CONSERVATIVE STOCHASTIC PDE AND FLUCTUATIONS OF THE SYMMETRIC SIMPLE EXCLUSION PROCESS

NICOLAS DIRR, BENJAMIN FEHRMAN, AND BENJAMIN GESS

Abstract. In this paper, we provide a continuum model for the fluctuations of the symmetric simple exclusion process about its hydrodynamic limit. The model is based on an approximating sequence of stochastic PDEs with nonlinear, conservative noise. In the small-noise limit, we show that the fluctuations of the solutions are to first-order the same as the fluctuations of the particle system. Furthermore, the SPDEs correctly simulate the rare events in the particle process. We prove that the solutions satisfy a zero-noise large deviations principle with rate function equal to that describing the deviations of the symmetric simple exclusion process.

1. Introduction

In this work we propose a nonlinear, conservative stochastic PDE as a continuum model incorporating fluctuation corrections for the symmetric simple exclusion process (SSEP). More precisely, for every $N \in \mathbb{N}$ let $\eta_t^N(x)$ be the SSEP on the discrete torus $\mathbb{Z}^d/N\mathbb{Z}^d$ with slowly varying initial state $\overline{\rho}_0(x/N)$ for some $\overline{\rho}_0 \in \mathrm{C}^4(\mathbb{T}^d;[0,1])$ on the d-dimensional unit torus $\mathbb{T}^d$ (for details see Kipnis and Landim [44, Chapter 2]). Then, the parabolically rescaled empirical measures

$$\pi_N := \frac{1}{N^d} \sum_{x \in (\mathbb{Z}^d/N\mathbb{Z}^d)} \delta_{\frac{x}{N}} \eta_{N^2 t}(x),$$

converge in probability as $N \to \infty$ to a deterministic measure that is absolutely continuous with respect to the Lebesgue measure, with density $\bar{\rho}$ solving

$$\partial_t \bar{\rho} = \Delta \bar{\rho} \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ \bar{\rho}(\cdot,0) = \bar{\rho}_0. \tag{1.1}$$

In this way, solutions to the heat equation (1.1) describe the SSEP dynamics up to zeroth order. To reach higher-order continuum approximations, it is necessary to incorporate fluctuations present in π_N. The non-equilibrium central limit fluctuations of π_N have been analyzed in $d = 1$ by Galves, Kipnis, and Spohn [31] and in $d \geq 2$ by Ravishankar [60], where it is shown that the measures

$$m_N := N^{\frac{d}{2}}(\pi_N - \mathbb{E}\pi_N),$$

converge as $N \to \infty$ to the generalized Ornstein–Uhlenbeck process v solving

$$\partial_t v = \Delta v - \nabla \cdot (\sqrt{\overline{\rho}(1-\overline{\rho})}\,\mathrm{d}\xi) \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ v = 0 \ \text{ on } \ \mathbb{T}^d \times \{0\}, \tag{1.2}$$

for $\mathrm{d}\xi$ a d-dimensional space-time white noise and for $\overline{\rho}$ solving (1.1). More recently, the third author and Konarovskyi [32] have proven a quantitative CLT, and shown an optimal rate of convergence for the measures m_N to v. Since solutions of (1.2) are not function-valued, the convergence is in distribution on the space $D([0,\infty);\mathcal{D}'(\mathbb{T}^d))$, see [60] and Jara and Landim [42, Theorem 2.2].

Introducing the fluctuation corrected continuum model $\bar{\rho}_N := \overline{\rho}\,\mathrm{d}x + N^{-d/2}v$ yields

$$\pi_N = \bar{\rho}_N + (\mathbb{E}\pi_N - \overline{\rho}\,\mathrm{d}x) + o(N^{-(\frac{d}{2}\wedge 1)}), \tag{1.3}$$

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where the exponent in (1.3) appears due to a discretization error of order N^{-1}. It is shown in [42] that $\mathbb{E}\pi_N$ is a solution to the discrete heat equation, and the estimates of [42, Theorem A.1] that compare $\mathbb{E}\pi_N$ to the solution of (1.1) prove with (1.3) that we have the first order expansion

$$\pi_N = \bar{\rho}_N + o(N^{-(\frac{d}{2}\wedge 1)}). \tag{1.4}$$

We do not expect that this rate is optimal. The actual order of convergence is expected to be higher, but this would require a quantified CLT for the SSEP and a careful analysis of the discretization errors, which is the subject of the subsequent work [32].

However, while it follows from (1.4) that the continuum model $\overline{\rho}_N$ correctly describes to first-order the central limit fluctuations of the π_N, the rare events of π_N are not correctly captured by the affine linear expansion $\bar{\rho}_N$. Indeed, in Quastel, Rezakhanlou, and Varadhan [59] and [44, Chapter 10] it has been shown that the dynamical (equilibrium) large deviations for π_N are described in terms of the rate function

$$I_{\rho_0}(\mu) = \frac{1}{2}\inf\left\{ \|g\|_{L^2}^2 : \mu = \rho\, dx,\, \partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\rho(1-\rho)}g) \text{ with } \rho(\cdot,0)=\rho_0\right\}. \tag{1.5}$$

In contrast $\overline{\rho}_N$ solves the linear equation, for the solution $\overline{\rho}$ of (1.1),

$$\partial_t\overline{\rho}_N = \Delta\overline{\rho}_N - N^{-\frac{d}{2}}\nabla\cdot(\sqrt{\overline{\rho}(1-\overline{\rho})}\,\mathrm{d}\xi) \text{ in } \mathbb{T}^d\times(0,T) \text{ with } \overline{\rho}_N=\rho_0 \text{ on } \mathbb{T}^d\times\{0\}, \tag{1.6}$$

and, as follows formally from (1.6), the rate function associated to rare events of $\bar{\rho}_N$ is given by

$$J_{\rho_0}(\mu) = \frac{1}{2}\inf\left\{ \|g\|_{L^2}^2 : \mu = \rho\, dx,\, \partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\bar{\rho}(1-\bar{\rho})}g) \text{ with } \rho(\cdot,0)=\rho_0\right\}. \tag{1.7}$$

Motivated by work of the second and third authors in the context of the zero range process [26], and by connections to macroscopic fluctuation theory (see, for example, Bertini, De Sole, Gabrielli, Jona Lasinio, and Landim [3] and Derrida [18]) and fluctuating hydrodynamics (see, for example, Hohenberg and Halperin [38], Landau and Lifshitz [48], Spohn [66], and Bouchet, Gawędzki, and Nardini [4]), in this work, we introduce the nonlinear, stochastic PDE

$$\partial_t\rho_N = \Delta\rho_N - N^{-\frac{d}{2}}\nabla\cdot\left(\sqrt{\rho_N(1-\rho_N)}\circ \mathrm{d}\xi^{K(N)}\right), \tag{1.8}$$

for spectral approximations $\{\mathrm{d}\xi^K\}_{K\in\mathbb{N}}$ of $\mathbb{R}^d$-valued space-time white noise defined in Section 2.1, where $\circ$ denotes Stratonovich integration—see Remark 2.2 below for a further discussion of the choice of Stratonovich noise. We prove that, under an appropriate $N\to\infty$, $K(N)\to\infty$ scaling, the solutions ρ_N not only yield a first order expansion

$$\pi_N = \rho_N + o(N^{-(\frac{d}{2}\wedge 1)}),$$

analogous to (1.4), but also display the correct rare event behavior, in the sense that the large deviations for the ρ_N are described by the rate function (1.5). That is, as $N\to\infty$,

$$\mathbb{P}[\rho_N\in A] \approx e^{-N^d \inf_{\mu\in A} I_{\rho_0}(\mu)}.$$

The introduction of spatially correlated noise in (1.8) is necessary in order to obtain function-valued solutions, and to prove the estimates of Proposition 2.10 below. This fact is due to the supercriticality of (1.8) when driven by space-time white noise, a point that is already present in the simpler equation (1.2) for which solutions exist only in a space of distributions.

We will henceforth let $\varepsilon\in(0,1)$ play the role of N^{-d} to emphasize the fact that the results apply to the full continuous limit $\varepsilon\to 0$, and we will consider the solutions of the equation

$$\partial_t\rho^\varepsilon = \Delta\rho^\varepsilon - \sqrt{\varepsilon}\nabla\cdot\left(\sqrt{\rho^\varepsilon(1-\rho^\varepsilon)}\circ \mathrm{d}\xi^{K(\varepsilon)}\right) \text{ in } \mathbb{T}^d\times(0,T) \text{ with } \rho^\varepsilon(\cdot,0)=\rho_0. \tag{1.9}$$

We will write (1.9) in the Itô formulation

$$\partial_t\rho^\varepsilon = \Delta\rho^\varepsilon - \sqrt{\varepsilon}\nabla\cdot\left(\sqrt{\rho^\varepsilon(1-\rho^\varepsilon)}\,\mathrm{d}\xi^{K(\varepsilon)}\right) + \frac{\varepsilon\langle\xi^{K(\varepsilon)}\rangle}{8}\Delta\Theta(\rho^\varepsilon), \tag{1.10}$$

for the spatially constant quadratic variation $\langle\xi^{K(\varepsilon)}\rangle$ of the probabilistically stationary noise $\xi^{K(\varepsilon)}$, and for $\Theta \in \mathrm{C}^1(0,1)$ satisfying $\Theta'(\xi) = \frac{(1-2\xi)^2}{\xi(1-\xi)}$. Equation (1.10) demonstrates two fundamental difficulties in treating (1.9). The first is the irregularity appearing in the noise coefficient due to the singularities of the square-root and its derivative near $\rho^\varepsilon \simeq 0$ and $\rho^\varepsilon \simeq 1$. The second is the logarithmic divergence of Θ near $\xi \simeq 0$ and $\xi \simeq 1$ and the fact that $\Theta(\rho^\varepsilon)$ is not known to be integrable. For these reasons, it is not even clear how to define a notion of weak solution to (1.9).

These issues were originally addressed by the second and third authors in [27], where a general class of SPDEs were treated including, for example, the Dean–Kawasaki equation with correlated Stratonovich noise

$$\partial_t \rho = \Delta\rho - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho}\circ \mathrm{d}\xi^{K(\varepsilon)}). \tag{1.11}$$

In this case, the Stratonovich-to-Itô correction presents similar difficulties and takes the form

$$\partial_t \rho = \Delta\rho - \sqrt{\varepsilon}\nabla\cdot\left(\sqrt{\rho}\,\mathrm{d}\xi^{K(\varepsilon)}\right) + \frac{\varepsilon\langle\xi^{K(\varepsilon)}\rangle}{8}\Delta\log(\rho^\varepsilon).$$

In [27] the authors proved the well-posedness of *stochastic kinetic solutions*, which are based on the equation's kinetic formulation and which renormalize the solutions away from their zero sets. We refer to Section 2.2 of this paper and [27, Section 3] for a derivation of the equation's kinetic form and a more complete explanation of the difficulties in treating equations like (1.10) and (1.11). In terms of (1.9), the equation must be localized away from both of the sets $\{\rho^\varepsilon \simeq 0\}$ and $\{\rho^\varepsilon \simeq 1\}$. The first main result of this work extends the methods of [27] to equations with noise coefficients like $\sqrt{\rho^\varepsilon(1-\rho^\varepsilon)}$ that contain multiple singularities.

Theorem (cf. Theorems 2.7, 2.16)**.** *Let $T\in(0,\infty)$, let $\varepsilon\in(0,1)$, let the noise $\{\xi^K\}_{K\in\mathbb{N}}$ satisfy Assumption* 3.1 *(see also Assumption* 2.1*) with respect to a filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, and let $\rho_0\in L^\infty(\Omega;L^\infty(\mathbb{T}^d;[0,1]))$ be $\mathcal{F}_0$-measurable. Then there exists a unique* stochastic kinetic solution *of* (1.9) *in the sense of Definition* 2.6*. Furthermore, $\mathbb{P}$-a.s. stochastic kinetic solutions $\rho_1^\varepsilon,\rho_2^\varepsilon$ corresponding to initial data $\rho_{0,1},\rho_{0,2}$ satisfy*

$$\sup_{t\in[0,T]}\|\rho_1^\varepsilon(\cdot,t)-\rho_2^\varepsilon(\cdot,t)\|_{L^1(\mathbb{T}^d)} \le \|\rho_{0,1}-\rho_{0,2}\|_{L^1(\mathbb{T}^d)}\,.$$

In the second main result of this paper, we identify a scaling regime such that the solutions of (1.9) satisfy a central limit theorem (CLT) equal to that of the SSEP. The fluctuations are characterized by a generalized Ornstein–Uhlenbeck process, which is a continuous $H^{-s}(\mathbb{T}^d)$-valued process, for every $s>\frac{d}{2}$, that solves (1.2). A primary difficulty in proving the CLT is that the solutions of (1.9) are defined in the renormalized sense of Definition 2.6, and the fundamentally nonlinear nature of this definition is incompatible with convergence in a space of distributions. For this reason, we first prove in Theorem 3.8 a quantitative CLT for solutions of (1.9) with the square root $\sqrt{\rho^\varepsilon(1-\rho^\varepsilon)}$ replaced by a smooth approximating nonlinearity $\sigma(\rho^\varepsilon)$. We then transfer the CLT for the approximating equation to the solutions of (1.9), which relies on the following novel L^∞-estimate that is based on a Moser iteration.

Theorem (cf. Theorem 3.9)**.** *Let $T\in(0,\infty)$, let $\varepsilon\in(0,1)$, let $\{\xi^K\}_{K\in\mathbb{N}}$ satisfy Assumption* 3.1 *(see also Definition* 2.1*), let $\rho_0\in L^\infty(\Omega;L^\infty(\mathbb{T}^d;[0,1]))$ be $\mathcal{F}_0$-measurable, let $M = ess\ sup_{x\in\mathbb{T}^d}\rho_0(x)$, and let $M'=\operatorname{ess\,inf}_{x\in\mathbb{T}^d}\rho_0(x)$. Then, if ρ^ε is the unique solution of* (1.9) *in the sense of Definition* 2.6 *below, there exist $c,\gamma\in(0,\infty)$ independent of ε and K but depending on T such that*

$$\mathbb{E}\Big[\,\|(\rho^\varepsilon-M)_+\|_{L^\infty(\mathbb{T}^d\times[0,T])}\Big]+\mathbb{E}\Big[\,\big\|(\rho^\varepsilon-M')_-\big\|_{L^\infty(\mathbb{T}^d\times[0,T])}\Big]\le c\big(\varepsilon K^{d+2}\big)^\gamma.$$

We now explain how the L^∞-estimate is used to establish a quantitative CLT for the original SPDE. Along appropriate scaling limits, for initial data taking values in $[\delta,1-\delta]$, the L^∞-estimate proves that the solutions of (1.9) take values in $[\delta/2, 1-\delta/2]$ with high probability. On this event, the pathwise uniqueness proof of Theorem 2.7 below shows that the solutions of (1.9) and the solutions of an approximating SPDE defined by a smooth $\sigma(\rho) \simeq \sqrt{\rho(1-\rho)}$ agree, which together with Theorem 3.8 proves the following quantitative CLT in probability. In the statement, we have fixed a specific choice of scaling limit $K(\varepsilon) \to 0$ as $\varepsilon \to 0$ for simplicity. See Theorem 3.10 for a general estimate that holds for arbitrary $K \in \mathbb{N}$ and $\varepsilon \in (0,1)$. This is the only result of the paper that requires the assumption that $\delta \leq \rho_0 \leq 1-\delta$ for some $\delta \in (0, 1/2)$.

Theorem (cf. Theorem 3.10, Corollary 3.11). *Let $T \in (0,\infty)$, let $\{\xi^K\}_{K\in\mathbb{N}}$ be the noise defined in Definition 3.1, let $\alpha_d = (\frac{1}{d+2} \wedge \frac{1}{2d})$ and let $K(\varepsilon) = \lfloor \varepsilon^{-\alpha_d} \rfloor$ for every $\varepsilon \in (0,1)$, and let $\rho_0 \in L^\infty(\mathbb{T}^d)$ satisfy $\delta \leq \rho_0 \leq 1-\delta$ for some $\delta \in (0, 1/2)$. Let ρ^ε be the solution of (1.9) corresponding to $(\varepsilon, K(\varepsilon))$ in the sense of Definition 2.6 below, let $\overline{\rho}$ be the solution of (1.1), and let $v^\varepsilon = \varepsilon^{-1/2}(\rho^\varepsilon - \overline{\rho})$. Then, for every $s > \frac{d}{2}$ there exist $c, \gamma \in (0,\infty)$ such that, for every $\eta \in (0,1)$,*

$$\mathbb{P}\Big[\|v^\varepsilon - v\|_{L^2([0,T];H^{-s}(\mathbb{T}^d))} \geq \eta \Big] \leq c\eta^{-2}\delta^{-2}\Big(\varepsilon^{\alpha_d} + \varepsilon^{\alpha_d(2s-(d+2))}\Big) + c\delta^{-1}\varepsilon^{\frac{\gamma}{2}},$$

for v the solution of (1.2) in the sense of Definition 3.6 below.

The third main result is the identification of an $\varepsilon \to 0$, $K(\varepsilon) \to \infty$ scaling regime such that the solutions ρ^ε of (1.9) satisfy a large deviations principle with rate function (1.5). The proof of the large deviations principle is based on the weak convergence approach to large deviations of Budhiraja, Dupuis, and Maroulas [7], Dupuis and Ellis [22], and Budhiraja and Dupuis [6].

Theorem (cf. Theorem 4.8). *Let $T \in (0,\infty)$, let $\{\xi^K\}_{K\in\mathbb{N}}$ be the noise defined in Definition 3.1, let $\{K(\varepsilon)\}_{\varepsilon\in(0,1)}$ be a sequence that satisfies, as $\varepsilon \to 0$,*

$$\varepsilon K(\varepsilon)^{d+2} \to 0 \ \text{ and } \ K(\varepsilon) \to \infty,$$

and for every $\rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$ let $\rho^\varepsilon(\rho_0)$ be the unique stochastic kinetic solution of (1.9) in the sense of Definition 2.6. Then, for every $\rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$ the solutions $\{\rho^\varepsilon(\rho_0)\}_{\varepsilon\in(0,1)}$ satisfy a large deviations principle on $L^2(\mathbb{T}^d \times [0,T];[0,1])$ with rate function I_{ρ_0} defined in (1.5). Furthermore, the solutions $\{\rho^\varepsilon(\rho_0)\}_{\rho_0\in L^\infty(\mathbb{T}^d;[0,1])}$ satisfy a uniform large deviations principle on $L^\infty(\mathbb{T}^d \times [0,T];[0,1])$ with respect to weakly $L^2(\mathbb{T}^d;[0,1])$-compact subsets of $L^\infty(\mathbb{T}^d;[0,1])$.

The LDP shows that the nonlinear SPDEs (1.9) satisfy a large deviations principle with rate function equal to that of the SSEP, and by (1.7) this shows on the level of large deviations that the nonlinear SPDEs (1.9) provide a more accurate description of the particle process than the linear SPDEs (1.6). This observation was previously made by the second and third authors [26] in the context of the large deviations of the zero range particle process, and it is the LDP analogue of the higher order central limit theorems of Cornalba and Fischer [14] and Cornalba, Fischer, Ingmanns, and Raithel [15], who show that certain discretized Dean–Kawasaki equations approximate the density fluctuations of weakly interacting diffusions to arbitrary order in the particle number—an approximation accuracy that cannot be obtained using the analogous linearized equations.

We finally remark that the linear diffusion is not essential for the methods and results of this work. For example, by adapting the methods of [26, 27] and the second author and Clini [13] to this setting, it would be possible to handle porous media type nonlinearities $\Phi(\rho) = \rho^m$, for every $m \in [1,\infty)$, and more generally those nonlinearities Φ satisfying the assumptions of [26, 27]. The primary difference is that this would lead to a new entropy estimate in Proposition 2.10, and would require a new treatment of nonlinearities in the proof of the CLT following the methods of [13].

1.1. **Comments on Applications and Numerics.** The use of stochastic interacting particle models (also referred to as Monte Carlo methods) in science and engineering leads to the need for accurate computation of rare events for high-dimensional stochastic systems. For some applications of interacting particle models in chemistry and material science see, for example, [12, 17, 50, 55, 58]. Interacting particle models like the SSEP provide simplified examples that nevertheless retain many of the features of more realistic models. For example, often the number of particles is quite large but the features one is interested in appear on a much coarser scale than the microscopic scale of individual particles. In such situations it is inefficient to use the particle model itself for Monte–Carlo simulations. An SPDE solved on a coarser time-space grid is more effective, but in order to investigate large deviations from the deterministic limit, this SPDE must model the fluctuations correctly on the chosen scale. If the deterministic limit path $\bar{\rho}$ is known, then both (1.6) and (1.9) are candidates for simulating the fluctuations, and, by the results of Section 3 below, they have identical fluctuations on the CLT scale. However, as argued above, only (1.9) recovers the correct large deviations behavior.

Before presenting numerical results, let us emphasize an important qualitative difference between the two SPDEs: in the case of additive noise (1.6) the fluctuations are symmetric around $\overline{\rho}$. Their strength does not depend on the actual excursion away from $\overline{\rho}$, and the expectation of the fluctuation field integrated against any smooth test function is zero. The situation is different for the nonlinear SPDE (1.9): if $\overline{\rho}$ is close to 1, for example, then excursions above, when $\rho^{\varepsilon} > \overline{\rho}$, are driven by weaker fluctuations than excursions below, so that the fluctuations are asymmetric. The same effect can be expected for the full particle system because of the exclusion rule: the closer the local density of particles is to 1, the more jumps are excluded because positions are already filled by particles, and, thus, fluctuations are damped.

In order to demonstrate this effect qualitatively, we aim to make the local density explore a large range of values, including those close to 0 and 1, where fluctuations are expected to be damped, but also those close to 1/2, where fluctuations are expected to be maximal. The scenario giving rise to Figure 1 is as follows: the evolution starts from uniformly distributed particles on the unit interval with periodic boundary conditions and the field $H = \sin(2\pi x)$ creates a rare event which consists in many particles gathering near $x = 1/4$ and few particles near $x = 3/4$. Following Kipnis and Landim [45, p.258], we modify the jump rates by weighting them with H to obtain the shifted generator

$$(L_{N,t}^{H} f)(\eta) = N^2 \sum_{|x-y|=1} \eta(x)(1-\eta(y)) e^{H(y/N)-H(x/N)} (f(\eta^{x,y}) - f(\eta)).$$

Due to the difference in the exponential, this leads to the gradient of H appearing in the exponential change of measure and the deterministic limit equation, see [45, p.258, (5.1)].

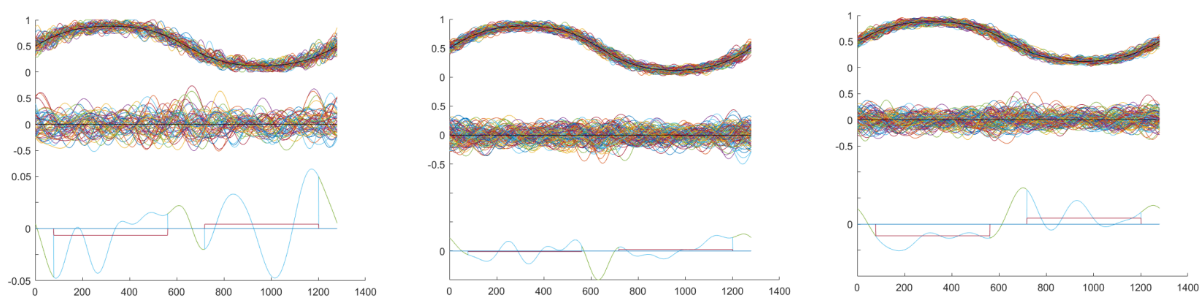

FIGURE 1. Particles (left), linear SPDE (centre), SPDE (1.9) (right). First row smoothed output, second row smoothed fluctuations, third row averaged fluctuations.

Numerically, we approximate the simple symmetric exclusion process by a Markov chain. The external field is chosen in such a way that all probabilities can be pre-computed. Nevertheless, the simulation is slow because we are on a diffusive timescale: one macroscopic time unit corresponds to N^2 microscopic time units and there are N particles. This corresponds to $O(N^3)$ Markov chain steps. The SPDEs are solved by a spectral method (Fast Fourier Transform), where the nonlinearity is resolved explicitly: we solve for each Fourier coefficient $\rho_k(t)$ of the density profile

$$\frac{\rho_k(t+\Delta t)-\rho_k(t)}{\Delta t} = -\Delta t(4\pi^2 k^2)\rho_k(t+\Delta t) + \sqrt{\varepsilon \Delta t}\left[\nabla \cdot \sqrt{\rho(t)(1-\rho(t))}\right]_k W_k,$$

where the divergence $\nabla \cdot \sqrt{\rho(1-\rho)}$ is computed by projecting $\sqrt{\rho(1-\rho)}$ into Fourier space, differentiating the Fourier expansion, and then transforming result back to physical space, where $[\cdot]_k$ denotes the $L^2(\mathbb{T}^d)$-projection on the k-th basis function in the Fourier basis, where the W_k, for $k \in \{-K, \ldots, K\}$, are independent centered Gaussians with variance 1, and where the parameter $\epsilon = N^{-1}$ is the inverse particle number as in (1.9) for $d = 1$. The numerical accuracy is determined by $M \gg K$, the number of nodes computed, and Δt, the time step. The following computations are done at time $t = 1$. The solutions are then smoothed by a projection on the span of the first 20 complex Fourier modes. The black curve is the deterministic solution $\overline{\rho}$. The second plot shows the fluctuation fields (1.6) smoothed by projecting on the span of the first 40 Fourier modes. This is all done with at least 100 independent realizations of $N = 1280$ particles. The last plot (not to scale) is the approximated pointwise expectation of the fluctuation fields, smoothed by taking the sliding average in space over 50 points.

