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# On bullwhip in a family of Order-Up-To policies with ARMA(2,2) demand and arbitrary lead-times

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## Abstract

A number of papers have recently appeared that investigate the “bullwhip effect” (the variance amplification of ordering decisions in the supply chain) produced by the order-up-to replenishment policy. An adapted policy, with a proportional inventory position feedback controller, has shown improved “bullwhip” behaviour. The dynamic behaviour of this so called “proportional order-up-to” policy has been investigated for arbitrary lead-times and several demand models such as i.i.d. demand and autoregressive moving average AR(1) and ARMA(1,1) models. It has been shown that, for a correct choice of the feedback parameter, the bullwhip effect can always be avoided. However, less attractive properties of this policy have also become clear.

Herein, we investigate the behaviour of the proportional order up to policy for ARMA(2,2) demand with arbitrary lead-times. In order to compensate for possible weaknesses of the proportional OUT policy we propose another replenishment rule that accounts for the characteristics of the demand in a superior manner. The characteristics of both policies are compared for several parameter settings of the ARMA(2,2) model. Finally, the consequences of our full-state-feedback order-up-to policy are discussed.

**Key words:** Bullwhip effect, ARMA(2,2) demand, Proportional Order-Up-To policy, Full-state-feedback Order-Up-To policy

## 1. Introduction

The bullwhip effect (the variance amplification of ordering decisions in the supply chain) is a widespread phenomenon throughout industry. In industrial supply chains a variance amplification of the ordering rates in the chain of more than 3 to 5 times can often be observed. The order-up-to policy is a standard ordering algorithm in many ERP systems that is used to achieve customer service and balance inventory and capacity investments (Gilbert, 2005). This policy is often used by companies to coordinate orders from suppliers, where setup costs may be reasonably ignored.

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Conceptually, the policy is easy to understand. Periodically, we review our inventory position and place an "order" to bring the inventory position "-up-to" a defined level. The influence of demand characteristics on the bullwhip effect generated by the order-up-to replenishment policy has been studied in a large number of papers. For instance, see Veinott (1965), Johnson and Thompson (1975), and more recently by Graves (1999), Lee, So and Tang (2000), Chen, Drezner, Ryan and Simchi-Levi (2000), Aviv (2003), Zhang (2004) and Gilbert (2005). This literature suggests that the value of the parameters of a demand model has a large influence on the bullwhip effect. For certain classes of demand patterns there is a bullwhip effect and for others there is not.

An adapted policy is the so-called "proportional order-up-to" policy. This policy exploits inventory feedback information in a more sophisticated manner and has shown improved 'bullwhip' behaviour. See, for instance, Disney and Towill (2003) and Dejonckheere, Disney, Lambrecht and Towill (2003). The dynamic behaviour of the proportional order-up-to policy has been investigated for arbitrary lead-times and several demand models such as i.i.d. demand and autoregressive, AR(1) and autoregressive moving average ARMA(1,1) models. Gaalman and Disney (2006) study ARMA(1,1) demand for unit lead-times; Gaalman and Disney (2005) investigate the general lead-time case. Gaalman (2006) investigates the general ARMA(*p*,*q*) demand model for unit lead-times and compares the proportional order-up-to policy with an alternative policy; the so-called "full-state-feedback" order-up-to policy. This literature indicates that with a correct choice of the feedback parameter the bullwhip effect can always be avoided. But at the same time less attractive properties of the proportional order-up-to policy have become clear.

Herein we investigate the behaviour of the proportional order-up-to policy for ARMA(2,2) demand with arbitrary lead-times. Although the behaviour for general ARMA(*p*,*q*) models might be scientifically interesting, for short term non-seasonal product demand forecasting ARMA(*p*,*q*) models with $p, q \leq 2$ are the most valuable. The proportional order-up-to policy will be compared with the adapted full-state-feedback policy in Gaalman (2006) for arbitrary lead-times.

The remainder of the paper is organized as follows. Section 2 describes the ARMA(2,2) demand process and the Kalman state space filter for optimal demand forecasts. Section 3 introduces the two policies and formulates a method to balance the trade-off between inventory and ordering variance. In section 4 expressions for the variances are derived. Section 5 discusses the characteristics of the proportional order-up-to and the full-state-feedback policy. Section 6 concludes.

## 2. The ARMA (2,2) demand model and the forecast system

Before we introduce the formulation of the 'problem' we will first shortly discuss the characteristics of the Auto Regressive Moving Average demand process as proposed by Box, Jenkins and Reinsel (1994). Suppose that the demand in the period ($t$, $t$+1) and observed at time $t$+1 can be described by the sum of a constant mean term and a stochastic normally distributed variable;

$$d_{t+1} = \overline{d} + z_{t+1}. \tag{1}$$

The stochastic variable ($z_t$) satisfies the stochastic ARMA(2,2) process

$$z_{t+1} - \phi_1 z_t - \phi_2 z_{t-1} = \eta_{t+1} - \theta_1 \eta_t - \theta_2 \eta_{t-1}, \tag{2}$$

where $\eta_{t+1}$ is a zero mean uncorrelated normally distributed random variable with variance $\Sigma_{\eta\eta}$. The autoregressive (AR) part of the demand process consists of the two parameters, $\phi_1$ and $\phi_2$. The moving average (MA) part has the parameters $\theta_1$ and $\theta_2$. For stable demand processes the parameters of the AR components should satisfy certain conditions, see (3). For the invertible conditions (Box, Jenkins and Reinsel, 1994) the MA parameters should satisfy the same type of conditions. This leads to

$$\begin{aligned} &p_1 + p_2 < 1,\ -p_1 + p_2 < 1,\ -1 < p_2 < 1 \\ &p_i = \phi_i \ or\ \theta_i\ ,\ i = 1,2. \end{aligned} \tag{3}$$

If the (possibly complex) eigenvalues of the AR and MA part are $\lambda_i^j, i = 1,2, j = \phi,\theta$ then the stationary or invertibility conditions correspond with the condition $\left|\lambda_i^j\right| < 1, i = 1,2$ and $j = \phi,\theta.$ The coefficients of the characteristic polynomials of the AR and MA part also satisfy $p_1 = \lambda_1^j + \lambda_2^j$, $p_2 = -\lambda_1^j \lambda_2^j$, where $\lambda^2 - p_1\lambda - p_2 = (\lambda - \lambda_i^j)(\lambda - \lambda_i^j)$. Complex eigenvalues are created when $p_1^2 + 4p_2 \le 0$. Usually, short term demand forecasting with ARMA(2,2) processes will have real eigenvalues.

