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Mobile-group Dominated Dynamics and Sequence of Mobility Transitions in Facilitated Random Walk Model of Glass

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Glassy dynamics has been a topic of ongoing debate for decades. We investigate a recently proposed facilitated random walk (FRW) model, which is a kinetically constrained model (KCM) with random constraints and is a coarse-grained version of the distinguishable-particle lattice model (DPLM) of glass. Dynamics in the FRW model with a finite density of particles are dominated by mobile groups of particles enjoying emergent facilitated dynamics. We observe a diffusion coefficient power-law with respect to the particle density, with the exponent identified as the dominant mobile group size. In addition, simplified FRW systems initiated with a few particles at the same site exhibit a sequence of mobility transitions, explaining the dynamical dominance of mobile groups. For such systems with two particles, the mobility transition point is calculated exactly.

I. INTRODUCTION

The nature of relaxation dynamics in glassy systems remains one of the central unsolved problems in condensed matter physics [1–6]. While many characteristic features of glass such as dynamical heterogeneity, kinetic arrest, and stretched exponential relaxation are intensively studied, their microscopic origins are still under debate. Molecular dynamics (MD) simulations [7, 8] are instrumental in studying glassy properties, and in particular, allow full access to individual particle dynamics. However, MD approaches face serious limitations at low temperatures due to their high computational cost in investigating the dynamics, although equilibrium properties are readily available using advanced algorithms [9]. Challenges can also arise due to the presence of molecular vibrations, which can obscure the dominant mechanisms governing slow relaxation.

To understand the underlying mechanisms of glassy phenomena, significant efforts have been made to reproduce and understand these behaviours in simplified settings. One such strategy involves the use of lattice models [10, 11], which abstract away continuous spatial degrees of freedom and allow computationally more efficient exploration of deeply supercooled regimes. A number of lattice models have been suggested, including those in Refs. [12–22]. Despite their simplicity, many lattice models have successfully reproduced key features of glassy dynamics, such as stretched exponential relaxation and dynamical heterogeneity. A few of these models belong to

the class of kinetically constrained models (KCMs), [12–14, 16] which employ simple deterministic kinetic constraints on particle movements to reproduce the slowed dynamics of glass.

We have recently proposed the facilitated random walk (FRW) model [23], which offers an alternative to conventional KCMs by employing stochastic rather than deterministic kinetic constraint rules. This element of randomness in the FRW can be physically relevant, as it reflects the inherently disordered nature of glassy materials. Moreover, the FRW model exhibits emergent dynamical facilitation behaviours among particle motions, analogous to directly imposed facilitation rules in the Fredrickson-Andersen (FA) spin facilitation model [12, 24], which is one of the first KCMs. Analogous to related studies [25, 26], rate parameters in the FRW are local and are updated as particles move.

Besides being a KCM, the FRW model is also a coarse-grained version of the distinguishable particle lattice model (DPLM) [18], which incorporates void-induced particle motions and exhibits key glassy phenomena such as Kovacs paradox [27], a wide range of kinetic fragility [28], Kauzmann paradox [29], fragile-to-strong transition [30], enhanced mobility in films [31], etc. We expect the FRW model to be able to reproduce most of these observations, in some cases after further generalisation such as incorporating temperature dependence.

The stochastic kinetic constraints in the FRW are intuitively understandable and are motivated by the energy-driven dynamics illustrated by the DPLM. Furthermore, the FRW is expected to inherit many glassy properties already demonstrated by the latter [18, 27–31]. The FRW captures hallmark glassy phenomena, including plateaus in mean squared displacement (MSD), stretched exponential relaxation in the self-intermediate scattering function (SISF), and dynamical heterogeneity [23]. The

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dynamics is not energetically driven. Rather, particles interact kinetically via a rate resampling and restoration scheme. In particular, the restoration rule is able to quench the disorder in the configuration space. At a low particle density, we also observe mobile groups of particles dominating relaxation, as evident from particle position-time graphs [23]. A mobile group extends over a few lattice sites. It can be more mobile than isolated particles because of emergent facilitation behaviours.

In this paper, we study a diffusion-coefficient power-law relationship with respect to particle density and investigate its relation to facilitated dynamics. An analogous diffusion-coefficient power law was first observed in the DPLM [18] and then observed in MD systems with partial swap [32]. This suggests that dynamics are dominated by mobile groups of particles or defects with facilitated dynamics and the power-law exponent estimates the dominant group size. We further study simple m -particle systems and observe m -dependent mobility transitions. In such systems, we initiate the system state with m particles at the same site and observe their dynamics. We provide an exact mobility threshold for two-particle systems by analysing a configuration tree. These mobility transitions have a strong impact on the dominant mobile group size. Our results suggest that the dynamics of glassy systems can be governed by the facilitated motions of these groups, providing an intuitive mechanism for the observed slowdown in the structural relaxation.

This paper is organised as follows. In Sec. II, we explain the definition of the FRW model. In Sec. III, we investigate the evolution of the bond states using an overlap function to explore the relaxation dynamics. In Sec. IV, we explain the diffusion-coefficient power law. We then explain how particle mobile group size relates to the exponent of the power law in Sec. V. The dominant mobile group size is explained in Sec. VI. We conclude in Sec. VII with a discussion of the broader implications of our results and potential avenues for future research.

II. FRW MODEL

The FRW model is a kinetically constrained lattice model introduced in our prior work [23]. Defined in one dimension (1D), the model evolves on a lattice of L sites with lattice constant $a = 1$ and no particle exclusion. With a total number of N particles, it has a particle density $\rho = N/L$.

All particles execute continuous-time random walks (CTRW) with randomly assigned hopping constraints. Specifically, the hopping rate $w_{ijk}(t)$ of particle k from site i to its nearest neighbour j is given by:

$$w_{ijk}(t) = \begin{cases} w_0 & \text{with probability } q, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

where $w_0 = 1$ and q is an unblocking probability. Detailed balance requires $w_{ijk} = w_{jik}$. These rates are dy-

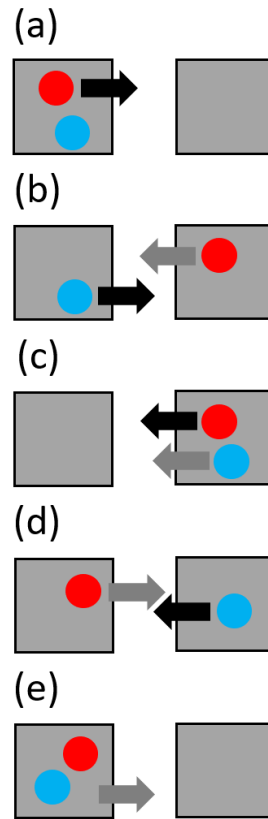


FIG. 1. A diagram showing 5 snapshots of a simple FRW lattice with two sites (grey squares) and two particles (coloured circles). The arrows (black and grey), occurring with an unblocking probability q , indicate the directions the particle is available to hop towards at rate w_0 . Black arrows denote independently sampled or resampled directions. Grey arrows indicate the availability of return hops as ensured by detailed balance. To enable an exact analysis of two-particle systems, we define the following. A joint (split) state occurs when two particles are at the same site (different sites). (a), (c) and (e) show joint states, while (b) and (d) represent split states. After the motions shown in (a)-(c), further hops in (c)-(e) demonstrate *backward* hops by the pair so that the state (e) is different from the state (a). Instead, if the blue particle were to hop before the red particle in (c)-(e) resulting in reversed hops, the original state as in (a) would be restored.

namically updated throughout the simulation according to two rules stated in [23] with further details:

- (i) **Rate resampling:** When particle k performs a non-reversing hop from site i to j , the hopping rates $w_{ijl}(t) = w_{jil}(t)$ for all other particles $l \neq k$ and $w_{jj'k}$ where j' is the other NN-site of j are resampled.
- (ii) **Rate restoration:** If particle k subsequently performs a reversed hop from j back to i , the previous rates before resampling are restored.

