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On the Lambert W function: EOQ applications and pedagogical considerations

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Abstract

The Lambert W function dates back to Euler's time and despite offering solutions to many operations management problems it is still relatively unknown. This may be due to the fact that it is only incorporated into specialist mathematical software and is not generally available in common spreadsheet applications. This obscurity is rather unfortunate as it is relatively easy to use when the significance and mechanics of the function are known. In order to illustrate the use of the Lambert W function we consider two Economic Order Quantity (EOQ) scenarios: an EOQ model with perishable inventory; and a Net Present Value (NPV) analysis of an EOQ problem with trim loss. Both scenarios are motivated by real world situations. Via these two examples we reflect upon the pedagogical aspects of using the Lambert W function, specifically at a postgraduate level, and provide guidance on the manipulation of equations containing exponentials. We present a Lambert W function 'look-up' table for classroom use (which we suggest is no more difficult to use than the standard normal table popular in operations management texts) and a Microsoft Excel 'Add-In' for self-study and practical use. We also illustrate the use of the Laplace transform to conduct NPV analyses of the EOQ model, and demonstrate the close relation between the Laplace transform and the Lambert W function. It is hoped that is paper will accelerate the resurgence of the use of the Lambert W function in our field, as it provides exact analytical solutions to many problems that currently are viewed as not having explicit solutions.

Keywords: Lambert W function, Economic Order Quantity, Net Present Value, perishable inventory, trim loss.

1. Introduction

The Lambert W function, $W[z]$, is the function that satisfies $W[z]e^{W[z]} = z$, where e is the natural exponential and z is a complex number. Named after Johann Heinrich Lambert, it is sometimes called the Omega function or the Product Log function [1] and is known as the 'golden ratio of the exponentials'. In general, $z \in \mathbb{C}$ and the Lambert W function is a multi-valued function. However, if we restrict $z \in \mathbb{R}$ then for $-e^{-1} \le z \le 0$ there are only two possible solutions of $W[z]$. The 'principle' branch satisfying $W[z] \ge -1$ is denoted by $W_0[z]$ and the 'alternative' branch satisfying $W[z] \le -1$ is denoted by $W_{-1}[z]$. For $W[z] > 0$ there is only one real solution, $W_0[z]$. Figure 1 illustrates the real solutions to the Lambert W function.

Although the Lambert W Function is not well known, it is available in modern computer software packages such as Maple and Mathematica. In Maple, it is known as 'LambertW' function, in Mathematica as the 'ProductLog' function. Corless et al [2] recently popularized the Lambert W function, beginning its resurgence as a useful, if under-appreciated function, and suggested the symbol 'W' after the pioneering work of Wright [3]. [2] show that the applications of the Lambert W function are wide ranging but often go unnoticed. They review many practical applications that include the jet fuel problem, combustion models, enzyme kinetics, molecular physics, water movement in soil, epidemics, and applications in computer science.

The Lambert W function also plays a role in the stability and evolution of the differential delay equations. [4] used the Lambert W function to identify the stability boundary for a

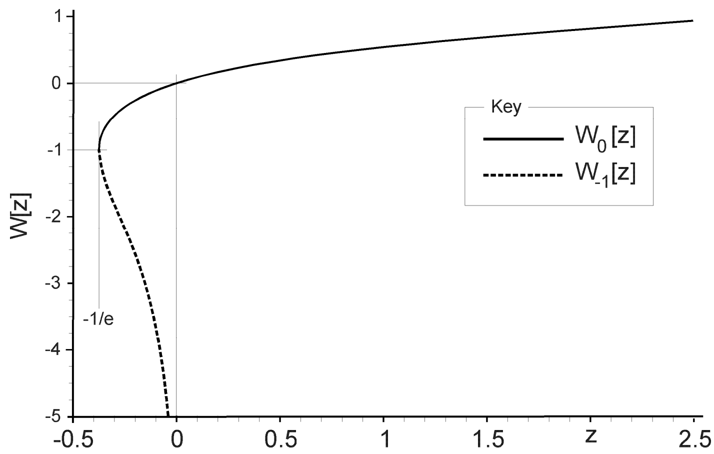


Figure 1. Real solutions of the Lambert W function

supply chain model in continuous time when a pure time delay represents the production and distribution delay. [5] characterises the time evolution of a continuous time production and inventory system. [6] use the Lambert W function in order to determine expressions for the bullwhip and inventory variance amplification produced by a continuous time Order-Up-To policy. These bullwhip and inventory variance expressions are needed in order to conduct an economic analysis of a system with random demand processes – see [7] for an example of how to achieve this in discrete time.

[8] has recently noted the usefulness of the Lambert W function for certain Economic Order Quantity (EOQ) problems. Specifically, [8] considers EOQ problems when the inventory deteriorates over time, when demand contains a stock dependent term and when the Net Present Value (NPV) of the cash flows is considered.

Herein we make a small additional contribution to literature on the Lambert W function applied to EOQ problems by studying two cases. First we consider the problem of EOQ with perishable inventory, and emphasize that there is a difference between deteriorating inventory and perishable inventory. In the deteriorating inventory scenario, the inventory physically decays and is destroyed over time, whilst perishable inventory loses value but it is not destroyed. In a second EOQ scenario we include net present value (NPV) in an EOQ problem with trim loss. The addition of the trim loss is a rather trivial extension of the classic EOQ problem [9]. Our focus here, however, is to derive the cash flows via the Laplace transform (rather than directly from the time domain description of the inventory levels and order placements as in [8]), which is novel and more interesting. We believe the Laplace transform approach is advantageous as descriptions of the cash flows are obtained in a rather concise manner.