The forecast of this ARMA process is relatively simple. Here we will give a state space formulation of the ARMA demand process, which enables the use of the Kalman filtering theory (Jazwinski, 1970) and will facilitate the derivation of expressions for the ordering and inventory variances. State space descriptions of time series are, and have been, a powerful tool in econometric forecasting. State space methods can represent a wide range of time series and have become a standard topic in many economic time series text books (see for example Harvey, 1989). A state space representation of a process is not unique and in standard text books several possibilities can be found. Our representation differs from the one commonly used in forecasting economic time series (see also Gaalman, 2006) because we have defined a slightly different state vector, which of course, we are free to do. For this demand process we define a *2*-dimensional (demand) state vector $y_t$ with components

$$y_t = \begin{pmatrix} y_{1:t+1} \\ y_{2:t+1} \end{pmatrix} = \begin{pmatrix} \phi_1 z_t + \phi_2 z_{t-1} - \theta_1\eta_t - \theta_2\eta_{t-1} \\ \phi_2 z_t - \theta_2 \eta_t \end{pmatrix}. \tag{4}$$

With this definition the demand model can directly be transformed into an observable canonical state space form

$$\begin{aligned} z_{t+1} &= My_{t+1} + \eta_{t+1}, \\ y_{t+1} &= Dy_t + E\eta_t, \end{aligned} \tag{5}$$

where $D$ is a (2×2) left companion matrix (see for instance Kailath, 1980) with $D = \begin{pmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{pmatrix}, M = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $E$ is a two dimensional vector with $E = \begin{pmatrix} \phi_1 - \theta_1 & \phi_2 - \theta_2 \end{pmatrix}^T$. Here

$(\cdots)^T$ is the transpose of a vector or matrix. Using the Kalman filter approach (Jazwinski, 1970), the one-period-ahead demand forecast at time $t$, $\hat{d}_{t+1,t}$ satisfies

$$\begin{aligned} \hat{d}_{t+1,t} &= \bar{d} + \hat{z}_{t+1,t},\ \hat{z}_{t+1,t} = M\hat{y}_{t+1,t}, \\ \hat{y}_{t+1,t} &= D\hat{y}_{t,t-1} + G(z_t - \hat{z}_{t,t-1}), z_t - \hat{z}_{t,t-1} = \eta_t, \end{aligned} \tag{6}$$

with

$$G = \begin{pmatrix} \phi_1 - \theta_1 & \phi_2 - \theta_2 \end{pmatrix}^T, \tag{7}$$

where $\hat{d}_{t+1,t}, \hat{z}_{t+1,t}$ and $\hat{y}_{t+1,t}$ (the state of the forecast system) are conditional expectations at time $t$ given all the observed realizations of the demand. Here we assume that an infinite amount of past demand information is available. $G$ is the so-called Kalman gain vector that can be calculated from the associated stationary matrix Riccati equation (see also Gaalman, 2006, for the general ARMA($p,q$) demand process). The forecast update procedure is similar to (general) exponential smoothing (Brown, 1963) and is an efficient way to calculate the demand forecasts. The first step to compute $\hat{y}_{t+1,t}$ is to forecast this value at $t$-1: $\hat{y}_{t+1,t-1} = D\hat{y}_{t,t-1}$. However, the forecast error measured at time $t$ also provides some additional information about $y_{t+1}$. So the forecast error is weighted with a gain factor and added to the initial forecast. Knowing the state of the forecast system $\hat{y}_{t+1,t}$ at time $t$, the forecasted demand at $j$ periods ahead can then be calculated as

$$\begin{aligned} \hat{d}_{t+1+j,t} &= \bar{d} + \hat{z}_{t+j+1,t},\ \hat{z}_{t+j+1,t} = M\hat{y}_{t+j+1,t}, \\ \hat{y}_{t+1+j,t} &= D^j \hat{y}_{t+1,t}. \end{aligned} \tag{8}$$

We have represented the demand forecast as a state space model. In the next section we consider the problem formulation.

## 3. Problem formulation

Consider the inventory balance equation with an arbitrary production / replenishment delay of $k$ time periods ($k$=0,1,..)

$$i_{t+1+k} = i_{t+k} + o_t - d_{t+1+k}, \tag{9}$$

where $i_t$ is the inventory at time $t$ (the start of period $t$+1), $o_t$ is the order decision made at time $t$ (just after the inventory $i_t$ is observed) that will arrive at the beginning of period $t$+$k$+1, and $d_{t+1+k}$ is the demand during period $t$+$k$+1 and (completely) realised at time $t+k+1$. When $k$=0 there is no physical ordering delay (lead-time), although the effect of the sequence of events within the ordering decision means that there is always a delay of one period. The proportional order-up-to policy (Gaalman and Disney, 2006) can be formulated as

$$o_t = \bar{d} - f(\hat{i}_{t+k,t} - \bar{i}) + \hat{z}_{t+k+1,t}. \tag{10}$$

This policy uses, at time *t*, the estimated inventory at *t*+*k*, $\hat{i}_{t+k,t}$ and the demand forecast $\hat{z}_{t+1+k,t}$, the conditional expectation of the demand in period $t+1+k$. Using (8) we can eliminate the demand forecast and introduce the forecast state $\hat{y}_{t+k+1,t}$

$$o_t = \bar{d} - f(\hat{i}_{t+k,t} - \bar{i}) + M\hat{y}_{t+k+1,t}. \tag{11}$$

Since $M = \begin{pmatrix} 1 & 0 \end{pmatrix}$ the "proportional" order-up-to policy uses only the first element of the forecast state $\hat{y}_{t+k+1,t}$. Thus information embedded in the demand process is only partly used. Based on this insight an alternative policy has been developed by Gaalman (2006) that uses all the components of the state of the system. Adapted for the arbitrary lead-times and ARMA(2,2) demand this so-called "full-state-feedback" policy becomes

$$o_t = \bar{d} - f(\hat{i}_{t+k,t} - \bar{i}) - F_y\hat{y}_{t+1+k,t},\; F_y = \frac{-f}{1-(1-f)\phi_1 - (1-f)^2\phi_2}\big(1,\; (1-f)\big). \tag{12}$$

The two feedback parameters of the vector $F_y$ are non-linear functions of the inventory feedback parameter $f$. We can observe that the autoregressive parameters also play a role. Since for a stable system $0 \le f < 2$, the absolute value of the second part of the vector $F_y$ is weighted less than the first part of the vector $F_y$. The denominator depends on the AR parameters and is positive in the stability region.

In order to study the dynamic behaviour of these two policies we will consider the stationary variances of the order rates and inventory levels. The literature suggests that the proportional order-up-to policy allows the bullwhip effect to be avoided with a suitable choice of the inventory feedback parameter $f$. This feedback parameter also influences the inventory variance and thus avoiding the bullwhip effect could result in large inventory variances. However, there is no explicit guidance on a suitable value of the feedback parameter. For this reason we will consider the following objective function

$$J = Q_{ii}\Sigma_{ii} + R_{oo}\Sigma_{oo}, \tag{13}$$

where $\Sigma_{ii}$, $\Sigma_{oo}$ are the stationary inventory and ordering variances and $Q_{ii}, R_{oo}$ are weighting parameters. In the next section we will derive the expressions for the variances.

## 4. Expressions for the variances

Both policies use the forecasted inventory, $\hat{i}_{t+k,t}$ and the demand state forecast, $\hat{y}_{t+1+k,t}$. So to calculate the ordering variance, the variance of these two variables as well as their co-variances are required. The variance of the forecast state can be obtained from the Kalman filter (6). Using $\hat{y}_{t+1+k,t} = D^k\hat{y}_{t+1,t}$ the recursion becomes

$$\begin{aligned} \hat{d}_{t+1+k,t} &= \bar{d} + \hat{z}_{t+1+k,t} = \bar{d} + M\hat{y}_{t+1+k,t}, \\ \hat{y}_{t+2+k,t+1} &= D\hat{y}_{t+1+k,t} + D^kG(\eta_{t+1}). \end{aligned} \tag{14}$$

To obtain the variance of the forecasted inventory we follow the same recursive approach as used in Kalman filtering. Consider the inventory balance equation as a stochastic process. Having estimated at $t$, $\hat{i}_{t+k,t}$ we want to find $\hat{i}_{t+k+1,t+1}$. In a similar manner to (6) we may formulate the update recursion as

$$\hat{i}_{t+k+1,t+1} = \hat{i}_{t+k,t} + o_t - \hat{d}_{t+1+k,t} - E(k)\eta_{t+1}. \tag{15}$$

Without going into details the 'inventory gain component' $E(k)$, satisfies

$$E(k) = [1 + M\sum_{j=0}^{k-1}(D^j)G],\ \ E(0) = 1. \tag{16}$$

We can observe that the structure of (15) is equal to the balance equation (9) and the gain component is dependent upon the lead-time. Moreover equation (15) holds regardless the policy used.

