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# Quasi-zero stiffness metastructure with continuous cosine-shaped beam for vibration isolation

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**Abstract:** In this paper, a monolithic quasi-zero stiffness beam with a continuous cosine shape is proposed for the construction of QZS metastructures for low-frequency vibration isolation. Owing to its simple continuous configuration, the QZS characteristic can be conveniently controlled and adjusted by geometric parameters. The metastructure based on the proposed QZS beam exhibits outstanding flexible designability and enables a broader range of material selections. A chained beam constraint method is first employed in conjunction with the finite element model and experimental validation to derive the existence of the QZS characteristic, and the variation trend of QZS characteristics is correlated with the critical design parameters for the QZS unit. Subsequently, this paper explores the designability of metastructures via the combination of QZS units and modular design to demonstrate the potential for satisfying customized requirements. Then, the vibration isolation of the proposed QZS metastructures is confirmed by a dynamic model, the finite element method, and experiments. When the payload mass matches the QZS plateau, the onset isolation frequency is much lower than that under the conditions of positive stiffness status. Furthermore, due to the monolithic configuration of the proposed QZS unit, carbon fiber reinforced polymer prepreg is explored for its fabrication through a mold-pressing process to overcome the strength and stability limitations of additive manufacturing materials. The nonlinear isolation performance is investigated under a relatively high acceleration level, and this composite QZS unit can achieve an onset isolation frequency below 5 Hz at an acceleration of 0.15 *g*. This work exhibits the potential of the QZS metastructure by employing the composite material for advanced lightweight low-frequency vibration isolation applications.

**Keywords:** Vibration isolation; Low-frequency; Metastructure; Quasi-zero stiffness (QZS); Cosine beam; Composite laminate

## 1. Introduction

Vibration isolation is a critical bottleneck in high-tech fields, including transportation [1-4], civil construction [5-7], aerospace [8-11], and precision manufacturing [12-18], where system accuracy and structural integrity depend on it. As mechanical systems become lighter and more sensitive, low- and ultra-low-frequency environmental disturbances increasingly challenge vibration isolation performance. Linear vibration theory shows that a system's effective isolation bandwidth is strictly limited by its natural frequency [19, 20]. Lowering stiffness extends low-frequency

1 isolation but risks static instability under high loads. This trade-off between stability and isolation performance is a  
2 fundamental bottleneck for linear isolators in ultra-low-frequency applications.

3 To address the aforementioned issues, quasi-zero stiffness (QZS) systems offer significant design advantages. Such  
4 a system typically introduces a negative-stiffness mechanism near the equilibrium position to cancel the elastic force  
5 of the positive-stiffness element, thereby enabling the overall stiffness to approach zero while maintaining high static  
6 load-bearing capacity [21]. This combination of high static stiffness and low dynamic stiffness allows for a significant  
7 reduction in natural frequency, enabling ultra-low-frequency and even broadband vibration isolation, while ensuring  
8 system stability. Conventional mechanical QZS systems typically employ springs as the primary positive-stiffness  
9 element, and introduce various negative-stiffness mechanisms, such as linear springs [22-27], buckled beams [28-  
10 31], X-shaped structures [32-34], magnetic configurations [35-43], bio-inspired structures [44-47], geometrically  
11 nonlinear structures [48-51], and cam-roller mechanisms [52-56], to counteract the restoring force of the  
12 positive-stiffness component. These QZS systems provide reliable low-frequency vibration isolation across complex  
13 environments, thanks to their wide stroke range, stable mechanical response, and high reliability. However, they  
14 typically use multiple discrete components, such as springs, linkages, cams, and permanent magnets, arranged in  
15 parallel or series. This leads to bulky, complex architectures and tight assembly tolerances, limiting use in space-  
16 constrained or weight-sensitive applications.

17 Additive manufacturing now enables monolithic mechanical metastructures or metamaterials with tailored QZS  
18 properties. These QZS metastructures offer installation-free assembly, miniaturization, and are often reported to  
19 possess modularity and programmability [57]. QZS behavior arises from designed internal elastic elements, primarily  
20 curved-beam or origami-inspired units [58-62]. Curved-beam QZS metastructures are better suited to additive  
21 manufacturing than origami-based ones, as their in-plane geometry can be precisely parametrized, enhancing both  
22 spatial programmability and fabrication accuracy. The elastic units in curved-beam QZS metastructures are typically  
23 designed as pre-shaped beams with specific configurations, enabling precise tuning of their nonlinear mechanical  
24 response. Based on whether distinct positive- and negative-stiffness components can be identified, these units can be  
25 classified into combined and monolithic types. Similar to conventional mechanical QZS systems, combined-type  
26 units achieve QZS properties through the parallel combination of positive- and negative-stiffness elements with  
27 matching stiffness characteristics. The negative-stiffness elements typically employ cosine-shaped beams [63-66] or  
28 inclined straight beams [67, 68], while the positive-stiffness units are often realized using curved beams [69-71] or  
29 folded straight beams [72, 73]. Combined-type QZS units deliver strong performance but also face key engineering

1 barriers. For instance, this combined-type complex design significantly restricts the available manufacturing methods,  
2 rendering them almost solely reliant on additive manufacturing and hardly adaptable to traditional fabrication  
3 processes. Moreover, the matching mechanism enables the adjustment of QZS properties through the synchronous  
4 control of the negative-stiffness element and the positive-stiffness element. However, this leads to the fact that the  
5 controllable method has insufficient flexibility to adapt to complex scenarios.

6 To further simplify the QZS unit, monolithic-type designs are proposed. These designs feature a single elastic beam  
7 with a specially tailored shape. By taking advantage of its large elastic deformation under load, the monolithic-type  
8 unit naturally exhibits a QZS nonlinear mechanical response. This design renders the QZS unit more compact while  
9 offering high geometric tunability and design flexibility. Moreover, monolithic QZS units generally have an in-plane  
10 configuration, which can be fabricated using conventional molding processes, so their fabrication is not restricted to  
11 additive manufacturing. Monolithic QZS units generally can be classified into two types based on whether the unit  
12 configuration is amenable to parametric expression: inversely tailored curved beams and parametrically designed  
13 curved beams. Inversely tailored curved beams are typically aimed at achieving QZS behavior, with their curved  
14 geometry iteratively adjusted using artificial intelligence algorithms or optimization methods, such as spline curves  
15 defined by multiple control points [74-77], inclined folded curved beams [78, 79], holey scattering integrated  
16 structures [80], and Fourier-series-defined curved beams [81]. However, inversely tailored curved beams are typically  
17 generated via complex mathematical descriptions and require high precision in geometric features. Even minor  
18 fabrication deviations can cause the load-displacement response to deviate from the design target. Moreover,  
19 establishing a reliable inverse design model necessitates thousands or even tens of thousands of high-fidelity finite  
20 element simulations to form the training set, which requires substantial computational resources and time in the  
21 preliminary stage.

22 Parametrically designed curved beams typically start from an existing geometric configuration and achieve precise  
23 control over stiffness by adjusting geometric parameters until QZS behavior emerges. Building on the classical  
24 negative-stiffness cosine beam, Dalela *et al.* [82] adopted a variable cross-section design by adding reinforcing  
25 material at the bottom of the cosine beam to achieve the QZS characteristic, and Zhao *et al.* [83] achieved QZS  
26 behavior by reducing the ratio of arch height to thickness. However, both approaches come at the cost of increased  
27 thickness, which reduces the elastic deformation capacity of the beam, leading to a narrower isolation stroke. Gao *et*  
28 *al.* [84] modified the conventional cosine beam by introducing a geometric twist angle along its axis to alter the stress  
29 distribution and buckling path under large deformation, enabling QZS behavior within a more compact structure. Lu

*et al.* [85] proposed an asymmetric configuration incorporating spatially rotated trigonometric functions to reduce the influence of thickness and achieve QZS behavior. Yang *et al.* [86] induced a unique segmented buckling deformation mode through local reinforcement, thereby achieving an ultra-long QZS stroke. However, the approaches mentioned above transform the beam shape from in-plane planar configurations to spatial configurations or truss-like topologies, which cannot be monolithically fabricated using traditional molding processes.

To achieve QZS characteristics while retaining the advantages of a planar configuration, inspired by the elastic boundary conditions [87, 88], auxiliary beams with specific geometries are used to connect in series at both ends of a negative-stiffness beam. Liu *et al.* [89] proposed a straight-arch-straight QZS beam, and this integrated design directly resolves the stiffness-matching issue from a physical configuration perspective. Zhang *et al.* [90] introduced a monolithic QZS preshaped beam featuring a cosine-shaped segment integrated with arc-shaped boundary segments, where geometric coupling between the two enables QZS behavior. In these configurations, the flexible constraints offered by auxiliary beams are utilized to regulate the buckling behavior of the cosine beam, thus precisely inducing a QZS plateau within a single plane. Due to the uniform and small thickness of the beam, significant elastic deformation is attained, leading to a wide QZS range. Furthermore, auxiliary beams can also be introduced in the middle of the negative-stiffness beam, and Liu *et al.* [91] introduced a folded beam into a simple inclined beam to achieve the QZS characteristic by leveraging the geometric nonlinearity of members under load. However, the series configuration introduces discontinuities and sharp inflection points in the beam, making it prone to stress concentration and plastic deformation. Therefore, it is necessary to design monolithic QZS units with simpler continuous configurations, ensuring large elastic deformation while reducing the complexity of parametric modeling and the requirements for fabrication precision.

In addition, the majority of works rely on the additive manufacturing process to handle the complex design of the QZS metastructures, particularly for the combined-type designs. This reliance results in a limited choice of materials. Although the hyperelastic polymer materials employed in the additive manufacturing process circumvent the issue of fracture, their stability and durability are poor, causing the isolation performance of the QZS metastructure to fall short of the desired level [75, 92]. Compared to the combined-type design, the monolithic-type design enables the utilization of other fabrication techniques, such as mold pressing. As a result, elastic metal materials with excellent durability can be chosen for the QZS metastructure [93]. Nevertheless, during the mold-pressing process, the thin metal plate demonstrates high resilience, which poses challenges in shape control for the QZS unit. Hence, it is also essential to explore other materials for the QZS metastructure to improve the stability of the isolation performance,

such as composite materials.

Based on the foregoing review, it can be concluded that despite the considerable progress achieved in existing monolithic QZS curved beams, they still encounter several common limitations. The key limitations of current monolithic QZS designs can be summarized as follows: inversely tailored curved beams typically introduce complex geometries, and parametric curved beams with serial joints undermine structural continuity. Accordingly, there is an urgent demand for a monolithic QZS unit that features a simpler continuous planar configuration, supports parametric control, and can be fabricated from high-performance materials via traditional manufacturing processes. In this paper, we propose a monolithic QZS beam featuring a continuous cosine shape, which is governed solely by three geometric parameters, allowing the QZS plateau to be conveniently controlled and adjusted. Benefiting from this simple configuration, identical beams can be arranged through various stacking arrangements to construct QZS metastructures, offering considerable design flexibility for complex operational scenarios. Furthermore, the planar configuration enables the QZS unit to be compatible with both additive manufacturing and conventional molding, expanding the material range beyond common 3D printing materials to allow adoption of high-stability, high-performance alternatives such as continuous fiber-reinforced composites. This work thus further explores the composite QZS unit to demonstrate this advantage.

The remainder of this paper is structured as follows: Section 2 presents the conceptual design of the QZS beam unit and the theoretical approach for static analysis; Section 3 analyzes the quasi-static results to demonstrate the feasibility of the proposed QZS metastructures; Section 4 examines the vibration isolation performance and further explores composite laminate implementation; Section 5 concludes with a summary of the key findings.

## 2. Design

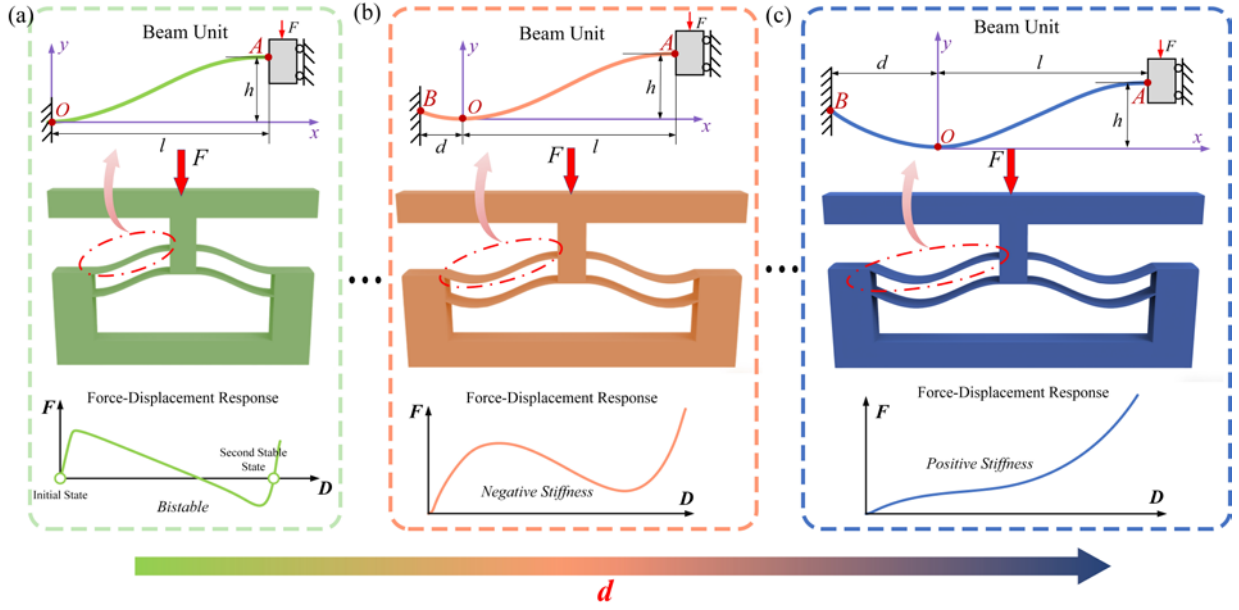
### 2.1 Conceptual design

The use of a cosine-shaped beam is widely recognized as an effective unit for constructing the metastructure with distinctive mechanical properties, such as negative stiffness, bi-stability, or multi-stability. As shown in Fig. 1(a), a typical bistable structure can be designed with four cosine-shaped beam units symmetrically arranged between two frames, and the shape of these beam units is described by,

$$y(x) = \frac{h}{2}(1 - \cos(\frac{\pi x}{l})), \quad x \in [0, l] \quad (1)$$

where  $l$  is the span length of the beam unit, and  $h$  is its apex height. If the height  $h$  and thickness  $t_b$  of the beam unit are appropriately selected, the force-displacement response of the structure under axial compression exhibits a distinct negative-stiffness characteristic. Moreover, the force-displacement curve will intersect the zero-force line.

1 This indicates that the entire structure has bi-stability and can reach and maintain stability at another stable state.