The restoration property applies to multi-step processes: a step-by-step reversal of particle hops fully reverses the previous rates step by step. A hop not involved in any step-by-step reversal is considered a non-reversing hop. Figure 1 shows time-consecutive snapshots of how particles could move on a simple lattice with two sites.

Technically, a straightforward way to enable possible multi-step rate restoration of constraints in the system would be to store the full history in memory during the simulation. For computational convenience, however, we choose instead to introduce and store an extended system state at each simulation step. The system state is then fully specified by the particle positions x_k , the bond states Φ_{ij} , and the particle states Ψ_k , which together encode all hopping constraints. In our simulations, Φ_{ij} and Ψ_k are 64-bit random numbers, so that any two distinct states are highly unlikely to share the same value. As Φ_{ij} and Ψ_k evolve under the resampling and restoration rules, the hopping rates change accordingly as detailed in [23].

In the FRW model, particles effectively represent voids in the DPLM. The model's control parameters, q and ρ , are analogous to temperature and void density respectively, as outlined in [23]. In real glasses, such voids are suggested to take the form of quasi-voids [33], which are believed to initiate particle string-like motions. In principle, ρ and q are not independent, since both depend on temperature and the type of material. In particular, q should effectively increase with ρ because a high local density of voids reduces atomic coordination and suppresses constraints. However, under deep supercooling, ρ is small. Therefore voids are largely isolated from one another, and the effects of ρ on q are almost negligible. In the present FRW model, we hence treat ρ and q as independent control parameters, for simplicity.

All simulation results shown in this work have been averaged over more than 10 independent runs. Equilibrium systems at any given particle density are constructed directly using exact equilibrium statistics [23]. Typical simulations use a lattice size ranging from $5000 \leq L \leq 50000$, with periodic boundary conditions. The simulations run for $10^5 \leq t_{max} \leq 5 \times 10^8$ and are averaged over 10 to 10000 independent realisations. A larger system size is used for systems with fewer particles or a larger dominant mobile group size α .

The FRW model is designed to perform exceptionally fast simulations compared to atomistic models or models defined in higher dimensions. The longest runs in Fig. 2, Fig. 4 and Fig. 6 took about 12 hours, 22 hours and 10 minutes, respectively. This efficiency grants access to the analysis of dynamics under significantly deeper supercooling within comparable simulation run times. For instance, within a day of simulation, the FRW could reach a diffusion-coefficient power law of $\alpha \simeq 7$ (see Fig. 4) corresponding to a much deeper supercooled state than the DPLM which could obtain reliable results showing $\alpha \simeq 2.7$ [18]. All simulations were run on single processor cores.

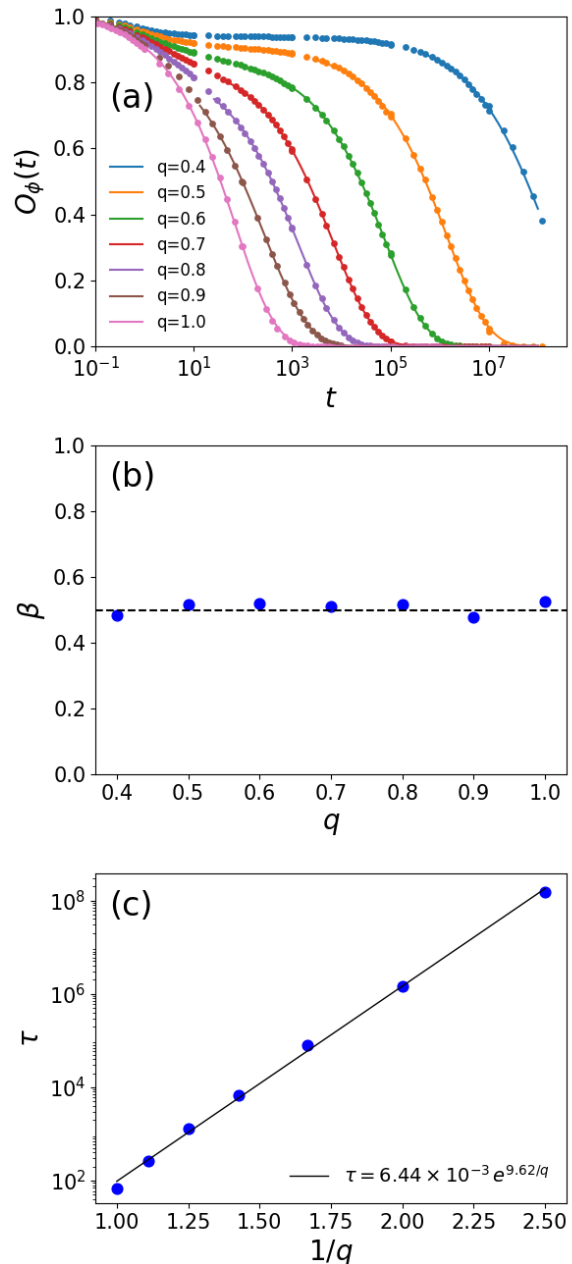


FIG. 2. (a) Overlap $O_\Phi(t)$ of bond states plotted against time t for particle density $\rho = 0.1$ and various unblocking probabilities q , ascending from $q = 0.4$ to $q = 1.0$ from top to bottom. Lines show the full fitting range of the fit to the second relaxation step using the KWW stretched exponential function. (b) Stretching exponent β against unblocking probability q . A dotted line represents $\beta = 0.5$. (c) Relaxation time τ against $1/q$ fitted by a straight line on semi-log scales.

III. BOND-STATE OVERLAP AND STRUCTURAL RELAXATION

We have previously demonstrated that the FRW model is able to show many typical glassy relaxation characteristics when considering the movement of particles [23].

Alternatively, we monitor changes in bond states Φ_{ij} between neighbouring sites i and j . Any change in Φ_{ij} implies that a particle has hopped from site i to j or vice versa, thereby indicating that the bond, representing a local region, has relaxed. We measure the overlap function $O_\Phi(t)$ defined for equilibrium systems as

$$O_\Phi(t) = \langle \delta_{\Phi_{ij}(0), \Phi_{ij}(t)} \rangle, \quad (2)$$

where δ is the Kronecker delta and the average is taken over all bonds, initial time and independent runs. The results for a small particle density $\rho = 0.1$ are shown in Fig. 2(a). We observe a two-step relaxation and the second relaxation step fits well with the Kohlrausch-Williams-Watts (KWW) stretched exponential function

$$A \exp[-(t/\tau)^\beta], \quad (3)$$

where $0 < A \leq 1$, τ is the structural relaxation time, and β is the stretching exponent. The fitted values of β and τ for each q are shown in Fig. 2(b) and (c) respectively. It is observed that the stretching exponent $\beta \simeq 0.5$ for all q . The near-linear relation in the semi-log plot of τ against $1/q$ suggests an Arrhenius-like relation with q , which is analogous to temperature. Note that introducing an appropriate relationship between q and temperature is non-trivial, and hence the current FRW does not exhibit a true Arrhenius relation. It follows that its fragility cannot be determined. The coefficient A ranges from 0.86 to 1. In Appendix A, we derive $\beta = 0.5$ analytically. The value of β is much smaller than 1 due to the FRW being defined in 1D, which leads to less efficient diffusion compared to systems in higher dimensions. In addition, β remains constant at 0.5 as q decreases. This feature is not typical of glass. This value of β has also been observed in other one-dimensional models [34, 35], but might not be relevant in two or three dimensions.

