

MULTIPLY-REFINED ENUMERATION OF ALTERNATING SIGN MATRICES

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ABSTRACT. Four natural boundary statistics and two natural bulk statistics are considered for alternating sign matrices (ASMs). Specifically, these statistics are the positions of the 1's in the first and last rows and columns of an ASM, and the numbers of generalized inversions and -1 's in an ASM. Previously-known and related results for the exact enumeration of ASMs with prescribed values of some of these statistics are discussed in detail. A quadratic relation which recursively determines the generating function associated with all six statistics is then obtained. This relation also leads to various new identities satisfied by generating functions associated with fewer than six of the statistics. The derivation of the relation involves combining the Desnanot–Jacobi determinant identity with the Izergin–Korepin formula for the partition function of the six-vertex model with domain-wall boundary conditions.

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1. INTRODUCTION

A major focus of attention throughout the history of alternating sign matrices (ASMs) has simply been the derivation of results related to their exact enumeration. Such results typically state that the number of ASMs which satisfy specific conditions, such as having prescribed values of certain statistics or being invariant under certain symmetry operations, is given by an explicit formula or generating function, is equal to the number of combinatorial objects of some other variety satisfying specific conditions, or is equal to a number which arises from a particular physical model.

A few examples of such results, with references to conjectures and initial proofs, are as follows: a formula for the total number of ASMs of any fixed size but with no further conditions applied (Mills, Robbins and Rumsey [112, 113], Zeilberger [157], and Kuperberg [100]); a formula for the number of ASMs with a prescribed boundary row or column (Mills, Robbins and Rumsey [112, 113], and Zeilberger [158]); formulae for numbers of ASMs invariant under certain natural symmetry operations (Robbins [133, 134], Kuperberg [101], Okada [121], and Razumov and Stroganov [129, 130]); equalities between numbers of certain ASMs and numbers of certain totally symmetric self-complementary plane partitions (Mills, Robbins and Rumsey [114], and Fonseca and Zinn-Justin [81]); equalities between numbers of certain ASMs and numbers of certain descending plane partitions (Mills, Robbins and Rumsey [113], and Behrend, Di Francesco and Zinn-Justin [16]); equalities between numbers of certain ASMs and numbers associated with a certain case of the $O(1)$ loop model (Razumov and Stroganov [126], and Cantini and Sportiello [34]). For reviews of some of these results, see, for example, Bressoud [24, 26], Bressoud and Propp [27], Di Francesco [63, Sec. 4], [64], Hone [86], Zeilberger [160], or Zinn-Justin [161].

The main result of this paper is of the type which provides a relation which determines the generating function associated with numbers of ASMs of any fixed size with prescribed values of particular statistics. In any ASM, the first and last rows and columns each contain a single entry of 1, with all of their other entries being 0's, so that the positions of these 1's provide four natural statistics which describe the boundary configuration of an ASM. The generating function under primary consideration in this paper is associated with these four boundary statistics, together with two statistics which depend on the bulk structure of an ASM, namely the number of generalized inversions (which will be defined in (2)) and the number of entries of -1 . A detailed account of the elementary properties of this generating function and these statistics will be given in Section 2.

All previously-known results for the exact enumeration of ASMs have involved fewer than six of these statistics. A review of such results will be given in Section 3. In particular, that section will aim to provide a comprehensive account of all previous appearances of these statistics in exact enumeration, together with an outline of some new results which can be obtained relatively straightforwardly from the previously-known results. Recently, a result involving all four of the boundary statistics, but neither of the bulk statistics, was obtained, independently of the work reported in this paper, by Ayyer and Romik [12, Thm. 2]. That result will be outlined in Section 3.3.

In ASM enumerations, it is common to refer to a certain order of refinement which is based on the number of boundary statistics involved. Hence, the primary generating functions used in this paper, and by Ayyer and Romik [12], can be described as quadruply-refined.

The main result of this paper, Theorem 1, will be stated in Section 4, and proved in Section 5. The result consists of a quadratic relation satisfied by the ASM generating function associated with all six statistics, and which enables this generating function to be obtained recursively for successive ASM sizes. Various corollaries of Theorem 1, consisting of new expressions or relations for ASM generating functions associated with fewer than six of the statistics, will also be obtained in Section 4.

The approach used in the proof of Theorem 1 is essentially as follows. First, certain standard techniques, which have played a crucial role in proofs of many of the other known enumerative results for ASMs, are used to obtain a determinantal expression related to ASM generating functions. More specifically, a bijection between ASMs and configurations of the statistical mechanical six-vertex, or square ice, model with domain-wall boundary conditions (DWBC) is used to derive a relation between ASM generating functions and the partition function of the model, and the Izergin–Korepin formula is used to provide a determinantal expression for this partition function. Next, the Desnanot–Jacobi determinant identity is applied to the matrix in the Izergin–Korepin formula, which leads to a quadratic relation involving the required quadruply-refined ASM generating function, and generating functions associated with no ASM boundary statistics or with two ASM boundary statistics, for all four pairs of adjacent boundaries.

The key step in this derivation is the use of the Desnanot–Jacobi identity together with the Izergin–Korepin formula, and the reason that this leads to a relation of the previously-described form can be understood relatively easily. The six-vertex model with DWBC, in the form used here, involves so-called spectral parameters u_1, \dots, u_n and v_1, \dots, v_n , and the Izergin–Korepin formula expresses the partition function of this model as an explicit prefactor multiplied by the determinant of an $n \times n$ matrix, whose entry in row i and column j depends only on spectral parameters u_i and v_j . Furthermore, in terms of generating functions for $n \times n$ ASMs, the parameters u_i and v_j are, in a certain sense, associated with row i and column j of the ASMs. For an $n \times n$ matrix M , the Desnanot–Jacobi identity, in the form used here, consists of a relation involving $\det M$ and five minors of M : the four connected $(n-1) \times (n-1)$ minors, corresponding to deletion of the first or last row and first or last column of M , and the central $(n-2) \times (n-2)$ minor, corresponding to deletion of the first and last rows and columns of M . Therefore, it seems that applying the Desnanot–Jacobi identity to the matrix in the Izergin–Korepin formula, and using an assignment of spectral parameters in which u_1, u_n, v_1 and v_n remain arbitrary, while u_2, \dots, u_{n-1} are equal and v_2, \dots, v_{n-1} are equal, should lead to an expression involving a quadruply-refined generating function for $n \times n$ ASMs, four doubly-refined generating functions for $(n-1) \times (n-1)$ ASMs, and an unrefined generating function for $(n-2) \times (n-2)$ ASMs. Indeed, this essentially does occur, with some further, subsidiary aspects of the resulting expression being related to the form of the prefactor in the Izergin–Korepin formula, and to the fact that each corner entry of an ASM is associated with two of the parameters u_1, u_n, v_1 or v_n .

The Desnanot–Jacobi identity has, in fact, been associated with ASMs since these matrices first arose. In particular, the identity is used in Dodgson’s condensation algorithm [66] for determinant evaluation, and it was through studies of this algorithm by Mills, Robbins and Rumsey that ASMs initially appeared. In particular, it was shown by Robbins and Rumsey [135, Eq. (27)] that if Dodgson’s algorithm is applied to an $n \times n$ matrix M using a modified form of the Desnanot–Jacobi identity containing a parameter λ , then the resulting so-called λ -determinant of M can be expressed naturally as a sum over $n \times n$ ASMs. The latter expression involves the two ASM bulk statistics which are used in this paper, and, as will be shown in Section 3.2, the λ -determinant shares certain features with the quadruply-refined ASM generating function studied in this paper. For example, the modified Desnanot–Jacobi identity satisfied by the λ -determinant is analogous to the quadratic relation satisfied by the quadruply-refined ASM generating function. For further information regarding Dodgson condensation, λ -determinants, and related matters, see, for example, Abeles [1], Bressoud [24, Sec. 3.5], Bressoud and Propp [27], Di Francesco [62], Langer [105], Propp [124], or Robbins and Rumsey [135].

It should also be noted that the Desnanot–Jacobi identity, the λ -modified Desnanot–Jacobi identity, and further related equations studied by Robbins and Rumsey [135], and others, are closely connected with the Laurent phenomenon, cluster algebras, and associated areas. For reviews of such connections, see, for example, Di Francesco [63, Sec. 5], Hone [86, Sec. 5], or Propp [123, Sec. 10].

Following a suggestion by a referee, a file `MultiplyRefinedEnumerationOfASMs.nb`, in which cases of the main results of this paper are demonstrated in Mathematica, has been provided at the author's web page.

2. DEFINITIONS AND BASIC PROPERTIES

2.1. Statistics. In this section, the standard definitions of ASMs and certain statistics for ASMs are given, and the elementary properties of these statistics are identified in detail.

An ASM, as first defined by Mills, Robbins and Rumsey [112, 113], is simply a square matrix in which each entry is 0, 1 or -1 , and along each row and column the nonzero entries alternate in sign and have a sum of 1.

It follows that, for any ASM A , each partial row sum $\sum_{j'=1}^j A_{ij'}$ and each partial column sum $\sum_{i'=1}^i A_{i'j}$ is 0 or 1. It can also be seen that any permutation matrix is an ASM, and that, in any ASM, the first and last rows and columns each contain a single 1, with all of their other entries being 0's.

For each positive integer n , denote the set of all $n \times n$ ASMs as $\text{ASM}(n)$. For example, for $n = 1, 2, 3$, these sets are

$$\begin{aligned} \text{ASM}(1) &= \{(1)\}, \\ \text{ASM}(2) &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \\ \text{ASM}(3) &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}. \quad (1) \end{aligned}$$

For any $A \in \text{ASM}(n)$, define statistics which depend on the bulk structure of A as

$$\nu(A) = \sum_{\substack{1 \leq i < i' \leq n \\ 1 \leq j' < j \leq n}} A_{ij} A_{i'j'}, \quad \mu(A) = \text{number of } -1\text{'s in } A, \quad (2)$$

and define statistics which describe the configuration of A at its top, right, bottom and left boundaries as, respectively,

$$\begin{aligned} \rho_{\text{T}}(A) &= \text{number of } 0\text{'s to the left of the } 1 \text{ in the top row of } A, \\ \rho_{\text{R}}(A) &= \text{number of } 0\text{'s below the } 1 \text{ in the right-most column of } A, \\ \rho_{\text{B}}(A) &= \text{number of } 0\text{'s to the right of the } 1 \text{ in the bottom row of } A, \\ \rho_{\text{L}}(A) &= \text{number of } 0\text{'s above the } 1 \text{ in the left-most column of } A. \end{aligned} \quad (3)$$

Note that certain generalizations of the statistics (2)–(3) will be defined in Section 3.5.

The statistic $\nu(A)$ in (2) is a nonnegative integer for any $A \in \text{ASM}(n)$, since it can be written as $\nu(A) = \sum_{i,j=1}^n (\sum_{i'=1}^{i-1} A_{i'j}) (\sum_{j'=1}^j A_{ij'})$, where each factor in the summand (being a partial row or column sum of an ASM) is 0 or 1. This statistic can also be written as $\nu(A) = \sum_{1 \leq i \leq i' \leq n; 1 \leq j' < j \leq n} A_{ij} A_{i'j'}$, where this can be obtained from the definition in (2) using the fact that each complete row and column sum of A is a constant. It will be seen

in Section 5.2 that, in terms of the configuration of the six-vertex model with DWBC which corresponds to A , $\nu(A)$ is simply the number of vertex configurations of type (1), or equally the number of vertex configurations of type (2).

If A is a permutation matrix, then it can be seen from (2) that $\nu(A)$ is the number of inversions in the permutation π given by $\delta_{\pi_i, j} = A_{ij}$. Accordingly, for any ASM A , $\nu(A)$ is referred to as the number of generalized inversions in A . This statistic was first defined and used by Robbins and Rumsey [135, Eq. (18)], who referred to it as the number of positive inversions in an ASM [135, p. 182]. A closely-related statistic, $\sum_{1 \leq i < i' \leq n; 1 \leq j' < j \leq n} A_{ij} A_{i'j'}$ = $\nu(A) + \mu(A)$, was previously defined and used by Mills, Robbins and Rumsey [113, p. 344], and provides an alternative generalized inversion number for each $A \in \text{ASM}(n)$.

Proceeding to the other statistics, $\mu(A)$ in (2) can also be written as $\mu(A) = (\sum_{i,j=1}^n (A_{ij})^2 - n)/2$, and the boundary statistics of (3) can be depicted diagrammatically as

$$\begin{array}{c}
 \leftarrow \rho_T(A) \rightarrow \\
 \uparrow \rho_L(A) \downarrow \\
 \left(\begin{array}{cccccccc}
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & & & & & & & & 0 \\
 0 & & & & & & & & 0 \\
 1 & & & & & & & & 0 \\
 0 & & & A & & & & & 1 \\
 0 & & & & & & & & 0 \\
 0 & & & & & & & & 0 \\
 0 & & & & & & & & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array} \right) \rho_R(A) \uparrow \downarrow \\
 \leftarrow \rho_B(A) \rightarrow
 \end{array} \quad (4)$$

It can be checked that the statistics of (2)–(3), regarded as functions on $\text{ASM}(n)$, have ranges

$$\begin{aligned}
 \nu(\text{ASM}(n)) &= \{0, 1, \dots, \frac{n(n-1)}{2}\}, \\
 \mu(\text{ASM}(n)) &= \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor\} = \{0, 1, \dots, \lfloor \frac{(n-1)^2}{4} \rfloor\}, \\
 \rho_T(\text{ASM}(n)) &= \rho_R(\text{ASM}(n)) = \rho_B(\text{ASM}(n)) = \rho_L(\text{ASM}(n)) = \{0, 1, \dots, n-1\}. \quad (5)
 \end{aligned}$$

The ASMs A which give the extreme values within the ranges in (5) are as follows. The only A with $\nu(A) = 0$ is the $n \times n$ identity matrix, and the only A with $\nu(A) = \frac{n(n-1)}{2}$ is the $n \times n$ antidiagonal matrix given by $A_{ij} = \delta_{i, n+1-j}$. The A with $\mu(A) = 0$ are the $n!$ $n \times n$ permutation matrices, and the A with $\mu(A) = \lfloor \frac{(n-1)^2}{4} \rfloor$ are given by $A_{ij} = (-1)^{i+j+k_n}$ if $k_n + 2 \leq i + j \leq 2n - k_n$ and $|i - j| \leq k_n$, and $A_{ij} = 0$ otherwise, where $k_n = \frac{n-1}{2}$ for n odd (giving a single such A), and $k_n = \frac{n}{2} - 1$ or $\frac{n}{2}$ for n even (giving two such A). The A for which a boundary statistic of (3) is 0 or $n-1$ are the ASMs with a 1 in a corner, which will be discussed further shortly.

It can be seen that transposition or anticlockwise quarter-turn rotation of an ASM give another ASM. It can also be checked easily that, for each $A \in \text{ASM}(n)$, the statistics (2)–(3)

behave under these operations according to

$$\begin{aligned} \nu(A) &= \nu(A^T) = \frac{n(n-1)}{2} - \nu(A^Q) - \mu(A), & \mu(A) &= \mu(A^T) = \mu(A^Q), \\ \rho_T(A) &= \rho_L(A^T) = n - 1 - \rho_L(A^Q), & \rho_R(A) &= \rho_B(A^T) = n - 1 - \rho_T(A^Q), \\ \rho_B(A) &= \rho_R(A^T) = n - 1 - \rho_R(A^Q), & \rho_L(A) &= \rho_T(A^T) = n - 1 - \rho_B(A^Q), \end{aligned} \quad (6)$$

where Q denotes anticlockwise quarter-turn rotation, i.e., $A_{ij}^Q = A_{j,n+1-i}$. By combining transposition and anticlockwise quarter-turn rotation, the behaviour of any of the eight operations of the dihedral group acting on ASMs can be obtained.

The properties of ASMs with a 1 in a corner can be described easily. For example, for the case of ASMs with a 1 in the top-left corner, the sets $\{A \in \text{ASM}(n) \mid A_{11} = 1\} = \{A \in \text{ASM}(n) \mid \rho_T(A) = \rho_L(A) = 0\}$ and $\text{ASM}(n-1)$ are in bijection for any $n \geq 2$, where an ASM from the first set is mapped to the second set by simply deleting the first row and first column. Furthermore, if A from the first set is mapped to A' in the second set, then $\nu(A') = \nu(A)$, $\mu(A') = \mu(A)$, $\rho_R(A') = \rho_R(A)$ and $\rho_B(A') = \rho_B(A)$.

The properties of ASMs in which a 1 on a boundary is separated from a corner by a single zero, i.e., ASMs in which a boundary statistic of (3) is 1 or $n-2$, can also be described relatively easily. For example, for the case of ASMs A with $A_{21} = 1$, the sets $\{A \in \text{ASM}(n) \mid A_{21} = A_{1,k+1} = 1\} = \{A \in \text{ASM}(n) \mid \rho_L(A) = 1, \rho_T(A) = k\}$ and $\{A \in \text{ASM}(n-1) \mid A_{11} = \dots = A_{1,k-1} = 0\} = \{A \in \text{ASM}(n-1) \mid \rho_T(A) \geq k-1\}$ are in bijection for any $n \geq 2$ and $k = 1, \dots, n-1$, where an ASM A from the first set is mapped to the second set by replacing $A_{2,k+1}$ by $A_{2,k+1} + 1$, while leaving all other entries unchanged, and then deleting the first row and first column. Furthermore, if A from the first set is mapped to A' in the second set, then $\nu(A') = \nu(A) - A_{2,k+1} - 1$ and $\mu(A') = \mu(A) + A_{2,k+1}$.

The previous bijection seems not to have appeared explicitly in the literature, although it is likely to have been known to several authors. In particular, it seems to have been used by Mills, Robbins and Rumsey [113, p. 344] in claiming that the case $k = 1$ of their Conjecture 2 (which will appear as the identity $\mathcal{A}_{n,1} = \frac{n}{2} \mathcal{A}_{n-1}$ in (33)) could be proved easily, and it also seems to have been used by Stroganov [150, p. 61] in claiming that a certain identity (which corresponds to the first equation of (33)) could be proved bijectively.

2.2. Generating functions. In this section, various ASM generating functions are defined, and some simple relations which they satisfy are derived. Each of these generating functions is described by a certain order of refinement, which corresponds to the number of boundary statistics of (3) (or simply the number of boundaries) with which it is associated. Each generating function is also associated with the two bulk statistics of (2), i.e., the numbers of generalized inversions and -1 's in an ASM.

For each positive integer n , define a quadruply-refined ASM generating function, which involves all six statistics of (2)–(3), and associated indeterminates x, y, z_1, z_2, z_3 and z_4 , as

$$Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)} z_1^{\rho_T(A)} z_2^{\rho_R(A)} z_3^{\rho_B(A)} z_4^{\rho_L(A)}. \quad (7)$$

It follows that x and y can be regarded as bulk parameters or weights, and that z_1, z_2, z_3 and z_4 can be regarded as boundary parameters or weights.

It can be seen, using (5) and (7), that $Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4)$ is a polynomial, with non-negative integer coefficients, of degree $\frac{n(n-1)}{2}$ in x , of degree $\lfloor \frac{(n-1)^2}{4} \rfloor$ in y , and of degree $n-1$ in each of z_1, z_2, z_3 and z_4 .

Examples of the quadruply-refined ASM generating function (7), for $n = 1, 2, 3$, are

$$\begin{aligned} Z_1^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) &= 1, \\ Z_2^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) &= 1 + x z_1 z_2 z_3 z_4, \\ Z_3^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) &= 1 + x z_1 z_4 + x z_2 z_3 + x^2 z_1 z_2 z_3^2 z_4^2 + x^2 z_1^2 z_2^2 z_3 z_4 + \\ &\quad x^3 z_1^2 z_2^2 z_3^2 z_4^2 + x y z_1 z_2 z_3 z_4, \end{aligned} \quad (8)$$

where the terms are written in orders which correspond to those used in (1).

Now define triply-refined, adjacent-boundary doubly-refined, opposite-boundary doubly-refined, singly-refined and unrefined ASM generating functions as, respectively,

$$\begin{aligned} Z_n^{\text{tri}}(x, y; z_1, z_2, z_3) &= Z_n^{\text{quad}}(x, y; z_1, 1, z_2, z_3) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)} z_1^{\rho_{\text{T}}(A)} z_2^{\rho_{\text{B}}(A)} z_3^{\rho_{\text{L}}(A)}, \\ Z_n^{\text{adj}}(x, y; z_1, z_2) &= Z_n^{\text{quad}}(x, y; z_1, 1, 1, z_2) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)} z_1^{\rho_{\text{T}}(A)} z_2^{\rho_{\text{L}}(A)}, \\ Z_n^{\text{opp}}(x, y; z_1, z_2) &= Z_n^{\text{quad}}(x, y; z_1, 1, z_2, 1) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)} z_1^{\rho_{\text{T}}(A)} z_2^{\rho_{\text{B}}(A)}, \\ Z_n(x, y; z) &= Z_n^{\text{quad}}(x, y; z, 1, 1, 1) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)} z^{\rho_{\text{T}}(A)}, \\ Z_n(x, y) &= Z_n^{\text{quad}}(x, y; 1, 1, 1, 1) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)}, \end{aligned} \quad (9)$$

where z is a further indeterminate.

Also define alternative quadruply-refined and alternative adjacent-boundary doubly-refined ASM generating functions as, respectively,

$$\begin{aligned} \tilde{Z}_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) &= (z_2 z_4)^{n-1} Z_n^{\text{quad}}(x, y; z_1, \frac{1}{z_2}, z_3, \frac{1}{z_4}) \\ &= \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)} z_1^{\rho_{\text{T}}(A)} z_2^{n-\rho_{\text{R}}(A)-1} z_3^{\rho_{\text{B}}(A)} z_4^{n-\rho_{\text{L}}(A)-1}, \\ \tilde{Z}_n^{\text{adj}}(x, y; z_1, z_2) &= Z_n^{\text{quad}}(x, y; z_1, z_2, 1, 1) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)} z_1^{\rho_{\text{T}}(A)} z_2^{\rho_{\text{R}}(A)}. \end{aligned} \quad (10)$$

Note that $\tilde{Z}_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4)$ is a generating function in which the positions of the 1's in the first and last columns of an ASM are measured relative to the opposite ends of the columns to those used in (3)–(4), i.e., in this generating function, the statistics associated with z_2 and z_4 are, respectively, the numbers of 0's above the 1 in the right-most column, and below the 1 in the left-most column of an ASM.

It can be seen immediately that some relations among generating functions of (9)–(10), involving specializations of boundary parameters to 1, are

$$\begin{aligned}
Z_n^{\text{tri}}(x, y; z_1, 1, z_2) &= Z_n^{\text{adj}}(x, y; z_1, z_2), \\
Z_n^{\text{tri}}(x, y; 1, z_1, z_2) &= \tilde{Z}_n^{\text{adj}}(x, y; z_1, z_2), \\
Z_n^{\text{tri}}(x, y; z_1, z_2, 1) &= Z_n^{\text{opp}}(x, y; z_1, z_2), \\
Z_n^{\text{adj}}(x, y; z, 1) &= \tilde{Z}_n^{\text{adj}}(x, y; z, 1) = Z_n^{\text{opp}}(x, y; z, 1) = Z_n(x, y; z), \\
Z_n(x, y; 1) &= Z_n(x, y).
\end{aligned} \tag{11}$$

Some elementary identities satisfied by the ASM generating functions of (7), (9) and (10), which follow from the ASM properties outlined in Section 2.1, will now be obtained.

Note that, in this and all subsequent sections, many of the identities which contain the positive integer n will be valid only for all $n \geq 2$, or for all $n \geq 3$. This will often be due to their containing terms (such as $Z_{n-1}(x, y)$ or $Z_{n-2}(x, y)$) which are not defined if n is taken to be 1 or 2.

Note also that several of the identities of this section, obtained here using simple combinatorial arguments, can alternatively be obtained as special cases of more general identities which will be derived in subsequent sections using other methods.