2

3 Fig. 1. Conceptual design: (a) A bistable structure featuring cosine-shaped beam units; (b) A negative-stiffness structure with extended  
4 cosine-shaped beam units regulated by a smaller value of  $d$ ; (c) A positive-stiffness structure with extended cosine-shaped beam units  
5 regulated by a larger value of  $d$ .

6 In this work, other possibilities are explored based on this typical bistable structure when the shape of this beam unit  
7 is further extended. As shown in Fig. 1(b), the shape of the beam unit is extended from the end (Point O) to the new  
8 end (Point B). This extension results in the beam unit being composed of two segments: the original segment OA and  
9 the additional segment BO. To describe the additional segment BO, its span length is defined as  $d$ . Therefore, the  
10 shape of this extended beam unit can be described by

$$11 \quad y(x) = \frac{h}{2} \left( 1 - \cos\left(\frac{\pi x}{l}\right) \right), \quad x \in [-d, l] \quad (2)$$

12 The fundamental prerequisite for the bistable beam unit is the restricted boundary condition at its two ends. If the  
13 boundary condition is not fully restricted, the mechanical properties of the structure may shift from bistability to a  
14 negative-stiffness characteristic. Here, the additional segment BO can be regarded as a weak boundary constraint for  
15 the bistable beam unit OA. Since the constraint is weakened, the structure with this extended cosine beam unit may  
16 exhibit a negative-stiffness feature without bistability, as shown in Fig. 1(b). Furthermore, if the beam unit is extended  
17 with a greater value of  $d$ , the mechanical properties of the structure may transition from the negative-stiffness feature  
18 to a positive-stiffness feature. Therefore, when the parameter  $d$  is appropriately selected, there exists an intermediate  
19 state between the negative-stiffness state and the positive-stiffness state, which is the quasi-zero stiffness (QZS) state.  
20 In this work, we aim to explore this possibility and identify the beam unit with a QZS characteristic, which can be

utilized to construct the quasi-zero stiffness metastructure for vibration isolation.

## 2.2 Chained beam constraint method

To design the QZS beam unit effectively, a model based on the chained beam constraint method (CBCM) is established [94]. When establishing this model, a coordinate transformation is first applied to the geometric description of the extended cosine-shaped beam, as shown in Fig. 2(a). In this new coordinate system, the origin  $O$  is the fixed end of the beam, at which the slope of the tangent is zero, and the other end is permitted to move along the  $y$ -axis direction. Here, the initial shape of the beam unit is as follows.

$$y = f(x) = \frac{h}{2}(1 - \cos(\frac{\pi x}{l})), \quad x \in [0, l + d] \quad (3)$$

For the purpose of using CBCM, the beam is divided into  $N$  elements at nodes  $O_i$  ( $i=2,3,\dots, N$ ) whose coordinates along the  $X$ -axis are  $x_i$ . Besides, we define two nodes at the fixed end and the tip of the beam, denoted as  $O_1$  ( $x_1=0$ ) and  $O_{N+1}$  ( $x_{N+1}=d+l$ ), respectively. Then the coordinates along the  $Y$ -axis, the slopes, and the curvatures of these  $(N+1)$  nodes are,

$$y_i = f(x_i) \quad (4)$$

$$\beta_i = \arctan f'(x_i) \quad (5)$$

$$K_i = \frac{f''(x_i)}{[1 + f'^2(x_i)]^{3/2}} \quad (6)$$

For this curved shape, the tip coordinates ( $X_0$  and  $Y_0$ ) and the tip slope  $\theta_0$  are determined by the shape of the beam, namely, as follows

$$X_0 = l + d, \quad Y_0 = f(d + l), \quad \theta_0 = \arctan f'(l + d) \quad (7)$$

For a given displacement  $\Delta$ , only the tip coordinate in the  $Y$ -axis is changed, as follows

$$Y_0 = f(d + l) - \Delta \quad (8)$$

The force  $F_0$  and  $P_0$  can be obtained by solving the CBCM equations (see Appendix A) for this given displacement  $\Delta$  using MATLAB, and then the force  $F$  required to actuate the whole structure with four beam units is calculated as

$$F = -4(P_0 \sin \theta_0 - F_0 \cos \theta_0) \quad (9)$$

Finally, the force-displacement ( $F$ - $\Delta$ ) curves can be predicted for the whole structure. The parameter  $d$  can be modified to find the value corresponding to the QZS characteristic. A convergence analysis is conducted to determine the number  $N$  of the discrete elements, as depicted in Fig. 2(b). Taking into account the computational cost and computational accuracy,  $N$  is chosen to be 200.

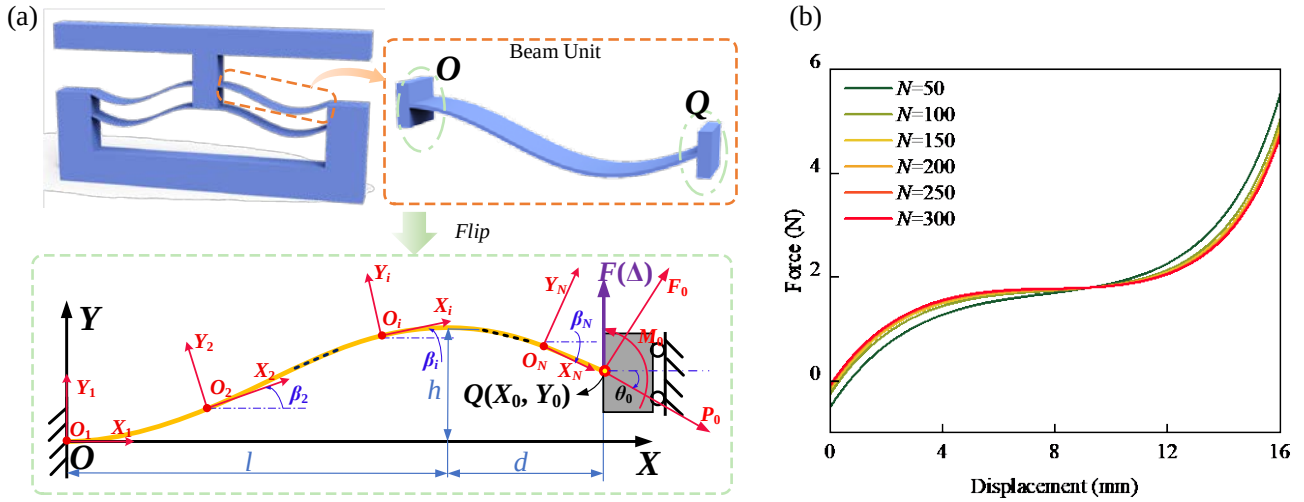


Fig. 2. (a) The typical metastructure with four beam units, and discretization analysis of the beam unit with  $N$  elements for the CBCM model; (b) The convergence analysis of the number  $N$  of elements used in the CBCM model, where  $l$  is 30 mm, and  $h$  is 9 mm.

### 2.3 Fabrication and quasi-static measurement

To validate the mechanical behavior of the proposed metastructure, a series of samples was manufactured using tough polylactic acid (PLA) via a fused deposition printer (Ultimaker S5). The processing parameters during the fabrication process were configured as follows: a nozzle temperature of 215 °C, a build plate temperature of 60 °C, a layer height of 0.2 mm, and a print speed of 50 mm/s. To strike a balance between cost and structural stiffness, the specimen body was printed with a 50% infill density, whereas the beams with small in-plane thickness were printed at a 100% infill density to ensure load-bearing integrity. To comprehend the fundamental mechanical properties of the utilized materials, three specimens were prepared and underwent uniaxial loading tests in accordance with the ASTM D638 standard on an MTS 810 testing machine. As depicted in Fig. 3, the engineering average stress-strain curves of the three specimens yield an elastic modulus of 2.32 GPa through a linear fit to the initial portion, and this value will be employed in the aforementioned model and subsequent finite element method.

To acquire the force-displacement response of the experimental specimens, quasi-static uniaxial compression tests were carried out on an Instron 5966 testing system under displacement control at a constant rate of 5 mm/min. The compression displacement was set to be twice the apex height  $h$  of the beam unit to fully induce the structure's nonlinear deformation. Load-displacement curves of all experimental samples were recorded throughout the tests for further analysis of the mechanical response.

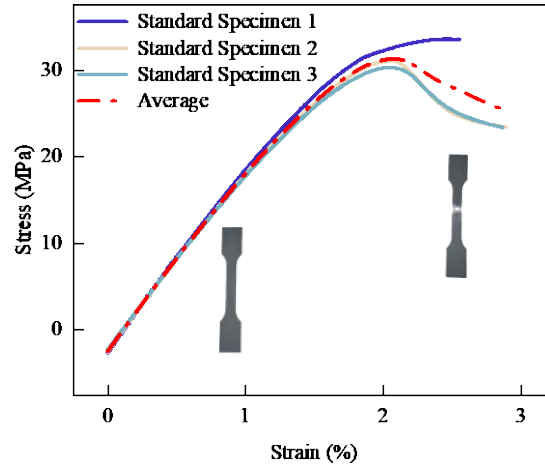


Fig. 3. Engineering stress-strain curves of the tough PLA material used in the fabrication process.

## 2.4 Finite element modeling

The CBCM model can effectively predict the basic mechanical properties of this beam-type metastructure, making it suitable for parametric sweeps and preliminary screening. To obtain more detailed mechanical responses, finite element modeling (FEM) is also conducted. This analysis is beneficial for accurately capturing three-dimensional stress states and large geometric deformations. The combined use of both methods enhances the reliability of the design.

The finite element models are established utilizing the ABAQUS/Standard commercial software, and the entire structure is modeled using 8-node brick elements with reduced integration (C3D8R). To balance accuracy and computational cost, a global mesh size of approximately 1.5 mm is applied to the main body, whereas local refinement decreases the element size to 0.1 mm in beam units. The typical finite element model with mesh is depicted in Fig. 4. To replicate the experimental compression tests, the bottom surface of the structure is fully constrained, and the top surface is kinematically coupled to a reference point (RP), with only a vertical degree of freedom retained. A vertical displacement is imposed on the RP to conduct quasi-static compression in a ‘static, general’ analysis step, as shown in Fig. 4, and the corresponding reaction force is extracted to construct the force-displacement response. The FEM results will be systematically compared with model predictions and experimental results to evaluate the mechanical behaviors of the metastructures featuring different beam units.

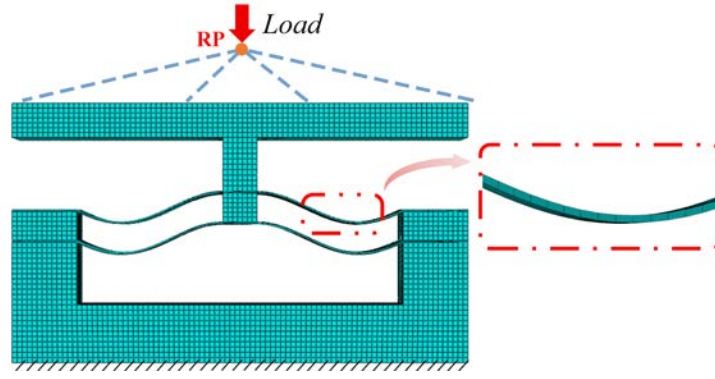


Fig. 4. The finite element model of a typical QZS metastructure with mesh and boundary conditions.

### 3. Quasi-static properties of QZS metastructure

#### 3.1 Existence validation of QZS characteristic

To verify the conceptual design, an example metastructure is employed to demonstrate the QZS characteristic, with its specific geometric parameters depicted in Fig. 5(a). The parameters  $l$  and  $h$ , which are utilized to control the shape of the beam unit, are 30 mm and 9 mm, respectively. Meanwhile, the parameter  $d$  is varied to identify the QZS characteristic. The in-plane thickness  $t_b$  of the beam is 0.6 mm, and the out-of-plane width  $w$  of the beam is 10 mm. Four beams are symmetrically arranged between the T-shaped top frame and the U-shaped bottom frame. Two symmetrical beams constitute a deformation layer, and the number  $n$  of such layers is 2 in this structure. The distance between the two layers is set at 9 mm to enhance the stability of the entire structure, and the distance between the beam end connected to the bottom frame and the bottom frame is set at 15 mm to guarantee sufficient space for the deformation process. The remaining geometric parameters for the two frames are selected in accordance with the stiffness requirement for the support function, as shown in Fig. 5(a).

As depicted in Fig. 5(b), the force-displacement curves corresponding to different values of  $d$  are predicted by the CBCM model. The results indicate that the mechanical properties shift from negative stiffness to positive stiffness as  $d$  increases from 9 mm to 17 mm. Moreover, a QZS characteristic is observed when  $d$  is 13 mm. To verify the existence of this QZS characteristic, three experimental samples with three different values of  $d$  (10 mm, 13 mm, and 16 mm) are fabricated, as depicted in Fig. 5(c). The force-displacement curves of these samples are determined through experiments, and the FEM results are also utilized to validate the predictions of the CBCM model, as shown in Fig. 5(b). The force-displacement curves obtained by the three methods exhibit good agreement, which confirms the conceptual design and the existence of the QZS characteristic. Additionally, the experimental results also verify the validity of the model and the FEM. Furthermore, when the QZS characteristic is manifested in this sample, the ratio of  $d$  to  $l$  is 0.433, and this will be examined further in the subsequent parametric analysis.

The strain energy evolution during deformation of the beam unit is also analyzed using the CBCM model to elucidate the physical role of parameter  $d$ . As illustrated in Fig. 5(d), the total strain energy  $U$  is the sum of the bending strain energy  $U_b$  and the axial strain energy  $U_a$ . The bending strain energy  $U_b$  dominates over the axial strain energy  $U_a$ ; hence the evolution of  $U$  is essentially governed by  $U_b$ . Increasing the parameter  $d$  from 10 mm to 16 mm markedly reduces both energy components  $U_a$  and  $U_b$ . Additionally, the variation trend of the  $U_b$ -displacement curve changes from slightly convex to concave, indicating that the mechanical property of stiffness transitions from negative to positive. Therefore, when a value of 13 mm is selected appropriately for parameter  $d$ , an intermediate state can be obtained, and the  $U_b$ -displacement curve presents an approximately linear profile, which corresponds to the QZS state. This can be attributed to the fact that the extended segment itself participates in bending deformation, which redistributes the deformation originally concentrated at the central arch crown across the entire beam, thereby preventing excessive accumulation of local bending energy. The aforementioned analysis based on strain energy elucidates how the parameter  $d$  suppresses the accumulation of axial and bending strain energy, ultimately inducing the formation of the QZS plateau.