To further characterise the relaxation dynamics beyond the overlap, we examine the displacement distribution $\text{prob}(\Delta x)$, where $\Delta x = x(t) - x(0)$. Figure 3 shows $\text{prob}(\Delta x)$ at $\rho = 0.1$ and $q = 0.5$ for various values of t . The tails of all curves deviate systematically from the Gaussian fits at every t examined, reflecting an existence of trapped particles and providing a signature of dynamical heterogeneity complementary to that seen in the bond-state overlap.

IV. DIFFUSION-COEFFICIENT POWER LAW AND DOMINANT MOBILE GROUP SIZE

We calculate the particle mean square displacement (MSD) in equilibrium systems defined as $\langle |x_k(t) - x_k(0)|^2 \rangle$ where $x_k(t)$ denotes the position of particle k at time t . From the diffusive regime of the MSD, we calculate the diffusion coefficient $D = (1/2d) \lim_{t \rightarrow \infty} \text{MSD}/t$, where $d = 1$ is the spatial dimension. We consider that time t is long enough for the dynamics to be in the diffusive regime when the slope

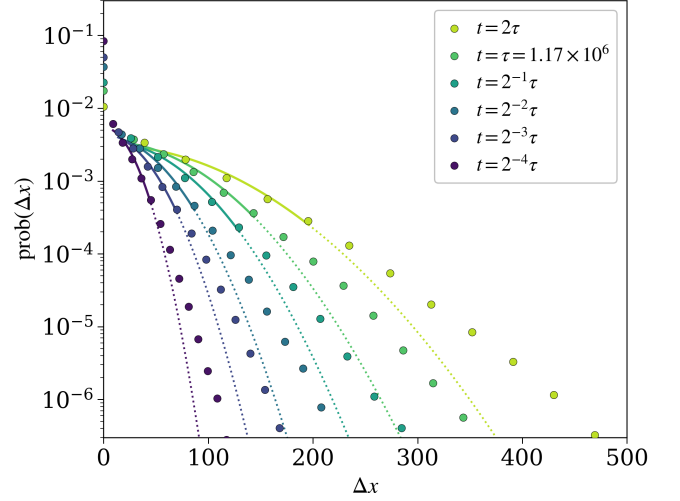


FIG. 3. Displacement distribution $\text{prob}(\Delta x)$ where $\Delta x = x(t) - x(0)$ at $\rho = 0.1$ for $q = 0.5$ plotted on semi-log scales for t ranging from $\tau/16$ to 2τ . Here, $\tau = 1.17 \times 10^6$ is the α -relaxation time plotted in Fig. 2(c). We perform Gaussian fits for a range of data as indicated by the solid line. Dotted extensions of the same fits highlight the clear deviation from Gaussian behaviour for both small and large Δx .

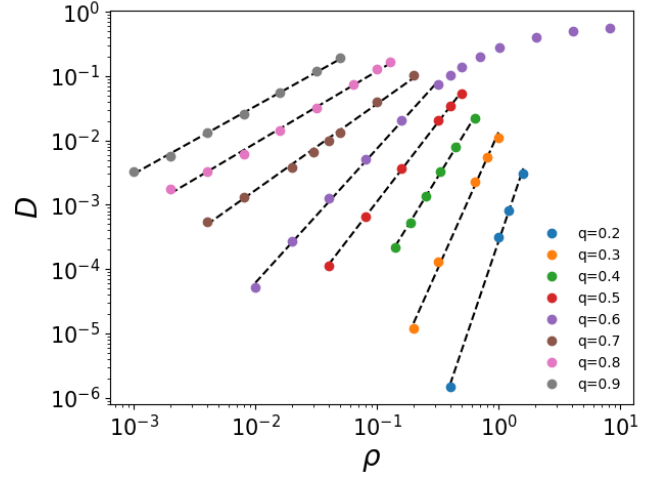


FIG. 4. Diffusion coefficient D versus particle density ρ for various unblocking probabilities q . At small ρ , data fit well to linear forms (dashed lines) in the log-log scales. The fitted lines also indicate the ranges of data used in the fits. For $q = 0.6$ as an example, data deviating from the power law at large ρ are also shown. Statistical errors are smaller than the symbol size and are therefore not shown.

of the MSD versus time curve in the log-log plot exceeds 0.96. Using this criterion, we compute D for various combinations of particle density ρ and unblocking probability q .

Figure 4 shows the diffusion coefficient D plotted against particle density ρ for several values of q . Note that particles in the FRW represent defects physically.

Following analogous observations in molecular dynamics (MD) simulations of an atomistic glass model that uses partial swap, we fit our data to a power law of the following form, appropriate for defects at a small density ρ [32],

$$D \propto \rho^{\alpha-1}, \quad (4)$$

where $\alpha \geq 1$. This is equivalent to the power-law for the diffusion coefficient D_a of atoms [18, 32],

$$D_a \propto \rho^\alpha, \quad (5)$$

noting that $D_a \propto \rho D$ at small ρ for defect-induced dynamics. This can be seen by noting that each hop of an FRW particle corresponds to a void displacing an atom: the void hops from site j to site i , while the atom previously at i moves to j . Hence $D_a(1 - \rho) = \rho D$ at small ρ where the particle density $(1 - \rho) \simeq 1$, reducing to $D_a \propto \rho D$. However, Eq. (5) is not applicable to the FRW as atoms are not explicitly simulated. The linear relationship observed on log-log scales in Fig. 4 clearly supports the power-law form in Eq. (4) for all values of q . The corresponding values of α extracted for each q are shown in Fig. 5 (blue circles).

The power-law in Eq. (4) can be intuitively understood through a simple chemical kinetics argument involving the density of mobile groups of FRW particles [18, 32]. Specifically, Eq. (4) suggests that a particle contributes to the dynamics only if it neighbours and is dynamically coupled with $\alpha - 1$ other particles. This thus implies mobile groups consisting of α particles that dominate the dynamics.

Equation (4) only holds at small ρ where mobile groups each of α particles are well isolated from each other. As ρ increases, the diffusion coefficient D no longer satisfies the power-law. Instead, it approaches a maximum value

$$D(\rho \rightarrow \infty) = qw_0. \quad (6)$$

This is validated by simulations as shown in Fig. 4, for instance, for $q = 0.6$. Equation (6) is derived as follows. At large ρ , the site occupancy n_i at every site i is also large, noting that each site can hold multiple particles in the FRW. Consequently, the combined hopping rate for any particle from site i to a nearest-neighbour site j , which averages at $n_i q$, is large. As a result, the rate resampling rule (see Sec. II) is invoked frequently, causing $w_{ijk}(t)$ for any particle k to rapidly alternate between 0 and w_0 . The hopping rates therefore become effectively unquenched and $w_{ijk}(t)$ can be accurately approximated by its annealed time average $\langle w_{ijk}(t) \rangle = qw_0$. Each particle k thus ends up with a uniform hopping rate qw_0 in each direction, giving a total hopping rate of

$$2qw_0. \quad (7)$$

The model thus reduces to a simple CTRW with diffusion coefficient $D = 2qw_0/(2d)$ in one dimension ($d = 1$), and we hence arrive at Eq. (6). Therefore, the diffusion-coefficient power law in Eq. (4) applies only at low ρ when $D \ll qw_0$.