Some symmetry relations, which can be derived by acting on $\text{ASM}(n)$ with transposition or anticlockwise quarter-turn rotation, and using (6), are

$$\begin{aligned}
Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) &= Z_n^{\text{quad}}(x, y; z_4, z_3, z_2, z_1) \\
&= x^{n(n-1)/2} (z_1 z_2 z_3 z_4)^{n-1} Z_n^{\text{quad}}\left(\frac{1}{x}, \frac{y}{x}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_1}\right), \\
\tilde{Z}_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) &= (z_1 z_2 z_3 z_4)^{n-1} \tilde{Z}_n^{\text{quad}}\left(x, y; \frac{1}{z_4}, \frac{1}{z_3}, \frac{1}{z_2}, \frac{1}{z_1}\right) \\
&= x^{n(n-1)/2} \tilde{Z}_n^{\text{quad}}\left(\frac{1}{x}, \frac{y}{x}; z_2, z_3, z_4, z_1\right), \\
Z_n^{\text{tri}}(x, y; z_1, z_2, z_3) &= x^{n(n-1)/2} (z_1 z_2 z_3)^{n-1} Z_n^{\text{tri}}\left(\frac{1}{x}, \frac{y}{x}; \frac{1}{z_2}, \frac{1}{z_1}, \frac{1}{z_3}\right), \\
Z_n^{\text{adj}}(x, y; z_1, z_2) &= Z_n^{\text{adj}}(x, y; z_2, z_1) \\
&= x^{n(n-1)/2} (z_1 z_2)^{n-1} \tilde{Z}_n^{\text{adj}}\left(\frac{1}{x}, \frac{y}{x}; \frac{1}{z_1}, \frac{1}{z_2}\right), \\
Z_n^{\text{opp}}(x, y; z_1, z_2) &= Z_n^{\text{opp}}(x, y; z_2, z_1) \\
&= x^{n(n-1)/2} (z_1 z_2)^{n-1} Z_n^{\text{opp}}\left(\frac{1}{x}, \frac{y}{x}; \frac{1}{z_1}, \frac{1}{z_2}\right), \\
Z_n(x, y; z) &= x^{n(n-1)/2} z^{n-1} Z_n\left(\frac{1}{x}, \frac{y}{x}; \frac{1}{z}\right), \\
Z_n(x, y) &= x^{n(n-1)/2} Z_n\left(\frac{1}{x}, \frac{y}{x}\right).
\end{aligned} \tag{12}$$

Some identities involving specializations of boundary parameters to 0, which follow from the properties of ASMs with a 1 in a corner, as discussed near the end of Section 2.1, are

$$\begin{aligned}
Z_n^{\text{quad}}(x, y; z_1, 0, z_2, z_3) &= Z_n^{\text{tri}}(x, y; z_1, 0, z_3) = Z_{n-1}^{\text{adj}}(x, y; z_1, z_3), \\
\tilde{Z}_n^{\text{adj}}(x, y; z, 0) &= Z_n^{\text{opp}}(x, y; z, 0) = Z_{n-1}(x, y; z),
\end{aligned}$$

$$Z_n^{\text{adj}}(x, y; z, 0) = Z_n(x, y; 0) = Z_{n-1}(x, y). \quad (13)$$

It will sometimes be useful to refer to boundary parameter coefficients in the adjacent-boundary doubly-refined and singly-refined ASM generating functions. In particular, define

$$\begin{aligned} Z_n^{\text{adj}}(x, y)_{k_1, k_2} &= \text{coefficient of } z_1^{k_1} z_2^{k_2} \text{ in } Z_n^{\text{adj}}(x, y; z_1, z_2), \\ Z_n(x, y)_k &= \text{coefficient of } z^k \text{ in } Z_n(x, y; z). \end{aligned} \quad (14)$$

These coefficients satisfy identities which correspond to identities of (11)–(13). For example,

$$\begin{aligned} \sum_{k_2=0}^{n-1} Z_n^{\text{adj}}(x, y)_{k_1, k_2} &= Z_n(x, y)_{k_1}, & \sum_{k=0}^{n-1} Z_n(x, y)_k &= Z_n(x, y), \\ Z_n^{\text{adj}}(x, y)_{k_1, k_2} &= Z_n^{\text{adj}}(x, y)_{k_2, k_1}, & Z_n(x, y)_k &= x^{n(n-1)/2} Z_n\left(\frac{1}{x}, \frac{y}{x}\right)_{n-1-k}, \\ Z_n^{\text{adj}}(x, y)_{k, 0} &= Z_{n-1}(x, y) \delta_{k, 0}, & Z_n^{\text{adj}}(x, y)_{k+1, n-1} &= x^{n-1} Z_{n-1}(x, y)_k, \\ Z_n(x, y)_0 &= Z_{n-1}(x, y). \end{aligned} \quad (15)$$

Furthermore, the properties of ASMs in which a 1 on a boundary is separated from a corner by a single zero, as discussed at the end of Section 2.1, lead to identities such as

$$\begin{aligned} Z_n^{\text{adj}}(x, y)_{k, 1} &= x Z_{n-1}(x, y)_{k-1} + y \sum_{i=k}^{n-2} Z_{n-1}(x, y)_i - y Z_{n-1}(x, y) \delta_{k, 0}, \\ Z_n(x, y)_1 &= x Z_{n-1}(x, y) + y \sum_{k=1}^{n-2} k Z_{n-1}(x, y)_k. \end{aligned} \quad (16)$$

3. PREVIOUSLY-KNOWN AND RELATED RESULTS

In this section, an account is given of results for the exact enumeration of $n \times n$ ASMs (for arbitrary, finite n), involving any of the six statistics of (2)–(3), or any of the associated generating functions of (7), (9) or (10).

Most of these results have appeared elsewhere in the literature (although possibly in different formulations, or with different notation), but some new results which are closely related to previously-known results are also presented. The main new results in this section are (19) and some cases of (20), for ASM generating functions in which the bulk parameters are related by $y = x + 1$, and (72), (73) and (75), for a generating function, which will be introduced in (67), associated with several rows (or several columns) of ASMs. Derivations of the new results will be given in this section and in Section 5.

Various cases for the values of the parameters x and y , associated with the bulk statistics (2), will first be considered separately: $y = 0$ in Section 3.1, $y = x + 1$ in Section 3.2, $x = y = 1$ in Section 3.3, and x and y arbitrary in Section 3.4. Further results will then be discussed in Sections 3.5–3.15.

3.1. Bulk parameter $y = 0$. The case in which the bulk parameter y is 0 corresponds simply to the enumeration of permutation matrices, with prescribed values of the inversion number of the associated permutations, and prescribed positions of 1's on the boundaries of the matrices.

Let x -numbers and the x -factorial be defined, as usual, as $[n]_x = 1 + x + \dots + x^{n-1}$, and $[n]_x! = [n]_x [n-1]_x \dots [1]_x$.

For $y = 0$, the quadruply-refined ASM generating function is explicitly

$$\begin{aligned}
Z_n^{\text{quad}}(x, 0; z_1, z_2, z_3, z_4) = & x^2 z_1 z_2 z_3 z_4 \sum_{0 \leq i < j \leq n-3} (x^{n+i-j-3} z_4^i z_2^{n-j-3} + x^{n-i+j-4} z_4^{n-i-3} z_2^j) \times \\
& \sum_{0 \leq i < j \leq n-3} (x^{n+i-j-3} z_1^i z_3^{n-j-3} + x^{n-i+j-4} z_1^{n-i-3} z_3^j) [n-4]_x! + \\
& (x z_4 z_1 [n-2]_{x z_4} [n-2]_{x z_1} + z_1 z_2 (x z_3 z_4)^{n-1} [n-2]_{x z_1} [n-2]_{x z_2} + \\
& x z_2 z_3 [n-2]_{x z_2} [n-2]_{x z_3} + z_3 z_4 (x z_1 z_2)^{n-1} [n-2]_{x z_3} [n-2]_{x z_4}) [n-3]_x! + \\
& (1 + x^{2n-3} (z_1 z_2 z_3 z_4)^{n-1}) [n-2]_x!, \quad (17)
\end{aligned}$$

and the ASM generating functions of (9) are explicitly

$$\begin{aligned}
Z_n^{\text{tri}}(x, 0; z_1, z_2, z_3) = & (x z_2 \sum_{0 \leq i < j \leq n-3} (x^{n+i-j-3} z_1^i z_2^{n-j-3} + x^{n-i+j-4} z_1^{n-i-3} z_2^j) + \\
& [n-2]_{x z_1} + z_2 (x z_1)^{n-2} [n-2]_{x z_2}) x z_1 z_3 [n-2]_{x z_3} [n-3]_x! + \\
& (z_1 (x z_2 z_3)^{n-1} [n-2]_{x z_1} + x z_2 [n-2]_{x z_2} + 1 + x^{2n-3} (z_1 z_2 z_3)^{n-1}) [n-2]_x! \\
Z_n^{\text{adj}}(x, 0; z_1, z_2) = & x z_1 z_2 [n-1]_{x z_1} [n-1]_{x z_2} [n-2]_x! + [n-1]_x!, \\
Z_n^{\text{opp}}(x, 0; z_1, z_2) = & \sum_{0 \leq i < j \leq n-1} (x^{n+i-j-1} z_1^i z_2^{n-j-1} + x^{n-i+j-2} z_1^{n-i-1} z_2^j) [n-2]_x!, \\
Z_n(x, 0; z) = & [n]_{xz} [n-1]_x!, \\
Z_n(x, 0) = & [n]_x!. \quad (18)
\end{aligned}$$

Each of these formulae is either a standard result for permutations, or a straightforward variation of such a result, and each formula can be derived using simple combinatorial arguments. See, for example, Stanley [142, Cor. 1.3.13] for a derivation of the last equation of (18).

Note that if (17) is shown first to be valid, then each equation of (18) can subsequently be obtained by setting certain boundary parameters in (17) to 1. Alternatively, if the last two equations of (18) are shown first to be valid, then (17) and the remaining equations of (18) can subsequently be obtained by using the general results (81), (83) (or (86)), (87) and (88), which will be given in Section 4.

In interpreting (17), note also that the seven main terms on the RHS of (17) (i.e., the term ending in $[n-4]_x!$, the four terms ending in $[n-3]_x!$ and the two terms ending in $[n-2]_x!$) correspond to sums over the sets of $n \times n$ permutation matrices A with $(A_{11}, A_{1n}, A_{nn}, A_{n1})$ equal to $(0, 0, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(1, 0, 1, 0)$ or $(0, 1, 0, 1)$, respectively. In subsequent formulae for quadruply-refined ASM generating functions (e.g., (19), (49) and (81)), the RHS will again consist of seven or eight main terms, but these terms will no longer correspond simply to sums over sets of ASMs with fixed values of the four corner entries.

3.2. Bulk parameters satisfying $y = x + 1$. The case in which the bulk parameters are related by $y = x + 1$ is closely related to x -determinants (or, in the notation usually used, λ -determinants) of matrices, domino tilings of an Aztec diamond, tournaments, and the free

fermion case of the six-vertex model. (Note that the sub-case $x = 1$ and $y = 2$, which corresponds to the so-called 2-enumeration of ASMs, is often considered separately.) Due to certain combinatorial and algebraic simplifications, it is again possible to obtain explicit formulae.

For $y = x + 1$, the quadruply-refined ASM generating function is given explicitly by

$$\begin{aligned}
& (xz_4z_1 + z_4 + z_1 - 1)(xz_1z_2 - xz_1 - xz_2 - 1) \times \\
& \quad (xz_2z_3 + z_2 + z_3 - 1)(xz_3z_4 - xz_3 - xz_4 - 1) Z_n^{\text{quad}}(x, x+1; z_1, z_2, z_3, z_4) = \\
& z_1z_2z_3z_4(xz_1z_3 + 1)(xz_2z_4 + 1) \left((xz_1 + 1)(xz_2 + 1)(xz_3 + 1)(xz_4 + 1) \right)^{n-2} (x+1)^{(n-4)(n-5)/2} - \\
& \quad \left((z_2 - 1)(z_3 - 1)z_4z_1 \left((xz_4 + 1)(xz_1 + 1) \right)^{n-2} + \right. \\
& \quad \quad \left. (z_4 - 1)(z_1 - 1)z_2z_3 \left((xz_2 + 1)(xz_3 + 1) \right)^{n-2} \right) \times \\
& \quad (xz_1z_2 - xz_1 - xz_2 - 1)(xz_3z_4 - xz_3 - xz_4 - 1)(x+1)^{(n-3)(n-4)/2} - \\
& \quad \left((z_3 - 1)(z_4 - 1)z_1z_2(z_3z_4)^{n-1} \left((xz_1 + 1)(xz_2 + 1) \right)^{n-2} + \right. \\
& \quad \quad \left. (z_1 - 1)(z_2 - 1)z_3z_4(z_1z_2)^{n-1} \left((xz_3 + 1)(xz_4 + 1) \right)^{n-2} \right) \times \\
& \quad (xz_4z_1 + z_4 + z_1 - 1)(xz_2z_3 + z_2 + z_3 - 1) x^n (x+1)^{(n-3)(n-4)/2} + \\
& \quad (z_1 - 1)(z_2 - 1)(z_3 - 1)(z_4 - 1) \left((xz_1z_2 - xz_1 - xz_2 - 1)(xz_3z_4 - xz_3 - xz_4 - 1) + \right. \\
& \quad \left. (xz_4z_1 + z_4 + z_1 - 1)(xz_2z_3 + z_2 + z_3 - 1)(z_1z_2z_3z_4)^{n-1} x^{2n-1} \right) (x+1)^{(n-2)(n-3)/2}, \quad (19)
\end{aligned}$$

and the ASM generating functions of (9) are given explicitly by

$$\begin{aligned}
& (xz_1z_3 + z_1 + z_3 - 1)(xz_2z_3 - xz_2 - xz_3 - 1) Z_n^{\text{tri}}(x, x+1; z_1, z_2, z_3) = \\
& \quad - z_1z_3(xz_1z_2 + 1) \left((xz_1 + 1)(xz_2 + 1) \right)^{n-2} (xz_3 + 1)^{n-1} (x+1)^{(n-3)(n-4)/2} + \\
& \quad (z_2 - 1)(z_3 - 1)(xz_1z_3 + z_1 + z_3 - 1)z_1(z_2z_3)^{n-1} (xz_1 + 1)^{n-2} x^n (x+1)^{(n-2)(n-3)/2} - \\
& \quad (z_1 - 1)(z_3 - 1)(xz_2z_3 - xz_2 - xz_3 - 1)(xz_2 + 1)^{n-2} (x+1)^{(n-2)(n-3)/2}, \\
& (xz_1z_2 + z_1 + z_2 - 1) Z_n^{\text{adj}}(x, x+1; z_1, z_2) = z_1z_2 \left((xz_1 + 1)(xz_2 + 1) \right)^{n-1} (x+1)^{(n-2)(n-3)/2} \\
& \quad - (z_1 - 1)(z_2 - 1) (x+1)^{(n-1)(n-2)/2}, \\
& Z_n^{\text{opp}}(x, x+1; z_1, z_2) = (xz_1z_2 + 1) \left((xz_1 + 1)(xz_2 + 1) \right)^{n-2} (x+1)^{(n-2)(n-3)/2}, \\
& Z_n(x, x+1; z) = (xz + 1)^{n-1} (x+1)^{(n-1)(n-2)/2}, \\
& Z_n(x, x+1) = (x+1)^{n(n-1)/2}. \quad (20)
\end{aligned}$$

Each of these formulae is either a previously-known result, or can be obtained relatively easily from such results. More specifically, the last three cases of (20) have appeared, in various forms, in the literature (see the references at the end of this section), but it seems that (19) and the first two cases of (20) have not. A formula which can be regarded as a generalization of the last three cases of (20) will be given near the end of Section 3.5.

A derivation of (19)–(20), based on a result of Robbins and Rumsey [135, Sec. 5] for x -determinants, will now be given, since this derivation involves the use of a quadratic relation satisfied by the quadruply-refined ASM generating function, and thereby shares some features with new material which will be presented in Section 4. An alternative derivation, based on the Izergin–Korepin formula for the partition function of the six-vertex model with DWBC and the Cauchy double alternant evaluation, will be given in Section 5.8.

For an $n \times n$ matrix of indeterminates, $(M_{ij})_{1 \leq i, j \leq n}$, define

$$\mathcal{Z}_n(x, M) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} (x+1)^{\mu(A)} \prod_{i,j=1}^n (M_{ij})^{A_{ij}}, \quad (21)$$

this being the so-called x -determinant of M , as introduced by Robbins and Rumsey [135, Sec. 5]. It can be seen that $\mathcal{Z}_n(-1, M)$ is the standard determinant of M , since the RHS of (21) is then a sum over all $n \times n$ permutation matrices A , with $\nu(A)$ being the number of inversions in the permutation associated with A .

It follows from a result of Robbins and Rumsey [135, Sec. 5] that

$$\mathcal{Z}_n(x, M) \mathcal{Z}_{n-2}(x, M_C) = \mathcal{Z}_{n-1}(x, M_{\text{TL}}) \mathcal{Z}_{n-1}(x, M_{\text{BR}}) + x \mathcal{Z}_{n-1}(x, M_{\text{TR}}) \mathcal{Z}_{n-1}(x, M_{\text{BL}}), \quad (22)$$

where M_{TL} , M_{TR} , M_{BR} and M_{BL} denote the $(n-1) \times (n-1)$ submatrices corresponding to the top-left, top-right, bottom-right and bottom-left corners of M , respectively, and M_C denotes the central $(n-2) \times (n-2)$ submatrix of M . For $x = -1$, (22) is the Desnanot–Jacobi determinant identity, which will be discussed in more detail in Section 5.5.

Taking M in (22) to be

$$M = \begin{pmatrix} 1 & z_1 & \dots & z_1^{n-2} & (z_1 z_2)^{n-1} \\ z_4 & 1 & \dots & 1 & z_2^{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ z_4^{n-2} & 1 & \dots & 1 & z_2 \\ (z_3 z_4)^{n-1} & z_3^{n-2} & \dots & z_3 & 1 \end{pmatrix}, \quad (23)$$

it follows, using definitions from (7), (9), (10) and (21), that the quadruply-refined ASM generating function satisfies the quadratic relation

$$\begin{aligned} Z_n^{\text{quad}}(x, x+1; z_1, z_2, z_3, z_4) Z_{n-2}(x, x+1) &= Z_{n-1}^{\text{adj}}(x, x+1; z_4, z_1) Z_{n-1}^{\text{adj}}(x, x+1; z_2, z_3) + \\ &\quad x z_1 z_2 z_3 z_4 \tilde{Z}_{n-1}^{\text{adj}}(x, x+1; z_1, z_2) \tilde{Z}_{n-1}^{\text{adj}}(x, x+1; z_3, z_4). \end{aligned} \quad (24)$$

Note that a quadratic relation for arbitrary x and y will be given in (81), and that a different quadratic relation for the present case can be obtained by setting $y = x+1$ in (81).

Setting $z_1 = z_2 = z_3 = z_4 = 1$ in (24) gives

$$Z_n(x, x+1) Z_{n-2}(x, x+1) = (x+1) Z_{n-1}(x, x+1)^2, \quad (25)$$

which, together with $Z_1(x, x+1) = 1$ and $Z_2(x, x+1) = x+1$, gives the formula in (20) for $Z_n(x, x+1)$.

Setting $z_2 = z_3 = z_4 = 1$ in (24), and relabelling z_1 as z , gives

$$Z_n(x, x+1; z) Z_{n-2}(x, x+1) = (xz+1) Z_{n-1}(x, x+1; z) Z_{n-1}(x, x+1), \quad (26)$$

which, together with $Z_1(x, x+1; z) = 1$ and the formula for $Z_n(x, x+1)$, gives the formula in (20) for $Z_n(x, x+1; z)$. The formula for $Z_n^{\text{opp}}(x, x+1; z_1, z_2)$ can be obtained similarly, by setting $z_2 = z_4 = 1$ in (24), and using the formulae for $Z_n(x, x+1; z)$ and $Z_n(x, x+1)$.

Setting $z_2 = z_3 = 1$ in (24), and relabelling z_4 as z_2 , gives

$$Z_n^{\text{adj}}(x, x+1; z_1, z_2) Z_{n-2}(x, x+1) = Z_{n-1}^{\text{adj}}(x, x+1; z_1, z_2) Z_{n-1}(x, x+1) + xz_1z_2 Z_{n-1}(x, x+1; z_1) Z_{n-1}(x, x+1; z_2), \quad (27)$$

which, together with $Z_1^{\text{adj}}(x, x+1; z_1, z_2) = 1$, gives

$$Z_n^{\text{adj}}(x, x+1; z_1, z_2) = Z_{n-1}(x, x+1) \left(1 + xz_1z_2 \sum_{i=1}^{n-1} \frac{Z_i(x, x+1; z_1) Z_i(x, x+1; z_2)}{Z_{i-1}(x, x+1) Z_i(x, x+1)} \right), \quad (28)$$

where, in the sum over i , $Z_0(x, x+1)$ is taken to be 1. Substituting the formulae for $Z_i(x, x+1; z)$ and $Z_i(x, x+1)$ into (28), and then performing the sum over i and simplifying, gives the formula in (20) for $Z_n^{\text{adj}}(x, x+1; z_1, z_2)$.

Finally, the formulae in (19)–(20) for $Z_n^{\text{quad}}(x, x+1; z_1, z_2, z_3, z_4)$ and $Z_n^{\text{tri}}(x, x+1; z_1, z_2, z_3)$ can be obtained using (24), together with the formulae for $Z_n^{\text{adj}}(x, x+1; z_1, z_2)$ and $Z_n(x, x+1)$, and a similar formula for $\tilde{Z}_n^{\text{adj}}(x, x+1; z_1, z_2)$ (which can be obtained from the formula for $Z_n^{\text{adj}}(x, x+1; z_1, z_2)$ and the relevant equation from (12)).

For further information regarding various aspects of the case $y = x + 1$ and closely related cases, including details of several alternative methods for obtaining formulae such as (19)–(20), see, for example, Bogoliubov, Pronko and Zvonarev [21, Sec. 5], Bosio and van Leeuwen [22], Bousquet-Mélou and Habsieger [23, Sec. 5], Bressoud [25], Brualdi and Kirkland [28], Brubaker, Bump and Friedberg [30, 31], Bump, McNamara and Nakasuji [32], Chapman [36], Ciucu [39], [40, Thm. 6.1], Colomo and Pronko [48, Secs. 4.2 & 5.2], [49, Sec. 4.3], Di Francesco [62], Elkies, Kuperberg, Larsen and Propp [69, 70], Eu and Fu [71], Ferrari and Spohn [72], Hamel and King [85, Cor. 5.1], Kuo [99, Sec. 3], Kuperberg [101, Thm. 3], Langer [105], Lascoux [106], McNamara [111], Mills, Robbins and Rumsey [113, Sec. 6], Okada [121, Thm. 2.4(1), third eq.], Rosengren [136, Sec. 9], Striker [144, Sec. 5], [146, Sec. 6], Tokuyama [152, Cor. 3.4], and Yang [155].

3.3. Bulk parameters $x = y = 1$. The case in which the bulk parameters x and y are both 1 involves numbers of ASMs with prescribed configurations on certain boundaries, but without prescribed values of bulk statistics. The cases of zero, one or two boundaries will be considered together first, followed by the cases of three or four boundaries.

Define unrefined, singly-refined, opposite-boundary doubly-refined and adjacent-boundary doubly-refined ASM numbers as, respectively,

$$\begin{aligned}
\mathcal{A}_n &= |\text{ASM}(n)|, \\
\mathcal{A}_{n,k} &= |\{A \in \text{ASM}(n) \mid A_{1,k+1} = 1\}|, \\
\mathcal{A}_{n,k_1,k_2}^{\text{opp}} &= |\{A \in \text{ASM}(n) \mid A_{1,k_1+1} = A_{n,n-k_2} = 1\}|, \\
\mathcal{A}_{n,k_1,k_2}^{\text{adj}} &= |\{A \in \text{ASM}(n) \mid A_{1,k_1+1} = A_{k_2+1,1} = 1\}|,
\end{aligned} \tag{29}$$

for $0 \leq k, k_1, k_2 \leq n-1$, with the numbers being 0 for k, k_1 or k_2 outside this range. These numbers are therefore related to generating functions of (9)–(10) by

$$\begin{aligned}
Z_n(1, 1) &= \mathcal{A}_n, \\
Z_n(1, 1; z) &= \sum_{k=0}^{n-1} \mathcal{A}_{n,k} z^k, \\
Z_n^{\text{opp}}(1, 1; z_1, z_2) &= \sum_{k_1, k_2=0}^{n-1} \mathcal{A}_{n,k_1,k_2}^{\text{opp}} z_1^{k_1} z_2^{k_2}, \\
Z_n^{\text{adj}}(1, 1; z_1, z_2) &= \sum_{k_1, k_2=0}^{n-1} \mathcal{A}_{n,k_1,k_2}^{\text{adj}} z_1^{k_1} z_2^{k_2}, \\
\tilde{Z}_n^{\text{adj}}(1, 1; z_1, z_2) &= \sum_{k_1, k_2=0}^{n-1} \mathcal{A}_{n,n-1-k_1,n-1-k_2}^{\text{adj}} z_1^{k_1} z_2^{k_2},
\end{aligned} \tag{30}$$

and to the boundary-parameter coefficients of (14) by

$$\mathcal{A}_{n,k} = Z_n(1, 1)_k, \quad \mathcal{A}_{n,k_1,k_2}^{\text{adj}} = Z_n^{\text{adj}}(1, 1)_{k_1,k_2}. \tag{31}$$

The ASM numbers of (29) satisfy various elementary identities related to their definitions, symmetry properties of ASMs or properties of ASMs with a 1 in a corner, and which correspond to identities, such as (11)–(13) and (15), satisfied by the generating functions. Some examples of these identities are

$$\begin{aligned}
\sum_{k=0}^{n-1} \mathcal{A}_{n,k} &= \mathcal{A}_n, \quad \mathcal{A}_{n,k} = \mathcal{A}_{n,n-1-k}, \quad \mathcal{A}_{n,0} = \mathcal{A}_{n-1}, \\
\sum_{k_2=0}^{n-1} \mathcal{A}_{n,k_1,k_2}^{\text{opp}} &= \sum_{k_2=0}^{n-1} \mathcal{A}_{n,k_1,k_2}^{\text{adj}} = \mathcal{A}_{n,k_1}, \\
\mathcal{A}_{n,k_1,k_2}^{\text{opp}} &= \mathcal{A}_{n,k_2,k_1}^{\text{opp}} = \mathcal{A}_{n,n-1-k_1,n-1-k_2}^{\text{opp}}, \quad \mathcal{A}_{n,k_1,k_2}^{\text{adj}} = \mathcal{A}_{n,k_2,k_1}^{\text{adj}}, \\
\mathcal{A}_{n,k,0}^{\text{adj}} &= \mathcal{A}_{n-1} \delta_{k,0}, \quad \mathcal{A}_{n,k,0}^{\text{opp}} = \mathcal{A}_{n,k+1,n-1}^{\text{adj}} = \mathcal{A}_{n-1,k}.
\end{aligned} \tag{32}$$

Furthermore, some identities related to properties of ASMs in which a 1 on a boundary is separated from a corner by a single zero, and which can be obtained from (16) and the first two identities of (32), are

$$\mathcal{A}_{n,k,1}^{\text{adj}} = \sum_{i=k-1}^{n-2} \mathcal{A}_{n-1,i} - \mathcal{A}_{n-1} \delta_{k,0}, \quad \mathcal{A}_{n,1} = \mathcal{A}_{n-1} + \sum_{k=1}^{n-2} k \mathcal{A}_{n-1,k} = \frac{n}{2} \mathcal{A}_{n-1}. \tag{33}$$

Proceeding to more general identities, for which derivations using combinatorial arguments are not currently known, formulae which give the ASM numbers of (29) explicitly are

$$\mathcal{A}_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}, \quad (34)$$

$$\mathcal{A}_{n,k} = \begin{cases} \frac{(n+k-1)!(2n-k-2)!}{k!(n-k-1)!(2n-2)!} \prod_{i=0}^{n-2} \frac{(3i+1)!}{(n+i-1)!}, & 0 \leq k \leq n-1, \\ 0, & \text{otherwise,} \end{cases} \quad (35)$$

$$\mathcal{A}_{n,k_1,k_2}^{\text{opp}} = \frac{1}{\mathcal{A}_{n-1}} \sum_{i=0}^{\min(k_1, n-k_2-1)} (\mathcal{A}_{n,k_1-i} \mathcal{A}_{n-1,k_2+i} + \mathcal{A}_{n-1,k_1-i-1} \mathcal{A}_{n,k_2+i} - \mathcal{A}_{n,k_1-i-1} \mathcal{A}_{n-1,k_2+i} - \mathcal{A}_{n-1,k_1-i-1} \mathcal{A}_{n,k_2+i+1}), \quad (36)$$

$$\mathcal{A}_{n,k_1,k_2}^{\text{adj}} = \begin{cases} \mathcal{A}_{n-1}, & k_1 = k_2 = 0, \\ \binom{k_1+k_2-2}{k_1-1} \mathcal{A}_{n-1} - \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \binom{k_1+k_2-i-j}{k_1-i} \mathcal{A}_{n,i-1,n-j}^{\text{opp}}, & 1 \leq k_1, k_2 \leq n-1, \\ 0, & \text{otherwise,} \end{cases} \quad (37)$$

