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The Value of Coordination in a Two Echelon Supply Chain: Sharing information, policies and parameters

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We study a coordination scheme in a two echelon supply chain. It involves sharing details of replenishment rules, lead-times, demand patterns and tuning the replenishment rules to exploit the supply chain's cost structure. We examine four different coordination strategies; naïve operation, local optimisation, global optimisation and altruistic behaviour on behalf of the retailer.

We assume the retailer and the manufacturer use the Order-Up-To policy to determine replenishment orders and end consumers demand is a stationary i.i.d. random variable. We derive the variance of the retailer's order rate and inventory levels and the variance of the manufacturer's order rate and inventory levels. We initially assume that costs in the supply chain are directly proportional to these variances (and later the standard deviations) and investigate the options available to the supply chain members for minimising costs.

Our results show that if the retailer takes responsibility for supply chain cost reduction and acts altruistically by dampening his order variability, then the performance enhancement is robust to both the actual costs in the supply chain and to a naïve or uncooperative manufacturer. Superior performance is achievable if firms coordinate their actions and if they find ways to re-allocate the supply chain gain.

Keywords.
Supply chains; Bullwhip; Inventory variance; Local optimisation, Global optimisation

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1. Introduction

We study a two-echelon supply chain model consisting of a retailer and a manufacturer reacting to stationary i.i.d. stochastic consumer demand. Both echelons implement a periodic review Order-Up-To (OUT) replenishment policy. The inventory position is reviewed every period (e.g. daily, weekly) and an order is placed to raise the inventory position up to an order-up-to or base stock level that determines the order (production) quantities.

When making replenishment decisions two primary factors must be considered. Firstly, a replenishment rule has an impact on order variability (as measured by the bullwhip effect, i.e. the ratio of the variance of orders over the variance of demand) shown to the manufacturer (or supplier). Secondly, the replenishment rule has an impact on the variance of the net stock (as measured by the net stock amplification, i.e., the ratio of net stock variance over the variance of demand).

It is well known that reducing the bullwhip effect has an adverse effect on the net stock amplification and consequently on customer service, Disney, Farasyn, Lambrecht, Towill and Van de Velde (2006a). In other words, bullwhip reduction comes at a price. This is a key trade-off faced by the actors in a supply chain. In order to capture both aspects mentioned above, we develop a generalized Order-Up-To policy that includes a proportional controller in order to be able to alter its dynamic response.

Both echelons implement the generalized OUT policy and consequently incur costs that we assume to be directly proportional to the long term variance of the inventory level and the long term variance of the replenishment orders. The combined total cost of the retailer and manufacturer has to be minimized. This can be done in several ways. One possibility is that both the retailer and the manufacturer act independently and minimize their own local costs. The objective of this paper however, is to ascertain whether a global optimization will result in superior performance. The supply chain coordination is not only realized through demand information sharing, but the actors also have to share information with respect to the parameters of the replenishment rules used and the lead-times. This is of course a far more advanced coordination scheme than what is often proposed in the literature.

The major contributions of this paper can be summarized as follows. We examine OUT policies in a multi echelon environment aiming at minimizing both the impact of bullwhip and net stock amplification (which determines the safety stock required) through a local or global optimization procedure. Several supply chain coordination mechanisms are proposed.

This paper is organized as follows. Section 2 reviews pertinent literature. In section 3 the generalized Order-Up-To policy is introduced. We use two metrics to measure performance; the order variance metric (bullwhip) and the inventory variance metric (indicating the safety stock needed). These measures are quantified for the retailer in section 4 and for the manufacturer in section 5. Combining these metrics into one objective function allows us to measure total supply chain performance. In section 6 we analyse the impact of supply chain members who are primarily concerned with optimizing their own single echelon objectives. In section 7, the supply chain members aim at a global optimum by coordinating the replenishment policies. A near optimal, but easy to implement policy is introduced in section 8. Section 9 presents the results from an investigation with an alternative objective function based on the standard deviation, rather than the variance, of the fluctuations in the net stock levels and order rates. Section 10 briefly discusses some of the practical issues involved in each of the four coordination strategies. Section 11 concludes.

2. Literature background

The problem introduced in section 1 is multi-facetted. There is no literature available dealing with all parts simultaneously. Important insights however, can be obtained from analysis of the separate elements of the problem. We therefore shortly review the literature on bullwhip quantification in multi-echelon environments, smoothing replenishment rules, the quantification of the inventory variance, information sharing and supply chain coordination.

Lee, Padmanabhan and Wang (2004) offer an excellent up-to date review of the bullwhip effect. Chen, Drezner, Ryan and Simchi-Levi (2000) quantify the bullwhip effect and show that the classical OUT policy with exponential smoothing and moving average forecasting for auto regressive demands always results in bullwhip. This work was extended by Dejonckheere, Disney, Lambrecht and Towill (2003a). Chatfield, Kim, Harrison and Hayya (2004) investigate the impact of stochastic lead-times, information quality and information sharing in the OUT policy via a simulation experiment.

Recent work on smoothing replenishment rules can be also found in Dejonckheere, Disney, Lambrecht and Towill (2003a, 2003b) and Balakrishnan, Geunes and Pangburn (2004). Hoberg, Bradley and Thonemann (2004) analyzed the effect of three inventory policies (inventory on-hand, installation-stock and echelon-stock policies) on supply chain performance. Various authors quantify the bullwhip under i.i.d., AR, ARMA and ARIMA demand processes (for example see, Zhang (2004) and Gilbert (2005)).

The issue of inventory variance also has a long history. Vassian (1955) developed a periodic review rule resulting in a minimum inventory variance to any sequence of forecasting errors. Magee (1956, 1958) also produced pioneering work on the development of smoothing rules (our analysis builds on the work of Magee). Other key papers were written by Deziel and Eilon (1967) and Simon (1952). More recent work can be found in Graves (1999), Disney and Towill (2003), Warburton (2004a, 2004b) and Disney, Farasyn, Lambrecht, Towill and Van de Velde (2006a).

Recently a lot of attention has been directed towards the value of information sharing in supply chains. We refer to Chen, Drezner, Ryan and Simchi-Levi (2000), Lee and Whang (2000) and Dejonckheere, Disney, Lambrecht and Towill (2004). They all conclude that the bullwhip effect can be reduced, but not eliminated by centralizing demand information. The advantages and issues involved in advanced supply chain coordination schemes are provided by Holweg et al (2005). Although the specific scheme we propose herein is not explicitly discussed in Holweg et al (2005), some of the issues involved are revealed there.

In his excellent review paper on supply chain coordination Cachon (2003) emphasises coordination actions and transfer payments that ensures each firm's objective becomes aligned with the supply chain's objective. This approach results in buyback contracts, revenue-sharing contracts, quantity-flexibility contracts, sales-rebate contracts and quantity-discount contracts. Cachon and Lariviere (2005) discuss revenue sharing contracts and their relationship to other types of contracts in supply chains with risk-neutral agents. Gan, Sethi and Yan (2004) investigate decision making by risk adverse agents in a supply chain. In this paper we suggest a method to improve supply chain performance by coordinating, or aligning, the parameters of replenishment rules. To our knowledge the only similar paper we know of is Hosoda and Disney (2006), where the role of the proportional controller was studied in relation to supply chain inventory costs with AR(1) demand.