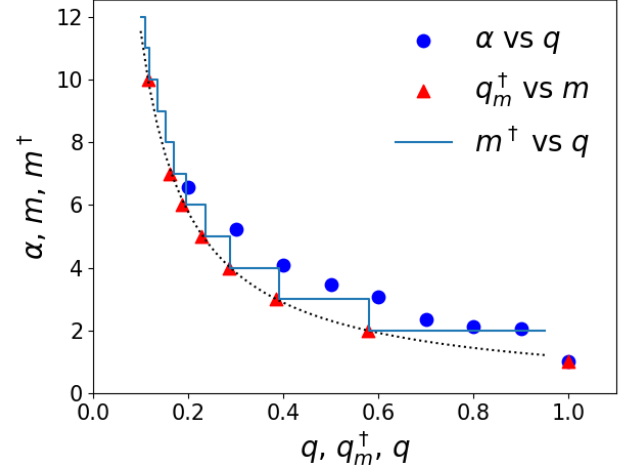


FIG. 5. Exponent α plotted against q (blue circles), measured using the diffusion-coefficient power-law in Eq. (4). Mobility thresholds q_m^+ for systems with $m = 1, 2, 3, 4, 5, 6, 7$ and 10 particles are shown in red. The dotted line plots Eq. (9), and the blue line plots Eq. (10). These empirical equations, however, deviate from the exact value $q_m^+ = 1$ at $m = 1$.

V. MOBILITY THRESHOLDS IN m -PARTICLE SYSTEMS

To understand the nature of mobile groups implied by the diffusion coefficient power-law in Eq. (4) and observed in particle position-time graphs in Ref. [23], we now investigate systems with only m particles all initialised at the same position $x = 0$. For each m , we find that there exists a sharp mobility threshold q_m^+ so that $q \geq q_m^+$ is needed for the group of particles to have a chance of displacing unboundedly at long time, i.e. in the mobile phase.

First, for the trivial case of $m = 1$, we have a mobility threshold $q_1^+ = 1$. This is because the model reduces to a simple CTRW at $q = 1$ where all particles are fully mobile. On the other hand, when $q < 1$, $D = 0$ as each particle is confined to a single-particle trap of average size $s_{\text{trap}} = 1/(1 - q) - 1$ [23]. Hence, D for a single-particle system exhibits a discontinuous transition at $q_1^+ = 1$.

For $m = 2$, we get exactly

$$q_2^+ = 1/\sqrt{3} \approx 0.577, \quad (8)$$

using a configuration tree analysis (see Appendix B). At $q = q_2^+$, some pairs can move unboundedly but subdiffusively with $D = 0$. The mobility transition from $D = 0$ to $D > 0$ at q_2^+ as q increases is a continuous one.

To visualise particle dynamics close to the transition, Fig. 6 plots x against $\sqrt{t}$, showing the motions of 100 independent samples of two-particle systems under three values of q close to q_2^+ . We see in Fig. 6(a) that when $q < q_2^+$, the particles are well trapped within a limited number of sites even at long time so that $D = 0$. In

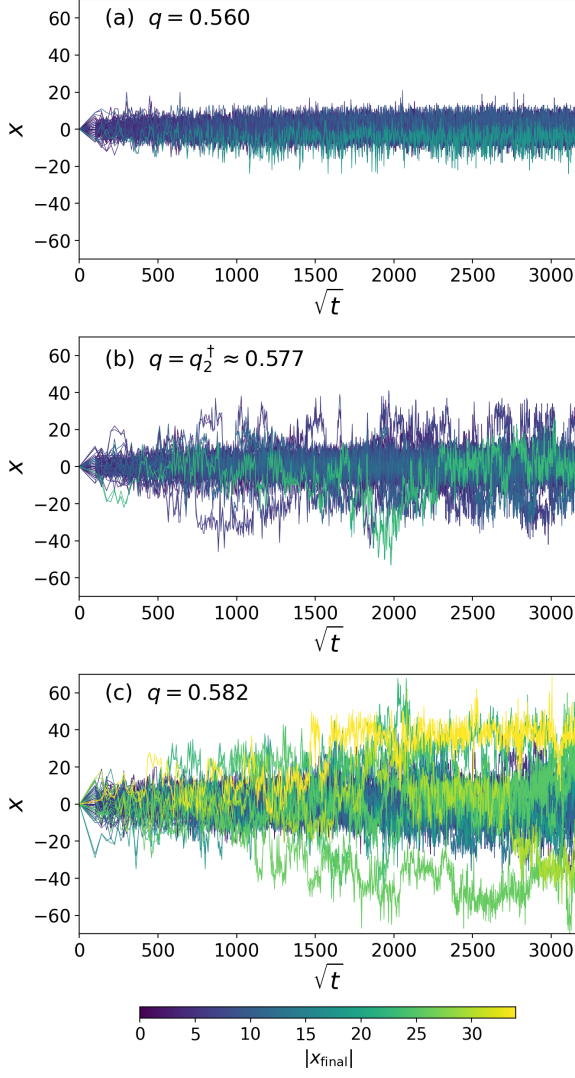


FIG. 6. Motion of particles in 100 samples of two-particle systems for unblocking probability $q = 0.560$ (a), $q_2^\dagger \simeq 0.577$ (b), and 0.582 (c) close to the mobility transition point $q_2^\dagger$. Particles are coloured by their final position x_{final} , as indicated by the colour bar. The larger the deviation of x_{final} from $x = 0$, the brighter the colour.

contrast, for $q > q_2^\dagger$ corresponding to the mobile phase, particle pairs can be mobile. However, even in this mobile phase, some particle pairs are immobile and will be forever trapped within a finite number of sites. This is analogous to percolation [36], where infinite clusters only exist in the percolating phase, but many clusters remain finite. The typical magnitude of displacement x of the mobile pairs appears to be proportional to $\sqrt{t}$, suggesting a diffusive regime with $D > 0$. For $q = q_2^\dagger$ plotted in Fig. 6(b), we see that some particle pairs are able to travel rather far but typical values of x increase more slowly than $\sqrt{t}$, consistent with a sub-diffusive regime with $D = 0$.

We now further verify the value of $q_2^\dagger$ quantitatively

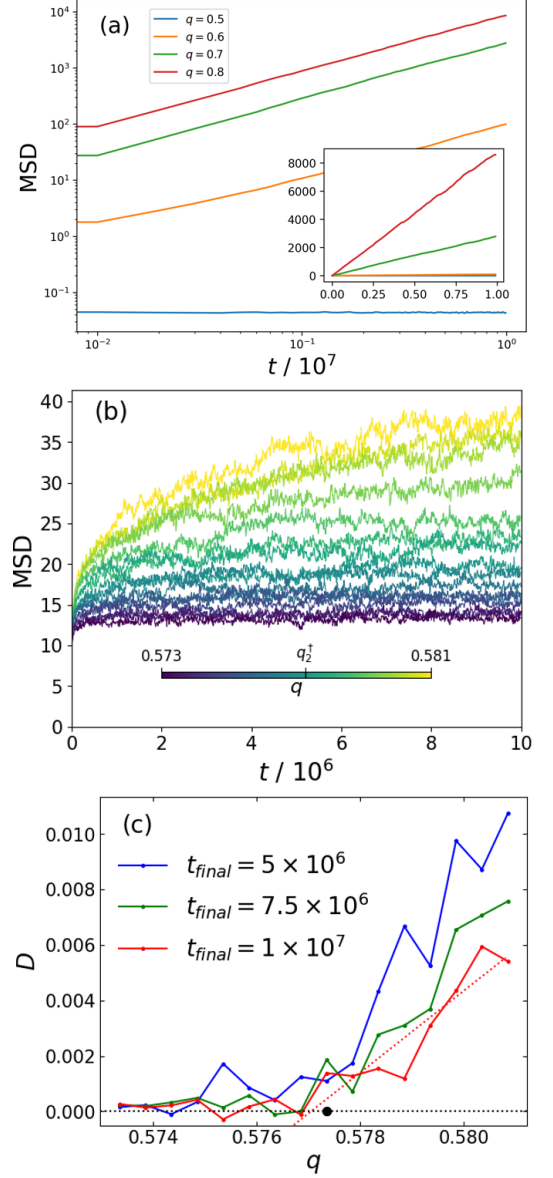


FIG. 7. (a) Particle MSD in two-particle systems at various values of q across a wide range below and above $q_2^\dagger$, plotted against time t on log-log scales. Inset: same data on linear scales. (b) MSD against t near the mobility threshold $q_2^\dagger = 1/\sqrt{3} \simeq 0.577$, with the value of q indicated by the colour bar centred on $q_2^\dagger$. (c) We calculate the diffusion coefficient D using the data in (b) and identify the start of the sudden increase from $D = 0$ as q increases. Red dotted line fits values of $D > 0$. Black circle shows $q_2^\dagger$.