The red line is the spatial average of the fluctuations in the regions where the density is expected to be above and below $1/2$ respectively. We indeed observe an asymmetry of the fluctuations: in the region where the density is expected to be above $1/2$ the fluctuations are on average negative and in the region with the density is expected to be below $1/2$ the fluctuations are on average positive.

Figure 2 below illustrates the qualitative advantage of the nonlinear SPDE (1.9). The number of nodes with noise is chosen in accordance with Corollary 3.11, that is, of the order $\varepsilon^{-1/2}$ for $\varepsilon \in (0,1)$ in (1.9). The number of grid points for the SPDEs is 1280, the time step is chosen to be inversely quadratic to the number of grid points in order to keep numerical errors low. The time $t = 1$ is chosen such that the fluctuations of the initial condition—independent Bernoulli random variables on each grid point—for the particle system do not play a role except for the conserved total mass, which the plot is adjusted for. The expectation is approximated by averaging over 100 independent realizations.

The plot in Figure 2 shows the fluctuation fields integrated against smooth, scalar function $\overline{\Psi}$ that is an approximation of $1_{(0,0.5)} - 1_{(0.5,1)}$. Precisely, we consider the integrated quantity

$$\varepsilon^{-1/2}\mathbb{E}\int_0^1 \overline{\Psi}(x)(\rho^\varepsilon(x,1) - \overline{\rho}(x,1))dx, \tag{1.12}$$

where $\overline{\rho}$ is the deterministic solution of (1.1) and ρ^ε is either the density of particles for $N = \varepsilon^{-1}$ or the solution of the SPDE (1.9) for $\varepsilon \in (0,1)$.

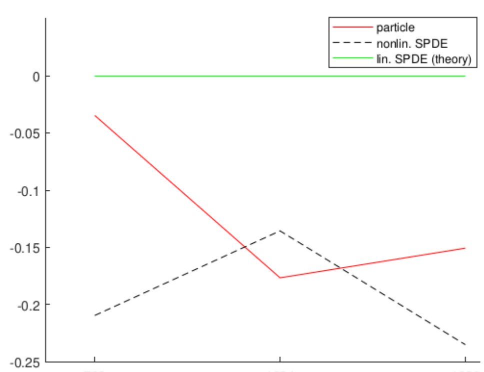

FIGURE 2. x-axis: the particle number $N = \varepsilon^{-1}$. y-axis: the value of (1.12) for ρ^ε the SSEP (red); for ρ^ε the solution of (1.9) (black); for ρ^ε the theoretical solution of (1.6) (green).

Due to the bias introduced by the field H above, for either the particle system or the nonlinear SPDE we expect that it is more likely for $\rho^\varepsilon < \overline{\rho}$ if $x \in [0, 1/2]$ and $\rho^\varepsilon > \overline{\rho}$ if $x \in [1/2, 1]$. We therefore expect that, for the particle system and nonlinear SPDEs, the quantity (1.12) is negative. Conversely, for the linear equations (1.6) for $N = \varepsilon^{-1}$ this expectation is zero. And indeed, Figure 2 illustrates qualitatively this asymmetry. In particular, for $N = 1280$, the result for the particle system (red line) is with high probability negative, and this is correctly captured by the nonlinear SPDE (1.9) (black dotted line). In contrast, the theoretical value for the linear SPDE (1.6) is zero, so it cannot capture the law of the particle model at orders beyond the CLT scale.

1.2. **Comments on the literature.** Stochastic PDEs with nonlinear, conservative noise have been studied in the context of stochastic scalar conservation laws by Lions, Perthame, and Souganidis [52, 53, 54], Friz and the third author [30], and the third author and Souganidis [33, 34]. These methods were later extended to parabolic-hyperbolic equations with conservative noise by Gess and Souganidis [35], the second and third authors [25, 27], and Dareiotis and the third author [16].

Large deviation principles for singular SPDEs have been obtained by Cerrai and Freidlin [9], Faris and Jona-Lasinio [24], Jona-Lasinio and Mitter [43], Hairer and Weber [37], and the second and third authors [26]. In particular, in the context of stochastic Allen-Cahn equations, it was shown in [37] that renormalization constants may enter into the large deviations rate functional. The constants are determined by the relative scaling of the noise intensity ε and ultraviolet cutoff K. In analogy with [26], the above results therefore identify a scaling regime in which the large deviations are unaffected by renormalization and for which the solutions of (1.9) correctly simulate the fluctuations of the particle process. Further applications of the weak convergence approach to proving large deviations principles for SPDEs include, for example, Cerrai and Debussche [8], Brzeźniak, Goldys, and Jegaraj [5], and Dong, Wu, Zhang, and Zhang [21].

The inference of the fluctuation correction from data, that is, from observations of the underlying particle system has been studied by Li, the first author, Embacher, Zimmer, and Reina in [49].

A central limit theorem for the stochastic heat equation has been obtained by Huang, Nualart, and Viitasaari [40], Huang, Nualart, Viitasaari, and Zheng [41], and for parabolic equations with multiplicative noise by Chen, Khoshnevisan, Nualart, and Pu [11]. A central limit theorem and moderate deviations principle has been obtained by Hu, Li, and Wang [39] and Guo, Zhang, and Zhuo [36] for a class of semilinear SPDEs. The authors are not aware of any prior works proving a central limit theorem for a SPDE with conservative noise.

After publication of the first version of this article, several groups have investigated the use of conservative SPDEs in the numerical approximation of particle systems: in an intriguing contribution [14], Cornalba and Fischer have shown that a system of independent Brownian motions can be approximated to arbitrary order by a discretization of the Dean–Kawasaki SPDE. A related result has been obtained by Djurdjevac, Kremp, and Perkowski in [20]. Subsequently, Cornalba, Fischer, Ingmanns, and Raithel in [15] have extended their theory to weakly interacting particles.

The fluctuations of the symmetric simple exclusion process about its hydrodynamic limit have been studied by Galves, Kipnis, and Spohn [31], Ravishankar [60], Kipnis and Varadhan [46], Ferrari, Presutti, Scacciatelli, and Vares [28, 29], and Rezakhanlou [62]. Furthermore, improving upon these qualitative results, an optimal quantitative central limit theorem for the SSEP was recently obtained by the third author and Konarovskyi [32]. Tagged particles in symmetric simple exclusion processes have been studied by Arratia [1], Sethuraman, Varadhan, and Yau [64], Varadhan [67], and Jara and Landim [42]. Large deviations principles for a general class of symmetric simple exclusion processes were obtained by Quastel, Rezakhanlou, and Varadhan [59].

1.3. **Organization of the paper.** In Section 2, we prove the well-posedness of stochastic kinetic solutions to (1.9). Section 2.1 defines the noise, Section 2.2 defines stochastic kinetic solutions and proves that they are unique, and Section 2.3 proves the existence of solutions. We prove the central limit theorem in Section 3, and we prove the large deviations principle in Section 4.

2. The well-posedness of the SPDE

We will now establish the existence and uniqueness of suitably defined renormalized kinetic solutions to equation (1.9). The section is split into three subsections. The first defines the randomness in the equation, the second defines stochastic kinetic solutions in Definition 2.6 and proves that they are unique, and the third proves that they exist.

2.1. The definition of the equation and noise.

We will first introduce the randomness in the controlled SPDE, for a spatially smooth control $g \in L^2([0,T]; H^1(\mathbb{T}^d))^d$,

$$\partial_t \rho = \Delta\rho - \nabla\cdot(\sqrt{\rho(1-\rho)} \circ \mathrm{d}\xi^F) - \nabla\cdot(\sqrt{\rho(1-\rho)}g) \ \text{ in } \ \mathbb{T}^d \times (0,\infty) \ \text{ with } \ \rho(\cdot,0)=\rho_0. \tag{2.1}$$

We will assume that the noise $\mathrm{d}\xi^F$ is colored in space and white in time, and that it is adapted to some filtration $\mathcal{F}_t$, and that the initial data is $\mathcal{F}_0$-measurable.

Assumption 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration on Ω, let $\{B^k\}_{k\in\mathbb{N}}$ be independent $\mathcal{F}_t$-adapted d-dimensional Brownian motions on Ω, and let $\{f_k \colon \mathbb{T}^d \to \mathbb{R}\}_{k\in\mathbb{N}}$ be twice continuously differentiable functions on $\mathbb{T}^d$. We define ξ^F to be the noise $\xi^F(x,t) = \sum_{k=1}^\infty f_k(x)B_t^k$ and assume that the sums $F_1 = \sum_{k=1}^\infty f_k^2$ and $F_3 = \sum_{k=1}^\infty |\nabla f_k|^2$ are finite and continuous on $\mathbb{T}^d$ and that the noise is probabilistically stationary in the sense that F_1 is constant on $\mathbb{T}^d$: this means that $\nabla F_1 = 2\sum_{k=1}^\infty f_k \nabla f_k = 0$. We also assume that $\rho_0 \in L^\infty(\Omega; L^\infty(\mathbb{T}^d; [0,1]))$ is $\mathcal{F}_0$-measurable.

The control term is necessary for the application of the weak approach to large deviations below. The reason for considering spatially smooth controls in the H^1-sense is that the Cameron–Martin space of the noise ξ^F is defined by spatially smooth functions of the form

$$g^F(x,t) = \sum_{k=1}^\infty f_k(x)g_k(t) \ \text{ with } \ \left\|g^F\right\|^2_{L^2(\mathbb{T}^d\times[0,T])^d} < \infty,$$

for maps $g_k \in L^2([0,T];\mathbb{R}^d)$. This stands in contrast to the Cameron–Martin space $L^2(\mathbb{T}^d \times [0,T];\mathbb{R}^d)$ of space-time white noise. The solutions of (2.1) describe the behavior of the system on the (potentially, rare) event that the noise is centered on the control g. In comparison to the SPDE (1.9) with $g=0$ and $s^\eta(\rho) = \sqrt{\rho(1-\rho)}$, these solutions describe the fluctuations of the process about the trajectory

$$\partial_t \overline{\rho} = \Delta\overline{\rho} - \nabla\cdot(s^\eta(\overline{\rho})g). \tag{2.2}$$

Showing the collapse, as $\varepsilon \to 0$, of the solutions to the SPDE (2.1) to the solution of the PDE (2.2) is essential to the application of the weak approach to large deviations, and it is for this reason that we prove the well-posedness of the controlled SPDE in Theorem 2.7 and Theorem 2.16 below and the collapse of the solutions to the solution of (2.2) in Proposition 4.6 below.

Remark 2.2. We now briefly remark on the choice of Stratonovich noise. On a technical level, the methods of this paper can be viewed as studying the corrected Itô equation, for $\theta \in [0,\infty)$,

$$\partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\rho(1-\rho)}\,\mathrm{d}\xi^F) - \nabla\cdot(\sqrt{\rho(1-\rho)}g) + \theta\frac{F_1}{4}\nabla\cdot\Big(\frac{(1-2\rho)^2}{\rho(1-\rho)}\nabla\rho\Big), \tag{2.3}$$

for the correction term defined by $\theta\frac{F_1}{4}\nabla\cdot\big(\frac{(1-2\rho)^2}{\rho(1-\rho)}\nabla\rho\big)$. Formally, the choice of Itô noise in (2.1) corresponds to $\theta = 0$, the choice of Stratonovich noise corresponds to $\theta = 1/2$, and the choice of Klimontovich noise corresponds to $\theta = 1$. The role of this choice is most apparent in the a priori estimates of Proposition 2.10 below and therefore in the existence of solutions, where the cancellation coming from the Stratonovich-to-Itô correction is used essentially to prove the L^p and entropy-dissipation estimates. The methods of this paper apply without change to equation (2.3) for any $\theta \in [1/2, \infty)$. The choice of Stratonovich noise $\theta = 1/2$ is the minimal amount of dissipation required for estimates of Proposition 2.10 below and the methods of this paper, which can be used to construct a unique invariant measure.

However, concerning the invariant measure, from the entropic perspective Klimontovich noise $\theta = 1$ is formally the correct choice. To see this, consider the entropy function $\Psi(\xi) = \big(\xi\log(\xi) - \xi\big) + \big((1-\xi)\log(1-\xi) - (1-\xi)\big)$, which is chosen to satisfy $\Psi''(\xi) = (\xi(1-\xi))^{-1}$ and which appears in the entropy estimate of Proposition 2.10 below. We have that the heat equation can be written in the form

$$\partial_t\rho = \Delta\rho = \nabla\cdot((\rho(1-\rho))\nabla\Psi'(\rho)),$$

where $m(\rho) = \rho(1-\rho)$ is the mobility of the SSEP. Following, for example, the first author, Stamatakis, and Zimmer [19], this allows to interpret the heat equation as a gradient flow of the entropy functional $\mathcal{S}(\rho) = \int_{\mathbb{T}^d}\Psi(\rho)$ with respect to the thermodynamic metric [19, Equation (1.5)] determined by the mobility. These dynamics come with a formal Gibbs measure determined by the entropy $\mathcal{S}(\rho)$, which is the formal invariant measure for (1.9) driven (again, due to the supercriticality, formally) by the choice of Klimontovich space-time white noise (see, for example, Öttinger [56, Section 1.2.5]). As stated above, the methods of this paper apply to both of these cases with spatially correlated noise and equally to any choice of $\theta \in [1/2, \infty)$. This choice does not change the results due to the relative scaling regime of the CLT and LDP in this work.

2.2. **Uniqueness of stochastic kinetic solutions.** We will now explain how the methods of the second and third authors [27] can be adapted to prove the well-posedness (2.1), which we now write in the Itô formulation

$$\partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\rho(1-\rho)}\,\mathrm{d}\xi^F) - \nabla\cdot(\sqrt{\rho(1-\rho)}g) + \frac{F_1}{8}\nabla\cdot\Big(\frac{(1-2\rho)^2}{\rho(1-\rho)}\nabla\rho\Big). \tag{2.4}$$

There are two essential difficulties in treating (2.4). The first is the singularity of the noise coefficient $\sqrt{\rho(1-\rho)}$. It is only $1/2$-Hölder continuous on $[0,1]$ and its derivative diverges algebraically on the sets $\{\rho \simeq 0\}$ and $\{\rho \simeq 1\}$. The second is the singularity in the Stratonovich-to-Itô correction, which is defined by

$$\nabla\cdot\Big(\frac{(1-2\rho)^2}{\rho(1-\rho)}\nabla\rho\Big) = \Delta\Theta(\rho) \text{ for } \Theta \text{ satisfying } \Theta(1/2) = 0 \text{ and } \Theta'(\rho) = \frac{(1-2\rho)^2}{\rho(1-\rho)},$$

and which diverges logarithmically on the same sets. In particular, since for a solution ρ of (2.4) the composition $\Theta(\rho)$ is not known to be integrable, there is not even an obvious concept of weak solution for (2.4).

It is for these reasons that it is necessary to develop a renormalized solution theory. The theory is based on localizing the solution away from the sets $\{\rho \simeq 0\}$ and $\{\rho \simeq 1\}$. The methods are based on the equation's kinetic formulation, which we introduce briefly here. A more complete introduction can be found in Perthame [57], Chen and Perthame [10], and [27, Section 2].

The kinetic formulation extends the entropy formulation of the equation, which is based on studying the equation satisfied by nonlinear functions of the solution: if $S\colon\mathbb{R}\to\mathbb{R}$ is convex, then it follows formally from Itô's formula (see, for example, Krylov [47, Theorem 3.1]) that, for every $\psi \in H^1(\mathbb{T}^d)$,

$$\begin{aligned}
\mathrm{d}\int_{\mathbb{T}^d} S(\rho)\psi(x) &= \int_{\mathbb{T}^d} S'(\rho)\psi(x)\,\mathrm{d}\rho + \frac{1}{2}\int_{\mathbb{T}^d} S''(\rho)\psi(x)\,\mathrm{d}\langle\rho\rangle \\
&= \int_{\mathbb{T}^d}\Big(S'(\rho)\psi(x)\Delta\rho - S'(\rho)\psi(x)\nabla\cdot(\sqrt{\rho(1-\rho)}\,\mathrm{d}\xi^F) + \frac{F_1}{8}S'(\rho)\psi(x)\nabla\cdot\Big(\frac{(1-2\rho)^2}{\rho(1-\rho)}\nabla\rho\Big)\Big) \\
&\quad - \int_{\mathbb{T}^d} S'(\rho)\psi(x)\nabla\cdot(\sqrt{\rho(1-\rho)}g) + \frac{1}{2}\int_{\mathbb{T}^d} S''(\rho)\psi(x)\,\mathrm{d}\langle\rho\rangle,
\end{aligned}$$

for the formal quadratic variation $\langle\rho\rangle$ of the process ρ. For the first term on the righthand side, we have after integrating by parts that

$$\begin{aligned}(2.5)\quad &\int_{\mathbb{T}^d}\Big(S'(\rho)\psi(x)\Delta\rho - S'(\rho)\psi(x)\nabla\cdot(\sqrt{\rho(1-\rho)}\,\mathrm{d}\xi^F) + \frac{F_1}{8}S'(\rho)\psi(x)\nabla\cdot\Big(\frac{(1-2\rho)^2}{\rho(1-\rho)}\nabla\rho\Big)\Big)\\ &= -\int_{\mathbb{T}^d}S'(\rho)\nabla\psi(x)\cdot\big(\nabla\rho + \frac{F_1(1-2\rho)^2}{8\rho(1-\rho)}\nabla\rho\big) - \int_{\mathbb{T}^d}S'(\rho)\psi(x)\nabla\cdot(\sqrt{\rho(1-\rho)}\,\mathrm{d}\xi^F)\\ &\quad - \int_{\mathbb{T}^d}S'(\rho)\psi(x)\nabla\cdot(\sqrt{\rho(1-\rho)}g) - \int_{\mathbb{T}^d}S''(\rho)\psi(x)|\nabla\rho|^2 - \frac{F_1}{8}\int_{\mathbb{T}^d}S''(\rho)\psi(x)\frac{(1-2\rho)^2}{\rho(1-\rho)}|\nabla\rho|^2.\end{aligned}$$

Furthermore, we have using Assumption 2.1 that the last term satisfies

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{T}^d}S''(\rho)\psi(x)\,\mathrm{d}\langle\rho\rangle = \frac{1}{2}\int_{\mathbb{T}^d}S''(\rho)\psi(x)\sum_{k=1}^{\infty}|\nabla(\sqrt{\rho(1-\rho)}f_k)|^2\\ &= \frac{1}{2}\Big(\int_{\mathbb{T}^d}S''(\rho)\psi(x)\sum_{k=1}^{\infty}\big(f_k^2|\nabla\sqrt{\rho(1-\rho)}|^2 + \sqrt{\rho(1-\rho)}\nabla\sqrt{\rho(1-\rho)}\cdot f_k\nabla f_k + (\rho(1-\rho))|\nabla f_k|^2\big)\Big)\\ &= \frac{F_1}{8}\int_{\mathbb{T}^d}S''(\rho)\psi(x)\frac{(1-2\rho)^2}{\rho(1-\rho)}|\nabla\rho|^2 + \frac{1}{2}\int_{\mathbb{T}^d}S''(\rho)\psi(x)F_3\rho(1-\rho).\end{aligned}$$

After observing the cancellation between the final term of (2.5) and the first term on the righthand side above, we obtain the equation

$$\begin{aligned}(2.6)\quad \mathrm{d}\int_{\mathbb{T}^d}S(\rho)\psi(x) &= -\int_{\mathbb{T}^d}S'(\rho)\nabla\psi(x)\cdot\big(\nabla\rho + \frac{F_1(1-2\rho)^2}{8\rho(1-\rho)}\nabla\rho\big) + \frac{1}{2}\int_{\mathbb{T}^d}S''(\rho)\psi(x)F_3\rho(1-\rho)\\ &\quad + \int_{\mathbb{T}^d}S'(\rho)\psi(x)\Big(\nabla\cdot(\sqrt{\rho(1-\rho)}\,\mathrm{d}\xi^F + \nabla\cdot(\sqrt{\rho(1-\rho)}g)\Big) - \int_{\mathbb{T}^d}S''(\rho)\psi(x)|\nabla\rho|^2,\end{aligned}$$

where, in general, $S(\rho)$ will only satisfy the *entropy inequality* that the lefthand side of (2.6) is less than or equal to the righthand side.

Notice that $\Psi(x,\eta) = \psi(x)S'(\eta)$ can be seen as a test function in (2.6), where the terms respectively involve $\nabla_x\Psi$, Ψ, and $\partial_\eta\Psi$ all evaluated at the point $(x,\rho(x,t))$. The kinetic formulation makes this observation precise by factoring out the dependence of (2.6) on the nonlinear function S, and obtaining a single equation with respect to an additional velocity variable $\eta\in\mathbb{R}$. To do this, we introduce the kinetic function $\overline{\chi}\colon\mathbb{R}^2\to\{-1,0,1\}$ as

$$\overline{\chi}(s,\eta) = \mathbf{1}_{\{0<\eta<s\}} - \mathbf{1}_{\{s<\eta<0\}},$$

and, using the nonnegativity of ρ, the kinetic function χ of ρ is

$$\chi = \overline{\chi}(\rho(x,t),\eta) = \mathbf{1}_{\{0<\eta<\rho(x,t)\}}. \tag{2.7}$$

It follows from the equality $S(\rho) = \int_{\mathbb{R}}S'(\eta)\chi\,\mathrm{d}\eta$ and the distributional equalities

$$\nabla_x\chi = \delta_0(\eta-\rho)\nabla\rho \ \text{ and } \ \partial_\eta\chi = \delta_0(\eta) - \delta_0(\eta-\rho),$$

for the one-dimensional Dirac delta distribution δ_0 at zero, that the test function $\Psi(x,\eta) = \psi(x)S'(\eta)$ can be factored out of (2.6) and the kinetic function is formally a distributional solution of the equation

$$\begin{aligned}\partial_t\chi &= \Delta_x\chi - \delta_0(\eta-\rho)\Big(\nabla\cdot(\sqrt{\rho(1-\rho)}\,\mathrm{d}\xi^F) + \nabla\cdot(\sqrt{\rho(1-\rho)}g)\Big) + \frac{F_1(1-2\eta)^2}{8\eta(1-\eta)}\Delta_x\chi\\ &\quad + \partial_\eta\Big(\delta_0(\eta-\rho)|\nabla\rho|^2\Big) - \frac{F_3}{2}\partial_\eta\Big(\delta_0(\eta-\rho)\eta(1-\eta)\Big).\end{aligned}$$

However, as in the case of the entropy inequality above, the kinetic function will not in general satisfy this equality exactly. Rather, there will $\mathbb{P}$-a.s. exist a nonnegative measure q on $\mathbb{T}^d\times\mathbb{R}\times[0,T]$ satisfying

$$\delta_0(\eta-\rho)|\nabla\rho|^2\leq q \text{ on } \mathbb{T}^d\times\mathbb{R}\times[0,T],$$

such that the kinetic function is a distributional solution of the equation

$$\partial_t\chi=\Delta_x\chi-\delta_0(\eta-\rho)\Big(\nabla\cdot(\sqrt{\rho(1-\rho)}\,\mathrm{d}\xi^F)+\nabla\cdot(\sqrt{\rho(1-\rho)}g)\Big)+\frac{F_1(1-2\eta)^2}{8\xi(1-\eta)}\Delta_x\chi$$
$$+\partial_\eta q-\frac{F_3}{2}\partial_\eta\big(\delta_0(\eta-\rho)\eta(1-\eta)\big).$$

The measure q quantifies the entropy inequality exactly, and allows to consider test functions that are not convex in the velocity variable. This point is essential for the solution theory developed in this work, for which it is necessary to localize the solution compactly in the interval $(0,1)$. We define a stochastic kinetic solution in Definition 2.6 below and prove that such solutions are unique and satisfy a pathwise L^1-contraction in Theorem 2.7.

Definition 2.3. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration on $(\Omega,\mathcal{F})$. A *parabolic defect measure* is a measurable map q from Ω to the space of nonnegative, finite Radon measures on $\mathbb{T}^d\times\mathbb{R}\times[0,T]$ such that, for every $\psi\in \mathrm{C}_c^\infty(\mathbb{T}^d\times\mathbb{R})$, the process

$$(\omega,t)\in\Omega\times[0,T]\to\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\psi(x,\xi)\,\mathrm{d}q(\omega)\ \text{ is }\mathcal{F}_t\text{-predictable.}$$

Remark 2.4. In what follows, we will often encounter derivatives of functions $\psi\in\mathrm{C}_c^\infty(\mathbb{T}^d\times\mathbb{R})$ evaluated at $\eta=\rho(x,t)$. We will write

$$(\nabla\psi)(x,\rho(x,t))=\nabla\psi(x,\eta)|_{\eta=\rho(x,t)},$$

for the gradient of ψ evaluated at $(x,\rho(x,t))$ as opposed to the gradient of the full composition.