3. The generalized Order-Up-To replenishment policy

We assume demand is a stationary i.i.d. random process with a positive mean, $\mu_D > 4\sigma_D$, greater than four standard deviations of the demand variance to ensure the likelihood of negative demand is negligible. For such a demand process the Order-Up-To (OUT) is an appropriate replenishment policy to generate orders on either the supplier or a production process. The order produced by the OUT policy is defined by

$$O_t = S_t - \textit{inventory position}, \tag{1}$$

where O_t is the ordering decision made at the end of period *t*. The inventory position equals the net stock (*NS*) plus the inventory on order (Work In Progress or WIP). The net stock equals inventory on hand minus backlog thus,

$$O_t = (Tp + 1 + a)\hat{D}_t - NS_t - WIP_t$$

$$O_t = \hat{D}^{\alpha}{}_t + (a\hat{D}_t - NS_t) + (Tp.\hat{D}_t - WIP_t) \tag{2}$$

where; $a\hat{D}_t$ can be viewed as a target net stock (safety stock), $Tp.\hat{D}_t$ as a target pipeline stock (on order inventory), where is *Tp* is the physical lead-time, and the unit of forecasted demand is to cover the review period. In (2) we have decomposed the original formula into three components: a demand forecast, a net stock discrepancy term and a WIP or pipeline discrepancy term, see Dejonckheere, Disney, Lambrecht and Towill (2003a). We will also introduce a proportional controller, 1/*Ti,* into the classical OUT policy in order be able to tune the dynamic response.

$$O_t = \hat{D}_t + \frac{a\hat{D}_t - NS_t}{Ti} + \frac{Tp.\hat{D}_t - WIP_t}{Ti} \tag{3}$$

We call (3) the generalized OUT replenishment policy. The proportional controller is a common control engineering technique used by hardware engineers to dampen the response of dynamic systems. Indeed, the first modern control system, the Maxwell Governor, which proved so useful in the industrial revolution for controlling the velocity of steam powered machinery, has a proportional controller in its velocity feedback loop. It is the most simple control engineering technique for this purpose, but others, such as the PI controller, also exist – see, for example Towill, Evans and Cheema (1997).

The proportional controller also has a long - but largely unnoticed - history in inventory control, for example see Magee (1956). An obvious alternative to the matched controllers in (3) would be to use unmatched, or independent, proportional controllers, as detailed in Disney and Towill (2002). Using this approach it is important to consider stability issues as the system has a much wider range of possible dynamic responses. However, for the objective function used in this paper (the sum of the inventory and order variances), it is not possible for unmatched controllers to show superior performance over the matched controllers and thus we ignore this case here.

As we have assumed the demand is stationary i.i.d. the best possible forecast for all future demands is $\hat{D}_t = \overline{D} = \mu_D$, that is, to set the forecast $\hat{D}_t$, to a time invariant constant $\overline{D}$, equal to the unconditional mean of the demand process, μ_D. Arguably this is a special case but it does allow us to proceed with the detailed analysis.

Expression (3) reduces to

$$O_t = \overline{D} + \frac{(a\overline{D} - NS_t + Tp.\overline{D} - WIP_t)}{Ti}. \tag{4}$$

After substitution (see Disney et al., (2006a)) we obtain

$$O_t = O_{t-1} + \frac{(D_t - O_{t-1})}{Ti}. \tag{5}$$

If Ti=1 expression (5) reduces to $O_t = D_t$. The order quantity is exactly equal to the consumer demand as the replenishment rule has simply "passed on orders". Importantly, for Ti>1 the generalized OUT policy will create a smoothed replenishment pattern and for Ti<1 bullwhip is created.

4. The retailer's order and inventory variance

Let us now start to construct our supply chain model by first considering the retailer. The retailer's demand from the consumers is a stationary i.i.d. random process. In section 6 we assume the retailer incurs two types of costs; one directly proportional to the long-run variance of the retailers inventory level and the other directly proportional to the long-run variance of the retailer replenishment orders placed on the supplier (manufacturer). So we wish to quantify these variances produced by the retailer's replenishment rule. This has already been achieved by Dejonckheere, Disney, Farasyn, Janssen, Lambrecht, Towill and Van de Velde (2002) and Disney, Farasyn, Lambrecht, Towill and Van de Velde (2006a) and we now summarise their results. The retailers order variance is given by

$$\frac{\sigma_{RO}^2}{\sigma_D^2} = \frac{1}{2Ti - 1} \tag{6}$$

in which σ_D^2 denotes the variance of consumer demand and σ_{RO}^2 is the variance of orders placed by the retailer on the manufacturer. Interestingly we note that under the assumptions of; stationary i.i.d demand, forecasts generated by conditional expectation, and matched feedback controllers, bullwhip is independent of the lead-time, *Tp*. If either of these assumptions are not meet, than the bullwhip effect does depend upon the lead-time, *Tp*. The classical OUT ($Ti = 1$) policy's order rate variance amplification ratio is unity. By using Ti>1 the generalised OUT policy will remove order variance amplification.

The variance of the retailer's net stock is (inventory on hand minus backlog) given by

$$\frac{\sigma_{RNS}^2}{\sigma_D^2} = 1 + Tp + \frac{(Ti - 1)^2}{2Ti - 1} = Tp + \left(Ti^2 \frac{1}{2Ti - 1} \right). \tag{7}$$

σ_{RNS}^2 denotes the variance of the retailers net stock. From (7) we observe that for this policy with stationary i.i.d. demand:

- If Ti =1, i.e. a "chase sales" strategy is adopted, then the inventory variance is 1 + *Tp*. In this case the retailer simply passes on the consumers demand to the manufacturer, that is, the retailer "passes on orders".
- If Ti >1 or Ti < 1 then inventory variance increases.
- Inventory variance is always greater than 1+*Tp,* highlighting the fallacy of a zero inventory target with our policy.
- Inventory variance contains a lead-time component, $1 + Tp$, and a smoothing component, $\frac{(Ti-1)^2}{2Ti-1}$.
- Decreasing the lead-time (*Tp*) reduces inventory variance.
- The longer the lead-time the smaller the relative importance of *Ti* on inventory variance.
- The inventory variance approaches *Tp*+*Ti*/2 asymptotically as the lead-time increases.

- For a "level scheduling" strategy, $Ti=\infty$, in which case the inventory variance is ∞.

When Tp=1 (a further assumption we will make later in order to ease the exposition of our investigation on the manufacturer in the supply chain) equation (7) reduces to

$$\frac{\sigma^2_{RNS,Tp=1}}{\sigma^2_D}=\frac{Ti(2+Ti)-1}{2Ti-1}. \tag{8}$$

For completeness we note that the variance of WIP is given by (9), which is of the same form as the smoothing component in the inventory variance equation.

$$\frac{\sigma^2_{WIP}}{\sigma^2_D}=\frac{(Ti-1)^2}{2Ti-1} \tag{9}$$

5. The manufacturer's order and inventory variance

The manufacturer responds to the retailer's orders and we assume that he uses our modified OUT policy (3) for scheduling production. Thus we may couple two generalised OUT policies together and investigate the manufacturers order and inventory variance. For simplicity, we assume, from this point on for the rest of the paper unless explicitly stated, a unit replenishment lead-time at both the retailer and the manufacturer. However, for completeness, the general lead-time case is detailed in the Appendix. Interested readers may use the expressions in the appendix to explore scenarios where the manufacturer's lead-time is greater than unity. It is remarkable to note that the retailer's lead-time does not influence the order variances at either echelon of the supply chain. In fact, the retailer's lead-time only affects the retailer's inventory levels, whilst the manufacturer's lead-time influences both the manufacturer's order and inventory variance.

The retailers order process, after passing through the generalised OUT policy, is now auto-correlated. Thus, we exploit this structural information to forecast both the demand over the lead-time and the demand in the period after the lead-time with conditional expectation. This alters the way in which the target pipeline content ($Tp.\hat{D}_t$ in (3)) and the target net stock ($a\hat{D}_t$ in (3)) is calculated in the generalised OUT policy, but has the advantage of generating optimal forecasts. These forecasts are optimal in the sense that they minimise the mean squared error between the forecast and its realisation over the lead-time and review period. More details can be found in the Appendix where a control theory derivation of the manufacturer's order and inventory variance is presented. Here it will suffice to just present our results. The manufacturers order variance for the case when $Tp=Mp$=1 is given by

$$\frac{\sigma^2_{MO}}{\sigma^2_D}=\frac{2Mi^2(Ti-1)^4-(Ti-1)Ti^2\left(2+(Ti-4)Ti\right)+MiTi\left(Ti\left(14+Ti\left(Ti(5+2Ti)-16\right)\right)-4\right)}{(2Mi-1)Ti^4(Mi+Ti-1)(2Ti-1)} \tag{10}$$

and the manufacturer's inventory variance when $Tp=Mp$=1 is given by

$$\frac{\sigma^2_{MNS}}{\sigma^2_D}=\frac{Mi^2(1-2Ti)^2+Ti^2(2Mi-1)}{(2Mi-1)Ti^4}=\frac{Mi^2(1-2Ti)^2}{(2Mi-1)Ti^4}+\frac{1}{Ti^2} \tag{11}$$

where, Ti is the retailer's proportional feedback controller, as before, and Mi is the corresponding feedback controller in the manufacturer's replenishment policy. The limit values of (10) and (11) contain some interesting insights. When Ti approaches infinity, the manufacturers order and inventory variances both go to zero. This is due to the fact that the retail orders are simply a constant (equal to the mean demand). When the manufacturers feedback gain, Mi, approaches infinity, the variance of the manufacturers net stock is infinite. This is because the net stock is now an accumulation of random variables and is non-stationary with no natural mean. Equation (11) shows that increasing Ti will reduce the manufacturers inventory requirements, but increasing Mi will increase the manufacturers inventory requirements.