It follows from (34)–(35) that the unrefined and singly-refined ASM numbers satisfy simple recursion relations, such as

$$\begin{aligned} \mathcal{A}_n &= \frac{(n-1)!(3n-2)!}{(2n-2)!(2n-1)!} \mathcal{A}_{n-1}, \\ \mathcal{A}_n \mathcal{A}_{n-2} &= \frac{3(3n-2)(3n-4)}{4(2n-1)(2n-3)} \mathcal{A}_{n-1}^2, \\ k(2n-k-1) \mathcal{A}_{n,k} &= (n-k)(n+k-1) \mathcal{A}_{n,k-1}. \end{aligned} \quad (38)$$

It also follows that the singly-refined ASM generating function at $x = y = 1$ can be expressed in terms of the Gaussian hypergeometric function as

$$Z_n(1, 1; z) = \mathcal{A}_{n-1} {}_2F_1 \left[\begin{matrix} 1-n, n \\ 2-2n \end{matrix}; z \right], \quad (39)$$

and satisfies the hypergeometric differential equation

$$z(1-z) \frac{d^2}{dz^2} Z_n(1, 1; z) + 2(1-n-z) \frac{d}{dz} Z_n(1, 1; z) + n(n-1) Z_n(1, 1; z) = 0. \quad (40)$$

Furthermore, the singly-refined ASM numbers satisfy, for $0 \leq k \leq n-1$,

$$\mathcal{A}_{n,k} = \sum_{i=0}^k (-1)^i \binom{n+k-1}{k-i} \mathcal{A}_{n,i}. \quad (41)$$

Recursion relations which involve the opposite-boundary and adjacent-boundary doubly-refined ASM numbers are

$$(\mathcal{A}_{n,k_1-1,k_2}^{\text{opp}} - \mathcal{A}_{n,k_1,k_2-1}^{\text{opp}}) \mathcal{A}_{n-1} = \mathcal{A}_{n,k_1-1} \mathcal{A}_{n-1,k_2-1} - \mathcal{A}_{n,k_1} \mathcal{A}_{n-1,k_2-1} - \mathcal{A}_{n-1,k_1-1} \mathcal{A}_{n,k_2-1} + \mathcal{A}_{n-1,k_1-1} \mathcal{A}_{n,k_2}, \quad (42)$$

$$\mathcal{A}_{n,k_1-1,k_2}^{\text{adj}} + \mathcal{A}_{n,k_1,k_2-1}^{\text{adj}} - \mathcal{A}_{n,k_1,k_2}^{\text{adj}} = \mathcal{A}_{n,k_1-1,n-k_2}^{\text{opp}} - (\delta_{k_1,1} - \delta_{k_1,0})(\delta_{k_2,1} - \delta_{k_2,0}) \mathcal{A}_{n-1}. \quad (43)$$

It follows that, in terms of generating functions, (42) can be written as

$$(z_1 - z_2) Z_n^{\text{opp}}(1, 1; z_1, z_2) \mathcal{A}_{n-1} = (z_1 - 1) z_2 Z_n(1, 1; z_1) Z_{n-1}(1, 1; z_2) - z_1 (z_2 - 1) Z_{n-1}(1, 1; z_1) Z_n(1, 1; z_2), \quad (44)$$

and (43) can be written as

$$(z_1 + z_2 - 1) Z_n^{\text{adj}}(1, 1; z_1, z_2) = z_1 z_2^n Z_n^{\text{opp}}(1, 1; z_1, \frac{1}{z_2}) - (z_1 - 1)(z_2 - 1) \mathcal{A}_{n-1}. \quad (45)$$

The relation (45) will be used in Section 4.3 for the derivation of results for the quadruply-refined ASM generating function at $x = y = 1$.

The opposite-boundary doubly-refined ASM generating function at $x = y = 1$ can also be expressed as

$$Z_n^{\text{opp}}(1, 1; z_1, z_2) = 3^{-n(n-1)/2} (q^2(z_1 + q)(z_2 + q))^{n-1} \times s_{(n-1, n-1, \dots, 2, 2, 1, 1)} \left(\frac{qz_1+1}{z_1+q}, \frac{qz_2+1}{z_2+q}, \underbrace{1, \dots, 1}_{2n-2} \right) \Big|_{q=e^{\pm 2\pi i/3}}, \quad (46)$$

where $s_{(n-1, n-1, \dots, 2, 2, 1, 1)} \left(\frac{qz_1+1}{z_1+q}, \frac{qz_2+1}{z_2+q}, 1, \dots, 1 \right)$ is the Schur function indexed by the double-staircase partition $(n-1, n-1, \dots, 2, 2, 1, 1)$, evaluated at the $2n$ parameters $\frac{qz_1+1}{z_1+q}, \frac{qz_2+1}{z_2+q}, 1, \dots, 1$. Setting $z_1 = 1$ or $z_2 = 1$ in (46) gives an expression for the singly-refined ASM generating function at $x = y = 1$, while setting $z_1 = z_2 = 1$ in (46), and using (30) and (32), it follows that

$$\mathcal{A}_n = 3^{-n(n-1)/2} \times (\text{number of semistandard Young tableaux of shape } (n-1, n-1, \dots, 2, 2, 1, 1) \text{ with entries from } \{1, \dots, 2n\}). \quad (47)$$

The product formula (34) for \mathcal{A}_n can be obtained from (47) using the hook-content formula for semistandard Young tableaux.

A further identity satisfied by the opposite-boundary doubly-refined and unrefined ASM numbers is

$$\det_{0 \leq k_1, k_2 \leq n-1} (\mathcal{A}_{n, k_1, k_2}^{\text{opp}}) = (-1)^{n(n+1)/2+1} (\mathcal{A}_{n-1})^{n-3}. \quad (48)$$

Additional expressions for some of the previous ASM numbers or associated ASM generating functions will be discussed in Section 3.13.

Proceeding now to the enumeration of ASMs with prescribed configurations on three or four boundaries, the alternative quadruply-refined ASM generating function at $x = y = 1$ is given by

$$\begin{aligned} & (z_4 z_1 - z_4 + 1)(z_1 z_2 - z_1 + 1)(z_2 z_3 - z_2 + 1)(z_3 z_4 - z_3 + 1) \tilde{Z}_n^{\text{quad}}(1, 1; z_1, z_2, z_3, z_4) = \\ & \frac{z_1 z_2 z_3 z_4 \det_{1 \leq i, j \leq 4} (z_i^{j-1} (z_i - 1)^{4-j} Z_{n-j+1}(1, 1; z_i))}{\mathcal{A}_{n-1} \mathcal{A}_{n-2} \mathcal{A}_{n-3} \prod_{1 \leq i < j \leq 4} (z_i - z_j)} + \\ & (z_2 - 1)(z_3 - 1)(z_4 z_1 - z_4 + 1)(z_1 z_2 - z_1 + 1)(z_3 z_4 - z_3 + 1)(z_2 z_4)^{n-1} Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_4}, z_1) + \\ & (z_3 - 1)(z_4 - 1)(z_1 z_2 - z_1 + 1)(z_2 z_3 - z_2 + 1)(z_4 z_1 - z_4 + 1)(z_1 z_3)^{n-1} Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_1}, z_2) + \\ & (z_4 - 1)(z_1 - 1)(z_2 z_3 - z_2 + 1)(z_3 z_4 - z_3 + 1)(z_1 z_2 - z_1 + 1)(z_2 z_4)^{n-1} Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_2}, z_3) + \\ & (z_1 - 1)(z_2 - 1)(z_3 z_4 - z_3 + 1)(z_4 z_1 - z_4 + 1)(z_2 z_3 - z_2 + 1)(z_1 z_3)^{n-1} Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_3}, z_4) - \end{aligned}$$

$$(z_1 - 1)(z_2 - 1)(z_3 - 1)(z_4 - 1) \left((z_1 z_2 - z_1 + 1)(z_3 z_4 - z_3 + 1)(z_2 z_4)^{n-1} + (z_2 z_3 - z_2 + 1)(z_4 z_1 - z_4 + 1)(z_1 z_3)^{n-1} \right) \mathcal{A}_{n-2}. \quad (49)$$

Setting $z_2 = 1$ in (49) (and then relabelling z_3 as z_2 and z_4 as z_3), it follows that the triply-refined ASM generating function at $x = y = 1$ is given by

$$\begin{aligned} (z_1 z_3 - z_3 + 1)(z_2 z_3 - z_2 + 1) z_3^{n-1} Z_n^{\text{tri}}(1, 1; z_1, z_2, \frac{1}{z_3}) = \\ \frac{z_1 z_3 \det_{1 \leq i, j \leq 3} (z_i^{j-1} (z_i - 1)^{3-j} Z_{n-j+1}(1, 1; z_i))}{\mathcal{A}_{n-1} \mathcal{A}_{n-2} \prod_{1 \leq i < j \leq 3} (z_i - z_j)} + \\ (z_2 - 1)(z_3 - 1)(z_1 z_3 - z_3 + 1) z_1 z_2^{n-1} Z_{n-1}(1, 1; z_1) + \\ (z_1 - 1)(z_3 - 1)(z_2 z_3 - z_2 + 1) z_3^{n-1} Z_{n-1}(1, 1; z_2). \quad (50) \end{aligned}$$

Note that the identities (44) and (45), satisfied by the doubly-refined ASM generating functions at $x = y = 1$, can be regarded as special cases of the identity (50) satisfied by the triply-refined ASM generating function. More specifically, setting $z_3 = 1$ in (50) gives (44), while setting $z_2 = 1$ in (50), and using (44), gives (45).

Note also that, by using (45) to replace each case of an adjacent-boundary doubly-refined ASM generating function in (49) by an opposite-boundary doubly-refined ASM generating function, (49) can be restated as

$$\begin{aligned} (z_4 z_1 - z_4 + 1)(z_1 z_2 - z_1 + 1)(z_2 z_3 - z_2 + 1)(z_3 z_4 - z_3 + 1) \tilde{Z}_n^{\text{quad}}(1, 1; z_1, z_2, z_3, z_4) = \\ \frac{z_1 z_2 z_3 z_4 \det_{1 \leq i, j \leq 4} (z_i^{j-1} (z_i - 1)^{4-j} Z_{n-j+1}(1, 1; z_i))}{\mathcal{A}_{n-1} \mathcal{A}_{n-2} \mathcal{A}_{n-3} \prod_{1 \leq i < j \leq 4} (z_i - z_j)} + \\ (z_2 - 1)(z_3 - 1)(z_1 z_2 - z_1 + 1)(z_3 z_4 - z_3 + 1) z_4 z_1 z_2^{n-1} Z_{n-1}^{\text{opp}}(1, 1; z_4, z_1) + \\ (z_3 - 1)(z_4 - 1)(z_2 z_3 - z_2 + 1)(z_4 z_1 - z_4 + 1) z_1 z_2 z_3^{n-1} Z_{n-1}^{\text{opp}}(1, 1; z_1, z_2) + \\ (z_4 - 1)(z_1 - 1)(z_3 z_4 - z_3 + 1)(z_1 z_2 - z_1 + 1) z_2 z_3 z_4^{n-1} Z_{n-1}^{\text{opp}}(1, 1; z_2, z_3) + \\ (z_1 - 1)(z_2 - 1)(z_4 z_1 - z_4 + 1)(z_2 z_3 - z_2 + 1) z_3 z_4 z_1^{n-1} Z_{n-1}^{\text{opp}}(1, 1; z_3, z_4) + \\ (z_1 - 1)(z_2 - 1)(z_3 - 1)(z_4 - 1) \left((z_1 z_2 - z_1 + 1)(z_3 z_4 - z_3 + 1)(z_2 z_4)^{n-1} + \right. \\ \left. (z_2 z_3 - z_2 + 1)(z_4 z_1 - z_4 + 1)(z_1 z_3)^{n-1} \right) \mathcal{A}_{n-2}. \quad (51) \end{aligned}$$

The first terms (i.e., the determinant terms) on the RHS of (49)–(51) are proportional to certain cases of a function which will be defined in (70). Specifically, using (70), the first terms on the RHS of (49) (or (51)) and (50) are $z_1 z_2 z_3 z_4 X_n(1, 1; z_1, z_2, z_3, z_4)$ and $z_1 z_3 X_n(1, 1; z_1, z_2, z_3)$, respectively. It will also be seen, in (75), that the function (70) at $x = y = 1$, and hence the first terms on the RHS of (49)–(51), are related to a certain Schur function.

Further expressions for $\tilde{Z}_n^{\text{quad}}(1, 1; z_1, z_2, z_3, z_4)$ and $Z_n^{\text{tri}}(1, 1; z_1, z_2, \frac{1}{z_3})$, which differ from (50)–(51) in the first terms on each RHS, will be obtained in Corollaries 10 and 11.

The origins of the results given in this section will now be outlined.

The product formula (34) was conjectured by Mills, Robbins and Rumsey [112, 113, Conj. 1], and first proved by Zeilberger [157] and, shortly thereafter, but using a different method, by Kuperberg [100]. The product formula (35) was first proved by Zeilberger [158], and confirms the validity of further conjectures of Mills, Robbins and Rumsey [112, 113, Conj. 2].

Alternative proofs of (34)–(35) have been given by Colomo and Pronko [48, Sec. 5.3], [49, Sec. 4.2], Fischer [74], and Stroganov [150, Sec. 4]. (See also Razumov and Stroganov [127, Sec. 2], [131, Sec. 2] for additional details related to the third of these proofs.)

The formulae (34)–(35) also follow from certain other known results. For example, as already indicated, (34) can be obtained from the result (47) and the hook-content formula. Alternatively, (34) can be obtained as a special case of a result of Rosengren [136, Cor. 8.4]. Also, (35) can be obtained by combining a result of Behrend, Di Francesco and Zinn-Justin [16, Thm. 1] that $\mathcal{A}_{n,k}$ is the number of descending plane partitions with each part at most n and exactly k parts equal to n , with a result of Mills, Robbins and Rumsey [112, Sec. 5] which gives a product formula for the number of such objects.

Among these various derivations of (34)–(35), all make essential use of the Izergin–Korepin formula [89, Eq. (5)] for the partition function of the six-vertex model with DWBC, or related properties of integrability, except for Zeilberger’s proof [157] of (34), which uses a result of Andrews [4] for the number of totally symmetric self-complementary plane partitions in a $2n \times 2n \times 2n$ box, and Fischer’s proof [74], which uses an operator formula obtained by Fischer [73, 75] (see also Colomo and Pronko [52, Eq. (3.3)]).

The hypergeometric function expression (39) is given by Colomo and Pronko [46, Eq. (2.16)], [48, Eq. (5.43)], [49, Eq. (4.19)], and the differential equation (40) is given by Stroganov [150, Eq. (26)].

The relation (41) was obtained by Fischer [74, Sec. 3], with a corresponding relation for descending plane partitions, containing a further parameter associated with the sum of the parts of a descending plane partition, having been obtained previously by Mills, Robbins and Rumsey [112, Sec. 5]. It was also shown by Fischer [74, Sec. 4] that the formula (35) follows from (41), the first three equations of (32), and $\mathcal{A}_{1,0} = 1$.

The relations (42)–(45) for the opposite-boundary and adjacent-boundary doubly-refined ASM numbers were first obtained by Stroganov [150, Sec. 5]. Furthermore, (43) is a special case of a formula obtained by Fischer [77, Thm. 1], and (44) is a special case of a relation, (62), which will be discussed in Section 3.4. As indicated elsewhere in this section, (44) and (45) are also special cases of (49).

The formula (36) follows easily from (42). The formula (37) was first obtained by Fischer [77, p. 570], and can be derived by dividing both sides of (45) by $z_1 + z_2 - 1$ and equating coefficients of $z_1^{k_1} z_2^{k_2}$ on each side.

The Schur function expression (46) was obtained by Di Francesco and Zinn-Justin [65, Eqs. (2.2) & (2.4)], using a result of Okada [121, Thm. 2.4(1), second equation]. A generalization of (46) will be given in (75), and a derivation of (75) will be given in Section 5.10.

The relation (47) was first obtained by Okada [121, Thm. 1.2 (A1)].

The identity (48) was obtained by Biane, Cantini and Sportiello [18, Thm. 1], by combining (46) with a certain Schur function identity, also obtained by Biane, Cantini and Sportiello [18, Thm. 2].

Some further aspects of the opposite-boundary doubly-refined ASM numbers are discussed by Fischer [75, Sec. 6].

Identities for the quadruply- and triply-refined ASM generating functions at $x = y = 1$, which are essentially equivalent to (49)–(50), were obtained recently by Ayyer and Romik [12, Thms. 1–3]. The forms of (49)–(50) given here were observed by Colomo [43], and can be derived by combining a result which will be given in (73), with a result that the partition function of the six-vertex model with DWBC is symmetric in all of its spectral parameters at its so-called combinatorial point. A derivation of (49) using this approach will be given in Section 5.10. Roughly speaking, this derivation involves associating spectral parameters t_1, t_2, t_3 and t_4 with the first row, last column, last row and first column, respectively, of an ASM. The symmetry of the partition function in all of its spectral parameters then enables t_2 and t_4 to be associated instead with rows, so that the result (73) (in the form (156)), which involves parameters associated with several rows, can be applied.

Note that, since setting certain boundary parameters in (49) to 1 gives (44), (45) and (50), Section 5.10 also provides derivations of these other identities. (For the identity (45), such a derivation essentially depends only the symmetry of the partition function in all of its spectral parameters, and not on (73).)

In the versions of (49)–(50) obtained by Ayyer and Romik [12, Thms. 1–3], slightly different quadruply-refined and triply-refined ASM generating functions are used, and matrices appear which are related by transposition and column operations to those in the first terms on the RHS of (49)–(50). In particular, the quadruply-refined generating function used there is $Z'_n(x, y; z_1, z_2, z_3, z_4) = \sum_{\substack{A \in \text{ASM}(n) \\ A_{11}=A_{1n}=A_{nn}=A_{n1}=0}} x^{\nu(A)} y^{\mu(A)} z_1^{\rho_{\text{T}}(A)-1} z_2^{n-2-\rho_{\text{R}}(A)} z_3^{\rho_{\text{B}}(A)-1} z_4^{n-2-\rho_{\text{L}}(A)}$ (at $x = y = 1$), which can be shown straightforwardly to be related to generating functions used here by

$$\begin{aligned} z_1 z_2 z_3 z_4 Z'_n(x, y; z_1, z_2, z_3, z_4) = & \\ & \tilde{Z}_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) - (z_2 z_4)^{n-1} Z_{n-1}^{\text{adj}}(x, y; \frac{1}{z_4}, z_1) - (z_2 z_4)^{n-1} Z_{n-1}^{\text{adj}}(x, y; \frac{1}{z_2}, z_3) - \\ & z_1 z_2^{n-2} (x z_3)^{n-1} \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_1, \frac{1}{z_2}) - z_3 z_4^{n-2} (x z_1)^{n-1} \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_3, \frac{1}{z_4}) + \\ & ((z_2 z_4)^{n-1} + x^{2n-3} (z_1 z_3)^{n-1}) Z_{n-2}(x, y). \end{aligned} \quad (52)$$

The matrices used in some of the formulae of Ayyer and Romik [12, Eqs. (1.8) & (1.11)] involve functions $\gamma_n(z)$ and $\delta_n(z)$ (see Ayyer and Romik [12, Eq. (1.7)] for definitions) which are related to the functions used in the matrices in the first terms on the RHS of (49)–(50) by

$$\begin{aligned} \gamma_n(z) &= \frac{4(n-1)!(3n-5)!}{(2n-3)!(2n-4)!} z^2 Z_{n-2}(1, 1; z) - \frac{4(2n-3)!(2n-5)!}{(n-3)!(3n-5)!} (z-1)^2 Z_n(1, 1; z), \\ \delta_n(z) &= \frac{4(n-1)!(n-4)!(3n-5)!(3n-8)!}{(2n-4)!^2 (2n-5)!(2n-7)!} z^3 Z_{n-3}(1, 1; z) - \frac{9(9n^2-30n+20)}{2n-5} z(z-1)^2 Z_{n-1}(1, 1; z). \end{aligned} \quad (53)$$

These relations can be confirmed (perhaps with the assistance of a computer to verify certain underlying identities) by applying the definitions of $\gamma_n(z)$, $\delta_n(z)$ and $Z_n(1, 1; z)$, and using the product formula (35) for the singly-refined ASM numbers which appear in these functions. The fact that the matrices used by Ayyer and Romik are related to those used here by (transposition and) column operations follows immediately from (53). See also Ayyer and Romik [12, Thm. 3 & App. A] for a discussion of this matter.

3.4. Arbitrary bulk parameters x and y . The case in which the bulk parameters x and y are arbitrary is of primary interest in this paper, and several new results will be presented in Section 4. In this section, the previously-known results for this case, which involve the unrefined, singly-refined and opposite-boundary doubly-refined ASM generating functions, are reviewed.

A determinant formula for the opposite-boundary doubly-refined ASM generating function is

$$Z_n^{\text{opp}}(x, y; z_1, z_2) = \det_{0 \leq i, j \leq n-1} (K_n(x, y; z_1, z_2)_{ij}), \quad (54)$$

where

$$K_n(x, y; z_1, z_2)_{ij} = -\delta_{i, j+1} + \begin{cases} \sum_{k=0}^{\min(i, j+1)} \binom{i-1}{i-k} \binom{j+1}{k} x^k y^{i-k}, & j \leq n-3, \\ \sum_{k=0}^i \sum_{l=0}^k \binom{i-1}{i-k} \binom{n-l-2}{k-l} x^k y^{i-k} z_2^{l+1}, & j = n-2, \\ \sum_{k=0}^i \sum_{l=0}^k \sum_{m=0}^l \binom{i-1}{i-k} \binom{n-l-2}{k-l} x^k y^{i-k} z_1^m z_2^{l-m}, & j = n-1. \end{cases} \quad (55)$$

Setting $z_2 = 1$ or $z_1 = z_2 = 1$ in (54)–(55), and using standard binomial coefficient identities, it follows that determinant formulae for the singly-refined and unrefined ASM generating functions are

$$Z_n(x, y; z) = \det_{0 \leq i, j \leq n-1} (K_n(x, y; z)_{ij}), \quad (56)$$

$$Z_n(x, y) = \det_{0 \leq i, j \leq n-1} (K_n(x, y)_{ij}), \quad (57)$$

where

$$\begin{aligned} K_n(x, y; z)_{ij} &= K_n(x, y; z, 1)_{ij} \\ &= -\delta_{i, j+1} + \begin{cases} \sum_{k=0}^{\min(i, j+1)} \binom{i-1}{i-k} \binom{j+1}{k} x^k y^{i-k}, & j \leq n-2, \\ \sum_{k=0}^i \sum_{l=0}^k \binom{i-1}{i-k} \binom{n-l-1}{k-l} x^k y^{i-k} z^l, & j = n-1, \end{cases} \end{aligned} \quad (58)$$

$$K_n(x, y)_{ij} = K_n(x, y; 1, 1)_{ij} = -\delta_{i, j+1} + \sum_{k=0}^{\min(i, j+1)} \binom{i-1}{i-k} \binom{j+1}{k} x^k y^{i-k}. \quad (59)$$

There are further simplifications if $x = y = 1$. For example,

$$K_n(1, 1)_{ij} = -\delta_{i, j+1} + \binom{i+j}{i}. \quad (60)$$

The opposite-boundary doubly-refined ASM generating function satisfies

$$\begin{aligned} (z_1 - z_2) (z_3 - z_4) Z_n^{\text{opp}}(x, y; z_1, z_2) Z_n^{\text{opp}}(x, y; z_3, z_4) - \\ (z_1 - z_3) (z_2 - z_4) Z_n^{\text{opp}}(x, y; z_1, z_3) Z_n^{\text{opp}}(x, y; z_2, z_4) + \\ (z_1 - z_4) (z_2 - z_3) Z_n^{\text{opp}}(x, y; z_1, z_4) Z_n^{\text{opp}}(x, y; z_2, z_3) = 0, \end{aligned} \quad (61)$$

and it can be expressed in terms of singly-refined and unrefined ASM generating functions as

$$\begin{aligned} (z_1 - z_2) Z_n^{\text{opp}}(x, y; z_1, z_2) Z_{n-1}(x, y) = (z_1 - 1) z_2 Z_n(x, y; z_1) Z_{n-1}(x, y; z_2) - \\ z_1 (z_2 - 1) Z_{n-1}(x, y; z_1) Z_n(x, y; z_2). \end{aligned} \quad (62)$$

The two previous identities are essentially equivalent, since (62) can be obtained from (61) by setting $z_3 = 1$ and $z_4 = 0$, and then applying relations from (11) and (13), while (61) can be obtained from (62) by expressing each of the six cases of $(z_i - z_j) Z_n^{\text{opp}}(x, y; z_i, z_j)$ in terms of singly-refined and unrefined ASM generating functions, and then checking that the resulting expression for the LHS of (61) vanishes.

The formula (54) was derived by Behrend, Di Francesco and Zinn-Justin [17, Eqs. (21)–(22)], as part of the proof of an equality [17, Thm. 1] between the opposite-boundary doubly-refined ASM generating function and a four-parameter generating function for descending plane partitions. (See also Section 3.12.) The formula (56) was derived by Behrend, Di Francesco and Zinn-Justin [16, Eqs. (97)–(98)], prior to the derivation of (54), as part of the proof of an equality [16, Thm. 1] (conjectured previously by Mills, Robbins and Rumsey [113, Conj. 3]) between the singly-refined ASM generating function and a three-parameter generating function for descending plane partitions.