Combining the two recursions into one total system recursion makes it possible to calculate the variances and co-variances. Let the 3-dimensional system state be given by $\hat{x}_{t+k,t} = \left(\hat{i}_{t+k,t}, \hat{y}_{t+1+k,t}\right)^T$ we then get

$$\hat{x}_{t+k+1,t+1} = A\hat{x}_{t+k,t} + Bo_t + C\eta_{t+1}, \tag{17}$$

with $A = \begin{pmatrix} 1 & -M \\ 0 & D \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} -E(k) \\ D^k G \end{pmatrix}$. Thus $A$ is a $(3\times 3)$ matrix, and both $B$ and $C$ are 3-dimensional vectors. Since we are interested in the inventory and ordering fluctuations by means of the variances, the next step is to transform (17) into a variance expression. Since the variances do not depend on mean values for simplicity reasons we will, from now on, ignore the mean values of the demand and the inventory norm. Furthermore, to yield a common notation for both policies we introduce vector $P_i, i = 1,2$ such that

$$o_t = -P_i\hat{x}_{t+k,t},\ \ P_i = FS_i + M_i, \tag{18}$$

where $F$ is the feedback vector. In the case of the proportional policy (case 1) $F = f$ and so: $S_1 = (1\ \ 0),\ M_1 = (0\ \ -M)$. For the full state feedback (case 2) $F = (f\ \ F_y)$ giving $S_2 = (I_{m+1}),\ M_2 = (0)$. Substitution of (18) into (17) gives

$$\hat{x}_{t+k+1,t+1} = (A - BP_i)\hat{x}_{t+k,t} + C\eta_{t+1}. \tag{19}$$

In order to calculate the variance of the system state we multiply the state with its transpose and take the expectation (see for instance Gaalman (1976) and Kwakernaak and Sivan (1972)). The stationary state variance equation exists when the stability condition is satisfied ($0 \le f < 2$) and the eigenvalues of $D$ are within the unit circle. This gives

$$\Sigma_{\hat{x}\hat{x}}(k) = (A - BP_i)\Sigma_{\hat{x}\hat{x}}(k)(A - BP_i)^T + C\Sigma_{\eta\eta}C^T, \tag{20}$$

where the system variance matrix $\Sigma_{\hat{x}\hat{x}}(k) = E\{(\hat{x}_{t+k,t})(\hat{x}_{t+k,t})^T\}$. Knowing the components of the state we can write the system variance as

$$\Sigma_{\hat{x}\hat{x}}(k) = \begin{pmatrix} \Sigma_{\hat{i}\hat{i}}(k) & \Sigma_{\hat{i}\hat{y}}(k) \\ \Sigma_{\hat{y}\hat{i}}(k) & \Sigma_{\hat{y}\hat{y}}(k) \end{pmatrix} \tag{21}$$

Note: In order to show explicitly that the variances depend on the lead-time we have introduced the notation $\Sigma_{\hat{x}\hat{x}} = \Sigma_{\hat{x}\hat{x}}(k)$. This will also be done for the ordering and inventory variances. Using (18), the variance of the ordering decision can then be expressed by

$$\Sigma_{oo}(k) = P_i\Sigma_{\hat{x}\hat{x}}(k)P_i^T. \tag{22}$$

From (20) the forecasted inventory variance $\Sigma_{\hat{i}\hat{i}}(k)$ can be calculated. However to investigate the trade-off between bullwhip and inventory, it is the inventory variance that is required (see equation (13)). The inventory $i_{t+k}$ can be written as

$$i_{t+k} = \hat{i}_{t+k,t} + (i_{t+k} - \hat{i}_{t+k,t}). \tag{23}$$

The second component of the right hand side is the inventory forecast error and is uncorrelated with the forecasted inventory. Using the characteristics of the demand model we get

$$(i_{t+k} - \hat{i}_{t+k,t}) = -\sum_{l=0}^{k-1} E(l)\eta_{t+k-l}. \tag{24}$$

This leads to the inventory variance expression

$$\Sigma_{ii}(k) = \Sigma_{\hat{i}\hat{i}}(k) + \left\{\sum_{l=0}^{k-1} E(l)^2\right\}\Sigma_{\eta\eta}. \tag{25}$$

The inventory variance consists of the forecasted inventory variance, influenced by the ordering policy, and a constant component related to the forecast errors over the lead-time. Having derived general expressions for the variances we are now able to find specific expressions for each policy.

## I. The proportional policy (*i*=1)

By working out the variance equation (20) the forecasted inventory variance can be found and this gives

$$\Sigma_{\hat{i}\hat{i}}(k) = \frac{E(k)^2\Sigma_{\eta\eta}}{f(2-f)}. \tag{26}$$

This expression has a similar structure to the case *k=0* where $\Sigma_{ii}(0) = \Sigma_{\hat{i}\hat{i}}(0)$ that has been derived in Gaalman (2006) for general ARMA($p$,$q$) demand and in Gaalman & Disney (2006) for ARMA (1,1) demand. Using $P_1$ in equation (22) the ordering variance of this policy satisfies

$$\Sigma_{oo}(k) = f^2\Sigma_{\hat{i}\hat{i}}(k) - 2f\Sigma_{\hat{z}\hat{i}}(k) + \Sigma_{\hat{z}\hat{z}}(k), \tag{27}$$

where $\Sigma_{\hat{z}\hat{z}}(k)$ is the variance of the forecasted demand and $\Sigma_{\hat{z}\hat{i}}(k)$ is the co-variance between the inventory and the forecasted demand. Since $\Sigma_{\hat{z}\hat{i}}(k) = M\Sigma_{\hat{y}\hat{i}}(k)$ and since we can calculate the co-variance of the forecast state and inventory using (20) then

$$\Sigma_{\hat{y}\hat{i}}(k) = -E(k)(1-(1-f)D)^{-1}(D^kG)\Sigma_{\eta\eta}, \tag{28}$$

and the second component of (27) can be written as

$$-2f\Sigma_{\hat{z}\hat{i}}(k) = -2fM\Sigma_{\hat{y}\hat{i}}(k) = 2fE(k)W(f,k)\Sigma_{\eta\eta}, \tag{29}$$

where $W(f,k)$ is a scalar satisfying

$$W(f,k) = M(1-(1-f)D)^{-1}(D^kG). \tag{30}$$

Since $D$ is a companion matrix, an explicit expression for the inverse exists (see for instance Kailath, 1980)

$$M(1-(1-f)D)^{-1} = \begin{pmatrix}1 & (1-f)\end{pmatrix}/\{1-\sum_{j=1}^{2}\phi_j(1-f)^j\} \tag{31}$$

The third component of (27) does not depend on the feedback parameter but it does depend on the lead-time. Using (18) and (20) we have the variance of the forecasted demand,

$$\Sigma_{\hat{z}\hat{z}}(k) = M\Sigma_{\hat{y}\hat{y}}(k)M^T, \Sigma_{\hat{y}\hat{y}}(k) = D\Sigma_{\hat{y}\hat{y}}(k)D^T + (D^kG)(D^kG)^T\Sigma_{\eta\eta}. \tag{32}$$

Using (26) and (29) we can write the ordering variance as

$$\Sigma_{oo}(k) = E(k)f\left\{\frac{E(k)}{(2-f)} + 2W(f,k)\right\}\Sigma_{\eta\eta} + \Sigma_{\hat{z}\hat{z}}(k). \tag{33}$$

The structure of this expression is similar to the unit lead-time case ($k$=0), that has been derived in Gaalman (2006). The variance consists of three components. The third component, the forecasted demand variance, is positive. The first component related to the inventory variance is always a positive increasing function of $f$ in the stability interval. If the demand is i.i.d. then the second component is zero and ordering variance is a positive increasing function of $f$. For the ARMA(2,2) process we consider here, the correlation between the forecasted demand and the forecasted inventory $\Sigma_{\hat{z}\hat{i}}(k)$ can be negative. That is $W(f,k)$ can be negative in the stability interval. This implies that the ordering variance is not necessarily always an increasing function of $f$. We will explore this further in section 4.