Both of the EOQ problems studied here are motivated by real-world situations, which provide authentic data and compelling observations. The perishable inventory problem follows Blackburn and Scudder's [10] study of a melon supply chain in California, USA, while the NPV of the EOQ with trim loss problem was motivated by the case of VT Foams in the UK [11]. In order to obtain numerical solutions to these EOQ problems it is necessary to have access to the real solutions of the Lambert W function. Hence, we present a 'look-up' table of the principle and alternative solutions of the Lambert W function for classroom use. We also present the Visual Basic code needed to write a Microsoft 'Add-In' for enumerating the Lambert W function in Excel. We believe that these pedagogical features will make this paper an interesting source for teaching advanced EOQ problems with 'Lambert W' solutions to postgraduate students in both engineering and business schools.

2. EOQ with perishable inventory

Blackburn and Scudder [10] recently investigated the supply chain design for melons. Interestingly, as soon as the melons are picked from the vine, they lose value as they respire. The temperature of the melon determines the rate of respiration and hence the rate of value loss. [10] carefully and clearly describe the melon supply chain. We do not repeat this description here; we simply refer readers to [10] for information on this aspect. [10] then go on to justify an appropriate cost function, (see Eq. 3 in [10]), which we repeat below.

$$TC = \tfrac{KD}{Q} + DV - \tfrac{D}{Q\alpha}\left(pVe^{-\beta t_j - \alpha t_r}\right)\left(1 - e^{-\alpha Q/p}\right) + cD + C_j \tag{1}$$

Ignoring the exogenous variable D and the constant C_j the optimisation problem becomes

$$\min TC = \tfrac{1}{Q}\left(K - \left(pe^{-\beta t_j - \alpha t_r}V/\alpha\right)\left(1 - e^{-\alpha Q/p}\right)\right) \text{ s.t. } \{Q, p\} \geq 0 . \tag{2}$$

This is equivalent to (Eq. 5 in [10]) when their small error with the second closing bracket of the RHS is corrected. [10] show that the optimal Q satisfies (Eq. 6 in [10])

$$Q = \left(\tfrac{p}{\alpha} - \kappa\right)e^{\alpha Q/p} - \tfrac{p}{\alpha} \tag{3}$$

where the constants in (Eq. 6 in [10]) have been replaced by $\kappa = e^{\beta t_j + \alpha t_r} K/V$ for ease of exposition. Failing to recognise that this equation has an exact solution via the Lambert W function, [10] provide an approximate solution with the following lower bound,

$$Q \geq \sqrt{2p\kappa/\alpha} . \tag{4}$$

This approximation is not necessary. There is an easily obtainable exact solution with the Lambert W function. The general approach to be taken is to transpose all the Q's to the right hand side (RHS) of the equation, with the goal of manipulating it into the form $y = xe^x$. The optimal Q can then be determined from the Lambert W function, $x = W[y]$. For the melon case, we depart from (3) and collect all the Q's on the RHS by dividing throughout by $e^{-\alpha Q/p}$,

$$\tfrac{p}{\alpha} - \kappa = \left(Q + \tfrac{p}{\alpha}\right)e^{-\alpha Q/p} . \tag{5}$$

Multiplying by $-\alpha/(pe)$ yields

$$\tfrac{\kappa\alpha}{pe} - \tfrac{1}{e} = \left(-\tfrac{Q\alpha}{p} - 1\right)e^{\left(-\frac{Q\alpha}{p} - 1\right)}, \tag{6}$$

which is in the form required by the Lambert W function as the solution will then be given by $x = W[y]$. Thus the exact solution is

$$\begin{aligned} -\tfrac{Q\alpha}{p} - 1 &= W_{-1}\left[\tfrac{\kappa\alpha}{pe} - \tfrac{1}{e}\right] \\ Q^* &= -\tfrac{p}{\alpha}\left(W_{-1}\left[\tfrac{\kappa\alpha}{pe} - \tfrac{1}{e}\right] + 1\right). \end{aligned} \tag{7}$$

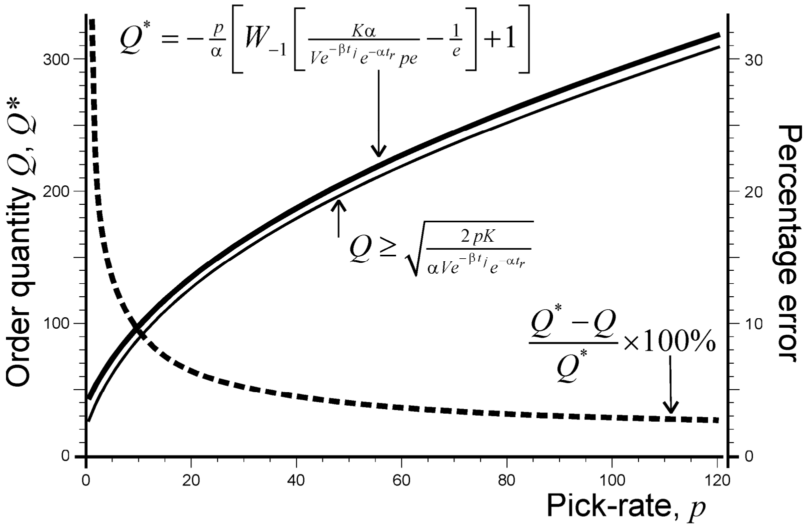


Figure 2. Blackburn and Scudder's lower bound and the exact Lambert W solution

We emphasize that (7) is an exact analytical solution for the optimal order quantity, Q^*. Notice that in (7), we have specified the alternative branch to the Lambert W function. This is because we know the optimal order quantity, the deterioration rate, and the picking rate are all positive $\left\{Q^*, \alpha, p\right\} \in \mathbb{R} \geq 0$. It then follows that $W\left[z\right] < -1$, which only occurs on $W_{-1}[z]$, the alternative branch, see Figure 1.