For an alternative version of (54), involving a transformation of the matrix $K_n(x, y; z_1, z_2)$, see Behrend, Di Francesco and Zinn-Justin [17, Eqs. (65)–(66)], and for alternative versions of (56) or (57), involving transformations of the matrices $K_n(x, y; z)$ or $K_n(x, y)$, see Behrend, Di Francesco and Zinn-Justin [16, Props. 1 & 4, Eqs. (65), (66), (87) & (88), & Sec. 4.1].

For additional information on determinants closely related to those of (56)–(57), including formulae for their evaluation or factorization in certain special cases, see, for example, Andrews [2], Andrews and Stanton [5], Ciucu, Eisenkölbl, Krattenthaler and Zare [41, Thms. 10–13], Ciucu and Krattenthaler [42], Colomo and Pronko [44, Eqs. (23)–(24)], [45, Eqs. (4.3)–(4.7)], de Gier [56, Sec. 2.1], Gessel and Xin [84, Sec. 5], Krattenthaler [96, e.g., Thms. 25–37], [97, Sec. 5.5], Lalonde [102, Thm. 3.1], Mills, Robbins and Rumsey [112], [113, p. 346], [115, Secs. 3–4], Robbins [134, Sec. 2], and Rosengren [138].

The case of (62) with $x = y = 1$ (as given in (44)) was first obtained by Stroganov [150, Sec. 5], and this relation for arbitrary x and y was first obtained, as an identity involving one- and two-point boundary correlation functions for the six-vertex model with DWBC, by Colomo and Pronko [47, Eq. (5.32)], [49, Eq. (3.32)]. An alternative proof of (62) was given by Behrend, Di Francesco and Zinn-Justin [17, Sec. 5]. This identity will also be obtained

from Theorem 1 of this paper, as Corollary 5, and it will be found to be a special case of the identity (73) given in Section 3.5.

3.5. ASM enumeration involving statistics associated with several rows or several columns. In this section, a generating function involving statistics associated with several rows (or several columns) of ASMs is defined, and (in (73)) a result is stated which, for the case of particular assignments of certain parameters, provides a determinantal expression for this generating function in terms of singly-refined and unrefined ASM generating functions. A derivation of this result will be given in Section 5.9.

For $A \in \text{ASM}(n)$, define ASM statistics, associated with row i or column j of A , as

$$\begin{aligned}\nu^{\text{row } i}(A) &= |\{j \in \{1, \dots, n\} \mid \sum_{i'=1}^{i-1} A_{i'j} = \sum_{j'=1}^j A_{ij'}\}|, \\ \nu^{\text{col } j}(A) &= |\{i \in \{1, \dots, n\} \mid \sum_{i'=1}^{i-1} A_{i'j} = \sum_{j'=1}^j A_{ij'}\}|, \\ \mu^{\text{row } i}(A) &= \text{number of } -1\text{'s in row } i \text{ of } A, \\ \mu^{\text{col } j}(A) &= \text{number of } -1\text{'s in column } j \text{ of } A.\end{aligned}\tag{63}$$

It follows from the defining properties of ASMs that if, for an ASM A , $\sum_{i'=1}^{i-1} A_{i'j} = \sum_{j'=1}^j A_{ij'}$ (or equivalently $\sum_{i'=1}^i A_{i'j} = \sum_{j'=1}^{j-1} A_{ij'}$), then $A_{ij} = 0$. Hence, $\nu^{\text{row } i}(A)$ and $\nu^{\text{col } j}(A)$ can be regarded as the numbers of certain 0's in row i and column j , respectively, of A . It will be seen in Section 5.2 that, in terms of the configuration of the six-vertex model with DWBC which corresponds to A , $\nu^{\text{row } i}(A)$ and $\nu^{\text{col } j}(A)$ are simply the numbers of vertex configurations of types (1) and (2) in row i and column j , respectively, of the grid.

It can also be seen that the statistics $\nu^{\text{row } i}(A)$ and $\nu^{\text{col } j}(A)$ can be written in various other ways, for example as

$$\begin{aligned}\nu^{\text{row } i}(A) &= \sum_{j=1}^n \left(\sum_{i'=1}^{i-1} \sum_{j'=1}^j A_{i'j} A_{ij'} + \sum_{i'=i}^n \sum_{j'=j+1}^n A_{i'j} A_{ij'} \right), \\ \nu^{\text{col } j}(A) &= \sum_{i=1}^n \left(\sum_{i'=1}^{i-1} \sum_{j'=1}^j A_{i'j} A_{ij'} + \sum_{i'=i}^n \sum_{j'=j+1}^n A_{i'j} A_{ij'} \right).\end{aligned}\tag{64}$$

It can be checked easily that, for any $A \in \text{ASM}(n)$, the statistics of (2)–(3) and those of (63) are related by

$$\begin{aligned}\sum_{i=1}^n \nu^{\text{row } i}(A) &= \sum_{j=1}^n \nu^{\text{col } j}(A) = 2\nu(A), \\ \nu^{\text{row } 1}(A) &= \rho_{\text{T}}(A), \quad \nu^{\text{col } n}(A) = \rho_{\text{R}}(A), \quad \nu^{\text{row } n}(A) = \rho_{\text{B}}(A), \quad \nu^{\text{col } 1}(A) = \rho_{\text{L}}(A), \\ \sum_{i=2}^{n-1} \mu^{\text{row } i}(A) &= \sum_{j=2}^{n-1} \mu^{\text{col } j}(A) = \mu(A), \\ \mu^{\text{row } 1}(A) &= \mu^{\text{col } n}(A) = \mu^{\text{row } n}(A) = \mu^{\text{col } 1}(A) = 0,\end{aligned}\tag{65}$$

and that the statistics of (63) behave under the operations of transposition or anticlockwise quarter-turn rotation of ASMs according to

$$\begin{aligned}\nu^{\text{row } i}(A) &= \nu^{\text{col } i}(A^T) = n - 1 - \nu^{\text{col } i}(A^Q) - 2\mu^{\text{row } i}(A), \\ \mu^{\text{row } i}(A) &= \mu^{\text{col } i}(A^T) = \mu^{\text{col } i}(A^Q),\end{aligned}\tag{66}$$

using the same notation as in (6).

For any $0 \leq m \leq n$ and $1 \leq k_1 < \dots < k_m \leq n$, define an ASM generating function associated with rows (or, as will be seen in (69), columns) k_1, \dots, k_m as

$$Z_n^{k_1, \dots, k_m}(x, y; z_1, \dots, z_m; w_1, \dots, w_m) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A) - \sum_{i=1}^m \mu^{\text{row } k_i(A)}} \prod_{i=1}^m z_i^{\nu^{\text{row } k_i(A)}} w_i^{\mu^{\text{row } k_i(A)}}, \quad (67)$$

for indeterminates x, y, z_1, \dots, z_m and w_1, \dots, w_m .

It can be seen that

$$Z_n^{k_1, \dots, k_m}(x, y; z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_m; w_1, \dots, w_{i-1}, y, w_{i+1}, \dots, w_m) = Z_n^{k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_m}(x, y; z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m; w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_m), \quad (68)$$

and that $Z_n^{k_1, \dots, k_m}(x, y; z_1, \dots, z_m; w_1, \dots, w_m)$ is the unrefined ASM generating function $Z_n(x, y)$ for $m = 0$, the singly-refined ASM generating function $Z_n(x, y; z_1)$ for $m = 1$ and $k_1 = 1$ or $k_1 = n$, and the opposite-boundary doubly-refined ASM generating function $Z_n^{\text{opp}}(x, y; z_1, z_2)$ for $m = 2$, $k_1 = 1$ and $k_2 = n$.

Using (66), it also follows that

$$Z_n^{k_1, \dots, k_m}(x, y; z_1, \dots, z_m; w_1, \dots, w_m) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A) - \sum_{j=1}^m \mu^{\text{col } k_j(A)}} \prod_{j=1}^m z_j^{\nu^{\text{col } k_j(A)}} w_j^{\mu^{\text{col } k_j(A)}} = x^{n(n-1)/2} (z_1 \dots z_m)^{n-1} Z_n^{k_1, \dots, k_m}\left(\frac{1}{x}, \frac{y}{x}, \frac{1}{z_1}, \dots, \frac{1}{z_m}; \frac{w_1}{xz_1^2}, \dots, \frac{w_m}{xz_m^2}\right). \quad (69)$$

Now define, for $0 \leq m \leq n$, a further function

$$X_n(x, y; z_1, \dots, z_m) = \begin{cases} Z_n(x, y), & m = 0, \\ \frac{\det_{1 \leq i, j \leq m} (z_i^{j-1} (z_i - 1)^{m-j} Z_{n-j+1}(x, y; z_i))}{\prod_{1 \leq i < j \leq m} (z_i - z_j) \prod_{i=1}^{m-1} Z_{n-i}(x, y)}, & 1 \leq m \leq n. \end{cases} \quad (70)$$

This definition is based on a definition of Colomo and Pronko [50, Eq. (6.6)], [52, Eq. (4.18)] of a closely-related function for the six-vertex model with DWBC.

Some properties of $X_n(x, y; z_1, \dots, z_m)$, which follow straightforwardly from (70) and the latter equations of (11), (12) and (13), are

$$\begin{aligned} X_n(x, y; z_1, \dots, z_m) &\text{ is symmetric in } z_1, \dots, z_m, \\ X_n(x, y; z_1, \dots, z_m) &= x^{n(n-1)/2} (z_1 \dots z_m)^{n-1} X_n\left(\frac{1}{x}, \frac{y}{x}, \frac{1}{z_1}, \dots, \frac{1}{z_m}\right), \\ X_n(x, y; z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_m) &= X_n(x, y; z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m), \\ X_n(x, y; z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_m) &= X_{n-1}(x, y; z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m), \\ X_n(x, y; z) &= Z_n(x, y; z). \end{aligned} \quad (71)$$

Certain further properties can be found by combining (70) with general identities for minors of a matrix. For example, for $1 \leq m \leq n$, and indeterminates $z_1, \dots, z_m, u_1, \dots, u_{m-1}$ and

v_1, \dots, v_{m-1} , the identity

$$\begin{aligned} & \prod_{1 \leq i < j \leq m} (z_i - z_j) \prod_{1 \leq i \leq j \leq m-1} (u_i - v_j) \times \\ & X_n(x, y; z_1, \dots, z_m) \prod_{i=1}^{m-1} X_n(x, y; u_1, \dots, u_{m-i}, v_{m-i}, \dots, v_{m-1}) = \\ & \det_{1 \leq i, j \leq m} \left(\prod_{k=1}^{m-j} (z_i - u_k) \prod_{k=m-j+1}^{m-1} (z_i - v_k) X_n(x, y; z_i, u_1, \dots, u_{m-j}, v_{m-j+1}, \dots, v_{m-1}) \right) \end{aligned} \quad (72)$$

is satisfied. This result can be obtained from (70) by applying a certain identity for minors of a matrix, which will be given in (154), to the $(3m - 2) \times m$ matrix N with entries $N_{ij} = f_j(z_i)$ for $1 \leq i \leq m$, $N_{ij} = f_j(u_{i-m})$ for $m + 1 \leq i \leq 2m - 1$, and $N_{ij} = f_j(v_{i-2m+1})$ for $2m \leq i \leq 3m - 2$, where $f_j(z) = z^{j-1}(z-1)^{m-j} Z_{n-j+1}(x, y; z)$ (although the actual form of the function $f_j(z)$ in this derivation is immaterial).

Note that the $1 \leq m \leq n$ case of (70) can be retrieved from (72), using only the last three properties in (71) and the $m = 0$ case of (70), by setting $u_1 = \dots = u_{m-1} = 1$ and $v_1 = \dots = v_{m-1} = 0$.

The main result of this section is that, for any $0 \leq m \leq n$ and $1 \leq k_1 < \dots < k_m \leq n$,

$$Z_n^{k_1, \dots, k_m}(x, y; z_1, \dots, z_m; xz_1^2 + (y-x-1)z_1 + 1, \dots, xz_m^2 + (y-x-1)z_m + 1) = X_n(x, y; z_1, \dots, z_m). \quad (73)$$

A derivation of (73), based on a result of Colomo and Pronko [50, Eq. (6.8)], [51, Eq. (A.13)], will be given in Section 5.9. It will also be shown in Section 5.9 that (73) can be derived by combining a general identity for minors of a matrix with the Izergin–Korepin formula for the partition function of the six-vertex model with DWBC.

Note that, using (68) and the third property of (71), the cases of (73) with $m < n$ all follow from the case $m = n$.

It can now be seen, using (67) and (73), that the function (70) is a polynomial in x, y and z_1, \dots, z_m with integer coefficients.

It follows from (73) that, for fixed n and m , the LHS of (73) is independent of k_1, \dots, k_m . This implies, for example, that the singly-refined and opposite-boundary doubly-refined ASM generating functions can be written, for any $1 \leq k \leq n$ and $1 \leq k_1 < k_2 \leq n$, as

$$\begin{aligned} Z_n(x, y; z) &= Z_n^k(x, y; z; xz^2 + (y-x-1)z + 1), \\ Z_n^{\text{opp}}(x, y; z_1, z_2) &= Z_n^{k_1, k_2}(x, y; z_1, z_2; xz_1^2 + (y-x-1)z_1 + 1, xz_2^2 + (y-x-1)z_2 + 1). \end{aligned} \quad (74)$$

It can also be seen that the case $m = 2$, $k_1 = 1$ and $k_2 = n$ of (73) gives the identity (62) satisfied by the opposite-boundary doubly-refined ASM generating function, and that the case $m = 2$ of (72) and the result $Z_n^{\text{opp}}(x, y; z_1, z_2) = X_n(x, y; z_1, z_2)$ give the identity (61).

For the case $y = x + 1$, using the last two formulae of (20) in (70) gives $X_n(x, x + 1; z_1, \dots, z_m) = \left(\prod_{i=1}^m (xz_i + 1) \right)^{n-m} (x + 1)^{(n-m)(n-m-1)/2} \det_{1 \leq i, j \leq m} (z_i^{j-1} ((z_i - 1)(xz_i + 1))^{m-j}) / \prod_{1 \leq i < j \leq m} (z_i - z_j) = \prod_{1 \leq i < j \leq m} (xz_i z_j + 1) \left(\prod_{i=1}^m (xz_i + 1) \right)^{n-m} (x + 1)^{(n-m)(n-m-1)/2}$. (In the last step, the determinant evaluation $\det_{1 \leq i, j \leq m} (z_i^{j-1} ((z_i - 1)(xz_i + 1))^{m-j}) = \prod_{1 \leq i < j \leq m} (z_i - z_j) (xz_i z_j + 1)$ can be obtained using the method of identification of factors.

See Krattenthaler [96, p. 5 & Sec. 2.4], [97, Sec. 4, Method 3]. For example, it follows that $xz_i z_j + 1$ is a factor of this determinant, for each $i \neq j$, by setting $z_j = -1/(xz_i)$, and observing that row j of the matrix is then row i divided by $(-xz_i^2)^{m-1}$.) This explicit expression for $X_n(x, x+1; z_1, \dots, z_m)$ can be seen to generalize the last three formulae of (20).

Finally, it will be shown in Section 5.10 that the function (70) at $x = y = 1$ can be expressed as

$$X_n(1, 1; z_1, \dots, z_m) = 3^{-n(n-1)/2} (-q)^{m(n-1)} ((z_1+q) \dots (z_m+q))^{n-1} \times \\ s_{(n-1, n-1, \dots, 2, 2, 1, 1)} \left(\frac{qz_1+1}{z_1+q}, \dots, \frac{qz_m+1}{z_m+q}, \underbrace{1, \dots, 1}_{2n-m} \right) \Big|_{q=e^{\pm 2\pi i/3}}, \quad (75)$$

where this uses the same notation as, and can be seen to generalize, the identity (46).

3.6. ASMs with several rows or columns closest to two opposite boundaries prescribed. In this section, the enumeration of ASMs with prescribed configurations of several rows or columns closest to two opposite boundaries (but not involving any further statistics) is discussed.

Let K_1 and K_2 be subsets of $\{1, \dots, n\}$ with $|K_1| + |K_2| \leq n$, and let A_{K_1} and A_{K_2} be matrices with n columns and $|K_1|$ or $|K_2|$ rows respectively, in which each entry is 0, 1 or -1 , along each row and column the nonzero entries alternate in sign, in each row the entries sum to 1, in column j of A_{K_1} the entries sum to 1 if $j \in K_1$ and the entries sum to 0 with the first nonzero entry (if there is one) being 1 if $j \notin K_1$, and in column j of A_{K_2} the entries sum to 1 if $j \in K_2$ and the entries sum to 0 with the first nonzero entry (if there is one) being -1 if $j \notin K_2$. (For example, the entries of these matrices could be taken as simply $(A_{K_1})_{ij} = \delta_{j, K_{1,i}}$ and $(A_{K_2})_{ij} = \delta_{j, K_{2,i}}$, where $K_{1,i}$ and $K_{2,i}$ are the i th smallest elements of K_1 and K_2 , respectively.) For a fixed choice of matrices A_{K_1} and A_{K_2} which satisfy these conditions, define

$$\mathcal{A}_{n, K_1, K_2} = \text{number of } n \times n \text{ ASMs whose first } |K_1| \text{ rows are given by } A_{K_1}, \\ \text{and last } |K_2| \text{ rows are given by } A_{K_2}, \quad (76)$$

where it can be checked easily that $\mathcal{A}_{n, K_1, K_2}$ is independent of the choice of A_{K_1} and A_{K_2} . (Note that if $K_1 = \emptyset$ or $K_2 = \emptyset$, then there is no restriction on the first or last rows, respectively, of the ASMs in (76).) Thus, $\mathcal{A}_{n, K_1, K_2}$ is the number of $n \times n$ ASMs in which certain rows or columns closest to two opposite boundaries are prescribed.

Alternatively, $\mathcal{A}_{n, K_1, K_2}$ can be written, without reference to A_{K_1} and A_{K_2} , as the number of $(n - |K_1| - |K_2|) \times n$ matrices in which each entry is 0, 1 or -1 , along each row and column the nonzero entries alternate in sign, in each row the entries sum to 1, and in column j the entries sum to 1 if $j \in \{1, \dots, n\} \setminus (K_1 \cup K_2)$, the entries sum to -1 if $j \in K_1 \cap K_2$, the entries sum to 0 with the first nonzero entry (if there is one) being -1 if $j \in K_1 \setminus K_2$, and the entries sum to 0 with the first nonzero entry (if there is one) being 1 if $j \in K_2 \setminus K_1$.

It can be seen that

$$\mathcal{A}_{n, K_1, K_2} = \mathcal{A}_{n, K_2, K_1} = \mathcal{A}_{n, \{n+1-k | k \in K_1\}, \{n+1-k | k \in K_2\}} \quad (77)$$

and that

$$\mathcal{A}_{n,\emptyset,\emptyset} = \mathcal{A}_n, \quad \mathcal{A}_{n,\{k+1\},\emptyset} = \mathcal{A}_{n,k}, \quad \mathcal{A}_{n,\{k_1+1\},\{n-k_2\}} = \mathcal{A}_{n,k_1,k_2}^{\text{opp}}, \quad (78)$$

for any $0 \leq k, k_1, k_2 \leq n-1$.

It can also be seen that $\mathcal{A}_{n,\{1,\dots,n\} \setminus K_1, \emptyset}$ or $\mathcal{A}_{n,\{1,\dots,n\} \setminus K_2, \emptyset}$ give the numbers of possible matrices A_{K_1} or A_{K_2} , respectively, which satisfy the conditions outlined above. Furthermore, for a fixed set K of positive integers, $\mathcal{A}_{n,\{1,\dots,n\} \setminus K, \emptyset}$ is independent of n , for $n \geq \max(K)$, with an operator formula for this number having been obtained by Fischer [73, Thm. 1]. Some related functions for the six-vertex model with DWBC have been studied by Colomo and Pronko [52].

The numbers \mathcal{A}_{n,K_1,K_2} have been studied by Fischer [76, 77], Fischer and Romik [78], and Karklinsky and Romik [91]. Results which have been obtained include linear relations between the numbers $\mathcal{A}_{n,K,\emptyset}$ with $|K| = 2$ and the opposite-boundary doubly-refined ASM numbers (Fischer [76, Eq. (5.3)], and Karklinsky and Romik [91, Eq. (7)]), explicit formulae for $\mathcal{A}_{n,K,\emptyset}$ with $|K| = 2$ (Fischer [76, Eq. (1.3)], and Karklinsky and Romik [91, Thm. 1]), and an expression for $\mathcal{A}_{n,K,\{k\}}$ with $|K| = 2$ in terms of numbers of $n \times n$ ASMs with prescribed configurations on three boundaries (Fischer [77, Thm. 1]). Also, related numbers have been defined (Fischer [76, Eq. (1.2)], [77, Eq. (2.9)]), and have been shown to give the numbers \mathcal{A}_{n,K_1,K_2} in certain cases (Fischer [76, Thm. 1]), and to satisfy certain linear relations (Fischer [76, Eq. (5.1) & p. 253, first equation], [77, Eqs. (1.5) & (1.6)]).

3.7. The cases $Z_n(1, 3)$ and $Z_n(1, 3; z)$. Results involving $Z_n(1, 3)$ and $Z_n(1, 3; z)$, some of which were previously conjectured by Mills, Robbins and Rumsey [113, Conj. 6–7], [114, Conj. 5], and which correspond to the so-called 3-enumeration of ASMs, have been obtained by Cantini [33, Sec. 4.1], Colomo and Pronko [46], [48, Secs. 4.4 & 5.4], [49, Sec. 4.3], Kuperberg [100, Thm. 2], [101, Thm. 3], Okada [121, Thm. 2.4(1), fourth eq.], and Stroganov [148].

3.8. The case $Z_n(1, 1; -1)$. It was observed by Di Francesco [60, Eqs. (2.7)–(2.8)], and shown explicitly by Williams [154, Thm. 4], that

$$Z_n(1, 1; -1) = \mathcal{V}_n^2, \quad (79)$$

where \mathcal{V}_n is the number of vertically-symmetric $n \times n$ ASMs (i.e., ASMs A with $A_{ij} = A_{i,n+1-j}$). Note that for n even, it can be seen easily (for example, using the second equation of (32)) that both sides of the equation are zero. For n odd, $\mathcal{V}_n = \prod_{i=1}^{(n-1)/2} (6i-2)! / (n+2i-1)!$, as conjectured by Robbins [133, 134], and first shown by Kuperberg [101, Thm. 2, second equation]. It is known that \mathcal{V}_n is also the number of descending plane partitions invariant under a certain operation (see Behrend, Di Francesco and Zinn-Justin [16, Sec. 4.2 & Eq. (102)]), and the number of totally symmetric self-complementary plane partitions invariant under a certain operation (see Ishikawa [88, Thm. 7.11(i)]), and it has been conjectured by Di Francesco [60, Eq. (2.8)], [61, Eq. (4.5)] that \mathcal{V}_n^2 (up to sign) and \mathcal{V}_n^4 are obtained when a parameter is set to -1 in certain generating functions for totally symmetric self-complementary and cyclically symmetric transpose-complementary plane partitions.

3.9. ASMs with a fixed number of generalized inversions. Expressions for the coefficients of x^p in $Z_n(x, 1)$, i.e., for the number of $n \times n$ ASMs with p generalized inversions, can be obtained using a result of Behrend [15, Cor. 14], and are given by Behrend, Di Francesco and Zinn-Justin [16, Eqs. (24)–(25)]. Some particular cases of p , for ASMs with fixed values of certain further statistics, are also considered by Behrend, Di Francesco and Zinn-Justin [16, pp. 337–338], [17, Sec. 7.1].

3.10. ASMs with a fixed number of -1 's. For the case of $Z_n(1, y)$, an expression for the coefficient of y^m , i.e., for the number of $n \times n$ ASMs with m -1 s, has been obtained by Cori, Duchon and Le Gac [53, 108], [109, Ch. 3] (and, for $m = 1, 2$, by Aval [6, Prop. 4]), a certain factorization property (previously conjectured by Mills, Robbins and Rumsey [113, Conj. 4 & Conj. 5], [115, pp. 50 & 54] and Robbins [134, Sec. 2]) has been established by Kuperberg [100, Thm. 3], [101, Thm. 4, first two eqs.], and a certain operator formula containing a parameter which corresponds to the number of -1 's in an ASM has been obtained by Fischer [75, Thm. 1].

Results involving the coefficient of y^1 in $Z_n(x, y; z)$, i.e., for a certain generating function for $n \times n$ ASMs with a single -1 , have been obtained by Lalonde [102, 104].

3.11. Objects in simple bijection with ASMs. Simple bijections are known between ASMs and various other combinatorial objects. Some examples, all of which are reviewed by Propp [123, Secs. 2–4 & 7], are bijections between $\text{ASM}(n)$ and sets of monotone (or Gog) triangles with bottom row $1, \dots, n$, $(n+1) \times (n+1)$ corner-sum matrices, $(n+1) \times (n+1)$ height-function matrices, 3-colourings of an $(n+1) \times (n+1)$ grid with certain boundary conditions, configurations of the six-vertex model on an $n \times n$ grid with DWBC, or fully packed loop configurations on an $n \times n$ grid with certain boundary conditions. Some further examples are a bijection between $\text{ASM}(n)$ and the set of sets of n osculating paths on an $n \times n$ grid in which all paths start and end on two adjacent boundaries (see, for example, Behrend [15, Secs. 2–4], Bousquet-Mélou and Habsieger [23, pp. 68–69], Bressoud [24, pp. 226–227], or Egecioglu, Redmond and Ryavec [68, pp. 35–36]), and a bijection between $\text{ASM}(n)$ and the set of alternating paths for any fixed fully packed loop configuration on an $n \times n$ grid (see Ng [120, Prop. 3.1]).

In each of these cases, the statistics for the other combinatorial object which correspond, under the bijection, to the ASM statistics (2)–(3) or (63) can be obtained relatively straightforwardly. For example, this will be done for configurations of the six-vertex model with DWBC in (113). Accordingly, all of the results reviewed or obtained in this paper for $\text{ASM}(n)$ with the statistics (2)–(3) or (63) could alternatively be expressed in terms of any of these other combinatorial objects and their associated statistics. However, it should be noted that certain statistics will seem more natural when expressed in terms of some objects than others.