Remark 2.5. If $G\in L^2([0,T]\times\Omega;L^2(\mathbb{T}^d))^d$ is an $\mathcal{F}_t$-predictable process, we will write $\int_0^T\int_{\mathbb{T}^d}G\cdot\xi^F$ for the stochastic integral understood in the Itô-sense. Due to the L^2-integrability and $\mathcal{F}_t$-predictability of G, this is a well-defined random variable. We will use the notation $\int_0^T\int_{\mathbb{T}^d}G\circ\xi^F$ to denote the corresponding Stratonovich integral.

Definition 2.6. Let $T\in(0,\infty)$, let the control $g\in L^2([0,T];H^1(\mathbb{T}^d))^d$, and let ξ^F and $\rho_0\in L^\infty(\Omega;L^\infty(\mathbb{T}^d;[0,1]))$ satisfy Assumption 2.1. A *stochastic kinetic solution* of (2.1) is a continuous $L^2(\mathbb{T}^d;[0,1])$-valued, $\mathcal{F}_t$-adapted process $\rho\in L^\infty(\Omega\times[0,T];L^\infty(\mathbb{T}^d;[0,1]))$ that satisfies the following two properties.

(1) *Preservation of mass*: $\mathbb{P}$-a.s. for every $t\in[0,T]$,

$$\|\rho(\cdot,t)\|_{L^1(\mathbb{T}^d)}=\|\rho_0\|_{L^1(\mathbb{T}^d)}\,.$$

(2) *Local regularity*: for every $\delta\in(0,1/2)$,

$$(\delta\vee\rho)\wedge(1-\delta)\in L^2(\Omega\times[0,T];H^1(\mathbb{T}^d)).\tag{2.8}$$

Furthermore, there exists a nonnegative parabolic defect measure q that $\mathbb{P}$-a.s. satisfies the following three properties.

(3) *The entropy condition*: $\mathbb{P}$-a.s. as measures on $\mathbb{T}^d\times\mathbb{R}\times[0,T]$,

$$\delta_0(\eta-\rho)|\nabla\rho|^2\leq q.$$

(4) *Optimal regularity*: the measure μ defined by

$$\mathrm{d}\mu=(\eta(1-\eta))^{-1}\,\mathrm{d}q \text{ is finite on } \mathbb{T}^d\times\mathbb{R}\times[0,T].$$

(5) *The equation*: for the kinetic function χ of ρ, for every $\psi\in C_c^\infty(\mathbb{T}^d\times(0,1))$ and $t\in[0,T]$,

$$\begin{aligned}&\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi(x,\eta,t)\psi(x,\eta)=\int_{\mathbb{T}^d}\int_{\mathbb{R}}\overline{\chi}(\rho_0)\psi(x,\eta)-\int_0^t\int_{\mathbb{T}^d}\nabla\rho\cdot(\nabla\psi)(x,\rho)\\&-\int_0^t\int_{\mathbb{T}^d}\nabla\cdot\Big(\sqrt{\rho(1-\rho)}\,\mathrm{d}\xi^F+\sqrt{\rho(1-\rho)}g\Big)\psi(x,\rho)-\frac{F_1}{8}\int_0^t\int_{\mathbb{T}^d}\frac{(1-2\rho)^2}{\rho(1-\rho)}\nabla\rho\cdot(\nabla\psi)(x,\rho)\\&-\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\eta\psi(x,\eta)\,\mathrm{d}q+\frac{1}{2}\int_0^t\int_{\mathbb{T}^d}F_3(\partial_\eta\psi)(x,\rho)\rho(1-\rho),\end{aligned}$$

for $\overline{\chi}(\rho_0)$ defined in (2.7), where the third integral on the righthand side is well-defined using the smoothness of the noise and control, Remark 2.5, the local regularity of ρ in (2.8), and the compact support of ψ in the velocity variable.

Theorem 2.7. *Let $T\in(0,\infty)$, let the control $g\in L^2([0,T];H^1(\mathbb{T}^d))^d$, and let ξ^F and $\rho_{0,1},\rho_{0,2}\in L^\infty(\Omega;L^\infty(\mathbb{T}^d;[0,1]))$ satisfy Assumption 2.1. If ρ_1,ρ_2 are pathwise kinetic solutions of (2.1) in the sense of Definition 2.6 then, $\mathbb{P}$-a.s.,*

$$\max_{t\in[0,T]}\|\rho_1(\cdot,t)-\rho_2(\cdot,t)\|_{L^1(\mathbb{T}^d)}\leq\|\rho_{0,1}-\rho_{0,2}\|_{L^1(\mathbb{T}^d)}.$$

Proof. For every $i\in\{1,2\}$ let χ_i denote the kinetic function of ρ_i and let q_i denote the corresponding kinetic measure defined in Definition 2.6. For every $\varepsilon\in(0,1)$ let κ^ε denote a standard convolution kernel of scale $\varepsilon\in(0,1)$ on $\mathbb{T}^d$, for every $\delta\in(0,1)$ let κ^δ denote a standard convolution kernel of scale $\delta\in(0,1)$ on $\mathbb{R}$, and let $\kappa^{\varepsilon,\delta}(x,y,\xi,\eta)=\kappa^\varepsilon(x-y)\kappa^\delta(\xi-\eta)$. It is furthermore necessary to introduce a cutoff in the velocity variable. For every $\beta\in(0,1/4)$ let $\zeta_\beta\colon\mathbb{R}\to[0,1]$ be a smooth function satisfying $\zeta_\beta(\xi)=0$ if $\xi\leq\beta$ or $\xi\geq1-\beta$, $\zeta_\beta(\xi)=1$ if $\xi\in[2\beta,1-2\beta]$, and $|\zeta_\beta'(\xi)|\leq c/\beta(1-\beta)$ for some $c\in(0,\infty)$ independent of β.

We write $\chi_{i,t}(x,\xi)=\chi_i(x,\xi,t)$ and make a similar convention for all other time-dependent quantities and we define the convolution $\chi_{i,t}^{\varepsilon,\delta}=(\chi_{i,t}*\kappa^{\varepsilon,\delta})$. The proof of uniqueness is based on the equality

$$\int_{\mathbb{T}^d}|\rho_1(y,t)-\rho_2(y,t)|\,\mathrm{d}y=\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\chi_1+\chi_2-2\chi_1\chi_2)\,\mathrm{d}y\,\mathrm{d}\eta,$$

which we approximate, for every $\beta\in(0,1/4)$, $\varepsilon\in(0,1)$, and $\delta\in(0,\beta/2)$ by the quantity

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\chi_{1,t}^{\varepsilon,\delta}(y,\eta)+\chi_{2,t}^{\varepsilon,\delta}(y,\eta)-2\chi_{1,t}^{\varepsilon,\delta}(y,\eta)\chi_{2,t}^{\varepsilon,\delta}(y,\eta))\zeta_\beta(\eta)\,\mathrm{d}y\,\mathrm{d}\eta.\tag{2.9}$$

In view of definition 2.6, the cutoff ζ_β and the restriction $\delta\in(0,\beta/2)$ guarantee that on the support of ζ_β the function $\kappa^{\varepsilon,\delta}(\cdot-y,\cdot-\eta)$ is an admissible test function. We will differentiate the three terms of (2.9) individually.

The singletons. For the first two terms on the righthand side of (2.9), it follows from the equation and the symmetry of the convolution kernel that, for every η in the support of ζ_β, $\delta\in(0,\beta/2)$, and $y\in\mathbb{T}^d$, for $\overline{\kappa}_{i,t}^{\varepsilon,\delta}(x,y,\eta)=\kappa^{\varepsilon,\delta}(x,y,\rho_i(x,t),\eta)$,

$$\begin{aligned}\mathrm{d}\chi_{i,t}^{\varepsilon,\delta}(y,\eta)&=\nabla_y\cdot(\nabla\rho_i*\overline{\kappa}_{i,t}^{\varepsilon,\delta})-(\overline{\kappa}_{i,t}^{\varepsilon,\delta}*\nabla\cdot\big(\sqrt{\rho_i(1-\rho_i)}\,\mathrm{d}\xi^F+\sqrt{\rho_i(1-\rho_i)}g\big))\\&\quad+\frac{F_1}{8}\nabla_y\cdot\Big(\frac{(1-2\rho_i)^2}{\rho_i(1-\rho_i)}\nabla\rho_i*\overline{\kappa}_{i,t}^{\varepsilon,\delta}\Big)+\partial_\eta(\kappa^{\varepsilon,\delta}*q_{i,t})-\partial_\eta(F_3\rho_i(1-\rho_i)*\overline{\kappa}_{i,t}^{\varepsilon,\delta}).\end{aligned}\tag{2.10}$$

Therefore, using (2.10) and integrating by parts, for every $i\in\{1,2\}$,

$$\mathrm{d}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_{i,t}^{\varepsilon,\delta}(y,\eta)\zeta_\beta(\eta)\,\mathrm{d}y\,\mathrm{d}\eta=\mathrm{d}I_{i,t}^{\mathrm{mart}}+\mathrm{d}I_{i,t}^{\mathrm{con}}+\mathrm{d}I_{i,t}^{\mathrm{cut}},\tag{2.11}$$

for the martingale term

$$\mathrm{d}I^{\mathrm{mart}}_{i,t} = -\int_{\mathbb{T}^d}\int_{\mathbb{R}} \big(\overline{\kappa}^{\varepsilon,\delta}_{i,t} * \nabla\cdot(\sqrt{\rho_i(x,t)(1-\rho_i(x,t))}\,\mathrm{d}\xi^F)\big)\zeta_\beta,$$

for the control term

$$\mathrm{d}I^{\mathrm{con}}_{i,t} = -\int_{\mathbb{T}^d}\int_{\mathbb{R}} \big(\overline{\kappa}^{\varepsilon,\delta}_{i,t} * \nabla\cdot(\sqrt{\rho_i(x,t)(1-\rho_i(x,t))}g)\big)\zeta_\beta,$$

and for the cutoff term

$$\mathrm{d}I^{\mathrm{cut}}_{i,t} = -\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\kappa^{\varepsilon,\delta} * q_{i,t})\zeta'_\beta + \frac{1}{2}\int_{(\mathbb{T}^d)^2}\int_{\mathbb{R}}\big(F_3\rho_i)(1-\rho_i) * \overline{\kappa}^{\varepsilon,\delta}_{i,t}\big)\zeta'_\beta,$$

which completes the initial analysis of the first two terms on the righthand side of (2.9).

The mixed term. For the mixed term on the righthand side of (2.9), we observe using the distributional equalities

$$\nabla_x\chi_i(x,\xi,t) = \delta_0(\xi-\rho_i(x,t))\nabla\rho_i \text{ and } \partial_\xi\chi_i(x,\xi,t) = \delta_0(\xi)-\delta_0(\xi-\rho_i(x,t)),$$

the stochastic product rule, and $\delta\in(0,{}^{\beta}\!/2)$ that

$$\mathrm{d}\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi^{\varepsilon,\delta}_{1,t}(y,\eta)\chi^{\varepsilon,\delta}_{2,t}(y,\eta)\zeta_\beta(\eta)\Big) = \mathrm{d}I^{\mathrm{meas}}_t + \mathrm{d}I^{\mathrm{mart}}_{\mathrm{mix},t} + \mathrm{d}I^{\mathrm{con}}_{\mathrm{mix},t} + \mathrm{d}I^{\mathrm{cor}}_t + \mathrm{d}I^{\mathrm{cut}}_{\mathrm{mix},t}, \tag{2.12}$$

for the measure term

$$\begin{aligned}\mathrm{d}I^{\mathrm{meas}}_t = &-2\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\nabla\rho_1 * \kappa^{\varepsilon,\delta}_{1,t})(\nabla\rho_2 * \kappa^{\varepsilon,\delta}_{2,t})\zeta_\beta\\ &+\int_{(\mathbb{T}^d)^2}\int_{\mathbb{R}}(\kappa^{\varepsilon,\delta} * q_{1,t})(y,\eta)\overline{\kappa}^{\varepsilon,\delta}_{2,t}(x,y,\eta)\zeta_\beta(\eta)\\ &+\int_{(\mathbb{T}^d)^2}\int_{\mathbb{R}}(\kappa^{\varepsilon,\delta} * q_{2,t})(y,\eta)\overline{\kappa}^{\varepsilon,\delta}_{1,t}(x,y,\eta)\zeta_\beta(\eta),\end{aligned}$$

for the martingale term

$$\begin{aligned}\mathrm{d}I^{\mathrm{mart}}_{\mathrm{mix},t} = &-\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi^{\varepsilon,\delta}_{2,t}\big(\overline{\kappa}^{\varepsilon,\delta}_{1,t} * \nabla\cdot\big(\sqrt{\rho_1(1-\rho_1)}\,\mathrm{d}\xi^F\big)\big)\zeta_\beta\\ &-\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi^{\varepsilon,\delta}_{1,t}\big(\overline{\kappa}^{\varepsilon,\delta}_{2,t} * \nabla_{x'}\cdot\big(\sqrt{\rho_2(1-\rho_2)}\,\mathrm{d}\xi^F\big)\big)\zeta_\beta,\end{aligned}$$

for the control term

$$\begin{aligned}\mathrm{d}I^{\mathrm{con}}_{\mathrm{mix},t} = &-\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi^{\varepsilon,\delta}_{2,t}\big(\overline{\kappa}^{\varepsilon,\delta}_{1,t} * \nabla\cdot\big(\sqrt{\rho_1(1-\rho_1)}g\big)\big)\zeta_\beta\\ &-\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi^{\varepsilon,\delta}_{1,t}\big(\overline{\kappa}^{\varepsilon,\delta}_{2,t} * \nabla_{x'}\cdot\big(\sqrt{\rho_2(1-\rho_2)}g\big)\big)\zeta_\beta,\end{aligned}$$

for the correction term, writing (x,ξ) for the convolution variables of ρ_1 and all related quantities and writing (x',ξ') for the convolution variables of ρ_2,

$$\begin{aligned}\mathrm{d}I_t^{\mathrm{cor}} &= -\frac{F_1}{8}\int_{(\mathbb{T}^d)^3}\int_{\mathbb{R}}\zeta_\beta\overline{\kappa}^{\varepsilon,\delta}_{1,t}\overline{\kappa}^{\varepsilon,\delta}_{2,t}\Big(\frac{(1-2\rho_1)^2}{\rho_1(1-\rho_1)}+\frac{(1-2\rho_2)^2}{\rho_2(1-\rho_2)}\Big)\nabla\rho_1\cdot\nabla\rho_2\\
&+\frac{1}{4}\int_{(\mathbb{T}^d)^3}\int_{\mathbb{R}}\zeta_\beta\overline{\kappa}^{\varepsilon,\delta}_{1,t}\overline{\kappa}^{\varepsilon,\delta}_{2,t}\Big(\sum_{k=1}^\infty f_k(x)f_k(x')\Big)\frac{(1-2\rho_1)(1-2\rho_2)}{\sqrt{\rho_1(1-\rho_1)}\sqrt{\rho_2(1-\rho_2)}}\nabla\rho_1\cdot\nabla\rho_2\\
&-\frac{1}{2}\int_{(\mathbb{T}^d)^3}\int_{\mathbb{R}}\zeta_\beta\overline{\kappa}^{\varepsilon,\delta}_{1,t}(x,y,\eta)\overline{\kappa}^{\varepsilon,\delta}_{2,t}(x',y,\eta)F_3(x)\rho_1(1-\rho_1)\\
&-\frac{1}{2}\int_{(\mathbb{T}^d)^3}\int_{\mathbb{R}}\zeta_\beta\overline{\kappa}^{\varepsilon,\delta}_{2,t}(x',y,\eta)\overline{\kappa}^{\varepsilon,\delta}_{1,t}(x,y,\eta)F_3(x')\rho_2(1-\rho_2)\\
&+\int_{(\mathbb{T}^d)^3}\int_{\mathbb{R}}\zeta_\beta\overline{\kappa}^{\varepsilon,\delta}_{1,t}\overline{\kappa}^{\varepsilon,\delta}_{2,t}\Big(\sum_{k=1}^\infty\nabla f_k(x)\cdot\nabla f_k(x')\Big)\sqrt{\rho_1(1-\rho_1)}\sqrt{\rho_2(1-\rho_2)},\end{aligned}$$

and for the cutoff term

$$\begin{aligned}\mathrm{d}I^{\mathrm{cut}}_{\mathrm{mix},t} &= -\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\kappa^{\varepsilon,\delta}*q_{1,t})\chi^{\varepsilon,\delta}_{2,t}\zeta'_\beta-\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\kappa^{\varepsilon,\delta}*q_{2,t})\chi^{\varepsilon,\delta}_{1,t}\zeta'_\beta\\
&+\frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\overline{\kappa}^{\varepsilon,\delta}_{1,t}*F_3\rho_1(1-\rho_1))\chi^{\varepsilon,\delta}_{2,t}\zeta'_\beta+\frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\overline{\kappa}^{\varepsilon,\delta}_{2,t}*F_3\rho_2(1-\rho_2))\chi^{\varepsilon,\delta}_{1,t}\zeta'_\beta.\end{aligned}$$

This completes the initial analysis of the mixed term. Returning to (2.9), we have from (2.11) and (2.12) that

$$\mathrm{d}\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\chi^{\varepsilon,\delta}_{1,t}+\chi^{\varepsilon,\delta}_{2,t}-2\chi^{\varepsilon,\delta}_{1,t}\chi^{\varepsilon,\delta}_{2,t})\zeta_\beta\Big)=-2\,\mathrm{d}I_t^{\mathrm{meas}}+\mathrm{d}I_t^{\mathrm{mart}}+\mathrm{d}I_t^{\mathrm{con}}-2\,\mathrm{d}I_t^{\mathrm{cor}}+\mathrm{d}I_t^{\mathrm{cut}},\tag{2.13}$$

for the cutoff, martingale, and control terms defined analogously to

$$\mathrm{d}I_t^{\mathrm{cut}}=\mathrm{d}I^{\mathrm{cut}}_{1,t}+\mathrm{d}I^{\mathrm{cut}}_{2,t}-2\,\mathrm{d}I^{\mathrm{cut}}_{\mathrm{mix},t}.$$

We will handle each of the four terms on the righthand side of (2.13) separately.

The measure term. It is an immediate consequence of Hölder's inequality and the entropy condition of Definition 2.6 that, $\mathbb{P}$-a.s.,

$$I_t^{\mathrm{meas}}=\int_0^t\mathrm{d}I_s^{\mathrm{meas}}\geq 0.\tag{2.14}$$

The martingale term. The analysis of the martingale term will use the following fact: if $F^\varepsilon_t, F\in L^2(\Omega\times[0,T])$ are $\mathcal{F}_t$-progressively measurable processes that satisfy that, as $\varepsilon\to0$, $F^\varepsilon_t\to F$ in $L^2(\Omega\times[0,T])$, then along a subsequence $\varepsilon\to0$ we have that, $\mathbb{P}$-a.s. for every $t\in[0,T]$,

$$\lim_{\varepsilon\to0}\int_0^t F^\varepsilon_s\,\mathrm{d}B_s=\int_0^t F_s\,\mathrm{d}B_s,\tag{2.15}$$

which follows from the Burkhölder–Davis–Gundy inequality (see, for example, Revuz and Yor [61, Chapter 4, Theorem 4.1]). It follows from the local H^1-regularity of the solutions, the definition of the convolutions, and (2.15) that, $\mathbb{P}$-a.s. along a subsequence $\varepsilon\to0$, for $I_t^{\mathrm{mart}}=\int_0^t\mathrm{d}I_s^{\mathrm{mart}}$,

$$\begin{aligned}\lim_{\varepsilon\to0}I_t^{\mathrm{mart}} &= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\overline{\kappa}^\delta_{s,1}(2\chi^\delta_{s,2}-1)\zeta_\beta(\eta)\nabla\cdot(\sqrt{\rho_1(1-\rho_1)}\,\mathrm{d}\xi^F)\,\mathrm{d}y\,\mathrm{d}\eta\\
&+\int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\overline{\kappa}^\delta_{s,2}(2\chi^\delta_{s,1}-1)\zeta_\beta(\eta)\nabla\cdot(\sqrt{\rho_2(1-\rho_2)}\,\mathrm{d}\xi^F)\,\mathrm{d}y\,\mathrm{d}\eta,\end{aligned}$$

for $\overline{\kappa}^\delta_{s,i}(y,\eta)=\kappa^\delta_1(\eta-\rho_i(y,t))$ and for $\chi^\delta_{s,i}(y,\eta)=(\chi^i_s(y,\cdot)*\kappa^\delta_1)(\eta)$. It follows from the local H^1-regularity of the ρ_i and the dominated convergence theorem that, after passing to a subsequence $\delta\to 0$, $\mathbb{P}$-a.s. for every $t\in[0,T]$,

$$\lim_{\delta\to 0}\Big|\int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\overline{\kappa}^\delta_{s,1}(2\chi^\delta_{s,2}-1)(\zeta_\beta(\eta)-\zeta_\beta(\rho_1))\nabla\cdot(\sqrt{\rho_1(1-\rho_1)}\,\mathrm{d}\xi^F)\Big|=0, \tag{2.16}$$

and similarly for the symmetric term obtained by swapping the roles of ρ_1 and ρ_2. It then follows from an explicit calculation that, whenever $2\delta<\rho_2(y,s)$,

$$\int_{\mathbb{R}^2}\kappa^\delta(\xi-\eta)\kappa^\delta(\eta-\xi')\chi^2_s(y,\xi')\,\mathrm{d}\eta\,\mathrm{d}\xi'=\begin{cases}0 & \text{if } \xi\le -2\delta \text{ or } \xi\ge\rho_2(y,s)+2\delta,\\ 1/2 & \text{if } \xi=0 \text{ or } \xi=\rho_2(y,s),\\ 1 & \text{if } 2\delta<\xi<\rho_2(y,s)-2\delta.\end{cases}$$

Then, using $\zeta_\beta(0)=0$, we have that

$$\lim_{\delta\to 0}\Big(\int_{\mathbb{R}}\overline{\kappa}^\delta_{s,1}(2\chi^\delta_{s,2}-1)\,\mathrm{d}\eta\Big)\zeta_\beta(\rho_1)=\Big(\mathbf{1}_{\{\rho_1=\rho_2\}}+2\mathbf{1}_{\{0\le\rho_1<\rho_2\}}-1\Big)\zeta_\beta(\rho_1), \tag{2.17}$$

and similarly for the symmetric term obtained by swapping the roles of ρ_1 and ρ_2. In combination (2.16), (2.17), and the equality $\mathrm{sgn}(\rho_2-\rho_1)=\mathbf{1}_{\{\rho_1=\rho_2\}}+2\mathbf{1}_{\{\rho_1<\rho_2\}}-1$ prove that, $\mathbb{P}$-a.s. there exist random subsequences $\varepsilon,\delta\to 0$ such that

$$\begin{aligned}\lim_{\delta,\varepsilon\to 0}\Big(I^{\mathrm{mart}}_t\Big)&=\int_0^t\int_{\mathbb{T}^d}\mathrm{sgn}(\rho_2-\rho_1)\zeta_\beta(\rho_1)\nabla\cdot(\sqrt{\rho_1(1-\rho_1)}\,\mathrm{d}\xi^F)\\&\quad-\int_0^t\int_{\mathbb{T}^d}\mathrm{sgn}(\rho_2-\rho_1)\zeta_\beta(\rho_2)\nabla\cdot(\sqrt{\rho_2(1-\rho_2)}\,\mathrm{d}\xi^F),\end{aligned} \tag{2.18}$$

where it is not necessary to define $\mathrm{sgn}(0)$ since by Stampacchia's lemma (see Evans [23, Chapter 5, Exercises 17,18]), almost everywhere on the set $\{\rho_1=\rho_2\}$,

$$\Big(\zeta_\beta(\rho_1)\nabla\cdot(\sqrt{\rho_1(1-\rho_1)}\,\mathrm{d}\xi^F)-\zeta_\beta(\rho_2)\nabla\cdot(\sqrt{\rho_2(1-\rho_2)}\,\mathrm{d}\xi^F)\Big)=0.$$