In the following sections, we develop several supply chain policies ranging from local optimization to global optimization.

6. The sequential optimisation scenario: The self-serving focus

In order to test various supply chain policies, we have to define a cost function to be used. We assume that the costs in the supply chain are linearly related to the variance of the order rate and inventory levels at each echelon in the supply chain. For example, we assume the inventory holding and shortage costs are linearly related to the inventory variance, and production / replenishment on-costs resulting from variable schedules are linearly related to the order variance. Furthermore we assume inventory variance is equally as costly as order variance.

This cost function may of course be subject of debate. The σ^2_{RNS} and σ^2_{MNS} terms are directly linked to the safety stock needed (see Disney et al. (2006a)) and consequently cover inventory holding costs of the safety stock and backorder costs. The σ^2_{RO} and σ^2_{MO} terms are linked to production switching costs or adjustments costs due to order rate changes (capacity adjustment costs). We consider order rate variance as equally important as inventory level variance. It is however perfectly possible that the bullwhip may be more costly then inventory variance amplification or vice versa, depending on the context. In this case we have to apply weights to these factors. This may change the shape of the cost curves that are derived in this paper. We refer to Disney, Towill and Van de Velde (2004), for an investigation on such weighting schemes. Alternative objective cost functions are also discussed in Kim and Ryan (2003), Disney and Grubbström (2004) and Chen and Disney (2003).

6.1. The selfish retailer

First, let us consider the retailer. If the retailer only incurs inventory related costs (that is the costs related to the order variance are constant or zero) then the retailer costs are given by (7) and the optimal behaviour of the retailer is to set Ti=1 as minimising (7) w.r.t. Ti results in Ti^*=1.

However if the retailer has both inventory and order related costs then his costs are given by (12),

$$R_{\pounds} = \frac{\sigma^2_{RO}}{\sigma^2_D} + \frac{\sigma^2_{RNS}}{\sigma^2_D} = \frac{Ti^2 + 2Ti}{2Ti - 1} \tag{12}$$

which is plotted in Figure 1 together with (6) and (7).

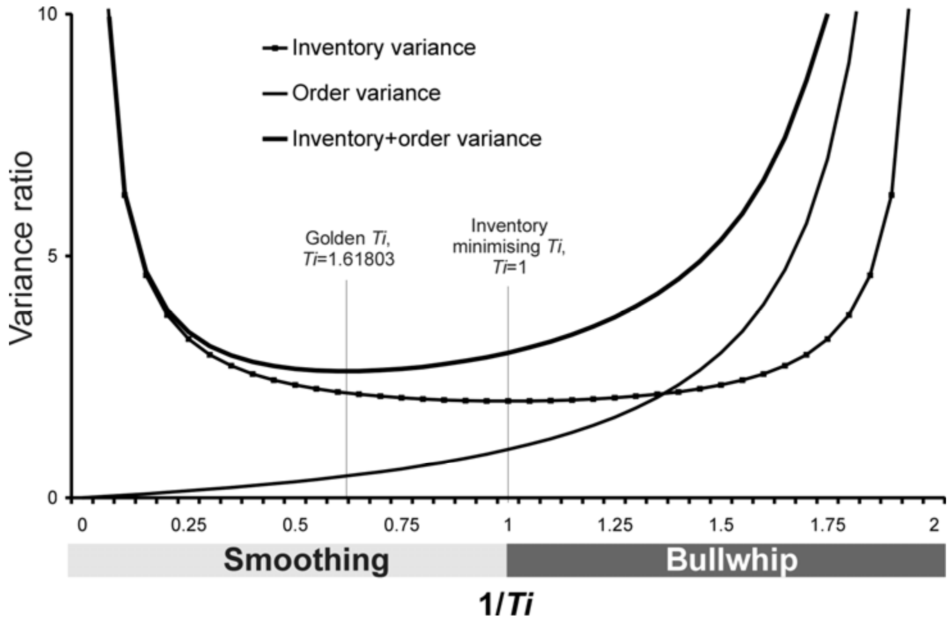

Figure 1. The i.i.d. one echelon variability trade-off

Now differentiating (12) w.r.t. Ti yields $\frac{2Ti(Ti-1)-2}{(1-2Ti)^2}$, solving for zero gradient and selecting the relevant root yields the optimum Ti to minimise the sum of bullwhip and inventory variance. It is $Ti^*=\frac{1+\sqrt{5}}{2}$, that we will recognise as the "Golden Ratio". So, the optimal Ti in this case has a long mathematical history.

For illustration we have simulated the "golden" response (i.e. Ti set to the Golden Ratio) to an i.i.d. random demand pattern when Tp=1, see Figure 2. The frequency histograms refer to a simulation 10,000 time periods in length. We can see that after 10,000 time periods the statistical process is reasonably close to the theoretical values of Bullwhip (0.447) and inventory variance (2.171). Simulating for a longer time period will obviously reduce this error.

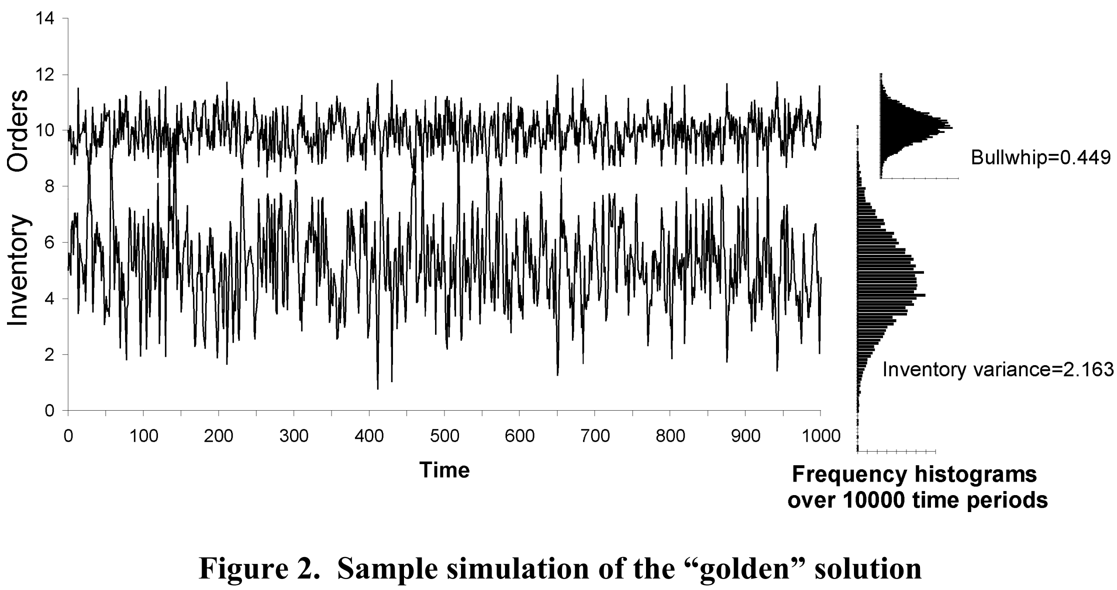

Figure 2. Sample simulation of the "golden" solution

6.2 The selfish manufacturer

Now let us consider the manufacturer. If the retailer has used Ti=1 to minimise the retailers inventory costs, the manufacturer faces a demand pattern that is exactly equal to the consumers demand as the retailer has simply "passed on the orders". The manufacturer's variance ratio and cost analysis in this case is exactly the same as the retailer's variance trade-off. Thus our previous remarks in Section 6.1 for the retailer hold for the manufacturer. That is the manufacturer's cost and the optimal feedback controller, Mi*, are given by (12) and the golden ratio respectively[1].