It should also be noted that, by applying some of these simple bijections between ASMs and other objects, further statistics or characteristics, which would not seem natural in terms of the ASMs themselves, may become apparent. For instance, a natural statistic

for the monotone triangle which corresponds to an ASM is the number of diagonals whose entries all have certain minimal values (see, for example, Zeilberger [157, Lem. 1] for a result, previously conjectured by Mills, Robbins and Rumsey [114, Conj. 7], involving this statistic), natural statistics for the 3-colouring of a grid which corresponds to an ASM are the numbers of appearances of each colour (see, for example, Rosengren [136, Cor. 8.4], [137, Sec. 3] for results involving these statistics), and a natural characteristic for the fully packed loop configuration which corresponds to an ASM is its associated link pattern (see, for example, Cantini and Sportiello [34, Eq. (23)] for a result, previously conjectured by Razumov and Stroganov [126], involving this characteristic).

In the case of the six-vertex model with DWBC, ASM generating functions are closely related to certain partition functions, expectation values, probabilities or correlation functions for the model, and the results of this paper could alternatively be expressed in terms of the latter quantities. For example, the quadruply-refined ASM generating function is closely associated with a certain four-point boundary correlation function for the six-vertex model with DWBC. For studies of various correlation functions, and related quantities, for the six-vertex model with DWBC, see, for example, Bogoliubov, Kitaev and Zvonarev [20], Bogoliubov, Pronko and Zvonarev [21], Colomo and Pronko [47, 49, 50, 52], Foda and Preston [79], and Motegi [116, 117].

3.12. Descending plane partitions. Descending plane partitions are certain combinatorial objects first defined by Andrews [2, 3]. It was shown by Behrend, Di Francesco and Zinn-Justin [17, Thm. 1] that the opposite-boundary doubly-refined ASM generating function $Z_n^{\text{opp}}(x, y; z_1, z_2)$ is equal to the generating function for descending plane partitions with largest part at most n , in which the statistics associated with x , y , z_1 and z_2 are, respectively, the number of nonspecial parts, the number of special parts (as first defined by Mills, Robbins and Rumsey [113, p. 344]), the number of n 's, and the number of $(n - 1)$'s plus the number of rows of length $n - 1$, in a descending plane partition. (Note, however, that no general bijection is currently known between $\text{ASM}(n)$ and the set of descending plane partitions with largest part at most n , and that no further statistics for descending plane partitions are currently known which, together with the previous four statistics, lead to generating functions which are equal to the adjacent-boundary doubly-refined, triply-refined or quadruply-refined ASM generating functions.)

It follows that certain known results for descending plane partitions correspond to results involving the opposite-boundary doubly-refined, singly-refined or unrefined ASM generating functions. For example, certain such results were outlined in Section 3.4.

Results are also known for the enumeration of descending plane partitions with a prescribed sum of parts. In particular, it follows from work of Mills, Robbins and Rumsey [112, Sec. 5] that, for $0 \leq k \leq n - 1$,

$$\begin{aligned} \sum_{D \in \text{DPP}(n)} q^{|D|} &= \prod_{i=0}^{n-1} \frac{[3i+1]_q!}{[n+i]_q!}, \\ \sum_{D \in \text{DPP}(n,k)} q^{|D|} &= \frac{q^{kn} [n+k-1]_q! [2n-k-2]_q!}{[k]_q! [n-k-1]_q! [2n-2]_q!} \prod_{i=0}^{n-2} \frac{[3i+1]_q!}{[n+i-1]_q!}, \end{aligned} \quad (80)$$

where $\text{DPP}(n)$ denotes the set of descending plane partitions with largest part at most n , $\text{DPP}(n, k)$ denotes the set of elements of $\text{DPP}(n)$ with exactly k parts equal to n , $|D|$ denotes the sum of parts of a descending plane partition D , and $[n]_q!$ is the usual q -factorial of n . As expected, each RHS of (80) reduces to each RHS of (34)–(35) for $q = 1$. However, no ASM statistic is currently known which corresponds to the sum of parts of a descending plane partition, and leads to generating functions matching those of (80).

For further information, and related results or conjectures, regarding descending plane partitions, see, for example, Andrews [2, 3], Ayyer [10], Behrend, Di Francesco and Zinn-Justin [16, 17], Bressoud [24], Bressoud and Propp [27], Krattenthaler [98], Lalonde [102, 103, 104], Mills, Robbins and Rumsey [112, 113, 115], Robbins [133, 134], and Striker [145].

3.13. Totally symmetric self-complementary plane partitions. Totally symmetric self-complementary plane partitions are certain combinatorial objects first defined by Mills, Robbins and Rumsey [114, Sec. 1], and Stanley [141, Sec. 2]. It was shown by Fonseca and Zinn-Justin [81], following conjectures of Mills, Robbins and Rumsey [114, Conj. 2 & 3], that the opposite-boundary doubly-refined ASM generating function with both bulk parameters set to 1, i.e., $Z_n^{\text{opp}}(1, 1; z_1, z_2)$, is equal to a generating function for totally symmetric self-complementary plane partitions in a $2n \times 2n \times 2n$ box in which the statistics associated with z_1 and z_2 can be certain pairs from among several statistics defined by Doran [67, Sec. 7], and Mills, Robbins and Rumsey [114, Sec. 3] (see also Robbins [133, p. 16]). The singly-refined case of this result was obtained previously by Razumov, Stroganov and Zinn-Justin [132, Sec. 5.5]. (Note, however, that no general bijection is currently known between $\text{ASM}(n)$ and the set of totally symmetric self-complementary plane partitions in a $2n \times 2n \times 2n$ box, and that no statistics for totally symmetric self-complementary plane partitions are currently known which have the same enumerative behaviour as the number of generalized inversions or the number of -1 's in an ASM.)

It follows that certain known results for totally symmetric self-complementary plane partitions, or obtained while studying these objects, correspond to results involving the ASM generating functions $Z_n^{\text{opp}}(1, 1; z_1, z_2)$ or $Z_n(1, 1; z)$, or the ASM numbers $\mathcal{A}_{n, k_1, k_2}^{\text{opp}}$, $\mathcal{A}_{n, k}$ or \mathcal{A}_n . For example, for $Z_n^{\text{opp}}(1, 1; z_1, z_2)$, $Z_n(1, 1; z)$ or \mathcal{A}_n , Pfaffian expressions follow from results of Ishikawa [87, Thms. 1.2 & 1.4, & Sec. 7] and Stembridge [143, Thm. 8.3], constant-term expressions follow from results of Ishikawa [87, Sec. 8], Krattenthaler [95, Thm.] and Zeilberger [156], [157, Sublems. 1.1 & 1.2], and integral expressions (which can easily be converted to constant-term expressions) follow from results of Fonseca and Zinn-Justin [81, Eqs. (4.9) & (4.14)] and Zinn-Justin and Di Francesco [162, Eqs. (37) & (39)]. Note that many of these results are expressed in terms of certain triangles of positive integers (specifically, monotone or Gog triangles for ASMs, and Magog triangles for totally symmetric self-complementary plane partitions), or closely related integer arrays. Also, many such results are stated in more general forms which contain additional parameters associated with certain entries of such arrays being prescribed to take certain values, or being bounded by certain values.

For the case of adjacent-boundary doubly-refined ASM enumeration, it has been conjectured by Cheballah [37, Conj. 4.3.1] that $\mathcal{A}_{n,k_1,k_2}^{\text{adj}}$ equals the number of totally symmetric self-complementary plane partitions in a $2n \times 2n \times 2n$ box for which a certain pair of statistics have values k_1 and k_2 .

For further information, and related results or conjectures, regarding totally symmetric self-complementary plane partitions, see, for example, Andrews [4], Ayyer, Cori and Gouyou-Beauchamps [11], Bressoud [24, Sec. 6.2], Bressoud and Propp [27], Biane and Cheballah [19], Cheballah [37], Cheballah and Biane [38], Di Francesco [60, 61], Doran [67], Fonseca [80, Secs. 3–4], Fonseca and Zinn-Justin [81], Ishikawa [87, 88], Krattenthaler [95], Mills, Robbins and Rumsey [114], Robbins [133], Stanley [141], Stembridge [143, Sec. 8], Striker [144, 146], Zeilberger [156, 157], Zinn-Justin [161], and Zinn-Justin and Di Francesco [162].

3.14. Loop models. Numbers of certain ASMs, or plane partitions, have been found to appear also as appropriately normalized entries of particular eigenvectors (or sums of such entries, or norms of such eigenvectors) associated with certain cases of integrable loop models (or associated quantum spin chains). A wide variety of such cases are known, and these often involve ASMs with prescribed link patterns of associated fully packed loop configurations, with prescribed values of some of the statistics of (2)-(3) or of related statistics or parameters, or subject to invariance under certain symmetry operations.

Of the many papers which study the confirmed or conjectured appearances of numbers of ASMs in such contexts, a few examples are Batchelor, de Gier and Nienhuis [13], Cantini and Sportiello [34, 35], Di Francesco [58, 59], Di Francesco and Zinn-Justin [65], Pasquier [122], Razumov and Stroganov [125, 126, 128], and Stroganov [147]. For reviews of some of these matters, see, for example, de Gier [54, 55, 56], or Zinn-Justin [161].

3.15. ASMs invariant under symmetry operations. This paper is focused on the enumeration of ASMs with prescribed values of the bulk statistics (2) and boundary statistics (3), but with no other conditions applied. However, other studies have focused on the enumeration of ASMs which are invariant under certain symmetry operations, or subject to related conditions, and which in some cases also have prescribed values of some of the statistics of (2)-(3). For results of this type (some of which remain conjectural), see, for example, Aval and Duchon [8, 9], Bousquet-Mélou and Habsieger [23], Bressoud [24, Sec. 6.1], Kuperberg [101], Okada [121], Robbins [133, 134], Razumov and Stroganov [127, 129, 130], Stanley [140], and Stroganov [149, 151].

Numbers of certain such ASMs have also been related (again, in some cases, only conjecturally) to numbers of certain descending plane partitions (see, for example, de Gier, Pyatov and Zinn-Justin [57, Prop. 3, first equation], and Mills, Robbins and Rumsey [113, Conj. 3S]), numbers of certain totally-symmetric self-complementary plane partitions (see, for example, Ishikawa [87, 88], and Mills, Robbins and Rumsey [114, Conjs. 4 & 6]), or appropriately normalized entries of eigenvectors associated with certain cases of loop models (see, for example, the references given in Section 3.14).

4. MAIN RESULTS

In this section, the main result of this paper (Theorem 1) is stated, and some of its consequences (Corollaries 2–11) are derived and discussed. The proof of Theorem 1 is deferred to Section 5.6.

In particular, new results are given for the quadruply-refined ASM generating function (Theorem 1 and Corollary 10), triply-refined ASM generating function (Corollaries 2, 4 and 11), adjacent-boundary doubly-refined ASM generating functions (Corollaries 3, 6 and 7), singly-refined ASM generating function (Corollary 8), and unrefined ASM generating function (Corollary 9), and a previously-known result is obtained for the opposite-boundary doubly-refined ASM generating function (Corollary 5).

4.1. Main theorem. The primary result of this paper is as follows.

Theorem 1. *The quadruply-refined ASM generating function satisfies*

$$\begin{aligned}
y(z_4 - z_2)(z_1 - z_3) Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) Z_{n-2}(x, y) = \\
& ((z_1 - 1)(z_2 - 1) - yz_1z_2)((z_3 - 1)(z_4 - 1) - yz_3z_4) Z_{n-1}^{\text{adj}}(x, y; z_4, z_1) Z_{n-1}^{\text{adj}}(x, y; z_2, z_3) - \\
& (x(z_4 - 1)(z_1 - 1) - y)(x(z_2 - 1)(z_3 - 1) - y) z_1z_2z_3z_4 \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_1, z_2) \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_3, z_4) - \\
& (z_2 - 1)(z_3 - 1)((z_4 - 1)(z_1 - 1) - yz_4z_1) Z_{n-1}^{\text{adj}}(x, y; z_4, z_1) Z_{n-2}(x, y) + \\
& (z_3 - 1)(z_4 - 1)(x(z_1 - 1)(z_2 - 1) - y) z_1z_2 (xz_3z_4)^{n-1} \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_1, z_2) Z_{n-2}(x, y) - \\
& (z_4 - 1)(z_1 - 1)((z_2 - 1)(z_3 - 1) - yz_2z_3) Z_{n-1}^{\text{adj}}(x, y; z_2, z_3) Z_{n-2}(x, y) + \\
& (z_1 - 1)(z_2 - 1)(x(z_3 - 1)(z_4 - 1) - y) z_3z_4 (xz_1z_2)^{n-1} \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_3, z_4) Z_{n-2}(x, y) + \\
& (z_1 - 1)(z_2 - 1)(z_3 - 1)(z_4 - 1)(1 - (x^2z_1z_2z_3z_4)^{n-1}) Z_{n-2}(x, y)^2. \quad (81)
\end{aligned}$$

The identity (81) holds for each $n \geq 3$. Furthermore, if $Z_0(x, y)$ is taken to be 1, then it can be seen, using $Z_1^{\text{adj}}(x, y; z_1, z_2) = \tilde{Z}_1^{\text{adj}}(x, y; z_1, z_2) = 1$ and $Z_2^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) = 1 + xz_1z_2z_3z_4$ (from (8)–(10)), that (81) also holds for $n = 2$.

The proof of (81), which will be given in Section 5.6, will involve using a relation between a certain generalized ASM generating function and the partition function of the six-vertex model with DWBC, and then combining the Desnanot–Jacobi determinant identity with the Izergin–Korepin formula for this partition function.

An alternative form of (81) will be obtained, and some further related results will be discussed, in Section 5.7.

It can be seen that (81) enables $Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4)$ to be obtained recursively, using the initial conditions (from (8)) $Z_1^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) = 1$ and $Z_2^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) = 1 + xz_1z_2z_3z_4$, and the definitions (from (9)–(10)) $Z_n^{\text{adj}}(x, y; z_1, z_2) = Z_n^{\text{quad}}(x, y; z_1, 1, 1, z_2)$, $\tilde{Z}_n^{\text{adj}}(x, y; z_1, z_2) = Z_n^{\text{quad}}(x, y; z_1, z_2, 1, 1)$ and $Z_n(x, y) = Z_n^{\text{quad}}(x, y; 1, 1, 1, 1)$. Accordingly, in this sense, $Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4)$, and all of the ASM generating functions of (9)–(10) which are defined in terms of $Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4)$, are determined by (81).

Note, however, that if the generating functions are computed recursively in this way, then, for each successive n , $Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4)$ should first be computed for arbitrary z_1, z_2, z_3 , and z_4 , with the factor $(z_1 - z_3)(z_4 - z_2)$ being explicitly cancelled from both sides of (81), so that division by zero is avoided when boundary parameters need to be set to 1 in subsequent computations. Alternatively, certain expressions in which this cancellation has effectively been done, will be given in Section 4.2.

By replacing z_2 and z_4 by $\frac{1}{z_2}$ and $\frac{1}{z_4}$ respectively, Theorem 1 can be restated for the alternative quadruply-refined ASM generating function of (10) as

$$\begin{aligned}
& y(z_1 - z_3)(z_2 - z_4) \tilde{Z}_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) Z_{n-2}(x, y) = \\
& ((z_1 - 1)(z_2 - 1) + yz_1)((z_3 - 1)(z_4 - 1) + yz_3) (z_2 z_4)^{n-1} \times \\
& \qquad \qquad \qquad Z_{n-1}^{\text{adj}}(x, y; \frac{1}{z_4}, z_1) Z_{n-1}^{\text{adj}}(x, y; \frac{1}{z_2}, z_3) - \\
& (x(z_4 - 1)(z_1 - 1) + yz_4)(x(z_2 - 1)(z_3 - 1) + yz_2) z_1 z_3 (z_2 z_4)^{n-2} \times \\
& \qquad \qquad \qquad \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_1, \frac{1}{z_2}) \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_3, \frac{1}{z_4}) - \\
& (z_2 - 1)(z_3 - 1)((z_4 - 1)(z_1 - 1) + yz_1) (z_2 z_4)^{n-1} Z_{n-1}^{\text{adj}}(x, y; \frac{1}{z_4}, z_1) Z_{n-2}(x, y) + \\
& (z_3 - 1)(z_4 - 1)(x(z_1 - 1)(z_2 - 1) + yz_2) z_1 z_2^{n-2} (x z_3)^{n-1} \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_1, \frac{1}{z_2}) Z_{n-2}(x, y) - \\
& (z_4 - 1)(z_1 - 1)((z_2 - 1)(z_3 - 1) + yz_3) (z_2 z_4)^{n-1} Z_{n-1}^{\text{adj}}(x, y; \frac{1}{z_2}, z_3) Z_{n-2}(x, y) + \\
& (z_1 - 1)(z_2 - 1)(x(z_3 - 1)(z_4 - 1) + yz_4) z_3 z_4^{n-2} (x z_1)^{n-1} \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_3, \frac{1}{z_4}) Z_{n-2}(x, y) + \\
& (z_1 - 1)(z_2 - 1)(z_3 - 1)(z_4 - 1)((z_2 z_4)^{n-1} - (x^2 z_1 z_3)^{n-1}) Z_{n-2}(x, y)^2. \quad (82)
\end{aligned}$$

4.2. Corollaries for arbitrary bulk parameters x and y . In this section, some consequences of Theorem 1, for the case in which the bulk parameters x and y remain arbitrary, are derived. These results are obtained from (81) using only definitions and elementary properties of ASM generating functions from Section 2.2, and the initial conditions (from (8)–(10)) $Z_1^{\text{adj}}(x, y; z_1, z_2) = \tilde{Z}_1^{\text{adj}}(x, y; z_1, z_2) = 1$ and $Z_2^{\text{adj}}(x, y; z_1, z_2) = \tilde{Z}_2^{\text{adj}}(x, y; z_1, z_2) = 1 + xz_1 z_2$. In particular, most of the results are obtained by setting boundary parameters to 1, using (11) for the corresponding specializations, and, in some cases, solving recursion relations.

Alternative forms of some of the results of this section will be given in Section 5.7.

Corollary 2. *The triply-refined ASM generating function satisfies*

$$\begin{aligned}
& (z_2 - z_1)(z_3 - 1) Z_n^{\text{tri}}(x, y; z_1, z_2, z_3) Z_{n-2}(x, y) = \\
& ((z_2 - 1)(z_3 - 1) - yz_2 z_3) z_1 Z_{n-1}^{\text{adj}}(x, y; z_1, z_3) Z_{n-1}(x, y; z_2) - \\
& (x(z_1 - 1)(z_3 - 1) - y) z_1 z_2 z_3 \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_2, z_3) Z_{n-1}(x, y; z_1) - \\
& (z_1 - 1)(z_3 - 1) z_2 Z_{n-1}(x, y; z_2) Z_{n-2}(x, y) + \\
& (z_2 - 1)(z_3 - 1) z_1 (x z_2 z_3)^{n-1} Z_{n-1}(x, y; z_1) Z_{n-2}(x, y). \quad (83)
\end{aligned}$$

Proof. Set $z_2 = 1$ in (81) (and then relabel z_3 as z_2 and z_4 as z_3). \square

Corollary 3. *The adjacent-boundary doubly-refined ASM generating functions satisfy the recursion relations*

$$\begin{aligned} (z_1 - 1)(z_2 - 1) Z_n^{\text{adj}}(x, y; z_1, z_2) Z_{n-2}(x, y) &= yz_1z_2 Z_{n-1}^{\text{adj}}(x, y; z_1, z_2) Z_{n-1}(x, y) + \\ &\quad (x(z_1 - 1)(z_2 - 1) - y) z_1z_2 Z_{n-1}(x, y; z_1) Z_{n-1}(x, y; z_2) + \\ &\quad (z_1 - 1)(z_2 - 1) Z_{n-1}(x, y) Z_{n-2}(x, y), \end{aligned} \quad (84)$$

$$\begin{aligned} (z_1 - 1)(z_2 - 1) \tilde{Z}_n^{\text{adj}}(x, y; z_1, z_2) Z_{n-2}(x, y) &= yz_1z_2 \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_1, z_2) Z_{n-1}(x, y) + \\ &\quad ((z_1 - 1)(z_2 - 1) - yz_1z_2) Z_{n-1}(x, y; z_1) Z_{n-1}(x, y; z_2) + \\ &\quad (z_1 - 1)(z_2 - 1) (xz_1z_2)^{n-1} Z_{n-1}(x, y) Z_{n-2}(x, y). \end{aligned} \quad (85)$$

Proof. To obtain (84), set $z_2 = 1$ in (83) (and then relabel z_3 as z_2). To obtain (85), set $z_1 = 1$ in (83) (and then relabel z_2 as z_1 and z_3 as z_2). \square

Note that, if $Z_0(x, y)$ is taken to be 1, then (84) and (85) hold for all $n \geq 2$.

Corollary 4. *The triply-refined ASM generating function also satisfies*

$$\begin{aligned} y(z_2 - z_1)z_3 Z_n^{\text{tri}}(x, y; z_1, z_2, z_3) Z_{n-1}(x, y) &= \\ &\quad (z_1 - 1)((z_2 - 1)(z_3 - 1) - yz_2z_3) Z_n^{\text{adj}}(x, y; z_1, z_3) Z_{n-1}(x, y; z_2) - \\ &\quad (z_2 - 1)(x(z_1 - 1)(z_3 - 1) - y) z_1z_3 \tilde{Z}_n^{\text{adj}}(x, y; z_2, z_3) Z_{n-1}(x, y; z_1) - \\ &\quad (z_1 - 1)(z_2 - 1)(z_3 - 1) Z_{n-1}(x, y; z_2) Z_{n-1}(x, y) + \\ &\quad (z_1 - 1)(z_2 - 1)(z_3 - 1) z_1z_2^{n-1} (xz_3)^n Z_{n-1}(x, y; z_1) Z_{n-1}(x, y). \end{aligned} \quad (86)$$

Proof. Use (84)–(85) to replace $Z_{n-1}^{\text{adj}}(x, y; z_1, z_3)$ and $\tilde{Z}_{n-1}^{\text{adj}}(x, y; z_2, z_3)$ in (83) by terms which instead contain $Z_n^{\text{adj}}(x, y; z_1, z_3)$ and $\tilde{Z}_n^{\text{adj}}(x, y; z_2, z_3)$, and then cancel an overall factor which contains a term $z_3 - 1$. \square

Corollary 5. *The opposite-boundary doubly-refined ASM generating function satisfies*

$$\begin{aligned} (z_1 - z_2) Z_n^{\text{opp}}(x, y; z_1, z_2) Z_{n-1}(x, y) &= (z_1 - 1) z_2 Z_n(x, y; z_1) Z_{n-1}(x, y; z_2) - \\ &\quad z_1 (z_2 - 1) Z_{n-1}(x, y; z_1) Z_n(x, y; z_2). \end{aligned} \quad (87)$$

Proof. Set $z_3 = 1$ in (86). \square

Note that Corollary 5 is a previously-known result, as already given in (62) and discussed in Section 3.4.

Corollary 6. *The adjacent-boundary doubly-refined ASM generating functions can be expressed as*

$$Z_n^{\text{adj}}(x, y; z_1, z_2) = Z_{n-1}(x, y) \left(1 + \sum_{i=1}^{n-1} \left(\frac{y z_1 z_2}{(z_1-1)(z_2-1)} \right)^{n-i} \times \left(1 + \frac{(x(z_1-1)(z_2-1)-y) Z_i(x, y; z_1) Z_i(x, y; z_2)}{y Z_{i-1}(x, y) Z_i(x, y)} \right) \right), \quad (88)$$

$$\tilde{Z}_n^{\text{adj}}(x, y; z_1, z_2) = Z_{n-1}(x, y) \left((x z_1 z_2)^{n-1} + \sum_{i=1}^{n-1} \left(\frac{y}{(z_1-1)(z_2-1)} \right)^{n-i} \times \left(x^{i-1} (z_1 z_2)^{n-1} + \frac{(z_1 z_2)^{n-i-1} ((z_1-1)(z_2-1)-y z_1 z_2) Z_i(x, y; z_1) Z_i(x, y; z_2)}{y Z_{i-1}(x, y) Z_i(x, y)} \right) \right), \quad (89)$$

where, in the sums over i , $Z_0(x, y)$ is taken to be 1.

Proof. It can be checked straightforwardly (again taking $Z_0(x, y) = 1$), that the initial conditions $Z_1^{\text{adj}}(x, y; z_1, z_2) = \tilde{Z}_1^{\text{adj}}(x, y; z_1, z_2) = 1$ and recursion relations (84)–(85) are satisfied by (88)–(89). \square

Corollary 7. *The two types of adjacent-boundary doubly-refined ASM generating function are related by*

$$\begin{aligned} ((z_1-1)(z_2-1)-y z_1 z_2) Z_n^{\text{adj}}(x, y; z_1, z_2) - (x(z_1-1)(z_2-1)-y) z_1 z_2 \tilde{Z}_n^{\text{adj}}(x, y; z_1, z_2) \\ = (z_1-1)(z_2-1)(1 - (x z_1 z_2)^n) Z_{n-1}(x, y). \end{aligned} \quad (90)$$

Proof. This can be obtained directly from (88)–(89). Alternatively, it can be obtained from (84)–(85), by showing that $((z_1-1)(z_2-1)-y z_1 z_2) Z_n^{\text{adj}}(x, y; z_1, z_2) - (z_1-1)(z_2-1) Z_{n-1}(x, y)$ and $(x(z_1-1)(z_2-1)-y) z_1 z_2 \tilde{Z}_n^{\text{adj}}(x, y; z_1, z_2) - (z_1-1)(z_2-1)(x z_1 z_2)^n Z_{n-1}(x, y)$ satisfy the same recursion relation and initial condition. \square

Note that, from (12), the two types of adjacent-boundary doubly-refined ASM generating function are also related by

$$Z_n^{\text{adj}}(x, y; z_1, z_2) = x^{n(n-1)/2} (z_1 z_2)^{n-1} \tilde{Z}_n^{\text{adj}}\left(\frac{1}{x}, \frac{y}{x}; \frac{1}{z_1}, \frac{1}{z_2}\right). \quad (91)$$

Combining (90) and (91), it follows that the adjacent-boundary doubly-refined ASM generating function satisfies

$$\begin{aligned} ((z_1-1)(z_2-1)-y z_1 z_2) Z_n^{\text{adj}}(x, y; z_1, z_2) = \\ (x(z_1-1)(z_2-1)-y) (z_1 z_2)^n x^{n(n-1)/2} Z_n^{\text{adj}}\left(\frac{1}{x}, \frac{y}{x}; \frac{1}{z_1}, \frac{1}{z_2}\right) + \\ (z_1-1)(z_2-1)(1 - (x z_1 z_2)^n) Z_{n-1}(x, y). \end{aligned} \quad (92)$$

By using (81) and (90), together with identities from (12), it is also possible to obtain a certain identity which involves $Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4)$ and $Z_n^{\text{quad}}(x, y; z_3, z_2, z_1, z_4)$ (or $Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4)$ and $Z_n^{\text{quad}}(x, y; z_1, z_4, z_3, z_2)$). A form of this identity will be given in (143).