For every $\beta\in(0,1/4)$ let $\Theta_\beta\colon[0,\infty)\to[0,\infty)$ satisfy $\Theta_\beta(0)=0$ and $\Theta'_\beta(\xi)=\zeta_\beta(\xi)\partial_\xi(\sqrt{\xi(1-\xi)})$. Returning to (2.18), we have that

$$\begin{aligned}\lim_{\delta,\varepsilon\to 0}\Big(I^{\mathrm{mart}}_t\Big)&=\int_0^t\int_{\mathbb{T}^d}\mathrm{sgn}(\rho_2-\rho_1)\nabla\cdot\Big(\Big(\Theta_\beta(\rho_1)-\Theta_\beta(\rho_2)\Big)\,\mathrm{d}\xi^F\Big)\\&\quad+\int_0^t\int_{\mathbb{T}^d}\mathrm{sgn}(\rho_2-\rho_1)\Big(\zeta_\beta(\rho_1)\sqrt{\rho_1(1-\rho_1)}-\Theta_\beta(\rho_1)\Big)\nabla\cdot\,\mathrm{d}\xi^F\\&\quad-\int_0^t\int_{\mathbb{T}^d}\mathrm{sgn}(\rho_2-\rho_1)\Big(\zeta_\beta(\rho_2)\sqrt{\rho_2(1-\rho_2)}-\Theta_\beta(\rho_2)\Big)\nabla\cdot\,\mathrm{d}\xi^F.\end{aligned} \tag{2.19}$$

For the first term on the righthand side of (2.19), for the convolution $\mathrm{sgn}^\delta=(\mathrm{sgn}*\kappa^\delta_1)$ for every $\delta\in(0,1)$, we have using (2.15) that, $\mathbb{P}$-a.s. along a random subsequence $\delta\to 0$, for every $t\in[0,T]$,

$$\begin{aligned}&\int_0^t\int_{\mathbb{T}^d}\mathrm{sgn}(\rho_2-\rho_1)\nabla\cdot\Big(\Big(\Theta_\beta(\rho_1)-\Theta_\beta(\rho_2)\Big)\,\mathrm{d}\xi^F\Big)\\&=\lim_{\delta\to 0}\int_0^t\int_{\mathbb{T}^d}\mathrm{sgn}^\delta(\rho_1-\rho_2)\nabla\cdot\Big(\Big(\Theta_\beta(\rho_1)-\Theta_\beta(\rho_2)\Big)\,\mathrm{d}\xi^F\Big)\\&=-\lim_{\delta\to 0}\int_0^T\int_{\mathbb{T}^d}(\mathrm{sgn}^\delta)'(\rho_1-\rho_2)\Big(\Theta_\beta(\rho_1)-\Theta_\beta(\rho_2)\Big)(\nabla\rho_1-\nabla\rho_2)\cdot\,\mathrm{d}\xi^F.\end{aligned} \tag{2.20}$$

It follows from the Lipschitz continuity of Θ_β that there exists $c \in (0,\infty)$ independent of $\delta \in (0,1)$ but depending on β such that

$$|(\text{sgn}^\delta)'(\rho_1-\rho_2)\big(\Theta_\beta(\rho_1)-\Theta_\beta(\rho_2)\big)| \leq c\mathbf{1}_{\{0<|\rho_1-\rho_2|<c\delta\}}. \tag{2.21}$$

We then have using the local H^1-regularity of the solutions, the dominated convergence theorem, (2.20), and (2.21) that, $\mathbb{P}$-a.s. for every $t \in [0,T]$,

$$\int_0^t\int_{\mathbb{T}^d} \text{sgn}(\rho_2-\rho_1)\nabla\cdot\Big(\Big(\Theta_{\beta,M}(\rho_1)-\Theta_{\beta,M}(\rho_2)\Big)\,\mathrm{d}\xi^F\Big) = 0. \tag{2.22}$$

For the the second two terms on the righthand side of (2.19), it follows from the definition of Θ_β and the dominated convergence theorem that, for every $i \in \{1,2\}$,

$$\lim_{\beta\to 0}\Theta_\beta(\rho_i) = \sqrt{\rho_i(1-\rho_i)} \text{ strongly in } L^2(\mathbb{T}^d\times[0,T]).$$

Therefore, using (2.15), there exists a random sequence $\beta \to 0$ such that, for every $i \in \{1,2\}$ and $t \in [0,T]$,

$$\lim_{\beta\to 0}\Big|\int_0^t\int_{\mathbb{T}^d}\text{sgn}(\rho_2-\rho_1)\Big(\zeta_\beta(\rho_1)\sqrt{\rho_1(1-\rho_1)}-\Theta_\beta(\rho_1)\Big)\nabla\cdot\,\mathrm{d}\xi^F\Big| = 0. \tag{2.23}$$

In combination, (2.18), (2.19), (2.22), and (2.23) prove that, $\mathbb{P}$-a.s. along random subsequences, for every $t \in [0,T]$,

$$\lim_{\beta\to 0}\Big(\lim_{\delta\to 0}\Big(\lim_{\varepsilon\to 0} I_t^{\text{mart}}\Big)\Big) = 0. \tag{2.24}$$

The control term. A repetition of the analysis leading to (2.18) shows that, for $I_t^{\text{con}} = \int_0^t \mathrm{d}I_s^{\text{con}}$,

$$\begin{aligned}\lim_{\delta,\varepsilon\to 0}\Big(I_t^{\text{con}}\Big) = &\int_0^t\int_{\mathbb{T}^d}\text{sgn}(\rho_2-\rho_1)\zeta_\beta(\rho_1)\nabla\cdot(\sqrt{\rho_1(1-\rho_1)}g)\\ &-\int_0^t\int_{\mathbb{T}^d}\text{sgn}(\rho_2-\rho_1)\zeta_\beta(\rho_2)\nabla\cdot(\sqrt{\rho_2(1-\rho_2)}g).\end{aligned}$$

The argument leading from (2.19) to (2.24) using the spatial smoothness of g proves that

$$\lim_{\beta\to 0}\Big(\lim_{\delta\to 0}\Big(\lim_{\varepsilon\to 0} I_t^{\text{con}}\Big)\Big) = 0. \tag{2.25}$$

The correction term. The local H^1-regularity of the solutions and the support of ζ_β proves that, for every $t \in [0,T]$, for $I_t^{\text{cor}} = \int_0^t \mathrm{d}I_s^{\text{cor}}$,

$$\begin{aligned}\lim_{\varepsilon\to 0} I_t^{\text{cor}} = &-\frac{F_1}{8}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\zeta_\beta\overline{\kappa}^\delta_{1,s}\overline{\kappa}^\delta_{2,s}\Big(\frac{(1-2\rho_1)}{\sqrt{\rho_1(1-\rho_1)}}-\frac{(1-2\rho_2)}{\sqrt{\rho_2(1-\rho_2)}}\Big)^2\nabla\rho_1\cdot\nabla\rho_2\\ &-\frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\zeta_\beta\overline{\kappa}^\delta_{1,s}\overline{\kappa}^\delta_{2,s}F_3\Big(\sqrt{\rho_1(1-\rho_1)}-\sqrt{\rho_2(1-\rho_2)}\Big)^2.\end{aligned}$$

Since on the support of $\zeta_\beta\overline{\kappa}^\delta_{1,s}\overline{\kappa}^\delta_{2,s}$ we have for every $\delta \in (0, \beta/2)$ and $i \in \{1,2\}$ that $\beta/2 \leq \rho_i \leq 1-\beta/2$, we have using the local Lipschitz continuity of the nonlinearities on $(0,1)$ that, for $c \in (0,\infty)$ independent of $\delta \in (0,1)$ but depending on $\beta \in (0,1)$, for every $t \in [0,T]$,

$$\begin{aligned}|\lim_{\varepsilon\to 0} I_t^{\text{cor}}| &\leq c\delta\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}}\zeta_\beta(\delta\overline{\kappa}^\delta_{1,s})\overline{\kappa}^\delta_{2,s}\nabla\rho_1\cdot\nabla\rho_2\\ &\leq c\delta\int_0^T\int_{\mathbb{T}^d}\mathbf{1}_{\{\beta/4\leq\rho_i\leq 1-\beta/4\;\forall\, i\in\{1,2\}\}}\nabla\rho_1\cdot\nabla\rho_2.\end{aligned}$$

It then follows from the local H^1-regularity of the solutions and the dominated convergence theorem that, for every $t \in [0,T]$,

$$\lim_{\delta\to 0}\left(\lim_{\varepsilon\to 0} I_t^{\mathrm{cor}}\right) = 0. \tag{2.26}$$

The cutoff term. The cutoff term takes the form, for every $t\in[0,T]$, for $I_t^{\mathrm{cut}} = \int_0^t \mathrm{d}I_s^{\mathrm{cut}}$,

$$\begin{aligned} I_t^{\mathrm{cut}} = & \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}(2\chi_{2,s}^{\varepsilon,\delta}-1)(\kappa^{\varepsilon,\delta}*q_{1,s})\zeta'_\beta + \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}(2\chi_{1,s}^{\varepsilon,\delta}-1)(\kappa^{\varepsilon,\delta}*q_{2,s})\zeta'_\beta \\ & -\frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\left(\overline{\kappa}_{1,s}^{\varepsilon,\delta}*F_3(\rho_1(1-\rho_1)\right)(2\chi_{2,s}^{\varepsilon,\delta}-1)\zeta'_\beta \\ & -\frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\left(\overline{\kappa}_{2,s}^{\varepsilon,\delta}*F_3\rho_2(1-\rho_2)\right)(2\chi_{1,s}^{\varepsilon,\delta}-1)\zeta'_\beta, \end{aligned}$$

and therefore, using the boundedness of the kinetic function and F_3 and the support of the convolution kernel, for some $c\in(0,\infty)$ independent of ε, δ, and β, for every $t\in[0,T]$,

$$\begin{aligned} \limsup_{\varepsilon,\delta\to 0}|I_t^{\mathrm{cut}}| \le\ & c\Big(\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}}|\zeta'_\beta(\xi)|\,\mathrm{d}q_1 + \int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}}|\zeta'_\beta(\xi')|\,\mathrm{d}q_2\Big) \\ & + c\Big(\int_0^T\int_{\mathbb{T}^d}\rho_1(1-\rho_1)|\zeta'_\beta(\rho_1)| + \int_0^T\int_{\mathbb{T}^d}\rho_2(1-\rho_2)|\zeta'_\beta(\rho_2)|\Big). \end{aligned}$$

Finally, using the bounds on ζ'_β, for some $c\in(0,\infty)$ independent of ε, δ, and β, for every $t\in[0,T]$,

$$\begin{aligned} \limsup_{\varepsilon,\delta\to 0}|I_t^{\mathrm{cut}}| \le\ & c\sum_{i=1}^2\Big(\int_0^T\int_\beta^{2\beta}\int_{\mathbb{T}^d}(\xi(1-\xi))^{-1}\,\mathrm{d}q_i + \int_0^T\int_{1-2\beta}^{1-\beta}\int_{\mathbb{T}^d}(\xi(1-\xi))^{-1}\,\mathrm{d}q_i\Big) \\ & + c\sum_{i=1}^2\Big(\int_0^T\int_{\mathbb{T}^d}\mathbf{1}_{\{\beta<\rho_i<2\beta\}} + \mathbf{1}_{\{1-2\beta<\rho_i<1-\beta\}}\Big). \end{aligned} \tag{2.27}$$

It is a consequence of the finiteness of the measures $\mathrm{d}\mu_i = (\xi(1-\xi))^{-1}\,\mathrm{d}q_i$ and the dominated convergence theorem that both terms on the righthand side of (2.27) vanish as $\beta\to 0$, and therefore that

$$\lim_{\beta\to 0}\left(\limsup_{\varepsilon,\delta\to 0}|I_t^{\mathrm{cut}}|\right) = 0. \tag{2.28}$$

We have using the fact that the kinetic functions are $\{0,1\}$-valued that, for every $t\in[0,T]$,

$$\lim_{\beta,\delta,\varepsilon\to 0}\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\chi_{1,t}^{\varepsilon,\delta}+\chi_{2,t}^{\varepsilon,\delta}-2\chi_{1,t}^{\varepsilon,\delta}\chi_{2,t}^{\varepsilon,\delta})\zeta_\beta = \int_{\mathbb{T}^d}\int_{\mathbb{R}}|\chi_1-\chi_2|^2 = \int_{\mathbb{T}^d}|\rho_1-\rho_2|.$$

After returning to (2.9) and (2.13), we have from (2.14), (2.24), (2.25), (2.26), and (2.28) that, $\mathbb{P}$-a.s. for every $t\in[0,T]$,

$$\|\rho_1(\cdot,t)-\rho_2(\cdot,t)\|_{L^1(\mathbb{T}^d)} \le \|\rho_{0,1}-\rho_{0,2}\|_{L^1(\mathbb{T}^d)},$$

which completes the proof. □

2.3. **Existence of renormalized kinetic solutions.** We will first show the existence of solutions to approximations of (1.9) defined by smoothed versions of the square root, and including an L^2-valued control term that will be important for the proof of the large deviations principle below. The reason for the specific form of the regularizations in (2.8) appears in the proof of the entropy estimate in Proposition 2.10 below.

Lemma 2.8. *Let $s\colon \mathbb{R}\to[0,1]$ be defined by*

$$s(x)=\sqrt{x(1-x)}\ \textit{if}\ x\in[0,1]\ \textit{and}\ s(x)=0\ \textit{if}\ x\notin[0,1].$$

Then there exists a sequence of smooth, compactly supported approximations $\{s^\eta\}_{\eta\in(0,1/4)}$ satisfying the properties that $s^\eta(x)=0$ if $x\notin[\eta,1-\eta]$, that, as $\eta\to 0$,

$$s^\eta\to s\ \ \textit{uniformly on}\ \mathbb{R}\ \textit{and}\ (s^\eta)'\to s'\ \textit{locally uniformly on}\ (0,1),$$

and that, for some $c\in(0,\infty)$ independent of $\eta\in(0,1)$,

$$s^\eta(x)\leq cs(x)\ \ \textit{and}\ \ |(s^\eta)'(x)|\leq c|s'(x)|\ \ \textit{for every}\ \ x\in(0,1).$$

Proof. For every $\eta\in(0,1/4)$ let $\tilde{s}^\eta\in \mathrm{C}(\mathbb{R})$ be defined by

$$\tilde{s}^\eta(x)=\sqrt{x(1-x)}-\sqrt{2\eta(1-2\eta)}\ \text{ for every } x\in[2\eta,1-2\eta] \text{ and }\ \tilde{s}^\eta(x)=0 \text{ otherwise,}$$

and for every $\varepsilon\in(0,1)$ let $\kappa^\varepsilon\in \mathrm{C}^\infty_c(\mathbb{R})$ denote a standard convolution kernel of scale $\varepsilon\in(0,1)$. Then, since $\tilde{s}^\eta(x)=s(x)-\sqrt{2\eta(1-2\eta)}$ and $(\tilde{s}^\eta)'(x)=s'(x)$ for every $x\in(2\eta,1-2\eta)$ and $\tilde{s}^\eta(x)=(\tilde{s}^\eta)'(x)=0$ if $x\notin[2\eta,1-2\eta]$, for every $\eta\in(0,1/4)$ there exists $\varepsilon_\eta\in(0,\eta)$ such that the functions defined by $s^\eta=(\tilde{s}^\eta*\kappa^{\varepsilon_\eta})$ satisfy the hypothesis of the lemma with $c=2$. This completes the proof. ☐

We will now prove the existence of solutions to the controlled Stratonovich equation

$$\partial_t\rho=\Delta\rho-\sqrt{\varepsilon}\nabla\cdot\big(s^\eta(\rho)\circ \mathrm{d}\xi^F\big)-\nabla\cdot\big(s^\eta(\rho)g\big) \text{ in } \mathbb{T}^d\times(0,T) \text{ with }\ \rho(\cdot,0)=\rho_0, \tag{2.29}$$

for noise ξ^F and initial data satisfying Assumption 2.1 and for an $\mathcal{F}_t$-predictable, $L^2_tH^1_x$-integrable control g. We will write (2.29) in its Itô-form

$$\partial_t\rho=\Delta\rho-\sqrt{\varepsilon}\nabla\cdot\big(s^\eta(\rho)\,\mathrm{d}\xi^F\big)-\nabla\cdot\big(s^\eta(\rho)g\big)+\frac{\varepsilon F_1}{2}\nabla\cdot\big((s^\eta)'(\rho)\nabla s^\eta(\rho))\big), \tag{2.30}$$

and understand solutions in the equation's weak-formulation. We will first define the notion of a weak solution in Definition 2.9, and establish stable with respect to $\eta\in(0,1)$ energy estimates for the solutions in Proposition 2.10. We prove the existence of solutions to (2.30) in Proposition 2.11.

Definition 2.9. Let $T\in(0,\infty)$, let ξ^F and ρ_0 satisfy Assumption 2.1, let $\{s^\eta\}_{\eta\in(0,1/4)}$ satisfy the conditions of Lemma 2.8, let $\varepsilon,\eta\in(0,1/4)$, and let $g\in L^2(\Omega\times[0,T];H^1(\mathbb{T}^d))^d$ be an $\mathcal{F}_t$-predictable process. A *weak solution* of (2.30) is a $\mathbb{P}$-a.s. $L^2(\mathbb{T}^d;[0,1])$-continuous, $\mathcal{F}_t$-adapted process $\rho\in L^2([0,T]\times\Omega;H^1(\mathbb{T}^d))$ that satisfies, for every $\psi\in H^1(\mathbb{T}^d)$ and $t\in[0,T]$,

$$\begin{aligned}\int_{\mathbb{T}^d}\rho(x,t)\psi(x)\,\mathrm{d}x&=\int_{\mathbb{T}^d}\rho_0(x)\psi(x)\,\mathrm{d}x-\int_0^t\int_{\mathbb{T}^d}\nabla\rho(x,s)\cdot\nabla\psi(x)\,\mathrm{d}x\,\mathrm{d}s\\&+\sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^d}s^\eta(\rho(x,s))\nabla\psi(x)\cdot\,\mathrm{d}\xi^K+\int_0^t\int_{\mathbb{T}^d}s^\eta(\rho(x,s))\nabla\psi(x)\cdot g(x,s)\,\mathrm{d}x\,\mathrm{d}s\\&-\frac{\varepsilon F_1}{2}\int_0^t\int_{\mathbb{T}^d}((s^\eta)'(\rho(x,s)))^2\nabla\rho(x,s)\cdot\nabla\psi(x)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{2.31}$$

Proposition 2.10. *Let $T\in(0,\infty)$, let ξ^F and ρ_0 satisfy Assumption 2.1, let $\{s^\eta\}_{\eta\in(0,1/4)}$ satisfy Lemma 2.8, let $\varepsilon,\eta\in(0,1/4)$, and let $g\in L^2(\Omega\times[0,T];H^1(\mathbb{T}^d))^d$ be an $\mathcal{F}_t$-predictable process. Then, if ρ is a weak solution of (2.30) in the sense of Definition 2.9, we have the following four estimates.*

(1) Preservation of mass*: $\mathbb{P}$-a.s. for every $t\in[0,T]$,*

$$\|\rho(\cdot,t)\|_{L^1(\mathbb{T}^d)}=\|\rho_0\|_{L^1(\mathbb{T}^d)}\,.$$

(2) The energy estimate*: for every $\alpha\in[1,\infty)$ there exists $c\in(0,\infty)$ independent of ε, η, and α such that, for $\|F_3\|=\max_{x\in\mathbb{T}^d}|F_3(x)|$,*

$$\mathbb{E}\Big[\max_{t\in[0,T]}\|\rho(\cdot,t)\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{T}^d)}+\int_0^T\int_{\mathbb{T}^d}|\nabla\rho^{\frac{\alpha+1}{2}}|^2\Big]$$
$$\leq c\mathbb{E}\Big[\int_{\mathbb{T}^d}\rho_0^{\alpha+1}\Big]+c\alpha^2\Big(\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|g|^2\Big]+\varepsilon\,\|F_3\|\,\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}\rho^{\alpha-1}\Big]\Big).$$

(3) The entropy estimate*: for $\Psi\colon(0,1)\to\mathbb{R}$ defined by*

$$\Psi(\xi)=\Big(\xi\log(\xi)-\xi\Big)+\Big((1-\xi)\log(1-\xi)-(1-\xi)\Big),$$

for $c\in(0,\infty)$ independent of ε and η,

$$\mathbb{E}\Big[\max_{t\in[0,T]}\Big(\int_{\mathbb{T}^d}\Psi(\rho)\Big)+\int_0^T\int_{\mathbb{T}^d}\frac{1}{\rho(1-\rho)}|\nabla\rho|^2\,\mathrm{d}t\Big]$$
$$\leq\mathbb{E}\Big[\int_{\mathbb{T}^d}\Psi(\rho_0)\Big]+c\Big(\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|g|^2\Big]+\varepsilon(F_1+\|F_3\|\,T)\Big).$$

(4) The time regularity*: for every $\beta\in(0,1/2)$, for $c\in(0,\infty)$ independent of ε and η,*

$$\mathbb{E}\Big[\,\|\rho\|^2_{W^{\beta,2}([0,T];H^{-1}(\mathbb{T}^d))}\,\Big]$$
$$\leq c\Big(1+\varepsilon^2F_1^2\,\|(s^\eta)'\|^4_{L^\infty(\mathbb{R})}\Big)\Big(\mathbb{E}\Big[\int_{\mathbb{T}^d}\rho_0^2\Big]+\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|g|^2\Big]+\varepsilon(F_1+\|F_3\|\,T\Big).$$

Proof. The L^1-estimate follows from taking $\psi=1$ in Definition 2.9 and the fact that ρ is $\mathbb{P}$-a.s. $[0,1]$-valued. The $L^{\alpha+1}$-estimate for $\alpha\in[1,\infty)$ is a consequence of Itô's formula (see, for example, [47, Theorem 3.1]) applied to the function $\rho^{\alpha+1}$. We have $\mathbb{P}$-a.s. that

$$\mathrm{d}\Big(\int_{\mathbb{T}^d}\rho^{\alpha+1}\Big)+\alpha(\alpha+1)\int_{\mathbb{T}^d}\rho^{\alpha-1}|\nabla\rho|^2\,\mathrm{d}t=\sqrt{\varepsilon}\alpha(\alpha+1)\int_{\mathbb{T}^d}s^\eta(\rho)\rho^{\alpha-1}\nabla\rho\cdot\,\mathrm{d}\xi^F$$
$$+\alpha(\alpha+1)\int_{\mathbb{T}^d}s^\eta(\rho)\rho^{\alpha-1}\nabla\rho\cdot g\,\mathrm{d}t+\frac{\varepsilon\alpha(\alpha+1)}{2}\int_{\mathbb{T}^d}F_3(s^\eta(\rho))^2\rho^{\alpha-1}\,\mathrm{d}t,\tag{2.32}$$

where here we have already observed the cancellation between the final term on the righthand side of (2.31) and part of the Itô-correction. Since $0\leq\rho\leq1$ we have that $\rho^{\alpha-1}\leq\rho^{\frac{\alpha-1}{2}}$ and therefore, using the boundedness of s^η, Hölder's inequality, and Young's inequality, for every $\delta\in(0,1)$,

$$\alpha(\alpha+1)\int_{\mathbb{T}^d}s^\eta(\rho)\rho^{\alpha-1}\nabla\rho\cdot g\leq\frac{\alpha(\alpha+1)\delta}{2}\int_{\mathbb{T}^d}\rho^{\alpha-1}|\nabla\rho|^2+\frac{\alpha(\alpha+1)}{2\delta}\int_{\mathbb{T}^d}|g|^2.$$

Returning to (2.32), after choosing δ sufficiently small, we have $\mathbb{P}$-a.s. for some $c\in(0,\infty)$ that

$$\max_{t\in[0,T]}\|\rho(\cdot,t)\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{T}^d)}+\alpha(\alpha+1)\int_0^T\int_{\mathbb{T}^d}\rho^{\alpha-1}|\nabla\rho|^2$$
$$\leq\|\rho_0\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{T}^d)}+c\alpha(\alpha+1)\Big(\int_0^T\int_{\mathbb{T}^d}|g|^2+\varepsilon\,\|F_3\|\int_0^T\int_{\mathbb{T}^d}\rho^{\alpha-1}\Big)$$
$$+c\sqrt{\varepsilon}\alpha(\alpha+1)\Big(\max_{t\in[0,T]}|\int_0^t\int_{\mathbb{T}^d}s^\eta(\rho)\rho^{\alpha-1}\nabla\rho\cdot\,\mathrm{d}\xi^F|\Big),\tag{2.33}$$

for $\|F\|_3=\max_{x\in\mathbb{T}^d}|F_3(x)|$. For the stochastic integral, let Θ^η be defined for every $\xi\in[0,1]$ by

$$\Theta^\eta(\xi)=\int_0^\xi s^\eta(\xi')(\xi')^{\alpha-1}\,\mathrm{d}\xi'\leq\int_0^\xi(\xi')^{\alpha-1}\,\mathrm{d}\xi'\leq\frac{1}{\alpha}\xi^\alpha,$$

and observe that, after integrating by parts, for every $t\in[0,T]$,

$$\int_0^t\int_{\mathbb{T}^d} s^\eta(\rho)\rho^{\alpha-1}\nabla\rho\cdot\,\mathrm{d}\xi^F=-\int_0^t\int_{\mathbb{T}^d}\Theta^\eta(\rho)\nabla\cdot\,\mathrm{d}\xi^F.$$

The Burkholder–Davis–Gundy inequality, Hölder's inequality, and Young's inequality prove that, for every $\delta\in(0,1)$, for some $c\in(0,\infty)$,

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}_0}\Big[\sqrt{\varepsilon}(\alpha+1)\max_{t\in[0,T]}\big|\int_0^t\int_{\mathbb{T}^d} s^\eta(\rho)\nabla\rho\cdot\,\mathrm{d}\xi^F\big|\Big]\\
&\leq c\sqrt{\varepsilon}(\alpha+1)\mathbb{E}_{\mathcal{F}_0}\Big[\Big(\int_0^T\sum_{i=1}^d\sum_{k=1}^\infty\Big(\int_{\mathbb{T}^d}\rho^\alpha\partial_i f_k\Big)^2\Big)^{\frac12}\Big]\\
&\leq c\sqrt{\varepsilon}(\alpha+1)\mathbb{E}_{\mathcal{F}_0}\Big[\Big(\max_{t\in[0,T]}\int_{\mathbb{T}^d}\rho^{\alpha+1}\Big)^{\frac12}\Big(\int_0^T\int_{\mathbb{T}^d}\rho^{\alpha-1}F_3\Big)^{\frac12}\Big]\\
&\leq\frac{\delta}{2}\mathbb{E}_{\mathcal{F}_0}\Big[\max_{t\in[0,T]}\int_{\mathbb{T}^d}\rho^{\alpha+1}\Big]+\frac{\varepsilon(\alpha+1)^2\left\|F_3\right\|}{2\delta}\mathbb{E}_{\mathcal{F}_0}\Big[\int_0^T\int_{\mathbb{T}^d}\rho^{\alpha-1}\Big].
\end{aligned}\tag{2.34}$$

Returning to (2.33), after choosing δ sufficienty small in (2.34) and using that $(\alpha+1)\leq 2\alpha$ for every $\alpha\in[1,\infty)$, for some $c\in(0,\infty)$ independent of ε, η, and α,

$$\begin{aligned}
&\mathbb{E}\Big[\max_{t\in[0,T]}\|\rho(\cdot,t)\|^{\alpha+1}_{L^{\alpha+1}(\mathbb{T}^d)}+\int_0^T\int_{\mathbb{T}^d}|\nabla\rho^{\frac{\alpha+1}{2}}|^2\Big]\\
&\leq c\mathbb{E}\Big[\int_{\mathbb{T}^d}\rho_0^{\alpha+1}\Big]+c\alpha^2\Big(\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|g|^2\Big]+\varepsilon\left\|F_3\right\|\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}\rho^{\alpha-1}\Big]\Big),
\end{aligned}$$

where in the final step we have used the equality $(\alpha+1)^2\rho^{\alpha-1}|\nabla\rho|^2=4|\nabla\rho^{\frac{\alpha+1}{2}}|^2$. This completes the proof of the second energy estimate.