2.1. Practical example

Let's compare our exact solution (7) to the lower bound given by (4). We assume, as did [10], that the following enumeration is relevant: The value of the melons at picking is, V=\$7, and the deterioration rate at a field temperature of 30°C, $\alpha = 0.03$ per hour. The batch transfer time, $t_r = \frac{1}{2}$ hour, the batch transfer cost, K=\$75, the time in the cold chain, $t_j = 5$ days, and the deterioration rate in the cold chain, $\beta = 0.02$ per day.

Allowing the picking rate to vary between 1 and 120 cartons per hour produces an optimal transfer batch quantity, Q^*, as shown by Figure 2, where we have plotted both the lower bound given in [10] and the exact Lambert W solution. The lower bound (Q) consistently underestimates the true optimal batch quantity $\left(Q^*\right)$ by 8 to 14 units. The percentage error, $\left(\left(Q^* - Q\right) / Q^*\right) \times 100\%$, has also been plotted in Figure 2.

3. Net present value of an EOQ problem with trim loss

VT Foams Ltd. [11] makes foam for other manufacturers to use inside products, usually furniture or vehicles. Typically their customers order rolls of foam with specific properties, such as a certain thickness, colour, density, pore size, fire or heat resistance, and other special characteristics. There are many different types of foam offered for sale by VT Foams, with the number of different types running into the hundreds. The foam is made by the process described in Figure 3 below.

When the foam 'sets', it sets like a loaf of bread. There is a crust on the outside and the top is dome shaped. The raw loaf of foam is trimmed and the top is cut off to remove the crust. This produces a uniform block of foam, with a consistent density of holes throughout its volume. This block of foam is then bent into a 'donut' and the ends are bonded together.

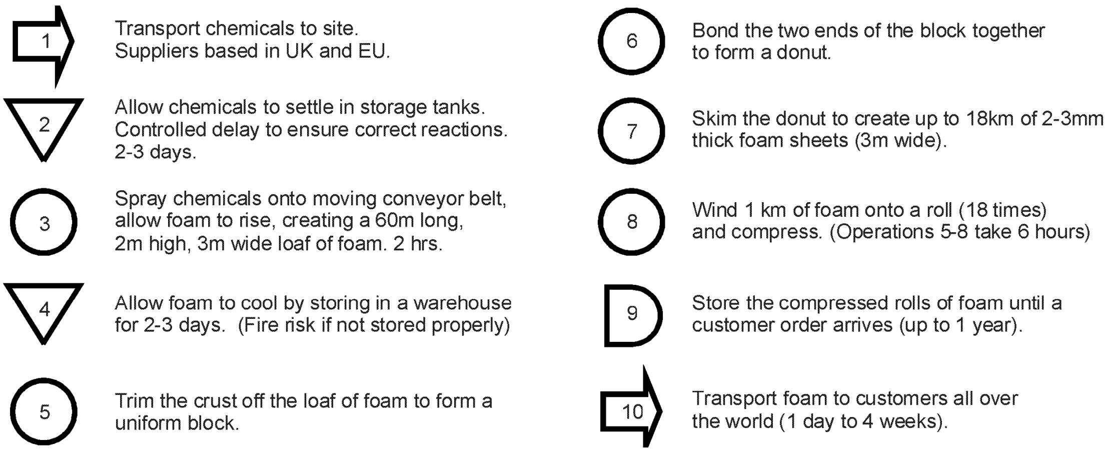
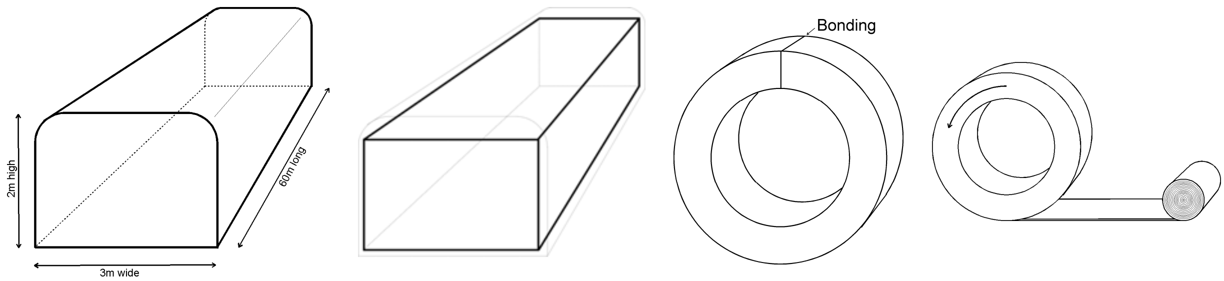


Figure 3. Overview process flow chart of VT Foams



The loaf The block The donut The skim

Figure 4. Making rolls of foam: The loaf, block, donut and skim

The donut is then skimmed (see Figure 4) to create rolls of continuous foam of a certain thickness. The thinner the slice of foam shaved of the donut, the longer the slice is. A 3mm slice will create about 18km of foam from a 2m high loaf. There is about 1km of 3mm foam on a typical roll that is sold to a customer, so that a loaf can supply up to 18 rolls of foam. Customers typically order one roll of foam at a time, the rest being stored as a finished product in a warehouse until they are later sold.

Now the loaf of foam is 60 meters long, and this is a fixed constant. If the loaf is any shorter, the ‘donut’ cannot be formed as the bend is too tight; and if the loaf is any longer, it will not fit into the skimming machine in the later stage. The width is also fixed as this is the height of the roll that will be shaved off the donut later, and can’t be changed. However the height of the loaf is a variable. If demand for that particular foam is high, more chemicals can be sprayed onto the conveyor belt and the height of the loaf will rise.