Corollary 8. *The boundary parameter coefficients in the singly-refined ASM generating function, as defined in (14), satisfy*

$$Z_n(x, y)_k = Z_{n-1}(x, y) \delta_{k,0} + Z_{n-1}(x, y) \sum_{i=0}^{k-1} \left(y^{i+1} \binom{k-1}{i} \binom{n-1}{i+1} + y^i \sum_{j_1=0}^{k-i-1} \sum_{j_2=0}^{n-i-2} \frac{Z_{n-i-1}(x, y)_{j_1} Z_{n-i-1}(x, y)_{j_2}}{Z_{n-i-1}(x, y) Z_{n-i-2}(x, y)} \left(x \binom{k-j_1-2}{i-1} \binom{n-j_2-2}{i} - y \binom{k-j_1-1}{i} \binom{n-j_2-1}{i+1} \right) \right), \quad (93)$$

where $Z_0(x, y)$, if it appears, is taken to be 1.

Proof. Expanding the factors $1/((z_1 - 1)(z_2 - 1))^{n-i}$ in (88) as binomial series, equating coefficients of $z_1^{k_1} z_2^{k_2}$ on both sides of (88), and using the definitions (14), gives

$$Z_n^{\text{adj}}(x, y)_{k_1, k_2} = Z_{n-1}(x, y) \delta_{k_1,0} \delta_{k_2,0} + Z_{n-1}(x, y) \sum_{i=0}^{\min(k_1, k_2)-1} \left(y^{i+1} \binom{k_1-1}{i} \binom{k_2-1}{i} + y^i \sum_{j_1=0}^{k_1-i-1} \sum_{j_2=0}^{k_2-i-1} \frac{Z_{n-i-1}(x, y)_{j_1} Z_{n-i-1}(x, y)_{j_2}}{Z_{n-i-1}(x, y) Z_{n-i-2}(x, y)} \left(x \binom{k_1-j_1-2}{i-1} \binom{k_2-j_2-2}{i-1} - y \binom{k_1-j_1-1}{i} \binom{k_2-j_2-1}{i} \right) \right). \quad (94)$$

Summing (94) over k_2 , using a standard binomial coefficient summation identity, and relabelling k_1 as k , then gives (93). \square

Corollary 9. *The unrefined ASM generating function satisfies*

$$Z_n(x, y) = Z_{n-1}(x, y) \left(1 + \sum_{i=0}^{n-2} \left(y^{i+1} \binom{n-1}{i+1} \right)^2 + \frac{xy^i \left(\sum_{j=0}^{n-i-2} \binom{n-j-2}{i} Z_{n-i-1}(x, y)_j \right)^2 - y^{i+1} \left(\sum_{j=0}^{n-i-2} \binom{n-j-1}{i+1} Z_{n-i-1}(x, y)_j \right)^2}{Z_{n-i-1}(x, y) Z_{n-i-2}(x, y)} \right), \quad (95)$$

where $Z_0(x, y)$ is taken to be 1.

Proof. Sum (93) over k , and use a standard binomial coefficient summation identity. \square

Note that various other expressions for $Z_n(x, y)_k$ and $Z_n(x, y)$, which provide alternatives to (93) and (95), can be obtained by taking the limits $z_1 \rightarrow 1$ and $z_2 \rightarrow 1$ of (88) in ways somewhat different from those used in Corollaries 8 and 9.

As observed in Section 4.1, the main result (81) enables all of the ASM generating functions of (7), (9) and (10) to be computed recursively. The corollaries of this section essentially

comprise special cases of (81) which apply to specific ASM generating functions, and can alternatively be used for their computation.

For example, it can be seen that (93) and (95) give $Z_n(x, y)_k$ and $Z_n(x, y)$ in terms of $Z_i(x, y)_j$ and $Z_i(x, y)$ for $i = 1, \dots, n-1$, and thereby enable the singly-refined and unrefined ASM generating functions to be computed recursively. The adjacent-boundary doubly-refined, opposite-boundary doubly-refined, triply-refined and quadruply-refined ASM generating functions can then be computed using (88)–(89), (87), (83) or (86), and (81), respectively.

Note that the opposite-boundary doubly-refined, singly-refined and unrefined ASM generating functions can instead be computed using the determinant formulae (54), (56) and (57).

4.3. Corollaries for bulk parameters $x = y = 1$. In this section, some consequences of Theorem 1, for the case in which the bulk parameters x and y are both set to 1, are derived. In contrast to the consequences of Theorem 1 given in Section 4.2 for the case of arbitrary x and y , which were obtained essentially using only (81), the consequences in this section are obtained using (81) together with the nontrivial relation (45) between the adjacent-boundary and opposite-boundary doubly-refined ASM generating functions at $x = y = 1$.

Corollary 10. *The alternative quadruply-refined ASM generating function at $x = y = 1$ satisfies*

$$\begin{aligned}
& (z_4 z_1 - z_4 + 1)(z_1 z_2 - z_1 + 1)(z_2 z_3 - z_2 + 1)(z_3 z_4 - z_3 + 1) \tilde{Z}_n^{\text{quad}}(1, 1; z_1, z_2, z_3, z_4) = \\
& \frac{z_1 z_2 z_3 z_4}{\mathcal{A}_{n-2}(z_1 - z_3)(z_2 - z_4)} \times \\
& ((z_1 z_2 - z_1 + 1)(z_1 z_2 - z_2 + 1)(z_3 z_4 - z_3 + 1)(z_3 z_4 - z_4 + 1) Z_{n-1}^{\text{opp}}(1, 1; z_4, z_1) Z_{n-1}^{\text{opp}}(1, 1; z_2, z_3) - \\
& (z_4 z_1 - z_4 + 1)(z_4 z_1 - z_1 + 1)(z_2 z_3 - z_2 + 1)(z_2 z_3 - z_3 + 1) Z_{n-1}^{\text{opp}}(1, 1; z_1, z_2) Z_{n-1}^{\text{opp}}(1, 1; z_3, z_4)) + \\
& (z_2 - 1)(z_3 - 1)(z_1 z_2 - z_1 + 1)(z_3 z_4 - z_3 + 1) z_4 z_1 z_2^{n-1} Z_{n-1}^{\text{opp}}(1, 1; z_4, z_1) + \\
& (z_3 - 1)(z_4 - 1)(z_2 z_3 - z_2 + 1)(z_4 z_1 - z_4 + 1) z_1 z_2 z_3^{n-1} Z_{n-1}^{\text{opp}}(1, 1; z_1, z_2) + \\
& (z_4 - 1)(z_1 - 1)(z_3 z_4 - z_3 + 1)(z_1 z_2 - z_1 + 1) z_2 z_3 z_4^{n-1} Z_{n-1}^{\text{opp}}(1, 1; z_2, z_3) + \\
& (z_1 - 1)(z_2 - 1)(z_4 z_1 - z_4 + 1)(z_2 z_3 - z_2 + 1) z_3 z_4 z_1^{n-1} Z_{n-1}^{\text{opp}}(1, 1; z_3, z_4) + \\
& (z_1 - 1)(z_2 - 1)(z_3 - 1)(z_4 - 1)((z_1 z_2 - z_1 + 1)(z_3 z_4 - z_3 + 1)(z_2 z_4)^{n-1} + \\
& (z_2 z_3 - z_2 + 1)(z_4 z_1 - z_4 + 1)(z_1 z_3)^{n-1}) \mathcal{A}_{n-2}. \quad (96)
\end{aligned}$$

Proof. Setting $x = y = 1$ in (82), and using (91), gives

$$\begin{aligned}
& (z_1 - z_3)(z_2 - z_4) \tilde{Z}_n^{\text{quad}}(1, 1; z_1, z_2, z_3, z_4) \mathcal{A}_{n-2} = \\
& (z_1 z_2 - z_2 + 1)(z_3 z_4 - z_4 + 1)(z_2 z_4)^{n-1} Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_4}, z_1) Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_2}, z_3) - \\
& (z_4 z_1 - z_1 + 1)(z_2 z_3 - z_3 + 1)(z_1 z_3)^{n-1} Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_1}, z_2) Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_3}, z_4) - \\
& (z_2 - 1)(z_3 - 1)(z_4 z_1 - z_4 + 1)(z_2 z_4)^{n-1} Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_4}, z_1) \mathcal{A}_{n-2} +
\end{aligned}$$

$$\begin{aligned}
& (z_3-1)(z_4-1)(z_1z_2-z_1+1)(z_1z_3)^{n-1} Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_1}, z_2) \mathcal{A}_{n-2} - \\
& (z_4-1)(z_1-1)(z_2z_3-z_2+1)(z_2z_4)^{n-1} Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_2}, z_3) \mathcal{A}_{n-2} + \\
& (z_1-1)(z_2-1)(z_3z_4-z_3+1)(z_1z_3)^{n-1} Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_3}, z_4) \mathcal{A}_{n-2} + \\
& (z_1-1)(z_2-1)(z_3-1)(z_4-1)((z_2z_4)^{n-1} - (z_1z_3)^{n-1}) \mathcal{A}_{n-2}^2. \quad (97)
\end{aligned}$$

Multiplying both sides of (97) by $(z_4z_1-z_4+1)(z_1z_2-z_1+1)(z_2z_3-z_2+1)(z_3z_4-z_3+1)/((z_1-z_3)(z_2-z_4)\mathcal{A}_{n-2})$, and using (45), in the form $(z_1z_2-z_1+1)z_1^{n-2}Z_{n-1}^{\text{adj}}(1, 1; \frac{1}{z_1}, z_2) = z_2Z_{n-1}^{\text{opp}}(1, 1; z_1, z_2) + z_1^{n-2}(z_1-1)(z_2-1)\mathcal{A}_{n-2}$, then gives (96). \square

Corollary 11. *The triply-refined ASM generating function at $x = y = 1$ satisfies*

$$\begin{aligned}
& (z_1z_3-z_3+1)(z_2z_3-z_2+1)z_3^{n-1}Z_n^{\text{tri}}(1, 1; z_1, z_2, \frac{1}{z_3}) = \\
& \frac{z_1z_3}{\mathcal{A}_{n-2}(z_1-z_2)(z_3-1)} \left((z_1z_3-z_1+1)(z_1z_3-z_3+1)z_2Z_{n-1}(1, 1; z_1)Z_{n-1}^{\text{opp}}(1, 1; z_2, z_3) - \right. \\
& \quad \left. (z_2z_3-z_2+1)(z_2z_3-z_3+1)z_1Z_{n-1}(1, 1; z_2)Z_{n-1}^{\text{opp}}(1, 1; z_1, z_3) \right) + \\
& (z_2-1)(z_3-1)(z_1z_3-z_3+1)z_1z_2^{n-1}Z_{n-1}(1, 1; z_1) + \\
& (z_1-1)(z_3-1)(z_2z_3-z_2+1)z_3^{n-1}Z_{n-1}(1, 1; z_2). \quad (98)
\end{aligned}$$

Proof. Set $z_2 = 1$ in (96) (and then relabel z_3 as z_2 and z_4 as z_3). \square

It can be seen that (96) and (98) provide alternative formulae to (49)–(51) for the quadruply- and triply-refined ASM generating functions at $x = y = 1$. In fact, (96) and (98) differ from (51) and (50), respectively, only in the first terms on each RHS. It can also be seen that the last five terms on the RHS of (96) could be replaced by the last five terms on the RHS of (49).

5. PROOFS

In this section, proofs of Theorem 1, and of several of the results of Section 3, are given. Preliminary results which are needed for these proofs are obtained or stated in Sections 5.1–5.5, while the main steps of the proofs are given in Sections 5.6 and 5.8–5.10. In Section 5.7, alternative forms of certain results of Section 4, and some further results, are discussed.

5.1. A generalized ASM generating function. In this section, a generating function which generalizes the quadruply-refined ASM generating function is introduced, and some of its elementary properties are identified. This generating function will be shown, in Section 5.3, to be proportional to a certain case of the partition function of the six-vertex model with DWBC.

The generalized ASM generating function involves the six statistics of (2)–(3), together with four further statistics associated with the entries in the corners of an ASM, and is

defined, for indeterminates $x, y, z_1, z_2, z_3, z_4, z_{41}, z_{12}, z_{23}$ and z_{34} , as

$$Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; z_{41}, z_{12}, z_{23}, z_{34}) = \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A)} z_1^{\rho_{\text{T}}(A)} z_2^{\rho_{\text{R}}(A)} z_3^{\rho_{\text{B}}(A)} z_4^{\rho_{\text{L}}(A)} z_{41}^{1-A_{11}} z_{12}^{1-A_{1n}} z_{23}^{1-A_{nn}} z_{34}^{1-A_{n1}}. \quad (99)$$

For example, for $n = 3$, the function is

$$\begin{aligned} Z_3^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; z_{41}, z_{12}, z_{23}, z_{34}) &= z_{12} z_{34} + x z_1 z_4 z_{41} z_{12} z_{34} + \\ & x z_2 z_3 z_{12} z_{23} z_{34} + x^2 z_1 z_2 z_3^2 z_4^2 z_{41} z_{12} z_{23} + x^2 z_1^2 z_2^2 z_3 z_4 z_{41} z_{23} z_{34} + \\ & x^3 z_1^2 z_2^2 z_3^2 z_4^2 z_{41} z_{23} + x y z_1 z_2 z_3 z_4 z_{41} z_{12} z_{23} z_{34}, \end{aligned} \quad (100)$$

where the terms are written in an order which corresponds to that used in (1).

It can be seen immediately that the quadruply-refined ASM generating function is

$$Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) = Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; 1, 1, 1, 1). \quad (101)$$

By acting on $\text{ASM}(n)$ with transposition or anticlockwise quarter-turn rotation, and using (6), it follows that

$$\begin{aligned} Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; z_{41}, z_{12}, z_{23}, z_{34}) &= Z_n^{\text{gen}}(x, y; z_4, z_3, z_2, z_1; z_{41}, z_{34}, z_{23}, z_{12}) \\ &= x^{n(n-1)/2} (z_1 z_2 z_3 z_4)^{n-1} Z_n^{\text{gen}}\left(\frac{1}{x}, \frac{y}{x}; \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_1}; z_{12}, z_{23}, z_{34}, z_{41}\right). \end{aligned} \quad (102)$$

The properties of ASMs with a 1 in the top-left corner, as outlined in Section 2.1, imply that

$$\begin{aligned} Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; 0, z_{12}, z_{23}, z_{34}) &= Z_n^{\text{gen}}(x, y; 0, z_2, z_3, z_4; z_{41}, z_{12}, z_{23}, z_{34}) \\ &= z_{12} z_{34} Z_{n-1}^{\text{gen}}(x, y; 1, z_2, z_3, 1; 1, 1, z_{23}, 1), \end{aligned} \quad (103)$$

and the fact that an ASM cannot contain a 1 in both its top-left and top-right corner, implies that

$$Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; 0, 0, z_{23}, z_{34}) = 0. \quad (104)$$

It can be seen from the definition (99) that $Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; z_{41}, z_{12}, z_{23}, z_{34})$ is linear in z_{41}, z_{12}, z_{23} and z_{34} . Therefore, using interpolation for these parameters at 0 and 1, it follows that

$$\begin{aligned} Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; z_{41}, z_{12}, z_{23}, z_{34}) &= \\ \sum_{i_1, i_2, i_3, i_4 \in \{0, 1\}} (1 - z_{41})^{1-i_1} z_{41}^{i_1} (1 - z_{12})^{1-i_2} z_{12}^{i_2} (1 - z_{23})^{1-i_3} z_{23}^{i_3} (1 - z_{34})^{1-i_4} z_{34}^{i_4} \times \\ & Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; i_1, i_2, i_3, i_4). \end{aligned} \quad (105)$$

Applying cases of (9)–(10) and (101)–(104) to (105) (as a result of which, for example, (102) and (104) imply that nine of the sixteen terms on the RHS of (105) vanish, while (9) and (101)–(103) give $Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; 0, 1, 0, 1) = Z_{n-2}(x, y)$), it now follows that the

generalized ASM generating function can be expressed entirely in terms of quadruply-refined, adjacent-boundary doubly-refined and unrefined ASM generating functions as

$$\begin{aligned}
Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; z_{41}, z_{12}, z_{23}, z_{34}) &= z_{41} z_{12} z_{23} z_{34} Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4) + \\
&\quad z_{41} z_{12} (1 - z_{23}) z_{34} Z_{n-1}^{\text{adj}}(x, y; z_4, z_1) + \\
&\quad z_{41} z_{12} z_{23} (1 - z_{34}) z_1 z_2 (x z_3 z_4)^{n-1} \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_1, z_2) + \\
&\quad (1 - z_{41}) z_{12} z_{23} z_{34} Z_{n-1}^{\text{adj}}(x, y; z_2, z_3) + \\
&\quad z_{41} (1 - z_{12}) z_{23} z_{34} z_3 z_4 (x z_1 z_2)^{n-1} \tilde{Z}_{n-1}^{\text{adj}}(x, y; z_3, z_4) + \\
&\quad (1 - z_{41}) z_{12} (1 - z_{23}) z_{34} Z_{n-2}(x, y) + \\
&\quad z_{41} (1 - z_{12}) z_{23} (1 - z_{34}) x^{2n-3} (z_1 z_2 z_3 z_4)^{n-1} Z_{n-2}(x, y). \quad (106)
\end{aligned}$$

Some special cases of (106), which will be used in Section 5.6, are

$$\begin{aligned}
Z_n^{\text{gen}}(x, y; z_1, 1, 1, z_2; z_{12}, 1, 1, 1) &= Z_n^{\text{gen}}(x, y; 1, z_1, z_2, 1; 1, 1, z_{12}, 1) \\
&= z_{12} Z_n^{\text{adj}}(x, y; z_1, z_2) + (1 - z_{12}) Z_{n-1}(x, y), \\
Z_n^{\text{gen}}(x, y; z_1, z_2, 1, 1; 1, z_{12}, 1, 1) &= Z_n^{\text{gen}}(x, y; 1, 1, z_1, z_2; 1, 1, 1, z_{12}) \\
&= z_{12} \tilde{Z}_n^{\text{adj}}(x, y; z_1, z_2) + (1 - z_{12}) (x z_1 z_2)^{n-1} Z_{n-1}(x, y). \quad (107)
\end{aligned}$$

5.2. The bijection between ASMs and configurations of the six-vertex model with DWBC. In this section, the set of configurations of the six-vertex model on an $n \times n$ grid with DWBC is described, and the details of a natural bijection between this set and $\text{ASM}(n)$ are summarized. This is standard material, with similar accounts having been given, for example, by Behrend, Di Francesco and Zinn-Justin [16, Secs. 2.1 & 3.1], [17, Sec. 5.1].

The six-vertex, or square ice, model is a much-studied integrable statistical mechanical model (see, for example, Baxter [14, Chaps. 8 & 9] for further information and references), with DWBC for the model having been introduced and first studied by Korepin [93]. The bijection between ASMs and configurations of the model with DWBC was first discussed by Elkies, Kuperberg, Larsen and Propp [70, Sec. 7], with the details having mostly been observed previously, but using different terminology, by Robbins and Rumsey [135, pp. 179–180]. There exist closely-related bijections between ASMs and certain sets of osculating lattice paths (see Section 3.11 for references), and between ASMs and certain fully-packed loop configurations (see, for example, Propp [123, Sec. 7]).

Let \mathcal{G}_n be the $n \times n$ undirected grid with vertex set $\{(i, j) \mid i, j = 0, \dots, n+1\} \setminus \{(0, 0), (0, n+1), (n+1, 0), (n+1, n+1)\}$, where (i, j) is taken to be in the i th row from the top and j th column from the left, and for which there are horizontal edges between (i, j) and $(i, j \pm 1)$, and vertical edges between (i, j) and $(i \pm 1, j)$, for each $i, j = 1, \dots, n$. This grid is shown in Figure 1. Each vertex or edge of \mathcal{G}_n can be described as either internal or external, i.e., the n^2 vertices of degree 4 are internal, the remaining $4n$ vertices of degree 1 are external, the $2n(n-1)$ edges which connect two internal vertices are internal, and the remaining $4n$ edges are external.

It can be shown easily that, for any $C \in 6VDW(n)$,

$$\mathcal{N}_{(1)}(C) = \mathcal{N}_{(2)}(C), \quad \mathcal{N}_{(3)}(C) = \mathcal{N}_{(4)}(C), \quad \mathcal{N}_{(5)}(C) = \mathcal{N}_{(6)}(C) + n, \quad (109)$$

$$\mathcal{N}_{(5)}^{\text{row } 1}(C) = \mathcal{N}_{(5)}^{\text{col } n}(C) = \mathcal{N}_{(5)}^{\text{row } n}(C) = \mathcal{N}_{(5)}^{\text{col } 1}(C) = 1. \quad (110)$$

(See, for example, Bressoud [24, p. 228], [25, p. 290] for a discussion of the first two equations of (109).) It can also be seen that, for any $C \in 6VDW(n)$ and $1 \leq i, j \leq n$,

$$C_{1j} \in \{1, 3, 5\}, \quad C_{in} \in \{2, 3, 5\}, \quad C_{nj} \in \{2, 4, 5\}, \quad C_{i1} \in \{1, 4, 5\}, \quad (111)$$

and therefore that

$$C_{11} \in \{1, 5\}, \quad C_{1n} \in \{3, 5\}, \quad C_{nn} \in \{2, 5\}, \quad C_{n1} \in \{4, 5\}. \quad (112)$$

It can be shown straightforwardly that there is a natural bijection between $ASM(n)$ and $6VDW(n)$, and that, for each $A \in ASM(n)$ and $C \in 6VDW(n)$ which correspond under this bijection, the statistics (2)–(3) and (63) satisfy

$$\begin{aligned} \nu(A) &= \mathcal{N}_{(1)}(C) (= \mathcal{N}_{(2)}(C)), & \mu(A) &= \mathcal{N}_{(6)}(C) (= \mathcal{N}_{(5)}(C) - n), \\ \rho_T(A) &= \mathcal{N}_{(1)}^{\text{row } 1}(C), & \rho_R(A) &= \mathcal{N}_{(2)}^{\text{col } n}(C), & \rho_B(A) &= \mathcal{N}_{(2)}^{\text{row } n}(C), & \rho_L(A) &= \mathcal{N}_{(1)}^{\text{col } 1}(C), \\ \nu^{\text{row } i}(A) &= \mathcal{N}_{(1)}^{\text{row } i}(C) + \mathcal{N}_{(2)}^{\text{row } i}(C), & \nu^{\text{col } j}(A) &= \mathcal{N}_{(1)}^{\text{col } j}(C) + \mathcal{N}_{(2)}^{\text{col } j}(C), \\ \mu^{\text{row } i}(A) &= \mathcal{N}_{(6)}^{\text{row } i}(C), & \mu^{\text{col } j}(A) &= \mathcal{N}_{(6)}^{\text{col } j}(C). \end{aligned} \quad (113)$$

The details of this bijection are as follows. To map $A \in ASM(n)$ to $C \in 6VDW(n)$, for each $i = 1, \dots, n$ and $j = 0, \dots, n$, assign a right or left arrow to the horizontal edge of \mathcal{G}_n between (i, j) and $(i, j + 1)$, according to whether the partial row sum $\sum_{j'=1}^j A_{ij'}$ is 0 or 1, respectively. Similarly, for each $i = 0, \dots, n$ and $j = 1, \dots, n$, assign an upward or downward arrow to the vertical edge of \mathcal{G}_n between (i, j) and $(i + 1, j)$, according to whether the partial column sum $\sum_{i'=1}^i A_{i'j}$ is 0 or 1, respectively. To map $C \in 6VDW(n)$ to $A \in ASM(n)$, let

$$A_{ij} = \begin{cases} 1, & C_{ij} = 5, \\ -1, & C_{ij} = 6, \\ 0, & C_{ij} \in \{1, 2, 3, 4\}. \end{cases} \quad (114)$$

The behaviour of the statistics μ , ρ_T , ρ_R , ρ_B and ρ_L under this bijection can be seen easily. To obtain the behaviour of the statistics ν , $\nu^{\text{row } i}$ and $\nu^{\text{col } j}$, note that, for each $A \in ASM(n)$ and $C \in 6VDW(n)$ which correspond under the bijection, and each $1 \leq i, j \leq n$, $\sum_{i'=1}^{i-1} A_{i'j} = \sum_{j'=1}^j A_{ij'} = 0$ (or equivalently $\sum_{i'=1}^i A_{i'j} = \sum_{j'=1}^{j-1} A_{ij'} = 0$) if and only if $C_{ij} = 1$, and $\sum_{i'=1}^{i-1} A_{i'j} = \sum_{j'=1}^j A_{ij'} = 1$ (or equivalently $\sum_{i'=1}^i A_{i'j} = \sum_{j'=1}^{j-1} A_{ij'} = 1$) if and only if $C_{ij} = 2$.

As examples of this bijection, in (1) and (108) the elements of $ASM(3)$ and $6VDW(3)$ are listed in an order for which respective elements correspond under the bijection.

5.3. The partition function of the six-vertex model with DWBC. In this section, the partition function of the six-vertex model with DWBC is introduced. A relation between this partition function, for certain assignments of its parameters, and the generalized ASM generating function (99), for certain assignments of its parameters, is then derived using the bijection of Section 5.2.

Let a weight $W_{(k)}(u, v)$ be associated with the vertex configuration of type k , where u and v are so-called spectral parameters.