For the entropy estimate, let $\Psi\colon[0,1]\to\mathbb{R}$ denote the bounded, continuous function

$$\Psi(\xi)=\Big(\xi\log(\xi)-\xi\Big)+\Big((1-\xi)\log(1-\xi)-(1-\xi)\Big),$$

for which we have that $\Psi'(\xi)=\log(\xi)-\log(1-\xi)$ and $\Psi''(\xi)=(\xi(1-\xi))^{-1}$ on $(0,1)$. For every $\delta\in(0,1)$ let $\Psi_\delta\colon[0,1]\to\mathbb{R}$ denote the unique smooth function satisfying $\Psi_\delta(1/2)=\Psi(1/2)$ and $\Psi_\delta''(\xi)=(\xi(1-\xi)+\delta)^{-1}$. It is then a consequence of Itô's formula (see, for example, [47, Theorem 3.1]) applied to the function $\Psi(\rho)$ that, $\mathbb{P}$-a.s. for every $t\in[0,T]$,

$$\begin{aligned}
&\mathrm{d}\Big(\frac12\int_{\mathbb{T}^d}\Psi_\delta(\rho)\Big)+\int_{\mathbb{T}^d}\frac{1}{\rho(1-\rho)+\delta}|\nabla\rho|^2\,\mathrm{d}t\\
&=\sqrt{\varepsilon}\int_{\mathbb{T}^d}\frac{s^\eta(\rho)}{\rho(1-\rho)+\delta}\nabla\rho\cdot\,\mathrm{d}\xi^F+\int_{\mathbb{T}^d}\frac{s^\eta(\rho)}{\rho(1-\rho)+\delta}\nabla\rho\cdot g\,\mathrm{d}t+\frac{\varepsilon}{2}\int_{\mathbb{T}^d}\frac{F_3(s^\eta(\rho))^2}{\rho(1-\rho)+\delta}\,\mathrm{d}t,
\end{aligned}$$

where here, as in the above, we have already observed the cancellation between the final term of (2.31) and part of the Itô-correction. It then follows from the fact that there exists $c\in(0,\infty)$ independent of η such that $s^\eta(x)\leq cs(x)$ and from Hölder's inequality and Young's inequality that, for some $c\in(0,\infty)$ independent of ε and η,

$$\begin{aligned}
&\max_{t\in[0,T]}\int_{\mathbb{T}^d}\Psi_\delta(\rho)+\int_0^T\int_{\mathbb{T}^d}\frac{1}{\rho(1-\rho)+\delta}|\nabla\rho|^2\,\mathrm{d}t\leq\int_{\mathbb{T}^d}\Psi_\delta(\rho_0)\\
&\quad+c\Big(\sup_{t\in[0,T]}\Big(\sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\frac{s^\eta(\rho)}{\rho(1-\rho)+\delta}\nabla\rho\cdot\,\mathrm{d}\xi^F\Big)+\int_0^T\int_{\mathbb{T}^d}|g|^2+\varepsilon\left\|F_3\right\|T\Big).
\end{aligned}$$

A repetition of the argument above relying on the Burkhölder–Davis–Gundy inquality, Hölder's inequality, Young's inequality, Jensen's inequality, and $s^\eta(x) \leq cs(x)$ then proves that, for some $c \in (0, \infty)$ independent of ε, η, and δ,

$$\mathbb{E}\Big[\max_{t\in[0,T]} |c\sqrt{\varepsilon}\int_0^t\int_{\mathbb{T}^d} \frac{s^\eta(\rho)}{\rho(1-\rho)+\delta}\nabla\rho \cdot \mathrm{d}\xi^F|\Big] \leq c\varepsilon F_1 + \frac{1}{2}\int_0^T\int_{\mathbb{T}^d} \frac{1}{(\rho(1-\rho))+\delta}|\nabla\rho|^2.$$

Therefore, for some $c \in (0, \infty)$ independent of ε, η, and δ,

$$\begin{aligned}&\mathbb{E}\Big[\max_{t\in[0,T]}\int_{\mathbb{T}^d}\Psi_\delta(\rho) + \int_0^T\int_{\mathbb{T}^d}\frac{1}{\rho(1-\rho)+\delta}|\nabla\rho|^2\,\mathrm{d}t\Big] \\ &\leq \mathbb{E}\Big[\int_{\mathbb{T}^d}\Psi_\delta(\rho_0)\Big] + c\Big(\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|g|^2\Big] + \varepsilon(F_1 + T\,\|F_3\|)\Big).\end{aligned}$$

The claim then follows after passing to the limit $\delta \to 0$ using the definition of Ψ_δ and the monotone convergence theorem.

It remains to estimate the time derivative. We observe distributionally that, $\mathbb{P}$-a.s. for every $t \in [0, T]$,

$$\rho(\cdot, t) = \rho_0(\cdot) + I_t^{\text{f.v.}}(\cdot) + I_t^{\text{mart}}(\cdot),$$

for the finite variation part

$$I_t^{\text{f.v.}}(\cdot) = \nabla\cdot\Big(\int_0^t\nabla\rho(\cdot,s)\,\mathrm{d}s - \int_0^t s^\eta(\rho(\cdot,s))g\,\mathrm{d}s + \frac{\varepsilon F_1}{2}\int_0^t((s^\eta)'(\rho(\cdot,s)))^2\nabla\rho(\cdot,s)\,\mathrm{d}s\Big),$$

and for the martingale part

$$I_t^{\text{mart}}(\cdot) = -\nabla\cdot\Big(\sqrt{\varepsilon}\int_0^t s^\eta(\rho(\cdot,s))\,\mathrm{d}\xi^F\Big).$$

We then have that using the boundedness of s^η that, for $c \in (0, \infty)$ independent of ε and η,

$$\begin{aligned}&\Big\|I^{\text{f.v.}}\Big\|^2_{W^{1,2}([0,T];H^{-1}(\mathbb{T}^d))} \\ &\leq c\Big(\Big(1+\varepsilon^2F_1^2\,\|(s^\eta)'\|^4_{L^\infty(\mathbb{R})}\Big)\,\|\nabla\rho\|^2_{L^2(\mathbb{T}^d\times[0,T])^d} + \|g\|^2_{L^2(\mathbb{T}^d\times[0,T])^d}\Big).\end{aligned}$$

For the martingale part, we have by definition of the fractional Sobolev norm and the definition of the noise that, for every $\beta \in (0, 1/2)$,

$$\mathbb{E}\,\|I^{\text{mart}}\|^2_{W^{\beta,2}([0,T];H^{-1}(\mathbb{T}^d))} = \varepsilon\int_0^T\int_0^T\frac{\mathbb{E}\Big[|\sum_{k=1}^\infty\int_s^t\Big(\int_{\mathbb{T}^d}f_k^2(s^\eta(\rho))^2\Big)^{\frac{1}{2}}\mathrm{d}B_t^k|^2\Big]}{|s-t|^{1+2\beta}}\,\mathrm{d}s\,\mathrm{d}t.$$

It follows from Hölder's inequality, the Burkhölder–Davis–Gundy inequality, the definition of the noise, and the boundedness of s^η that, for some $c \in (0, \infty)$ independent of ε and η, for every $s \leq t \in [0, T]$,

$$\mathbb{E}\Big[|\sum_{k=1}^\infty\int_s^t\Big(\int_{\mathbb{T}^d}f_k^2(s^\eta(\rho))^2\Big)^{\frac{1}{2}}\mathrm{d}B_t^k|^2\Big] \leq cF_1|s-t|.$$

Therefore, since $\beta \in (0, 1/2)$, there exists $c \in (0, \infty)$ independent of ε and η but depending on T and β such that

$$\mathbb{E}\,\|I^{\text{mart}}\|^2_{W^{\beta,2}([0,T];H^{-1}(\mathbb{T}^d))} \leq c\varepsilon F_1.$$

It then follows from the embedding of $W^{2,1}$ into $W^{\beta,2}$ and the energy estimate that

$$\begin{aligned}&\mathbb{E}\Big[\,\|\rho\|^2_{W^{\beta,2}([0,T];H^{-1}(\mathbb{T}^d))}\,\Big]\\&\leq c\Big(1+\varepsilon^2F_1^2\,\|(s^\eta)'\|^4_{L^\infty(\mathbb{R})}\,\Big)\Big(\mathbb{E}\Big[\int_{\mathbb{T}^d}\rho_0^2\Big]+\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|g|^2\Big]+\varepsilon(F_1+\|F_3\|\,T)\Big),\end{aligned}$$

which completes the proof. □

Proposition 2.11. *Let $T\in(0,\infty)$, let ξ^F and ρ_0 satisfy Assumption 2.1, let $\{s^\eta\}_{\eta\in(0,1/4)}$ satisfy the conditions of Lemma 2.8, let $\varepsilon,\eta\in(0,1)$, and let $g\in L^2(\Omega\times[0,T];H^1(\mathbb{T}^d))^d$ be an $\mathcal{F}_t$-predictable process. Then there exists a weak solution of (2.30) in the sense of Definition 2.9.*

Proof. The proof of existence is a consequence of the smoothness and definition of the coefficients and noise, a standard Galerkin approximation, the estimates of Proposition 2.10, and the Aubin–Lions–Simon lemma [2, 51, 65]. Precisely, it follows from Proposition 2.10 that the approximating solutions are bounded in $L^2(\mathbb{T}^d\times[0,T])$, $L^2([0,T];H^1(\mathbb{T}^d))$, and $W^{\beta,2}([0,T];H^{-1}(\mathbb{T}^d))$ for $\beta\in(0,1/2)$. It then follows from Simon [65, Corollary 5], the compact embedding of $H^1(\mathbb{T}^d)$ into $L^2(\mathbb{T}^d)$, and the continuous embedding of $L^2(\mathbb{T}^d)$ into $H^{-1}(\mathbb{T}^d)$ that the approximating solutions are precompact in the strong topology of $L^2(\mathbb{T}^d\times[0,T])$ and the weak topology of $L^2([0,T];H^1(\mathbb{T}^d))$, which is sufficient to pass to the limit in (2.31). The fact that the solution is $[0,1]$-valued follows by applying Itô's formula to the functions $(\rho)_-$ and $(\rho-1)_+$, which completes the proof. □

We will now prove the existence of stochastic kinetic solutions solutions to the equation

$$\partial_t\rho=\Delta\rho-\sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho(1-\rho)}\circ \mathrm{d}\xi^F)+\nabla\cdot(\sqrt{\rho(1-\rho)}g),\tag{2.35}$$

in the sense of Definition 2.6. The essential difficulty is that the singularity appearing due to the Stratonovich-to-Itô correction makes it intractable to obtain stable estimates on the time-derivative of the solution. This issue also appears in the estimate on the time regularity in Proposition 2.10, which diverges in the limit $\eta\to 0$. It is for this reason that we instead consider renormalizations of the solution that localize it away from the sets $\{\rho\simeq 0\}$ and $\{\rho\simeq 1\}$. These renormalizations are defined in Definition 2.12 and are used in Proposition 2.14 to define an equivalent metric for the strong topology on $L^2(\mathbb{T}^d\times[0,T];[0,1])$ that is used in Proposition 2.15 to prove the tightness of the approximating probability laws and to prove existence in Theorem 4.8.

Definition 2.12. For every $\delta\in(0,1/4)$ let $\theta_\delta\in \mathrm{C}^\infty_c(\mathbb{R};[0,1])$ be a function satisfying that $\theta_\delta(\xi)=0$ if $\xi\leq\delta$ or $\xi\geq 1-\delta$ and that $\theta_\delta(\xi)=1$ if $2\delta\leq\xi\leq 1-2\delta$. For every $\delta\in(0,1/4)$ let $\Theta_\delta\colon[0,\infty)\to[0,1]$ be defined by $\Theta_\delta(0)=0$ and $\Theta'_\delta(\xi)=\theta_\delta(\xi)$.

Proposition 2.13. *Let $T\in(0,\infty)$, let ξ^F and ρ_0 satisfy Assumption 2.1, let $\varepsilon\in(0,1)$, let $g\in L^2(\Omega;L^2(\mathbb{T}^d\times[0,T])^d)$ be an $\mathcal{F}_t$-predictable process, and for every $\eta\in(0,1/4)$ let ρ^η be the solution of (2.30) constructed in Proposition 2.11. Then, for every $\delta\in(0,1/4)$ and $s>\frac{d+2}{2}$, there exists $c\in(0,\infty)$ independent of η and ε such that*

$$\begin{aligned}\mathbb{E}\,\|\Theta_\delta(\rho^\eta)\|_{W^{\beta,1}([0,T];H^{-s}(\mathbb{T}^d))}&\leq c\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|\nabla\rho^\eta|+s^\eta(\rho^\eta)|g|+(1+\varepsilon F_1)|\nabla\rho^\eta|^2\Big]\\&\quad+c\sqrt{\varepsilon F_1}\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}s^\eta(\rho^\eta)^2+|\nabla\rho^\eta|^2\Big]^{\frac{1}{2}}.\end{aligned}$$

Proof. For the functions $\{s^\eta\}_{\eta\in(0,1/4)}$ satisfying the conditions of Lemma 2.8, it follows from Itô's formula that distributionally, $\mathbb{P}$-a.s. for every $t\in[0,T]$,

$$\Theta_\delta(\rho^\eta)=\Theta_\delta(\rho_0)+I_t^{\text{f.v.}}+I_t^{\text{mart.}},$$

for the finite variation part

$$I_t^{\text{f.v.}} = \nabla\cdot\Big(\int_0^t \nabla\Theta_\delta(\rho^\eta) - \int_0^t s^\eta(\rho^\eta)g\,\mathrm{d}s + \frac{\varepsilon F_1}{2}\int_0^t ((s^\eta)'(\rho^\eta))^2\nabla\Theta_\delta(\rho^\eta)\,\mathrm{d}s\Big)$$
$$- \int_0^t \Theta_\delta''(\rho^\eta)|\nabla\rho^\eta|^2 - \frac{\varepsilon F_1}{2}\int_0^t ((s^\eta)'(\rho^\eta))^2\Theta_\delta''(\rho^\eta)|\nabla\rho^\eta|^2,$$

and for the martingale part

$$I_t^{\text{mart.}} = \sqrt{\varepsilon}\nabla\cdot\Big(\int_0^t \Theta_\delta'(\rho^\eta)s^\eta(\rho^\eta)\,\mathrm{d}\xi^F\Big) - \sqrt{\varepsilon}\int_0^t \Theta_\delta''(\rho^\eta)s^\eta(\rho^\eta)\nabla\rho^\eta\cdot\mathrm{d}\xi^F.$$

There are two essential differences in this case, when compared to the proof of time-regularity in Proposition 2.10. The first is that Θ_δ' and Θ_δ'' are bounded by a constant depending on δ and are supported on the set $\delta \le \xi \le 1-\delta$ on which $|(s^\eta)'(\xi)|$ remains bounded independently of η. The second are the terms arising from the Itô-correction, which are only L^1-integrable. It is for this reason that here we obtain estimates in H^{-s} for $s > \frac{d+2}{2}$, since by the Sobolev embedding theorem we have that H^s embeds into $W^{1,\infty}$.

Repeating the methods of Proposition 2.10, it follows from the definition of Θ_δ that there exists $c\in(0,\infty)$ depending on δ and $\sup_{\xi\in[\delta,1-\delta]}|(s^\eta)'(\xi)|$ but independent of η and ε such that

$$\Big\|I_t^{\text{f.v.}}\Big\|_{W^{1,1}([0,T];H^{-s}(\mathbb{T}^d))} \le c\Big(\int_0^T\int_{\mathbb{T}^d}|\nabla\rho^\eta| + s^\eta(\rho^\eta)|g| + (1+\varepsilon F_1)|\nabla\rho^\eta|^2\Big),$$

and for the martingale term, it follows from the Burkhölder–Davis–Gundy inequality and Hölder's inequality that, for every $\beta\in(0,1/2)$, for $c\in(0,\infty)$ depending on δ but independent of ε and η,

$$\begin{aligned}
&\mathbb{E}\left\|I^{\text{mart}}\right\|^2_{W^{\beta,1}([0,T];H^{-s}(\mathbb{T}^d))}\\
&= \sqrt{\varepsilon}\int_0^T\int_{\mathbb{T}^d}\int_0^T \frac{\mathbb{E}\Big[\big|\sum_{k=1}^\infty\int_s^t\Big(\int_{\mathbb{T}^d}|f_k|(s^\eta(\rho^\eta)+|\nabla\rho^\eta|)\Big)\,\mathrm{d}B_r^k\big|\Big]}{|s-t|^{1+2\beta}}\,\mathrm{d}s\,\mathrm{d}t\\
&\le c\sqrt{\varepsilon F_1}\int_0^T\int_0^T\frac{\mathbb{E}\Big[\int_s^t\int_{\mathbb{T}^d}s^\eta(\rho^\eta)^2+|\nabla\rho^\eta|^2\,\mathrm{d}r\Big]}{|s-t|^{1+2\beta}}\,\mathrm{d}s\,\mathrm{d}t\\
&\le c\sqrt{\varepsilon F_1}\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}s^\eta(\rho^\eta)^2+|\nabla\rho^\eta|^2\Big]^{\frac{1}{2}},
\end{aligned}$$

which, together with the embedding of $W^{1,1}$ into $W^{\beta,1}$ and the L^2-energy estimate for $\nabla\rho^\eta$, completes the proof. ☐

Proposition 2.14. *Let $D\colon L^\infty(\mathbb{T}^d\times[0,T];[0,1])\times L^\infty(\mathbb{T}^d\times[0,T];[0,1])\to[0,1]$ be defined by*

$$D(f,g) = \sum_{k=1}^\infty 2^{-k}\frac{\left\|\Theta_{1/5k}(f)-\Theta_{1/5k}(g)\right\|_{L^2(\mathbb{T}^d\times[0,T])}}{1+\left\|\Theta_{1/5k}(f)-\Theta_{1/5k}(g)\right\|_{L^2(\mathbb{T}^d\times[0,T])}}.$$

Then D defines a metric on $L^2(\mathbb{T}^d\times[0,T];[0,1])$ that is equivalent to the strong norm-induced metric on $L^2(\mathbb{T}^d\times[0,T];[0,1])$.

Proof. It follows from Definition 2.12 that $f=g$ in $L^2(\mathbb{T}^d\times[0,T];[0,1])$ if and only if $\Theta_{1/5k}(f) = \Theta_{1/5k}(g)$ for every $k\in\mathbb{N}$. This completes the proof that $D(f,g)=0$ if and only if $f=g$. The symmetry is a consequence of the definition of D and the symmetry of the norm-induced metric and the triangle inequality is a consequence of the triangle inequality for the norm-induced metric and the concavity of the function $\xi\mapsto\xi(1+\xi)^{-1}$ for $\xi\in[0,\infty)$.

In order to prove that D is equivalent to the strong norm-induced metric it suffices to prove that they determine the same convergent sequences. It follows from the definition of the Θ_δ that, for every $k \in \mathbb{N}$,

$$\left\|\Theta_{1/5k}(f)-\Theta_{1/5k}(g)\right\|_{L^2(\mathbb{T}^d\times[0,T])} \leq \|f-g\|_{L^2(\mathbb{T}^d\times[0,T])}.$$

Therefore, if a sequence is convergent with respect to the norm-induced metric it also converges with respect to D. Conversely, for every $k \in \mathbb{N}$, it follows from Definition 2.12 that, for some $c \in (0,\infty)$ independent of $k \in \mathbb{N}$,

$$\|f-g\|_{L^2(\mathbb{T}^d\times[0,T])} \leq \left\|\Theta_{1/5k}(f)-\Theta_{1/5k}(g)\right\|_{L^2(\mathbb{T}^d\times[0,T])} + \frac{c}{k},$$

from which it follows that a convergent sequence with respect to D must also converge with respect to the norm-induced metric. This completes the proof. □

Proposition 2.15. *Let $T \in (0,\infty)$, let ξ^F and ρ_0 satisfy Assumption* 2.1*, let $\varepsilon \in (0,1)$, let $g \in L^2(\Omega\times[0,T];H^1(\mathbb{T}^d))^d$ be an $\mathcal{F}_t$-predictable process, and for every $\eta \in (0,1/4)$ let ρ^η be the solution of* (2.30) *constructed in Proposition* 2.11*. Then the laws of $\{\rho^\eta\}_{\eta\in(0,1/4)}$ are tight on $L^2(\mathbb{T}^d\times[0,T])$.*

Proof. We will first show that, for every $k \in \mathbb{N}$, the laws of the $\{\Theta_{1/5k}(\rho^\eta)\}_{\eta\in(0,1/4)}$ are tight on $L^2(\mathbb{T}^d\times[0,T])$. It is a consequence of Definition 2.12, and in particular the boundedness and Lipschitz continuity of Θ_δ, that for every $\kappa \in \mathbb{N}$ there exists $c \in (0,\infty)$ depending on k such that

$$\sup_{\eta\in(0,1/4)} \mathbb{E}\left\|\Theta_{1/5k}(\rho^\eta)\right\|_{L^2([0,T];H^1(\mathbb{T}^d))} \leq c\Big(\mathbb{E}\Big[\int_{\mathbb{T}^d}\rho_0^2+\int_0^T\int_{\mathbb{T}^d}|g|^2\Big]+\varepsilon(F_1+\|F_3\|\,T)\Big).$$

The tightness of the $\{\Theta_{1/5k}(\rho^\eta)\}_{\eta\in(0,1/4)}$ is then a consequence of Proposition 2.13, the compact embedding of $H^1(\mathbb{T}^d)$ into $L^2(\mathbb{T}^d)$, the continuous embedding of $L^2(\mathbb{T}^d)$ into $H^{-s}(\mathbb{T}^d)$ for every $s > \frac{d+2}{2}$, and the Aubin–Lions–Simon lemma [2, 51, 65].