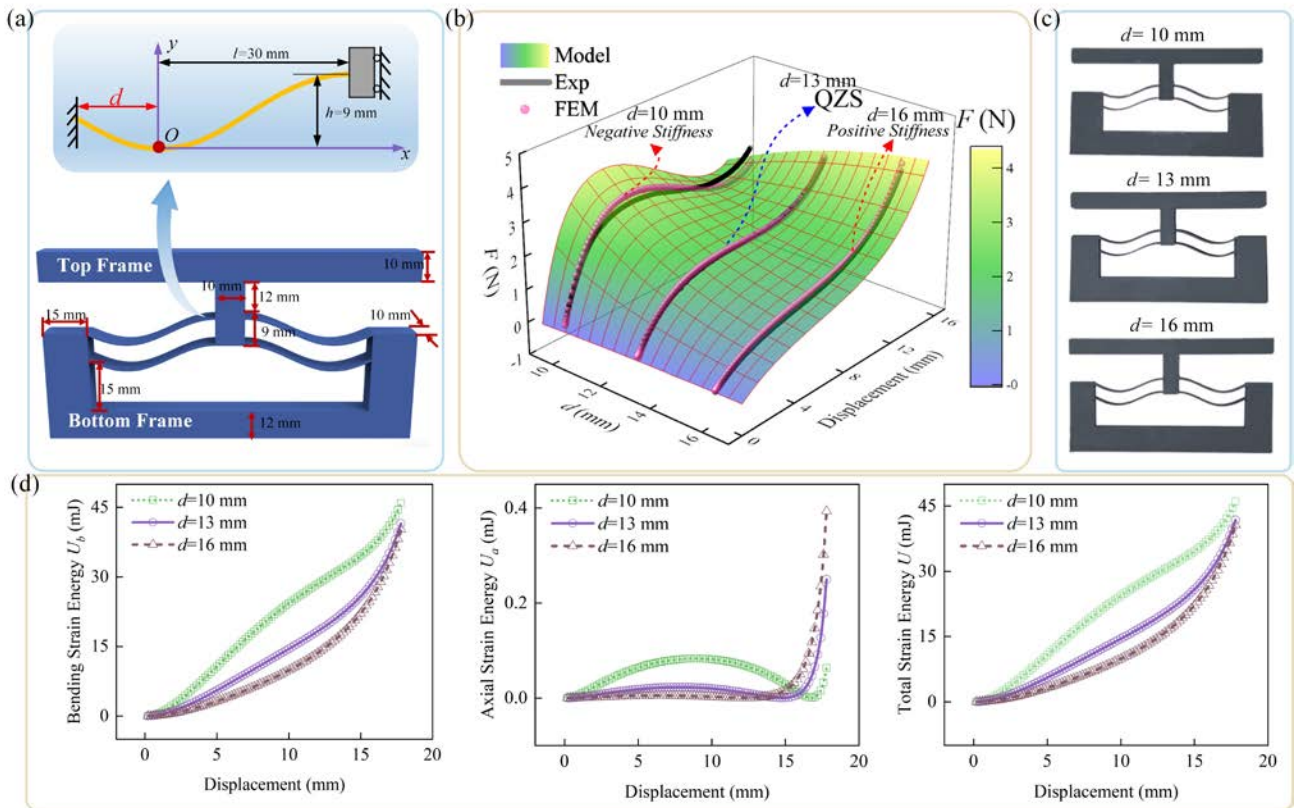


Fig. 5. (a) Geometric configuration and parameter definitions of the metastructure; (b) Force-displacement responses predicted by the CBCM model and compared with FEM predictions and experimental results for representative values of  $d$ ; (c) Fabricated experimental specimens with three values of  $d$ . (d) Energy evolution associated with displacement and parameter  $d$ : bending strain energy  $U_b$ , axial

strain energy  $U_a$ , and total potential energy  $U$ .

### 3.2 Typical QZS characteristics

The conceptual design of the QZS metastructure has been verified, and in this section, the mechanical characteristics of this typical QZS metastructure (with  $h=9$  mm,  $l=30$  mm, and  $d=13$  mm) are presented. As depicted in Fig. 6(a), the force-displacement curves obtained from the three methods show excellent agreement, and these curves display characteristic nonlinear response features. In the initial loading stage, the force gradually increases with displacement and then enters a wide and stable plateau region where the rate of force change nearly reaches zero. This region is a direct indication of the QZS characteristic, which could be termed the QZS plateau. Here, the force corresponding to this QZS plateau is defined as the QZS payload  $F_{QZS}$ , which is approximately 1.8 N. After passing through the QZS plateau, the force shows a significant upward trend with the increase in displacement, thus restoring a positive stiffness characteristic within the structure.

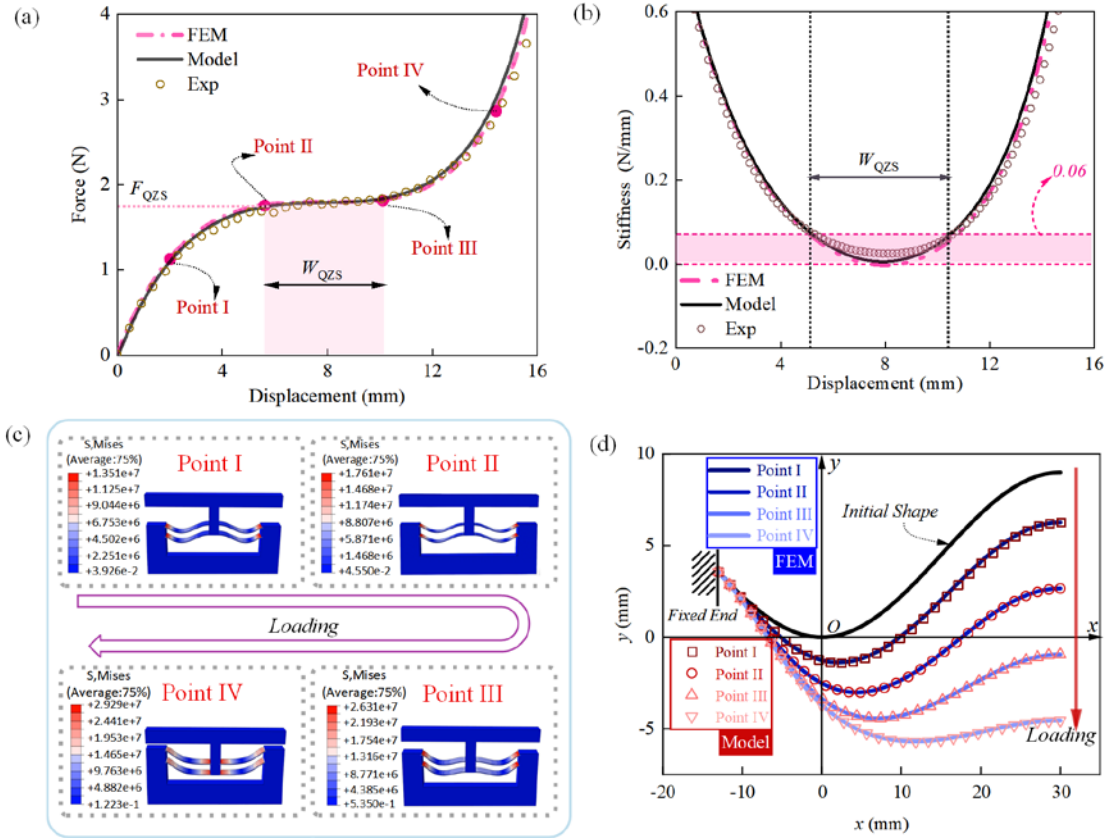


Fig. 6. Mechanical characteristics of the typical QZS metastructure: (a) Force-displacement curves obtained from experiment, FEM, and the CBCM model; (b) The stiffness-displacement curves obtained from the force-displacement results using a differential method; (c) The stress contours predicted by FEM corresponding to critical points on the load-displacement curve; (d) The deformation processes of the beam unit predicted by FEM and the CBCM model.

To quantitatively characterize the evolution of stiffness, the instantaneous stiffness is calculated from the force-displacement curve using a differential method, as shown in Fig. 6(b). The stiffness decreases to nearly zero and then

increases again, forming a QZS plateau. A stiffness threshold of 0.06 N/mm is selected to define the QZS plateau, resulting in an approximate width of the QZS plateau,  $W_{QZS}$ , of 4.75 mm. The adoption of this stiffness threshold is attributed not only to the fact that this value is sufficiently low to approximate zero, but also to the incorporation of nonlinear dynamic response effects, which will be discussed in subsequent sections. Based on the position of this QZS plateau, two critical points (Point II and Point III) corresponding to the QZS plateau are also marked on the force-displacement curve, as illustrated in Fig. 6(a). Additionally, two points (Point I and Point IV) are chosen from two positive-stiffness regions on the force-displacement curve. The transient shapes and stress states of these four critical points are predicted by FEM, as shown in Fig. 6(c). The maximum stress is located on the extended segment and subsequently examined to guarantee that it does not surpass the ultimate stress threshold in the QZS plateau, thus verifying the structural integrity. Additionally, the deformation process of the beam unit is also predicted by the model and compared with the FEM results, as presented in Fig. 6(d). The deformation process reveals that the beam is gradually flattened under the load, and the QZS characteristic is achieved through the nonlinear deformation mechanism.

### 3.3 Parametric analysis

The existence of the QZS characteristic and the typical QZS characteristic has been demonstrated by an example. Moreover, more results from a parametric analysis will be discussed in this section to illustrate the key geometric and configurational parameters of this type of QZS metastructure. The CBCM model and FEM are used to conduct a parametric analysis, including parameters  $l$  and  $h$ , and other parameters are kept as before. The values of  $d$  corresponding to the different combinations of  $h$  and  $l$  when the QZS characteristics exist are shown in Fig. 7(a). It is found that when  $l$  is constant, the value of  $d$  hardly changes with  $h$ , indicating that the value of  $d$  is hardly affected by  $h$  and is determined by  $l$ . The ratio of  $d$  to  $l$ , which is 0.433 as derived from the previous typical example, applies to other combinations of  $l$  and  $h$ . When  $l$  is known, this ratio can be used to estimate the value of  $d$ . Subsequently, further fine-tuning is carried out around this reference value to determine the exact value of  $d$ . Then, the payload  $F_{QZS}$  corresponding to the QZS plateau is associated with the different combinations of  $h$  and  $l$ , as depicted in Fig. 7(b). The  $F_{QZS}$  decreases non-linearly with  $l$  when  $h$  remains constant. Similarly, the  $F_{QZS}$  is also enhanced by  $h$  when  $l$  is fixed. Therefore, a combination featuring a higher  $h$  and a shorter  $l$  is a potential option for improving the  $F_{QZS}$ . To verify the above parametric analysis regarding  $h$  and  $l$ , two sets of experimental samples were fabricated and tested. The experimental force-displacement curves are then compared with the results of the CBCM model and the FEM. As shown in Fig. 8(a), when the values of  $h$  are 5 mm, 9 mm, and 13 mm, respectively, and  $l$  is 30 mm, the

corresponding values of  $d$  are 12.6 mm, 13.0 mm, and 13.3 mm. Thus, the ratio of  $d$  to  $l$  remains relatively stable within a range from 0.42 to 0.443. As previously analyzed, the payload  $F_{QZS}$  is enhanced by  $h$ . Moreover, the width of the QZS plateau narrows when  $h$  is small. As depicted in Fig. 8(b), with  $h$  fixed at 9 mm and  $l$  being 20 mm, 30 mm, and 50 mm, the corresponding values of  $d$  are 8.9 mm, 13.0 mm, and 21.0 mm, respectively. Therefore, the ratio of  $d$  to  $l$  falls within a range from 0.42 to 0.445. The payload  $F_{QZS}$  decreases as  $l$  increases, which is consistent with the previous parametric analysis.

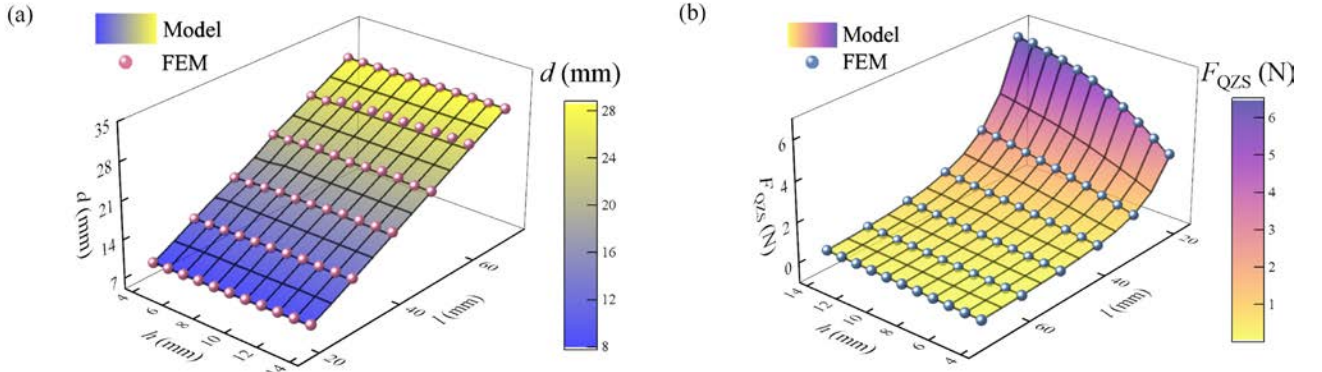


Fig. 7. Parametric analysis performed by FEM and the CBCM model: (a) The values of  $d$  corresponding to the different combinations of  $h$  and  $l$  when the QZS characteristic exists; (b) The QZS payload  $F_{QZS}$  associated with different combinations of  $h$  and  $l$ .

The shape of the beam unit is primarily determined by  $h$ ,  $l$ , and  $d$ . Nevertheless, the QZS characteristics are influenced by other secondary parameters, such as the out-of-plane width  $w$  and the in-plane thickness  $t_b$ . A representative configuration ( $h=9$  mm,  $l=30$  mm,  $d=13$  mm) is utilized to demonstrate their impacts. Three equidistant values of  $w$  are chosen, and the corresponding load-displacement responses are presented in Fig. 8(c). It is noted that the  $F_{QZS}$  exhibits an almost linear increase with respect to  $w$ . A similar effect can be observed for the parameter  $n$ , the number of deformation layers, as shown in Fig. 8(d). The in-plane thickness  $t_b$  of the beam is varied at three values, and the resulting load-displacement responses are depicted in Fig. 8(e). The  $F_{QZS}$  shows a nonlinear increase with  $t_b$ . However, an increase in  $t_b$  will lead to a significant rise in the maximum stress level, which poses challenges for safety considerations.

The above results provide a design strategy for the proposed QZS beam unit. For a specific combination of  $h$  and  $l$ , there is a corresponding value of  $d$  that exhibits a strong correlation with  $l$  but not with  $h$ . The value of  $d$  can be determined based on a reference ratio of  $d$  to  $l$ , which is 0.43, and then further adjusted according to the magnitude of  $h$  and  $l$ . For the QZS metastructure, the payload  $F_{QZS}$  increases as the value of  $h$  increases, but decreases as the value of  $l$  increases. Moreover, the width  $W_{QZS}$  can be enhanced by increasing  $h$  and  $l$ . Once the shape of the beam unit is determined, the out-of-plane thickness  $w$  and the number of beam units are optional variables that can be used

to finely tune  $F_{QZS}$  almost without altering  $W_{QZS}$ . In comparison, the in-plane thickness  $t_b$  can enhance the load-bearing ability, yet it will also increase the risk of plastic deformation simultaneously.