The partition function for the case of the six-vertex model of relevance here depends on these weights, and on spectral parameters u_i and v_j associated with row i and column j of \mathcal{G}_n , for each $1 \leq i, j \leq n$. Specifically, this partition function is defined as

$$Z_n^{6V}(u_1, \dots, u_n; v_1, \dots, v_n) = \sum_{C \in 6VDW(n)} \prod_{i,j=1}^n W_{(C_{ij})}(u_i, v_j). \quad (115)$$

Let the weights now satisfy

$$\begin{aligned} W_{(1)}(u, v) = W_{(2)}(u, v) = a(u, v), \quad W_{(3)}(u, v) = W_{(4)}(u, v) = b(u, v), \\ W_{(5)}(u, v) = W_{(6)}(u, v) = c(u, v), \end{aligned} \quad (116)$$

for functions a , b and c .

Taking the spectral parameters in (115) to be

$$u_2 = \dots = u_{n-1} = r, \quad v_2 = \dots = v_{n-1} = s, \quad u_1 = t_1, \quad v_n = t_2, \quad u_n = t_3, \quad v_1 = t_4, \quad (117)$$

for indeterminates r , s and t_1, \dots, t_4 , and using (109)–(112), gives

$$\begin{aligned} Z_n^{6V}(t_1, r, \dots, r, t_3; t_4, s, \dots, s, t_2) = \sum_{C \in 6VDW(n)} a(r, s)^{2\mathcal{N}_{(1)}(C)} b(r, s)^{2\mathcal{N}_{(3)}(C)} c(r, s)^{2\mathcal{N}_{(6)}(C)+n} \times \\ \left(\frac{a(t_1, s)}{a(r, s)}\right)^{\mathcal{N}_{(1)}^{\text{row } 1}(C) - \mathcal{N}_{(1)}^{11}(C)} \left(\frac{b(t_1, s)}{b(r, s)}\right)^{\mathcal{N}_{(3)}^{\text{row } 1}(C) - \mathcal{N}_{(3)}^{1n}(C)} \left(\frac{c(t_1, s)}{c(r, s)}\right)^{1 - \mathcal{N}_{(5)}^{11}(C) - \mathcal{N}_{(5)}^{1n}(C)} \times \\ \left(\frac{a(r, t_2)}{a(r, s)}\right)^{\mathcal{N}_{(2)}^{\text{col } n}(C) - \mathcal{N}_{(2)}^{nn}(C)} \left(\frac{b(r, t_2)}{b(r, s)}\right)^{\mathcal{N}_{(3)}^{\text{col } n}(C) - \mathcal{N}_{(3)}^{1n}(C)} \left(\frac{c(r, t_2)}{c(r, s)}\right)^{1 - \mathcal{N}_{(5)}^{1n}(C) - \mathcal{N}_{(5)}^{nn}(C)} \times \\ \left(\frac{a(t_3, s)}{a(r, s)}\right)^{\mathcal{N}_{(2)}^{\text{row } n}(C) - \mathcal{N}_{(2)}^{nn}(C)} \left(\frac{b(t_3, s)}{b(r, s)}\right)^{\mathcal{N}_{(4)}^{\text{row } n}(C) - \mathcal{N}_{(4)}^{n1}(C)} \left(\frac{c(t_3, s)}{c(r, s)}\right)^{1 - \mathcal{N}_{(5)}^{n1}(C) - \mathcal{N}_{(5)}^{nn}(C)} \times \\ \left(\frac{a(r, t_4)}{a(r, s)}\right)^{\mathcal{N}_{(1)}^{\text{col } 1}(C) - \mathcal{N}_{(1)}^{11}(C)} \left(\frac{b(r, t_4)}{b(r, s)}\right)^{\mathcal{N}_{(4)}^{\text{col } 1}(C) - \mathcal{N}_{(4)}^{n1}(C)} \left(\frac{c(r, t_4)}{c(r, s)}\right)^{1 - \mathcal{N}_{(5)}^{11}(C) - \mathcal{N}_{(5)}^{n1}(C)} \times \\ \left(\frac{a(t_1, t_4)}{a(r, s)}\right)^{\mathcal{N}_{(1)}^{11}(C)} \left(\frac{c(t_1, t_4)}{c(r, s)}\right)^{\mathcal{N}_{(5)}^{11}(C)} \left(\frac{b(t_1, t_2)}{b(r, s)}\right)^{\mathcal{N}_{(3)}^{1n}(C)} \left(\frac{c(t_1, t_2)}{c(r, s)}\right)^{\mathcal{N}_{(5)}^{1n}(C)} \times \\ \left(\frac{a(t_3, t_2)}{a(r, s)}\right)^{\mathcal{N}_{(2)}^{nn}(C)} \left(\frac{c(t_3, t_2)}{c(r, s)}\right)^{\mathcal{N}_{(5)}^{nn}(C)} \left(\frac{b(t_3, t_4)}{b(r, s)}\right)^{\mathcal{N}_{(4)}^{n1}(C)} \left(\frac{c(t_3, t_4)}{c(r, s)}\right)^{\mathcal{N}_{(5)}^{n1}(C)}. \quad (118) \end{aligned}$$

It can be seen that

$$\mathcal{N}_{(1)}(C) + \mathcal{N}_{(3)}(C) + \mathcal{N}_{(6)}(C) = \frac{n(n-1)}{2}, \quad (119)$$

$$\begin{aligned} \mathcal{N}_{(1)}^{\text{row } 1}(C) + \mathcal{N}_{(3)}^{\text{row } 1}(C) &= \mathcal{N}_{(2)}^{\text{col } n}(C) + \mathcal{N}_{(3)}^{\text{col } n}(C) = \\ \mathcal{N}_{(2)}^{\text{row } n}(C) + \mathcal{N}_{(4)}^{\text{row } n}(C) &= \mathcal{N}_{(1)}^{\text{col } 1}(C) + \mathcal{N}_{(4)}^{\text{col } 1}(C) = n - 1, \end{aligned} \quad (120)$$

$$\begin{aligned} \mathcal{N}_{(1)}^{11}(C) + \mathcal{N}_{(5)}^{11}(C) &= \mathcal{N}_{(3)}^{1n}(C) + \mathcal{N}_{(5)}^{1n}(C) = \\ \mathcal{N}_{(2)}^{nn}(C) + \mathcal{N}_{(5)}^{nn}(C) &= \mathcal{N}_{(4)}^{n1}(C) + \mathcal{N}_{(5)}^{n1}(C) = 1, \end{aligned} \quad (121)$$

where (119) follows from (109) and $\sum_{k=1}^6 \mathcal{N}_{(k)}(C) = n^2$, (120) follows from (110)–(111) and $\sum_{k=1}^6 \mathcal{N}_{(k)}^{\text{row } i}(C) = \sum_{k=1}^6 \mathcal{N}_{(k)}^{\text{col } j}(C) = n$, and (121) follows from (112).

By using (119)–(121) to eliminate $\mathcal{N}_{(3)}(C)$, $\mathcal{N}_{(3)}^{\text{row } 1}(C)$, $\mathcal{N}_{(3)}^{\text{col } n}(C)$, $\mathcal{N}_{(4)}^{\text{row } n}(C)$, $\mathcal{N}_{(4)}^{\text{col } 1}(C)$, $\mathcal{N}_{(5)}^{11}(C)$, $\mathcal{N}_{(5)}^{1n}(C)$, $\mathcal{N}_{(5)}^{nn}(C)$ and $\mathcal{N}_{(5)}^{n1}(C)$ from (118), and then using the bijection of Section 5.2 between $\text{ASM}(n)$ and $6\text{VDW}(n)$, the behaviour (113) of the statistics (2)–(3) under this bijection, and the definition (99) of the generalized ASM generating function, it now follows that (118) can be written as

$$\begin{aligned} Z_n^{6V}(t_1, r, \dots, r, t_3; t_4, s, \dots, s, t_2) &= \\ b(r, s)^{n(n-1)} c(r, s)^n &\left(\frac{b(t_1, s) b(r, t_2) b(t_3, s) b(r, t_4)}{b(r, s)^4} \right)^{n-1} \frac{c(t_1, t_4) c(t_1, t_2) c(t_3, t_2) c(t_3, t_4)}{c(t_1, s) c(r, t_2) c(t_3, s) c(r, t_4)} \times \\ Z_n^{\text{gen}} \left(\left(\frac{a(r, s)}{b(r, s)} \right)^2, \left(\frac{c(r, s)}{b(r, s)} \right)^2, \frac{a(t_1, s) b(r, s)}{a(r, s) b(t_1, s)}, \frac{a(r, t_2) b(r, s)}{a(r, s) b(r, t_2)}, \frac{a(t_3, s) b(r, s)}{a(r, s) b(t_3, s)}, \frac{a(r, t_4) b(r, s)}{a(r, s) b(r, t_4)}, \frac{a(r, s) a(t_1, t_4) c(t_1, s) c(r, t_4)}{a(t_1, s) a(r, t_4) c(r, s) c(t_1, t_4)}, \right. \\ &\left. \frac{b(r, s) b(t_1, t_2) c(t_1, s) c(r, t_2)}{b(t_1, s) b(r, t_2) c(r, s) c(t_1, t_2)}, \frac{a(r, s) a(t_3, t_2) c(t_3, s) c(r, t_2)}{a(t_3, s) a(r, t_2) c(r, s) c(t_3, t_2)}, \frac{b(r, s) b(t_3, t_4) c(t_3, s) c(r, t_4)}{b(t_3, s) b(r, t_4) c(r, s) c(t_3, t_4)} \right). \end{aligned} \quad (122)$$

5.4. The Izergin–Korepin determinant formula. In this section, the Izergin–Korepin formula for the partition function (115), with certain assignments of the weights (116), is stated.

It was found by Izergin [89, Eq. (5)], using certain results of Korepin [93], that if the weights (116) satisfy the Yang–Baxter equation (see, for example, Baxter [14, pp. 187–189]), then the partition function (115) is given by an explicit formula involving the determinant of a certain $n \times n$ matrix.

Let the functions a , b and c in (116) be given by

$$\begin{aligned} a(u, v) &= u q^{1/2} - v q^{-1/2}, \\ b(u, v) &= v q^{1/2} - u q^{-1/2} = a(v, u), \\ c(u, v) &= (q - q^{-1}) u^{1/2} v^{1/2}, \end{aligned} \quad (123)$$

where q is a further indeterminate, often known as the crossing parameter. (Note that q will be present in many subsequent equations, even though it may not appear explicitly.)

It can be seen that these functions satisfy

$$\frac{a(u, v)^2 + b(u, v)^2 - c(u, v)^2}{a(u, v) b(u, v)} = -(q + q^{-1}). \quad (124)$$

The fact that the LHS of (124) (which is often denoted as 2Δ) is independent of u and v implies that the six-vertex model weights given by (116) and (123) satisfy the Yang–Baxter equation (see, for example, Baxter [14, Eq. (9.6.14)]).

The resulting Izergin–Korepin determinant formula is then given by the following result.

Theorem (Izergin). *The partition function (115), with weights given by (116) and (123), satisfies*

$$Z_n^{6V}(u_1, \dots, u_n; v_1, \dots, v_n) = \frac{\prod_{i,j=1}^n a(u_i, v_j) b(u_i, v_j)}{\prod_{1 \leq i < j \leq n} (u_i - u_j)(v_j - v_i)} \det_{1 \leq i, j \leq n} \left(\frac{c(u_i, v_j)}{a(u_i, v_j) b(u_i, v_j)} \right). \quad (125)$$

This theorem can be proved by showing that each side of (125) satisfies, and is uniquely determined by, four particular properties. Specifically, each side is symmetric in u_1, \dots, u_n and in v_1, \dots, v_n (which can be obtained for the LHS using the Yang–Baxter equation, and is immediate for the RHS), is a polynomial of degree $2n - 1$ in each of $u_1^{1/2}, \dots, u_n^{1/2}, v_1^{1/2}, \dots, v_n^{1/2}$, satisfies the same recursion relation for cases in which $u_i = q^{\pm 1} v_j$ for some i and j , and satisfies the same initial condition at $n = 1$. For further details of this proof, see, for example, Izergin, Coker and Korepin [90, Sec. 5], Fonseca and Zinn-Justin [81, Sec. B.1], Kuperberg [100, Sec. 1], or Zinn-Justin [161, Secs. 2.5.2–2.5.3]. For an alternative proof, see Bogoliubov, Pronko and Zvonarev [21, Sec. 4].

Note that, although the determinant and the denominator of the prefactor on the RHS of (125) both vanish if $u_i = u_j$ or $v_i = v_j$ for some $i \neq j$, the RHS has a well-defined limit in these cases, as a polynomial in $u_1^{1/2}, \dots, u_n^{1/2}, v_1^{1/2}, \dots, v_n^{1/2}$. Accordingly, it will be valid to use (125) to derive properties of the partition function for the assignments (117), as will be done in Section 5.6.

5.5. The Desnanot–Jacobi identity. In this section, the Desnanot–Jacobi determinant identity is stated, and discussed briefly.

For a square matrix M , let M_{TL} , M_{TR} , M_{BR} , M_{BL} and M_{C} be the submatrices of M defined in Section 3.2. The Desnanot–Jacobi identity, in the form which will be used here, can be stated as follows.

Theorem (Desnanot, Jacobi). *For any square matrix M ,*

$$\det M \det M_{\text{C}} = \det M_{\text{TL}} \det M_{\text{BR}} - \det M_{\text{TR}} \det M_{\text{BL}}. \quad (126)$$

For proofs of (126) using various methods, see, for example, Bressoud [24, Sec. 3.5], Fulmek [82, Sec. 5.1], Fulmek and Kleber [83], or Zeilberger [159]. Cases of (126) for $n \times n$ matrices M with small values of n were published by Desnanot in 1819 (see Muir [118, Eqs. (A)–(G), (A′)–(G′), pp. 139–142]). The further attribution to Jacobi is based on the fact that, for arbitrary n , (126) corresponds to the case $m = 2$ of the identity, published by Jacobi in 1834 (see Muir [118, Eq. (XX. 4), p. 208]), that for any $n \times n$ matrix M and any $m \leq n$, each $m \times m$ minor of the matrix of $(n - 1) \times (n - 1)$ minors of M equals the complementary minor of M multiplied by $(\det M)^{m-1}$. For proofs of the Jacobi identity using various methods, see, for example, Brualdi and Schneider [29, Sec. 4], Knuth [92,

Eq. (3.16)], Leclerc [110, Sec. 3.2], Muir [119, Sec. 175], or Turnbull [153, pp. 77–79]. Some closely related identities will be given in (153) and (154).

For a discussion of alternative forms of (126), and of further determinant identities of which (126) is a special case, see, for example, Behrend, Di Francesco and Zinn-Justin [17, Sec. 4].

5.6. Proof of Theorem 1. In this section, the main steps in the proof of Theorem 1 are given. These involve using the relation (122) between the generalized ASM generating function and the partition function of the six-vertex model with DWBC, the Izergin–Korepin formula (125), and the Desnanot–Jacobi identity (126).

By applying the Desnanot–Jacobi identity (126) to the matrix $\left(\frac{c(u_i, v_j)}{a(u_i, v_j) b(u_i, v_j)}\right)_{1 \leq i, j \leq n}$, and then applying the Izergin–Korepin formula (125) to each of the six determinants which appear, it follows that the partition function (115), with weights given by (116) and (123), satisfies

$$\begin{aligned} (u_1 - u_n)(v_n - v_1) Z_n^{6V}(u_1, \dots, u_n; v_1, \dots, v_n) Z_{n-2}^{6V}(u_2, \dots, u_{n-1}; v_2, \dots, v_{n-1}) = \\ a(u_1, v_n) b(u_1, v_n) a(u_n, v_1) b(u_n, v_1) Z_{n-1}^{6V}(u_1, \dots, u_{n-1}; v_1, \dots, v_{n-1}) \times \\ Z_{n-1}^{6V}(u_2, \dots, u_n; v_2, \dots, v_n) \\ - a(u_1, v_1) b(u_1, v_1) a(u_n, v_n) b(u_n, v_n) Z_{n-1}^{6V}(u_1, \dots, u_{n-1}; v_2, \dots, v_n) \times \\ Z_{n-1}^{6V}(u_2, \dots, u_n; v_1, \dots, v_{n-1}), \end{aligned} \quad (127)$$

for any $u_1, \dots, u_n, v_1, \dots, v_n$. In fact, it can be seen, from the previous derivation, that (127) is satisfied by any function which has the form of (125), for arbitrary functions a , b and c .

Certain forms of the Desnanot–Jacobi identity have previously been combined with certain cases of the Izergin–Korepin formula by Korepin and Zinn-Justin [94, Sec. 3] (see also Sogo [139, Sec. 4]), and by Behrend, Di Francesco and Zinn-Justin [17, Sec. 5.4].

Now let parameters x , y , z_1 , z_2 , z_3 , z_4 , z_{41} , z_{12} , z_{23} , z_{34} , z'_{41} , z'_{12} , z'_{23} and z'_{34} be given in terms of parameters q , r , s , t_1 , t_2 , t_3 and t_4 by

$$\begin{aligned} x &= \left(\frac{a(r,s)}{b(r,s)}\right)^2, & y &= \left(\frac{c(r,s)}{b(r,s)}\right)^2, \\ z_1 &= \frac{a(t_1,s)b(r,s)}{a(r,s)b(t_1,s)}, & z_2 &= \frac{a(r,t_2)b(r,s)}{a(r,s)b(r,t_2)}, & z_3 &= \frac{a(t_3,s)b(r,s)}{a(r,s)b(t_3,s)}, & z_4 &= \frac{a(r,t_4)b(r,s)}{a(r,s)b(r,t_4)}, \\ z_{41} &= \frac{a(r,s)a(t_1,t_4)}{a(t_1,s)a(r,t_4)}, & z_{12} &= \frac{b(r,s)b(t_1,t_2)}{b(t_1,s)b(r,t_2)}, & z_{23} &= \frac{a(r,s)a(t_3,t_2)}{a(t_3,s)a(r,t_2)}, & z_{34} &= \frac{b(r,s)b(t_3,t_4)}{b(t_3,s)b(r,t_4)}, \\ z'_{41} &= \frac{b(r,s)b(t_1,t_4)}{b(t_1,s)b(r,t_4)}, & z'_{12} &= \frac{a(r,s)a(t_1,t_2)}{a(t_1,s)a(r,t_2)}, & z'_{23} &= \frac{b(r,s)b(t_3,t_2)}{b(t_3,s)b(r,t_2)}, & z'_{34} &= \frac{a(r,s)a(t_3,t_4)}{a(t_3,s)a(r,t_4)}, \end{aligned} \quad (128)$$

where the functions a , b and c are given by (123).

The parameterizations (128) allow arbitrary x , y and z_1, \dots, z_4 to be expressed in terms of q , r , s and t_1, \dots, t_4 . In particular, it follows, using (123), (124) and (128), that q can first be taken to be a solution of

$$x^{1/2} q^2 + (x - y + 1) q + x^{1/2} = 0, \quad (129)$$

that r and s can then be taken to satisfy

$$(x^{1/2} + q)r = (x^{1/2}q + 1)s, \quad (130)$$

and that t_1, \dots, t_4 are then given by

$$t_1 = \frac{s(x^{1/2}z_1q+1)}{x^{1/2}z_1+q}, \quad t_2 = \frac{r(x^{1/2}z_2+q)}{x^{1/2}z_2q+1}, \quad t_3 = \frac{s(x^{1/2}z_3q+1)}{x^{1/2}z_3+q}, \quad t_4 = \frac{r(x^{1/2}z_4+q)}{x^{1/2}z_4q+1}. \quad (131)$$

It can be checked easily that the parameterizations (128) imply that $z_{41}, z_{12}, z_{23}, z_{34}, z'_{41}, z'_{12}, z'_{23}$ and z'_{34} can be expressed in terms of x, y and z_1, \dots, z_4 as

$$z_{41} = 1 - \frac{(z_4-1)(z_1-1)}{y z_4 z_1}, \quad z_{12} = 1 - \frac{x(z_1-1)(z_2-1)}{y}, \quad z_{23} = 1 - \frac{(z_2-1)(z_3-1)}{y z_2 z_3}, \quad z_{34} = 1 - \frac{x(z_3-1)(z_4-1)}{y}, \quad (132)$$

and

$$z'_{41} = 1 - \frac{x(z_4-1)(z_1-1)}{y}, \quad z'_{12} = 1 - \frac{(z_1-1)(z_2-1)}{y z_1 z_2}, \quad z'_{23} = 1 - \frac{x(z_2-1)(z_3-1)}{y}, \quad z'_{34} = 1 - \frac{(z_3-1)(z_4-1)}{y z_3 z_4}. \quad (133)$$

Thus, each z_{ij} and z'_{ij} has the form $f(y, z_i, z_j) = 1 - \frac{(z_i-1)(z_j-1)}{y z_i z_j}$ or $f(\frac{y}{x}, \frac{1}{z_i}, \frac{1}{z_j})$.

The parameterizations (128) also give

$$(t_2 - t_4)(t_1 - t_3) = \frac{a(r,s)^2 b(t_1,s) b(r,t_2) b(t_3,s) b(r,t_4)}{(b(r,s) c(r,s))^2} (z_4 - z_2)(z_1 - z_3). \quad (134)$$

Assigning the parameters in (127) according to (117), applying the relation (122) to each of the six cases of the partition function in (127), and using (128) and (134), it now follows that

$$\begin{aligned} (z_4 - z_2)(z_1 - z_3) Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; z_{41}, z_{12}, z_{23}, z_{34}) Z_{n-2}^{\text{gen}}(x, y; 1, 1, 1, 1; 1, 1, 1, 1) = \\ y z_1 z_2 z_3 z_4 (z_{12} z'_{12} z_{34} z'_{34} Z_{n-1}^{\text{gen}}(x, y; z_1, 1, 1, z_4; z_{41}, 1, 1, 1) \times \\ Z_{n-1}^{\text{gen}}(x, y; 1, z_2, z_3, 1; 1, 1, z_{23}, 1) \\ - z_{41} z'_{41} z_{23} z'_{23} Z_{n-1}^{\text{gen}}(x, y; z_1, z_2, 1, 1; 1, z_{12}, 1, 1) \times \\ Z_{n-1}^{\text{gen}}(x, y; 1, 1, z_3, z_4; 1, 1, 1, z_{34})), \quad (135) \end{aligned}$$

where x, y and z_1, \dots, z_4 are arbitrary, while the remaining parameters are given by (132)–(133).

Finally, the required identity (81) follows from (135) by applying (106), including the special cases (107), to each of the six cases of generalized ASM generating functions, and using (132)–(133) to eliminate $z_{41}, z_{12}, z_{23}, z_{34}, z'_{41}, z'_{12}, z'_{23}$ and z'_{34} .

5.7. Alternative statement of some results of Sections 4.1 and 4.2. In this section, it is shown that some of the results of Sections 4.1 and 4.2 can be stated, slightly more compactly, in terms of a certain function which will be defined in (136) (and which has essentially already appeared in (135)). Also, an additional symmetry property for ASM generating functions is obtained, and further quadratic relations satisfied by these functions are discussed briefly.

Define a function

$$\begin{aligned}
Y_n(x, y; z_1, z_2, z_3, z_4) = & \sum_{A \in \text{ASM}(n)} x^{\nu(A)} y^{\mu(A) + A_{11} + A_{1n} + A_{nn} + A_{n1}} \times \\
& z_1^{\rho_T(A) + A_{11} - 1} z_2^{\rho_R(A) + A_{nn} - 1} z_3^{\rho_B(A) + A_{nn} - 1} z_4^{\rho_L(A) + A_{11} - 1} \times \\
& (yz_4z_1 - (z_4 - 1)(z_1 - 1))^{1 - A_{11}} (y - x(z_1 - 1)(z_2 - 1))^{1 - A_{1n}} \times \\
& (yz_2z_3 - (z_2 - 1)(z_3 - 1))^{1 - A_{nn}} (y - x(z_3 - 1)(z_4 - 1))^{1 - A_{n1}}. \quad (136)
\end{aligned}$$

It follows that $Y_n(x, y; z_1, z_2, z_3, z_4)$ is a polynomial in x, y and z_1, \dots, z_4 , with integer coefficients.

Also define

$$Y_n(x, y; z_1, z_2) = Y_n(x, y; z_1, z_2, 1, 1). \quad (137)$$

It can be seen, using the definition (99) of the generalized ASM generating function, that

$$\begin{aligned}
Y_n(x, y; z_1, z_2, z_3, z_4) = & y^4 Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; \\
& 1 - \frac{(z_4 - 1)(z_1 - 1)}{y z_4 z_1}, 1 - \frac{x(z_1 - 1)(z_2 - 1)}{y}, 1 - \frac{(z_2 - 1)(z_3 - 1)}{y z_2 z_3}, 1 - \frac{x(z_3 - 1)(z_4 - 1)}{y}), \quad (138)
\end{aligned}$$

i.e., $Y_n(x, y; z_1, z_2, z_3, z_4) = y^4 Z_n^{\text{gen}}(x, y; z_1, z_2, z_3, z_4; z_{41}, z_{12}, z_{23}, z_{34})$, with z_{41}, z_{12}, z_{23} and z_{34} given by (132).

Therefore, by applying (106) to the RHS of (138), $Y_n(x, y; z_1, z_2, z_3, z_4)$ can be expressed in terms of quadruply-refined, adjacent-boundary doubly-refined and unrefined ASM generating functions.