The higher the loaf that is made, the smaller the percentage of foam that is trimmed to create the uniform block. In a 2m high loaf, about 20% of the material cost is due to the wastage from the trimming activities. However, if a 1m high loaf is made, 40% of the material is lost in the trimming. About 60% of the final cost of the product is due to its material cost. The amount of waste trimmed off the loaf to create the block can be assumed to be a constant, regardless of the height of the loaf. Obviously, the department that makes the loaf wants to minimise this waste by making loafs as high as possible. However, this creates more finished goods to store later in the process (higher loafs create more rolls of foam in inventory). This conflict results in the classic Economic Order Quantity trade-off.

Also of interest here is the very long interval between successive batches. It is not uncommon for this company to make a batch that will satisfy a year's worth of customer demand. Given this long time scale, we propose that the NPV of the cash flows should be considered in the EOQ analysis.

3.1 EOQ with trim loss

First we will ignore the time value of money and briefly analyse the VT Foams Ltd. case as a classical EOQ problem. Let TC be the total annual cost to be minimised by changing Q, where Q is directly linked to the height of the loaf, the decision variable in this case. D is the annual demand for rolls of the foam variant, and h is the annual inventory holding (storage) cost per roll. k is the production set-up cost associated with the loaf, block, donut and the Q rolls. y is the cost of the lost yield due to the trimming of the loaf to create the uniform block. c is the direct cost to produce a unit of the foam (not including the holding, trim loss or order placement / set-up cost).

Under the usual assumptions for EOQ models, (see Hopp and Spearman [12] for a concise list that are also relevant here), the average inventory holding over the whole year is $Q/2$ and the average number of replenishment orders per year is D/Q Hence, the annual direct cost is given by Dc, the annual inventory cost is $Qh/2$, the annual set-up cost is Dk/Q and the annual cost of yield loss from the trimming is Dy/Q. Thus the annual costs are given by

$$\text{Total Annual Costs}\left(TC\right) = Dc + \tfrac{Qh}{2} + \tfrac{D}{Q}\left(k+y\right). \tag{8}$$

Differentiating (8) with respect to Q yields $\left(dTC/dQ\right) = \left(h/2\right) - \left(D\left(k+y\right)\right)Q^{-2}$. Setting $\left(dTC/dQ\right) = 0$ and solving for the positive root yields the optimal value of Q, Q^*, which is

$$Q^* = \sqrt{2D\left(k+y\right)/h}\,. \tag{9}$$

It is easy to show that the second derivative of (8) is always positive for relevant parameter settings, hence (9) is a minimum. Thus, as the optimal height of the loaf is related to Q, then the loaf height optimisation is simply an EOQ problem where the cost of the trimming is incorporated into the set-up cost of the traditional EOQ problem.

3.2. NPV of the cash flows in the EOQ with trim loss problem

Let NPV_{EOQ} be the NPV of the cash flows resulting from the EOQ decision for a particular product variety of foam. Our objective is to maximise NPV_{EOQ} by changing Q. All notations and assumptions from above also hold here. The production orders are placed Q/D periods apart and the inventory levels fall linearly by D per period of time. Using the Laplace transform and some rather basic control engineering knowledge (see Nise [13], or Buck and Hill [14]) we may develop the block diagram in Figure 5 to describe the cash flows in this EOQ system. The use of the Laplace transform to capture this information is possible as the NPV of a cash flow over time, $f(t)$ is given by it's the Laplace transform, $F(s)$, when the Laplace operator, s, has been replaced by the continuous discount rate, r, Grubbström [15].

$$NPV = \left[F\left(s\right) = \int_0^\infty e^{-st} f\left(t\right)dt\right]_{s=r}. \tag{10}$$

The Laplace transform approach is rather scalable as complex cash flows can be easily handled. In Figure 5 we have used; the impulse response δ, the standard input that drives

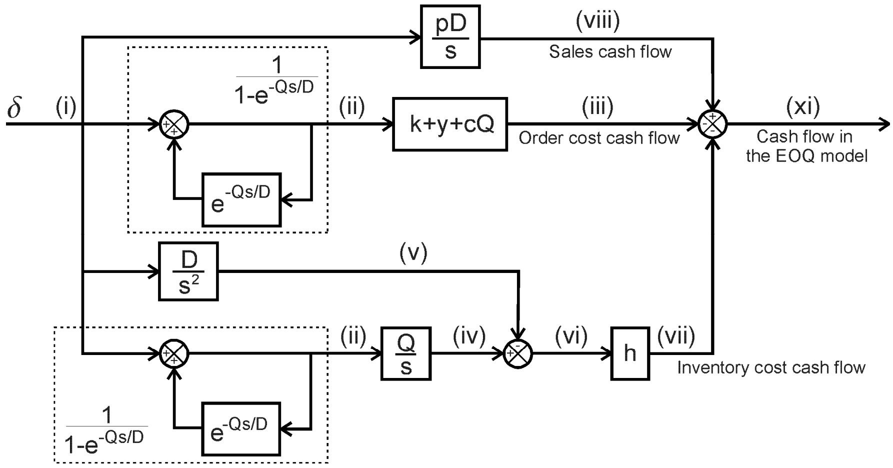


Figure 5. Block diagram of the cash flows in the EOQ with trim loss model

the whole system, the integrator s^{-1} to convert the impulse into a unit step, that has been scaled by pD to represent the sales cash flow. The feedback loop $(1-e^{-Qs/D})^{-1}$ converts the single impulse responseinto a sequence of repeating impulse responses that occur Q/D periods apart. These are scaled by $(k+y+cQ)$ to represent the order cost cash flow. The inventory cost cash flow uses; the ramp function s^{-2} with a slope is D, a repeating impulse generated by the feedback loop $(1-e^{-Qs/D})^{-1}$ which is then converted into a "staircase" by the integrator s^{-1} where each step is scaled by Q. The scaled ramp and scaled staircase is then joined and scaled by h to generate the inventory holding cost cash flow. The three flows can then be combined the give the complete cash flow of the EOQ model. Figure 6 gives an illustration of the individual signals that constitute the NPV.