from two-particle system simulations. We simulate 10000 independent systems with both particles starting at $x = 0$ until time $t = 10^7$ for each value of q . We plot in Fig. 7(a) the MSD against time for a wide range of q , both above and below $q_2^\dagger$. The log-log plot shows that there is no diffusion when $q = 0.5 \ll q_2^\dagger$. The early-time plateau in the log-log view corresponds to a sub-diffusive caging-

like regime in which the two particles repeatedly fail to escape their current neighbourhood, performing return hops until they encounter a configuration that facilitates further movement. The linearity of MSD with time when $q \gg q_2^\dagger$, as shown in the inset, is consistent with long-time diffusive growth. Figure 7(b) plots the measured MSD against time for each studied q close to $q_2^\dagger$. As seen, the MSD at long time tends to level off for $q < q_2^\dagger$ while it continues to increase for $q > q_2^\dagger$. Note that the levelling off for $q < q_2^\dagger$ signifies permanent particle trapping and no eventual rise occurs. This is in contrast to standard many-particle FRW in which distant mobile groups of a sufficiently large group size must eventually arrive to mobilise temporarily trapped particles, so that MSD always rises at long time [23].

We measure the particle diffusion coefficient D from the MSD shown in Figure 7(b). To minimise impacts of initial transients, we use $D = (1/2d) \lim_{t \rightarrow \infty} d(\text{MSD})/dt$ and linear regression to estimate the derivative over the final quarter of the MSD data. Figure 7(c) plots the values of D against q obtained. For example, for $t_{\text{final}} = 10^7$ (red curve in Fig. 7(c)), the data is taken from $t = 0.75 \times t_{\text{final}} = 7.5 \times 10^6$ to $t = t_{\text{final}} = 10^7$. Here we see a sharp increase in D from $D \simeq 0$ at $q \gtrsim q_2^\dagger$, supporting the value of $q_2^\dagger$ in Eq. (8). Figure 7(c) also shows that a similar conclusion can be obtained if data of shorter time duration is used. The apparent dependence of the diffusion coefficient on t_{final} is due to finite-time transient effects. At q near the mobility threshold $q_2^\dagger$, the system exhibits a longer crossover from the initial transients to the final diffusive regime. For any $q \gg q_2^\dagger$, such effects are minimal as observed in Figure 6a. Technically, we perform another linear regression of D against q using the data with noticeably positive D , as shown by the red dotted line in Fig. 7(c). The regression line is extrapolated back to $D = 0$ to get an estimate of the mobility threshold, which is very close to the exact value of $q_2^\dagger$ in Eq. (8).

For $m \geq 3$, we similarly simulate m -particle systems and observe analogous dependence of D on q . Then, $q_m^\dagger$ is similarly estimated. Our results are shown in red triangles in Fig. 5, which plots m against $q_m^\dagger$ for easier comparison with other data. Exact values, only available for $q_1^\dagger$ and $q_2^\dagger$, are used. We observe that our data approximately follow the empirical relation

$$q_m^\dagger = \frac{c}{m} \quad (9)$$

for $m > 1$. A convenient estimate of the constant c is $2/\sqrt{3} \simeq 1.155$, which gives the exact value for $q_2^\dagger$ and well approximates the other measured values of $q_m^\dagger$.

VI. CRITICAL MOBILE GROUP SIZE

At unblocking probability q , only groups beyond a critical size $m^\dagger$ are mobile, where $m^\dagger = \min \{m : q > q_m^\dagger\}$,

i.e. $m^\dagger$ is the smallest integer satisfying $q > q_m^\dagger$. It is convenient to estimate $m^\dagger$ by inverting Eq. (9) and we get

$$m^\dagger = \left\lfloor \frac{c}{q} \right\rfloor + 1. \quad (10)$$

for $m^\dagger > 1$ where $\lfloor \cdot \rfloor$ denotes the floor function. In Fig. 5, we have also plotted $m^\dagger$ against q using Eq. (10) (solid line). It appears as a staircase decreasing function inverting the dependence of $q_m^\dagger$ on m as well as generalising to all values of q .

From Fig. 5, we observe that the critical mobile group size $m^\dagger$ is only slightly smaller than the exponent α which represents the dominant mobile group size. A non-integral α should represent an average over two group sizes which dominate the dynamics. The difference between $m^\dagger$ and α is expected to narrow down when approaching the small ρ limit, as will now be explained. As $\rho \rightarrow 0$, mobile groups of size $m^\dagger$ should completely dominate the dynamics. Their population scales with $\rho^{m^\dagger}$. Whereas larger mobile groups with, e.g., $m^\dagger + 1$ particles are relatively rare with their population scaling at $\rho^{m^\dagger+1}$, despite a greater mobility. This ensures their negligible contribution. We anticipate that α , representing the dominant group size, will asymptotically approach $m^\dagger$ as $\rho \rightarrow 0$ across all q values. In fact, α exhibits some sign of a stepwise pattern, reflecting a close relation to $m^\dagger$. However, at larger ρ , the population of $(m^\dagger + 1)$ -groups becomes non-negligible as the ratio between $\rho^{m^\dagger}$ and $\rho^{m^\dagger+1}$ is not vanishingly small. Due to the higher mobility, these larger groups can contribute significantly and may even dominate the dynamics, resulting in $\alpha \geq m^\dagger$.

VII. DISCUSSION

The present paper builds on [23], which introduced the FRW model and reported its basic glassy phenomenology (MSD plateaus, stretched relaxation in the SISF, dynamical heterogeneity). The new contributions of this work include: (i) the bond-state overlap $O_\Phi(t)$ as a relaxation probe distinct from the SISF; (ii) the diffusion-coefficient power law $D \propto \rho^{\alpha-1}$ and its identification of the dominant mobile group size α ; (iii) the relation between α and the critical mobile group size $m^\dagger$; (iv) the sequence of mobility thresholds $q_m^\dagger$ from the systematic study of m -particle systems, and (v) the exact result $q_2^\dagger = 1/\sqrt{3}$ derived from the configuration-tree analysis of Appendix B.

Our results rigorously establish the phenomenon of dynamic facilitation in the FRW. The single-particle and two-particle mobility thresholds $q_1^\dagger = 1$ and $q_2^\dagger = 1/\sqrt{3}$ define a regime where $q_2^\dagger < q < q_1^\dagger$. In this regime, an isolated particle is trapped, but a pair of particles is mobile, demonstrating that facilitation between particles is the primary mechanism behind the system's mobility. However, this exact analytic approach for determining

thresholds could not be generalised readily to systems with three or more particles.

