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OSCILLATION OF THE PERTURBED HILL EQUATION AND THE LOWER SPECTRUM OF RADIALLY PERIODIC SCHRÖDINGER OPERATORS IN THE PLANE

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Abstract. Generalizing the classical result of Kneser, we show that the Sturm-Liouville equation with periodic coefficients and an added perturbation term $-c^2/r^2$ is oscillatory or non-oscillatory (for $r \to \infty$) at the infimum of the essential spectrum, depending on whether c^2 surpasses or stays below a critical threshold. An explicit characterization of this threshold value is given. Then this oscillation criterion is applied to the spectral analysis of two-dimensional rotation symmetric Schrödinger operators with radially periodic potentials, revealing the surprising fact that (except in the trivial case of a constant potential) these operators always have infinitely many eigenvalues below the essential spectrum.

1. Introduction

It is a famous result in the oscillation theory of Sturm-Liouville equations that $-1/4$ is a critical threshold in

$$-u'' + c(x)\,u = 0;$$

this equation being oscillatory at ∞ if $c(x) < (-1/4 - \varepsilon)\,x^{-2}$, and non-oscillatory at ∞ if $c(x) > (-1/4 + \varepsilon)\,x^{-2}$ for some $\varepsilon > 0$ and sufficiently large x ([8]; for a quick proof cf. [5], Chapt. XI, Ex. 1.2). This has later been generalized by Hille to include logarithmic corrections, which in particular settled the limiting case $\lim_{x\to\infty} x^2\,c(x) = -1/4$ to be non-oscillatory (cf. [13], §2.8).

Note that the equation

$$-u'' + \frac{c}{r^2}\,u = 0$$

may be interpreted as the constant-coefficient Sturm-Liouville equation $-u'' = 0$, with an added perturbation c/r^2, such as arises naturally from a separation of polar coordinates for a rotation symmetric Schrödinger operator in higher dimensions (cf. [14], Sect. 17.F). With this motivation in mind, it may not be inappropriate to ask about the oscillation behaviour of the eigenvalue equation for more general Sturm-Liouville equations perturbed by a c/r^2 term: Will r^{-2} decay still be the borderline case? And what will be the critical value of the coupling constant? An auspicious

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candidate to replace the constant-coefficient equation as unperturbed reference is the Sturm-Liouville equation with periodic coefficients,

$$-(p\,u')' + q\,u = \lambda\,w\,u, \qquad \lambda \in \mathbb{R},$$

which shares certain qualitative properties of the constant-coefficient case; thus it is well known that the spectrum of the corresponding self-adjoint Sturm-Liouville operator

$$h = \frac{1}{w}\left(-\frac{d}{dx}\,p\,\frac{d}{dx} + q\right)$$

consists of purely absolutely continuous bands, often separated by spectral gaps. In particular, at $\lambda = \lambda_0 := \inf \sigma_e(h)$, the equation is disconjugate, possessing a fundamental system composed of a positive periodic solution, and an unbounded solution which has exactly one zero (cf. [4], Thm. 3.1.2(i)).

However, the perturbed equation

$$-(p\,u')' + q\,u + \frac{c}{r^2}\,u = \lambda_0\,w\,u$$

does not immediately subordinate itself to any one of the classical oscillation criteria, as collected e.g. in [13] or [15]. More specifically, in studies of the perturbed Hill equation

$$-u'' + q\,u + \tilde{q}\,u = \lambda_0\,w\,u$$

(such as [1], [10]) assumptions of the type

$$\int^{\infty} (1 + |x|)\,\tilde{q}(x)\,dx < \infty$$

on the perturbation $\tilde{q}$ are common. In a series of works, Rofe-Beketov [11], [12] has devoted some attention to the case of Hill's equation with a c/r^2 perturbation. He gives Kneser-type critical values of the coupling constant for the discrete spectrum to be finite or infinite not only at the bottom of the essential spectrum, but also at each of the end-points of the spectral gaps; due to the growth of these critical values, sufficiently remote gaps contain only a finite number of eigenvalues ([11], p. 153). Moreover, he states that these results extend at least in part to the case of an almost-periodic unperturbed equation ([12], Thm. 1).