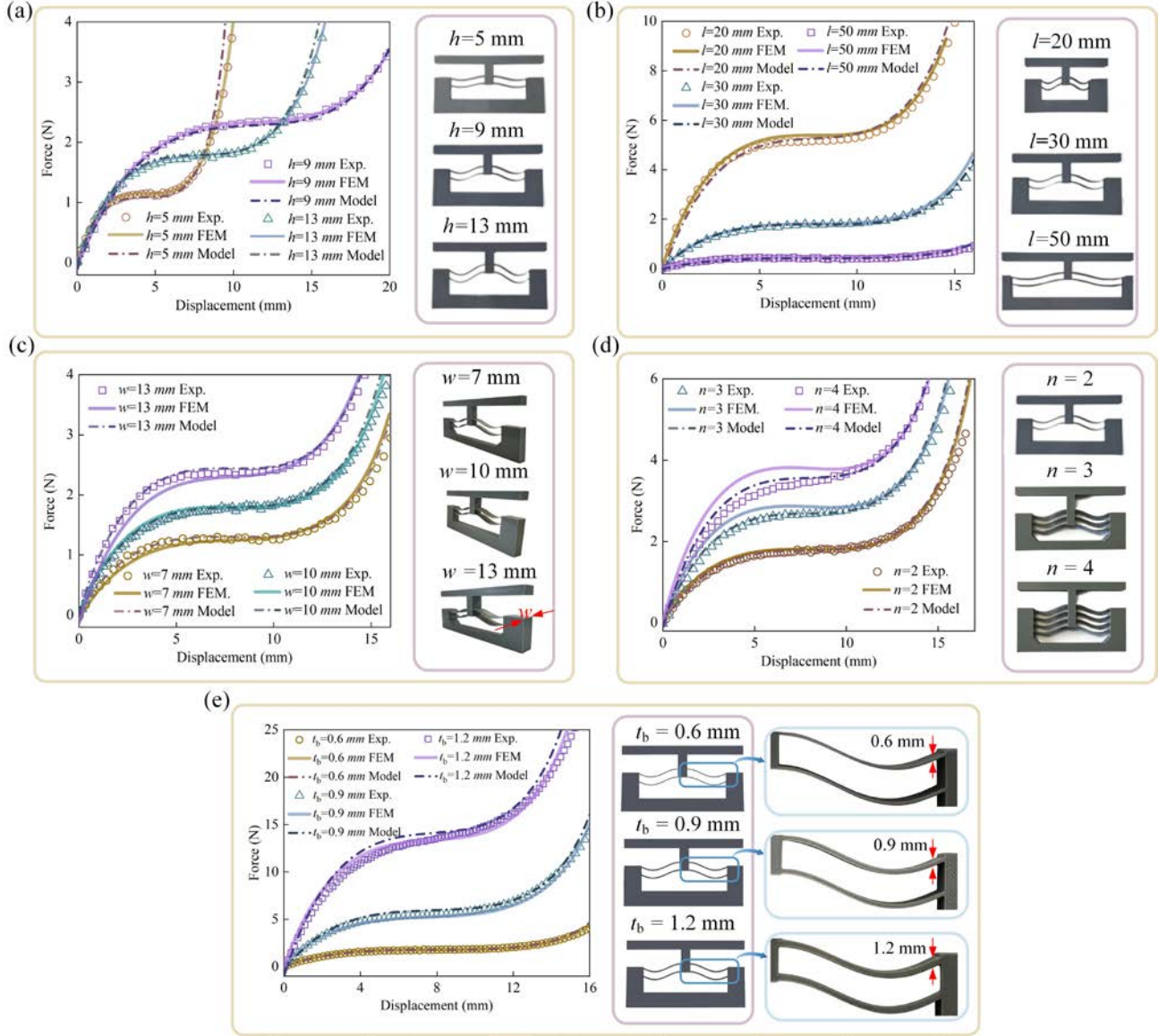


Fig. 8. The force-displacement curves of the QZS metastructures with different geometric parameters, obtained by three methods: experiments, FEM, and the CBCM model: (a) The height  $h$  of the beam; (b) The length  $l$  of the beam ; (c) The out-of-plane width  $w$  of the beam; (d) The number  $n$  of layers; (e) The in-plane thickness  $t_b$  of the beam.

### 3.4 Designability of QZS metastructure

To demonstrate the designability of QZS metastructures using the proposed monolithic QZS beam, a series of design possibilities is explored further. The existing QZS metastructure can be utilized as the QZS unit to build the multi-layer QZS metastructure by using different stacking methods. Three samples are designed and analyzed by FEM and experiments to illustrate the design flexibility, as illustrated in Fig. 9(a). Here, the parameters of the beam unit are as follows:  $h=8$  mm;  $l=20$  mm;  $d=8.3$  mm;  $t_b=0.6$  mm;  $w=10$  mm. The metastructure composed of two QZS units

connected in parallel allows for an almost double increase in the payload  $F_{QZS}$  compared to the single QZS unit, and the metastructure composed of two QZS units connected in series can extend the width of the QZS plateau. By employing this QZS unit, a three-layer metastructure featuring three-step QZS plateaus can also be explored.

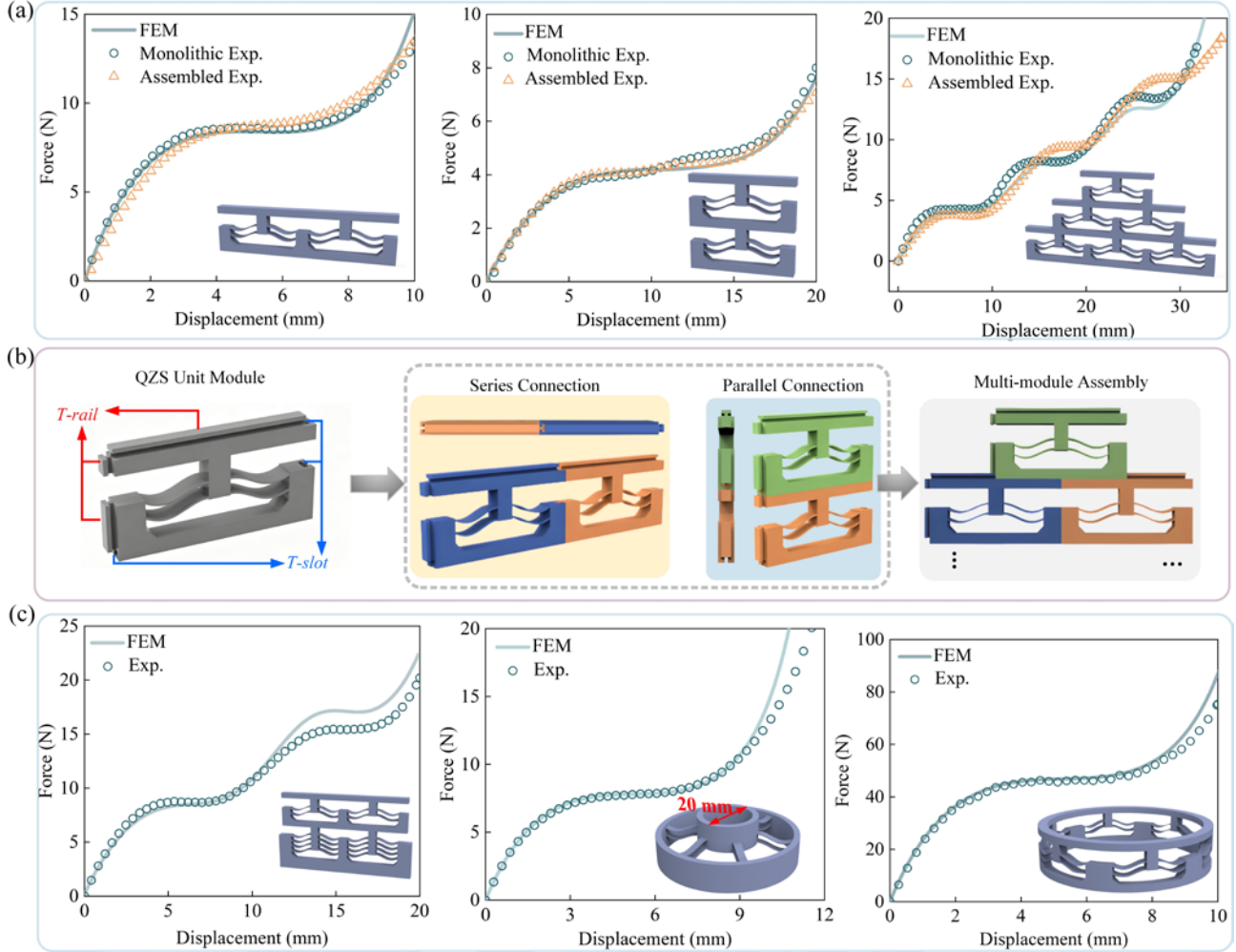


Fig. 9. (a) Force-displacement curves of metastructures constructed by QZS units obtained by FEM and experiments, including monolithic specimens and modularly assembled specimens; (b) Modular design strategy and reconfigurable QZS metastructures; (c) Force-displacement responses of different QZS units with multi-layer and spatial configurations obtained by FEM and experiments. To overcome the size limitations of monolithic fabrication and enable on-demand assembly, an interlocking modular design strategy is explored further, as shown in Fig. 9(b). Each independent QZS unit module integrates mating T-shaped guide rails and T-shaped slots on its outer frame, allowing horizontal series connection and vertical parallel connection. To verify the feasibility of the modular assembly strategy, three assembly scenarios were experimentally characterized and compared with monolithic specimens fabricated via integrated manufacturing, as illustrated in Fig. 9(a). In the first two scenarios, the measured curves of the assembled specimens agree basically with those of the monolithic specimens and the FEM predictions, indicating that the interlocking T-joints are reliable and exhibit no

obvious joint slippage or performance degradation. However, in the scenario involving three modules, the results obtained via the two methods differ significantly. This discrepancy can be attributed to inherent manufacturing imperfections and unstable load transmission induced by inter-module clearances. Therefore, fabrication accuracy and assembly accuracy should be carefully considered for this modular design.

Furthermore, the QZS unit is further extended to multi-layer and spatial configurations to accommodate complex requirements, as shown in Fig. 9(c). By tuning the number of layers  $N$  within the QZS unit, multi-stepped QZS characteristics can be realized without increasing the overall structural dimension. The QZS beams can be positioned between two circular frames to form a spatial QZS unit with self-stabilizing ability, and the payload capacity can be adjusted by the number of beam units, even if it is an odd number. A cylindrical spatial unit formed by bending six units along a circumference not only achieves a high load capacity but also exhibits good structural robustness. These QZS units can also be designed as modular units to construct QZS metastructures with customized QZS plateaus.

Integrating the above FEM results and experimental validations demonstrates that these QZS metastructures constructed by the proposed QZS beam feature design flexibility, modular scalability, and topological reconfigurability, rendering them promising candidates for task-specific applications.

## 4. Dynamic analysis of the QZS metastructure

### 4.1 Dynamic analysis for the QZS unit

To evaluate the vibration isolation performance of the QZS unit shown in Fig. 5(a), a dynamic model is established first. Considering a single-degree-of-freedom nonlinear vibration isolation system under excitation  $y(t)=Y_0\cos(\omega t)$ , its dynamic equation can be expressed as,

$$m\ddot{z}(t) + c\dot{z}(t) + f_{QZS}(z) = ma_0 \cos(\omega t) \quad (10)$$

where  $z$  is the relative displacement, and  $c$  is the damping coefficient. With asymmetric disturbances neglected, the nonlinear restoring force  $f_{QZS}(z)$  is reduced to an odd-power polynomial,

$$f_{QZS}(z) = k_1 z + k_3 z^3 + k_5 z^5 \quad (11)$$

Here,  $k_1$ ,  $k_3$ , and  $k_5$  represent stiffness coefficients, which are acquired by fitting the force-displacement curves at the loaded static equilibrium state under the payload mass  $m$ . The characteristic stiffness  $K_{ref}$  is defined as the initial linear stiffness under small deformation. To decouple the nonlinear performance of the structure from its cross-sectional dimensions and to evaluate the intrinsic contribution of its geometry, the beam thickness  $t_b$  is selected as the characteristic length for displacement normalization.

1 The introduction of dimensionless parameters leads to

$$2 \quad \ddot{x} + 2\zeta\dot{x} + a_1x + a_3x^3 + a_5x^5 = \hat{a}\cos(\Omega\tau) \quad (12)$$

$$3 \quad \text{where } x = \frac{z}{t_b}, \quad a_i = \frac{k_i t_b^{i-1}}{K_{ref}} (i=1, 3, 5), \quad \omega_n = \sqrt{\frac{K_{ref}}{m}}, \quad \Omega = \frac{\omega}{\omega_n}, \quad \zeta = \frac{c}{2\sqrt{mK_{ref}}}, \quad \tau = \omega_n t, \quad \hat{a} = \frac{a_0}{\omega_n^2 t_b}.$$

4 The derivation of the HBM equations is presented in Appendix B. Finally, the acceleration transmissibility  $T$  is  
5 obtained as:

$$6 \quad T = 20\log\left(\frac{\Omega^2 X_{abs}}{\hat{a}}\right) = 20\log_{10}\left(\sqrt{\frac{K_{eq}^2 + (2\zeta\Omega)^2}{(K_{eq} - \Omega^2)^2 + (2\zeta\Omega)^2}}\right) \quad (13)$$

7 Employing the above dynamic model, the vibration isolation performance of the proposed QZS unit is analyzed  
8 qualitatively. The static restoring force obtained from finite element analysis is normalized into the dimensionless  
9 force-displacement ( $f$ - $x$ ) curve shown in Fig. 10(a). This curve exhibits a broad, flat plateau near the static equilibrium  
10 position ( $x=0$ ) and symmetric progressive hardening under large deformation. The sensitivity of the system to payload  
11 mass or static load deviation is first evaluated at a base acceleration  $\hat{a}$  of 0.1 and a damping ratio  $\zeta$  of 0.1. As shown  
12 in Fig. 10(b), the dynamic isolation performance is highly sensitive to payload deviation. At the ideal design point  
13 where the load equals  $F_{QZS}$ , the resonance frequency is extremely low and close to zero, giving the best isolation  
14 performance. However, even a small deviation of  $\pm 2\%$ , corresponding to loads of  $0.98F_{QZS}$  and  $1.02F_{QZS}$ , shifts the  
15 resonance peak noticeably to the right, and the peak transmissibility rises to about 4 dB. When the deviation expands  
16 to  $\pm 10\%$ , the resonance frequency further shifts to farther, and the peak transmissibility jumps to nearly 8 dB. This  
17 high sensitivity arises because a static load deviation moves the static equilibrium away from the QZS condition,  
18 causing the local tangent stiffness to rise rapidly and degenerate into ordinary positive stiffness. This indicates that  
19 precise mass matching or payload regulation is essential in practical engineering applications.