Consider now each of the three cash flows in turn. Using standard block diagram manipulation techniques, we can determine the relationship between the sales cash flow and the Dirac delta function, the unit impulse, δ. The Laplace transform of the sales cash flow is given by pD/s. Setting the Laplace operator (s) to the discount rate yields the PV of the sales

$$PV_{\text{Sales}} = \tfrac{pD}{s}. \tag{11}$$

The PV of the order costs can also be determined from the block diagram, which is

$$PV_{\text{Order Cost}} = \tfrac{k+y+cQ}{1-e^{-Qs/D}}. \tag{12}$$

In a similar manner we can also obtain the PV function of the inventory cost cash flow,

$$PV_{\text{Inventory Cost}} = \tfrac{Qh}{s\left(1-e^{-Qs/D}\right)} + \tfrac{Dh}{s^2}. \tag{13}$$

The NPV of all three cash flows in the EOQ model is given by

$$NPV_{EOQ} = \frac{pD}{s} - \frac{k+y+cQ}{1-e^{-Qs/D}} - \frac{hQ}{\left(1-e^{-Qs/D}\right)s} - \frac{Dh}{s^2}. \tag{14}$$

Differentiating (14) w.r.t. Q yields,

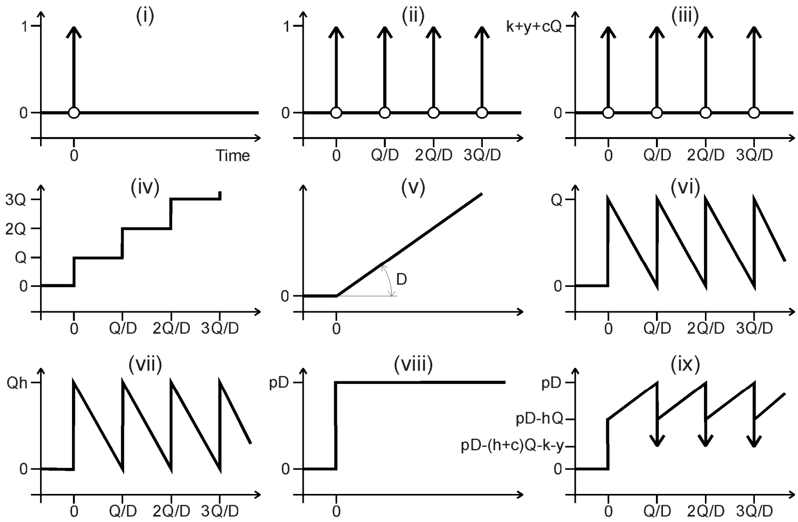


Figure 6. Block diagram signal that constitute the cash flows generated by the EOQ model

$$\frac{\mathrm{d}NPV_{EOQ}}{\mathrm{d}Q}=\frac{e^{-Qs/D}\left(D\left(h+cs\right)+\left(k+y\right)s^2+Qs\left(h+cs\right)\right)-D\left(h+cs\right)}{Ds\left(e^{-2Qs/D}-2e^{-Qs/D}+1\right)}, \tag{15}$$

where it is easy to see that the stationary points are at

$$e^{-Qs/D}\left(Qs\left(h+cs\right)+D\left(h+cs\right)+\left(k+y\right)s^2\right)=D\left(h+cs\right). \tag{16}$$

Multiplying by $-\left(D\left(h+cs\right)\right)^{-1}$ gives

$$e^{-Qs/D}\left(-\tfrac{Qs}{D}-1-\tfrac{(k+y)s^2}{D(h+cs)}\right)=-1. \tag{17}$$

Finally, multiplying by $e^{-1-\frac{(k+y)s^2}{D(h+cs)}}$ formats the stationary point in the required Lambert W form,

$$\left(-\tfrac{Qs}{D}-1-\tfrac{(k+y)s^2}{D(h+cs)}\right)e^{-\frac{Qs}{D}-1-\frac{(k+y)s^2}{D(h+cs)}}=-e^{-1-\frac{(k+y)s^2}{D(h+cs)}}, \tag{18}$$

where can now identify the following 'W' terms,

$$W[z]=\left(-\tfrac{Qs}{D}-1-\tfrac{(k+y)s^2}{D(h+cs)}\right);\ z=-e^{-1-\frac{(k+y)s^2}{D(h+cs)}}. \tag{19}$$

Re-arranging Equation (19) for Q^* yields

$$Q^*=\frac{-\left(k+y\right)s}{\left(h+cs\right)}-\frac{D}{s}\left(1+W_{-1}\left[-e^{-1-\frac{(k+y)s^2}{D(h+cs)}}\right]\right). \tag{20}$$

In (20) we have selected the alternative branch of the Lambert W function. Note that in our problem $\left\{k,y,c,s,h,D,Q^*\right\}\in\mathbb{R}\geq 0$, which implies that $W[-e^{-1-((k+y)s^2)/(D(h+cs))}]<-1$. Now the