It can be seen immediately from (136)–(137) that

$$\begin{aligned}
Y_n(x, y; z_1, 1, z_2, 1) &= y^4 Z_n^{\text{opp}}(x, y; z_1, z_2), \\
Y_n(x, y; z, 1) &= y^4 Z_n(x, y; z), \\
Y_n(x, y; 1, 1) &= y^4 Z_n(x, y), \quad (139)
\end{aligned}$$

and by acting on $\text{ASM}(n)$ in (136) with transposition or anticlockwise quarter-turn rotation and using (6) (or by applying (102) to (138)), it follows that

$$\begin{aligned}
Y_n(x, y; z_1, z_2, z_3, z_4) &= Y_n(x, y; z_4, z_3, z_2, z_1) \\
&= x^{n(n-1)/2+4} (z_1 z_2 z_3 z_4)^{n-1} Y_n\left(\frac{1}{x}, \frac{y}{x}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_1}\right). \quad (140)
\end{aligned}$$

It can also be shown, using (136) and the properties of ASMs in which a 1 on a boundary is in a corner or separated from a corner by a single zero (as discussed in Section 2.1), that

$$Y_n(x, y; z_1, z_2, z_3, 0) = \left(1 + \frac{x(z_1 - 1)}{y}\right) \left(1 + \frac{x(z_3 - 1)}{y}\right) Y_{n-1}(x, y; z_1, z_2, z_3, 1). \quad (141)$$

Taking $x, y, z_1, \dots, z_4, z_{41}, z_{12}, z_{23}$ and z_{34} to be parameterized by (128), and using (122), (132) and (138), gives

$$b(r, s)^{(n-1)(n-4)+8} c(r, s)^{n-8} (b(t_1, s) b(r, t_2) b(t_3, s) b(r, t_4))^{n-1} t_1^{1/2} t_2^{1/2} t_3^{1/2} t_4^{1/2} (rs)^{-1} \times \\ Y_n \left(\left(\frac{a(r, s)}{b(r, s)} \right)^2, \left(\frac{c(r, s)}{b(r, s)} \right)^2, \frac{a(t_1, s) b(r, s)}{a(r, s) b(t_1, s)}, \frac{a(r, t_2) b(r, s)}{a(r, s) b(r, t_2)}, \frac{a(t_3, s) b(r, s)}{a(r, s) b(t_3, s)}, \frac{a(r, t_4) b(r, s)}{a(r, s) b(r, t_4)} \right) = \\ Z_n^{6V}(t_1, r, \dots, r, t_3; t_4, s, \dots, s, t_2), \quad (142)$$

where the weights in the partition function are given by (116), and the functions a, b and c are given by (123).

It now follows from the symmetry of the partition function on the RHS of (142) in t_1 and t_3 , and in t_2 and t_4 , as discussed after (125), that

$$Y_n(x, y; z_1, z_2, z_3, z_4) = Y_n(x, y; z_3, z_2, z_1, z_4) = Y_n(x, y; z_1, z_4, z_3, z_2). \quad (143)$$

In contrast to (140), there does not seem to be a simple combinatorial derivation of (143). (Note that the first equality of (140) together with either equality of (143) gives the other equality of (143).)

It can be seen that (143), together with (106), provides an identity (referred to after (92)) involving $Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4)$ and $Z_n^{\text{quad}}(x, y; z_3, z_2, z_1, z_4)$ (or $Z_n^{\text{quad}}(x, y; z_1, z_2, z_3, z_4)$ and $Z_n^{\text{quad}}(x, y; z_1, z_4, z_3, z_2)$).

It follows from (135) and (138), together with (132)–(133), (137) and (139)–(140), that

$$y^7(z_4 - z_2)(z_1 - z_3) Y_n(x, y; z_1, z_2, z_3, z_4) Z_{n-2}(x, y) = \\ (x(z_1 - 1)(z_2 - 1) - y) ((z_1 - 1)(z_2 - 1) - y z_1 z_2) (x(z_3 - 1)(z_4 - 1) - y) \times \\ ((z_3 - 1)(z_4 - 1) - y z_3 z_4) Y_{n-1}(x, y; z_1, z_4) Y_{n-1}(x, y; z_2, z_3) - \\ (x(z_4 - 1)(z_1 - 1) - y) ((z_4 - 1)(z_1 - 1) - y z_4 z_1) (x(z_2 - 1)(z_3 - 1) - y) \times \\ ((z_2 - 1)(z_3 - 1) - y z_2 z_3) Y_{n-1}(x, y; z_1, z_2) Y_{n-1}(x, y; z_3, z_4). \quad (144)$$

This can be regarded as a restatement of (81), where (106) and (138) enable the conversion between (81) and (144).

Setting $z_2 = 1$ in (144), and relabelling z_3 as z_2 and z_4 as z_3 , gives

$$y(z_1 - z_2)(z_3 - 1) Y_n(x, y; z_1, 1, z_2, z_3) Z_{n-2}(x, y) = \\ (x(z_2 - 1)(z_3 - 1) - y) ((z_2 - 1)(z_3 - 1) - y z_2 z_3) z_1 Y_{n-1}(x, y; z_1, z_3) Z_{n-1}(x, y; z_2) - \\ (x(z_3 - 1)(z_1 - 1) - y) ((z_3 - 1)(z_1 - 1) - y z_3 z_1) z_2 Y_{n-1}(x, y; z_2, z_3) Z_{n-1}(x, y; z_1), \quad (145)$$

which can be regarded as a restatement of (83).

Setting $z_3 = 0$ in (144), using (141), replacing n by $n + 1$, and relabelling z_3 as z_2 and z_4 as z_3 gives

$$\begin{aligned} y^2(z_1 - z_2) z_3 Y_n(x, y; z_1, 1, z_2, z_3) Z_{n-1}(x, y) = \\ (z_1 - 1)(x(z_2 - 1)(z_3 - 1) - y)((z_2 - 1)(z_3 - 1) - y z_2 z_3) Y_n(x, y; z_1, z_3) Z_{n-1}(x, y; z_2) - \\ (z_2 - 1)(x(z_1 - 1)(z_3 - 1) - y)((z_1 - 1)(z_3 - 1) - y z_1 z_3) Y_n(x, y; z_2, z_3) Z_{n-1}(x, y; z_1), \end{aligned} \quad (146)$$

which can be regarded as a restatement of (86).

It can be seen that various other identities from Section 4 involving quadruply-refined, triply-refined or adjacent-boundary doubly-refined ASM generating functions could similarly be restated in terms of the functions (136)–(137).

Finally, note that some further quadratic identities satisfied by ASM generating functions can be obtained from (144), or by combining certain variations of the Desnanot–Jacobi identity with the Izergin–Korepin formula, for certain assignments of the spectral parameters, and then using (142).

An example of such an identity is

$$\begin{aligned} (z_1 - z_2)(z_3 - z_4) Y_n(x, y; z_1, w_1, z_2, w_2) Y_n(x, y; z_3, w_1, z_4, w_2) - \\ (z_1 - z_3)(z_2 - z_4) Y_n(x, y; z_1, w_1, z_3, w_2) Y_n(x, y; z_2, w_1, z_4, w_2) + \\ (z_1 - z_4)(z_2 - z_3) Y_n(x, y; z_1, w_1, z_4, w_2) Y_n(x, y; z_2, w_1, z_3, w_2) = 0. \end{aligned} \quad (147)$$

This reduces to (61) for $w_1 = w_2 = 1$, and can be obtained by applying (144) to each of the six cases of $(z_i - z_j)Y_n(x, y; z_i, w_1, z_j, w_2)$ on the LHS, and then checking that the overall expression on the LHS vanishes.

5.8. Derivations of (19) and (20). In this section, the explicit expressions (19) and (20) for the ASM generating functions at $y = x + 1$ are derived using a method different from that used in Section 3.2, which instead involves the six-vertex model with DWBC.

Let the crossing parameter q , for the weights (123), be given by $q = \pm i$, this corresponding to the so-called free fermion case of the six-vertex model. It can be shown that, for this assignment of q , the partition function (115), with weights given by (116) and (123), is explicitly

$$Z_n^{6V}(u_1, \dots, u_n; v_1, \dots, v_n) \Big|_{q=\pm i} = (\pm 2)^n i^{n^2} \prod_{i=1}^n u_i^{1/2} v_i^{1/2} \prod_{1 \leq i < j \leq n} (u_i + u_j)(v_i + v_j). \quad (148)$$

This result can be obtained by combining the Izergin–Korepin formula (125) (in which $q = \pm i$ gives $a(u, v)b(u, v) = -(u^2 + v^2)$) with the Cauchy double alternant evaluation

$$\det_{1 \leq i, j \leq n} \left(\frac{1}{\alpha_i + \beta_j} \right) = \frac{\prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)(\beta_i - \beta_j)}{\prod_{1 \leq i, j \leq n} (\alpha_i + \beta_j)}. \quad (149)$$

For previous appearances of (148), see, for example, Bogoliubov, Pronko and Zvonarev [21, Eq. (58)], or Okada [121, Thm. 2.4(1), third eq.], and for information regarding (149), see, for example, Muir [118, p. 345], [119, Sec. 353].

Assigning the spectral parameters in (148) according to (117), applying the relation (122), and using the first ten parameterizations of (128), which enable arbitrary x and z_1, \dots, z_4 to be obtained, with $y, z_{41}, z_{12}, z_{23}$ and z_{34} then being given by $y = x + 1$ and (132), it follows that

$$\begin{aligned} Z_n^{\text{gen}}(x, x+1; z_1, z_2, z_3, z_4; 1 - \frac{(z_4-1)(z_1-1)}{(x+1)z_4z_1}, 1 - \frac{x(z_1-1)(z_2-1)}{x+1}, 1 - \frac{(z_2-1)(z_3-1)}{(x+1)z_2z_3}, 1 - \frac{x(z_3-1)(z_4-1)}{x+1}) \\ = (xz_1z_3+1)(xz_2z_4+1)((xz_1+1)(xz_2+1)(xz_3+1)(xz_4+1))^{n-2} \times \\ (x+1)^{(n-2)(n-3)/2-2n+3}. \end{aligned} \quad (150)$$

(Using (138), the function in (150) could be written as $(x+1)^{-4}Y_n(x, x+1; z_1, z_2, z_3, z_4)$.)

The required expressions (19)–(20) now follow from (150) by applying (106)–(107), and, in some cases, setting boundary parameters to 1.

5.9. Derivation of (73). In this section, a derivation is given of the identity (73) satisfied by the generating function (67) associated with several rows (or several columns) of ASMs.

For $0 \leq m \leq n$ and $1 \leq k_1 < \dots < k_m \leq n$, and indeterminates r, s and t_1, \dots, t_m , consider the partition function $Z_n^{6V}(r, \dots, r, t_1, r, \dots, r, t_m, r, \dots, r; s, \dots, s)$, as given by (115)–(116), with arbitrary functions a, b and c , where t_i appears in position k_i within $r, \dots, r, t_1, r, \dots, r, t_m, r, \dots, r$. Using the bijection of Section 5.2 between $\text{ASM}(n)$ and $6\text{VDW}(n)$, and the behaviour (113) of the relevant statistics under this bijection, it can be checked (by applying a process similar to that used in Section 5.2 for the derivation of (122)) that this partition function is related to the ASM generating function of (67) by

$$\begin{aligned} Z_n^{6V}(r, \dots, r, t_1, r, \dots, r, t_m, r, \dots, r; s, \dots, s) = \\ b(r, s)^{(n-1)(n-m)} c(r, s)^{n-m} \prod_{i=1}^m b(t_i, s)^{n-1} c(t_i, s) \times \\ Z_n^{k_1, \dots, k_m} \left(\left(\frac{a(r, s)}{b(r, s)} \right)^2, \left(\frac{c(r, s)}{b(r, s)} \right)^2, \frac{a(t_1, s)b(r, s)}{a(r, s)b(t_1, s)}, \dots, \frac{a(t_m, s)b(r, s)}{a(r, s)b(t_m, s)}, \left(\frac{c(t_1, s)}{b(t_1, s)} \right)^2, \dots, \left(\frac{c(t_m, s)}{b(t_m, s)} \right)^2 \right). \end{aligned} \quad (151)$$

(Note that the case of (122) obtained by setting $t_2 = t_4 = s$, and then relabelling t_3 as t_2 , matches the case $m = 2$, $k_1 = 1$ and $k_2 = n$ of (151).)

It follows from a result of Colomo and Pronko [50, Eq. (6.8)], [51, Eq. (A.13)] (with a special case stated previously by Colomo and Pronko [49, Eq. (5.8)]) that if the functions a, b and c are given by (123), then, for $1 \leq m \leq n$,

$$\begin{aligned} (a(r, s) b(r, s))^{m(m-1)/2} \prod_{1 \leq i < j \leq m} (t_i - t_j) \times \\ Z_n^{6V}(r, \dots, r, t_1, r, \dots, r, t_m, r, \dots, r; s, \dots, s) \prod_{i=1}^{m-1} Z_{n-i}^{6V}(r, \dots, r; s, \dots, s) \\ = \det_{1 \leq i, j \leq m} \left((a(t_i, s) b(t_i, s))^{j-1} (t_i - r)^{m-j} Z_{n-j+1}^{6V}(t_i, r, \dots, r; s, \dots, s) \right). \end{aligned} \quad (152)$$

Due to the symmetry of the partition function in each set of spectral parameters (as discussed after (125)), the positions of t_1, \dots, t_m within $r, \dots, r, t_1, r, \dots, r, t_m, r, \dots, r$ on the LHS, and within t_i, r, \dots, r on the RHS, are immaterial in (152). A result related to (152), in

the case $q = e^{\pm 2\pi i/3}$ (where q is the crossing parameter which appears in (123)), has been obtained by Ayer and Romik [12, Thm. 6 & App. A].

An alternative derivation of (152) (in which a general identity for minors of a matrix is combined with the Izergin–Korepin formula (125), analogously to the derivation of (127)) will now be given, before continuing with the main derivation of (73).

It can be established that, for any $1 \leq m \leq n$, any $(n + m - 1) \times n$ matrix M , any $1 \leq k_1 < \dots < k_m < l_1 < \dots < l_{m-1} \leq n + m - 1$, and any $1 \leq p_1 < \dots < p_{m-1} \leq n$,

$$\det M^{l_1, \dots, l_{m-1}} \prod_{i=1}^{m-1} \det M_{p_1, \dots, p_i}^{k_1, \dots, k_m, l_1, \dots, l_{i-1}} = \det_{1 \leq i, j \leq m} \left(\det M_{p_1, \dots, p_{j-1}}^{k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_m, l_1, \dots, l_{j-1}} \right), \quad (153)$$

where $M_{j_1, \dots, j_c}^{i_1, \dots, i_r}$ denotes the submatrix of M in which rows i_1, \dots, i_r and columns j_1, \dots, j_c have been deleted. The identity (153) is closely related to various identities for compound determinants, as outlined, for example, by Leclerc [110, Sec. 3]. In particular, using a certain polarization of Bazin’s theorem, as given by Leclerc [110, Prop. 3.4], it follows that, for any $1 \leq m \leq n$, any $(n + 2m - 2) \times n$ matrix N , and any $1 \leq k_1 < \dots < k_m < l_1 < \dots < l_{m-1} < q_1 < \dots < q_{m-1} \leq n + 2m - 2$,

$$\det N^{l_1, \dots, l_{m-1}, q_1, \dots, q_{m-1}} \prod_{i=1}^{m-1} \det N^{k_1, \dots, k_m, l_1, \dots, l_{i-1}, q_{i+1}, \dots, q_{m-1}} = \det_{1 \leq i, j \leq m} \left(\det N^{k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_m, l_1, \dots, l_{j-1}, q_j, \dots, q_{m-1}} \right), \quad (154)$$

where the same notation as previously is used for the deletion of rows from a matrix. The identity (153) can be obtained from (154) by appending $m - 1$ rows to the bottom of an $(n + m - 1) \times n$ matrix M to form a matrix N , where the i th appended row (i.e., row $n + m - 1 + i$ of N) contains a 1 in column p_i and 0’s elsewhere, and then applying (154) to N , choosing $q_i = n + m - 1 + i$.

By applying (153) to the matrix $\left(\frac{c(u_i, v_j)}{a(u_i, v_j)b(u_i, v_j)} \right)_{1 \leq i \leq n+m-1; 1 \leq j \leq n}$, and then applying the Izergin–Korepin formula (125) to each of the minors which appear, it follows that, for any $1 \leq m \leq n$, any $1 \leq k_1 < \dots < k_m < l_1 < \dots < l_{m-1} \leq n + m - 1$, any $1 \leq p_1 < \dots < p_{m-1} \leq n$, and indeterminates u_1, \dots, u_{n+m-1} and v_1, \dots, v_n , the partition function (115), with weights given by (116) and (123), satisfies

$$\begin{aligned} & \prod_{1 \leq i \leq j \leq m-1} a(u_{l_j}, v_{p_i}) b(u_{l_j}, v_{p_i}) \prod_{1 \leq i < j \leq m} (u_{k_i} - u_{k_j}) \times \\ & Z_n^{6V}(u^{l_1, \dots, l_{m-1}}; v_1, \dots, v_n) \prod_{i=1}^{m-1} Z_{n-i}^{6V}(u^{k_1, \dots, k_m, l_1, \dots, l_{i-1}}; v^{p_1, \dots, p_i}) = \\ & \det_{1 \leq i, j \leq m} \left(\prod_{j'=1}^{j-1} a(u_{k_i}, v_{p_{j'}}) b(u_{k_i}, v_{p_{j'}}) \prod_{j'=j}^{m-1} (u_{k_i} - u_{l_{j'}}) \times \right. \\ & \left. Z_{n-j+1}^{6V}(u^{k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_m, l_1, \dots, l_{j-1}}; v^{p_1, \dots, p_{j-1}}) \right), \quad (155) \end{aligned}$$

where u^{i_1, \dots, i_r} and v^{j_1, \dots, j_c} denote the subsequences of u_1, \dots, u_{n+m-1} and v_1, \dots, v_n in which u_{i_1}, \dots, u_{i_r} and v_{j_1}, \dots, v_{j_c} have been deleted. The identity (152) can now be obtained from (155) by considering $1 \leq k_1 < \dots < k_m \leq n$, choosing arbitrary $1 \leq p_1 < \dots < p_{m-1} \leq n$, and setting $l_i = n + i$, $u_i = r$ for $i \notin \{k_1, \dots, k_m\}$, $u_{k_i} = t_i$, and $v_j = s$ (and also using

the symmetry of the partition function to reassign positions of t_i in the partition functions in the determinant on the RHS).

It can be seen from the previous derivation that (152) and (155) are, in fact, satisfied by any function which has the form of (125), for arbitrary functions a , b and c .

The main derivation of (73) will now be completed. Using (151) to express each case of a partition function in (152) as a case of the ASM generating function (67), noting (as already done after (68)) that $Z_n^{k_1, \dots, k_m}(x, y; z_1, \dots, z_m; w_1, \dots, w_m)$ is $Z_n(x, y)$ for $m = 0$, or $Z_n(x, y; z_1)$ for $m = 1$ and $k_1 = 1$, and then applying the definition (70), gives

$$Z_n^{k_1, \dots, k_m} \left(\left(\frac{a(r,s)}{b(r,s)} \right)^2, \left(\frac{c(r,s)}{b(r,s)} \right)^2, \frac{a(t_1,s)b(r,s)}{a(r,s)b(t_1,s)}, \dots, \frac{a(t_m,s)b(r,s)}{a(r,s)b(t_m,s)}, \left(\frac{c(t_1,s)}{b(t_1,s)} \right)^2, \dots, \left(\frac{c(t_m,s)}{b(t_m,s)} \right)^2 \right) = X_n \left(\left(\frac{a(r,s)}{b(r,s)} \right)^2, \left(\frac{c(r,s)}{b(r,s)} \right)^2, \frac{a(t_1,s)b(r,s)}{a(r,s)b(t_1,s)}, \dots, \frac{a(t_m,s)b(r,s)}{a(r,s)b(t_m,s)} \right). \quad (156)$$

The required result (73) now follows immediately from (156) by parameterizing x , y , z_1, \dots, z_m and w_1, \dots, w_m in terms of q , r , s and t_1, \dots, t_m as $x = \left(\frac{a(r,s)}{b(r,s)} \right)^2$, $y = \left(\frac{c(r,s)}{b(r,s)} \right)^2$, $z_i = \frac{a(t_i,s)b(r,s)}{a(r,s)b(t_i,s)}$ and $w_i = \left(\frac{c(t_i,s)}{b(t_i,s)} \right)^2$, and observing that this parameterization enables arbitrary x , y and z_1, \dots, z_m to be obtained, with w_1, \dots, w_m then being given by $w_i = xz_i^2 + (y-x-1)z_i + 1$.

5.10. Derivations of (49) and (75). In this section, an expression for the partition function of the six-vertex model with DWBC at a certain value of its crossing parameter is given. This result, together with certain other previously-stated results, is then used to obtain derivations of the identity (49) satisfied by the alternative quadruply-refined ASM generating function at $x = y = 1$, and of the identity (75) satisfied by the function (70) at $x = y = 1$.

Let the crossing parameter q , for the weights (123), be given by $q = e^{\pm 2\pi i/3}$, this corresponding to the so-called combinatorial point of the six-vertex model. It can be shown that, for this assignment of q , the partition function (115), with weights given by (116) and (123), can be expressed as

$$Z_n^{6V}(u_1, \dots, u_n; u_{n+1}, \dots, u_{2n}) \Big|_{q=e^{\pm 2\pi i/3}} = (\pm 1)^n i^{n^2} 3^{n/2} u_1^{1/2} \dots u_{2n}^{1/2} s_{(n-1, n-1, \dots, 2, 2, 1, 1)}(u_1, \dots, u_{2n}), \quad (157)$$

where $s_{(n-1, n-1, \dots, 2, 2, 1, 1)}(u_1, \dots, u_{2n})$ is the Schur function indexed by the double-staircase partition $(n-1, n-1, \dots, 2, 2, 1, 1)$, evaluated at the spectral parameters u_1, \dots, u_{2n} .

An important consequence of (157) is that $Z_n^{6V}(u_1, \dots, u_n; u_{n+1}, \dots, u_{2n}) \Big|_{q=e^{\pm 2\pi i/3}}$ is symmetric in all spectral parameters u_1, \dots, u_{2n} .

The result (157) was first obtained by Okada [121, Thm. 2.4(1), second equation], and Stroganov [150, Eq. (17)]. For further related information, derivations and results, see, for example, Aval [7], Fonseca and Zinn-Justin [81, Sec. B.2], Lascoux [107, p. 4], Razumov and Stroganov [127, Sec. 2], or Zinn-Justin [161, Sec. 2.5.6].

Proceeding to the derivation of (49), using the symmetry of the LHS of (157) in all of its spectral parameters, it follows that

$$Z_n^{6V}(t_1, r, \dots, r, t_3; t_4, r, \dots, r, t_2) \Big|_{q=e^{\pm 2\pi i/3}} = Z_n^{6V}(t_1, t_2, t_3, t_4, r, \dots, r; r, \dots, r) \Big|_{q=e^{\pm 2\pi i/3}}. \quad (158)$$

Applying (122) and (151) (with $r = s = 1$, $m = 4$ and $k_i = i$) to the LHS and RHS, respectively, of (158), and then using (123) (with $q = e^{\pm 2\pi i/3}$) for the functions a , b and c (and noting that $a(1, 1) = b(1, 1)$ and $b(1, 1)^2 = c(1, 1)^2$), gives

$$\begin{aligned} & \left(\frac{b(1, t_2) b(1, t_4)}{b(t_2, 1) b(t_4, 1)} \right)^{n-1} Z_n^{\text{gen}} \left(1, 1; \frac{a(t_1, 1)}{b(t_1, 1)}, \frac{a(1, t_2)}{b(1, t_2)}, \frac{a(t_3, 1)}{b(t_3, 1)}, \frac{a(1, t_4)}{b(1, t_4)}; \right. \\ & \quad \left. \frac{a(1, 1) a(t_1, t_4)}{a(t_1, 1) a(1, t_4)}, \frac{b(1, 1) b(t_1, t_2)}{b(t_1, 1) b(1, t_2)}, \frac{a(1, 1) a(t_3, t_2)}{a(t_3, 1) a(1, t_2)}, \frac{b(1, 1) b(t_3, t_4)}{b(t_3, 1) b(1, t_4)} \right) \Big|_{q=e^{\pm 2\pi i/3}} = \\ & Z_n^{1,2,3,4} \left(1, 1; \frac{a(t_1, 1)}{b(t_1, 1)}, \frac{a(t_2, 1)}{b(t_2, 1)}, \frac{a(t_3, 1)}{b(t_3, 1)}, \frac{a(t_4, 1)}{b(t_4, 1)}; \left(\frac{c(t_1, 1)}{b(t_1, 1)} \right)^2, \left(\frac{c(t_2, 1)}{b(t_2, 1)} \right)^2, \left(\frac{c(t_3, 1)}{b(t_3, 1)} \right)^2, \left(\frac{c(t_4, 1)}{b(t_4, 1)} \right)^2 \right) \Big|_{q=e^{\pm 2\pi i/3}}. \quad (159) \end{aligned}$$

It follows from (159), by applying (156) to the RHS, and then setting $t_i = \frac{qz_i+1}{z_i+q}$, or equivalently $z_i = \frac{qt_i-1}{q-t_i} = \frac{a(t_i, 1)}{b(t_i, 1)}$, for each $1 \leq i \leq 4$, that

$$\begin{aligned} & (z_2 z_4)^{n-1} Z_n^{\text{gen}} \left(1, 1; z_1, \frac{1}{z_2}, z_3, \frac{1}{z_4}; 1 + \frac{(z_4-1)(z_1-1)}{z_1}, 1 + \frac{(z_1-1)(z_2-1)}{z_2}, 1 + \frac{(z_2-1)(z_3-1)}{z_3}, 1 + \frac{(z_3-1)(z_4-1)}{z_4} \right) \\ & \quad = X_n(1, 1; z_1, z_2, z_3, z_4). \quad (160) \end{aligned}$$

The required result (49) now follows from (160) by applying (106) to the LHS, and using the first definition of (10), the definition (70) and the identity (91).

Proceeding to the derivation of (75), setting $r = s = 1$, $q = e^{\pm 2\pi i/3}$ and $k_i = i$ in (152), and then applying (157) to the partition function $Z_n^{6V}(t_1, \dots, t_m, 1, \dots, 1; 1, \dots, 1)$, applying (151) (with $m = 0$ or $m = 1$) to the remaining cases of partition functions, and using (70) and (123), gives

$$\begin{aligned} & s_{(n-1, n-1, \dots, 2, 2, 1, 1)}(t_1, \dots, t_m, \underbrace{1, \dots, 1}_{2n-m}) = \\ & \quad 3^{n(n-1)/2} b(1, 1)^{-m(n-1)} (b(t_1, 1) \dots b(t_m, 1))^{n-1} X_n(1, 1; \frac{a(t_1, 1)}{b(t_1, 1)}, \dots, \frac{a(t_m, 1)}{b(t_m, 1)}) \Big|_{q=e^{\pm 2\pi i/3}}. \quad (161) \end{aligned}$$

(Alternatively, this can be obtained by combining (123), (151), (156) and (157).)

The required result (75) now follows from (161) by setting $t_i = \frac{qz_i+1}{z_i+q}$, or equivalently $z_i = \frac{qt_i-1}{q-t_i} = \frac{a(t_i, 1)}{b(t_i, 1)}$, for each $1 \leq i \leq m$.

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