In this note we present an oscillation – non-oscillation criterion for the perturbed Sturm-Liouville equation with periodic coefficients, giving a Kneser-type critical value for the coupling constant c (Theorem 1). In particular, this provides an independent simple proof for Rofe-Beketov's results in so far as they concern the infimum of the essential spectrum in the periodic case. The key idea of the proof of Theorem 1 is to introduce an equivalent Riccati-type equation, which enables us to study the behaviour of solutions of the perturbed equation relative to the periodic solution of the unperturbed equation. The direction field of the Riccati equation can easily be discussed in general terms outside a critical interval; inside this interval, averaging over a period interval turns the right-hand side of the equation essentially into a parabola (plus higher order terms), and the problem of oscillation or non-oscillation reduces to the question of whether this parabola has zeros or not, which shows in a fairly transparent way how the Kneser-type constant arises.

In Theorem 2, we then apply this criterion to rotation symmetric Schrödinger operators in the plane with radially periodic potentials, making the somewhat surprising and apparently novel observation that (except in the trivial case of a constant potential) these operators always have infinitely many eigenvalues below the essential spectrum. The corresponding eigenfunctions are all rotationally symmetric. This phenomenon, resulting from a very subtle resonance between the concentric ripples of the potential, is restricted to the 2-dimensional case. Theorem 2 gives an affirmative answer to a question posed in [6], Remark 2.

2. An oscillation – non-oscillation criterion

Theorem 1. *Let $p \in L^\infty(\mathbb{R})$, $q, w, 1/p \in L^1_{\mathrm{loc}}(\mathbb{R})$, $p, w > 0$, q real-valued, and p, w, q α-periodic, $\alpha > 0$. Let λ_0 be the infimum of the essential spectrum of the maximal Sturm-Liouville operator on $(-\infty, \infty)$,*

$$h = \frac{1}{w}\left(-\frac{d}{dx}\, p\, \frac{d}{dx} + q\right),$$

and u_0 a positive periodic solution of the equation

$$-(p\,u')' + q\,u = \lambda_0\, w\, u. \tag{1}$$

Set $K := \dfrac{1}{4}\left(\dfrac{1}{\alpha}\displaystyle\int_0^\alpha \frac{1}{p\,u_0^2}\right)^{-1}\left(\dfrac{1}{\alpha}\displaystyle\int_0^\alpha u_0^2\right)^{-1}$; then the perturbed equation

$$-(p\,u')' + q\,u - \frac{c^2}{r^2}\,u = \lambda_0\, w\, u \tag{2}$$

is oscillatory at ∞ if $c^2 > K$, and non-oscillatory at ∞ if $c^2 < K$.

Remarks. 1. The existence of a positive periodic solution u_0 of equation (1) is a well-known result of Floquet theory; cf. [4], Thm 3.1.2 (i). (In [3] and [4], $q, w \in C(\mathbb{R})$, $p \in C^1(\mathbb{R})$ is assumed for convenience; however, the statements and their proofs directly carry over to the present case. Instead of $u \in C^2((0,\infty))$, we have $u, pu' \in AC_{\mathrm{loc}}((0,\infty))$.) As u_0 is clearly not square-integrable, we have the limit point case at $\pm\infty$, and hence the maximal operator h is self-adjoint.

2. Clearly $K > 0$. When the negative perturbation $-c^2/r^2$ is replaced by a positive perturbation c^2/r^2, the resulting equation

$$-(p\,u')' + q\,u + \frac{c^2}{r^2}\,u = \lambda_0\, w\, u$$

is non-oscillatory at ∞ for all $c \in \mathbb{R}$ by Theorem 1 and the Sturm comparison theorem (cf. [3], Thm 4.2.1).

3. If we specialize $p \equiv w \equiv 1$ and apply the results of oscillation theory ([2], XII.7.53, 55), Theorem 1 coincides with the findings of Rofe-Beketov [12], Thm. 1, on the finiteness or infinity, respectively, of the number of eigenvalues below λ_0.