20 At a fixed damping ratio  $\zeta$  of 0.1, the influence of the dimensionless excitation  $\hat{a}$  is shown in Figs. 10(c) and 10(d).  
21 Under weak excitation, the dynamic response amplitude is small, the system remains in the QZS state, and the  
22 resonance peak is smooth with a very low onset isolation frequency. As  $\hat{a}$  increases, the nonlinear geometric  
23 hardening dominated by the odd-order coefficients  $a_3$  and  $a_5$  causes the resonance peak to bend to the right, forming  
24 an unstable branch bounded by two bifurcation points. The effective QZS working stroke is determined by the  
25 bifurcation criterion  $\partial\Omega/\partial X=0$ , which is used to derive the maximum critical dynamic amplitude that completely  
26 avoids nonlinear jump instability. This critical amplitude is given by

$$X_{cri} = \sqrt{\frac{8\zeta}{3\sqrt{3}a_3}} \quad (14)$$

Since the system undergoes symmetric reciprocating vibration about the static equilibrium position, the QZS effective working range without nonlinear jump, as determined by the dynamic analysis, is

$$\Delta X = 2X_{crit} \quad (15)$$

This establishes a dynamic criterion for the QZS working stroke, as illustrated in Fig. 10(d). This working stroke is calculated as 4.74 mm, which is almost consistent with the width of the QZS plateau shown in Fig. 6(b). This consistency verifies that the stiffness threshold adopted to define the QZS plateau is reasonable.

As shown in Fig. 10(e), when the damping ratio  $\zeta$  increases from 0.02 to 0.15 under  $\hat{a}$  is 0.3, the peak transmissibility decreases from 18.5 dB to 4.2 dB. When  $\zeta$  is higher than 0.10, the full-band response returns to a stable state. In the isolation region, smaller damping gives a better high-frequency attenuation rate, so a trade-off between resonance suppression and high-frequency isolation performance must be considered in practical design.

As discussed in the previous static parametric study, the length  $l$  and the height  $h$  are the primary factors governing the QZS characteristics. To accentuate these two parameters' influence on the transmissibility, an excitation amplitude  $\hat{a}$  of 0.3 and a damping ratio  $\zeta$  of 0.05 are adopted. Here, the thickness  $t_b$  of the beam is employed to achieve the dimensionless form of these two parameters. Fig. 10(f) demonstrates that as  $l/t_b$  increases, the bending flexibility of the beam increases gradually, which widens the QZS plateau and slows the stiffness recovery slope under large deformation. As shown in Fig. 10(g), increasing  $l/t_b$  effectively suppresses the rightward shift of the resonance peak and narrows the interval of the unstable branch, thereby lowering the onset isolation frequency and enhancing the operating stability. As illustrated in Fig. 10(h) and Fig. 10(i), increasing  $h/t_b$  simultaneously extends the QZS plateau, which is also beneficial for suppressing the resonance peak and narrowing the interval of the unstable branch.

In summary, the above dimensionless theoretical analysis establishes a quantitative guideline for designing the QZS unit. Adjusting  $h$  and  $l$  can regulate the nonlinear hardening and suppress jump instability. At the same time, keeping the operating excitation within the QZS working stroke ( $X \leq X_{crit}$ ) and matching the optimal damping ratio can ensure that the system achieves highly stable, jump-free low-frequency vibration isolation. Therefore, maximizing the width of the QZS plateau is an effective approach to maintain the isolation bandwidth to handle high-level excitation.

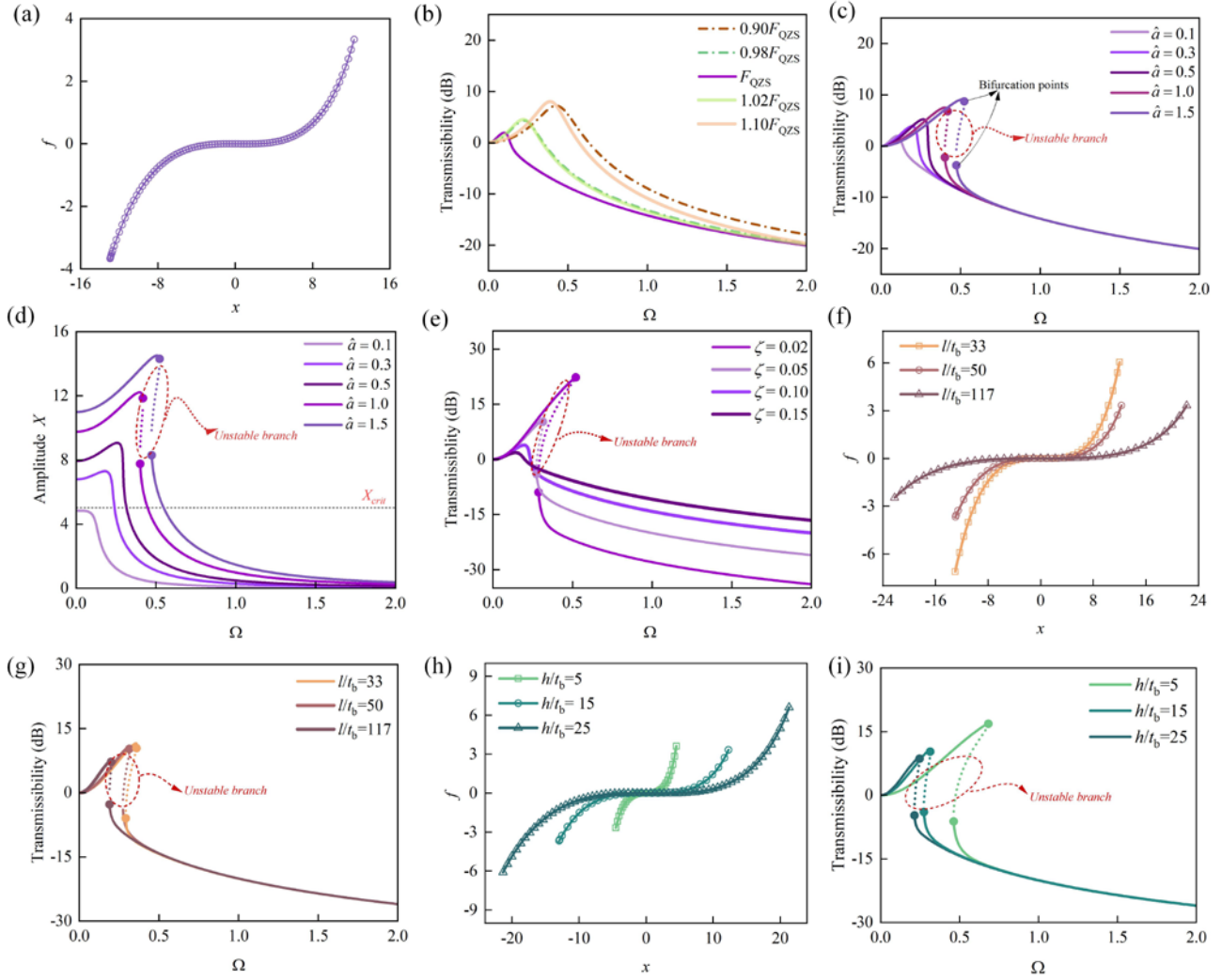


Fig. 10 (a) The dimensionless force-displacement ( $f$ - $x$ ) curve of the QZS unit used in the dynamic model; (b) The transmissibility curves of the QZS unit under different static payload deviations; (c) The transmissibility curves of the QZS unit with different excitations; (d) The amplitude response curves with varying excitations; (e) The transmissibility curves of the QZS unit with different damping ratios; (f) The dimensionless force-displacement ( $f$ - $x$ ) curves with varying  $l/t_b$  values; (g) The transmissibility curves with varying  $l/t_b$  values; (h) The dimensionless force-displacement ( $f$ - $x$ ) curves with varying  $h/t_b$  values,  $h$  is the height of the beam; (i) The transmissibility curves with varying  $h/t_b$  values.

## 4.2 Experimental verification

### 4.2.1 Vibration isolators

To verify the feasibility of the proposed QZS metastructures as vibration isolators, three typical metastructures are developed and fabricated to be vibration isolators, which are termed Type I, Type II, and Type III, respectively, as shown in Fig. 11. Here, all metastructures are constructed by the same beam unit, whose geometric parameters are as follows:  $h$  is 6 mm;  $l$  is 20 mm;  $d$  is 8.3 mm;  $w$  is 10 mm, but the in-plane thickness of the beam unit is 1.2 mm for Type I and is 0.6 mm for Type II and Type III. The sizes of the frames for the three types are carefully selected to be sufficiently large to provide effective support.

Type I is composed of four QZS units connected in parallel. To enhance stability under high load, the four QZS units are assembled into a cubic configuration, with stiffening ribs added to each unit's U-shaped frame to maintain stable boundary conditions. Type II consists of two layers connected in series. Each layer is a QZS unit comprising eight beams connected in parallel and arranged in a centrosymmetric configuration. This design not only widens the QZS plateau but also provides a higher load-bearing capacity. Type III is constructed by connecting one QZS unit cell in series with two parallel-connected QZS units. This configuration achieves a dual-level QZS plateau, making it adaptable for two payloads to cope with different application scenarios. It is noted that two metastructures of this type should be utilized in parallel to enhance the stability in practical applications.

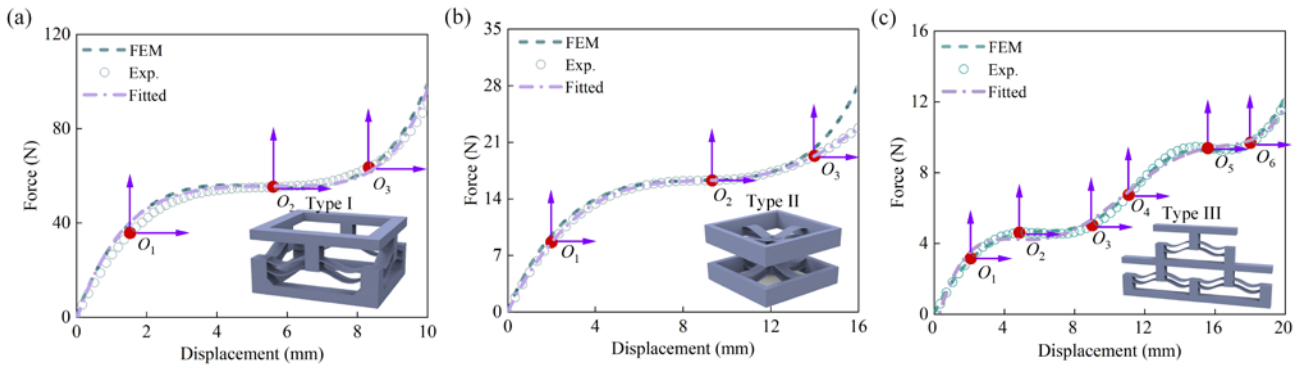


Fig. 11 Three types of metastructures for vibration isolators and the corresponding force-displacement curves, which are obtained by FEM and experiments: (a) Type I consisting of four QZS units connected in parallel under a cubic configuration; (b) Type II with two layers in series comprising eight beam units connected in parallel under a centrosymmetric configuration; (c) Type III constructed by connecting one QZS unit in series with two parallel-connected QZS units.

As shown in Fig. 11, the force-displacement curves of three types of isolators are obtained by experiment and compared with FEM predictions. Both Type I and Type II isolators show a three-stage characteristic in their load-displacement curves: an initial linear positive-stiffness stage, a QZS plateau, and a nonlinear hardening stage due to large geometric deformation. The experimental results indicate that the Type I isolator exhibits a load capacity corresponding to the QZS plateau of approximately 57.0 N. This is because of the thicker beam unit and the parallel configuration. For the Type II isolator, the load capacity is reduced to 16.0 N as a result of the smaller thickness of the beam unit. However, the width of the QZS plateau is widened, thanks to the series connection. Unlike the previous two isolators, the force-displacement curve of the Type III isolator contains two QZS plateaus, with different load capacities of approximately 4.5 N and 9.4 N, respectively. However, its load capacity is inferior to the previous two types due to the relatively small quantity of the beam unit.

#### 4.2.2 Dynamic experiments

To systematically evaluate the vibration transmissibility of the above three QZS isolators, a dynamic experimental testing setup is established, as shown in Fig. 12. The testing setup consists of an air-floating shaker, a zero calibrator,

a controller that can also function as a signal generator, a power amplifier, and a laptop with software used to control the excitation level and record the data. The experimental sample is mounted on the shaker, while a steel block is placed on top of it. Two high-precision accelerometers are bonded to the shaker and the top of the block. These accelerometers are used to measure the input excitation signal of the shaker and the output signal through the isolator, respectively. The mass, which encompasses both the block and the accelerometer, is the inertial mass of this system; in the meantime, the corresponding gravitational force serves as the payload. To investigate the isolation performance under different payload conditions, the mass of the block can be adjusted.

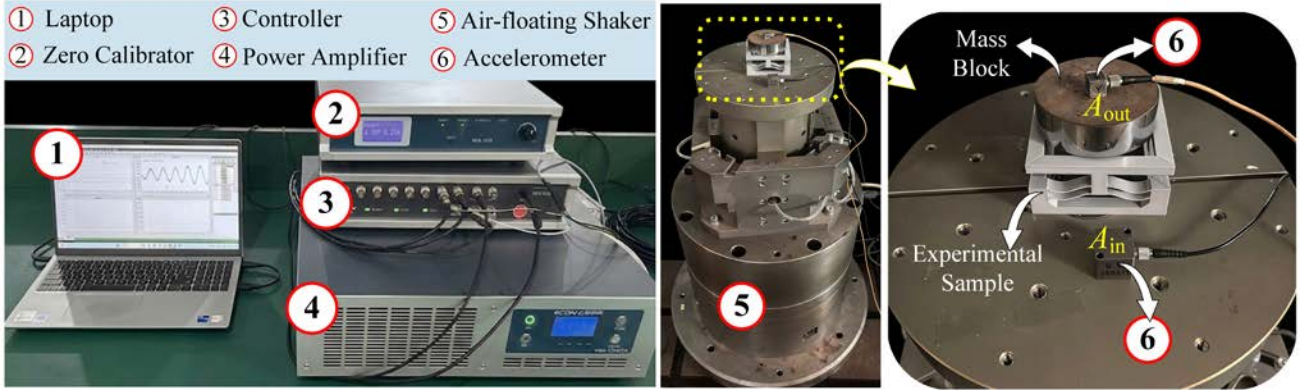


Fig. 12 The dynamic experimental system for frequency sweep test to obtain the frequency-transmissibility responses, and the partial enlarged detail of the experimental sample mounted on the shaker.

To evaluate the isolation performance, a frequency-sweep test is carried out for each experimental sample. The sweeping rate is set at 2 Hz/min within the frequency range from 1 Hz to 30 Hz. Considering that the 3D-printed samples are prone to fatigue or damage under strong dynamic loads, the input excitation amplitude was set at a relatively low level of 0.01  $g$  ( $g=9.8\text{m/s}^2$ ) to ensure the stability and repeatability of the experiment. The transmissibility  $T$ , defined as the ratio of the output acceleration amplitude  $A_{\text{out}}$  to the input acceleration amplitude  $A_{\text{in}}$ , is employed to describe the isolation performance, as follows,

$$T = 20\log_{10}(A_{\text{out}}/A_{\text{in}}) \quad (16)$$

The frequency-transmissibility curves of each experimental sample, which are the key results, will be discussed and compared with the FEM predictions and the results given by the dynamic model. Additionally, the acceleration signal in the time domain at a specific frequency is recorded as auxiliary results.