However, if the retailer has used the golden Ti in the inventory and WIP feedback loops, then the manufacturer's demand process has changed, and thus, his actions now have different consequences. Let us illustrate further. Using (10) and (11), with Ti equal to the golden ratio, the manufacturer's order and inventory variance (and their sum) is shown in Figure 3.

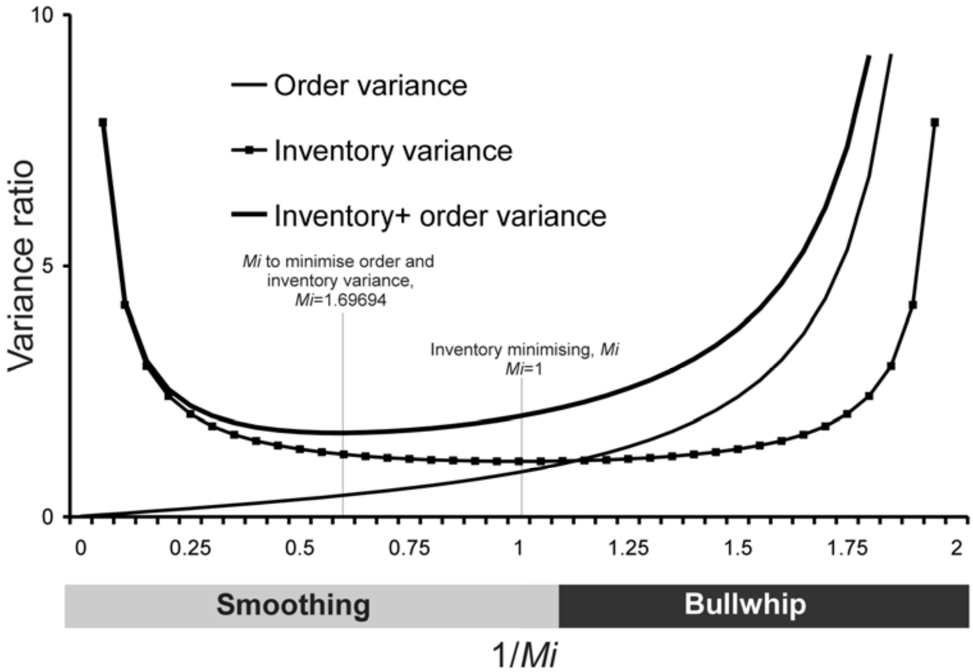

Figure 3. The manufacturer's variance trade-off with a golden retailer

Figure 3 shows there is a much greater region of order smoothing in the manufacturers replenishment policy with a golden retailer as the manufacturers order variance is less than unity for all Mi>0.939219. The manufacturer's local cost when both inventory and order variance costs are present is given by

$$M_{£} = \frac{16\left(\sqrt{5} - 5 + Mi\left(10 + 36\sqrt{5} + Mi\left(49\sqrt{5} + 50Mi - 25\right)\right)\right)}{5\left(1+\sqrt{5}\right)^4 \left(2Mi - 1\right)\left(\sqrt{5} + 2Mi - 1\right)}. \tag{13}$$

Differentiating (13) w.r.t. Mi, solving for zero gradient and selecting the relevant root yields

$$Mi^*_{O+NS,Ti=\Phi} = 1.69694. \tag{14}$$

[1] *With the appropriate change in notation; Ti→Mi*

However, if the manufacturer's cost function consists of costs associated with the variance of the inventory levels only, then the cost function to be minimised is

$$M_{£} = \frac{16Mi\left(3+\sqrt{5}+5Mi\right)-8\left(3+\sqrt{5}\right)}{\left(1+\sqrt{5}\right)^{4}\left(2Mi-1\right)}. \tag{15}$$

Again minimising (14) w.r.t Mi yields,

$$Mi^{*}_{NS,Ti=\Phi} = 1. \tag{16}$$

Summarising our results from the sequential local optimisation of the supply chain we have developed the following table that details the feedback gains and the resulting costs.

		Manufacturer incurs	
		Inventory variance costs only	**Inventory and order variance costs**
Retailer incurs	**Inventory variance costs only**	$Ti^* = 1$ $Mi^* = 1$ $R_{£} = 2$ $M_{£} = 2$ $SC_{£} = 4$	$Ti^* = 1$ $Mi^* = 1.61803$ $R_{£} = 2$ $M_{£} = 2.61803$ $SC_{£} = 4.61803$
	Inventory and order variance costs	$Ti^* = 1.61803$ $Mi^* = 1$ $R_{£} = 2.61803$ $M_{£} = 1.11146$ $SC_{£} = 3.72946$	$Ti^* = 1.61803$ $Mi^* = 1.69694$ $R_{£} = 2.61803$ $M_{£} = 1.661384$ $SC_{£} = 4.299418$

Table 1. The self-serving solutions

7. The global optimisation problem: Supply chain coordination

In this section we will show that the self-serving focus results in poor performance by considering what happens when supply chain members coordinate the replenishment decisions. Equations (6), (7), (10) and (11) allow us to explore the complete solution space for feedback design in our supply chain. More specifically, we can compute the values of the feedback controllers that the supply chain parties can use to improve overall costs. For example, the following contour plots (Figure 4) illustrate the performance of the supply chain for all possible settings.

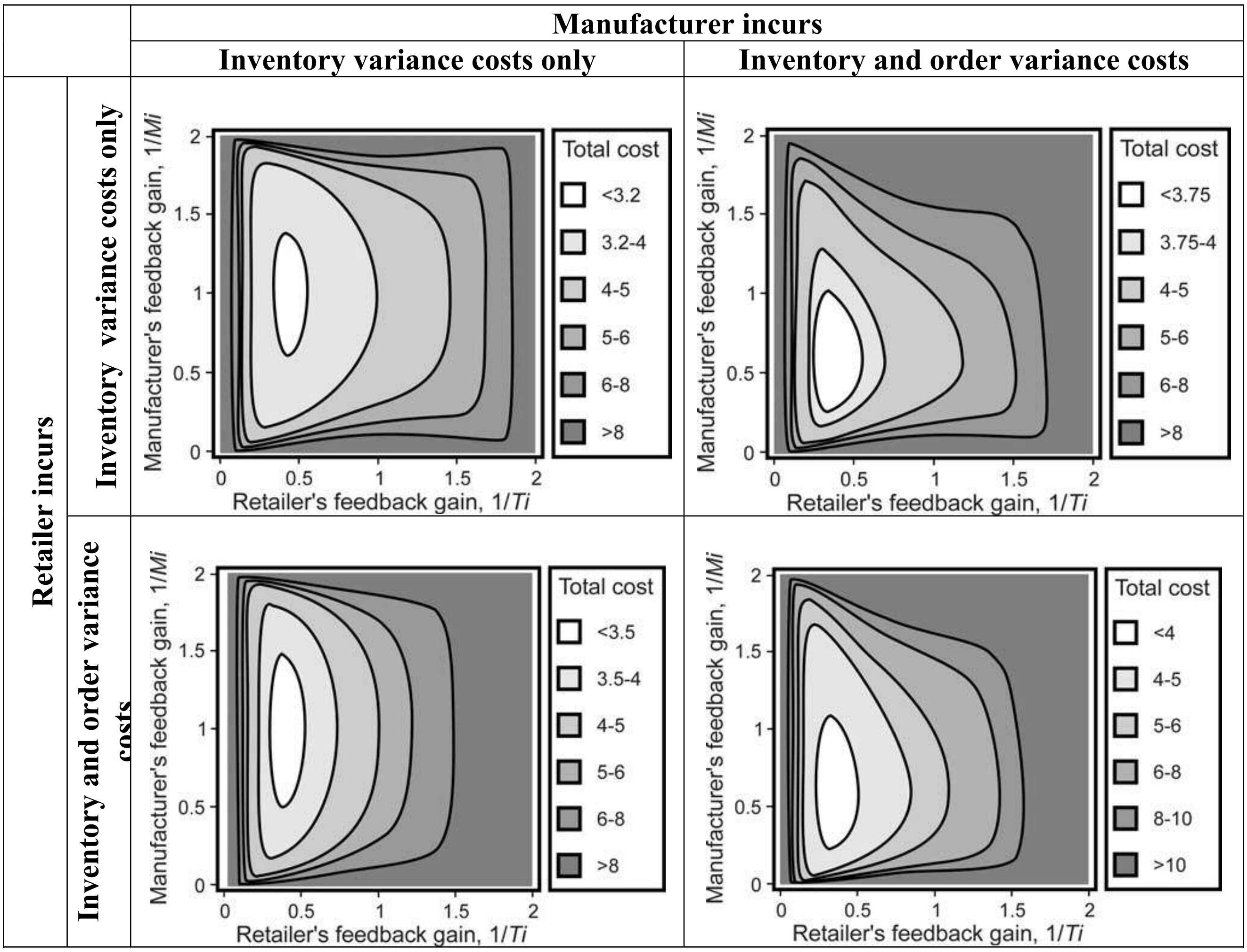

Figure 4. The complete supply chain total cost solution space with costs related to the variance of order rates and inventory levels

Using numerical techniques we have been able to find the actual optimal settings for *Ti* and *Mi* to minimise global supply chain costs as shown in Table 2.