**II. The full-state-feedback policy (*i*=2)**

In general, closed forms for the variance expressions in the full-state-feedback policy can be obtained, but they are very complex and lengthy. However, they are fairly compact in the state space notation. For example, the forecasted inventory variance is given by the (1,1) element of the variance matrix (20) and turns out to be

$$\Sigma_{\hat{i}\hat{i}}(k) = \{E(k)^2 \Sigma_{\eta\eta} - 2(1-f)(M + F_{\hat{y}})\Sigma_{\hat{y}\hat{i}}(k) + (M + F_{\hat{y}})\Sigma_{\hat{y}\hat{y}}(k)(M + F_{\hat{y}})^T\} / (f(2-f)). \tag{34}$$

The inventory variance can be obtained by using (34) in (25). Equation (34) also requires the co-variance between the inventory and the forecasted demand and the variance of the forecast state. The co-variance is given by

$$\Sigma_{\hat{y}i}(k) = (1-f)D\Sigma_{\hat{y}i}(k) - D\Sigma_{\hat{y}\hat{y}}(k)(M + F_{\hat{y}})^T - E(k)(D^k G)\Sigma_{\eta\eta}, \tag{35}$$

which can be written as

$$\Sigma_{\hat{y}i}(k) = -(I - (1-f)D)^{-1}\{E(k)(D^k G)\Sigma_{\eta\eta} + D\Sigma_{\hat{y}\hat{y}}(k)(M + F_{\hat{y}})^T\}. \tag{36}$$

From (21) the variance of the order rate in the full-state-feedback policy is given by

$$\Sigma_{oo}(k) = f^2 \Sigma_{\hat{i}\hat{i}}(k) + 2fF_{\hat{y}}\Sigma_{\hat{y}\hat{i}}(k) + F_{\hat{y}}\Sigma_{\hat{y}\hat{y}}(k)F_{\hat{y}}^T. \tag{37}$$

## 4. Comparison of the proportional order-up-to and the full-state policies

The expressions found in section 3 can be analysed and the characteristics of the variances can be analysed. The variance expressions have a number of interesting properties. These properties will be demonstrated using an ARMA(2,0) with $\phi_1 = 0.6$ and $\phi_2 = -0.9$ as an example. Generally an ARMA(2,0) process does not necessarily possess all possible characteristics of the ARMA(2,2) process. However the ARMA(2,0) case is fairly rich in that it possesses nearly all of the characteristics of the ARMA(2,2) case. First we will consider the characteristics of the proportional policy and then the characteristics of the full-state-feedback policy.

I. **The proportional order-up-to policy**

*The inventory variance.*

The forecasted inventory variance is a convex and symmetric function (around $f$ = 1) over the stability interval $0 \le f < 2$, with a (absolute) minimum of $E(k)^2 \Sigma_{\eta\eta}$ at $f$ = 1, and asymptotes to $+\infty$ at $f$ = 0 and 2. Since only a positive constant is added (see equation 25) the inventory variance has the same characteristics. In figure I:a the inventory variance of the demand process is shown for several lead-times (*k*=0,1,3,8, and 20). From (25) we can also conclude that the minimum of the inventory variance is a non-decreasing function of the lead-time.

*The ordering variance.*

Assuming that $E(k) \neq 0$ the ordering variance has an asymptote to $+\infty$ at $f$ = 2. So there is always an interval for which the bullwhip effect exists $(\Sigma_{oo} > \Sigma_{zz})$. At $f$ = 0 the order variance equals the variance of the forecasted demand, $\Sigma_{\hat{z}\hat{z}}(k)$. For $f$ = 1, the proportional order-up-to policy equals the (standard) order-up-to policy. The ordering variance has three possible patterns as function of $f$ in the stability interval ($0 \leq f < 2$):

- The order variance is an increasing function in $f$. See $k = 0$ and $k = 20$ in figure I.b. The demand variance is 5.84. For $k = 0$ the critical bullwhip point $(\Sigma_{oo} = \Sigma_{zz})$ happens at $f = 0.68$ and for $k = 20$ at $f = 1.77$.
- The order variance has a unique minimum. The necessary and sufficient condition for this is that $E(k)\left(\frac{E(k)}{4} + \frac{E(k+1) - E(k)}{1 - \phi_1 - \phi_2}\right) < 0$. The minimum can lie in either the 0 < $f$ < 1 region or in the 1 < $f$ < 2 region. In figure I.b, for $k = 1$ the minimum is at $f = 0.52$. For large $k$, $E(k)$ converges to the limit value $E(k \to \infty) = \left(\frac{1 - \theta_1 - \theta_2}{1 - \phi_1 - \phi_2}\right)$. Due to the stability and the invertibility conditions both the nominator and denominator are always positive; as is $E(k \to \infty)$. So for sufficiently large $k$ the difference between $E(k+1)$ and $E(k)$ is small, and the necessary and sufficient condition for a minimum will be positive. This means that no minimum will exist (see for instance the case of $k = 20$ in figure I.b). From (14) it can be shown that the forecasted demand state, $\hat{y}_{t+1+k,t}$, goes to zero for large $k$. Thus, the forecasted demand then approaches the mean demand $\bar{d}$. By this the proportional order-up-to policy reduces to a simple inventory feedback policy (see equation 10), which is similar to the case of i.i.d. demand. For this situation it is well known that the ordering variance is an increasing function in $f$ in the stability interval.
- The order variance has both a maximum and minimum in the stability region. The necessary and sufficient conditions are that the second derivative of the ordering variance with respect to $f$ is zero and the first derivative is negative in the stability interval. However, explicit expressions are in general difficult to obtain. This pattern is an extension compared with the two patterns described above that are possible for ARMA(1,1) demands, (Gaalman and Disney, 2005). It can be shown that value of the feedback parameter at the (local) maximum is always less than the value of the feedback parameter at the (local) minimum. In figure I:b two examples at $k = 3$ and $k = 8$ are given, with the maximum at $f = 0.55$ and $f = 0.60$, and the minimum at $f = 1.26$ and $f = 1.46$ respectively. We note that this example suggests that the difference between the (local) maximum variance and the minimum variance is small. In addition the minimum lies in the $1 < f < 2$ range. However other demand process parameters indicate that this is not always the case. Furthermore, in numerical experiments we always found the maximum to be in the region of $0 < f < 1$.

When (the ‘inventory gain component’) $E(k) = 0$ the ordering variance equals the forecasted demand $(\Sigma_{oo}(k) = \Sigma_{\hat{z}\hat{z}}(k))$. Remarkably the variance is independent of the feedback parameter $f$. In this case the disturbance in the forecasted balance equation (15) is zero, and since the forecasted demand is compensated by the proportional policy the forecasted inventory is zero (in the stationary situation, see (38)).

$$\hat{i}_{t+k+1,t+1} = \hat{i}_{t+k,t} + o_t - \hat{d}_{t+1+k,t} = (1-f)\hat{i}_{t+k,t}. \tag{38}$$

This means that the forecasted inventory variance and the co-variance with the forecasted demand are zero. This specific property of the proportional rule in combination with ARMA demand and arbitrary lead–times has also been reported in Gaalman and Disney (2005). For ARMA(1,1) demand, $E(k)$ could only be zero for even $k$'s. However for ARMA(2,2) demand $E(k)$ can be zero for both odd and even $k$'s.

