2\widetilde{W}}{\partial Z^2}\right) + (1-\nu)\left(3-2\frac{a}{\delta}\right)\frac{\partial\widetilde{W}}{\partial Z}\frac{\partial^2\widetilde{W}}{\partial Z^2}\right\}.$$

The coupled Eqs. (4.20)–(4.21) are the governing equations for the dynamics of the nonlinear rod in the weakly nonlinear case. As expected, eliminating the quadratic terms on the left-hand side of (4.20)–(4.21), gives the coupled Eqs. (4.9)–(4.10) corresponding to the linear case. Since (4.20)–(4.21) involve the constant a, they include the effect of physical nonlinearity.

5. Material model defined in Eq. (2.6)

In this section, for the material model defined in (2.6), we present the rod equations obtained by following a similar process as in the previous section. Using (3.5) the Frobenius norm of $\mathbf{S}$ is calculated in non-dimensional form as

$$|\mathbf{S}| = \left(2S_{RR}^2 + S_{ZZ}^2 + 2S_{RZ}^2\right)^{1/2}. \tag{5.1}$$

Inverting the dimensionless form of the constitutive relation (2.6), we can write the nonzero components of the stress $\mathbf{S}$ in terms of the strain $\mathbf{E}$ as

$$\mathbf{S} = \frac{\mathbf{E}}{(\delta^2 - |\mathbf{E}|^2)^{1/2}}, \tag{5.2}$$

where

$$|\mathbf{E}| = \left(2E_{RR}^2 + E_{ZZ}^2 + 2E_{RZ}^2\right)^{1/2}. \tag{5.3}$$

Substituting (3.17) into (5.3) we obtain the Frobenius norm in terms of U, $\frac{\partial U}{\partial Z}$ and $\frac{\partial W}{\partial Z}$ as

$$|\mathbf{E}|^2 = 2\phi^2 + \psi^2 + \frac{1}{2}\gamma^2 R^2\left(\frac{\partial U}{\partial Z}\right)^2(1+2\psi+2\phi) + \frac{1}{4}\gamma^4 R^4\left(\frac{\partial U}{\partial Z}\right)^4, \tag{5.4}$$

where ϕ and ψ are as in (4.4). It follows from (5.2) that the explicit forms of the stress components are

$$S_{RR} = \frac{\phi}{(\delta^2 - |\mathbf{E}|^2)^{1/2}}, \qquad S_{RZ} = \frac{\frac{1}{2}\gamma R(1+U)\frac{\partial U}{\partial Z}}{(\delta^2 - |\mathbf{E}|^2)^{1/2}}, \qquad S_{ZZ} = \frac{\psi + \frac{1}{2}\gamma^2 R^2\left(\frac{\partial U}{\partial Z}\right)^2}{(\delta^2 - |\mathbf{E}|^2)^{1/2}}, \tag{5.5}$$

where $|\mathbf{E}|^2$ is given by (5.4). Substituting these expressions into (3.20) and using the resulting equations in the nonlinear Eqs. (3.18)–(3.19) we could obtain the governing equations for the nonlinear rod corresponding to the constitutive relation (5.2). However, since the integrals involved are difficult to calculate explicitly and the resulting equations are highly complicated, we omit this step here.

5.1. Weakly nonlinear case

Using (4.14) in (5.4) and (5.5), and Taylor expansion in γ^2, after some lengthy calculations, we obtain the stress components as

$$\begin{aligned} S_{RR} &= \frac{\gamma^2}{\delta}\widetilde{U} + \frac{\gamma^4}{2\delta}\widetilde{U}^2 + \cdots, \\ S_{RZ} &= \frac{\gamma^3}{2\delta}R\frac{\partial\widetilde{U}}{\partial Z} + \frac{\gamma^5}{2\delta}R\,\widetilde{U}\,\frac{\partial\widetilde{U}}{\partial Z} + \cdots, \\ S_{ZZ} &= \frac{\gamma^2}{\delta}\frac{\partial\widetilde{W}}{\partial Z} + \frac{\gamma^4}{2\delta}\left(\frac{\partial\widetilde{W}}{\partial Z}\right)^2 + \cdots. \end{aligned} \tag{5.6}$$