Our simulations reveal a sequence of mobility transitions for m -particle systems, characterised by monotonically decreasing thresholds $q_m^\dagger$. For any given q , we thus compute the critical group size $m^\dagger$ required for mobility. In the standard FRW system with a finite particle density, these sharp transitions are not directly observable as particle groups of all sizes are simultaneously present. However, the transitions dictate the size of the mobile particle groups that dominate the dynamics. In these systems, the dominant group size is estimated from the exponent α of the diffusion coefficient power-law. We find that α is consistently slightly higher than $m^\dagger$.

The FRW model is ergodic in the limit of an infinite lattice ($L \rightarrow \infty$) for any finite density ρ . Although isolated particles or sub-critical particle groups can be transiently trapped, mobile groups larger than $m^\dagger$ must exist somewhere on a sufficiently large lattice, and their presence ensures eventual accessibility of all configurations over long times. At finite L and small q , implying a large $m^\dagger$, however, the expected number of such groups scales as $L\rho^{m^\dagger}$, which can be vanishingly small. Some sample systems without mobile groups have been shown in the trajectory plots in [23], which are visibly non-ergodic. All simulations reported here are performed with a sufficiently large lattice size, where the system is ergodic. In contrast, the m -particle systems investigated in this work are not ergodic. In these isolated systems where $\rho = 0$, any group of m particles has a finite probability of becoming permanently trapped within a bounded region when $q < 1$.

Our results highlight the central role of mobile groups in governing the dynamics. The strong correlation between results from m -particle systems and finite-density simulations strongly supports the importance of mobility transitions in dictating the dynamics. The enhanced dynamics of larger mobile groups further emphasises the importance of facilitated particle motion. With its minimalistic rules exhibiting emergent facilitation, the FRW model provides a clear possible framework for understanding the dominance of facilitated dynamics in glass.

The structural relaxation of the FRW model is analysed using the bond-state overlap function $O_\Phi(t)$, in contrast to the particle self-intermediate scattering function (SISF) employed in our previous work [23]. This choice is motivated by the fundamental nature of the FRW as a defect model, where particles represent voids and bond states capture local atomic configurations. Here, a hop of a particle across a bond changes its bond state, effectively signifying a local atomic rearrangement. Consequently, $O_\Phi(t)$ serves as a direct indicator of local relaxation. In contrast, the SISF in FRW tracks the relaxation of void positions, which may not be directly comparable to the SISF of atomistic models like the DPLM or other MD simulations. This distinction may explain why SISF results exhibiting typical glassy features are only obtained at wavelengths of several particle diam-

eters in FRW [23], analogous to the study of another lattice model [37]. However, the relaxation described by $O_\Phi(t)$ has stretching exponents β admitting a value of 0.5 and does not decrease further even as q and ρ decrease corresponding to deeper supercooling. This non-realistic feature, not observed in the DPLM [27], likely stems from the energetically trivial nature of the current FRW. A generalisation to an energy-driven FRW in the future may help resolve the issue. The DPLM has previously demonstrated that its fragility can be tuned by varying the interaction distribution between atoms [28]. It is possible for an energy-driven FRW to exhibit this same tunable fragility using an analogous approach, to model both fragile and strong behaviour.

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Appendix A: Exact stretched exponential relaxation of bond-state overlap

We now explain several cases in which we can exactly derive the relaxation of the bond-state overlap function and the stretching exponent β . In particular, cases 1 and 3 to be presented below explain the value $\beta \simeq 0.5$ reported in Fig. 2(b) for $\rho \ll 1$ and all values of q . We analyse the bond-state overlap $O_\Phi(t)$ for FRW defined on a 1D lattice of size L .

Relaxation is predominantly induced by mobile random walkers, as trapped particles have limited impact. For unblocking probability $q = 1$, the dominant random walkers are individual particles. For $q < 1$, they instead take the form of mobile groups of particles that move together through facilitation. We consider the centre of mass of a mobile group as the position of the random walker. The overlap $O_\Phi(t)$ is defined in Eq. (2) and represents the average probability that a specific bond, say bond ij , has the same bond state at time t as at time 0. At time t , assume that a given random walker k has relaxed n_k distinct bonds in the system. The probability that this walker has relaxed bond ij is therefore n_k/L . Consequently, the probability that bond ij is not relaxed by any walker in the system is

$$O_\Phi(t) = \left\langle \prod_k \left(1 - \frac{n_k}{L}\right) \right\rangle, \quad (\text{A1})$$

where the product is over all walkers in the system. In the large system limit, the product is self-averaging and $L \gg n_k$. We get

$$O_\Phi(t) \approx \prod_k e^{-n_k/L} = \exp\left(-\frac{1}{L} \sum_k n_k\right). \quad (\text{A2})$$

Let N_w be the total number of walkers. We define the average number of bonds relaxed per walker as $\bar{n} = (\sum_k n_k)/N_w$. We also define the walker density as $\rho_w = N_w/L$. The overlap hence simplifies to

$$O_\Phi(t) = e^{-\rho_w \bar{n}}. \quad (\text{A3})$$

Noting that $\bar{n}$ differs significantly for different values of q and ρ , we explain three cases for which exact results are available.

Case 1: Low-density free particles

We consider $\rho \ll 1$ and $q \approx 1$. The walkers are low-density free particles and the diffusion coefficient equals $D = w_0$ for simple 1D random walk. Assuming that particle k is displaced by $s_k(t)$ at time t , the number of bonds relaxed n_k is simply the magnitude of its displacement, i.e. $n_k = |s_k(t)|$. Note that n_k does not equal the maximum displacement within time t because return hops restore the bonds to the un-relaxed states. Then, $\bar{n} = \langle n_k \rangle = \langle |s_k| \rangle$ and we obtain [38]

$$\bar{n} = \sqrt{\frac{4D}{\pi}} t^{1/2}. \quad (\text{A4})$$

Substituting into Eq. (A3) gives

$$O_\Phi(t) = \exp\left(-\frac{2}{\sqrt{\pi}} \rho \sqrt{D} t^{1/2}\right). \quad (\text{A5})$$

This is the KWW form in Eq. (3) and the exponent of t shows that $\beta = 1/2$. Note that the $1/2$ exponent is inherited from Eq. (A4). The stretched relaxation is thus a consequence of 1D diffusion in which relatively few sites are visited, albeit repeatedly.

Case 2: High-density mobile particles

Consider the conditions $\rho \gg 1$ and $\rho q \gg 1$, which have been studied in Sec. IV with Eq. (7) giving the total particle hopping rate in both directions. Therefore, for a small time t ,

$$\bar{n} = 2q w_0 t. \quad (\text{A6})$$

Substituting into Eq. (A3) gives

$$O_\Phi(t) = \exp(-2\rho q w_0 t), \quad (\text{A7})$$

indicating that $\beta = 1$ when compared to the KWW form, corresponding to a simple exponential relaxation. We believe that a stretching exponent close to 1 extracted from the particle SISF in [23] under similar conditions can be similarly explained.

Case 3: Mobile groups of multiple particles

Finally, we consider the case where the walker represents a dominant mobile group of size $m^* \geq 2$, which occurs at $\rho \ll 1$ and $q < 1$. Here, m^* is expected to be close to the rounded-up integral value of the exponent α from the diffusion coefficient power-law in Eq. (4) and should be bounded below by the critical mobile group size, i.e. $m^* \geq m^\dagger$. Let the number density and diffusion coefficient of these mobile groups be ρ^* and D^* , respectively. The number of bonds n_k relaxed by such a random walker at time t is simply the difference between its maximum displacement to the right and to the left, because state-reversing motions in general are negligible at long time for mobile groups. This gives [39]

$$\bar{n} = 2\sqrt{\frac{D^*}{\pi}} t^{1/2}. \quad (\text{A8})$$

Substituting $\bar{n}$ and $\rho_w = \rho^*$ into Eq. (A3), we get

$$O_\Phi(t) = \exp\left(-2\rho^* \sqrt{\frac{D^*}{\pi}} t^{1/2}\right), \quad (\text{A9})$$

which again implies the KWW form with a stretching exponent of $\beta = 1/2$.