*The trade-off between the ordering and inventory variance.*

Since the objective function is the weighted sum of the ordering and inventory variances two asymptotes (at $f = 0$ and $f = 2$) exist at which the function becomes infinite. So, at least one (absolute) minimum exists in the stability interval. If the ordering variance is increasing or has a unique minimum then only one solution exists. If the ordering variance has a maximum and minimum then, depending of the weighting parameters $Q_{ii}, R_{oo}$, a minimum or a maximum with two minima may exist. In figure I:c we have chosen $Q_{ii} = R_{oo} = 1$. For $k = 0$, $k = 8$ and $k = 20$ a single unique minima exists at $f = 0.45$, $f = 1.20$ and $f = 0.50$ respectively for each case. For $k = 1$ the unique minimum lies at $f = 0.70$. For $k = 3$ we have an absolute minimum at $f = 1.40$, a (local) maximum at $f = 0.40$ and a local minimum at $f = 0.16$. From this example we can conclude that the absolute minimum is strongly influenced by the lead-time. In addition we can observe that, in the case for increasing lead-times the value of the feedback parameter initially increases and may even become larger than one before it then decreases. As indicated above for infinite lead-times, the forecasted demand process becomes similar to the i.i.d. demand case. The optimal trade-off has now a unique minimum at $f$=0.618034, the Golden Ratio. The minimum can also lie in the unattractive interval, $1 < f < 2$. This is unattractive because the difference between the (forecasted) inventory and the norm value is over-compensated. This example suggests that differences between minimum and maximum of the trade-off function are small. Again, for other demand process parameters, this may not necessarily hold.

**II. The full-state-feedback policy**

*The inventory variance.*

The forecasted inventory variance has asymptotes to $+\infty$ for $f = 0$ and $2$ (see equation (34)). The variance has a unique minimum at $f = 1$. It is decreasing in the $0 < f < 1$ region and increasing in the $1 < f < 2$ region. However the variance is not symmetric with respect to $f = 1$. The inventory variance has the same characteristics as the forecasted inventory variance as can be seen from (25).

In figure II:a the inventory variance of the demand process is shown for several lead-times k=0, 1, 3, 8 and 20. Again we can conclude that the minimum of the variance is a non-decreasing function of the lead-time. In addition, we can observe that for increasing $k$ the inventory variance becomes more symmetric.

*The ordering variance.*

The ordering variance always increases over the stability interval. There is an asymptote to $+\infty$ at $f = 2$ as there was for the proportional policy. For $f = 0$ the variance is zero for the full-state-feedback policy. This is in contrast to the proportional policy were it was equal to the variance of the forecasted demand. The examples in figure II:b confirm these properties. For $f = 1$ the full-state-feedback policy is equal to proportional order-up-to policy (and the traditional order-up-to policy).

When $E(k) = 0$ the disturbance in the forecasted balance equation (15) is zero. However, in contrast with the proportional policy, the full-state-feedback policy does not compensate for the forecasted demand completely. Thus the forecasted inventory is not zero (see equation 39) and remains dependent upon the feedback parameter $f$ . This also holds for the ordering variance.

$$\begin{aligned} &\hat{i}_{t+k+1,t+1} = \hat{i}_{t+k,t} + o_t - M\hat{y}_{t+1+k,t} = (1-f)\hat{i}_{t+k,t} - (1+F_{1:y})\hat{y}_{1:t+1+k,t} - F_{2:y}\hat{y}_{2:t+1+k,t} \\ &\left(F_{1:y} \quad F_{2:y}\right) = \frac{-f}{1-(1-f)\phi_1 - (1-f)^2\phi_2}\left(1, \ (1-f)\right). \end{aligned} \tag{39}$$

*The trade-off between the ordering and inventory variance.*

Since the inventory variance is;

- always a decreasing function of $f$ in the interval $0 < f < 1$,
- an increasing function in $1 < f < 2$ ,
- and the ordering variances is always an increasing function in $0 < f < 2$ ,
- then the objective function has a unique minimum in the $0 < f < 1$ region.

The condition for the minimum is independent of the parameters of the demand process and of the lead-times. Thus the derivation given in Gaalman (2006) for the case $k = 0$ holds. The condition is

$$f = (-\alpha + \sqrt{\alpha^2 + 4\alpha})/2, \tag{40}$$

where $\alpha = Q_{ii} / R_{oo}$ . In figure II:c, as α=1, the minimum lies at the Golden Ratio, $f$=0.618034. If the lead-time is infinite in the full-state-feedback policy, the state of the demand forecast model is zero and the demand forecast (see (14)) is the mean demand. Thus the feedback parameters of the forecasted demand $F_y$ have no influence on the ordering decisions and the full-state-feedback policy is equal to the proportional order-up-to policy. So, in this limiting case $(k \to \infty)$, the proportional policy has the same (attractive) behavior as the full-state-feedback policy.

If we compare both policies then we observe that three patterns of the ordering variance for the proportional policy are possible. Since local optima (minima or maxima) can exist, the finding of a suitable trade-off between the inventory variance and the ordering variance using numerical techniques is complicated. This also holds if, instead of the sum of the variances, expected costs are minimised. However, finding the trade-off for the full-state-feedback policy is less complex. Only one good balance exists and the feedback parameter lies always in the $0 < f < 1$ region. The full-state-feedback policy can be seen as the best policy within the class linear policies. Since the proportional policy has only one feedback parameter it is more restricted when finding the trade-off point. This explains why the proportional policy always has larger minimum values at the trade-off in the objective function and we sometimes find a feedback parameter in the $1 < f < 2$ region.

## 6. Conclusions

In this paper we have compared the proportional order-up-to policy and the full-state-feedback policy for ARMA(2,2) demand processes and arbitrary lead-times. We have shown that the full-state feedback policy is more attractive than the proportional order-up-to policy. Since the ordering variance is an increasing function of the feedback parameter, the identification of a bullwhip effect in the $0 < f < 1$ region is relatively simple and a suitable trade-off can always be found in this region. Moreover the optimal trade-off between the inventory and ordering variances is independent of the lead-time and the parameters of the demand model. In addition, no 'degeneration' in the order rate occurs when $E(k) = 0$. Thus, if the proportional policy is used in order to smooth the order rate and, by this, control the bullwhip effect, the full-state-feedback policy will improve upon its performance. In the future we plan to explore the behavior of both policies in real life scenarios.