4. It seems to be a rather subtle point to decide whether equation (2) is oscillatory or not in the borderline case $c^2 = K$.

Proof of Theorem 1. Multiplying u_0 by a suitable positive constant, we may assume that $\sup p\,u_0^2 = 1$. Also we set $\mu := \min u_0 > 0$. For any non-trivial solution u of the perturbed equation (2), we introduce the function

$$\zeta(r) := r\,u_0^2(r)\left(\frac{(p\,u_0')(r)}{u_0(r)} - \frac{(p\,u')(r)}{u(r)}\right).$$

If $0 < a < b$ are two consecutive zeros of u, or $b = \infty$ if u has no zeros in (a, ∞), then $\zeta|(a, b)$ is a maximal solution of the Riccati-type ordinary differential equation

$$\zeta' = \frac{1}{r}\left(\frac{\zeta^2}{p\,u_0^2} + \zeta + u_0^2\,c^2\right), \tag{3}$$

and $\zeta(r) \to -\infty$ $(r \to a+)$, $\zeta(r) \to \infty$ $(r \to b-)$ if $b < \infty$.

Furthermore, if $\zeta(r) \geq 0$ for some $r \in (a, b)$, then

$$\zeta'(r) \geq \frac{1}{r}\left(\frac{\zeta^2}{p\,u_0^2} + u_0^2\,c^2\right) \geq \frac{1}{r}\,(\zeta^2 + \mu^2 c^2);$$

thus ζ is strictly increasing on (r, b). Writing $\zeta(\rho) = \mu c\, z(\mu c \log \rho)$ $(\rho \in (a, b))$, we find $z' \geq z^2 + 1$, and comparing with the tan function, we infer that $b \leq r\,\exp\left(\frac{\pi}{2\mu c}\right)$.

Similarly, if $\zeta(r) \leq -1$ for some $r \in (a, b)$, then for $\mathrm{Z} := \zeta + 1$,

$$\begin{aligned}\mathrm{Z}'(r) &= \frac{1}{r}\left(\frac{\mathrm{Z}^2}{p\,u_0^2} - \left(\frac{2}{p\,u_0^2} - 1\right)\mathrm{Z} + \frac{1}{p\,u_0^2} - 1 + u_0^2\,c^2\right)\\ &\geq \frac{1}{r}\,(\mathrm{Z}^2 + \mu^2 c^2),\end{aligned}$$

and hence ζ is strictly increasing on (a, r). Writing $\zeta(\rho) = \mu c\, z(\mu c \log \rho) - 1$ $(\rho \in (a, b))$ and comparing with the tan function as above, we find $a \geq r\,\exp\left(-\frac{\pi}{2\mu c}\right)$.

Consequently, if $\zeta(r_0) \geq 0$ for some $r_0 > 0$, then the interval $[r_0, r_0\,\exp(\pi/(\mu c))]$ contains at least one zero of u, and some subsequent point r_1 such that $\zeta(r_1) \in [-1, 0)$. Therefore in order to decide whether or not u has infinitely many zeros, it is sufficient to concentrate our attention on the behaviour of ζ when it takes values in the interval $[-1, 0)$: the number of zeros is finite if and only if ζ is trapped in this interval after some point.

1st case: $c^2 > K$**.** We show that equation (2) is oscillatory at ∞. By contraposition, assume that there is a solution u with only finitely many zeros. From the above considerations it follows that for sufficiently large $R_0 > 0$, the corresponding function ζ satisfies $\zeta(r) \in [-1, 0)$ $(r > R_0)$. In particular, with $C := \int_0^\alpha (\frac{1}{p\,u_0^2} + 1 + u_0^2\,c^2)$, we have

$$|\zeta(\rho_1) - \zeta(\rho_2)| \leq \int_r^{r+\alpha} |\zeta'| \leq \frac{C}{r} \qquad (r > R_0; \rho_1, \rho_2 \in [r, r + \alpha]).$$