Substituting these into (3.20) and assuming the expansions given in (4.16) are valid, we calculate the linear terms as

$$\Lambda_1^{(1)} = \frac{1}{\delta}\frac{\partial\widetilde{W}}{\partial Z}, \qquad \Lambda_2^{(1)} = \frac{1}{\delta}\widetilde{U}, \qquad \Lambda_3^{(2)} = \frac{1}{4\delta}\frac{\partial\widetilde{U}}{\partial Z}, \qquad \Lambda_4^{(2)} = \frac{1}{2\delta}\frac{\partial\widetilde{W}}{\partial Z} \tag{5.7}$$

and the quadratic terms as

$$\Lambda_1^{(2)} = \frac{1}{2\,\delta}\left(\frac{\partial\widetilde{W}}{\partial Z}\right)^2 \qquad \Lambda_2^{(2)} = \frac{1}{2\,\delta}\widetilde{U}^2. \tag{5.8}$$

Eqs. (4.18)–(4.19) are still valid in the present weakly nonlinear case corresponding to (5.2). Substituting (5.7) and (5.8) into (4.18)–(4.19) gives

$$\frac{\gamma^2}{2}\frac{\partial^2\widetilde{U}}{\partial Z^2} - 4\,\widetilde{U} - 6\gamma^2\widetilde{U}^2 = \gamma^2\delta\frac{\partial^2\widetilde{U}}{\partial t^2}, \tag{5.9}$$

$$\frac{\partial^2\widetilde{W}}{\partial Z^2} + 3\gamma^2\frac{\partial\widetilde{W}}{\partial Z}\frac{\partial^2\widetilde{W}}{\partial Z^2} = \delta\frac{\partial^2\widetilde{W}}{\partial t^2}. \tag{5.10}$$

These equations govern the dynamics of the nonlinear rod in the weakly nonlinear case when the constitutive relation of the material is given by (2.6) and consequently by (5.2). What is special about these equations is that they are uncoupled and (5.10) is non-dispersive. We again observe that the effects of the inertia terms decrease for decaying values of δ.

5.2. Linear case

If we neglect the nonlinear terms in (5.9)–(5.10) we get the linear equations

$$\frac{\gamma^2}{2}\frac{\partial^2\widetilde{U}}{\partial Z^2} - 4\,\widetilde{U} = \gamma^2\delta\frac{\partial^2\widetilde{U}}{\partial t^2}, \tag{5.11}$$

$$\frac{\partial^2\widetilde{W}}{\partial Z^2} = \delta\frac{\partial^2\widetilde{W}}{\partial t^2}, \tag{5.12}$$

which are uncoupled and (5.12) is non-dispersive. This is a surprising result in the sense that the equations being uncoupled and the one describing the axial motion being non-dispersive is opposite to what is observed in (4.9)–(4.10).

6. Conclusion

We conclude with a few words about the differences between the rod equations derived in Sections 4 and 5. In Section 4, we considered an elastic rod whose material model is given by (2.5). For that material, we were able to obtain a set of two coupled dispersive equations governing the motion for the weakly nonlinear and the linear cases. However, this is not the case for the material model (2.6) considered in Section 5 because the corresponding equations are uncoupled. The reason for this difference can be most easily understood by comparing the linear Eqs. (5.11)–(5.12) with the linear Eqs. (4.9)–(4.10). If $\nu = 0$ is taken in (4.9)–(4.10), it is observed that these equations transform exactly into (5.11)–(5.12). This is a natural consequence of the fact that the material model given by (2.6) does not include the Poisson effect. Recall that the Poisson ratio is a measure of the Poisson effect and the Poisson effect in the material model (2.5) arises as a consequence of the presence of the $(\mathrm{tr}\,\mathbf{S})$ term. A similar comparison can be made between (4.20)–(4.21) and (5.9)–(5.10), which correspond to the weakly nonlinear case. In this case, it will be seen that taking only $\nu = 0$ in (4.20)–(4.21) is not sufficient to obtain Eqs. (5.9)–(5.10). To arrive at Eqs. (5.9)–(5.10) from Eqs. (4.20)–(4.21), it is necessary to take both $\nu = 0$ and $a = 0$ in (4.20)–(4.21). This shows that the material model (2.6) has a rather limited applicability compared to the model (2.5) due to both the absence of the $(\mathrm{tr}\,\mathbf{S})$ term, which introduces the Poisson effect, and the absence of the parameter a, which characterizes physical nonlinearity.

CRediT authorship contribution statement

H.A. Erbay: Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization. **Y. Şengül:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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