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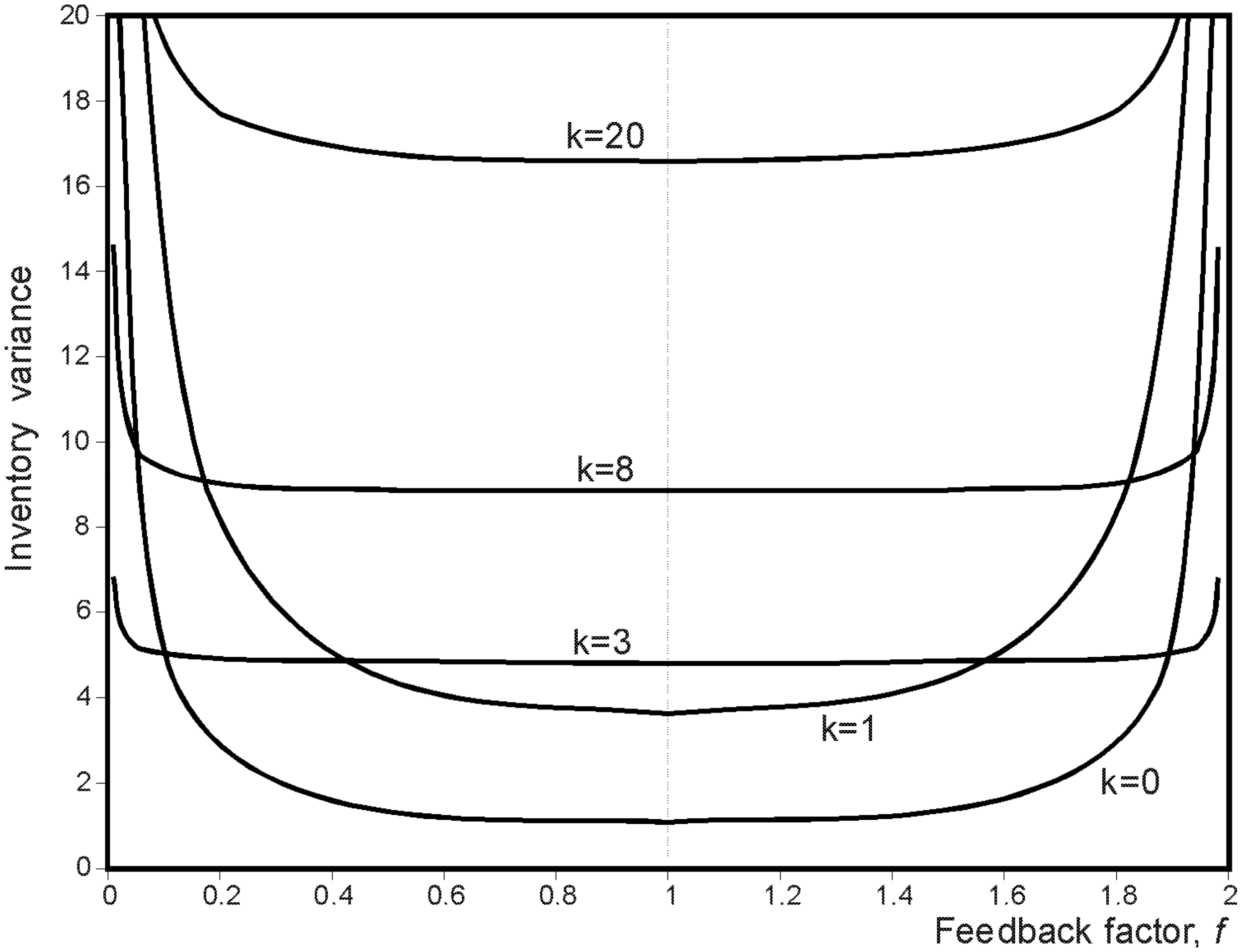


Figure I: a. The inventory variance for the proportional order-up-to policy for ARMA(2,0) demand when $\phi_1 = 0.6$ and $\phi_2 = -0.9$.

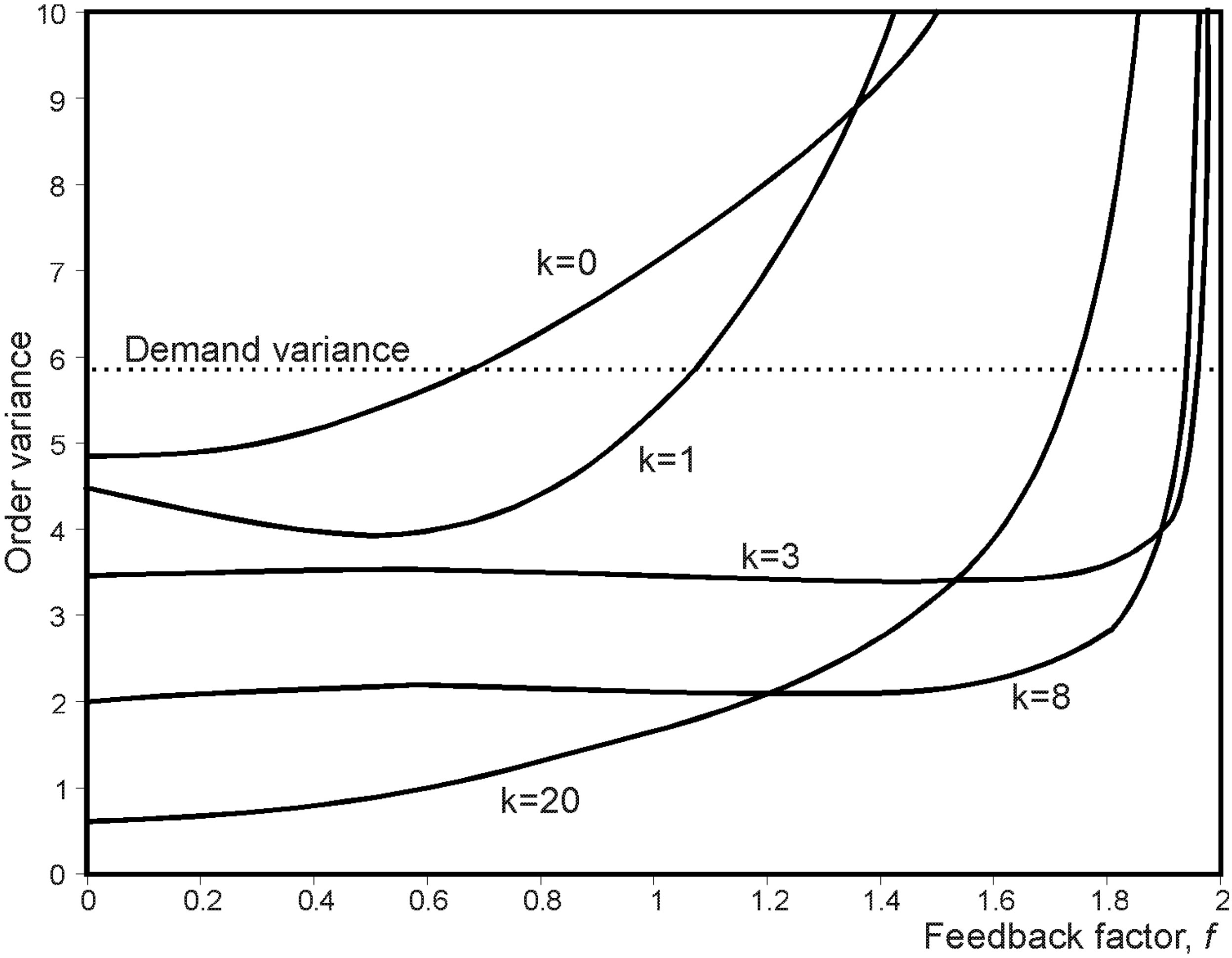


Figure I:b. The order variance for the proportional order-up-to policy for ARMA(2,0) demand when $\phi_1 = 0.6$ and $\phi_2 = -0.9$.