Set $K_c := \dfrac{1}{\alpha}\displaystyle\int_0^\alpha u_0^2 \cdot (c^2 - K) > 0$, and choose

$$R_1 \geq \max\left\{R_0, \frac{2}{K_c}\left(\frac{2C}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2} + \alpha\right)\right\}.$$

For $r > R_1$ define $\overline{\zeta}(r) := \frac{1}{\alpha}\int_r^{r+\alpha} \zeta \in [-1, 0)$; then

$$\begin{aligned}\overline{\zeta}'(r) &= \frac{1}{\alpha}(\zeta(r+\alpha) - \zeta(r)) = \frac{1}{\alpha}\int_r^{r+\alpha} \frac{1}{\rho}\left(\frac{\zeta^2}{p\,u_0^2} + \zeta + u_0^2\,c^2\right)d\rho \\ &\geq \frac{1}{r+\alpha}\left(\frac{1}{\alpha}\int_r^{r+\alpha}\frac{\zeta^2}{p\,u_0^2} + \overline{\zeta}(r) + \frac{c^2}{\alpha}\int_0^\alpha u_0^2\right) + \frac{\alpha}{r\,(r+\alpha)}\,\overline{\zeta}(r) \\ &= \frac{1}{r+\alpha}\left(\left[\left(\frac{1}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2}\right)^{1/2}\overline{\zeta}(r) + \frac{1}{2}\left(\frac{1}{\alpha}\int_0^\alpha\frac{1}{p\,u_0^2}\right)^{-1/2}\right]^2\right. \\ &\qquad \left. + \frac{c^2}{\alpha}\int_0^\alpha u_0^2 - \frac{1}{4}\left(\frac{1}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2}\right)^{-1} + \frac{1}{\alpha}\int_r^{r+\alpha}\frac{\zeta + \overline{\zeta}(r)}{p\,u_0^2}\,(\zeta - \overline{\zeta}(r)) + \frac{\alpha}{r}\,\overline{\zeta}(r)\right) \\ &\geq \frac{1}{r+\alpha}\left(K_c - \frac{1}{r}\left(\frac{2C}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2} + \alpha\right)\right) \geq \frac{K_c}{2\,(r+\alpha)} \qquad (r > R_1),\end{aligned}$$

using in the last but one step that since ζ is continuous, there is $\rho_0 \in [r, r+\alpha]$ such that $\overline{\zeta}(r) = \zeta(\rho_0)$, and therefore $|\zeta(\rho) - \overline{\zeta}(r)| \leq C/r \quad (\rho \in [r, r+\alpha])$.

We conclude that

$$\overline{\zeta}(r) \geq \overline{\zeta}(R_1) + \frac{K_c}{2}\log\frac{r+\alpha}{R_1+\alpha} \to \infty \qquad (r \to \infty),$$

a contradiction.

2nd case: $c^2 < K$**.** Let C be the same constant as in the 1st case,

$$K_c := \frac{1}{\alpha}\int_0^\alpha u_0^2 \cdot (K - c^2) > 0,$$

and

$$R_0 := \left(\frac{2}{K_c} + 1\right)\left(\frac{2C}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2} + \alpha\right).$$

Let $\zeta : [R_0, R_\infty) \to \mathbb{R}$ (with $R_\infty \in (R_0, \infty]$) be the right-maximal solution of the initial value problem (3), $\zeta(R_0) = -1$. If $R_\infty < \infty$, then $\lim_{r\to R_\infty} \zeta(r) = \infty$.

We observe that for $\rho_1, \rho_2 \in [R_0, R_\infty)$, $\rho_1 < \rho_2$, such that $\zeta([\rho_1, \rho_2]) \subset [-1, 0]$, we have

$$|\zeta(\rho_2) - \zeta(\rho_1)| \leq \frac{1}{R_0}\int_{\rho_1}^{\rho_2}\left(\frac{1}{p\,u_0^2} + 1 + u_0^2 c^2\right).$$

Hence, if $r \in [R_0, R_\infty)$ is a point such that $\zeta(r) \in [-1, -C/R_0]$, then $r < R_\infty - \alpha$ and $\zeta(\rho) \in [-1, 0]$ $(\rho \in [r, r+\alpha])$.