Appendix B: Two-particle configuration-tree theory for exact mobility transition

In this section, we consider a system with only two particles, red and blue, both initially at site $i = 0$. We define a joint state as whenever both particles reside at the same site and a split state occurs otherwise. We will first derive the exact mobility threshold of a simplified case in Appendix B1, and then show in Appendix B2 that the general case presents the same answer.

1. Constraint of pairwise sequential hops

Let us assume that whenever a particle hops, the other particle follows immediately after. We call such a pair of hops a hop sequence. This forces the system to always be at joint states after an even number of hops. The possible evolution of configurations in terms of only the joint states is shown in Fig. 8(a). The configuration space is naturally arranged as a tree. Each edge represents a hop sequence of the two particles, rather than a single hop. The initial state where both particles are at site $i = 0$ is represented by the root node (black circle). It has four children, two blue nodes and two red nodes. The colour of the node is defined by the colour of the particle that hops to the corresponding site first. For example, the blue particle hops followed by the red particle leads to a blue node. Since there are two neighbouring sites and

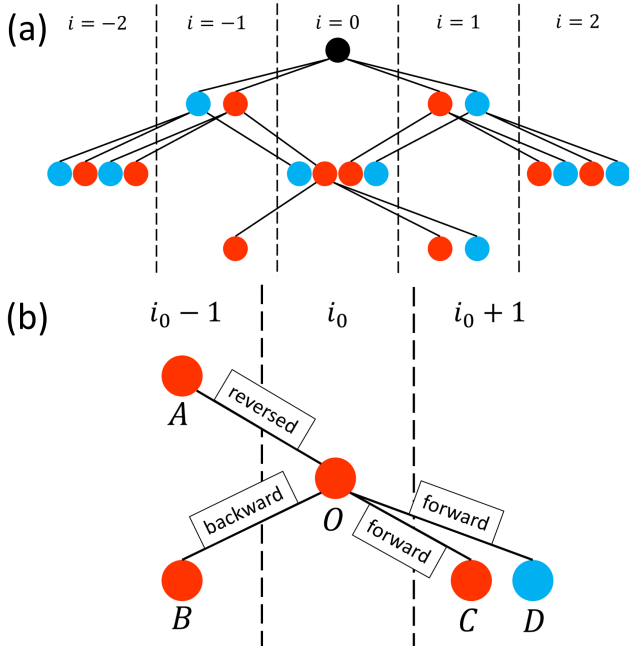


FIG. 8. (a) The configuration tree of the joint states of a two-particle system. All nodes at level 0 (black circle), 1 and 2 are shown as well as examples of a few level-3 nodes. Each node represents one joint state and each edge denotes a hop sequence by both particles. The colour of the node (red or blue) denotes which particle hops first when descending the tree. (b) A subtree of the tree in (a). Node O has a parent A and descendants B , C and D . A transition from O to A corresponds to a step-by-step reversed sequence of a pair of hops, cancelling previous tree-descending motions. A transition to B corresponds to a backward sequence with the other particle hopping first, while a transition to C or D describes forward sequences.

two node types, we arrive at 4 nodes at the first level of the configuration tree.

From the first level onwards, each node only has 3 descendants since one edge connects to a reversed sequence, where the system is restored to the previous configuration (see Fig. 8(b)). We define a forward sequence as a pair of hops of the two particles in the same direction as the preceding tree-descending sequence. A forward sequence can lead to a blue node or a red node. Conversely, a backward sequence is defined as a sequence of hops in the direction opposite to the preceding sequence. There is only one possible backward sequence to descend the tree, corresponding to one child node of the same colour as the current node. The other sequence of hops is instead an up-tree reversed sequence.

As an example, Fig. 1 shows a step-by-step process of particle hops with even steps representable by the evolution of nodes in Fig. 8(a). Labelling the left site by $i = 0$, Fig. 1(a) corresponds to the black node. After a forward sequence illustrated by Fig. 1(b)-(c), the system arrives at the level-1 red node at site $i = 1$. Then a backward sequence shown by Fig. 1(d)-(e) leads to the level-2

red node at site $i = 0$. Note that the system states in Fig. 1(a) and (e) are different in terms of both the bond state and the particle states, despite having identical particle occupancy at both sites.

Quantitatively, each edge in the tree, representing a possible transition between joint states, is unblocked with a probability q^2 with each factor q corresponding to the unblocking probability of one of the two particles. Consequently, the average number of unblocked descendants of any node beyond the root is $3q^2$. The system is only mobile when this branching process of the unblocked subtree has a finite chance of persisting indefinitely, which occurs when the average number of descendants per node exceeds 1. Thus, the exact mobility threshold $q_2^\dagger$ follows $3q_2^{\dagger 2} = 1$ which gives

$$q_2^\dagger = 1/\sqrt{3} \simeq 0.577 \quad (\text{B1})$$

as already stated in the main text.

2. General dynamics

We now relax the constraint above that a hop must be immediately followed by a similar hop of the other particle. Each tree node in Fig. 8 is then generalised to denote a set of states corresponding to a joint state and a set of split states. From any joint state, we now define the motions leading to the associated split states of the same node that do not result in a traverse to another node. Specifically, from a joint state at i_0 denoted by, e.g., a red node (see Fig. 8(b)): (1) the red particle executes one or more backward hops; and/or (2) either particle executes one or more forward hops. These sequences leave the system in a split state that is associated with the original red node at i_0 . While this defines an infinite number of split states associated with a node, only a small number of them are relevant for $q < 1$ because particles cannot move far without mutual facilitation. In addition, each split state is uniquely associated with a node in the tree.

We now explain how a system moves to another node under the general dynamics. This can occur even if only split states are involved. Consider, for example, a node at position $i = 0$, where the particles are at a split state at sites $i \geq 0$. We consider the leading (trailing) particle at $i > 0$ ($i = 0$). The node progresses every time the trailing particle moves from site $i = 0$ towards the leading particle and the new node is determined by the position of the trailing particle. More importantly, it is easy to see that a split state is accessible (i.e. connected by unblocked hops to the root) only if the associated joint state in the same node is accessible. The problem of determining the accessibility of any state reduces to first solving for the accessibility of joint states. The subsequent application to a split state amounts only to a few additional single-particle hops, which are negligible compared to the accessibility of high-level nodes. Equation

Eq. (B1) thus also holds exactly for the general FRW dynamics.