		Manufacturer incurs	
		Inventory variance costs only	**Inventory and order variance costs**
Retailer incurs	**Inventory variance costs only**	$Ti^* = 2.28782$ $Mi^* = 1$ $R_£ = 2.46828$ $M_£ = 0.657735$ $SC_£ = 3.12156$	$Ti^* = 2.62241$ $Mi^* = 1.7603$ $R_£ = 2.6201$ $M_£ = 0.939129$ $SC_£ = 3.55922$
	Inventory and order variance costs	$Ti^* = 2.56065$ $Mi^* = 1$ $R_£ = 2.833627$ $M_£ = 0.547575$ $SC_£ = 3.3812$	$Ti^* = 2.87954$ $Mi^* = 1.76846$ $R_£ =2 .951647$ $M_£ = 0.829541$ $SC_£ = 3.78119$

Table 2. The global optimum supply chain solutions

By comparing the $SC_£$ in Table 1 and 2, we observe that supply chain gains are realized by a global optimization (for all scenarios). The coordinating mechanism dominates the non-coordinating mechanism. The supply chain gain, however, is allocated to the manufacturer and the retailer incurs higher costs. This of course will not result in a coordination policy. Cachon (2003) describes what is needed to coordinate the supply chain: "if a coordinating contract can allocate rents arbitrarily, then there always exists a contract that Pareto dominates a non-coordinating contract, i.e. each firm's profit is no worse off and at least one firm is strictly better off with the coordinating contract".

Therefore part of the supply chain gain has to be allocated to the retailer so that the retailer has an economic incentive to cooperate. For all scenarios the gain is large enough to compensate the cost increase of the retailer. That means that the cost of the retailer, after allocation, is not worse and the manufacturer is strictly better off with coordination.

In the global solution, Ti is larger than in the local optimization solution. Consequently the retailer will incur a larger variance of the inventory level which results in a higher level of safety stock to guarantee a given level of customer service. The extra investment in safety stock has to be compensated by the supply chain gain in order to motivate the retailer to participate. This can be realized by a lower price charged by the manufacturer.

In Table 3 we quantify a naïve solution of $Ti = Mi = 1$ to benchmark a practice quite often found in industry. In this case the members of the supply chain are interested in minimizing investment in inventory and consequently follow a naïve JIT strategy.

We can see that the naïve solution (that is not accommodating for the supply chain cost structures and failing to tune replenishment rules accordingly) is always more costly than the case where supply chain players act rationally and minimize their local costs. Superior performance occurs when supply chain players "think outside the box" and act to minimize global supply chain costs.

<table>
<tr><td colspan="2" rowspan="2">Ti=1
Mi=1</td><th colspan="2">Manufacturer incurs</th></tr>
<tr><th>Inventory variance costs only</th><th>Inventory and order variance costs</th></tr>
<tr><th rowspan="2">Retailer incurs</th><th>Inventory variance costs only</th><td>$R_£ = 2$
$M_£ = 2$
$SC_£ = 4$</td><td>$R_£ = 2$
$M_£ = 3$
$SC_£ = 5$</td></tr>
<tr><th>Inventory and order variance costs</th><td>$R_£ = 3$
$M_£ = 2$
$SC_£ = 5$</td><td>$R_£ = 3$
$M_£ = 3$
$SC_£ = 6$</td></tr>
</table>

Table 3. The naïve solution

8. The altruistic retailer

As indicated by Cachon (2003), supply chain members may consider the coordination mechanism (and the corresponding contracts) costly and complicated. Our global optimization policy requires that (i) forecasts are generated with conditional expectation, (ii) a proportional OUT policy is employed throughout the supply chain (iii) that the feedback controllers are globally optimized. This may be hard to implement. We therefore suggest in this section an "easier to implement" policy. We will compare the results of this strategy with the three previous policies (the local optimum solution, the global optimum solution and the naïve solution).

We assume that the manufacturer is following a low inventory policy and sets $Mi = 1$. The retailer however, is willing and able to alter his replenishment rule to incorporate a proportional controller in the feedback rule. We call this policy the "altruistic retailer" policy. The results are shown in Table 4.

As can be seen from Table 4, the altruistic retailer can obtain near optimal performance for the supply chain by a proper reaction (i.e., by tuning the feedback controllers of the replenishment rule) to the non-cooperative manufacturer. In fact the performance is within 5% of the true optimum when the manufacturer has both inventory and order variance costs; there is no difference in the costs if the manufacturer has only inventory variance costs.

Mi=1		**Manufacturer incurs**	
		Inventory variance costs only	**Inventory and order variance costs**
Retailer incurs	**Inventory variance costs only**	$Ti = 2.28782$ $R_£ = 2.46828$ $M_£ = 0.657735$ $SC_£ = 3.12156$	$Ti = 2.87386$ $R_£ = 2.739587$ $M_£ = 0.990137$ $SC_£ = 3.72972$
	Inventory and order variance costs	$Ti = 2.56065$ $R_£ = 2.833627$ $M_£ = 0.547575$ $SC_£ = 3.3812$	$Ti = 3.09894$ $R_£ = 3.039953$ $M_£ = 0.890773$ $SC_£ = 3.930725$

Table 4. The altruistic retailer

9. Linking the variance ratios to costs in the supply chain

In this section we will quickly explore an alternative cost function based on a more traditional, OR / inventory theory approach. Some costs may be assumed to be constant or independent of the inventory levels or orders, but we simply ignore them here. Such costs may be materials, energy and administration overheads, for example. However to capture the costs that may reasonably be assumed to be dependent on the inventory levels and order rates we will assume;

- a linear system with normally distributed demand,
- piece-wise linear, convex inventory holding costs (of $\begin{cases} H(NS_t) \text{if } NS_t > 0 \\ 0 \text{ otherwise} \end{cases}$, thus H is a the unit cost of holding a unit in inventory per period),
- piece-wise linear, convex backlog costs (of $\begin{cases} B(-NS_t) \text{ if } NS_t \le 0 \\ 0 \text{ otherwise} \end{cases}$, thus B is the cost of a backlog per unit per period),
- and we set the safety stock target $k^* = \sigma_{NS}\sqrt{2}\left(erf^{-1}\left[(B-H)(B+H)^{-1}\right]\right)$ to achieve the economic stock-out probability[2],

then the inventory related costs will be given by (Disney et al (2006b)),

[2] erf^{-1} is the inverse error function.

$$I_{\$} = \underset{\min}{\text{Holding} + \text{Backlog}}\,\text{costs} = \frac{\sigma_{NS}\left(B+H\right)e^{-erf^{-1}\left[\frac{2B}{B+H}-1\right]^2}}{\sqrt{2\pi}}. \tag{17}$$

We notice from (17) that the inventory holding and backlog costs are linearly related to the standard deviation of the net stock levels.