In particular, $-1 \leq \zeta(\rho) \leq -1 + C/R_0 \leq -1 - \zeta_0 \leq \zeta_0$ $(\rho \in [R_0, R_0+\alpha])$, where

$$\zeta_0 := -\frac{1}{2}\left(\frac{1}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2}\right)^{-1} < 0;$$

thus $\overline{\zeta}(r) := \frac{1}{\alpha}\int_r^{r+\alpha}\zeta$ $(r \in [R_0, R_\infty - \alpha))$ is defined on a non-empty interval and satisfies $\overline{\zeta}(R_0) \in [-1, \zeta_0]$.

Now we show that $\overline{\zeta}(r) \leq \zeta_0$ $(r \in [R_0, R_\infty - \alpha))$. By contraposition, assume

$$r_0 := \sup\{r \geq R_0 \,|\, \overline{\zeta}(\rho) \in [-1, \zeta_0] \quad (\rho \in [R_0, r])\} < R_\infty - \alpha.$$

Then $\overline{\zeta}'(r_0) \geq 0$ and $\overline{\zeta}(r_0) = \zeta_0$; by the continuity of ζ, there is $\rho_0 \in [r_0, r_0+\alpha]$ such that $\zeta(\rho_0) = \overline{\zeta}(r_0) = \zeta_0$.

By the above observation we find $\zeta(\rho) \in [-1,0]$ ($\rho \in [r_0, r_0+\alpha]$). Estimating the error term as in the 1st case, we have:

$$\begin{aligned}
\overline{\zeta}'(r_0) &= \frac{1}{\alpha}\int_{r_0}^{r_0+\alpha} \frac{1}{\rho}\left(\frac{\zeta^2}{p\,u_0^2} + \zeta + u_0^2\,c^2\right) d\rho \\
&\le \frac{1}{r_0}\left(\frac{1}{\alpha}\int_{r_0}^{r_0+\alpha} \frac{\zeta^2}{p\,u_0^2} + \overline{\zeta}(r) + \frac{c^2}{\alpha}\int_0^\alpha u_0^2\right) - \frac{\alpha}{r_0\,(r_0+\alpha)}\,\overline{\zeta}(r_0) \\
&= \frac{1}{r_0}\left(\left[\left(\frac{1}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2}\right)^{1/2} \overline{\zeta}(r_0) + \frac{1}{2}\left(\frac{1}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2}\right)^{-1/2}\right]^2\right. \\
&\qquad + \frac{c^2}{\alpha}\int_0^\alpha u_0^2 - \frac{1}{4}\left(\frac{1}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2}\right)^{-1} \\
&\qquad\qquad \left. + \frac{1}{\alpha}\int_{r_0}^{r_0+\alpha} \frac{\zeta+\overline{\zeta}(r_0)}{p\,u_0^2}\,(\zeta - \overline{\zeta}(r_0)) - \frac{\alpha}{r_0+\alpha}\,\overline{\zeta}(r_0)\right) \\
&\le \frac{1}{r_0}\left(\frac{1}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2}\,(\overline{\zeta}(r_0) - \zeta_0)^2 - K_c + \frac{1}{r_0}\left(\frac{2C}{\alpha}\int_0^\alpha \frac{1}{p\,u_0^2} + \alpha\right)\right), \\
&\le -\frac{K_c}{2r_0} < 0,
\end{aligned}$$

a contradiction.

Hence, $\overline{\zeta}(r) \le \zeta_0$ ($r \in [R_0, R_\infty - \alpha)$), which implies $\zeta(r) \le 0$ ($r \in [R_0, R_\infty)$).