We will now deduce the tightness of the $\{\rho^\eta\}_{\eta\in(0,1)}$. Let $n \in \mathbb{N}$ be arbitrary and, using the tightness of the renormalized functions, for every $k \in \mathbb{N}$ let C_k be a compact subset of $L^2(\mathbb{T}^d\times[0,T])$ with respect to the usual norm-induced metric satisfying for every $\eta \in (0,1/4)$ that $\mathbb{P}[\Theta_{1/5k}(\rho^\eta) \notin C_k] \leq \frac{1}{2^k n}$. For every $k \in \mathbb{N}$ let $F_k\colon L^2(\mathbb{T}^d\times[0,T]) \to L^2(\mathbb{T}^d\times[0,T])$ be defined by $F_k(f) = \Theta_{1/5k}(f)$. Since it follows from the Lipschitz continuity of $\Theta_{1/5K}$ that F_k is continuous with respect to the usual norm-induced metric, let D_k be the closed subset $D_k = F_k^{-1}(C_k)$ for every $k \in \mathbb{N}$ and let $D = \cap_{k=1}^\infty D_k$. It follows from Proposition 2.14 that D is a compact subset of $L^2(\mathbb{T}^d\times[0,T])$, and it follows that, for every $\eta \in (0,1)$,

$$\mathbb{P}[\rho^\eta \notin D] \leq \sum_{k=1}^\infty \mathbb{P}[\Theta_{1/5k}(\rho^\eta)\notin C_k] \leq \sum_{k=1}^\infty \frac{1}{2^k n} \leq n^{-1}.$$

Since $n \in \mathbb{N}$ was arbitrary, this completes the proof. □

Theorem 2.16. *Let $T \in (0,\infty)$, let ξ^F and ρ_0 satisfy Assumption* 2.1*, let $\varepsilon \in (0,1)$, and let $g \in L^2(\Omega\times[0,T];H^1(\mathbb{T}^d))^d$ be an $\mathcal{F}_t$-predictable process. Then there exists a stochastic kinetic solution of* (2.35) *in the sense of Definition* 2.6*. Furthermore, the solution satisfies the estimates of Proposition* 2.10 *and Theorem* 3.9*.*

Proof. The proof is virtually identical to [27, Theorem 5.29], where in this case the metric defined in Proposition 2.14 plays the role of the metric defined in [27, Definition 5.23] and the entropy estimate of Proposition 2.10 yields the optimal regularity of the measure. □

3. The central limit theorem

In this section, we will study the fluctuations of the equation

$$\partial_t\rho^\varepsilon = \Delta\rho^\varepsilon - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho^\varepsilon(1-\rho^\varepsilon)}\circ \mathrm{d}\xi^K) \text{ in } \mathbb{T}^d\times(0,T) \text{ with } \rho(\cdot,0)=\rho_0, \tag{3.1}$$

for ξ^K defined in Assumption 3.1 below, about the hydrodynamic limit

$$\partial_t\overline{\rho} = \Delta\overline{\rho} \text{ in } \mathbb{T}^d\times(0,T) \text{ with } \overline{\rho}(\cdot,0)=\rho_0. \tag{3.2}$$

Precisely, we will identify an $\varepsilon\to 0$, $K(\varepsilon)\to\infty$ scaling regime such that the random variables

$$v^\varepsilon = \varepsilon^{-\frac{1}{2}}(\rho^\varepsilon - \overline{\rho}), \tag{3.3}$$

converge in probability in $L^2([0,T];H^{-s}(\mathbb{T}^d))$, for every $s>\frac{d}{2}$, to the generalized Ornstein–Uhlenbeck process

$$\partial_t v = \Delta v - \nabla\cdot(\sqrt{\overline{\rho}(1-\overline{\rho})}\,\mathrm{d}\xi) \text{ in } \mathbb{T}^d\times(0,\infty) \text{ with } v=0 \text{ on } \mathbb{T}^d\times\{0\}, \tag{3.4}$$

for $\mathrm{d}\xi$ a d-dimensional space-time white noise. An essential difficulty in proving a central limit theorem for the solutions of (3.1) is that the equation is only satisfied in the renormalized sense of Definition 2.6, which involves studying the equation satisfied by a nonlinear function of the solution. The nonlinearity is incompatible with the convergence of the fluctuations in the space $H^{-s}(\mathbb{T}^d)$.

It is for this reason that we first establish a strong CLT for equation (3.1) with the square root replaced by a smooth noise coefficient σ,

$$\partial_t\rho^\varepsilon = \Delta\rho^\varepsilon - \sqrt{\varepsilon}\nabla\cdot(\sigma(\rho^\varepsilon)\circ \mathrm{d}\xi^K) \text{ in } \mathbb{T}^d\times(0,T) \text{ with } \rho(\cdot,0)=\rho_0, \tag{3.5}$$

satisfying $\sigma(0)=\sigma(1)=0$ and $\sigma\in \mathrm{C}^2_c((0,1))\cap\mathrm{C}([0,1];[0,1])$. We then extend this CLT to equation (3.1) for initial data ρ_0 satisfying $\delta\le\rho_0\le 1-\delta$ for some $\delta\in(0,1/2)$ using the L^∞-estimate of Theorem 3.9 below, after approximating the square root by a smooth σ that agrees with the square root on $[\delta/2, 1-\delta/2]$ and proving using the pathwise uniqueness proof of Theorem 2.7 that the solutions of (3.1) and (3.5) agree for this choice of σ on the event that both solutions remain outside the $\delta/2$-neighborhood of zero and one.

3.1. **A quantitative LLN for the approximating SPDE.**

In this section, we will establish a qualitative law of large numbers for the solutions of (3.1) in Theorem 3.2, and a quantitative law of large numbers for the solutions of the approximating SPDE (3.5) in Proposition 3.5 that depends on the regularity of σ. The well-posedness of (3.5) is explained in Definition 3.3 and Proposition 3.4. Lastly, in Assumption 3.1 we introduce a particular choice of noise $\{\xi^K\}_{K\in\mathbb{N}}$. We emphasize that our methods do not rely on this choice, and would yield a quantitative rate of convergence for any sequence satisfying Assumption 2.1 with N_K replaced by F_1 and M_K by $\|F_3\|$.

Assumption 3.1. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration on $(\Omega,\mathcal{F})$, let $(B^k,W^k)_{k\in\mathbb{Z}^d}$ be independent, d-dimensional, $\mathcal{F}_t$-adapted Brownian motions taking values in the space $\mathrm{C}^\infty([0,\infty);(\mathbb{R}^d)^\infty)$ equipped with the metric topology of coordinate-wise convergence, and for every $k\in\mathbb{Z}^d$ let $e_k=\sqrt{2}\sin(k\cdot x)$ and $e'_k=\sqrt{2}\cos(k\cdot x)$. For every $K\in\mathbb{N}$ let ξ^K be defined by

$$\xi^K(x,t) = \sum_{|k|\le K}\Big(e_k(x)B^k_t + e'_k(x)W^k_t\Big).$$

We observe that this noise is a special case of Assumption 2.1 where we have that

$$F_1 = N_K = \#\{k\in\mathbb{Z}^d\colon |k|\le K\} \text{ and } F_3 = M_K = \sum_{|k|\le K}|k|^2,$$

and that $N_K\le cK^d$ and $M_K\le cK^{d+2}$ for some $c\in(0,\infty)$ independent of $K\in\mathbb{N}$. Finally, let $\rho_0\in L^\infty(\Omega;L^\infty(\mathbb{T}^d;[0,1]))$ be $\mathcal{F}_0$-measurable.

Theorem 3.2. *Let $T \in (0,\infty)$, let $\{\xi^K\}_{K\in\mathbb{N}}$ satisfy Assumption 3.1, let $\rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$, and let $\{K(\varepsilon)\}_{\varepsilon\in(0,1)}$ be a sequence that satisfies $\varepsilon K(\varepsilon)^{d+2} \to 0$ as $\varepsilon \to 0$. Then, for the solutions ρ^ε of (3.1) in the sense of Definition 2.6, $\mathbb{P}$-a.s. as $\varepsilon \to 0$,*

$$\rho^\varepsilon \to \overline{\rho} \;\; \text{strongly in} \;\; L^2(\mathbb{T}^d \times [0,T]),$$

for $\overline{\rho}$ the unique solution of (3.2) with initial data ρ_0.

Proof. The proof is a simplified version of Proposition 4.6 with $g = 0$ and ρ_0 deterministic. Since we will not require this result in what follows, we postpone the details until the proof of Proposition 4.6 below. □

Definition 3.3. Let $T \in (0,\infty)$, let $\sigma \in \mathrm{C}^2_c((0,1)) \cap \mathrm{C}([0,1];[0,1])$ with $\sigma(0) = \sigma(1) = 0$, let $\{\xi^K\}_{K\in\mathbb{N}}$ satisfy Assumption 3.1, let $\varepsilon \in (0,1)$, let $K \in \mathbb{N}$, and let $\rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$. A *weak solution* of (3.5) is a continuous $L^2(\mathbb{T}^d;[0,1])$-valued, $\mathcal{F}_t$-adapted process $\rho \in L^\infty(\Omega \times [0,T];L^\infty(\mathbb{T}^d,[0,1]))$ that satisfies the following two properties.

(1) *Regularity*: we have that
$$\rho \in L^2(\Omega \times [0,T];H^1(\mathbb{T}^d)).$$
(2) *The equation*: we have $\mathbb{P}$-a.s. that, for every $\psi \in \mathrm{C}^\infty(\mathbb{T}^d)$ and $t \in [0,T]$,
$$\int_{\mathbb{T}^d} \rho(x,t)\psi(x) = \int_{\mathbb{T}^d} \rho_0(x)\psi(x) + \sqrt{\varepsilon}\int_{\mathbb{T}^d} \sigma(\rho)\nabla\psi \cdot \mathrm{d}\xi^F - \frac{\varepsilon N_K}{2}\int_{\mathbb{T}^d} (\sigma'(\rho))^2 \nabla\rho\cdot\nabla\psi.$$

Proposition 3.4. *Let $T \in (0,\infty)$, let $\sigma \in \mathrm{C}^2_c((0,1)) \cap \mathrm{C}([0,1];[0,1])$ with $\sigma(0) = \sigma(1) = 0$, let $\{\xi^K\}_{K\in\mathbb{N}}$ satisfy Assumption 3.1, let $\varepsilon \in (0,1)$, let $K \in \mathbb{N}$, and let $\rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$. Then there exists a unique solution of (3.5) in the sense of Definition 3.3 that satisfies the estimates of Proposition 2.10. Furthermore, the solutions defined by Definition 2.6 and Definition 3.3 coincide.*

Proof. The existence is a consequence of Proposition 2.11, and the uniqueness follows from a simplified version of Theorem 2.7. □

Proposition 3.5. *Let $T \in (0,\infty)$, let $\sigma \in \mathrm{C}^2_c((0,1)) \cap \mathrm{C}([0,1];[0,1])$ with $\sigma(0) = \sigma(1) = 0$, let $\{\xi^K\}_{K\in\mathbb{N}}$ satisfy Assumption 3.1, let $\varepsilon \in (0,1)$, let $K \in \mathbb{N}$, and let $\rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$. Then, for the solution ρ of (3.5) in the sense of Definition 3.3 and the solution $\overline{\rho}$ of (3.2),*

$$\begin{aligned}&\mathbb{E}\Big[\sup\nolimits_{t\in[0,T]} \|\rho - \overline{\rho}\|^2_{L^2(\mathbb{T}^d)} + \int_0^T\int_{\mathbb{T}^d} |\nabla(\rho-\overline{\rho})|^2\Big] \\ &\leq c\Big(\varepsilon^2 N_K^2 \left\|\sigma'\right\|^2_{L^\infty((0,1))}\int_0^T\int_{\mathbb{T}^d}|\nabla\sigma(\rho)|^2 + \varepsilon(N_K + M_K T)\Big).\end{aligned}$$

Proof. Let $w = \rho - \overline{\rho}$ and observe that w is a solution of the equation

$$\partial_t w = \Delta w - \sqrt{\varepsilon}\nabla\cdot(\sigma(\rho)\xi^K) - \frac{\varepsilon N_K}{2}\nabla\cdot(\sigma'(\rho)\nabla\sigma(\rho)) \;\; \text{in} \;\; \mathbb{T}^d \times (0,T) \;\; \text{with} \;\; w(\cdot,0) = 0.$$

The claim then follows from a repetition of the arguments leading to the L^2-estimate of Proposition 2.10 using the boundedness of σ, Hölder's inequality, and Young's inequality. □

3.2. **The CLT for the approximating SPDE.** We will first establish a strong CLT for the solutions of (3.5) defined by a smooth and bounded σ. We will prove that the fluctuations converge strongly in $L^2([0,T];H^{-s}(\mathbb{T}^d))$, for every $s > \frac{d}{2}$, to the Ornstein–Uhlenbeck process

$$\partial_t v = \Delta v - \nabla\cdot(\sigma(\overline{\rho})\,\mathrm{d}\xi) \;\; \text{in} \;\; \mathbb{T}^d \times (0,T) \;\; \text{with} \;\; v = 0 \;\; \text{on} \;\; \mathbb{T}^d \times \{0\}, \tag{3.6}$$

for $\mathrm{d}\xi$ a d-dimensional space-time white noise. We explain the well-posedness of (3.6) in Definition 3.6 and Proposition 3.7. Observe that Definition 3.6 and Proposition 3.7 do not require any smoothness of σ, and therefore they apply to the square root $\sigma(\xi) = \sqrt{\xi(1-\xi)}$. We then prove a quantitative CLT for the solutions of (3.5) in Theorem 3.8 that does depend on the smoothness of σ.

Definition 3.6. Let $T \in (0,\infty)$, let $\mathrm{d}\xi$ be a d-dimensional space-time white noise, let $\sigma \in \mathrm{C}([0,1];[0,1])$ satisfy $\sigma(0)=\sigma(1)=0$, and let $\rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$. Let $\overline{\rho}$ be the weak solution of (3.3) with initial data ρ_0. A strong solution of (3.4) with $v_0=0$ is an $\mathcal{F}_t$-adapted and almost surely continuous $H^{-s}(\mathbb{T}^d)$-valued process $v \in L^2([0,T]\times\Omega;H^{-s}(\mathbb{T}^d))$, for every $s>\frac{d}{2}$, that almost surely satisfies, for every $\psi \in \mathrm{C}^\infty(\mathbb{T}^d)$ and $t\in[0,T]$,

$$\langle v(t),\psi\rangle_s = \int_0^t \langle v(r),\Delta\psi\rangle_s \,\mathrm{d}r + \int_0^t\int_{\mathbb{T}^d} \sigma(\overline{\rho})\nabla\psi\cdot\,\mathrm{d}\xi,$$

where $\langle\cdot,\cdot\rangle_s\colon H^{-s}(\mathbb{T}^d)\times H^s(\mathbb{T}^d)\to\mathbb{R}$ is the pairing between $H^{-s}(\mathbb{T}^d)$ and $H^s(\mathbb{T}^d)$.

Proposition 3.7. *Let $T\in(0,\infty)$, let $\sigma\in\mathrm{C}([0,1];[0,1])$ satisfy $\sigma(0)=\sigma(1)=0$, let $\mathrm{d}\xi$ be an $\mathbb{R}^d$-valued space-time white noise, and let $\rho_0\in L^\infty(\mathbb{T}^d;[0,1])$. Then there exists a unique solution of* (3.4) *in the sense of Definition* 3.6*.*

Proof. Let the noise $\{\xi^K\}_{K\in\mathbb{N}}$ be defined by Assumption 3.1 and let $\overline{\rho}$ be the unique solution of (3.2) with initial data ρ_0. Simplified versions of Theorem 2.7 and Proposition 2.11 (or, Theorem 2.16) prove that, for every $K\in\mathbb{N}$, there exists a unique continuous $L^2(\mathbb{T}^d)$-valued, $\mathcal{F}_t$-adapted process $v_K\in L^2([0,T]\times\Omega;L^2(\mathbb{T}^d))$ that satisfies the SPDE with additive noise

$$\partial_t v_K = \Delta v_K - \nabla\cdot(\sigma(\overline{\rho})\,\mathrm{d}\xi^K) \;\text{ in }\; \mathbb{T}^d\times(0,T) \;\text{ with }\; v_K(\cdot,0)=0.$$

Since it follows that $\int_{\mathbb{T}^d} v_K(x,t)=0$ for every $t\in[0,T]$, let $s>\frac{d+2}{2}$, let $z_K=(-\Delta)^{-\frac{s}{2}}v_K$, and observe using the methods of Proposition 2.10 that

$$\begin{aligned}
(3.7)\qquad & \mathbb{E}\Big[\max_{t\in[0,T]}\|z_K(\cdot,t)\|^2_{L^2(\mathbb{T}^d)} + \int_0^T\int_{\mathbb{T}^d}|\nabla z_K|^2\Big] \\
&\le \mathbb{E}\Big[\max_{t\in[0,T]}\Big|\int_0^t\int_{\mathbb{T}^d}\sigma(\overline{\rho})((-\Delta)^{-\frac{s}{2}}\nabla z_K)\cdot\,\mathrm{d}\xi^K\Big|\Big] \\
&\quad + \frac{1}{2}\mathbb{E}\Big[\sum_{|k|\le K}\sum_{i=1}^d\int_0^T\int_{\mathbb{T}^d}\Big(|(-\Delta)^{-\frac{s}{2}}\partial_i(\sigma(\overline{\rho})e_k)|^2 + |(-\Delta)^{-\frac{s}{2}}\partial_i(\sigma(\overline{\rho})e_k')|^2\Big)\Big].
\end{aligned}$$

For the first term on the righthand side of (3.7), the Burkhölder–Davis–Gundy inequality, Hölder's inequality, Young's inequality, $s>\frac{d+2}{2}\ge\frac{3}{2}$, the orthonormality of the $L^2(\mathbb{T}^d)$-basis $\{e_k,e_k'\}_{k\in\mathbb{Z}^d}$, and the boundedness of σ prove that, for some $c\in(0,\infty)$ independent of K,

$$\begin{aligned}
&\mathbb{E}\Big[\max_{t\in[0,T]}\Big|\int_0^t\int_{\mathbb{T}^d}\sigma(\overline{\rho})((-\Delta)^{-\frac{s}{2}}\nabla z_K)\cdot\,\mathrm{d}\xi^K\Big|\Big] \\
&\le c\mathbb{E}\Big\|\sigma(\overline{\rho})((-\Delta)^{-\frac{s}{2}}\nabla z_K)\Big\|_{L^2(\mathbb{T}^d\times[0,T])^d} \le c\mathbb{E}\,\|z_K\|_{L^2(\mathbb{T}^d\times[0,T])}.
\end{aligned}$$

For the final term on the righthand side of (3.7), it follows from $s>\frac{d+2}{2}\ge\frac{3}{2}$ and the orthonormality of the $L^2(\mathbb{T}^d)$-basis $\{e_k,e_k'\}_{k\in\mathbb{Z}^d}$ that, for some $c\in(0,\infty)$ independent of K,

$$\sum_{|k|\le K}\sum_{i=1}^d\int_0^T\int_{\mathbb{T}^d}\Big(|(-\Delta)^{-\frac{s}{2}}\partial_i(\sigma(\overline{\rho})e_k)|^2+|(-\Delta)^{-\frac{s}{2}}\partial_i(\sigma(\overline{\rho})e_k')|^2\Big)\le c\,\|\sigma(\overline{\rho})\|^2_{L^2}.$$

Returning to (3.7), it follows that, for some $c\in(0,\infty)$ independent of K,

$$\mathbb{E}\Big[\max_{t\in[0,T]}\|z_K(\cdot,t)\|^2_{L^2(\mathbb{T}^d)}+\int_0^T\int_{\mathbb{T}^d}|\nabla z_K|^2\Big]\le c\mathbb{E}\Big[\,\|z_K\|_{L^2}+\|\sigma(\overline{\rho})\|^2_{L^2}\Big],$$

from which it follows from Young's inequality, Grönwall's inequality, and the definition of z_K that there exists $c \in (0,\infty)$ independent of K but depending on T such that

$$\mathbb{E}\Big[\,\|v_K\|^2_{L^\infty([0,T];H^{-s}(\mathbb{T}^d))}+\|v_K\|^2_{L^2([0,T];H^{-s+1}(\mathbb{T}^d))}\Big]\le c(1+\|\sigma(\overline{\rho})\|^2_{L^2(\mathbb{T}^d\times[0,T])}). \tag{3.8}$$

Lastly, to estimate the time-regularity, we observe distributionally that

$$v_K(\cdot,t)=\int_0^t \Delta v_K-\int_0^t \nabla\cdot(\sigma(\overline{\rho})\,\mathrm{d}\xi^K)=I_t^{\text{f.v.}}(\cdot)+I_t^{\text{mart.}}(\cdot), \tag{3.9}$$

where it follows from (3.8) that the finite-variation part satisfies, for some $c \in (0,\infty)$ independent of K,

$$\left\|I_\cdot^{\text{f.v.}}(\cdot)\right\|_{W^{1,2}([0,T];H^{-(s+1)}(\mathbb{T}^d))}\le c\,\|v_K\|_{L^2([0,T];H^{-s}(\mathbb{T}^d))}\,, \tag{3.10}$$

and, following the methods of Proposition 2.10, it follows from $0 \le \sigma \le 1$ and the Burkhölder–Davis–Gundy inequality that for every $\beta \in (0, 1/2)$ there exists $c \in (0,\infty)$ depending on β and T but independent of K such that

$$\begin{aligned}
&\mathbb{E}\left\|I_\cdot^{\text{mart.}}(\cdot)\right\|^2_{W^{\beta,2}([0,T];H^{-(s+1)}(\mathbb{T}^d))}\\
&=\mathbb{E}\int_0^T\int_0^T|s-t|^{-(1+2\beta)}\Big\|\sum_{|k|\le K}\int_s^t\int_{\mathbb{T}^d}\Big(\sigma(\overline{\rho})e_k\,\mathrm{d}B_t^k+\sigma(\overline{\rho})e_k'\,\mathrm{d}W_t^k\Big)\Big\|^2_{H^{-s}(\mathbb{T}^d)}\\
&\le c\mathbb{E}\int_0^T\int_0^T|s-t|^{-(1+2\beta)}\sum_{|k|\le K}\int_s^t\|\sigma(\overline{\rho})e_k\|^2_{H^{-s}(\mathbb{T}^d)}+\left\|\sigma(\overline{\rho})e_k'\right\|^2_{H^{-s}(\mathbb{T}^d)}\\
&\le c\int_0^T\int_0^T|s-t|^{-2\beta}\le c.
\end{aligned} \tag{3.11}$$

Returning to (3.9), the embedding of $W^{1,2}$ into $W^{\beta,2}$ for $\beta \in (0, 1)$, (3.10), and (3.11) prove that, for every $\beta \in (0, 1/2)$ there exists $c \in (0,\infty)$ depending on β and T but independent of K such that

$$\mathbb{E}\Big[\,\|v_K\|_{L^2([0,T];H^{-(s+1)}(\mathbb{T}^d))}\Big]\le c(1+\|v_K\|_{L^2([0,T];H^{-(s+1)}(\mathbb{T}^d))}).$$

It now follows from the compact embedding of H^{-s} into $H^{-s'}$ whenever $s < s' \in (0,\infty)$, the Aubin–Lions–Simon lemma [2, 51, 65], and specifically [65, Corollary 5], and estimates (3.8) and (3.10) that the laws of the $\{v_K\}_{K\in\mathbb{N}}$ are tight on $L^2([0,T];H^{-s}(\mathbb{T}^d))$ for every $s > \frac{d}{2}$. A simplified version of Theorem 2.16 proves that, after passing to the limit $K \to 0$, the $\{v_K\}_{K\in\mathbb{N}}$ converge in law on $L^2([0,T];H^{-s}(\mathbb{T}^d))$ to an element $v \in L^2([0,T]\times\Omega;H^{-s}(\mathbb{T}^d))$, for every $s > \frac{d}{2}$, satisfying, for every $\psi \in \mathrm{C}^\infty(\mathbb{T}^d)$ and $\delta \in (0,1)$, for almost every $t \in [0,T]$,

$$\langle v(t),\psi\rangle_s=\int_0^t\langle v(r),\Delta\psi\rangle_s\,\mathrm{d}r+\int_0^t\int_{\mathbb{T}^d}\sigma(\overline{\rho})\nabla\psi\cdot\mathrm{d}\xi. \tag{3.12}$$

Since both terms on the righthand side of (3.12) are continuous in time, this implies that, for every $k \in \mathbb{Z}^d$, the pairings $t \mapsto \langle v(t), e_k\rangle_s$ and $t \mapsto \langle v(t), e_k'\rangle_s$ admit continuous modifications in $L^2([0,T])$. The $\mathbb{P}$-a.s. boundedness of v in $L^2([0,T];H^{-s}(\mathbb{T}^d))$, for every $s > \frac{d}{2}$, then implies that v admits a continuous, H^{-s}-valued modification still denoted v and that v satisfies (3.12) for every $\psi \in \mathrm{C}^\infty(\mathbb{T}^d)$ and $t \in [0,T]$. This completes the proof of existence. In order to prove uniqueness, observe that if v and $\tilde{v}$ are two solutions then $w = v - \tilde{v}$ is a distributional solution of the heat equation that $\mathbb{P}$-a.s. satisfies, for every $k \in \mathbb{Z}^d$, for every $t \in [0,T]$,

$$\langle w(t),e_k\rangle_s=-|k|^2\int_0^t\langle w(r),e_k\rangle_s\,\mathrm{d}r\;\text{ and }\;\langle w(t),e_k'\rangle_s=-|k|^2\int_0^t\langle w(r),e_k'\rangle_s\,\mathrm{d}r.$$