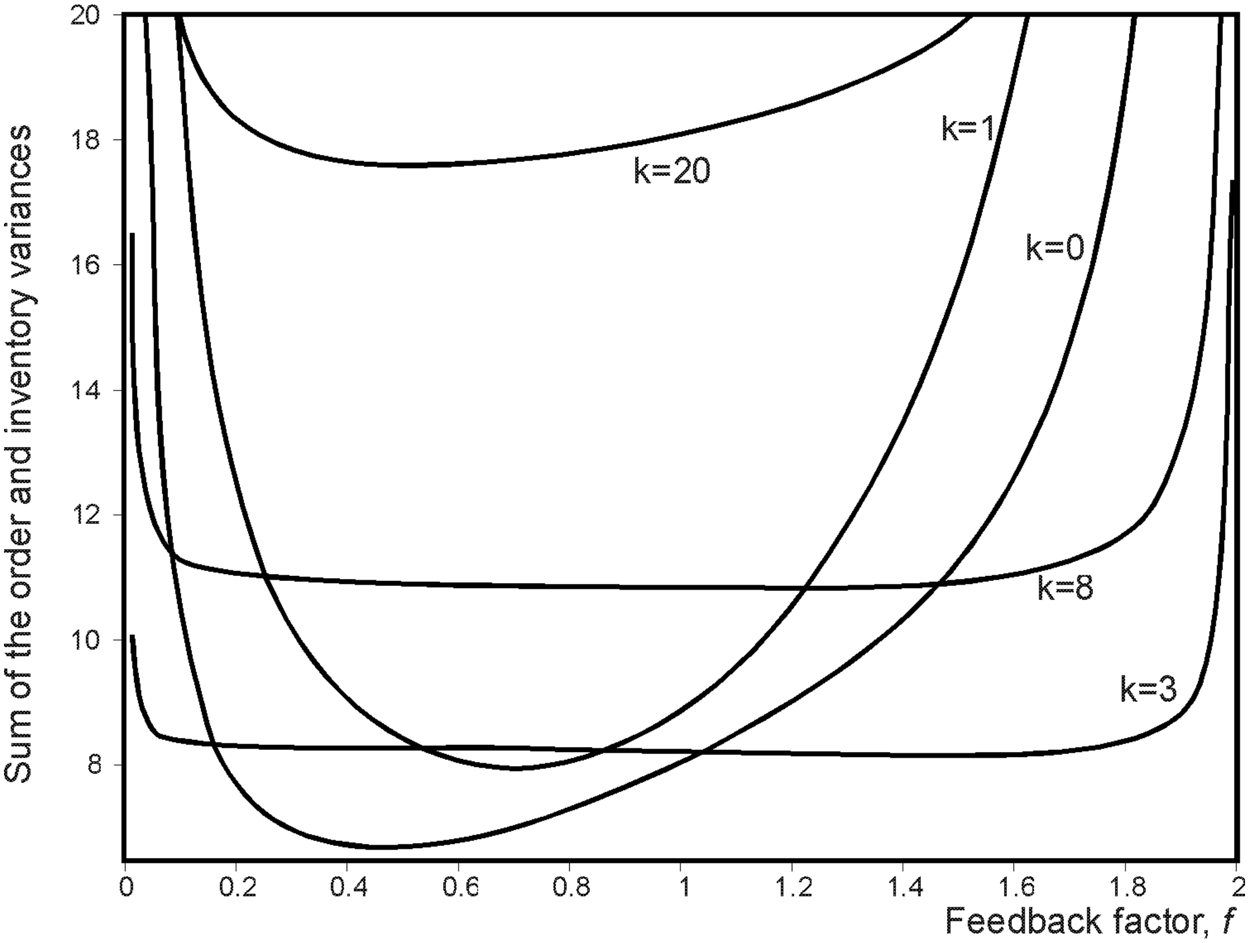


Figure I:c. The variance trade-off for the proportional order-up-to policy for ARMA(2,0) demand when $\phi_1 = 0.6$ and $\phi_2 = -0.9$.

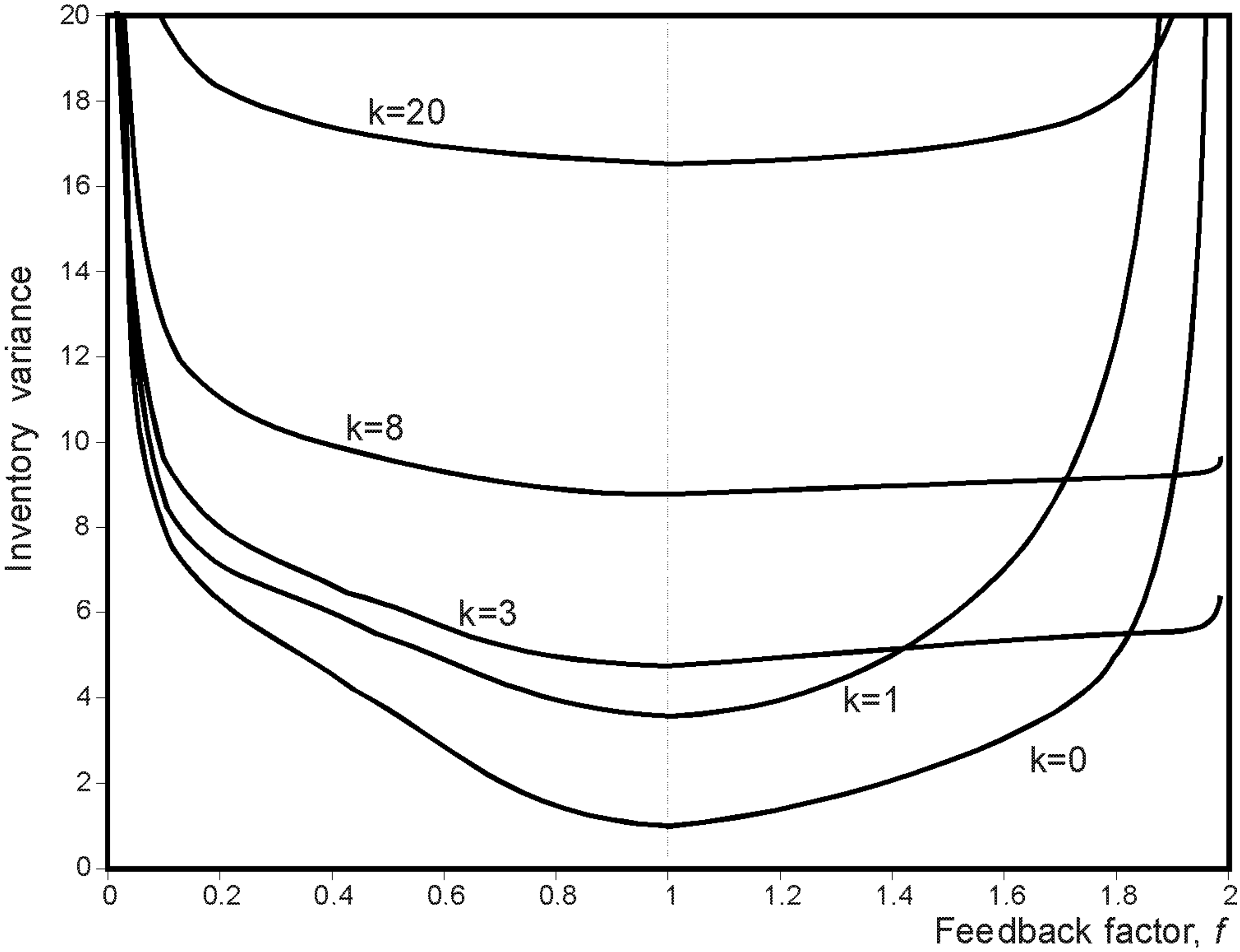


Figure II: a. The inventory variance for the full state feedback order-up-to policy for ARMA(2,0) demand when $\phi_1 = 0.6$ and $\phi_2 = -0.9$.