In a similar manner, if there are;

- piece-wise linear, convex lost capacity costs (of $\begin{cases} N\left(S+\overline{D}-O_t\right) \text{if } O_t \le \left(S+\overline{D}\right) \\ 0 \text{ otherwise} \end{cases}$, where S is the slack capacity above the mean demand rate and N is the cost per unit per period of not producing to the available production capacity)
- piece-wise linear, convex over-time costs (of $\begin{cases} P\left(O_t-S-\overline{D}\right) \text{if } O_t > \left(S+\overline{D}\right) \\ 0 \text{ otherwise} \end{cases}$, where P is the cost per unit per period of producing in over-time)
- and we invest in enough capacity (above / below average demand) to achieve an economic over-time probability, $S^* = \sigma_O\sqrt{2}\left(erf^{-1}\left[\left(N-P\right)\left(N+P\right)^{-1}\right]\right)$,

then the bullwhip related costs are equal to (Disney et al (2006b),

$$C_{\$} = \underset{\min}{\text{Lost capacity} + \text{Over-time}}\,\text{costs} = \frac{\sigma_O\left(N+P\right)e^{-erf^{-1}\left[1-\frac{2N}{N+P}\right]^2}}{\sqrt{2\pi}}. \tag{18}$$

Eq (18) shows us that the bullwhip costs are linearly related to the standard deviation of the order rates. Thus, it is interesting to investigate an objective function where the standard deviations, rather than the variances, of the inventory levels and order rates are used as building blocks.

For ease of exposition, we will restrict ourselves here to the case where the standard deviations of the inventory levels and order rates are equally weighted; that is when $H=N$ and $B=P$ (or interestingly when $H=P$ and $B=N$) at both echelons. This change to the objective function has a number of implications, but we note that major conclusions we have drawn so-far remain qualitatively unchanged.

Consider first the local optimisation collaboration scheme; the self serving solutions outlined in section 6. The objective function for the retailer when only inventory costs are present becomes

$$R_{£} = \sigma_{RNS} = \sqrt{Tp + \frac{Ti^2}{2Ti-1}}, \tag{19}$$

and the feedback gain that minimises the retailers cost is Ti=1. This is the same result as before when we considered the variance of the inventory levels and it implies that the manufacturer faces i.i.d. demand when the retailer considers only his inventory cost are important. However, when the retailer has costs related to the standard deviation of both inventory levels and order rates the objective function becomes

$$R_{£} = \sigma_{RO} + \sigma_{RNS} = \sqrt{\frac{1}{2Ti-1}} + \sqrt{Tp + \frac{Ti^2}{2Ti-1}}\,. \tag{20}$$

Minimising (18) w.r.t. Ti results in the following expression for Ti^*,

$$Ti^* = \frac{1}{2}\left(\begin{array}{l} 1+\sqrt{1+2^{2/3}\left(Tp(1+Tp)\right)^{1/3}} + \\ \sqrt{2-2^{2/3}\left(Tp(1+Tp)\right)^{1/3} + \dfrac{2+4Tp}{\sqrt{1+2^{2/3}\left(Tp(1+Tp)\right)^{1/3}}}} \end{array} \right). \tag{21}$$

We notice that the golden ratio solution no longer exists and the optimal Ti^* is now dependent on the retailers lead-time, T_p.

Analytical analysis of the manufacturers standard deviation costs is rather lengthy, and thus we resort to a numerical investigations for the case of Tp=Mp=1, although we remind readers the expressions for the general lead-time case are shown in the appendix. The complete solution space is portrayed graphically in Figure 5. Figure 5 is very similar to Figure 4; enough so, that our major conclusions (altruistic behaviour on behalf of the retailer with either a smart or naïve manufacturer, improves overall supply chain performance) remain unchanged, although absolute numbers are slightly different.

Table 5, details the four specific supply chain cooperation strategies for the different cost structures. Again, the internal relationships and its relationship to the self-serving solutions remain intact.

			Manufacturer incurs							
			Inventory standard deviation costs only				**Inventory and order standard deviation costs**			
			Naive	Local	Altruistic	Global	Naive	Local	Altruistic	Global
Retailer incurs	**Inventory standard deviation costs only**	Ti	1	1	3.54149	3.54149	1	1	5.52083	5.0266
		Mi	1	1	1	1	1	2.29663	1	1.80792
		$R£$	1.41421	1.41421	1.74981	1.74981	1.41421	1.41421	2.00881	2.27938
		$M£$	1.41421	1.41421	0.561212	0.56121	2.41421	2.09849	0.859025	0.86359
		$SC£$	2.82843	2.82843	2.31102	2.31102	3.82843	3.51271	2.86783	2.8162
	Inventory and order standard deviation costs	Ti	1	2.29663	4.60159	4.60159	1	2.29663	6.50865	6.10356
		Mi	1	1	1	1	1	2.11951	1	1.73876
		$R£$	2.41421	2.09849	2.24157	2.24157	2.41421	2.09849	2.4157	2.37816
		$M£$	1.41421	0.80851	0.44196	0.44196	2.41421	1.45936	0.75344	0.74684
		$SC£$	3.82843	2.907	2.68577	2.68577	4.82843	3.55786	3.16914	3.125

Table 5. The supply chain solutions when costs are linearly related to the standard deviation

To summarise all of our investigations considered in this paper, we have standardised (by defining the naïve designs costs as 100%) all of the costs in the different supply chain collaboration schemes with both the variance and standard deviation versions of our objective function. These are shown in Table 6. Again we highlight that the head-line results are similar for both cost functions (variance or standard deviations).

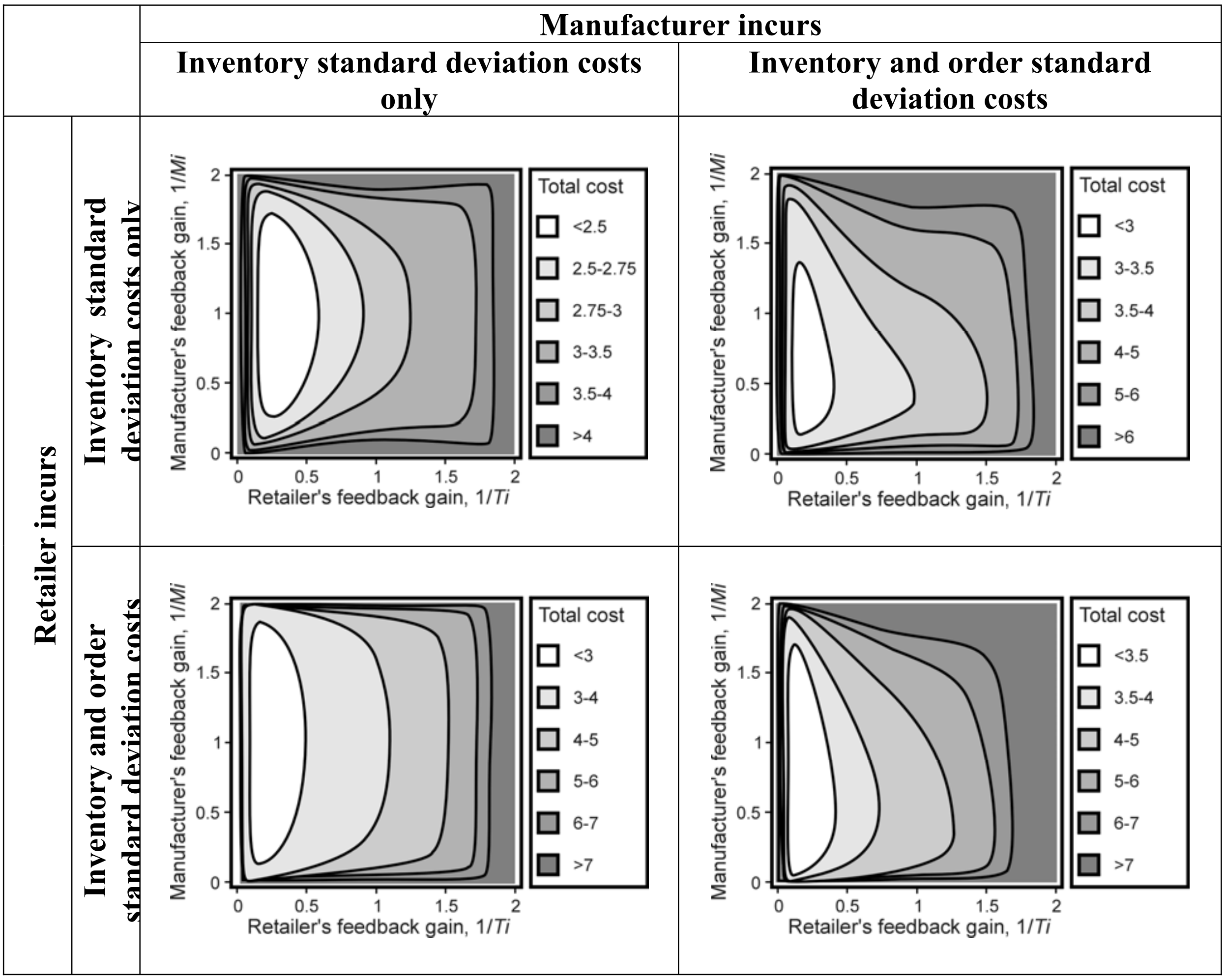

Figure 5. The complete chain total cost solution space with costs related to the standard deviation of order rates and inventory levels

For the naïve strategy, players in the supply only need to operate with standard replenishment software and have a standard trading relationship with their customers and suppliers. It is an easy option. But the naïve strategy results in an inefficient use of inventory and capacity. So to improve their performance, players should make re-engineering efforts to minimise lead-times and additionally create the most accurate forecasts they can achieve. These changes will directly improve business performance. Indeed, these efforts are required for all supply chains and will reduce inventory requirements in supply chains.