Grönwall's inequality proves that $\langle w(t), e_k\rangle_s = \langle w(t), e_k'\rangle_s = 0$ for every $k \in \mathbb{Z}^d$ and $t \in [0,T]$. Since $\{e_k, e_k'\}_{k\in\mathbb{Z}^d}$ is an $L^2(\mathbb{T}^d)$-basis, this implies $\mathbb{P}$-a.s. that $w = 0$ in $L^2([0,T]; H^{-s}(\mathbb{T}^d))$, for every $s > \frac{d}{2}$, and completes the proof of uniqueness. □

Theorem 3.8. *Let $T \in (0,\infty)$, let $\{\xi^K\}_{K\in\mathbb{N}}$ satisfy Assumption 3.1, let $\sigma \in \mathrm{C}([0,1];[0,1]) \cap \mathrm{C}^2_c((0,1))$ satisfy $\sigma(0) = \sigma(1) = 0$, let $K \in \mathbb{N}$, let $\varepsilon \in (0,1)$, and let $\rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$. Let ρ^ε be the solution of (3.5) in the sense of Definition 3.3, let $\overline{\rho}$ be the solution of (3.2), let $v^\varepsilon = \varepsilon^{-1/2}(\rho^\varepsilon - \overline{\rho})$, and let v be the solution of (3.4) in the sense of Definition 3.6. Then, for every $s > \frac{d}{2}$ there exists $c \in (0,\infty)$ independent of ε and K such that*

$$
\begin{aligned}
&\mathbb{E}\Big[\, \|v^\varepsilon - v\|^2_{L^2([0,T];H^{-s}(\mathbb{T}^d))} \Big] \\
&\leq c\Big(\Big(K^{-1} + \varepsilon N_K^2 \left\|\sigma'\right\|^2_{L^\infty((0,1))}\Big) \|\nabla\sigma(\overline{\rho})\|^2_{L^2} + K^{(-2s+(d+2))} \|\sigma(\overline{\rho})\|^2_{L^2}\Big) \\
&\quad + c\,\|\sigma(\rho^\varepsilon) - \sigma(\overline{\rho})\|^2_{L^2(\mathbb{T}^d\times[0,T])}\,.
\end{aligned}
$$

Proof. Let $w^\varepsilon = v^\varepsilon - v$, let $s > \frac{d+2}{2}$, and let $z^\varepsilon = (-\Delta)^{-\frac{s}{2}} w^\varepsilon$. We observe that since $s > \frac{d+2}{2}$ we have $\mathbb{P}$-a.s. that $z^\varepsilon \in L^2([0,T]; H^1(\mathbb{T}^d))$ and that z^ε solves

$$
\begin{aligned}
\partial_t z^\varepsilon &= \Delta z^\varepsilon - (-\Delta)^{-\frac{s}{2}}\nabla\cdot(\sigma(\rho^\varepsilon)\xi^K) \\
&\quad + (-\Delta)^{-\frac{s}{2}}\nabla\cdot(\sigma(\overline{\rho})\xi) + \frac{\sqrt{\varepsilon}N_K}{2}(-\Delta)^{-\frac{s}{2}}\nabla\cdot((\sigma'(\rho^\varepsilon))^2\nabla\rho^\varepsilon).
\end{aligned}
$$

A repetition of the L^2-energy estimate of Proposition 2.10 and the definition of ξ^K prove that, for $\Theta\colon [0,1] \to \mathbb{R}$ satisfying $\Theta(0) = 0$ and $\Theta'(\xi) = (\sigma'(\xi))^2$,

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d} |\nabla z^\varepsilon|^2\Big] \leq -\frac{\sqrt{\varepsilon}N_K}{2}\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}((-\Delta)^{-\frac{s}{2}}\nabla\Theta(\rho))\cdot\nabla z^\varepsilon\Big] \\
&+ \sum_{|k|\leq K}\int_0^T\int_{\mathbb{T}^d}((-\Delta)^{-\frac{s}{2}}\nabla\cdot((\sigma(\rho^\varepsilon) - \sigma(\overline{\rho}))e_k))^2 + ((-\Delta)^{-\frac{s}{2}}\nabla\cdot((\sigma(\rho^\varepsilon) - \sigma(\overline{\rho}))e_k'))^2 \\
&+ \sum_{|k|> K}\int_0^T\int_{\mathbb{T}^d}((-\Delta)^{-\frac{s}{2}}\nabla\cdot(\sigma(\overline{\rho})e_k))^2 + ((-\Delta)^{-\frac{s}{2}}\nabla\cdot(\sigma(\overline{\rho})e_k'))^2.
\end{aligned} \tag{3.13}
$$

Hölder's inequality and Young's inequality prove that the first term on the righthand side of (3.13) satisfies, for some $c \in (0,\infty)$,

$$
\begin{aligned}
&\frac{\sqrt{\varepsilon}N_K}{2}\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}((-\Delta)^{-\frac{s}{2}}\nabla\Theta(\rho))\cdot\nabla z^\varepsilon\Big] \\
&\leq c\varepsilon N_K^2 \left\|\sigma'\right\|^2_{L^\infty((0,1))}\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|\nabla\sigma(\rho^\varepsilon)|^2\Big] + \frac{1}{4}\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|\nabla z^\varepsilon|^2\Big].
\end{aligned} \tag{3.14}
$$

The orthonormality of the L^2-basis $\{e_k, e_k'\}_{k\in\mathbb{Z}^d}$ and $s > \frac{d+2}{2} > \frac{3}{2}$ prove that the final two terms on the righthand side of (3.13) satisfy, for some $c \in (0,\infty)$,

$$
\begin{aligned}
&\sum_{|k|\leq K}\int_0^T\int_{\mathbb{T}^d}((-\Delta)^{-\frac{s}{2}}\nabla\cdot((\sigma(\rho^\varepsilon) - \sigma(\overline{\rho}))e_k))^2 + ((-\Delta)^{-\frac{s}{2}}\nabla\cdot((\sigma(\rho^\varepsilon) - \sigma(\overline{\rho}))e_k'))^2 \\
&\leq c\,\|\sigma(\rho^\varepsilon) - \sigma(\overline{\rho})\|^2_{L^2(\mathbb{T}^d\times[0,T])}\,,
\end{aligned} \tag{3.15}
$$

and, a calculation using the definition of the H^{-s}-norm proves that, for some $c\in(0,\infty)$,

$$\begin{aligned}\sum_{|k|>K}\int_0^T\int_{\mathbb{T}^d}&((-\Delta)^{-\frac{s}{2}}\nabla\cdot(\sigma(\overline{\rho})e_k))^2+((-\Delta)^{-\frac{s}{2}}\nabla\cdot(\sigma(\overline{\rho})e_k'))^2\\&\leq c\sum_{|k|>K}\Big(\|\sigma(\overline{\rho})e_k\|^2_{H^{-s+1}(\mathbb{T}^d\times[0,T])}+\|\sigma(\overline{\rho})e_k'\|^2_{H^{-s+1}(\mathbb{T}^d\times[0,T])}\Big)\\&\leq cK^{-1}\|\nabla\sigma(\overline{\rho})\|^2_{L^2(\mathbb{T}^d\times[0,T])}+cK^{(-2s+(d+2))}\|\sigma(\overline{\rho})\|^2_{L^2(\mathbb{T}^d\times[0,T])}.\end{aligned}\tag{3.16}$$

Returning to (3.13), the claim now follows from estimates (3.14), (3.15), and (3.16). □

3.3. **The CLT for the SPDE with singular coefficients.** We will first establish an L^∞-estimate for the solutions of (3.1) with singular coefficients, and then use this estimate in Theorem 3.10 to establish the quantitative CLT in probability.

Theorem 3.9. *Let $T\in(0,\infty)$, let $\varepsilon\in(0,1)$, let ξ^F satisfy Assumption 2.1, let $\rho_0\in L^\infty(\mathbb{T}^d;[0,1])$, let $M=\text{ess sup}_{x\in\mathbb{T}^d}\rho_0(x)$, and let $M'=\operatorname{ess\,inf}_{x\in\mathbb{T}^d}\rho_0(x)$. Then, for the solution ρ^ε of (3.1) in the sense of Definition 2.6, there exist $c,\gamma\in(0,\infty)$ independent of ε but depending on T such that*

$$\mathbb{E}\Big[\|(\rho^\varepsilon-M)_+\|_{L^\infty(\mathbb{T}^d\times[0,T])}\Big]+\mathbb{E}\Big[\|(\rho^\varepsilon-M')_-\|_{L^\infty(\mathbb{T}^d\times[0,T])}\Big]\leq c\Big(\varepsilon\|F_3\|\Big)^\gamma.$$

Proof. Let $M=\text{ess sup}_{x\in\mathbb{T}^d}\rho_0(x)$ and let $\psi=(\rho-M)_+$. A repetition of the proof of the energy estimates in Proposition 2.10 proves that, for $\mathbb{E}_{\mathcal{F}_0}[\cdot]=\mathbb{E}[\cdot|\mathcal{F}_0]$, for every bounded stopping time τ taking values in $[0,T]$, we have $\mathbb{P}$-a.s. that, for $c\in(0,\infty)$ independent of ε and α,

$$\mathbb{E}_{\mathcal{F}_0}\Big[\max_{t\in[0,\tau]}\int_{\mathbb{T}^d}\psi^{\alpha+1}+\int_0^\tau\int_{\mathbb{T}^d}|\nabla\psi^{\frac{\alpha+1}{2}}|^2\Big]\leq c\alpha^2\varepsilon\|F_3\|\,\mathbb{E}_{\mathcal{F}_0}\Big[\int_0^\tau\int_{\mathbb{T}^d}\psi^{\alpha-1}\Big].\tag{3.17}$$

It then follows from [61, Chapter 4, Proposition 4.7, Exercise 4.30], (3.17), and Hölder's inequality that, for every $\alpha\in[1,\infty)$, for $n_\alpha=(\alpha+1)^{-1}$,

$$\begin{aligned}&\mathbb{E}\Big[\Big(\max_{t\in[0,\tau]}\int_{\mathbb{T}^d}\psi^{\alpha+1}+\int_0^\tau\int_{\mathbb{T}^d}|\nabla\psi^{\frac{\alpha+1}{2}}|^2\Big)^{\frac{1}{\alpha+1}}\Big]\\&\leq\frac{n_\alpha^{-n_\alpha}}{1-n_\alpha}(c\alpha^2\varepsilon\|F_3\|)^{n_\alpha}\mathbb{E}\Big[\|\psi\|_{L^{\alpha-1}(\mathbb{T}^d\times[0,T])}\Big]^{\frac{\alpha-1}{\alpha+1}}.\end{aligned}\tag{3.18}$$

We are now prepared to conclude using a Moser iteration. We first use interpolation and the Sobolev inequality to deduce that for

$$\theta=\frac{d}{2+d}\;\text{ and }\;q=\frac{(2+d)(\alpha+1)}{d},$$

we have that, for $c\in(0,\infty)$ independent of ε and α but depending on T,

$$\begin{aligned}&\|\psi\|_{L^q(\mathbb{T}^d\times[0,T])}\leq\|\psi\|^\theta_{L^\infty([0,T];L^{\alpha+1}(\mathbb{T}^d))}\|\psi\|^{1-\theta}_{L^{\alpha+1}([0,T];L^{\frac{2_*}{2}(\alpha+1)}(\mathbb{T}^d))}\\&=\|\psi\|^\theta_{L^\infty([0,T];L^{\alpha+1}(\mathbb{T}^d))}\Big\|\psi^{\frac{\alpha+1}{2}}\Big\|^{\frac{2(1-\theta)}{\alpha+1}}_{L^2([0,T];L^{2_*}(\mathbb{T}^d))}\\&\leq\|\psi\|^\theta_{L^\infty([0,T];L^{\alpha+1}(\mathbb{T}^d))}\Big(c\Big(\|\psi\|^{\frac{\alpha+1}{2}}_{L^\infty([0,T];L^{\alpha+1}(\mathbb{T}^d))}+\Big\|\nabla\psi^{\frac{\alpha+1}{2}}\Big\|_{L^2([0,T];L^2(\mathbb{T}^d))}\Big)\Big)^{\frac{2(1-\theta)}{\alpha+1}}.\end{aligned}$$

Hölder's inequality, the inequality $(x+y)^2 \leq 2(x^2+y^2)$ for all $x, y \in [0,\infty)$, $\theta \in (0,1)$, $\alpha \in [1,\infty)$, and (3.18) prove that, for $c \in (0,\infty)$ independent of ε and α,

$$\begin{aligned}
(3.19)\quad & \mathbb{E}\Big[\, \|\psi\|_{L^q(\mathbb{T}^d\times[0,T])} \Big] \\
\leq & \mathbb{E}\Big[\, \|\psi\|_{L^\infty([0,T];L^{\alpha+1}(\mathbb{T}^d))} \Big]^\theta \mathbb{E}\Big[\Big(c\,\|\psi\|^{\alpha+1}_{L^\infty([0,T];L^{\alpha+1}(\mathbb{T}^d))} + c\,\Big\|\nabla\psi^{\frac{\alpha+1}{2}}\Big\|^2_{L^2([0,T];L^2(\mathbb{T}^d))}\Big)^{\frac{1}{\alpha+1}}\Big]^{1-\theta} \\
\leq & \mathbb{E}\Big[\Big(c\,\|\psi\|^{\alpha+1}_{L^\infty([0,T];L^{\alpha+1}(\mathbb{T}^d))} + c\,\Big\|\nabla\psi^{\frac{\alpha+1}{2}}\Big\|^2_{L^2([0,T];L^2(\mathbb{T}^d))}\Big)^{\frac{1}{\alpha+1}}\Big] \\
\leq & \frac{n_\alpha^{-n_\alpha}}{1-n_\alpha}(c\alpha^2\varepsilon\,\|F_3\|)^{n_\alpha}\mathbb{E}\Big[\, \|\psi\|_{L^{\alpha-1}(\mathbb{T}^d\times[0,T])} \Big]^{\frac{\alpha-1}{\alpha+1}}.
\end{aligned}$$

We proceed inductively by defining

$$\alpha_0 = 0 \;\text{ and }\; \alpha_k = \frac{2+d}{d}\Big(\alpha_{k-1}+2\Big) \;\text{ for every }\; k\in\mathbb{N},$$

which implies that $\alpha_k > (1+\frac{2}{d})^k$ for every $k \in \mathbb{N}$, and we let $\beta_k = \alpha_{k-1}+1$. Observe using (3.19) and the definition of q that, for every $k \in \mathbb{N}$, for $c \in (0,\infty)$ independent of ε, α, and k,

$$\begin{aligned}
(3.20)\qquad & \mathbb{E}\Big[\, \|\psi\|_{L^{\alpha_k}(\mathbb{T}^d\times[0,T])} \Big] \leq \frac{n_{\beta_k}^{-n_{\beta_k}}}{1-n_{\beta_k}}(c\beta_k^2\varepsilon\,\|F_3\|)^{n_{\beta_k}}\mathbb{E}\Big[\, \|\psi\|_{L^{\alpha_{k-1}}(\mathbb{T}^d\times[0,T])} \Big]^{\frac{\beta_k-1}{\beta_k+1}} \\
& \leq \prod_{r=1}^k \Big(\frac{n_{\beta_r}^{-n_{\beta_r}}}{1-n_{\beta_r}}(c\beta_r)^{2n_{\beta_r}}\Big)^{\prod_{s=r+1}^k \frac{\beta_s-1}{\beta_s+1}} \times \Big(\varepsilon\,\|F_3\|\Big)^{\sum_{r=1}^k n_{\beta_r}\prod_{s=r+1}^k \frac{\beta_s-1}{\beta_s+1}}.
\end{aligned}$$

Since by definition $\beta_k > (1+\frac{2}{d})^{k-1}+1$ for every $k \in [2,\infty)$, we have that

$$\liminf_{N\to\infty} \log\Big(\prod_{s=2}^N \frac{\beta_s-1}{\beta_s+1}\Big) = \liminf_{N\to\infty}\sum_{s=2}^N \log\Big(1-\frac{2}{\beta_s+1}\Big) > -\infty.$$

Therefore, by the dominated convergence theorem, there exists $\gamma \in (0,1)$ such that

$$\lim_{k\to\infty}\Big(\sum_{r=1}^k n_{\beta_r}\prod_{s=r+1}^k \frac{\beta_s-1}{\beta_s+1}\Big) = \gamma.$$

We have similarly that, by definition of n_{β_r},

$$\limsup_{k\to\infty}\prod_{r=1}^k \Big(\frac{n_{\beta_r}^{-n_{\beta_r}}}{1-n_{\beta_r}}(c\beta_r)^{4n_{\beta_r}}\Big)^{\prod_{s=r+1}^k \frac{\beta_s-1}{\beta_s+1}} \leq \limsup_{k\to\infty}\prod_{r=1}^k\Big(\frac{(1+\beta_r)^{1+n_{\beta_r}}}{\beta_r}(c\beta_r)^{4n_{\beta_r}}\Big),$$

and that, for every $k \in \mathbb{N}$,

$$\begin{aligned}
&\log\Big(\prod_{r=1}^k\Big(\frac{(1+\beta_r)^{1+n_{\beta_r}}}{\beta_r}(c\beta_r)^{4n_{\beta_r}}\Big)\Big) \\
&= \sum_{r=1}^k\Big(\log(1+\beta_r^{-1}) + n_{\beta_r}\log(1+\beta_r) + 4n_{\beta_r}\log(c\beta_r)\Big).
\end{aligned}$$

It follows from the facts that $\beta_r \geq \alpha_{r-1}+1$, $\alpha_0 = 0$, and $\alpha_r > (1+\frac{2}{d})^r$ that

$$\limsup_{k\to\infty}\Big(\sum_{r=1}^k\Big(\log(1+\beta_r^{-1}) + n_{\beta_r}\log(1+\beta_r) + 4n_{\beta_r}\log(c\beta_r)\Big)\Big) < \infty,$$

which proves that there exists $c \in (0,\infty)$ independent of ε and η such that

$$\limsup_{k\to\infty}\prod_{r=1}^{k}\Big(\frac{n_{\beta_r}^{-n_{\beta_r}}}{1-n_{\beta_r}}(c\beta_r)^{4n_{\beta_r}}\Big)^{\prod_{s=r+1}^{k}\frac{\beta_s-1}{\beta_s+1}}\leq c,$$

and, therefore, after passing to the limit in $k\to\infty$ in (3.20), we conclude that

$$\mathbb{E}\Big[\,\|\psi\|_{L^\infty(\mathbb{T}^d\times[0,T])}\,\Big]\leq c\Big(\varepsilon\,\|F_3\|\,\Big)^\gamma,$$

which completes the proof of the upper bound. The lower bound is obtained by letting $M'=\operatorname{ess\,inf}_{x\in\mathbb{T}^d}\rho_0(x)$ and considering $\psi(x,t)=(\rho(x,t)-M')_-=-\max(0,(M'-\rho))$. □

Theorem 3.10. *Let $T\in(0,\infty)$, let $\{\xi^K\}_{K\in\mathbb{N}}$ be the noise defined in Definition 3.1, let $K\in\mathbb{N}$, let $\varepsilon\in(0,1)$, and let $\rho_0\in L^\infty(\mathbb{T}^d)$ satisfy $\delta\leq\rho_0\leq 1-\delta$ for some $\delta\in(0,1/2)$. Let ρ^ε be the solution of (3.1) in the sense of Definition 2.6, let $\overline{\rho}$ be the solution of (3.2), let $v^\varepsilon=\varepsilon^{-1/2}(\rho^\varepsilon-\overline{\rho})$, and let v be the solution of (3.4) in the sense of Definition 3.6 with $\sigma(\xi)=\sqrt{\xi(1-\xi)}$. Then, for every $s>\frac{d}{2}$ there exist $c,\gamma\in(0,\infty)$ such that, for every $\eta\in(0,1)$,*

$$\begin{aligned}\mathbb{P}\Big[\,\|v^\varepsilon-v\|_{L^2([0,T];H^{-s}(\mathbb{T}^d))}\geq\eta\Big]&\leq c\delta^{-1}\Big(\varepsilon M_K\Big)^\gamma\\&\quad+c\eta^{-2}\Big(K^{-1}+\varepsilon N_K^2\delta^{-1}+K^{(-2s+(d+2))}+\delta^{-2}\varepsilon^2N_K^2+\delta^{-1}\varepsilon M_K\Big).\end{aligned}$$

Proof. Let $\sigma\in\mathrm{C}^2_c((0,1))\cap\mathrm{C}([0,1];[0,1])$ be an arbitrary function satisfying that

$$\sigma(\xi)=\sqrt{\xi(1-\xi)}\text{ for every }\xi\in[\delta/2,1-\delta/2]\text{ and }|\sigma'(\xi)|\leq 2\delta^{-\frac12}\text{ for every }\xi\in(0,1),\tag{3.21}$$

and satisfying that

$$|\sigma'(\xi)|\leq\frac{2}{\sqrt{\xi(1-\xi)}}\;\text{ for every }\;\xi\in(0,1).\tag{3.22}$$

For every $\varepsilon\in(0,1)$ let $\tilde{v}^\varepsilon=\varepsilon^{-1/2}(\tilde{\rho}^\varepsilon-\overline{\rho})$ for the $\tilde{\rho}^\varepsilon$ the solution of (3.5) defined by this σ and let v denote the Ornstein–Uhlenbeck process

$$\partial_t v=\Delta v-\nabla\cdot(\sqrt{\overline{\rho}(1-\overline{\rho})}\,\mathrm{d}\xi)=\Delta v-\nabla\cdot(\sigma(\overline{\rho})\,\mathrm{d}\xi)\;\text{ in }\;\mathbb{T}^d\times(0,T)\;\text{ with }\;v=0\;\text{ on }\;\mathbb{T}^d\times\{0\},$$

where here we use the fact that, since the comparison principle proves that $\delta\leq\overline{\rho}\leq1-\delta$ on $\mathbb{T}^d\times[0,T]$, we have by (3.21) that $\sigma(\overline{\rho})=\sqrt{\overline{\rho}(1-\overline{\rho})}$. We then observe through an exact repetition of the pathwise uniqueness proof of Theorem 2.7 that

$$\rho^\varepsilon=\tilde{\rho}^\varepsilon\;\text{ in }\;L^2(\mathbb{T}^d\times[0,T]),$$

on the event $(\mathcal{S}\cap\tilde{\mathcal{S}})\subseteq\Omega$ defined by

$$\mathcal{S}=\{\|\rho^\varepsilon\|_{L^\infty(\mathbb{T}^d\times[0,T])}\leq1-\delta/2\text{ and }\|1-\rho^\varepsilon\|_{L^\infty(\mathbb{T}^d\times[0,T])}\leq1-\delta/2\},$$

and

$$\tilde{\mathcal{S}}=\{\|\tilde{\rho}^\varepsilon\|_{L^\infty(\mathbb{T}^d\times[0,T])}\leq1-\delta/2\text{ and }\|1-\tilde{\rho}^\varepsilon\|_{L^\infty(\mathbb{T}^d\times[0,T])}\leq1-\delta/2\}.$$

Therefore, for every $\eta\in(0,1)$, we have that

$$\begin{aligned}&\mathbb{P}\Big[\,\|v^\varepsilon-v\|_{L^2([0,T];H^{-s}(\mathbb{T}^d))}\geq\eta\Big]\\&\leq\mathbb{P}\Big[\,\|\tilde{v}^\varepsilon-v\|_{L^2([0,T];H^{-s}(\mathbb{T}^d))}>\eta\Big]+\mathbb{P}[\mathcal{S}^c]+\mathbb{P}[\tilde{\mathcal{S}}^c].\end{aligned}\tag{3.23}$$

Since it follows from (3.22), $\rho_0\in L^\infty(\mathbb{T}^d;[0,1])$, and the entropy estimate of Proposition 2.10 that there exists $c\in(0,\infty)$ independent of $\varepsilon\in(0,1)$ such that

$$\mathbb{E}\Big[\int_0^T\int_{\mathbb{T}^d}|\nabla\sigma(\rho^\varepsilon)|^2\Big]\leq c,$$

Theorem 3.8, Chebyshev's inequality, (3.21), and $0 \leq \sigma \leq 1$ prove that, for some $c \in (0,\infty)$ independent of ε, η, and K,

$$\begin{aligned}&\mathbb{P}\Big[\, \|\tilde{v}^\varepsilon - v\|_{L^2([0,T];H^{-s}(\mathbb{T}^d))} > \eta\Big] \\ &\leq c\eta^{-2}\Big(K^{-1} + \varepsilon N_K^2 \left\|\sigma'\right\|^2_{L^\infty((0,1))} + K^{(-2s+(d+2))} + \|\sigma(\rho^\varepsilon) - \sigma(\overline{\rho})\|^2_{L^2(\mathbb{T}^d\times[0,T])}\Big) \\ &\leq c\eta^{-2}\Big(K^{-1} + \varepsilon N_K^2\delta^{-1} + K^{(-2s+(d+2))} + \delta^{-1}\|\rho^\varepsilon - \overline{\rho}\|^2_{L^2(\mathbb{T}^d\times[0,T])}\Big).\end{aligned}$$