#### 4.2.3 Dynamic finite element modeling

To further investigate the isolator's dynamic performance, a dynamic finite element model is developed. Taking the Type I isolator as an example, its finite element model is shown in Fig. 13. The upper and lower surfaces of the structure are controlled by two reference points (RP1 and RP2), respectively. The entire analysis process needs two

steps, including a ‘static, general’ step and a ‘dynamic, implicit’ step. In the ‘static, general’ step, RP2 is fully fixed, and a vertical load is applied to RP1 to induce the compression state under payload. An equivalent lumped mass is also applied to RP1. In the ‘dynamic, implicit’ step, the vertical displacement degree of freedom of RP2 is activated, and an acceleration signal is applied to RP2 to simulate the vibration excitation. The frequency-transmissibility curves are obtained by monitoring the acceleration signals of two reference points. Furthermore, the Rayleigh proportional damping is utilized in this model, and the coefficients are calculated to achieve the desired damping ratio at the identified natural frequencies, based on either experimental measurements or assumed values.

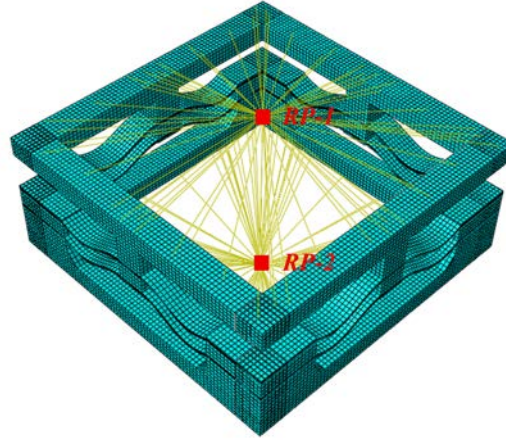


Fig. 13 Dynamic element model of Type I isolator

#### 4.2.4 Results and discussion

This section presents a comprehensive comparison of the experimental, numerical, and theoretical dynamic responses of the three QZS vibration isolators. To compare with the experimental practical conditions with different masses or payloads, the restoring force  $f$  used in the dynamic analysis is expressed as:

$$f(z) = k_1 z + k_2 z^2 + k_3 z^3 + k_4 z^4 + k_5 z^5 \quad (17)$$

where  $k_i$  ( $i=1\sim5$ ) is the coefficient obtained by fitting the experimental force-displacement curves. These coefficients are directly determined by the operating equilibrium positions. Taking Type I as an example, as depicted in Fig. 11(a), the system reaches three equilibrium positions, denoted as  $O_1$ ,  $O_2$ , and  $O_3$ , respectively, under different payloads provided by masses of 3.6 kg, 5.7 kg, and 6.4 kg. The coefficients under different payload conditions are obtained by shifting the original point to the equilibrium positions. These coefficients for all three isolators are provided in Appendix C (Tables C1, C2, C3). Utilizing these coefficients, the dynamic equation Eq (10) can be solved numerically by MATLAB’s ode45 solver, and the obtained transmissibility curves can be compared with experimental results for validation.

For the Type I QZS isolator, dynamic tests are conducted at three equilibrium positions:  $O_1$ ,  $O_2$ , and  $O_3$ . The starting

point of the QZS plateau ( $O_2$ ) is selected as the experimental equilibrium position, and this selection is maintained throughout the subsequent tests. The measured transmissibility-frequency curves for the three positions are compared with the FEM and analytical predictions in Fig. 15(a). Under a relatively small payload at position  $O_1$ , the isolator operates in the positive-stiffness regime. The experimental results show a resonance peak of 32.0 dB and an onset isolation frequency of 15.4 Hz. When the payload is increased to the QZS plateau ( $O_2$ ), the system enters a state of low dynamic stiffness, leading to substantially improved isolation performance. Specifically, the resonance peak drops to 18.8 dB, and the onset frequency of effective isolation decreases to 3.3 Hz. When the payload is further increased to position  $O_3$ , the isolator exhibited geometric hardening due to large deformation, leading to a return to positive-stiffness response, with a resonance peak of 28.9 dB and an onset isolation frequency of 9.0 Hz.

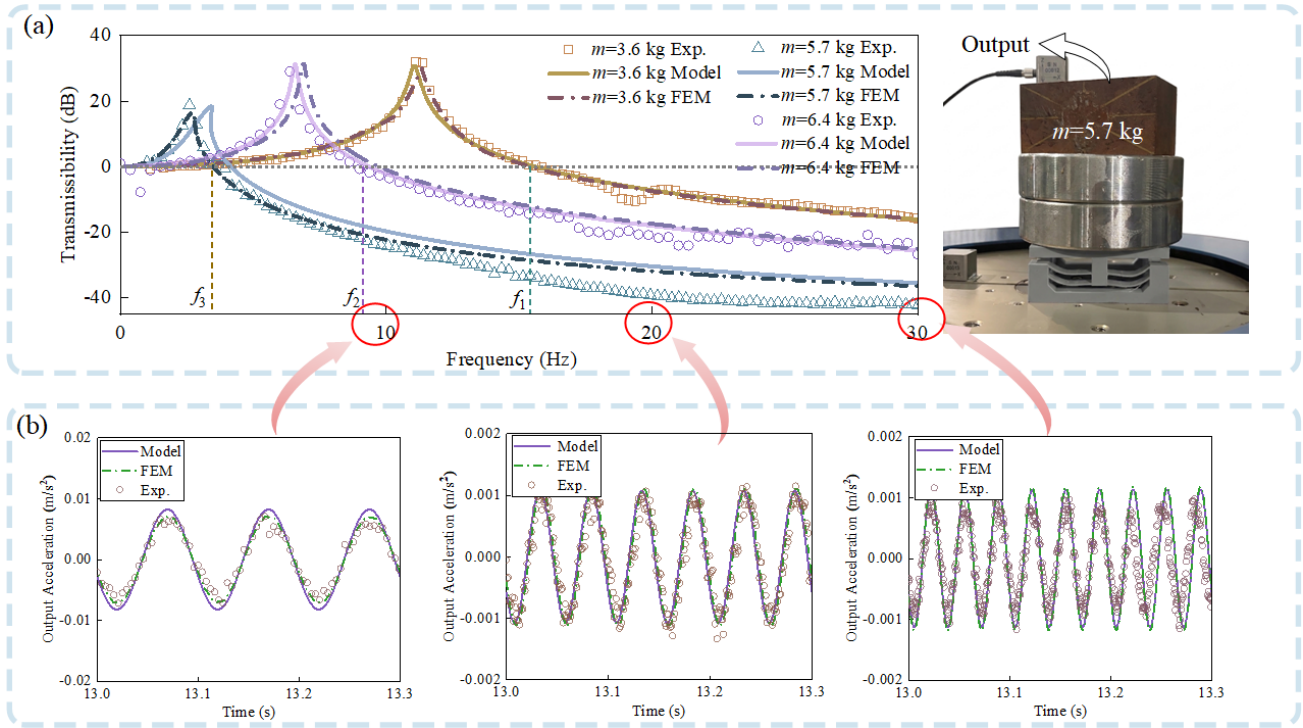


Fig. 14. (a) Transmissibility-frequency curves of the Type I isolator under three static payloads, and the experimental equilibrium QZS state; (b) The output acceleration signals in the time domain under three fixed frequency excitations, and the frequencies are 10 Hz, 20 Hz, and 30 Hz, respectively.

To further illustrate the dynamic performance of the isolator under its QZS operating state ( $O_2$ ), tests at specific frequencies (10 Hz, 20 Hz, and 30 Hz) are performed following the sweep-frequency analysis. The time-domain output acceleration signals are recorded and compared with the results given by the FEM and the model, as shown in Fig. 14(b). The results show good agreement among all three methods in terms of response amplitude and attenuation characteristics across the three excitation frequencies. The experimental results show that under an excitation frequency of 10 Hz, the isolator achieves a vibration isolation efficiency of approximately 94%, and it is

further reduced to 99 % at a frequency of 30 Hz, demonstrating its excellent performance.

For the Type II QZS isolator, the masses of 0.8 kg, 1.6 kg, and 1.9 kg were used in the frequency-sweep tests, respectively, and the frequency-transmissibility curves are presented in Fig. 15(a). The experimental results indicate that under loads of 0.8 kg and 1.9 kg, the isolator exhibited resonance peaks of 25.3 dB and 23.7 dB, with onset isolation frequencies of 15.4 Hz and 9.5 Hz, respectively. Under the payload of 1.6 kg corresponding to the QZS state, the experimental results showed a transmissibility peak of 13.2 dB and an onset isolation frequency of 3.2 Hz. The above results have verified the feasibility of the proposed QZS metastructure as a vibration isolator when the applied external static load is matched with the QZS payload.

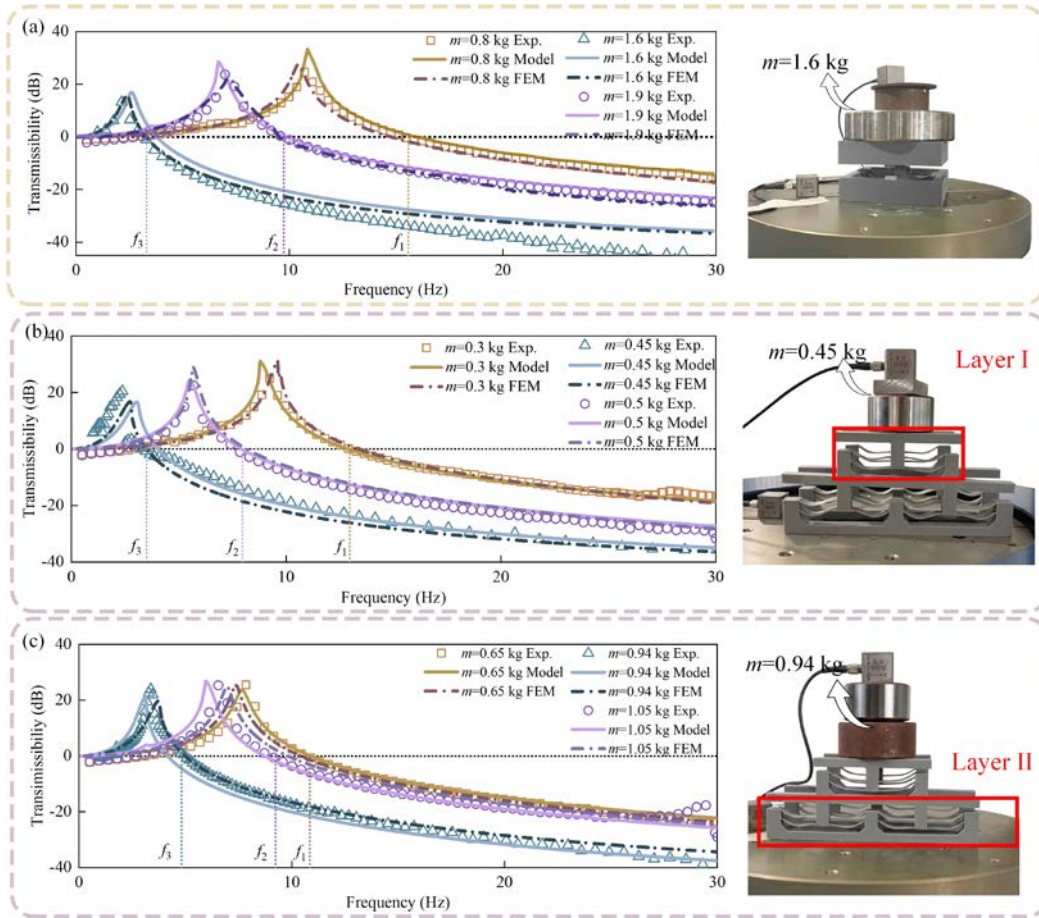


Fig. 15. Transmissibility-frequency curves under different payloads, and the experimental equilibrium QZS state: (a) Type II isolator; (b) Corresponding to layer I of Type III isolator; (c) Corresponding to layer II of Type III isolator.

For the Type III QZS isolator with two QZS plateaus at loads of 4.5 N and 9.4 N, the payload masses of 0.45 kg and 0.94 kg are employed to target these two QZS plateaus during the sweep-frequency tests. For comparative analysis, equilibrium points are selected in the positive-stiffness regimes on both sides of each QZS plateau, so payload masses of 0.3 kg, 0.5 kg, 0.65 kg, and 1.05 kg are applied accordingly, and the results are presented in Figs 15(b) and (c). They indicate that under the two aforementioned QZS operating conditions, the two-layer isolator exhibited

transmissibility peaks of 20.4 dB and 22.6 dB, with corresponding onset isolation frequencies of 3.3 Hz and 4.8 Hz. To illustrate the validity of the model and FEM, three onset isolation frequencies ( $f_1$ ,  $f_2$ , and  $f_3$ , shown in Figs. 15 and 16) of the three types of isolators are systematically compiled and summarized in Table 1. The average error of the two methods is 5.0 % and 3.7 %, respectively.

Table 1. Comparison of the onset isolation frequencies for different isolators obtained by experiments, the model, and FEM.

| QZS<br>isolator     | Onset isolation frequency (Hz) |       |      |       |       |     |             |       |     |
|---------------------|--------------------------------|-------|------|-------|-------|-----|-------------|-------|-----|
|                     | $f_1$                          |       |      | $f_2$ |       |     | $f_3$ (QZS) |       |     |
|                     | Exp.                           | Model | FEM  | Exp.  | Model | FEM | Exp.        | Model | FEM |
| Type I              | 15.4                           | 15.4  | 15.6 | 9.0   | 9.0   | 9.6 | 3.3         | 3.8   | 3.4 |
| Type II             | 15.5                           | 15.7  | 15.0 | 9.6   | 9.8   | 9.5 | 3.1         | 3.2   | 3.3 |
| Type III (layer I)  | 13.1                           | 12.9  | 13.2 | 7.7   | 7.9   | 8.1 | 3.3         | 3.8   | 3.5 |
| Type III (layer II) | 10.5                           | 10.9  | 9.8  | 8.8   | 8.6   | 9.0 | 4.7         | 4.1   | 4.6 |

To further investigate the nonlinear dynamic response under stronger excitation, forward and backward frequency-sweep experiments were conducted on the Type I and Type II isolators at base accelerations of 0.03  $g$  and 0.08  $g$ . The measured transmissibility curves are compared with the model predictions in Fig. 16. In the model predictions, the resonance peak bends noticeably toward higher frequencies, forming an unstable branch. In experimental observations, the response is unable to track the unstable branch. Instead, a distinct jump phenomenon and sweep-direction hysteresis emerge between the forward jump frequency  $f_F$  and the backward jump frequency  $f_B$ . As the acceleration increases from 0.03  $g$  to 0.08  $g$ , the rightward bending of the resonance peak becomes more significant for both isolators, and the unstable jump region between  $f_F$  and  $f_B$  broadens considerably, indicating that the vibration isolation performance is no longer stable. Comparing the two isolators, the Type II, which benefits from a wider QZS plateau and a smaller effective inertia mass due to the series configuration, exhibits noticeably weaker nonlinear hardening and a narrower gap between the forward and backward jump frequencies than the Type I under both excitation levels. This experimentally confirms that widening the QZS plateau is a key design criterion for suppressing nonlinear jump instability and maintaining a stable low-frequency isolation bandwidth under strong vibration environments.