More perceptive “players” will try to understand their own cost structures, demand patterns and tune their replenishment rules in order to *minimise their own local costs*. This may be an appropriate strategy if a player;

- has a very large customer or supply base,
- is geographically or culturally separated,
- unwilling or unable to collaborate with others.

In order to be tune the OUT policy in the manner we have investigated here, some adjustments to computer code or decision support systems may be required. For example a grocery retailer we have worked with has actually re-coded their in-house, bespoke replenishment system to incorporate a proportional controller in the WIP and inventory feedback loops. Other companies we have worked with who have standard ERP software have exploited spreadsheet based decision support systems to assist replenishment analysts when they conduct final conformation of replenishment decisions.

Retailer incurs		Manufacturer incurs: Inventory costs only			Manufacturer incurs: Inventory and order costs		
		Supply chain design	Variance costs	Standard deviation costs	Supply chain design	Variance costs	Standard deviation costs
Inventory costs only		Naïve	100	100	Naïve	100	100
		Self serving	100	100	Self serving	92.36	91.75
		Altruistic	78.04	81.71	Altruistic	74.59	74.91
		Globally optimal	78.04	81.71	Globally optimal	71.18	73.56
Inventory and order costs		Supply chain design	Variance costs	Standard deviation costs	Supply chain design	Variance costs	Standard deviation costs
		Naïve	100	100	Naïve	100	100
		Self serving	74.59	75.93	Self serving	71.66	73.69
		Altruistic	67.62	70.15	Altruistic	65.51	65.63
		Globally optimal	67.62	70.15	Globally optimal	63.02	64.72

Table 6. Standardised cost summary of all our investigations

10. Practical considerations

The global optimisation strategy requires players to first intimately understand their own business, as well as other players cost structures, demand patterns and replenishment rules and to be able to tune their replenishment rules appropriately. Furthermore, some agreement has to be reached to re-allocate the supply chain "gains" between the players. This should be possible as the global optimisation strategy Pareto dominates the self serving solution. However in a supply chain with an extended vendor base, it may be difficult to gain the commitment from the large number of suppliers and the re-engineering effort will increase substantially.

The altruistic retailer strategy, although not as efficient as the global optimal strategy has good performance with considerably less effort as there are no supply chain re-engineering requirements essential to the manufacturer. It only requires the retailer to understand cost structures, demand rates, lead-times and replenishment rules throughout the supply chain. It may also not even be necessary to have a formal re-allocation of the supply chain gain. This could be redistributed though traditional pricing polices and the willingness of manufacturers to accept cost reductions over time.

Other attractions may also exist for the retailer to behave altruistically. For example, the UK grocery company who has reduced the bullwhip produced by their replenishment rules (via incorporating a proportional controller discussed therein) did so in order to reduce the workload variability in their warehouse and transportation activities. Furthermore, from a queuing theory viewpoint, bullwhip reduction may actually have a compensating effect on inventory requirements due to reduced manufacturing lead-times, Boute et al (2006). This may help to offset the predicted extra inventory investment at the retailer. This will be especially important for retailers concerned about maintaining a wide product range with correspondingly large requirements for shelf space.

11. Summary

We have developed a discrete time dynamic model of a two-echelon supply chain using a z-transform methodology. The supply chain implements a modified OUT policy to place replenishment / production orders. We have modified the OUT policy by incorporating a proportional controller in the inventory and the "orders placed but not yet received" feedback loops. Using this model we have quantified the order and inventory variance at both echelons of the supply chain.

Four different cost scenarios were constructed using the variance expressions and four different optimisation strategies where undertaken; a naïve solution, a local optimum solution, an altruistic retailer solution, and a global optimum solution. The naïve solution resulted in the worst performance. Interestingly, when total supply chain costs are considered, the classic OUT policy is not optimal in a multi-echelon scenario, even with only inventory variance costs.

We demonstrated this with three different optimisation strategies. The first of these was a local optimisation strategy where the retailer first tuned his replenishment rule to minimise his costs and the manufacturer then tuned his replenishment rule accordingly. This improved the performance of the supply chain from a global perspective, but was not globally optimal. This global optimal was identified from our variance ratios numerically and plotted graphically. Inspection of these results, lead us to the final scenario where action as only taken by the retailer. Although this scenario is not optimal, reasonable performance from the supply chain could be achieved from the altruistic retailer, without re-engineering efforts at the manufacturer.

In order to achieve this coordination scheme, supply chain players need to share information about demand patterns, replenishment policies, parameter settings and lead-times. We concede that this may be difficult to achieve. However, our experience suggests that both retailers and manufacturers may have other incentives over and above the supply chain gains we have discussed here to undertake such seemingly altruistic behaviour.

We have assumed in our analysis, a linear system and thus some level of operational proficiency is required in the supply chain. It would be interesting to conduct some analysis of the impact of non-linear aspects on our findings. This might include; random lead-times, capacity constraints and lost sales, amongst others. The approach of Chatfield et al (2004) maybe useful here. Other extensions to this research could include an investigation of the impact of more echelons, or more manufacturers.

Finally it is interesting to note that in our simple two-echelon supply chain model, the best result comes from the players acting in the best interest of the supply chain, and not by the players acting solely in their own immediate interest. This is in contrast to Adam Smith, for example, who argued that the best result for a group resulted from each individual player acting solely in own interests. It is however congruent with the arguments of John Nash. Superior performance is achievable if firms coordinate their actions.

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Appendix A. Derivation of the variance ratios

The retailer's variance ratios

Using basic control theory we may construct the following block diagram of our supply chain model. Rearranging the block diagram (Figure A.1) using standard techniques (we refer readers to Nise (1995) for a general introduction to control theory) yields the transfer function of the retailers order rate. It is;

$$\frac{RO\{z\}}{D\{z\}}=\frac{z}{1+Ti(z-1)}. \tag{A.1}$$

The inverse z-transform of (A.1) yields the time domain solution

$$ro_t = (Ti-1)^t Ti^{-1-t}, \tag{A.2}$$

where t is the time index. Using Tsypkin's Relation, we know $\frac{\sigma^2_{Output}}{\sigma^2_{Input}} = \sum_{t=0}^{\infty} f^2(t)$ (Tsypkin, 1962), which we may apply directly to (A.2) to yield (A.3). We refer to Disney and Towill (2003) for more details on this technique.