In particular, $R_\infty = \infty$; the solution of (2) corresponding to ζ has no zeros in $[R_0, \infty)$, and (2) is non-oscillatory at ∞. □

3. The lower spectrum of rotation symmetric Schrödinger operators in the plane

Theorem 2. *Let $V \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ be rotation symmetric: $V(x) = q(|x|)$ ($x \in \mathbb{R}^2$), $q \in L^2_{\mathrm{loc}}(\mathbb{R})$ α-periodic, $\alpha > 0$, and bounded below.*

If V is not constant, then the Schrödinger operator $H = -\Delta + V$ has infinite lower spectrum.

Remark. Following [9], we call the discrete spectrum of H in the interval $(-\infty, \inf \sigma_e(H))$ the *lower spectrum of H*. $H|C_0^\infty(\mathbb{R}^2)$ is essentially self-adjoint by Kato's theorem [7].

Proof of Theorem 2. By separation of polar coordinates, H is unitarily equivalent to a direct sum of self-adjoint Sturm-Liouville operators on the half-axis $(0,\infty)$, formally given as (cf. [14], p. 286)

$$h_l = -\frac{d^2}{dr^2} + q(r) + \frac{l^2 - 1/4}{r^2} \qquad (r > 0; l \in \mathbb{Z}).$$

Let λ_0 be the infimum of the essential spectrum of the Hill operator

$$h = -\frac{d^2}{dx^2} + q(x) \qquad (x \in \mathbb{R}).$$

Since h is in the limit point case at ∞ (cf. Remark 1), and the angular momentum term $(l^2 - 1/4)/r^2$ is bounded near ∞, h_l is in the limit point case at ∞. For $|l| \ge 1$,

h_l is in the limit point case at 0 as well ([14], Thm. 6.4 a), and hence essentially self-adjoint on $C_0^\infty(0,\infty)$. Consequently,

$$(h_l\,\varphi,\varphi) = \int_0^\infty (|\varphi'|^2 + q\,|\varphi|^2 + \frac{l^2-1/4}{r^2}\,|\varphi|^2) \geq (h\,\varphi,\varphi) \qquad (\varphi \in C_0^\infty(0,\infty)),$$

and so $h_l \geq h \geq \lambda_0$. Thus the parts h_l, $|l| \geq 1$, do not contribute any spectrum below λ_0.

Now consider the operator h_0. For each $\lambda \in \mathbb{R}$, the equation

$$-u'' - \frac{u}{4r^2} - \lambda\,u = 0$$

(which can be solved explicitly in terms of Bessel functions) is non-oscillatory at 0; thus by Sturm's comparison theorem and the lower boundedness of q,

$$-u'' + q\,u - \frac{u}{4r^2} = \lambda\,u \tag{4}$$

is non-oscillatory at 0 for all $\lambda \in \mathbb{R}$.

On the other hand, $-u'' + q\,u = \lambda\,u$ is non-oscillatory at ∞ if $\lambda < \lambda_0$, and oscillatory at ∞ if $\lambda > \lambda_0$ (cf. [2], XIII.7.40, [14], Thm. 14.9 c), and since $1/(4r^2) \to 0$ $(r \to \infty)$, the same holds true for equation (4). As a consequence, $\lambda_0 = \inf \sigma_e(h_0)$.

Let u_0 be a positive periodic solution of $-u'' + q\,u = \lambda_0\,u$; clearly u_0 is not constant, since q is not constant. Hence by the Schwarz inequality,

$$K := \frac{1}{4}\left(\frac{1}{\alpha}\int_0^\alpha \frac{1}{u_0^2}\right)^{-1}\left(\frac{1}{\alpha}\int_0^\alpha u_0^2\right)^{-1} < \frac{1}{4}\left(\frac{1}{\alpha}\int_0^\alpha 1\right)^{-2} = \frac{1}{4};$$

by Theorem 1, the equation (4), with $\lambda = \lambda_0$, is oscillatory. Oscillation theory ([2], XIII.7.51, [14], Thm. 14.7 b) implies that h_0, and therefore H, has an infinity of eigenvalues below $\lambda_0 = \inf \sigma_e(H)$. $\square$

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