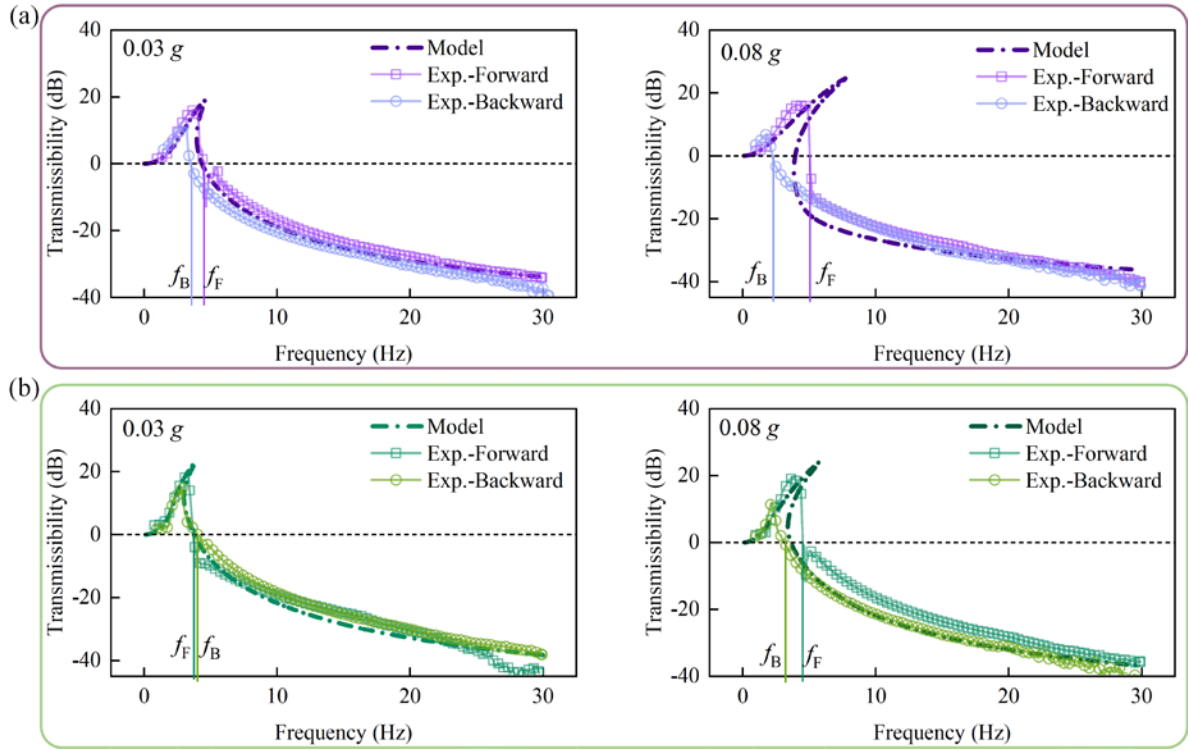


Fig. 16. Transmissibility-frequency curves under forward and backward frequency sweeps obtained by the model and experiments: (a) Type I isolator; (b) Type II isolator.

### 4.3 Composite QZS unit

To address the inherent limitations of 3D-printed prototypes, such as inadequate material properties and the inability to withstand high excitation, this section further expands the application of the proposed QZS unit in the field of composite materials. These composite materials offer excellent fatigue resistance, creep resistance, and high specific strength, which can ensure long-term stability in isolation performance while realizing a lightweight design. Benefiting from the continuous configuration of the beam, the composite QZS unit can be obtained through a molding process. Building on the findings before, a pair of dedicated molds is designed based on the specified parameters:  $h$  is 9 mm,  $l$  is 40 mm,  $d$  is 17 mm, and  $w$  is 30 mm. The carbon fiber reinforced polymer (CFRP) prepreg is tailored into the desired size, and then stacked to obtain an uncured composite laminate. This uncured laminate is placed between two molds, and a vacuum bag molding process is performed to cure the pre-shaped laminate. Finally, the cured laminate is assembled with supports to obtain a composite QZS unit, as shown in Fig. 17(a).

A static experiment is conducted on this composite QZS unit, and its force–displacement curve shows an obvious QZS feature that is consistent with the FEM prediction, as depicted in Fig. 17(b). To further investigate the effect of layup design, three stacking sequences ( $[0/90]_s$ ,  $[90/0]_s$ , and  $[45/-45]_s$ ) were also fabricated and tested. The results reveal that the load-bearing capacity can be tuned over a wide range by adjusting the ply angles. The  $[0/90]_s$  layup,

1 with 0° fibers aligned along the beam length, provides high bending stiffness and raises the QZS plateau load to  
2 approximately 11.8 N. In contrast, the [90/0]<sub>s</sub> layup reduces the load-bearing capacity to about 2.5 N. The [45/-45]<sub>s</sub>  
3 layup adopted in this work offers moderate bending stiffness and produces a smooth, stable QZS plateau with a load  
4 capacity of approximately 3.2 N, making it well suited for vibration isolation measurements. These results verify that  
5 the proposed continuous cosine beam constitutes a scalable design paradigm, whose applicability is not restricted to  
6 specific materials. Furthermore, the load-bearing performance of the composite QZS unit can be flexibly customized  
7 via layup design.

8 Owing to the CFRP material's excellent fatigue resistance, repeated tests under a payload of 320 g, corresponding to  
9 the QZS plateau, can be conducted for this composite QZS unit under high excitation levels. The model, based on  
10 the fitting force-displacement curve shown in Fig. 17(c), is also employed to predict transmissibility-frequency  
11 responses. Here, the stiffness coefficients are as follows:  $k_1$  is 1.41 N/m;  $k_3$  is  $4.79 \times 10^6$  N/m<sup>3</sup>;  $k_5$  is  $9.66 \times 10^9$  N/m<sup>5</sup>.  
12 As shown in Fig. 17(d), the results of frequency-sweep tests under different acceleration levels (0.01 g, 0.05 g, 0.10  
13 g, and 0.15 g) are compared with the predictions of the model. When the acceleration level is 0.01 g, the frequency-  
14 transmissibility curve exhibits a linear feature, and the experimental results demonstrate that the onset isolation  
15 frequency is around 2.8 Hz regardless of the sweeping direction, with the data showing excellent agreement with the  
16 model. When the acceleration level is increased, the nonlinear feature with the rightward transmissibility peak  
17 gradually emerges, as analyzed previously. The onset isolation frequencies corresponding to the forward and  
18 backward sweeping directions are termed  $f_F$  and  $f_B$ .  $f_F$  increases with the acceleration level, but  $f_B$  is relatively stable.  
19 The region between  $f_F$  and  $f_B$  is unstable, which is broadened by the acceleration level.

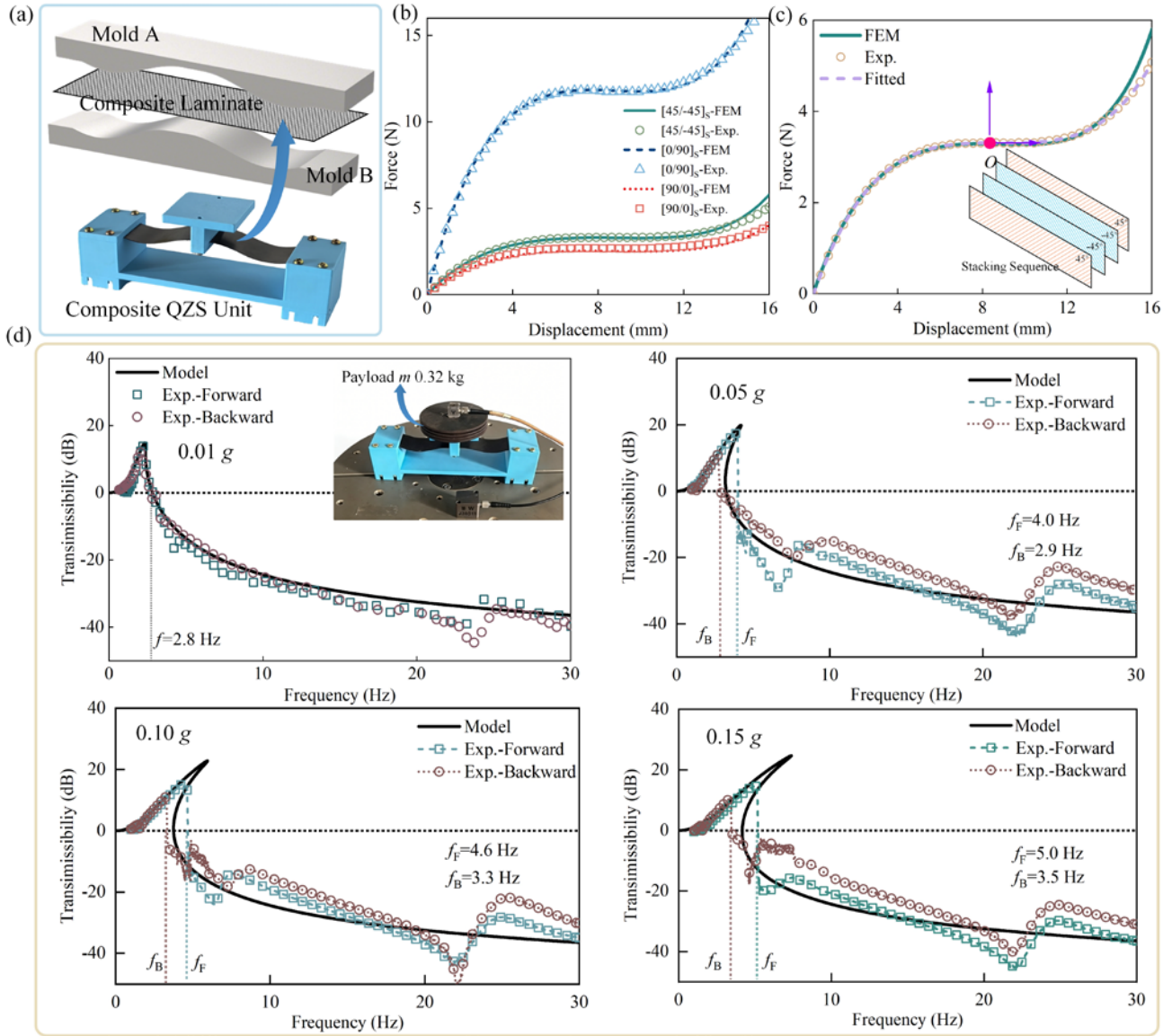


Fig. 17. (a) The fabrication schematic diagram of the composite QZS unit; (b) Force-displacement curves of the composite QZS unit with different stacking sequences obtained by FEM and experiments; (c) Force-displacement curves of the composite QZS unit with  $[45/-45]_s$ ; (d) The frequency-transmissibility curves of the composite QZS unit under different acceleration levels, and the results are obtained by experiments in two sweeping directions and by the model.

The experimental results are in agreement with the model's predictions within the low-frequency range. However, near 23 Hz, the experimental transmissibility curve exhibits a significant local fluctuation. As verified by FEM, the second natural frequency of this composite QZS unit with twisting mode coincides with the vicinity of 23 Hz. Therefore, this fluctuation is the resonant response to this second natural frequency, which can not be captured by this single-degree-of-freedom model. However, the model has given reasonable results about the first natural resonance response.

Table 2 presents a comprehensive comparison between the proposed QZS isolator with continuous cosine-shaped

beam units and other typical isolators with monolithic beam units in terms of performance and materials. The results highlight the advantages of the proposed design, including a low onset isolation frequency and structural simplicity. Future work will focus on the CFRP-based QZS isolator, with the aims of enhancing its load-bearing capacity and fatigue resistance, while further lowering the onset isolation frequency and broadening the effective isolation bandwidth.

Table 2. Comparison between the proposed QZS isolator and representative QZS isolators with monolithic beam units

| Ref. No.  | Structural mode               | Payload (kg) | Onset isolation frequency(Hz) | Materials |
|-----------|-------------------------------|--------------|-------------------------------|-----------|
| [82]      | Bottom reinforced cosine beam | 0.3          | 9.5                           | TPU       |
| [83]      | Cosine-shaped curved beam     | 10.0         | 8.7                           | Resin     |
| [84]      | Twisted cosine beam           | 1.72         | 12                            | TPU       |
| [89]      | Straight-arch-straight beam   | 0.7          | 5                             | Nylon     |
| [91]      | Inclined folded beam          | 0.8          | 10                            | Resin     |
| This work | Continuous cosine beam        | 5.7          | 3.3(Type I)                   | PLA       |
|           |                               | 0.32         | 2.8                           | CFRP      |

## 5. Conclusions

In this work, a monolithic QZS beam with a continuous cosine shape, which is described solely by three geometric parameters  $l$ ,  $h$ , and  $d$ , is proposed for constructing QZS metastructures for low-frequency vibration isolation. From the static analysis, it is found that the ratio of the extended length  $d$  to the length  $l$  can serve as a criterion for the QZS characteristic, and the QZS characteristic exists when this ratio is approximately 0.43, regardless of the value of the height  $h$ . The payload corresponding to the QZS plateau decreases as  $l$  increases but increases as  $h$  increases, and the width of the QZS plateau increases with both  $h$  and  $l$ . After the shape of the QZS beam unit is determined, the out-of-plane width  $w$  and the in-plane thickness  $t_b$  can be effectively utilized to adjust the payload. Due to the simple continuous configuration, the QZS metastructure based on this beam demonstrates strong designability and reconfigurability. The load capacity and QZS plateaus of the metastructure can be customized by QZS unit arrangement strategies and modular design.

The feasibility of the QZS unit as a vibration isolator is qualitatively analyzed. It is found that nonlinear features, such as jump instabilities, are enhanced by the excitation amplitude and can be controlled by the damping ratio and the width of the QZS plateau. Subsequently, the vibration isolation performance is verified through three types of the proposed QZS metastructures. The results indicate that the onset isolation frequency, corresponding to the QZS plateau, is significantly lower than half of that in the positive-stiffness status, and the isolation efficiency can reach approximately 94 % when the frequency is 10 Hz. Furthermore, the two-step QZS plateau strategy is verified to be

effective for adapting to varying loading conditions. Additionally, it is observed that wider QZS plateaus can suppress the degradation of vibration isolation performance instability at high excitation levels.