The quantitative law of large numbers of Proposition 3.5, (3.21), and $N_K \leq M_K$ then prove that, for $c \in (0,\infty)$ independent of ε but depending on T,

$$\mathbb{E}\big[\, \|\rho^\varepsilon - \overline{\rho}\|^2_{L^2(\mathbb{T}^d\times[0,T])}\,\big] \leq c\Big(\delta^{-1}\varepsilon^2 N_K^2 + \varepsilon M_K\Big).$$

Therefore,

$$\begin{aligned}(3.24)\qquad &\mathbb{P}\Big[\, \|\tilde{v}^\varepsilon - v\|_{L^2([0,T];H^{-s}(\mathbb{T}^d))} > \eta\Big] \\ &\leq c\eta^{-2}\Big(K^{-1} + \varepsilon N_K^2\delta^{-1} + K^{(-2s+(d+2))} + \delta^{-2}\varepsilon^2 N_K^2 + \delta^{-1}\varepsilon M_K\Big).\end{aligned}$$

For the final two terms on the righthand side of (3.23), Theorem 3.9, which applies equally to the approximating equation thanks to (3.21) and (3.22), and Chebyshev's inequality prove that, for some $c, \gamma \in (0,\infty)$ independent of ε and K,

$$\mathbb{P}[\mathcal{S}^c] + \mathbb{P}[\tilde{\mathcal{S}}^c] \leq c\delta^{-1}\Big(\varepsilon M_K\Big)^\gamma. \tag{3.25}$$

Returning to (3.23), it follows from (3.24) and (3.25) that

$$\begin{aligned}&\mathbb{P}\Big[\, \|v^\varepsilon - v\|_{L^2([0,T];H^{-s}(\mathbb{T}^d))} \geq \eta\Big] \\ &\leq c\eta^{-2}\Big(K^{-1} + \varepsilon N_K^2\delta^{-1} + K^{(-2s+(d+2))} + \delta^{-2}\varepsilon^2 N_K^2 + \delta^{-1}\varepsilon M_K\Big) + c\delta^{-1}\Big(\varepsilon M_K\Big)^\gamma,\end{aligned}$$

which completes the proof. □

Corollary 3.11. *Let $T \in (0,\infty)$, let $\{\xi^K\}_{K\in\mathbb{N}}$ be the noise defined in Definition 3.1, let $\alpha_d = (\frac{1}{d+2} \wedge \frac{1}{2d})$ and let $K(\varepsilon) = \lfloor \varepsilon^{-\alpha_d}\rfloor$ for every $\varepsilon \in (0,1)$, and let $\rho_0 \in L^\infty(\mathbb{T}^d)$ satisfy $\delta \leq \rho_0 \leq 1-\delta$ for some $\delta \in (0, 1/2)$. Let ρ^ε be the solution of (3.1) corresponding to $(\varepsilon, K(\varepsilon))$ in the sense of Definition 2.6, let $\overline{\rho}$ be the solution of (3.2), and let $v^\varepsilon = \varepsilon^{-1/2}(\rho^\varepsilon - \overline{\rho})$. Then, for every $s > \frac{d}{2}$ there exist $c, \gamma \in (0,\infty)$ such that, for every $\eta \in (0,1)$,*

$$\mathbb{P}\Big[\, \|v^\varepsilon - v\|_{L^2([0,T];H^{-s}(\mathbb{T}^d))} \geq \eta\Big] \leq c\eta^{-2}\delta^{-2}\Big(\varepsilon^{\alpha_d} + \varepsilon^{\alpha_d(2s-(d+2))}\Big) + c\delta^{-1}\varepsilon^{\frac{\gamma}{2}}.$$

Proof. The proof is an immediate consequence of Theorem 3.10 and the bounds $N_K \leq cK^d$ and $M_K \leq cK^{d+2}$ for some $c \in (0,\infty)$ independent of $K \in \mathbb{N}$. □

4. The large deviations principle

In this section, we will identify a scaling regime for which the solutions of the SPDE

$$\partial_t\rho_\varepsilon = \Delta\rho_\varepsilon - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho_\varepsilon(1-\rho_\varepsilon)} \circ \mathrm{d}\xi^{K(\varepsilon)}), \tag{4.1}$$

for the spectral approximations $\{\xi^K\}_{K\in\mathbb{N}}$ defined in Assumption 3.1, satisfy a large deviations principle with rate functions I_{ρ_0} defined by

$$I_{\rho_0}(\mu) = \frac{1}{2}\big\{\, \|g\|^2_{L^2} : \mu = \rho\, dx,\, \partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\rho(1-\rho)}g) \text{ with } \rho(\cdot,0) = \rho_0\big\}. \tag{4.2}$$

We emphasize that these techniques do not rely on the specific choice of noise $\{\xi^K\}_{K\in\mathbb{N}}$ and would apply without change, for example, to spatial convolutions $\{\,\mathrm{d}\xi^\delta = (\,\mathrm{d}\xi * \kappa^\varepsilon)\}_{\delta\in(0,1)}$ of space-time white noise, or to a general sequence satisfying Assumption 2.1 converging in distribution to a space-time white noise. We make this choice in order to precisely quantify the scaling in ε and K. In what follows, we first analyze the skeleton equation appearing in the definition of the rate function in Section 4.1 and prove the LDP in Section 4.2.

4.1. **The skeleton equation.** In this section, we will prove the well-posedness of the skeleton equation

$$\partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\rho(1-\rho)}g) \text{ in } \mathbb{T}^d\times(0,T) \text{ with } \rho=\rho_0 \text{ on } \mathbb{T}^d\times\{0\}, \tag{4.3}$$

for $L^2(\mathbb{T}^d\times[0,T])^d$-integrable controls g and initial data ρ_0 in the space $L^\infty(\mathbb{T}^d;[0,1])$. In Definition 4.1, we define a weak solution to (4.3). In Proposition 4.2, we prove that weak solutions are unique. In Proposition 4.3, we prove that weak solutions exist. Finally, in Proposition 4.4, we prove the strong continuity of the solution ρ with respect to weak convergence of the control g.

Definition 4.1. Let $T\in(0,\infty)$, $\rho_0\in L^\infty(\mathbb{T}^d;[0,1])$, and $g\in L^2(\mathbb{T}^d\times[0,T])^d$. A *weak solution* of (4.3) is a function $\rho\in L^2([0,T];H^1(\mathbb{T}^d))\cap \mathrm{C}([0,T];L^2(\mathbb{T}^d;[0,1]))$ that satisfies, for every $t\in[0,T]$ and $\psi\in \mathrm{C}^\infty(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d}\rho(x,t)\psi(x)\,\mathrm{d}x = \int_{\mathbb{T}^d}\rho_0\psi\,\mathrm{d}x - \int_0^t\int_{\mathbb{T}^d}\nabla\rho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^d}\sqrt{\rho(1-\rho)}g\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}s.$$

Proposition 4.2. *Let $T\in(0,\infty)$, let $\rho_{0,1},\rho_{0,2}\in L^\infty(\mathbb{T}^d;[0,1])$, and let $g\in L^2(\mathbb{T}^d\times[0,T])^d$. Let $\rho_1,\rho_2\in L^2([0,T];H^1(\mathbb{T}^d))$ be weak solutions of* (4.3) *in the sense of Definition* 4.1 *with initial data $\rho_{0,1},\rho_{0,2}$ and control g. Then,*

$$\max_{t\in[0,T]}\|\rho_1(\cdot,t)-\rho_2(\cdot,t)\|_{L^1(\mathbb{T}^d)} \le \|\rho_{0,1}-\rho_{0,2}\|_{L^1(\mathbb{T}^d)}.$$

Proof. For every $\varepsilon\in(0,1)$ let η^ε_d be a standard convolution kernel on $\mathbb{T}^d$ of scale ε and let η^ε_1 be a standard convolution kernel on $\mathbb{R}$ of scale ε. For every $i\in\{1,2\}$ let $\rho^\varepsilon_i=\rho_i*\eta^\varepsilon_d$. Let $a\colon\mathbb{R}\to[0,\infty)$ denote the absolute value function $a(x)=|x|$ and for every $\delta\in(0,1)$ let $a^\delta=a*\eta^\delta_1$.

Definition 4.1 implies that, for every $i\in\{1,2\}$, as functions in $L^2(\mathbb{T}^d\times[0,T])$,

$$\partial_t\rho^\varepsilon_i(x,t) = -\int_{\mathbb{T}^d}\nabla_y\eta^\varepsilon_d(y-x)\cdot\nabla\rho_i(y,t) + \sqrt{\rho_i(y,t)(1-\rho_i(y,t))}g(y,t)\cdot\nabla_y\eta^\varepsilon_d(y-x)\,\mathrm{d}y. \tag{4.4}$$

Therefore, for every $\varepsilon,\delta\in(0,1)$,

$$\partial_t\int_{\mathbb{T}^d}a^\delta(\rho^\varepsilon_1-\rho^\varepsilon_2)\,\mathrm{d}x = \int_{\mathbb{T}^d}\mathrm{sgn}^\delta(\rho^\varepsilon_1-\rho^\varepsilon_2)\partial_t(\rho^\varepsilon_1-\rho^\varepsilon_2)\,\mathrm{d}x,$$

where $\mathrm{sgn}^\delta=\mathrm{sgn}*\eta^\delta_1$ for the left-continuous sign function $\mathrm{sgn}\colon\mathbb{R}\to\{-1,1\}$. It follows from (4.4) that, after integrating by parts in the x-variable and passing to the limit $\varepsilon\to0$ using the H^1-regularity and boundedness of weak solutions, for every $\delta\in(0,1)$,

$$\begin{aligned}\partial_t\int_{\mathbb{T}^d}a^\delta(\rho_1(x,t)-\rho_2(x,t))\,\mathrm{d}x &= -\int_{\mathbb{T}^d}2\eta^\delta_1(\rho_1-\rho_2)|\nabla(\rho_1-\rho_2)|^2\,\mathrm{d}x\\ &+\int_{\mathbb{T}^d}2\eta^\delta_1(\rho_1-\rho_2)\Big(\sqrt{\rho_1(1-\rho_1)}-\sqrt{\rho_2(1-\rho_2)}\Big)\nabla\Big(\rho_1-\rho_2\Big)\cdot g\,\mathrm{d}x.\end{aligned}$$

Hölder's inequality and Young's inequality prove that, for every $\delta\in(0,1)$,

$$\partial_t\int_{\mathbb{T}^d}a^\delta(\rho_1-\rho_2)\,\mathrm{d}x \le \int_{\mathbb{T}^d}\eta^\delta_1(\rho_1-\rho_2)\Big(\sqrt{\rho_1(1-\rho_1)}-\sqrt{\rho_2(1-\rho_2)}\Big)^2|g|^2\,\mathrm{d}x.$$

Since there exists $c\in(0,\infty)$ independent of $\delta\in(0,1)$ such that $|\eta^\delta_1|\le c/\delta$, the definition of the convolution kernel and the Hölder regularity of the function $x\in[0,1]\mapsto\sqrt{x(1-x)}$ prove that there exists $c \in (0,\infty)$ such that

$$|\eta_1^\delta(\rho_1-\rho_2)\Big(\sqrt{\rho_1(1-\rho_1)}-\sqrt{\rho_2(1-\rho_2)}\Big)^2| \leq c\mathbf{1}_{\{0<|\rho_1-\rho_2|<c\delta\}}.$$

Therefore, for $c \in (0,\infty)$ independent of $\delta \in (0,1)$,

$$\partial_t \int_{\mathbb{T}^d} a^\delta(\rho_1(x,t)-\rho_2(x,t))\,\mathrm{d}x \leq c\int_{\{0<|\rho_1-\rho_2|<c\delta\}} |g|^2\,\mathrm{d}x.$$

The L^2-integrability of g, the definition of a, and the dominated convergence theorem prove that, in the sense of distributions, after passing to the limit $\delta \to 0$,

$$\partial_t \int_{\mathbb{T}^d} |\rho_1(x,t)-\rho_2(x,t)|\,\mathrm{d}x \leq 0,$$

which completes the proof. □

Proposition 4.3. *Let $T \in (0,\infty)$. For every $\rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$ and $g \in L^2(\mathbb{T}^d \times [0,T])^d$ there exists a weak solution of* (4.3) *in the sense of Definition* 4.1*. Furthermore, for some $c \in (0,\infty)$,*

$$\|\rho\|^2_{L^\infty([0,T];L^2(\mathbb{T}^d))} + \|\nabla\rho\|^2_{L^2(\mathbb{T}^d\times[0,T])} \leq c\Big(\|\rho_0\|^2_{L^2(\mathbb{T}^d)} + \|g\|^2_{L^2(\mathbb{T}^d\times[0,T])^d}\Big),$$

and

$$\|\partial_t\rho\|_{L^2([0,T];H^{-1}(\mathbb{T}^d))} \leq c(\|\rho_0\|_{L^2(\mathbb{T}^d)} + \|g\|_{L^2(\mathbb{T}^d\times[0,T])^d}).$$

Proof. Let $s\colon \mathbb{R} \to [0,1/2]$ be defined by

$$s(x) = \sqrt{x(1-x)} \;\text{ if }\; x \in [0,1] \;\text{ and }\; s(x) = 0 \;\text{ if }\; x \notin [0,1].$$

Let $S\colon L^2(\mathbb{T}^d \times [0,T]) \to L^2([0,T];H^1(\mathbb{T}^d))$ denote the solution map

$$\partial_t S(\rho) = \Delta S(\rho) - \nabla\cdot(s(\rho)g) \;\text{ in }\; \mathbb{T}^d \times (0,T) \;\text{ with }\; S(v) = \rho_0 \;\text{ on }\; \mathbb{T}^d \times \{0\}.$$

The boundedness of s, Hölder's inequality, and Young's inequality prove that, for some $c \in (0,\infty)$,

$$\|S(\rho)\|_{L^2([0,T];H^1(\mathbb{T}^d))} \leq c(\|\rho_0\|_{L^2(\mathbb{T}^d)} + \|g\|_{L^2(\mathbb{T}^d\times[0,T])^d}), \tag{4.5}$$

and it follows from (4.5) and the equation that, for some $c \in (0,\infty)$,

$$\|\partial_t S(\rho)\|_{L^2([0,T];H^{-1}(\mathbb{T}^d))} \leq c(\|\rho_0\|_{L^2(\mathbb{T}^d)} + \|g\|_{L^2(\mathbb{T}^d\times[0,T])^d}). \tag{4.6}$$

The compact embedding $H^1(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d)$, (4.5), (4.6), and the Aubin–Lions–Simons lemma [2, 51, 65] prove that the image of S lies in a compact subset of $L^2(\mathbb{T}^d \times [0,T])$. The Schauder fixed point theorem proves that there exists a weak solution $\rho \in L^2([0,T];H^1(\mathbb{T}^d))$ of the equation

$$\partial_t\rho = \Delta\rho - \nabla\cdot(s(\rho)g) \;\text{ in }\; \mathbb{T}^d \times (0,T) \;\text{ with }\; \rho = \rho_0 \;\text{ on }\; \mathbb{T}^d \times \{0\}.$$

A repetition of the regularization argument used in the proof of Proposition 4.2 proves using the definition of s that, for the Dirac delta-distribution δ_0,

$$\partial_t \int_{\mathbb{T}^d} (\rho-1)_+\,\mathrm{d}x = -\int_{\mathbb{T}^d} \delta_0(\rho-1)|\nabla\rho|^2 + \int_{\mathbb{T}^d} \delta_0(\rho-1)s(\rho)g\,\mathrm{d}x \leq 0,$$

which, since $\rho_0 \leq 1$, proves that $\rho \leq 1$ almost surely. Similarly,

$$\partial_t \int_{\mathbb{T}^d} (\rho)_-\,\mathrm{d}x = \int_{\mathbb{T}^d} \delta_0(\rho)|\nabla\rho|^2 - \int_{\mathbb{T}^d} \delta_0(\rho)s(\rho)g\,\mathrm{d}x \geq 0,$$

which, since $\rho_0 \geq 0$, proves that $\rho \geq 0$ almost surely. We conclude that $0 \leq \rho \leq 1$ almost surely, and therefore using the definition of s that ρ is a weak solution of

$$\partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\rho(1-\rho)}g) \;\text{ in }\; \mathbb{T}^d \times (0,T) \;\text{ with }\; \rho = \rho_0 \;\text{ on }\; \mathbb{T}^d \times \{0\},$$

in the sense of Definition 4.1. The estimate is a consequence of (4.5) and the weak-lower semicontinuity of the Sobolev norm. The proof of $L^2(\mathbb{T}^d)$-continuity follows analogously to the conclusion of Proposition 3.7, which completes the proof. ☐

Proposition 4.4. *Let $T \in (0,\infty)$, let $\{\rho_0^n\}_{n\in\mathbb{N}}, \rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$, and let $\{g_n\}_{n\in\mathbb{N}}, g \in L^2(\mathbb{T}^d \times [0,T])^d$. Assume that, as $n\to\infty$, $\rho_0^n \rightharpoonup \rho_0$ weakly in $L^2(\mathbb{T}^d)$ and $g_n \rightharpoonup g$ weakly in $L^2(\mathbb{T}^d\times[0,T])^d$. Then, for the solutions $\{\rho_n\}_{n\in\mathbb{N}}, \rho$ of* (4.3) *in the sense of Definition* 4.1 *with controls $\{g_n\}_{n\in\mathbb{N}}$ and g and initial data ρ_0^n and ρ_0, as $n\to\infty$,*

$$\rho_n \to \rho \ \text{ strongly in } \ L^2(\mathbb{T}^d\times[0,T]).$$

Proof. The proof is an immediate consequence of Definition 4.1, Proposition 4.3, the Aubin-Lions-Simon Lemma [2, 51, 65], the compact embedding of $H^1(\mathbb{T}^d)$ into $L^2(\mathbb{T}^d)$, and the continuous embedding of $L^2(\mathbb{T}^d)$ into $H^{-1}(\mathbb{T}^d)$. ☐

4.2. **The large deviations principle.** The LDP is based on the weak convergence approach to large deviations [7]. For this, it is essentially necessary to prove three conditions. The first is to prove the existence of a measurable solution map taking initial data and noise living in the space $L^\infty(\mathbb{T}^d;[0,1]) \times \mathrm{C}([0,\infty);(\mathbb{R}^d)^\infty)$ to the solution space $L^\infty([0,T];L^2(\mathbb{T}^d;[0,1]))$.

The second of these is to prove the collapse of the controlled SPDE

$$\partial_t\rho = \Delta\rho - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho(1-\rho)}\circ\xi^K) - \nabla\cdot(\sqrt{\rho(1-\rho)}P_K g), \tag{4.7}$$

for $P_K g$ the Fourier projection of an arbitrary controls $g \in L^2(\mathbb{T}^d\times[0,T])^d$ onto the modes $\{e_k, e_k'\}_{\{|k|\le K\}}$, to the skeleton equation

$$\partial_t\rho = \Delta\rho - \nabla\cdot(\sqrt{\rho(1-\rho)}g). \tag{4.8}$$

Notice here that the Fourier projections $P_K g \in L^2([0,T];H^1(\mathbb{T}^d))^d$, and therefore that the results of Theorem 2.7 and Theorem 2.16 apply to equation (4.7).

The third condition is to prove the compactness of the level sets of the controlled skeleton equation, in the sense that it is necessary to prove that families of solutions of (4.8) corresponding to $L^2(\mathbb{T}^d\times[0,T])^d$-bounded families of controls g are relatively compact in $L^2(\mathbb{T}^d\times[0,T])$. These three properties are established respectively in Proposition 4.5, Proposition 4.6, and Proposition 4.7. The LDP in Theorem 4.8 is then a consequence of [7, Theorem 6].

Proposition 4.5. *Let $T\in(0,\infty)$, let $\{\xi^K\}_{K\in\mathbb{N}}$ and ρ_0 satisfy Assumption* 3.1*, and for every $\varepsilon\in(0,1)$ and $K\in\mathbb{N}$ let $\rho^{\varepsilon,K}(\rho_0) \in L^\infty(\Omega\times[0,T];L^\infty(\mathbb{T}^d;[0,1]))$ be the unique solution of* (4.1) *with initial condition ρ_0 in the sense of Definition* 2.6*. Then there exists a measurable map*

$$S_{\varepsilon,K}\colon L^\infty(\mathbb{T}^d;[0,1])\times C([0,T];(\mathbb{R}^d)^\infty) \to L^\infty([0,T];L^2(\mathbb{T}^d;[0,1])),$$

such that, for every $\rho_0\in L^\infty(\mathbb{T}^d;[0,1])$, $\mathbb{P}$-a.s., $S_{\varepsilon,K}(\rho_0,(B^k,W^k)_{k\in\mathbb{Z}^d}) = \rho^{\varepsilon,K}(\rho_0)$, for the Brownian motions $(B^k,W^k)_{k\in\mathbb{Z}^d}$ defined in Assumption 3.1*.*

Proof. The proof is a straightforward consequence of the pathwise L^1-contraction property of Theorem 2.7 and is a small modification of [26, Theorem 23]. ☐

Proposition 4.6. *Let $T\in(0,\infty)$, let $\{\rho_0\}_{\varepsilon\in(0,1)}, \rho_0 \in L^\infty(\mathbb{T}^d;[0,1])$ and let $\{g^\varepsilon\}_{\varepsilon\in(0,1)}, g$ be $\mathcal{F}_t$-predictable, $\mathbb{R}^d$-valued processes such that, as $\varepsilon\to0$,*

$$\rho_0^\varepsilon \rightharpoonup \rho_0 \ \text{ weakly in } \ L^2(\mathbb{T}^d) \ \text{ and } \ g^\varepsilon \rightharpoonup g \ \text{ in law},$$

and let $\{K(\varepsilon)\}_{\varepsilon\in(0,1)}$ be a sequence that satisfies $\varepsilon K(\varepsilon)^{d+2}\to0$ and $K(\varepsilon)\to\infty$ as $\varepsilon\to0$. Then, the solutions ρ^ε of (4.7) *in the sense of Definition* 2.6 *with parameters $(\varepsilon,K(\varepsilon))$, initial data ρ_0^ε, and control g^ε satisfy, as $\varepsilon\to0$,*

$$\rho^\varepsilon \rightharpoonup \rho \ \text{ in law on } \ L^2(\mathbb{T}^d\times[0,T]),$$

for ρ the unique solution of (4.3) *in the sense of Definition* 4.1 *with initial data ρ_0 and control g.*

Proof. The proof is a small modification of [26, Theorem 28]. The estimates of Proposition 2.10 establish the tightness of the probability laws of the solutions $\{\rho^\varepsilon\}_{\varepsilon\in(0,1)}$ on $L^2(\mathbb{T}^d\times[0,T])$ and of their gradients $\{\nabla\rho^\varepsilon\}_{\varepsilon\in(0,1)}$ on $L^2(\mathbb{T}^d\times[0,T])^d$. We then pass to the $\varepsilon\to 0$ limit in (4.7) using Prokhorov's theorem, the Skorokhod representation theorem, and the choice of scaling limit $\varepsilon K(\varepsilon)^{d+2}\to 0$ as $\varepsilon\to 0$. □

Proposition 4.7. *Let $T, M\in(0,\infty)$ and let $\mathcal{S}_M\subseteq L^2(\mathbb{T}^d\times[0,T])$ be defined by*

$$\mathcal{S}_M=\{\rho\colon \exists\, g\in B_M(L^2(\mathbb{T}^d\times[0,T])^d), \rho_0\in L^2(\mathbb{T}^d) \text{ such that } \rho \text{ solves } (4.3)\},$$

for $B_M(L^2(\mathbb{T}^d\times[0,T])^d)$ the ball of radius M in $L^2(\mathbb{T}^d\times[0,T])^d$. Then $\mathcal{S}_M$ is compact in the strong topology of $L^2(\mathbb{T}^d\times[0,T])$.

Proof. The proof is a consequence of the estimates of Proposition 4.3, the uniform L^2-boundedness of the controls, and the Aubin–Lions–Simons lemma [2, 51, 65]. □

Theorem 4.8. *Let $T\in(0,\infty)$ and assume that $\{K(\varepsilon)\}_{\varepsilon\in(0,1)}$ satisfy, as $\varepsilon\to 0$,*

$$\varepsilon K(\varepsilon)^{d+2}\to 0 \quad\text{and}\quad K(\varepsilon)\to\infty.$$

Then the rate functions $\{I_{\rho_0}\}_{\rho_0\in L^\infty(\mathbb{T}^d;[0,1])}$ defined in (4.2) are good rate functions with compact level sets on compact sets. For every $\rho_0\in L^\infty(\mathbb{T}^d;[0,1])$ the solutions $\{\rho^\varepsilon(\rho_0)\}_{\varepsilon\in(0,1)}$ of (4.1) satisfy a large deviations principle with rate function I_{ρ_0} on $L^2(\mathbb{T}^d\times[0,T];[0,1])$. Furthermore, the solutions satisfy a uniform large deviations principle with respect to weakly $L^2(\mathbb{T}^d;[0,1])$-compact subsets of $L^\infty(\mathbb{T}^d;[0,1])$.

Proof. The statement is now a consequence of [7, Theorem 6], Proposition 4.5, Proposition 4.6, Proposition 4.7, and the equivalence of uniform Laplace and large deviations principles with respect to weakly compact subsets of the initial data [63, Theorem 4.3]. □

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