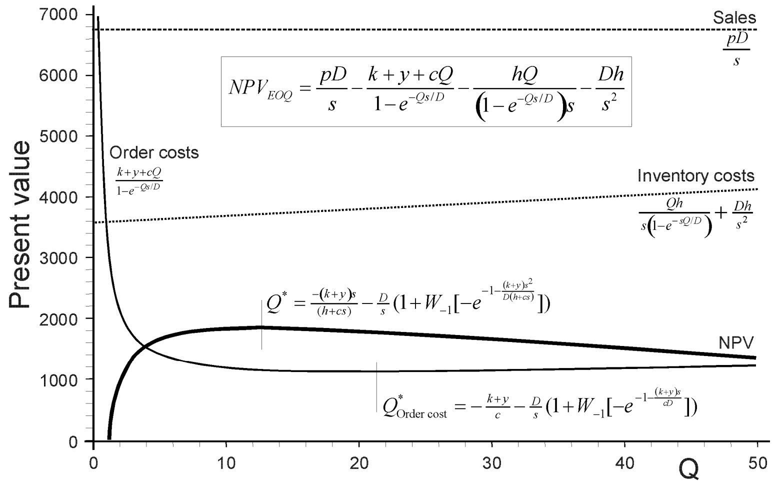


Figure 7. Net present value costs in the EOQ with trim loss against the batch size Q

exponential term $-e^{-1} \le -e^{-1-\frac{(k+y)s^2}{D(h+cs)}} \le 0$, which in turn means that $W[-e^{-1-\frac{(k+y)s^2}{D(h+cs)}}] < -1$ will only happen if the alternative branch is selected, and hence our choice in (20).

The NPV and the three individual costs have been plotted as a function of Q in Figure 7. The curves refer to the practically relevant case of sales price p = 75, demand D = 18, discount rate s = 0.2, order placement cost k = 25, direct (variable) order cost c = 10, trim loss per order y = 2, inventory holding cost h = 4. Interestingly we can see that when the individual cash flows are considered there is even a minimum in the order costs, a result that was also found by [8]. This is something that did not happen in the classical EOQ approach. We leave it to interested readers to derive the optimal Q to minimise the order costs. It is rather easy with the approach we have discussed here. Whilst the inventory cost NPV curve looks linear plotted in Figure 7, it is in fact a curve. It's derivative is $(h/2s)$ at $Q=0$ and $h/2$ at $Q=\infty$.

Finally, we note that even though the NPV curve implies that small deviations in Q^* will not result in a significant loss in NPV, the batch size given by the classical EOQ formula is $Q^*_{\text{Classical}} = 31.82$, whilst that provided by the NPV EOQ approach is $Q^*_{NPV} = 12.44$. This is implies that the VT Foams should reduce its batch size and produce more frequently. The NPV of producing to Q = 12 is £1,849.70, and to Q = 31 is £1,665.70, a fall of 10% in NPV. Given the number of batches produced per year this is a significant cost difference.

4. Pedagogical considerations

In order to arrive at these exact solutions we did not have to change anything in the way we set up the equations. It was only at the point of solving the equation did the 'Lambert W' function emerge. We only had to manipulate the equations, and the most difficult aspect of that is to recall how to treat exponential functions. This should be easily achievable by postgraduate students after a short refresher, see Table 1. The other slight complication is the selection of the relevant branch of the Lambert W function. In essence however, it is no more complicated or abstract that selecting the positive square root in the classic EOQ procedure. Indeed, selecting the wrong branch will result in a negative Q, prompting one to consider the other branch.

The Lambert W function does however require enumeration. Whilst this is easy to do in specialist mathematical software such as Maple or Mathematica, access to this software is rather limited. In order to overcome this is a class-room setting, we have provided a 'look-up' table for the real solutions to the Lambert W function in the Appendix. This may be

$e^0 = 1$	$e^{-x} = \frac{1}{e^x}$	Each value of x determines a unique value of e^x
$e^1 \approx 2.718281828459045$	$e^x \times e^y = e^{x+y}\ \forall \{x, y\} \in \mathbb{R}$	If $x > y$ then $e^x > e^y$
$\lim_{x\to\infty}[e^x] = \infty$	$(e^x)^k = e^{xk}\ \forall x \in \mathbb{R}\ \&\ k \in \mathbb{Z}$	$\frac{d(e^x)}{dx} = e^x$
$\lim_{x\to-\infty}[e^x] = 0_+$	$e^x > 0\ \forall x \in \mathbb{R}$	$e^{i\pi} = -1$

Table 1. Properties of the exponential function

printed out by interested readers and become part of their teaching materials. It is intuitive and no more difficult to use than the standard normal table used in many operations management / industrial engineering texts. However, a much more useful iterative procedure for determining the real solutions to the Lambert W function has been proposed by Johnson [16], see (21). It is based on Newton's Method and converges rather quickly (usually within 5-10 iterations, so there is no need to do limitless iterations).

$$w_{i+1} = \frac{ze^{-w_i} + w_i^2}{w_i + 1};\ W[z] = \lim_{i\to\infty} w_i \tag{21}$$

The principle branch, $W_0[z]$, can be found by using $w_0 = 0$ when $-e^{-1} \le z \le 10$, when $z > 10$ use $w_0 = \log[z] - \log[\log[z]]$. For the alternative branch, $W_{-1}[z]$, use $w_0 = -2$ if $-e^{-1} \le z < -0.1$ or $w_0 = \log[-z] - \log[-\log[-z]]$ if $0.1 \le z < 0$. [16] also proposes another iterative procedure (see (22)) for use when z is near $-e^{-1}$ that converges more rapidly than (21).

$$w_{i+1} = -1 + (w_i + 1)\sqrt{\frac{z + 1/e}{w_i e^{w_i} + 1/e}};\ W[z] = \lim_{i\to\infty} w_i \tag{22}$$