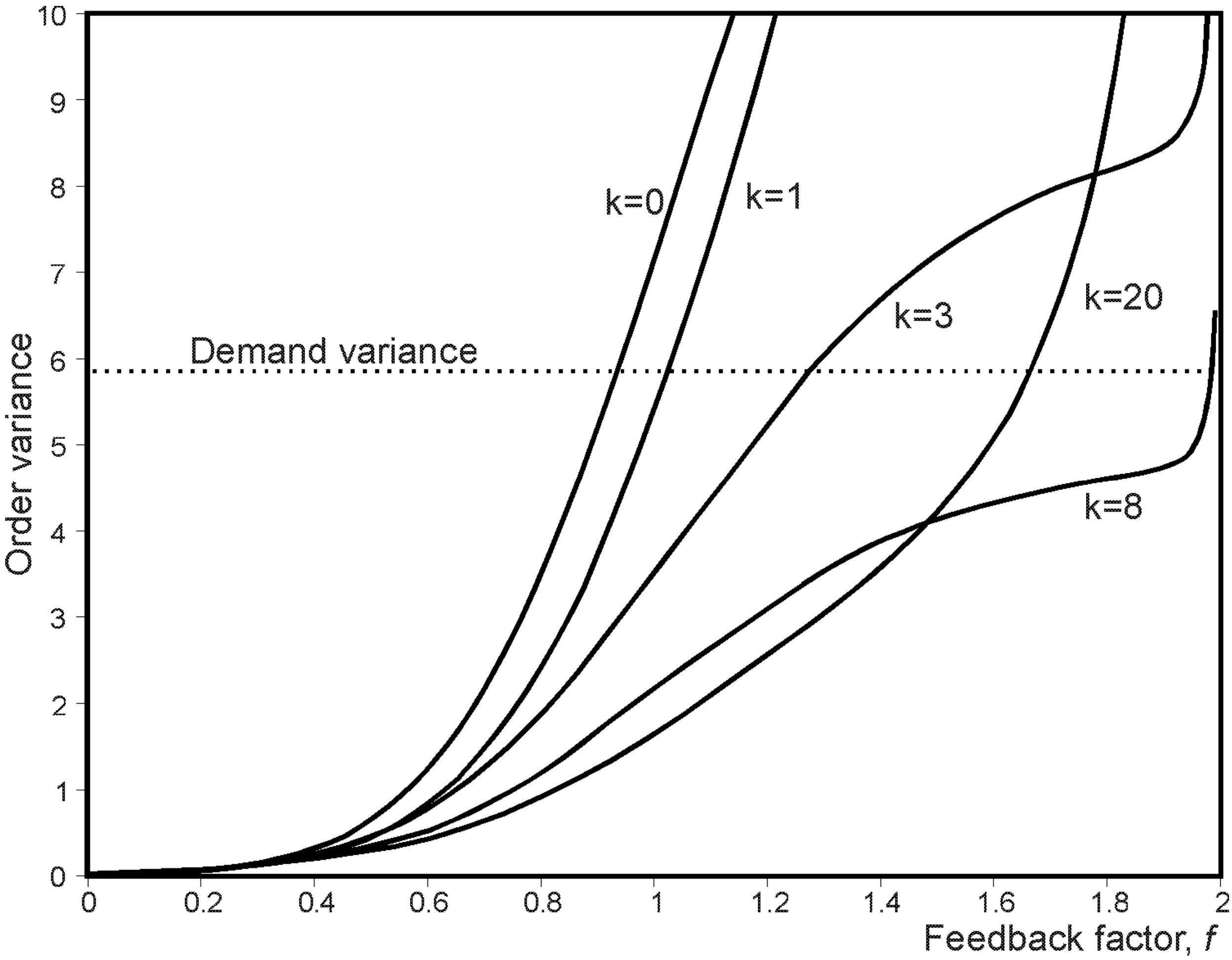


Figure II:b. The order variance for the full state feedback order-up-to policy for ARMA(2,0) demand when $\phi_1 = 0.6$ and $\phi_2 = -0.9$.

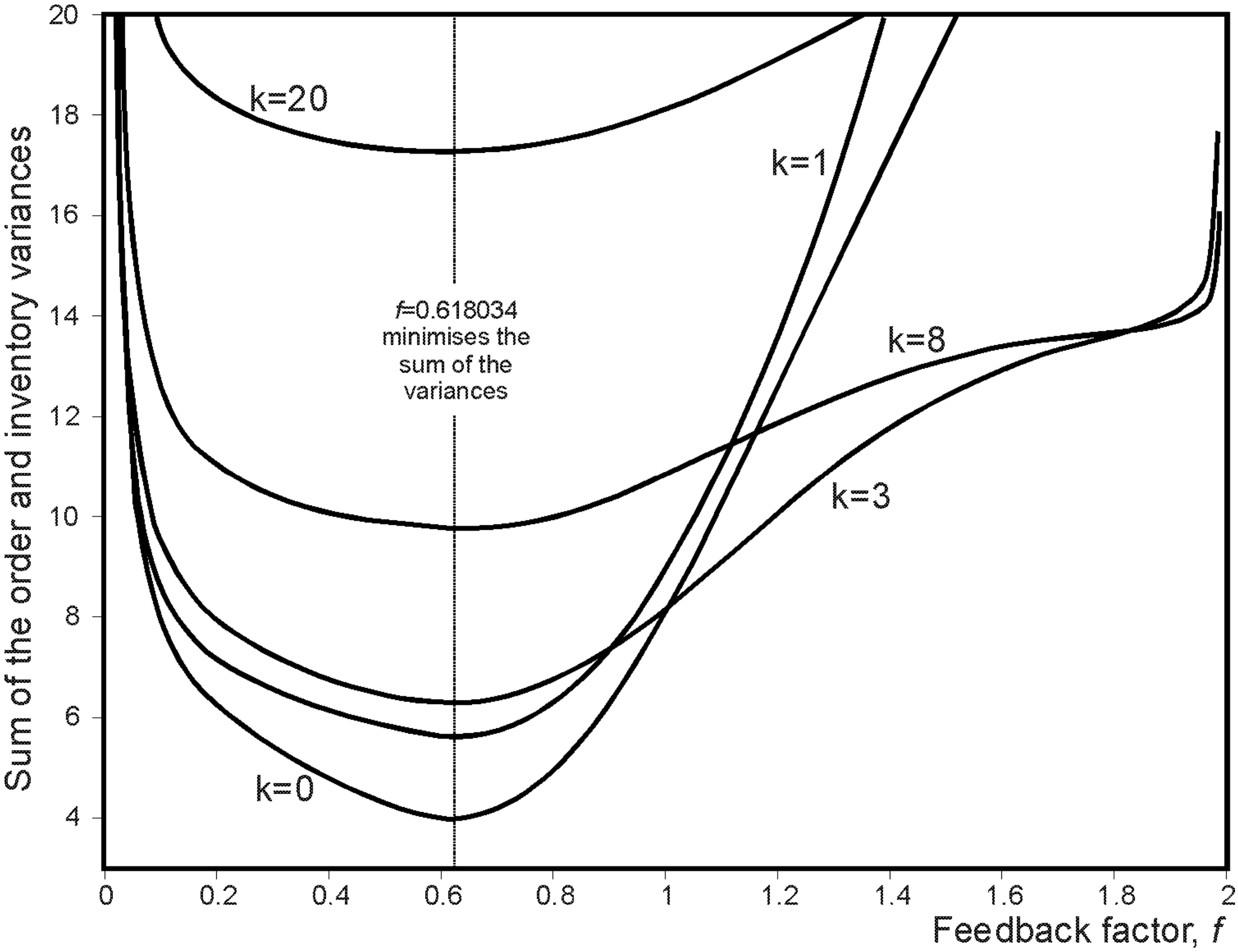


Figure II:c. The variance trade-off for the full state feedback order-up-to policy for ARMA(2,0) demand when $\phi_1 = 0.6$ and $\phi_2 = -0.9$.