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- [1] G. Biroli and J. P. Garrahan, Perspective: The glass transition, *The Journal of Chemical Physics* **138**, 12A301 (2013).
 - [2] F. Arceri, F. P. Landes, L. Berthier, and G. Biroli, Glasses and aging, a statistical mechanics perspective on, in *Statistical and Nonlinear Physics* (Springer, 2022) pp. 229–296.
 - [3] W. H. Wang, Dynamic relaxations and relaxation-property relationships in metallic glasses, *Progress in Materials Science* **106**, 100561 (2019).
 - [4] C. Rodriguez-Tinoco, M. Gonzalez-Silveira, M. A. Ramos, and J. Rodriguez-Viejo, Ultrastable glasses: new perspectives for an old problem, *La Rivista del Nuovo Cimento* **45**, 325 (2022).
 - [5] J. C. Dyre, Solid-that-flows picture of glass-forming liquids, *The Journal of Physical Chemistry Letters* **15**, 1603 (2024).
 - [6] Z.-Y. Zhou, Q. Yang, and H.-B. Yu, Toward atomic-scale understanding of structure-dynamics-properties relations for metallic glasses, *Progress in Materials Science* **145**, 101311 (2024).
 - [7] J.-L. Barrat, J. Baschnagel, and A. Lyulin, Molecular dynamics simulations of glassy polymers, *Soft Matter* **6**, 3430 (2010).
 - [8] J.-L. Barrat and L. Berthier, Computer simulations of the glass transition and glassy materials, *Comptes Rendus. Physique* **24**, 57 (2023).
 - [9] A. Ninarello, L. Berthier, and D. Coslovich, Models and algorithms for the next generation of glass transition studies, *Physical Review X* **7**, 021039 (2017).
 - [10] F. Ritort and P. Sollich, Glassy dynamics of kinetically constrained models, *Advances in Physics* **52**, 219 (2003).
 - [11] J. P. Garrahan, P. Sollich, and C. Toninelli, Kinetically constrained models, in *Dynamical Heterogeneities in Glasses, Colloids and Granular Media*, edited by L. Berthier, G. Biroli, J.-P. Bouchaud, L. Cipelletti, and W. van Saarloos (Oxford University Press, 2011).
 - [12] G. H. Fredrickson and H. C. Andersen, Kinetic ising model of the glass transition, *Physical review letters* **53**, 1244 (1984).
 - [13] R. G. Palmer, D. L. Stein, E. Abrahams, and P. W. Anderson, Models of hierarchically constrained dynamics for glassy relaxation, *Physical Review Letters* **53**, 958 (1984).
 - [14] W. Kob and H. C. Andersen, Kinetic lattice-gas model of cage effects in high-density liquids and a test of mode-coupling theory of the ideal-glass transition, *Physical Review E* **48**, 4364 (1993).
 - [15] G. Biroli and M. Mézard, Lattice glass models, *Physical Review Letters* **88**, 025501 (2001).
 - [16] D. Chandler and J. P. Garrahan, Thermodynamics of coarse-grained models of supercooled liquids, *The Journal of chemical physics* **123**, 044511 (2005).
 - [17] N. B. Tito, J. E. Lipson, and S. T. Milner, Lattice model of dynamic heterogeneity and kinetic arrest in glass-forming liquids, *Soft Matter* **9**, 3173 (2013).
 - [18] L.-H. Zhang and C.-H. Lam, Emergent facilitation behavior in a distinguishable-particle lattice model of glass, *Physical Review B* **95**, 184202 (2017).
 - [19] A. Tahaei, G. Biroli, M. Ozawa, M. Popović, and M. Wyart, Scaling description of dynamical heterogeneity and avalanches of relaxation in glass-forming liquids, *Physical Review X* **13**, 031034 (2023).
 - [20] M. Ozawa and G. Biroli, Elasticity, facilitation, and dynamic heterogeneity in glass-forming liquids, *Phys. Rev. Lett.* **130**, 138201 (2023).
 - [21] M. R. Hasyim and K. K. Mandadapu, Emergent facilitation and glassy dynamics in supercooled liquids, *Proceedings of the National Academy of Sciences* **121**, e2322592121 (2024).
 - [22] Y. Nishikawa and L. Berthier, Collective relaxation dynamics in a three-dimensional lattice glass model, *Physical Review Letters* **132**, 067101 (2024).
 - [23] L. S. I. Lam, H.-Y. Deng, W.-B. Zhang, U. Nwankwo, C. Xiao, C.-T. Yip, C.-S. Lee, H. Ruan, and C.-H. Lam, Emergent facilitation by random constraints in a facilitated random walk model of glass, *Phys. Rev. E* **111**, 044120 (2025).
 - [24] G. H. Fredrickson and H. C. Andersen, Facilitated kinetic ising models and the glass transition, *The Journal of chemical physics* **83**, 5822 (1985).
 - [25] R. L. Jack, P. Sollich, and P. Mayer, Subdiffusive motion in kinetically constrained models, *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics* **78**, 061107 (2008).
 - [26] R. Gutiérrez, J. P. Garrahan, and R. L. Jack, Accelerated relaxation and suppressed dynamic heterogeneity in a kinetically constrained (east) model with swaps, *Journal of Statistical Mechanics: Theory and Experiment* **2019**, 094006 (2019).
 - [27] M. Lulli, C.-S. Lee, H.-Y. Deng, C.-T. Yip, and C.-H. Lam, Spatial heterogeneities in structural temperature cause kovacs’ expansion gap paradox in aging of glasses, *Physical Review Letters* **124**, 095501 (2020).
 - [28] C.-S. Lee, M. Lulli, L.-H. Zhang, H.-Y. Deng, and C.-H. Lam, Fragile glasses associated with a dramatic drop of entropy under supercooling, *Physical Review Letters* **125**, 265703 (2020).
 - [29] X.-Y. Gao, C.-Y. Ong, C.-S. Lee, C.-T. Yip, H.-Y. Deng, and C.-H. Lam, Kauzmann paradox: A possible crossover due to diminishing local excitations, *Physical Review B* **107**, 174206 (2023).
 - [30] C.-Y. Ong, C.-S. Lee, X.-Y. Gao, Q. Zhai, Z. Yu, R. Shi, H.-Y. Deng, and C.-H. Lam, Relating fragile-to-strong transition to fragile glass via lattice model simulations, *Physical Review E* **109**, 054124 (2024).
 - [31] Q. Zhai, H.-Y. Deng, X.-Y. Gao, L. S. I. Lam, S. Yang, K. Yan, and C.-H. Lam, Penetration of surface effects on structural relaxation and particle hops in glassy films, *The Journal of Chemical Physics* **162**, 234502 (2025).
 - [32] G. Gopinath, C.-S. Lee, X.-Y. Gao, X.-D. An, C.-H. Chan, C.-T. Yip, H.-Y. Deng, and C.-H. Lam, Diffusion-coefficient power laws and defect-driven glassy dynamics in swap acceleration, *Physical Review Letters* **129**, 168002 (2022).
 - [33] C.-T. Yip, M. Isobe, C.-H. Chan, S. Ren, K.-P. Wong,

- Q. Huo, C.-S. Lee, Y.-H. Tsang, Y. Han, and C.-H. Lam, Direct evidence of void-induced structural relaxations in colloidal glass formers, [Physical Review Letters](#) **125**, 258001 (2020).
- [34] E. Teomy and Y. Shokef, Relation between structure of blocked clusters and relaxation dynamics in kinetically constrained models, *Phys. Rev. E* **92**, 032133 (2015).
- [35] M. Bramson and J. L. Lebowitz, Asymptotic behavior of densities in diffusion-dominated annihilation reactions, *Physical review letters* **61**, 2397 (1988).
- [36] D. Stauffer, A. Aharony, and S. Redner, *Introduction to percolation theory* (American Institute of Physics, 1993).
- [37] R. K. Darst, D. R. Reichman, and G. Biroli, Dynamical heterogeneity in lattice glass models, [The Journal of Chemical Physics](#) **132**, 044510 (2010).
- [38] J. Hizak and R. Logozar, A derivation of the mean absolute distance in one-dimensional random walk, *Tehnički glasnik* **5**, 10 (2011).
- [39] E. Ben-Naim and P. Krapivsky, Statistical properties of sites visited by independent random walks, *Journal of Statistical Mechanics: Theory and Experiment* **2022**, 103208 (2022).