Here the principle branch, $W_0[z]$, can be found by using $w_0 = 0$. For the alternative branch, $W_{-1}[z]$, use $w_0 = -2$. It is a trivial matter to incorporate both of these iterative procedures into a User Defined Function in Microsoft Excel. The Visual Basic code is shown below in Table 2, and provides both the $W_0[z]$ and $W_{-1}[z]$ branches. This can be saved as a 'Microsoft Excel Add-In'. After such a procedure is undertaken, then "=LambertW(mode,z)" can be used within Microsoft Excel to enumerate the Lambert W Function. Interested readers may also email the authors to request an electronic copy if they so wish.

```
Function LambertW(mode As Integer, z As Double)
Dim Wo As Double
Dim Wnew As Double

Wnew = 0
If mode = 0 Then
   If z > 10 Then
   Wo = Log(z) - Log(Log(z))
   Else
   Wo = 0
   End If
Else
   If z < -0.1 Then
   Wo = -2
   Else
   Wo = Log(-z) - Log(-Log(-z))
   End If
End If
If z < -0.35 Then
   For grandloop = 1 To 10000
      Wnew = -1+(Wo+1) * ((z+(1/ Exp(1))) / (Wo*Exp(Wo) + (1/Exp(1))))^ 0.5
      If Wo = Wnew Then grandloop = 10000
      Else Wo = Wnew
      End If
   Next grandloop
Else
   For grandloop = 1 To 10000
      Wnew = ((z * Exp(-Wo)) + Wo ^ 2) / (Wo + 1)
      If Wo = Wnew Then grandloop = 10000
      Else Wo = Wnew
      End If
   Next grandloop
End If

LambertW = Wnew
End Function
```

Table 2. Visual Basic Code to enumerate the real solutions to the Lambert W function

5. Conclusions

We made a small, but we believe important, contribution to Blackburn and Scudder's [10] EOQ problem with perishable inventory by improving upon their lower bound for the optimum order quantity. We accomplished this by exploiting the properties of the Lambert W function. Recognising that the Lambert W Function is useful also for other EOQ problems with exponential terms, we have illustrated a rather general approach that exploits the Laplace transform to optimise the NPV of an EOQ problem with trim loss. The relation between the Laplace transform and the Lambert W function emerged as an interesting feature. The parallelism between these two functions was first pointed out in [17].

Both of our EOQ problems were motivated by real-world scenarios. Acknowledging that the Lambert W function is not well known, we have provided two pedagogical tools for the operations management and industrial engineering teacher. One of these tools is a standard 'look-up' table for class-room use, the other is a Microsoft Excel 'Add-In' for self study and professional purposes. Arguably, together these tools render the 'Lambert W' solutions no more difficult to use and teach than other inventory problems. Certainly they are no more difficult than a normal distribution table, which is ubiquitous in operations management texts.

6. References

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Appendix: The real solutions to the Lambert W function

Numerical solutions for $W_{-1}[z]$ and $W_0[z]$ are provided in Table 3.