Due to its monolithic configuration, the QZS unit is explored with a carbon fiber-reinforced composite laminate with high specific strength and fatigue resistance. The nonlinear isolation performance under a relatively high excitation level is investigated. It is found that the onset isolation frequency in the forward sweeping direction is higher than that in the backward sweeping direction, and this difference is augmented by the excitation level. Nevertheless, at an acceleration of 0.15  $g$ , the onset isolation frequency is below 5 Hz regardless of the sweeping direction.

The proposed QZS metastructure offers a novel strategy for low-frequency vibration isolation. The simple shape of the beam unit enables the exploration of composite materials and presents a reliable, high-capacity, and lightweight solution for low-frequency protection in complex vibration environments. In the future, more efforts will be devoted to this composite QZS metastructure to investigate its durability and stability as isolators.

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## Appendix A

This appendix presents the beam constraint model (BCM) equations for each element, along with the associated discretization procedure. A total of  $6N$  equations are derived, covering all chain-beam constraints, equilibrium equations, and displacement boundary conditions.

The nominal length of each element can be calculated as:

$$L_{xi} = x_{i+1} - x_i, \quad L_{yi} = y_{i+1} - y_i \quad (A1)$$

where

$$L_i = L_{xi} \cos \beta_i + L_{yi} \sin \beta_i \quad (A2)$$

The thickness and the approximate curvature of the  $i$ -th element are normalized with respect to its own length as:

$$t_i = \frac{t_b}{L_i}, \quad \kappa_i \approx \frac{K_i + K_{i+1}}{2} L_i \quad (A3)$$

The discretization introduces  $6N$  intermediate parameters, i.e., the deflection and the load parameters of each element.

The parameters of the  $i$ -th element are normalized with respect to  $L_i$  as:

$$p_i = \frac{P_i L_i^2}{EI}, \quad f_i = \frac{F_i L_i^2}{EI}, \quad m_i = \frac{M_i L_i}{EI}, \quad \lambda_i = \frac{\Lambda_i}{L_i}, \quad \delta_i = \frac{\Delta_i}{L_i}, \quad \alpha_i = \alpha_i \quad (A4)$$

where  $F_i$ ,  $P_i$ , and  $M_i$  are denoted the radial force, the tangential force, and the end moment applied on the free end of the  $i$ -th element, and  $\Lambda_i$ ,  $\Delta_i$ , and  $\alpha_i$  are the corresponding tangential and radial deflections and the end slope, all are measured with respect to its local coordinate frame  $O_i X_i Y_i$ ;  $E$  and  $I$  are the Young's modulus of the material and the area moment of inertia of the beam cross-section, respectively.

To obtain the above  $6N$  unknown quantities,  $6N$  equations are required, as follows:

(i) Beam constraint model (BCM) equations for each element:

$$\begin{bmatrix} f_i \\ m_i \end{bmatrix} = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} \delta_i \\ \alpha_i \end{bmatrix} + p_i \begin{bmatrix} 6/5 & -1/10 \\ -1/10 & 2/15 \end{bmatrix} \begin{bmatrix} \delta_i \\ \alpha_i \end{bmatrix} + p_i^2 \begin{bmatrix} -1/700 & 1/1400 \\ 1/1400 & -11/6300 \end{bmatrix} \begin{bmatrix} \delta_i \\ \alpha_i \end{bmatrix} + p_i \begin{bmatrix} \kappa_i/2 \\ \kappa_i/12 \end{bmatrix} \quad (A5)$$

$$\begin{aligned} \lambda_i &= \frac{t_i^2 p_i}{12} - \frac{\kappa_i}{2} \delta_i - \frac{\kappa_i}{12} \alpha_i - \frac{1}{2} \begin{bmatrix} \delta_i & \alpha_i \end{bmatrix} \begin{bmatrix} 6/5 & -1/10 \\ -1/10 & 6/5 \end{bmatrix} \begin{bmatrix} \delta_i \\ \alpha_i \end{bmatrix} - \\ & p_i \begin{bmatrix} \delta_i & \alpha_i \end{bmatrix} \begin{bmatrix} -1/700 & 1/1400 \\ 1/1400 & -11/6300 \end{bmatrix} \begin{bmatrix} \delta_i \\ \alpha_i \end{bmatrix} + p_i \frac{\kappa_i}{360} \alpha_i + p_i \frac{\kappa_i^2}{720} \end{aligned} \quad (A6)$$

(ii) Static equilibrium equations:

1 Static equilibrium between the beam tip and the 1st element:

$$2 \quad P_0 = \frac{p_1 EI}{L_1^2}, \quad F_0 = \frac{f_1 EI}{L_1^2}, \quad M_0 = \frac{m_1 EI}{L_1} \quad (A7)$$

3 And static equilibrium between 1st and  $i$ -th elements ( $i=2, \dots, N$ ):

$$4 \quad \begin{bmatrix} \cos \theta_i & \sin \theta_i & 0 \\ -\sin \theta_i & \cos \theta_i & 0 \\ (1 + \lambda_i) & -(0.5\kappa_i + \delta_i) & 1 \end{bmatrix} \begin{bmatrix} f_i \\ p_i \\ m_i \end{bmatrix} = \begin{bmatrix} f_1 L_i^2 / L_1^2 \\ p_1 L_i^2 / L_1^2 \\ m_{i-1} L_i / L_{i-1} \end{bmatrix} \quad (A8)$$

5 where  $\theta_i$  is the rotation angle of the  $i$ -th element's coordinate frame to the global coordinate frame that can be directly  
6 expressed by the unknowns:

$$7 \quad \theta_1 = 0, \quad \theta_i = \beta_i + \sum_{k=1}^{i-1} \alpha_k \quad (i=2, \dots, N) \quad (A9)$$

8 (iii) Geometric constraint equations

$$9 \quad \begin{cases} \sum_{i=1}^N [(1 + \lambda_i) L_i \cos \theta_i - (0.5\kappa_i + \delta_i) L_i \sin \theta_i] = X_0 \\ \sum_{i=1}^N [(1 + \lambda_i) L_i \sin \theta_i + (0.5\kappa_i + \delta_i) L_i \cos \theta_i] = Y_0 \\ \beta_{N+1} + \sum_{i=1}^N \alpha_i = \theta_0 \end{cases} \quad (A10)$$

10 (iv) Strain energy evaluation: Based on the solved internal forces and deflections of each discrete element, the strain  
11 energy stored in a single beam is calculated by integrating the elastic strain energy densities. The axial strain energy  
12  $U_a$  resulting from the mid-plane tension/compression is given by:

$$13 \quad U_a = \sum_{i=1}^N \frac{P_i^2 L_i}{2EA} = \sum_{i=1}^N \frac{1}{2} EA \frac{\Lambda_i^2}{L_i} \quad (A11)$$

14 where  $A$  is the cross-sectional area.

15 Considering the linear distribution of the bending moment along each element, the bending strain energy  $U_b$  is derived  
16 by exact analytical integration:

$$17 \quad U_b = \sum_{i=1}^N \int_0^{L_i} \frac{(M_i - F_i s)^2}{2EI} ds = \sum_{i=1}^N \frac{L_i}{2EI} (M_i^2 - M_i F_i L_i + \frac{1}{3} F_i^2 L_i^2) \quad (A12)$$

18 The total strain energy of a single cosine beam is the sum of the axial and bending strain energies:

$$19 \quad U = U_s + U_b \quad (A13)$$

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## Appendix B

This appendix presents the harmonic balance method (HBM) utilized to solve the dynamic equation. To solve the dynamic equation, the harmonic balance method (HBM) is utilized. It is assumed that the steady-state response of the system takes on a single-harmonic form,

$$x(\tau) = X \cos(\Omega \tau + \phi) = X \cos \theta \quad (\text{B1})$$

where  $X$  is the dimensionless response amplitude,  $\phi$  is the phase difference, and  $\theta = \Omega \tau + \phi$ . Then, equation (12) is given by

$$(-\Omega^2 X + a_1 X + \frac{3}{4} a_3 X^3 + \frac{5}{8} a_5 X^5) \cos \theta - (2\zeta \Omega X) \sin \theta = \Omega^2 a \cos \phi \cos \theta + \Omega^2 a \sin \phi \sin \theta \quad (\text{B2})$$

Therefore,

$$\begin{cases} X(K_{eq} - \Omega^2) = \Omega^2 a \cos \phi \\ -2\zeta \Omega X = \Omega^2 a \sin \phi \end{cases} \quad (\text{B3})$$

where the amplitude-dependent equivalent linear stiffness  $K_{eq}$  is defined as,

$$K_{eq}(X) = a_1 + \frac{3}{4} a_3 X^2 + \frac{5}{8} a_5 X^4 \quad (\text{B4})$$

Adding the two squared equations eliminates  $\phi$  and yields an implicit equation for the response amplitude  $X$ ,

$$X^2[(K_{eq} - \Omega^2)^2 + (2\zeta \Omega)^2] = (\Omega^2 a)^2 \quad (\text{B5})$$

The absolute displacement of the mass,  $x_{abs}$ , is the vector sum of its relative displacement,  $x$ , and the base displacement,  $y$ . The total force transmitted to the mass, comprising the damping force and the restoring force, has an amplitude,

$$F_{tr} = \sqrt{(K_{eq} X)^2 + (2\zeta \Omega X)^2} = X \sqrt{K_{eq}^2 + (2\zeta \Omega)^2} \quad (\text{B6})$$

According to Newton's second law, the absolute displacement amplitude satisfies  $X_{abs} = F_{tr} / \Omega^2$ . Substituting the expression for  $X$  yields,

$$X_{abs} = \frac{\hat{a}}{\sqrt{(K_{eq} - \Omega^2)^2 + (2\zeta \Omega)^2}} \cdot \frac{\sqrt{K_{eq}^2 + (2\zeta \Omega)^2}}{\Omega^2} \quad (\text{B7})$$

The acceleration transmissibility  $T$  is obtained as:

$$T = 20 \log\left(\frac{\Omega^2 X_{abs}}{\hat{a}}\right) = 20 \log_{10}\left(\sqrt{\frac{K_{eq}^2 + (2\zeta \Omega)^2}{(K_{eq} - \Omega^2)^2 + (2\zeta \Omega)^2}}\right) \quad (\text{B8})$$

## Appendix C

This appendix presents the essential nonlinear stiffness and damping coefficients utilized to solve the dynamic equations, along with their corresponding operating equilibrium positions. A series of tabulated data are provided, covering the specific force-displacement fitting parameters ( $k_1$  to  $k_5$ ) and damping coefficients ( $c$ ) for the Type I, Type II, and Type III isolators under different payload conditions.

Table C1. Coefficients used in the model for the Type I Isolator under different payload conditions.

| Equilibrium position | Mass $m$ (kg) | Coefficients of the restoring force $f$ |                           |                           |                           |                           | Damping coefficient $c$ (N·s/m) |
|----------------------|---------------|-----------------------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------------|
|                      |               | $k_1$ (N/m)                             | $k_2$ (N/m <sup>2</sup> ) | $k_3$ (N/m <sup>3</sup> ) | $k_4$ (N/m <sup>4</sup> ) | $k_5$ (N/m <sup>5</sup> ) |                                 |
| $O_1$                | 3.6           | $0.17 \times 10^5$                      | $-0.68 \times 10^7$       | $1.51 \times 10^9$        | $-1.87 \times 10^{11}$    | $1.01 \times 10^{13}$     | 4.53                            |
| $O_2$                | 5.7           | $0.02 \times 10^5$                      | $-0.10 \times 10^7$       | $0.28 \times 10^9$        | $-0.62 \times 10^{11}$    | $1.01 \times 10^{13}$     | 7.61                            |
| $O_3$                | 6.4           | $0.11 \times 10^5$                      | $-0.57 \times 10^7$       | $1.43 \times 10^9$        | $-1.82 \times 10^{11}$    | $1.01 \times 10^{13}$     | 8.03                            |

Table C2. Coefficients used in the model for the Type II Isolator under different payload conditions.

| Equilibrium position | Mass $m$ (kg) | Coefficients of the restoring force $f$ |                           |                           |                           |                           | Damping coefficient $c$ (N·s/m) |
|----------------------|---------------|-----------------------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------------|
|                      |               | $k_1$ (N/m)                             | $k_2$ (N/m <sup>2</sup> ) | $k_3$ (N/m <sup>3</sup> ) | $k_4$ (N/m <sup>4</sup> ) | $k_5$ (N/m <sup>5</sup> ) |                                 |
| $O_1$                | 0.8           | $3.67 \times 10^3$                      | $-1.03 \times 10^5$       | $1.68 \times 10^8$        | $-1.34 \times 10^{10}$    | $4.07 \times 10^{11}$     | 2.18                            |
| $O_2$                | 1.6           | $0.44 \times 10^3$                      | $-0.48 \times 10^5$       | $4.35 \times 10^6$        | $1.85 \times 10^9$        | $4.07 \times 10^{11}$     | 3.90                            |
| $O_3$                | 1.9           | $1.28 \times 10^3$                      | $4.96 \times 10^5$        | $1.12 \times 10^8$        | $-1.10 \times 10^{10}$    | $4.07 \times 10^{11}$     | 4.83                            |

Table C3. Coefficients used in the model for the Type III Isolator under different payload conditions.

| Equilibrium position | Mass $m$ (kg) | Coefficients of the restoring force $f$ |                           |                           |                           |                           | Damping coefficient $c$ (N·s/m) |
|----------------------|---------------|-----------------------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------------|
|                      |               | $k_1$ (N/m)                             | $k_2$ (N/m <sup>2</sup> ) | $k_3$ (N/m <sup>3</sup> ) | $k_4$ (N/m <sup>4</sup> ) | $k_5$ (N/m <sup>5</sup> ) |                                 |
| $O_1$                | 0.3           | $1.06 \times 10^3$                      | $-4.62 \times 10^5$       | $8.27 \times 10^7$        | $-5.54 \times 10^9$       | $1.26 \times 10^{11}$     | 0.22                            |
| $O_2$                | 0.45          | $0.14 \times 10^3$                      | $0.93 \times 10^5$        | $0.92 \times 10^7$        | $-2.73 \times 10^9$       | $1.26 \times 10^{11}$     | 0.29                            |
| $O_3$                | 0.5           | $0.66 \times 10^3$                      | $7.50 \times 10^5$        | $-1.09 \times 10^7$       | $-1.07 \times 10^9$       | $1.26 \times 10^{11}$     | 0.32                            |
| $O_4$                | 0.65          | $0.80 \times 10^3$                      | $0.65 \times 10^3$        | $-1.45 \times 10^7$       | $9.14 \times 10^7$        | $1.26 \times 10^{11}$     | 1.29                            |
| $O_5$                | 0.94          | $0.32 \times 10^3$                      | $-0.86 \times 10^5$       | $0.57 \times 10^7$        | $2.52 \times 10^9$        | $1.26 \times 10^{11}$     | 1.59                            |
| $O_6$                | 1.05          | $1.60 \times 10^3$                      | $5.50 \times 10^5$        | $8.92 \times 10^7$        | $5.72 \times 10^9$        | $1.26 \times 10^{11}$     | 3.50                            |