$$\frac{\sigma^2_{RO}}{\sigma^2_D} = \frac{1}{2Ti-1} \tag{A.3}$$

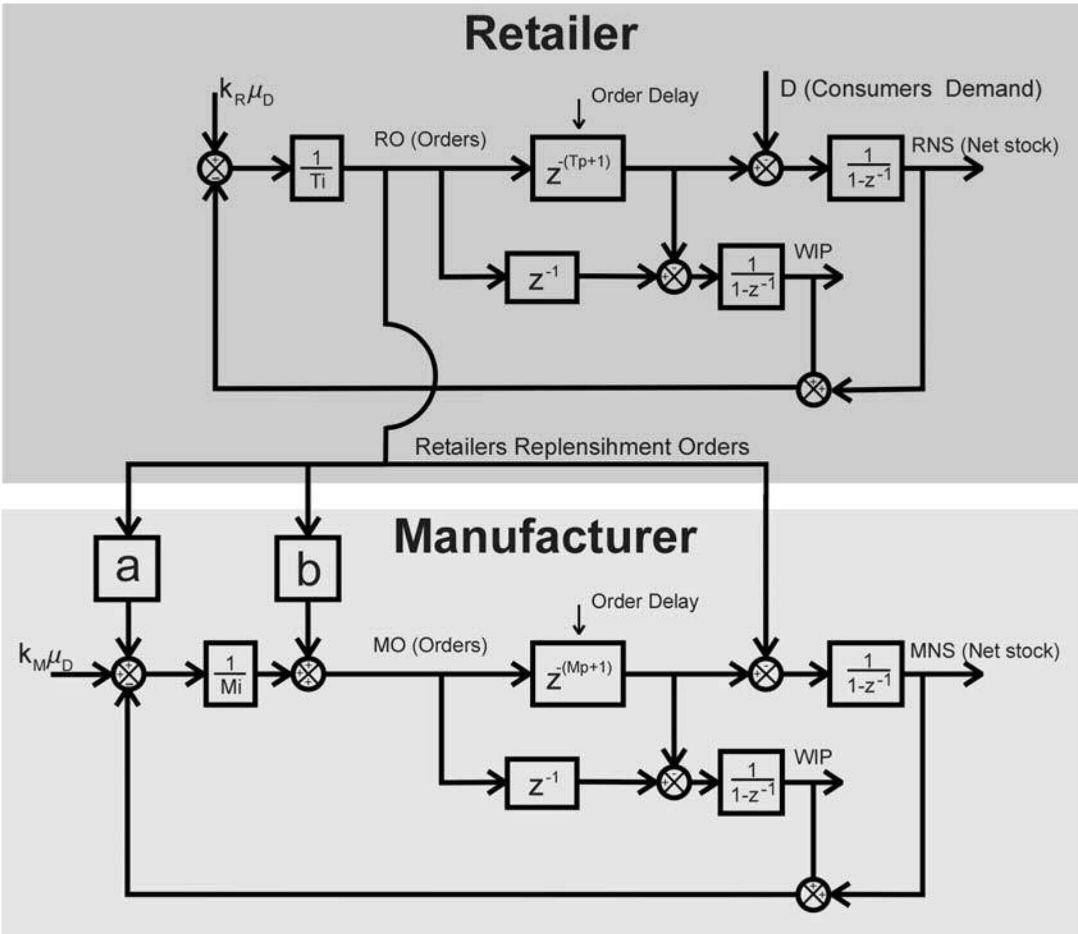

Figure A1. Block diagram of our supply chain model

The retailer's inventory levels are similarly described by

$$\frac{RNS\{z\}}{D\{z\}} = \frac{z\left(Ti-1-Tiz+z^{-Tp}\right)}{(1+Ti(z-1))(z-1)} \tag{A.4}$$

The time domain solution is

$$rns_t = \left(1-\left(\frac{Ti-1}{Ti}\right)^{t-Tp}\right)h\left[t-Tp\right]-1 \tag{A.5}$$

where h[.] is the Heaviside step function. Using Tsypkin's relation, the retailer inventory variance is given by

$$\frac{\sigma^2_{RNS}}{\sigma^2_D} = 1 + Tp + \frac{(Ti-1)^2}{2Ti-1} = Tp + \left(Ti^2 \frac{1}{2Ti-1}\right). \tag{A.6}$$

The manufacturer's variance ratios

In Figure A.1 there are two constants; *a* and *b*. They are given by

$$a = \left(\frac{Ti-1}{Ti}\right)^{1+Mp} \text{ and } b = (1-Ti)\left(\left(\frac{Ti-1}{Ti}\right)^{Mp}-1\right). \tag{A.7}$$

Rearranging the block diagram yields the manufacturers order rate which is given by

$$\frac{MO\{z\}}{D\{z\}} = \frac{Ti(1+Ti(z-1))z+(Mi-Ti)(Ti-1)^{1+Mp}Ti^{-Mp}(z-1)z}{Ti(1+Mi(z-1))(1+Ti(z-1))}. \tag{A.8}$$

Taking the inverse z-transform, surrenders the manufacturers time domain response,

$$mo_t = \frac{Ti^{-2-Mp}\left(Mi(Ti-1)^{1+Mp+t}Ti^{-t}+\left(\frac{Mi-1}{Mi}\right)^t Ti\left(Ti^{1+Mp}-(Ti-1)^{1+Mp}\right)\right)}{Mi}. \tag{A.9}$$

and the order variance is given by

$$\frac{\sigma^2_{MO}}{\sigma^2_D} = \frac{\left(Ti^{-2(1+Mp)}\left(\begin{array}{l}2(Mi-Ti)(Ti-1)^{1+Mp}Ti^{1+Mp}(2Ti-1)+\\ Ti^{2+2Mp}(Mi+Ti-1)(2Ti-1)+2(Mi-Ti)^2(Ti-1)^{2+2Mp}\end{array}\right)\right)}{\left((2Mi-1)(Mi+Ti-1)(2Ti-1)\right)}. \tag{A.10}$$

The manufacturer's net stock time domain response is given by

$$mns_t = \sum_{n=0}^{t}\left(\left(mo_{n-Mp-1}h[n-Mp-1]\right)-ro_t\right). \tag{A.11}$$

After substitution (A.10) the sum becomes

$$mns_t = \sum_{n=0}^{t}\left(\frac{1}{Mi}\left(Ti^{-2-Mp}\left(\begin{array}{l} Mi(Ti-1)^n Ti^{1+Mp-n} + \\ \left(\frac{Mi-1}{Mi}\right)^{n-1-Mp} Ti\left(Ti^{1+Mp}-(Ti-1)^{1+Mp}\right)h[n-1-Mp]\end{array}\right)\right)-\frac{\left(\frac{Ti-1}{Ti}\right)^n}{Ti}\right).$$

(A.12)

which, as Mp is an integer and $Mp\geq 0$, (A.12) converges to,

$$mns_t = \left(-1+\left(\frac{Ti-1}{Ti}\right)^{1+t}+\left(\begin{array}{l} 1-\left(\frac{Mi-1}{Mi}\right)^{-Mp+t}+\left(\frac{Ti-1}{Ti}\right)^{1+Mp}-\left(\frac{Ti-1}{Ti}\right)^{1+t}- \\ \frac{\left(\frac{Mi-1}{Mi}\right)^{Mp}(Ti-1)^{1+Mp}\left(\frac{(Mi-1)Ti}{Mi}\right)^{-Mp}}{Ti}+ \\ \frac{\left(\frac{Mi-1}{Mi}\right)^{t}(Ti-1)^{1+Mp}\left(\frac{(Mi-1)Ti}{Mi}\right)^{-Mp}}{Ti}\end{array}\right)h[t-1-Mp]\right)$$

(A.13)

Now, the required variance expression of the manufacturer's net stock is given by,

$$\frac{\sigma_{MNS}^2}{\sigma_D^2} = \sum_{t=0}^{\infty} mns_t = \sum_{t=0}^{Mp}\left(\left(\frac{Ti-1}{Ti}\right)^{1+t}-1\right)^2+\sum_{t=Mp+1}^{\infty}\left(\begin{array}{l}\left(\frac{Ti-1}{Ti}\right)^{1+Mp}-\left(\frac{Mi-1}{Mi}\right)^{t-Mp}- \\ \frac{\left(\frac{Mi-1}{Mi}\right)^{Mp}(Ti-1)^{1+Mp}\left(\frac{(Mi-1)Ti}{Mi}\right)^{-Mp}}{Ti}+ \\ \frac{\left(\frac{Mi-1}{Mi}\right)^{t}(Ti-1)^{1+Mp}\left(\frac{(Mi-1)Ti}{Mi}\right)^{-Mp}}{Ti}\end{array}\right)^2.$$

(A.14)

Which is equivalent to

$$\frac{\sigma_{MNS}^2}{\sigma_D^2} = Ti^{-2(1+Mp)}\left(\begin{array}{l}\frac{(Mi-1)^2\left((Ti-1)^{1+Mp}-Ti^{1+Mp}\right)^2}{2Mi-1}+ \\ \frac{1}{2Ti-1}\left(Ti^{Mp}\left(\begin{array}{l}2(Ti-1)^{2+Mp}Ti(2Ti-1)-(Ti-1)^{4+2Mp}Ti^{-Mp}- \\ Ti^{2+Mp}(2+Mp-2MpTi+3(Ti-2)Ti)\end{array}\right)\right)\end{array}\right). \quad (A.15)$$

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