z	$W_{-1}[z]$	$W_0[z]$	z	$W_0[z]$	x	$W_0[z]$	x	$W_0[z]$
-1/*e* = -0.36788	-1	-1	0.01	0.009901	0.8	0.490068	4.5	1.267237
-0.365	-1.13066	-0.87982	0.02	0.019612	0.85	0.510279	4.75	1.297615
-0.36	-1.22277	-0.80608	0.03	0.029138	0.9	0.529833	5	1.326724
-0.355	-1.29124	-0.75624	0.04	0.03849	0.95	0.548774	5.25	1.354670
-0.35	-1.34972	-0.71664	0.05	0.047672	1	0.567143	5.5	1.381545
-0.345	-1.40239	-0.68311	0.06	0.056693	1.05	0.584975	5.75	1.407432
-0.34	-1.4512	-0.65369	0.07	0.065558	1.1	0.602304	6	1.432404
-0.335	-1.49726	-0.62731	0.08	0.074273	1.15	0.619158	6.25	1.456526
-0.33	-1.54127	-0.60327	0.09	0.082844	1.2	0.635564	6.5	1.479856
-0.325	-1.58368	-0.5811	0.1	0.091277	1.25	0.651548	6.75	1.502447
-0.32	-1.62485	-0.56049	0.11	0.099574	1.3	0.667132	7	1.524345
-0.315	-1.66502	-0.54118	0.12	0.107743	1.35	0.682337	7.25	1.545593
-0.31	-1.70439	-0.52299	0.13	0.115786	1.4	0.697182	7.5	1.566230
-0.305	-1.74312	-0.50577	0.14	0.123709	1.45	0.711684	7.75	1.586292
-0.3	-1.78134	-0.4894	0.15	0.131515	1.5	0.725861	8	1.605811
-0.295	-1.81915	-0.47379	0.16	0.139208	1.55	0.739728	8.25	1.624817
-0.29	-1.85665	-0.45886	0.17	0.146791	1.6	0.753298	8.5	1.643337
-0.285	-1.89391	-0.44453	0.18	0.154268	1.65	0.766585	8.75	1.661395
-0.28	-1.93101	-0.43076	0.19	0.161642	1.7	0.779601	9	1.679016
-0.275	-1.968	-0.41749	0.2	0.168916	1.75	0.792358	9.25	1.696220
-0.27	-2.00495	-0.40468	0.21	0.176093	1.8	0.804866	9.5	1.713028
-0.265	-2.04191	-0.3923	0.22	0.183177	1.85	0.817136	9.75	1.729458
-0.26	-2.07892	-0.38031	0.23	0.190169	1.9	0.829176	10	1.745528
-0.255	-2.11604	-0.36869	0.24	0.197072	1.95	0.840997	10.25	1.761252
-0.25	-2.15329	-0.3574	0.25	0.203888	2	0.852606	10.5	1.776647
-0.245	-2.19073	-0.34643	0.26	0.210621	2.05	0.86401	10.75	1.791726
-0.24	-2.2284	-0.33576	0.27	0.217272	2.1	0.875219	11	1.806502
-0.235	-2.26633	-0.32537	0.28	0.223843	2.15	0.886238	11.25	1.820988
-0.23	-2.30457	-0.31523	0.29	0.230337	2.2	0.897074	11.5	1.835195
-0.225	-2.34315	-0.30535	0.3	0.236755	2.25	0.907734	11.75	1.849135
-0.22	-2.38212	-0.29569	0.31	0.2431	2.3	0.918224	12	1.862816
-0.215	-2.42151	-0.28626	0.32	0.249373	2.35	0.928548	12.25	1.876250
-0.21	-2.46136	-0.27703	0.33	0.255575	2.4	0.938714	12.5	1.889445
-0.205	-2.50173	-0.26801	0.34	0.26171	2.45	0.948725	12.75	1.902409
-0.2	-2.54264	-0.25917	0.35	0.267777	2.5	0.958586	13	1.915152
-0.195	-2.58415	-0.25051	0.36	0.27378	2.55	0.968303	13.25	1.927680
-0.19	-2.62631	-0.24203	0.37	0.279719	2.6	0.97788	13.5	1.940001
-0.185	-2.66917	-0.23371	0.38	0.285595	2.65	0.98732	13.75	1.952121
-0.18	-2.71277	-0.22554	0.39	0.291411	2.7	0.996629	14	1.964049
-0.175	-2.75718	-0.21752	0.4	0.297168	2.75	1.005809	14.25	1.975789
-0.17	-2.80245	-0.20965	0.41	0.302866	2.8	1.014864	14.5	1.987347
-0.165	-2.84866	-0.20192	0.42	0.308508	2.85	1.023799	14.75	1.998730
-0.16	-2.89587	-0.19432	0.43	0.314094	2.9	1.032616	15	2.009943
-0.155	-2.94415	-0.18684	0.44	0.319625	2.95	1.041318	15.25	2.020991
-0.15	-2.99359	-0.17949	0.45	0.325104	3	1.049909	15.5	2.031879
-0.145	-3.04429	-0.17226	0.46	0.33053	3.05	1.058391	15.75	2.042611
-0.14	-3.09633	-0.16514	0.47	0.335905	3.1	1.066768	16	2.053192
-0.135	-3.14983	-0.15813	0.48	0.34123	3.15	1.075042	16.25	2.063627
-0.13	-3.2049	-0.15122	0.49	0.346506	3.2	1.083216	16.5	2.073919
-0.125	-3.26169	-0.14442	0.5	0.351734	3.25	1.091292	16.75	2.084073
-0.12	-3.32033	-0.13772	0.51	0.356914	3.3	1.099273	17	2.094092
-0.115	-3.38099	-0.13111	0.52	0.362049	3.35	1.107161	17.25	2.103980
-0.11	-3.44387	-0.1246	0.53	0.367138	3.4	1.114958	17.5	2.113741
-0.105	-3.50918	-0.11817	0.54	0.372183	3.45	1.122667	17.75	2.123377
-0.1	-3.57715	-0.11183	0.55	0.377184	3.5	1.130289	18	2.132892
-0.095	-3.64808	-0.10558	0.56	0.382143	3.55	1.137827	18.25	2.142289
-0.09	-3.72228	-0.09941	0.57	0.387059	3.6	1.145283	18.5	2.151571
-0.085	-3.80014	-0.09331	0.58	0.391934	3.65	1.152657	18.75	2.160742
-0.08	-3.88211	-0.0873	0.59	0.396769	3.7	1.159953	19	2.169802
-0.075	-3.96871	-0.08136	0.6	0.401564	3.75	1.167172	19.25	2.178756
-0.07	-4.06059	-0.07549	0.61	0.406319	3.8	1.174316	19.5	2.187606
-0.065	-4.15853	-0.06969	0.62	0.411037	3.85	1.181385	19.75	2.196354
-0.06	-4.2635	-0.06396	0.63	0.415716	3.9	1.188383	20	2.205003
-0.055	-4.37672	-0.0583	0.64	0.420359	3.95	1.19531	20.25	2.213554
-0.05	-4.49976	-0.05271	0.65	0.424965	4	1.202168	20.5	2.222011
-0.045	-4.63465	-0.04717	0.66	0.429535	4.05	1.208958	20.75	2.230376
-0.04	-4.78419	-0.0417	0.67	0.434071	4.1	1.215682	21	2.238649
-0.035	-4.95225	-0.03629	0.68	0.438571	4.15	1.222341	21.25	2.246834
-0.03	-5.14448	-0.03094	0.69	0.443037	4.2	1.228936	21.5	2.254932
-0.025	-5.36964	-0.02565	0.7	0.44747	4.25	1.235469	21.75	2.262946
-0.02	-5.64232	-0.02041	0.71	0.45187	4.3	1.24194	22	2.270876
-0.015	-5.98976	-0.01523	0.72	0.456238	4.35	1.248352	22.25	2.278725
-0.01	-6.47278	-0.0101	0.73	0.460573	4.4	1.254704	22.5	2.286495
-0.005	-7.284	-0.00503	0.74	0.464877	4.45	1.260999	22.75	2.294186
0	$-\infty$	0	0.75	0.46915	4.5	1.267238	23	2.301801

Table 3. Enumeration of the real solutions to the Lambert W function